The topology of terminal quartic 3-folds

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Chapter 1

Introduction

1.1 Main results

Consider a possibly singular complex projective variety Y of dimension 3. The 3-fold Y is said to be $\mathbb{Q}$-factorial if an integral multiple of every Weil divisor on Y is a Cartier divisor. $\mathbb{Q}$-factoriality is a subtle property, which is not local in the analytic topology. For instance, an ordinary double point is not locally analytically $\mathbb{Q}$-factorial, yet a nodal quartic 3-fold $Y_4^3 \subset \mathbb{P}^4$ is known to be $\mathbb{Q}$-factorial if it has less than 8 ordinary double points. The topology of a quartic 3-fold Y with isolated singularities is well understood when Y is $\mathbb{Q