

Intertemporal Substitution and Hyperbolic Discounting

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Abstract:

Evidence from behavioral experiments suggests that intertemporal preferences reflect a hyperbolic discount function. This paper shows that in contrast to exponential discounting, the elasticity of intertemporal substitution for hyperbolic consumers depends on the persistence of the change in the intertemporal relative price. In particular, lasting changes in the real interest rate are likely to generate a smaller degree of intertemporal substitution in consumption than temporary changes. This result holds for both sophisticated and naive hyperbolic consumers. It provides a novel testable implication of hyperbolic discounting and a new perspective on intertemporal substitution.

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Intertemporal Substitution and Hyperbolic Discounting¹

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1 Introduction

Temptations are often irresistible, even when they lead to unintended and with hindsight regrettable behavior. This inclination for immediate gratification reflects a bias in intertemporal preferences towards present rewards. Behavioral evidence indicates that intertemporal discount rates decline with the delay in rewards and are well-described by a hyperbolic discount function (e.g. Ainslie 1992). This paper builds on the seminal contributions by Laibson (1996, 1997) and shows that in contrast to exponential discounting, the elasticity of intertemporal substitution for hyperbolic consumers depends on the duration of the change in the intertemporal relative price. This holds for both sophisticated consumers, who realize that they have dynamically inconsistent preferences and rationally anticipate their future behavior, and naive consumers, who do not foresee their future self-control problems and corresponding present bias. The result provides a novel testable implication of hyperbolic discounting and is relevant for intertemporal substitution effects in micro-founded business cycle models and macroeconomic policy.

Intuitively, the intertemporal substitution of consumption depends on the difference between the real interest rate and the (effective)

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discount rate. With hyperbolic discounting the discount rate declines as the time horizon increases and the effective discount rate is a consumption-weighted average of the high short-run and the low long-run discount rate. For a short change in the interest rate, future intertemporal trade-offs are not affected so the effective discount rate remains constant. But a lasting interest rate change generally influences the effective discount rate, which alters the effect of the interest rate on intertemporal substitution. In particular, when the income effect dominates the substitution effect of a permanent increase in the real interest rate, the consumption rate rises, which increases the effective discount rate towards the higher, short-run discount rate. This partially offsets the increase in the real interest rate and diminishes the degree of intertemporal substitution.

The theoretical literature has identified several ways in which hyperbolic and exponential consumers differ. Laibson (1998) provides a useful overview. One interesting distinction is that hyperbolic discounting helps to explain the empirical anomaly that the elasticity of intertemporal substitution is less than the inverse of the coefficient of relative risk aversion. This was first shown by Laibson (1996) for a permanent change in the real interest rate with sophisticated consumers in discrete time. The present paper finds that this result extends to naive consumers and to continuous time. Moreover, it shows that in all these cases the result no longer holds for a short change in the interest rate. In other words, the degree of intertemporal substitution under hyperbolic discounting depends on the duration of the intertemporal price change. For plausible levels of risk aversion, the elasticity of intertemporal substitution is smaller for more persistent changes.

This implication of hyperbolic discounting already applies to the basic infinite-horizon model with one liquid asset and no financial market imperfections, for which the consumption behavior of hyperbolic and exponential agents is otherwise indistinguishable. But the argument is also relevant for more realistic ‘buffer-stock’

models. In addition, it provides a potential explanation for the wide range of estimates of the elasticity of intertemporal substitution in the empirical literature, including the possibility of negative elasticity values for naive hyperbolic consumers with a low degree of self control. All in all, the paper sheds new light on the importance of persistence for intertemporal substitution.

The remainder of this paper is organized as follows. Section 2 presents the basic discrete-time model with a (quasi-)hyperbolic discount function and sophisticated consumers. The effect of the persistence of the interest rate on intertemporal substitution is analyzed in section 3. The results are extended to naive consumers, a buffer-stock model and continuous time in section 4. Empirical and policy implications are addressed in section 5, which concludes.

2 Hyperbolic Discounting

Intertemporal discounting has been studied extensively in psychology. Experiments regarding human (and animal) behavior show that the rate of time preference depends on the time interval τ between the moment of choice and the actual events. Imminent outcomes are discounted at a higher rate than payoffs in the distant future. This was first described by Herrnstein's Matching Law and later refined to the generalized hyperbolic discount function $\phi_h(\tau) = (1 + \alpha\tau)^{-\gamma/\alpha}$ (see Ainslie 1992). The corresponding discount rate $\gamma/(1 + \alpha\tau)$ decreases in the delay τ , which is consistent with behavioral data. (e.g. Thaler 1981, Benzion, Rapoport and Yagil 1989). An axiomatic derivation of the generalized hyperbolic discount function is provided by Loewenstein and Prelec (1992).

With hyperbolic discounting intertemporal preferences feature a systematic bias towards immediate gratification. Intertemporal choices in the distant future are evaluated at a lower discount rate than immediate choices, which gives rise to dynamic inconsistency.

Since the currently optimal plan may no longer be optimal in the future, it is useful to model an individual as distinct ‘temporal selves’ who are each in control for one period. Generally, the optimal decision for the current self depends on the anticipated behavior of future selves. A ‘sophisticated’ person has rational expectations of future behavior, whereas a ‘naive’ person wrongly believes that future selves will act in the interest of the current self (Strotz 1956, Pollak 1968).

Laibson (1996) analyzes a standard consumption model with a ‘quasi-hyperbolic’ discount function that was first used by Phelps and Pollak (1968)

to model imperfect intergenerational altruism. In particular, it is assumed that each temporal self t maximizes life-time utility

$$U_t = u(C_t) + \beta \sum_{i=1}^{\infty} \delta^i u(C_{t+i}) \quad (1)$$

where $u(C)$ is the instantaneous utility from consumption C , β is the degree of self-control which reduces the ‘present bias’ in intertemporal preferences ($0 < \beta \leq 1$), and δ is the intertemporal discount factor ($0 < \delta \leq 1$). Note that the quasi-hyperbolic specification conveniently nests exponential discounting as the special case in which the present bias parameter $\beta = 1$.

The quasi-hyperbolic discount function in (1) mimics the hyperbolic shape of behavioral discount functions remarkably well. This is illustrated in figure 1. It shows the discount function constructed from Thaler’s (1981) survey data in which subjects were asked for certain payoffs at various delays that would be just as attractive as an immediate reward of \$250. In addition, figure 1 shows the conventional exponential discount function $\phi_e(\tau) = \delta_e^\tau$ and the quasi-hyperbolic specification $\phi_q(\tau) = \beta\delta^\tau$, where τ is time measured in years. The parameters, $\delta_e = 0.768$, $\beta = 0.867$ and $\delta = 0.831$, are estimated by nonlinear least squares using Thaler’s (1981) data.³

³To be precise, the data are from Thaler (1981), Table 2, Panel A, for the amount of \$250. The standard errors of the coefficient estimates are 0.044, 0.011 and 0.008,

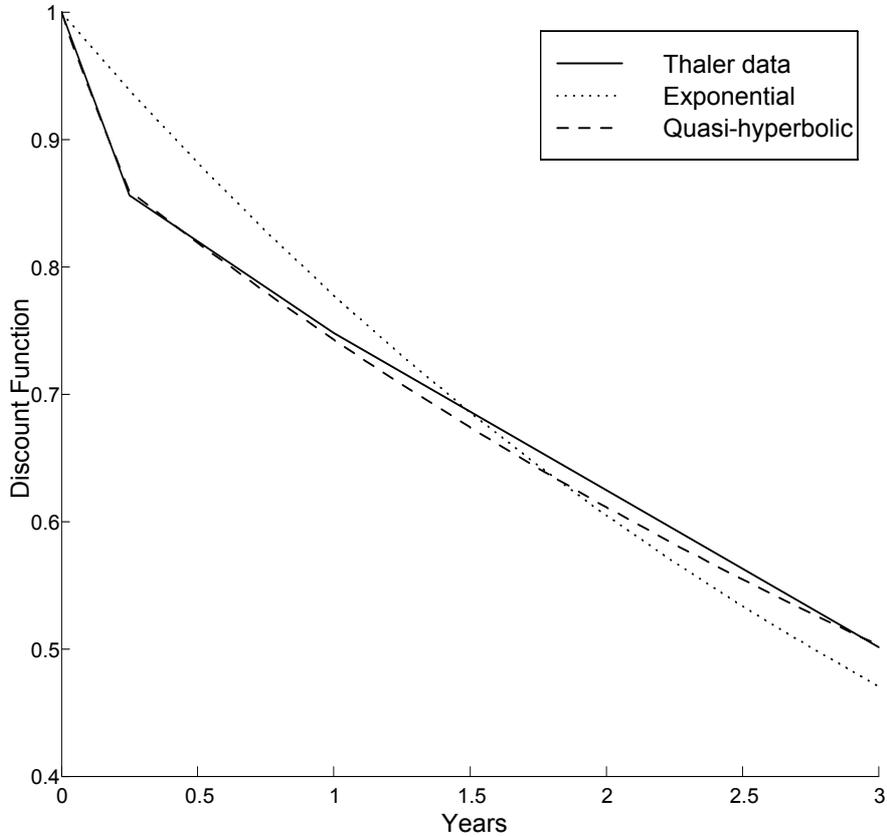


Figure 1: Discount functions estimated using Thaler (1981) data.

Following Laibson (1996), utility is assumed to be iso-elastic:

$$u(C) = \frac{C^{1-\rho} - 1}{1-\rho} \quad (2)$$

where ρ is the coefficient of relative risk aversion ($\rho > 0$). Each self s is endowed with life-time wealth W_s and is in control to choose the consumption level C_s . The subsequent period, self $s+1$ inherits the remaining wealth level

$$W_{s+1} = R(W_s - C_s) \quad (3)$$

respectively. Note that β is significantly different from 1, so the hypothesis of exponential discounting is rejected. This also holds for the other amounts in the Panel. Using all the Thaler (1981) data and applying more elaborate econometric techniques, Keller and Strazzera (2002) also reject the exponential discount function.

where R is the gross real interest rate. The consumer invests in one (liquid) asset and faces no credit market imperfections, so $0 \leq C_s \leq W_s$. Finally, it is assumed that each self s is sophisticated and rationally anticipates the behavior of future selves. Extensions to this basic model are discussed in section 4.

3 Intertemporal Substitution

First, the key finding by Laibson (1996) is discussed. Subsequently, it is shown that this result hinges on a lasting intertemporal price change, so the degree of intertemporal substitution depends on the persistence of the intertemporal price.

The intertemporal substitution of consumption is described by the intertemporal Euler equation for self s :⁴

$$u'(C_s) = R [\lambda\beta\delta + (1 - \lambda)\delta] u'(C_{s+1}) \quad (4)$$

This is the same as the Euler equation with exponential discounting except that the discount factor δ is replaced by the effective discount factor $\delta_H \equiv \lambda\beta\delta + (1 - \lambda)\delta$, where λ is the fraction of lifetime wealth consumed each period: $\lambda = C_s/W_s$. The standard exponential case is obtained for $\beta = 1$. The Euler equation shows that the intertemporal substitution of consumption depends on the real interest rate R and the effective discount rate δ_H . The latter is a weighted average of the short and long run discount factors $\beta\delta$ and δ , where the weights are the (next period) consumption rate and savings rate, respectively. The consumption rate λ satisfies

$$\lambda = 1 - (R^{1-\rho}\delta)^{1/\rho} [1 - (1 - \beta)\lambda]^{1/\rho} \quad (5)$$

This implicitly defines a unique λ , but typically no closed-form solution exists.⁵ For $\beta = 1$, the outcome under exponential

⁴The derivations for this section are in appendix A.1. Note that (4), (5) and (6) correspond to equations (11), (9) and (14) in Laibson (1996), respectively.

⁵An exception is logarithmic utility ($\rho = 1$), in which case $\lambda = \frac{1-\delta}{1-(1-\beta)\delta}$.

discounting emerges: $\lambda_E = 1 - (R^{1-\rho}\delta)^{1/\rho}$. Since hyperbolic discounters have a lower degree of self-control ($\beta < 1$), they consume at a higher rate than exponential consumers: $\lambda > \lambda_E$.

Suppose there is an unanticipated permanent change in the intertemporal relative price of current consumption, R . This affects the intertemporal consumption ratio C_{t+1}/C_t , which is described by the ‘permanent’ elasticity of intertemporal substitution

$$\bar{\sigma} \equiv \frac{\partial (C_{t+1}/C_t)}{\partial R} \frac{R}{C_{t+1}/C_t} = \frac{\partial \ln (C_{t+1}/C_t)}{\partial \ln R}$$

Using (4) and (2), it follows that

$$\frac{\partial \ln (C_{t+1}/C_t)}{\partial r} = \frac{1}{\rho} - \frac{1}{\rho} \frac{(1-\beta)}{1-(1-\beta)\lambda} \frac{\partial \lambda}{\partial r} \quad (6)$$

where $r \equiv \ln R$ denotes the continuously compounded real interest rate. Note that with exponential discounting ($\beta = 1$), the elasticity of intertemporal substitution equals the inverse of the coefficient of relative risk aversion ($\bar{\sigma}_E = 1/\rho$). But with hyperbolic discounting this typically no longer holds. Using (5) to compute $\partial \lambda / \partial r$ and simplifying yields the permanent elasticity of intertemporal substitution for sophisticated hyperbolic discounters:

$$\bar{\sigma}_S = \frac{1}{\rho + (\rho - 1)(1 - \lambda)(1 - \beta) / \beta} \quad (7)$$

In a hyperbolic economy ($\beta < 1$), $\bar{\sigma}_S < 1/\rho$ if $\rho > 1$.⁶ Intuitively, when the income effect dominates the substitution effect of a permanent increase in the real interest rate r , the consumption rate λ rises. This puts greater emphasis on the short-run discount factor $\beta\delta$, which reduces the effective discount factor δ_H . This reduction partially offsets the effect of the increase in the interest rate, thereby diminishing the degree of intertemporal substitution. But, when the substitution effect dominates ($\rho < 1$), the consumption rate declines after the interest rate rise, which increases the effective discount

⁶This is Proposition 5 in Laibson (1996).

factor and reinforces the effect of the interest rate on intertemporal substitution, so $\bar{\sigma}_S > 1/\rho$.

As a consequence, for hyperbolic consumers the elasticity of intertemporal substitution is generally not equal to the inverse of the coefficient of relative risk aversion. The difference could be quantitatively significant. This is illustrated by table 1, which shows $\bar{\sigma}_S$ for several values of ρ , δ and β , assuming $r = 0.04$. The column with $\beta = 1$ gives the exponential outcome $1/\rho$. However, the results above assume that the intertemporal price change is permanent and the literature has not analyzed the effect of temporary changes.

Suppose now there is an unanticipated one-period change in the gross real interest rate such that it equals R_t in period t and returns to the initial level R in all future periods ($t + 1, t + 2, \dots$). This means that the behavior of all future selves $s \in \{t + 1, t + 2, \dots\}$ is still described by $C_s = \lambda W_s$, (3), (4) and (5). But for self t , $W_{t+1} = R_t (W_t - C_t)$ so that the Euler equation (4) becomes

$$u'(C_t) = R_t [\lambda\beta\delta + (1 - \lambda)\delta] u'(C_{t+1}) \quad (8)$$

where λ is the consumption rate of future selves $s \in \{t + 1, t + 2, \dots\}$, which is independent of R_t so that $\partial\lambda/\partial r_t = 0$. As a result, in a hyperbolic economy with sophisticated consumers the elasticity of intertemporal substitution in response to a one-period change in r_t equals

$$\sigma_S = \frac{\partial(C_{t+1}/C_t)}{\partial R_t} \frac{R_t}{C_{t+1}/C_t} = \frac{1}{\rho} \quad (9)$$

This is identical to an exponential economy. The reason is that the intertemporal Euler equation is observationally equivalent for hyperbolic and exponential consumers, so a one-period change in the interest rate leads to the same degree of intertemporal substitution.

Comparing (7) and (9) it is clear that for hyperbolic consumers ($\beta < 1$), the elasticities of intertemporal substitution for permanent and one-period changes in the interest rate differ, except when

$\rho = 1$. More precisely, $\bar{\sigma}_S > \sigma_S > 1$ if $\rho < 1$ and $\bar{\sigma}_S < \sigma_S < 1$ if $\rho > 1$. Intuitively, the effective discount factor of hyperbolic consumers is not affected by a one-period change in the real interest rate but only by a lasting change. In particular, when the income effect dominates the substitution effect ($\rho > 1$), the future consumption rate λ rises in response to a permanent increase in the real interest rate r , which increases the weight on the lower short-run discount factor and reduces the effective discount factor δ_H , thereby decreasing the effect of the higher interest rate. As a result, the degree of intertemporal substitution for hyperbolic consumers is smaller for persistent changes in the interest rate, which is in contrast to an exponential economy. The effect could be significant, as is illustrated by table 1, where the column $\beta = 1$ provides the short run elasticity σ_S .

4 Discussion

This section discusses the robustness of the result that hyperbolic consumers exhibit an elasticity of intertemporal substitution that typically depends on the persistence of the interest rate. In particular, it is shown that this result also holds for naive consumers, imperfect credit markets and continuous time.

4.1 Naive Consumers

Consider the basic consumption model in section 2 with the quasi-hyperbolic discount function, one liquid asset and no credit market imperfections, but now suppose the consumer is naive and incorrectly believes that future selves will act in the interest of the current self. More precisely, each self t maximizes life-time utility U_t (1) and thinks that future selves $s \in \{t + 1, t + 2, \dots\}$ also maximize U_t (instead of U_s). Although the current self t knows that it is a hyperbolic discounter with an inclination for immediate gratification, it naively believes that future selves do not have

present-biased preferences but behave as exponential discounters ($\beta = 1$).

The consumption rate for naive hyperbolic discounters equals:⁷

$$\lambda_N = \frac{1 - (R^{1-\rho}\delta)^{1/\rho}}{1 - (1 - \beta^{1/\rho})(R^{1-\rho}\delta)^{1/\rho}} \quad (10)$$

In contrast to the sophisticated case, the naive consumption rate always has a closed-form solution.⁸ Setting $\beta = 1$ gives the consumption rate under exponential discounting: $\lambda_E = 1 - (R^{1-\rho}\delta)^{1/\rho}$, which is the *intended* future consumption rate of the naive hyperbolic consumer. However, future selves are also affected by the self-control problem, which gives rise to unanticipated present-biased preferences ($\beta < 1$) in every period and causes the naive hyperbolic discounter to consume more than intended ($\lambda_N > \lambda_E$).

The permanent elasticity of intertemporal substitution for naive hyperbolic discounters also differs from the exponential case:

$$\bar{\sigma}_N = \frac{1}{\rho} - \frac{\rho - 1}{\rho} \frac{\lambda_N - \lambda_E}{\lambda_E} \quad (11)$$

A hyperbolic economy with naive consumers features $\bar{\sigma}_N < 1/\rho$ if $\rho > 1$, like the sophisticated case. For $\beta = 1$, $\lambda_N = \lambda_E$ and the exponential result $\bar{\sigma}_E = 1/\rho$ emerges. The difference between $\bar{\sigma}_N$ and $\bar{\sigma}_E$ could be significant, as is illustrated in table 2. Comparing the results with table 1 suggests that $\bar{\sigma}_N \leq \bar{\sigma}_S$ for $\rho > 1$. Interestingly, naive hyperbolic discounters could even have a negative permanent elasticity of intertemporal substitution for $\rho > 1$ and β sufficiently small.⁹ For instance, for $\rho = 3$, $\beta = 0.2$, $\delta = 0.9$ and $r = 4\%$, $\bar{\sigma}_N = -0.093$. Intuitively, when the income effect dominates the substitution effect and the degree of self-control is small enough, an increase in the interest

⁷The derivation of the results for naive consumers is in appendix A.2

⁸In the case of logarithmic utility ($\rho = 1$), naive and sophisticated behavior coincide ($\lambda_N = \frac{1-\delta}{1-(1-\beta)\delta} = \lambda$), which is consistent with Pollak (1968).

⁹This follows from $\lim_{\beta \rightarrow 0} \bar{\sigma}_N = [1 - \rho(1 - \lambda_E)] / \rho\lambda_E$.

rate could raise current consumption so much that wealth drops and the intertemporal consumption ratio C_{t+1}/C_t actually declines. Regarding comparative statics, tables 1 and 2 suggest that for both sophisticates and naifs, $\bar{\sigma}$ tends to decrease in ρ , slightly decrease in δ and increase in β . In the case of logarithmic utility ($\rho = 1$), $\bar{\sigma}_N = \bar{\sigma}_S = \bar{\sigma}_E = 1$.

The one-period elasticity of intertemporal substitution for naive hyperbolic discounters is identical to the exponential outcome:

$$\sigma_N = \frac{1}{\rho} \quad (12)$$

Comparing (11) and (12) shows that again, the degree of intertemporal substitution depends on the persistence of the real interest rate. In particular, $\bar{\sigma}_N < \sigma_N < 1$ for $\rho > 1$, which means that shorter interest rate changes have a larger effect on intertemporal substitution.

4.2 Buffer-Stock Model

So far, the paper has considered a deterministic model in which consumers have access to perfect credit markets. In practice, income is stochastic and consumers face liquidity constraints. In particular, suppose that labor income Y_t is stochastic and that the consumer cannot borrow against uncertain future income so that $C_t \leq X_t$, where X_t is cash-on-hand in period t , which satisfies $X_t = R(X_{t-1} - C_{t-1}) + Y_t$. Harris and Laibson (2001) show that the hyperbolic Euler relation for sophisticated consumers in such a ‘buffer-stock’ model similar to Carroll (1997) equals:¹⁰

$$u'(c(X_t)) \geq E_t R [c'(X_{t+1}) \beta \delta + (1 - c'(X_{t+1}) \delta)] u'(c(X_{t+1})) \quad (13)$$

where $c(X_t)$ is the consumption function. For periods in which the liquidity constraint is non-binding so that $c(X_t) < X_t$, (13)

¹⁰To be precise, this is the ‘strong’ hyperbolic Euler relation formally derived by Harris and Laibson (2001) and it assumes that the consumption function $c(\cdot)$ is Lipschitz continuous, which holds in a neighborhood of $\beta = 1$.

holds with equality. This resembles the Euler equation (4), but the fraction of life-time wealth consumed λ is now replaced by the marginal propensity to consume out of cash-on-hand $c'(X_{t+1})$ because of the borrowing constraint.

Intertemporal substitution in response to a permanent change in the real interest rate is given by

$$\frac{\partial \ln(C_{t+1}/C_t)}{\partial r} = \frac{1}{\rho} - \frac{1}{\rho} \frac{(1-\beta)}{1 - (1-\beta)c'(X_{t+1})} \frac{\partial c'(X_{t+1})}{\partial r}$$

which is the buffer-stock equivalent of (6). For $\rho > 1$, the income effect dominates the substitution effect, so $\partial c'(X_{t+1})/\partial r > 0$ and $\bar{\sigma}_S < 1/\rho$ (Laibson 1998, p. 867). Following the same approach as in section 3, (13) can be used to find that $\sigma_S = 1/\rho$ whenever the consumer is not liquidity constrained. As a result, the conclusions of section 3 hold more generally.

4.3 Continuous-Time Model

As a further robustness check, consider a continuous-time version of the basic model in section 2. Harris and Laibson (2004) have adapted the quasi-hyperbolic discount function to continuous time.¹¹ They assume that time can be divided into the ‘present’ and the ‘future’, which arrives with a stochastic hazard rate. The discount function is exponential and the additional present bias factor β applies to future utility flows. As the hazard rate increases, the model converges to an ‘instantaneous gratification’ model in which the present is infinitesimally short.

To complete the description of the continuous-time version of the basic model, the change in life-time wealth $W(t)$ is given by

$$\dot{W} = rW(t) - C(t)$$

¹¹Barro (1999) and Luttmmer and Mariotti (2003) present alternative approaches to modeling hyperbolic discounting in continuous time, but the model by Harris and Laibson (2004) provides greater analytical tractability, with closed-form solutions for iso-elastic utility.

where $\dot{W} \equiv dW(t)/dt$. Finally, following Harris and Laibson (2004) it is assumed that $\beta > 1 - \rho$ and $\gamma > (1 - \rho)r$. These conditions are satisfied for plausible parameter values for ρ , β , γ and r .

The elasticity of intertemporal substitution in response to an unanticipated permanent change in the real interest rate r equals¹²

$$\bar{\sigma}_c \equiv \frac{d\dot{C}/C}{dr} = \frac{1}{\rho + (\rho - 1)(1 - \beta)/\beta} \quad (14)$$

For $\beta = 1$ the familiar exponential result $\bar{\sigma}_E = 1/\rho$ emerges. With hyperbolic discounting ($\beta < 1$), $\bar{\sigma}_c < 1/\rho$ for $\rho > 1$, just like in the discrete-time model with sophisticated consumers. In fact, comparing (7) to (14) shows that $\bar{\sigma}_c$ is very similar to $\bar{\sigma}_S$. Although the deviation from the exponential outcome $\bar{\sigma}_E$ is larger in continuous time (to be precise, $|\bar{\sigma}_c - 1/\rho| > |\bar{\sigma}_S - 1/\rho|$), the quantitative difference with the discrete-time results tends to be small, as is clear from a numerical comparison of $\bar{\sigma}_S$ and $\bar{\sigma}_c$ in table 1 and 3, respectively. For logarithmic utility ($\rho = 1$), $\bar{\sigma}_c = \bar{\sigma}_S = \bar{\sigma}_E = 1$.

The ‘instantaneous’ elasticity of intertemporal substitution in response to an unanticipated, infinitesimally short change in the real interest rate $r(t)$ from r to r_s is again equal to the exponential elasticity:

$$\sigma_c \equiv \frac{d\dot{C}/C}{dr_s} = \frac{1}{\rho} \quad (15)$$

As a result, in the continuous-time hyperbolic model the elasticity of intertemporal substitution also generally depends on the duration of the intertemporal price change. In particular, for the empirically likely case in which $\rho > 1$, the degree of intertemporal substitution is smaller for more persistent changes.

¹²The results for the continuous-time model are derived in appendix A.3.

5 Conclusion

Intertemporal substitution tends to be a key mechanism in micro-founded business cycle models. This paper shows that intertemporal substitution for consumers with a hyperbolic discount function differs significantly from the behavior of exponential consumers. With an exponential discount function, the elasticity of intertemporal substitution is independent of the duration of the change in the intertemporal price ratio. In contrast, for hyperbolic discounters, the intertemporal substitution elasticity typically depends on the persistence of the intertemporal price change.

For a short change in the real interest rate, the elasticity of intertemporal substitution with iso-elastic utility equals the inverse of the coefficient of relative risk aversion for both exponential and hyperbolic discounters. Essentially, this is the structural preference parameter that measures the curvature of the intertemporal indifference curves. However, for a lasting change in the interest rate, the degree of intertemporal substitution is generally different for hyperbolic consumers because the effective discount rate is affected. The reason is that a persistent interest rate change typically influences the future consumption rate, which shifts the weight between the high short-run and the low long-run hyperbolic discount rate. This adjustment in the effective discount rate alters the effect of a lasting interest rate change on intertemporal substitution. For plausible values of risk aversion, the elasticity of intertemporal substitution for hyperbolic consumers is smaller when the change in the real interest rate is more persistent.

These results hold both for sophisticated hyperbolic discounters, who rationally anticipate the dynamic inconsistency of their preferences, and for naive consumers, who do not realize that the ‘present bias’ in their intertemporal preferences continues to exert itself in the future. In addition, the result is shown for the standard quasi-hyperbolic discrete-time model and for the continuous-time ‘instantaneous gratification’ model. The finding that hyper-

hyperbolic discounters display a different degree of intertemporal substitution than exponential consumers already holds for a basic model with a single liquid asset and perfect credit markets. So, it does not rely on the presence of (partial) commitment devices, such as illiquid assets, that is usually required to distinguish (sophisticated) hyperbolic from exponential consumers. The result is also relevant in more realistic ‘buffer-stock’ models that feature stochastic income and liquidity constraints. Although the focus of the paper is on the intertemporal consumption decision, a similar argument applies to the intertemporal substitution of leisure. The sensitivity of the elasticity of intertemporal substitution to the duration of the intertemporal price change appears to be a robust feature of hyperbolic discounting and provides a novel testable implication.

There is a large empirical literature on intertemporal substitution, including Mankiw, Rotemberg and Summers (1985), Hall (1988), Attanasio and Weber (1995) and Mulligan (2002). Such empirical studies have obtained a remarkably wide range of estimates for the elasticity of intertemporal substitution, with a typical parameter value of about 0.3. Although a large variety of parameter estimates would not be expected with exponential discounting, under hyperbolic discounting it is natural to get different estimates depending on the persistence of the interest rate in the sample. In fact, it is not unusual to find empirical elasticity estimates that are negative, which is difficult to reconcile with the standard model of exponential discounting. But interestingly, naive hyperbolic consumers with plausible risk aversion and a sufficiently low degree of self control could exhibit a negative elasticity of intertemporal substitution for a lasting interest rate change. Thus, hyperbolic discounting could explain empirical findings on intertemporal substitution that are puzzling under exponential discounting.

The fact that the degree of intertemporal substitution for hyperbolic agents tends to depend on the duration of intertemporal price changes could have important implications for the understanding of

business cycle fluctuations and the formulation of macroeconomic policy. For instance, it suggests that shorter changes that affect the intertemporal price ratio are likely to be more important for intertemporal substitution, and that permanent policy measures (e.g. adjustments to tax rates) could have significantly smaller intertemporal effects than predicted by models with exponential discounting or by empirical estimates based on temporary policies. It would be an interesting avenue for future research to explore these issues further and to pursue the testable implication suggested by this paper, leading to a better understanding of intertemporal substitution and the empirical relevance of hyperbolic discounting.

A Appendix

This appendix contains the derivation of the basic hyperbolic model with sophisticated consumers presented in section 2. In addition, it derives the results for naive hyperbolic consumers, presented in section 4.1, and the continuous-time version of the model, discussed in section 4.3.

A.1 Sophisticated Consumers

This section provides a derivation of the equations in section 2. It largely follows Laibson (1996), with the exception of the derivation of σ_S which is new.

Derivation of (4):

Each self s faces the same infinite-horizon optimization problem without credit market imperfections. Let λ denote the fraction of life-time wealth W_s that is consumed by self s , so that $C_s = \lambda W_s$. It is shown below that λ is constant for iso-elastic utility (2). Using (1), the optimal life-time utility of self s can be written as

$$U_s = u(C_s) + \beta\delta V(W_{s+1}) \quad (16)$$

where

$$V(W_{s+1}) = \sum_{i=s}^{\infty} \delta^{i-s} u(\lambda W_{i+1})$$

Using (3), the continuation-value function satisfies

$$V(W_{s+1}) = u(\lambda W_{s+1}) + \delta V(R(1-\lambda)W_{s+1}) \quad (17)$$

Maximizing (16) with respect to C_s subject to (3) yields the first order condition for each self s :

$$u'(C_s) = R\beta\delta V'(W_{s+1}) \quad (18)$$

Differentiate (17) and substitute for $V'(W_{s+2})$ using (18) to get

$$\begin{aligned} u'(C_s) &= R\beta\delta [\lambda u'(C_{s+1}) + R(1-\lambda)\delta V'(W_{s+2})] \\ &= R[\lambda\beta\delta + (1-\lambda)\delta] u'(C_{s+1}) \end{aligned}$$

This is the quasi-hyperbolic intertemporal Euler equation (4) and it corresponds to equation (11) in Laibson (1996).

Derivation of (5):

Using (2), the Euler equation (4) can be written as

$$C_{s+1}/C_s = (R\delta)^{1/\rho} [1 - (1 - \beta)\lambda]^{1/\rho} \quad (19)$$

Using $C_s = \lambda W_s$ and (3), $C_{s+1}/C_s = R(1 - \lambda)$. Substituting into (19) and rearranging yields:

$$\lambda = 1 - (R^{1-\rho}\delta)^{1/\rho} [1 - (1 - \beta)\lambda]^{1/\rho}$$

This is condition (5) for the optimal consumption rate and it corresponds to equation (9) in Laibson (1996).

Derivation of (6):

Rearranging (19) and taking logs gives

$$\ln(C_{t+1}/C_t) = \frac{1}{\rho} \{ \ln R + \ln \delta + \ln [1 - (1 - \beta)\lambda] \}$$

Differentiating yields (6):

$$\bar{\sigma}_S = \frac{\partial \ln(C_{t+1}/C_t)}{\partial r} = \frac{1}{\rho} - \frac{1}{\rho} \frac{(1 - \beta)}{1 - (1 - \beta)\lambda} \frac{\partial \lambda}{\partial r}$$

which corresponds to equation (14) in Laibson (1996).

Derivation of (7):

Totally differentiating (5), using $R = e^r$ and simplifying gives

$$\partial \lambda = -\frac{1 - \rho}{\rho} (1 - \lambda) \partial r + \frac{1}{\rho} (1 - \beta) \frac{1 - \lambda}{1 - (1 - \beta)\lambda} \partial \lambda$$

Rearranging produces

$$\frac{\partial \lambda}{\partial r} = \frac{-(1 - \rho)(1 - \lambda)[1 - (1 - \beta)\lambda]}{\rho[1 - (1 - \beta)\lambda] - (1 - \beta)(1 - \lambda)}$$

Substituting into (6) yields

$$\bar{\sigma} = \frac{1}{\rho} - \frac{1}{\rho} \frac{(1 - \beta)(\rho - 1)(1 - \lambda)}{\rho[1 - (1 - \beta)\lambda] - (1 - \beta)(1 - \lambda)}$$

which corresponds to equation (15) in Laibson (1996). Further simplifying gives (7):

$$\bar{\sigma}_S = \frac{\beta}{\rho\beta + \rho(1-\beta) - \rho(1-\beta)\lambda - (1-\beta)(1-\lambda)} = \frac{1}{\rho + (\rho-1)(1-\lambda)(1-\beta)/\beta}$$

Derivation of (9):

Now suppose that the gross real interest rate equals R_t in period t and R in all future periods ($t+1, t+2, \dots$). This means that $C_s = \lambda W_s$, (3), (4) and (5) still hold for all future selves $s \in \{t+1, t+2, \dots\}$. However, in period t , $W_{t+1} = R_t(W_t - C_t)$. Retracing the derivations above shows that (18) becomes $u'(C_s) = R_t\beta\delta V'(W_{s+1})$, so the Euler equation for self t becomes

$$u'(C_t) = R_t[\lambda\beta\delta + (1-\lambda)\delta]u'(C_{t+1})$$

Note that λ is the consumption rate for future selves $s \in \{t+1, t+2, \dots\}$, which is still implicitly defined by (5) and independent of R_t . This means that $\partial\lambda/\partial r_t = 0$. Using (2) and taking logs,

$$\ln(C_{t+1}/C_t) = \frac{1}{\rho} \{\ln R_t + \ln \delta + \ln [1 - (1-\beta)\lambda]\}$$

Differentiating and using $r_t \equiv \ln R_t$ yields (9):

$$\sigma_S \equiv \frac{\partial \ln(C_{t+1}/C_t)}{\partial r_t} = \frac{1}{\rho}$$

A.2 Naive Consumers

This section derives the results for the basic hyperbolic model when consumers are naive. In particular, each naive self t maximizes U_t in (1) believing that future selves $s \in \{t+1, t+2, \dots\}$ are exponential discounters that maximize U_s with $\beta = 1$. Since it seems that an analysis of the basic consumption model with naive hyperbolic discounters has not appeared elsewhere, the finite-horizon case is considered first, although it is not required for the derivation of the infinite-horizon results presented in section 4.1.

A.2.1 Finite Horizon

Suppose the consumer is born at time $t = 0$ and lives until $t = T$. The current self t sets C_t to maximize $U_t = u(C_t) + \beta\delta V(W_{t+1}; T - t)$ subject to (3), where $V(W_{t+1}; T - t)$ is the continuation-value function for self t :

$$\begin{aligned} V(W_{t+1}; T - t) &\equiv \max_{\{C_{t+i}\}_{i=1}^{T-t}} \sum_{i=1}^{T-t} \delta^{i-1} u(C_{t+i}) \\ &= \max_{C_{t+1}} [u(C_{t+1}) + \delta V(W_{t+2}; T - t - 1)] \end{aligned}$$

However, the naive current self t thinks that the optimization problem for future selves $t + i$ ($i = 1, \dots, T - t$) is different and that they will set C_{t+i} to obtain $V(W_{t+1}; T - t)$. So, the first order conditions for present and *intended* future consumption are

$$\begin{aligned} u'(C_t) &= R\beta\delta V'(W_{t+1}; T - t) \\ u'(C_{t+i}) &= R\delta V'(W_{t+i+1}; T - t - i) \quad \text{for } i = 1, 2, \dots, T - t - 1 \end{aligned}$$

Differentiating the continuation-value function and using the first order condition for $t + i$,

$$\begin{aligned} V'(W_{t+i}; T - t - i + 1) &= u'(C_{t+i}) \frac{\partial C_{t+i}}{\partial W_{t+i}} \\ &\quad + R\delta V'(W_{t+i+1}; T - t - i) \left(1 - \frac{\partial C_{t+i}}{\partial W_{t+i}}\right) \\ &= u'(C_{t+i}) \end{aligned}$$

This is in fact the envelope theorem. Thus, the first order conditions for intended consumption reduce to

$$u'(C_t) = R\beta\delta u'(C_{t+1}) \tag{20}$$

$$u'(C_{t+i}) = R\delta u'(C_{t+i+1}) \tag{21}$$

So, for future periods $t + i$, the naive consumer intends to behave as an exponential discounter. The naive beliefs about future consumption can be obtained recursively. Let $\tilde{\lambda}_{T-\tau} = C_\tau/W_\tau$ denote the intended future consumption rate. In the final period,

$C_T = W_T = R(W_{T-1} - C_{T-1})$ so that $\tilde{\lambda}_0 = \frac{C_T}{W_T} = 1$. Substituting $C_{\tau+1} = \tilde{\lambda}_{T-\tau-1}W_{\tau+1}$ and (3) into (21) and using (2), $C_\tau = (\delta R^{1-\rho})^{-\frac{1}{\rho}} \tilde{\lambda}_{T-\tau-1} (W_\tau - C_\tau)$ which implies that $C_\tau = \frac{\tilde{\lambda}_{T-\tau-1}}{(\delta R^{1-\rho})^{1/\rho} + \tilde{\lambda}_{T-\tau-1}} W_\tau$. Thus, the intended consumption rates $\tilde{\lambda}_{T-\tau}$ for $\tau = t, t+1, \dots, T-1$ can be obtained using the recursion formula

$$\tilde{\lambda}_{T-\tau} = \frac{\tilde{\lambda}_{T-\tau-1}}{(\delta R^{1-\rho})^{1/\rho} + \tilde{\lambda}_{T-\tau-1}} \quad (22)$$

But in the current period t the present-bias in intertemporal preferences prevails. Substituting intended future consumption $C_{t+1} = \tilde{\lambda}_{T-t-1}W_{t+1}$ and (3) into (20) and using (2), $C_t = (\beta\delta R^{1-\rho})^{-\frac{1}{\rho}} \tilde{\lambda}_{T-t-1} (W_t - C_t)$ which implies that $C_t = \frac{\tilde{\lambda}_{T-t-1}}{(\beta\delta R^{1-\rho})^{1/\rho} + \tilde{\lambda}_{T-t-1}} W_t$. Therefore, the consumption rate of the naive hyperbolic consumer equals

$$\lambda_{T-t} = \frac{\tilde{\lambda}_{T-t-1}}{(\beta\delta R^{1-\rho})^{1/\rho} + \tilde{\lambda}_{T-t-1}} \quad (23)$$

where the sequence $\{\tilde{\lambda}_{T-\tau}\}_{\tau=t}^{T-1}$ is given by the recursion (22) with $\tilde{\lambda}_0 = 1$ and $\lambda_0 = 1$. Note that for $\beta = 1$, (23) reduces to (22), which is the recursion formula for exponential discounting. A lower degree of self-control β yields a higher consumption rate λ_{T-t} for every period t . As a result, the naive consumer is running down life-time wealth faster and has less wealth remaining to consume at the end of his life-time than an exponential consumer.

The consumption profile of naive hyperbolic discounters depends on the parameters δ , β , R and ρ .¹³ A higher discount factor δ tends to give a consumption profile that is initially more upward (or less downward) sloping. Intuitively, a higher δ means that the consumer is more patient and values future consumption relatively more. Similarly, a higher degree of self control β tends to increase the initial slope of the hyperbolic consumption profile and make the

¹³ The properties for sophisticated hyperbolic consumers are very similar.

difference between hyperbolic and exponential consumers smaller. A higher gross real interest rate R results in a more upward sloping profile because the relative price of future consumption is lower. For $\rho = 1$, the income and substitution effects offset each other and the consumption rate λ_i is independent of the interest rate R . It is straightforward to verify that $\tilde{\lambda}_i = 1 / \sum_{k=0}^i \delta^k$ and the naive hyperbolic consumption rate equals $\lambda_i = 1 / (1 + \beta \delta \sum_{k=0}^{i-1} \delta^k)$ in that case.

Interestingly, hyperbolic discounting is able to generate hump-shaped consumption profiles similar to those observed empirically, even in the basic hyperbolic model without liquidity constraints, ‘buffer stock’ behavior or changes in demographics and labor supply. Intuitively, when the interest rate R is sufficiently high to generate an initially upward sloping consumption profile, hyperbolic consumers spend so much of their wealth due to their lack of self control that they have to reduce consumption when they get old.

A.2.2 Infinite Horizon

Derivation of (10):

The naive hyperbolic consumer maximizes (16), where $V(W_{s+1})$ is now the anticipated continuation-value function for future selves, which are believed to be exponential discounters without present-biased preferences. Substituting $\beta = 1$ into (5) gives the anticipated consumption rate for future selves

$$\lambda_E = 1 - (R^{1-\rho} \delta)^{1/\rho} \quad (24)$$

which corresponds to the exponential outcome. So, the anticipated continuation-value function satisfies

$$V(W_{s+1}) = u(\lambda_E W_{s+1}) + \delta V(R(1 - \lambda_E) W_{s+1}) \quad (25)$$

The first order condition for the current self $s = t$ is still given by (18). However, for future selves $s \in \{t + 1, t + 2, \dots\}$, which are believed to be exponential discounters with $\beta = 1$, the anticipated

first order condition is

$$u'(C_s) = R\delta V'(W_{s+1}) \quad (26)$$

Differentiating (25) and substituting for $V'(W_{t+2})$ using (26), (18) yields

$$\begin{aligned} u'(C_t) &= R\beta\delta [\lambda_E u'(C_{t+1}) + R(1 - \lambda_E) \delta V'(W_{t+2})] \\ &= R\beta\delta u'(C_{t+1}) \end{aligned} \quad (27)$$

Using (2) and substituting $C_{t+1} = \lambda_E W_{t+1}$ and (3), (27) gives

$$C_t = (R\beta\delta)^{-1/\rho} \lambda_E R (W_t - C_t)$$

Substituting (24) and solving for C_t gives

$$C_t = \frac{1 - (R^{1-\rho}\delta)^{1/\rho}}{1 - (1 - \beta^{1/\rho})(R^{1-\rho}\delta)^{1/\rho}} W_t$$

As a result, the naive hyperbolic consumption rate equals (10):

$$\lambda_N = \frac{1 - (R^{1-\rho}\delta)^{1/\rho}}{1 - (1 - \beta^{1/\rho})(R^{1-\rho}\delta)^{1/\rho}}$$

Comparing this to (24) shows that the current consumption rate λ_N exceeds the intended future consumption rate λ_E of the naive hyperbolic discounter ($\beta < 1$). But when the future arrives, the naive consumer faces the same self-control problem and chooses the consumption rate λ_N again, which is higher than intended. So, (10) describes the actual consumption rate of naive hyperbolic discounters in every period.

Note that (10) corresponds to the limiting case of the finite horizon outcome as $T \rightarrow \infty$. In particular, $\lim_{T \rightarrow \infty} \tilde{\lambda}_{T-t} = 1 - (R^{1-\rho}\delta)^{1/\rho}$ using (22), so λ_{T-t} in (23) converges to λ_N .

Derivation of (11):

Since the naive intertemporal Euler equation (27) only describes

intended behavior, an alternative approach is required to compute the *actual* consumption ratio C_{t+1}/C_t . The consumption rate of the naive consumer is the same in every period, so $C_{t+1}/C_t = W_{t+1}/W_t = R(1 - \lambda_N)$, using (3). Differentiating yields

$$\bar{\sigma}_N = \frac{\partial (C_{t+1}/C_t)}{\partial R} \frac{R}{C_{t+1}/C_t} = 1 - \frac{R}{1 - \lambda_N} \frac{\partial \lambda_N}{\partial R}$$

Differentiating (10) and simplifying gives

$$\frac{\partial \lambda_N}{\partial R} = \frac{\rho - 1}{\rho} \frac{(R^{1-\rho} \beta \delta)^{1/\rho}}{\left(1 - \left(1 - \beta^{1/\rho}\right) (R^{1-\rho} \delta)^{1/\rho}\right)^2} \frac{1}{R} = \frac{\rho - 1}{\rho} \frac{\lambda_N}{\lambda_E} \frac{1 - \lambda_N}{R}$$

Substituting for $\partial \lambda_N / \partial R$ and rearranging produces (11):

$$\bar{\sigma}_N = \frac{1}{\rho} - \frac{\rho - 1}{\rho} \frac{\lambda_N - \lambda_E}{\lambda_E}$$

Derivation of (12):

Now suppose that the gross real interest rate equals R_t in period t and R in all future periods ($t + 1, t + 2, \dots$). This means that $C_s = \lambda_E W_s$, (24), (3) and (26) still hold for all future selves $s \in \{t + 1, t + 2, \dots\}$. However, in period t , $W_{t+1} = R_t (W_t - C_t)$. Retracing the derivations above shows that (18) becomes $u'(C_t) = R_t \beta \delta V'(W_{t+1})$, so the naive *intended* intertemporal Euler equation equals

$$u'(C_t) = R_t \beta \delta u'(C_{t+1})$$

Using (2) and substituting $C_{t+1} = \lambda_E W_{t+1}$ and $W_{t+1} = R_t (W_t - C_t)$, gives

$$C_t = (R_t \beta \delta)^{-1/\rho} \lambda_E R_t (W_t - C_t)$$

Solving for C_t gives

$$C_t = \frac{\lambda_E (R_t^{1-\rho} \beta \delta)^{-1/\rho}}{1 + \lambda_E (R_t^{1-\rho} \beta \delta)^{-1/\rho}} W_t$$

Substituting this into $W_{t+1} = R_t (W_t - C_t)$ and $C_{t+1} = \lambda_E W_{t+1}$

$$C_{t+1} = \frac{\lambda_E R_t}{1 + \lambda_E (R_t^{1-\rho} \beta \delta)^{-1/\rho}} W_t$$

As a consequence,

$$\frac{C_{t+1}}{C_t} = (R_t \beta \delta)^{1/\rho}$$

Hence, the one-period elasticity of intertemporal substitution for naive hyperbolic discounters equals (12):

$$\sigma_N = \frac{\partial (C_{t+1}/C_t)}{\partial R_t} \frac{R_t}{C_{t+1}/C_t} = \frac{1}{\rho}$$

A.3 Continuous Time

This section derives the results for the continuous-time version of the basic hyperbolic model. First, there is a heuristic derivation of the optimality condition. For a rigorous derivation, see Harris and Laibson (2004) who consider a more general model with labor income, liquidity constraints and stochastic asset returns. Subsequently, the elasticities of intertemporal substitution $\bar{\sigma}_c$ and σ_c are derived.

Derivation of optimality condition:

Suppose first that each period lasts dt . The continuous time model is the limiting case $dt \rightarrow 0$. The change in life-time wealth $W(s)$ for self s is given by (cf (3) where $R = e^r$)

$$dW = (rW(s) - C(s)) dt \quad (28)$$

The optimal life-time utility can be written as the sum of the present utility flow and the value of future utility flows discounted by the long run discount rate γ and the present-biased discount factor β (cf (16) where $\delta = e^{-\gamma}$):

$$U(s) = \int u(C(s)) dt + \beta e^{-\gamma dt} V(W(s) + dW) \quad (29)$$

where the continuation-value function equals

$$V(W(s)) = \int_s^\infty e^{-\gamma(t-s)} u(C(t)) dt \quad (30)$$

Maximizing (29) with respect to $C(s)$ subject to (28) yields the first order condition

$$u'(C) dt = \beta e^{-\gamma dt} V'(W + dW) dt$$

Simplifying and taking the limit $dt \rightarrow 0$ yields the optimality condition for the instantaneous gratification model:

$$u'(C) = \beta V'(W) \quad (31)$$

This corresponds to equation (15) in Harris and Laibson (2004) for the case in which consumers are not liquidity constrained.

Derivation of (14):

Each self s faces the same infinite-horizon optimization problem without credit market imperfections. Let λ denote the fraction of life-time wealth $W(s)$ that is consumed by self s . It is shown below that λ is constant for iso-elastic utility (2). Using (28) and $C(t) = \lambda W(t)$, life-time wealth is described by

$$W(t) = W(s) e^{-(\lambda-r)(t-s)}$$

Substituting into (30) and differentiating with respect to $W(s)$ gives

$$V'(W(s)) = \lambda \int_s^\infty e^{-(\gamma+\lambda-r)(t-s)} u'(C(t)) dt$$

So, the optimality condition (31) becomes

$$u'(C(s)) = \beta \lambda \int_s^\infty e^{-(\gamma+\lambda-r)(t-s)} u'(C(t)) dt$$

Differentiating with respect to s yields

$$\begin{aligned} u''(C(s)) \frac{dC}{ds} &= -\beta \lambda u'(C(s)) + (\gamma + \lambda - r) \beta \lambda \int_s^\infty e^{-(\gamma+\lambda-r)(t-s)} u'(C(t)) dt \\ &= -[\beta \lambda - (\gamma + \lambda - r)] u'(C(s)) \end{aligned}$$

Rearrange, assuming iso-elastic utility (2) with coefficient of relative risk aversion $-\frac{u''(C)C}{u'(C)} = \rho$, to get

$$\frac{\dot{C}}{C} = \frac{1}{\rho} [r - \gamma - (1 - \beta) \lambda] \quad (32)$$

where $\dot{C} \equiv dC(s)/ds$. Using $C = \lambda W$ and $\dot{W} = (r - \lambda) W$ gives $\dot{C}/C = r - \lambda$. Substituting into (32) and solving for λ yields¹⁴

$$\lambda = \frac{(\rho - 1)r + \gamma}{\rho - (1 - \beta)} \quad (33)$$

The consumption rate is increasing in the long run discount rate γ and decreasing in the degree of self control β . Note that $\beta = 1$ gives the exponential result $\lambda = \left(1 - \frac{1}{\rho}\right)r + \frac{1}{\rho}\gamma$. For log utility ($\rho = 1$), $\lambda = \gamma/\beta$ and the consumption rate is independent of the interest rate because of the offsetting income and substitution effects.¹⁵ Substitute (33) into (32) and simplify to get the growth rate of consumption:

$$\frac{\dot{C}}{C} = \frac{\beta r - \gamma}{\rho - (1 - \beta)} \equiv g \quad (34)$$

Note that $\beta = 1$ yields the exponential outcome $g_E = \frac{1}{\rho}(r - \gamma)$. Differentiating (34) with respect to r and further rearranging produces (14):

$$\bar{\sigma}_c = \frac{d\dot{C}/C}{dr} = \frac{\beta}{\rho - (1 - \beta)} = \frac{1}{\rho + (\rho - 1)(1 - \beta)/\beta}$$

Derivation of (15):

First, consider a temporary change in the real interest rate r such

¹⁴ Alternatively, the intertemporal budget constraint $W(s) = \int_s^\infty e^{-r(t-s)} C(t) dt = \int_s^\infty e^{-(r-g)(t-s)} C(s) dt = \frac{1}{r-g} C(s)$ could be used to get $\lambda = r - g$, which also yields (33). Note that the condition for life-time wealth $W(s)$ to be bounded is $r > g$, or equivalently $\lambda = \frac{\gamma - (1-\rho)r}{\beta - (1-\rho)} > 0$, which holds because of the assumptions that $\gamma > (1 - \rho)r$ and $\beta > 1 - \rho$.

¹⁵This is consistent with Barro (1999) who finds that $\lambda \approx \gamma/\beta$ for log utility using a different solution approach relying on several approximations.

that $r(t) = r_s$ for $t \in [s, \tau]$, and $r(t) = r$ for $t > \tau$. The computation of the instantaneous elasticity of intertemporal substitution σ_c is based on the limiting case $\tau \rightarrow s$. Note that $C(s) = \lambda W(s)$, (34) and (33) still hold for all selves $t > \tau$. However, during $t \in [s, \tau]$, $dW = (r_s W(t) - C(t)) dt$. Using $C(t) = \lambda(t) W(t)$ and $W(t) = W(s) e^{-(\lambda(t)-r(t))(t-s)}$, the optimality condition becomes

$$u'(C(s)) = \beta \int_s^\infty e^{-(\gamma+\lambda(t)-r(t))(t-s)} \lambda(t) u'(C(t)) dt$$

Differentiating with respect to s , rearranging and assuming iso-elastic utility (2) produces

$$\frac{\dot{C}}{C} = \frac{1}{\rho} [r(s) - \gamma - (1 - \beta) \lambda(s)] \quad (35)$$

To compute $\lambda(s)$, substitute $C(t) = C(s) e^{\int_s^t g(v) dv}$ into the intertemporal budget constraint:

$$\begin{aligned} W(s) &= \int_s^\infty e^{-\int_s^t r(v) dv} C(t) dt = \int_s^\infty e^{-\int_s^t (r(v)-g(v)) dv} C(s) dt \\ &= \left[\int_s^\tau e^{-\int_s^t (r_s-g(v)) dv} dt + e^{-\int_s^\tau (r_s-g(v)) dv} \int_\tau^\infty e^{-(r-g)(t-\tau)} dt \right] C(s) \end{aligned}$$

Taking the limit as $\tau \rightarrow s$ gives the consumption rate $\lambda(s)$ for an infinitesimally short change in the interest rate:

$$\lambda(s) = C(s) / W(s) = \frac{1}{\int_\tau^\infty e^{-(r-g)(t-\tau)} dt} = r - g$$

This implies that $d\lambda(s) / dr_s = 0$. Although the consumption rate $\lambda(s)$ is not affected, the growth rate $g(s)$ changes in line with the temporary interest rate change. Substituting $\lambda(s)$ into (35), simplifying and using (34) yields

$$\frac{\dot{C}}{C} = \frac{1}{\rho} (r_s - r) + g \equiv g_s$$

Differentiating with respect to r_s gives (15):

$$\sigma_c = \frac{d\dot{C}/C}{dr_s} = \frac{1}{\rho}$$

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Table 1: Permanent Elasticity of Intertemporal Substitution
for Sophisticated Hyperbolic Discounters

$\bar{\sigma}_S$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$
$\rho = 2$					
$\delta = 0.8$	0.192	0.308	0.390	0.451	0.500
$\delta = 0.9$	0.180	0.298	0.383	0.448	0.500
$\delta = 1$	0.170	0.289	0.377	0.446	0.500
$\rho = 3$					
$\delta = 0.8$	0.100	0.177	0.239	0.290	0.333
$\delta = 0.9$	0.097	0.173	0.236	0.288	0.333
$\delta = 1$	0.093	0.169	0.233	0.287	0.333
$\rho = 5$					
$\delta = 0.8$	0.051	0.095	0.134	0.169	0.200
$\delta = 0.9$	0.050	0.094	0.133	0.168	0.200
$\delta = 1$	0.049	0.093	0.132	0.168	0.200

Note: Author's calculations based on (7) for $r = 4\%$.

Table 2: Permanent Elasticity of Intertemporal Substitution
for Naive Hyperbolic Discounters

$\bar{\sigma}_N$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$
$\rho = 2$					
$\delta = 0.8$	0.030	0.262	0.377	0.449	0.500
$\delta = 0.9$	-0.029	0.240	0.367	0.446	0.500
$\delta = 1$	-0.091	0.218	0.358	0.442	0.500
$\rho = 3$					
$\delta = 0.8$	-0.067	0.125	0.223	0.287	0.333
$\delta = 0.9$	-0.094	0.114	0.218	0.285	0.333
$\delta = 1$	-0.119	0.104	0.213	0.282	0.333
$\rho = 5$					
$\delta = 0.8$	-0.074	0.053	0.121	0.167	0.200
$\delta = 0.9$	-0.083	0.049	0.119	0.166	0.200
$\delta = 1$	-0.091	0.045	0.117	0.165	0.200

Note: Author's calculations based on (11) for $r = 4\%$.

Table 3: Permanent Elasticity of Intertemporal Substitution for Sophisticated Hyperbolic Discounters in Continuous Time

$\bar{\sigma}_c$	$\beta = 0.2$	$\beta = 0.4$	$\beta = 0.6$	$\beta = 0.8$	$\beta = 1$
$\rho = 2$	0.167	0.286	0.375	0.444	0.500
$\rho = 3$	0.091	0.167	0.231	0.286	0.333
$\rho = 5$	0.048	0.091	0.130	0.167	0.200

Note: Author's calculations based on (14).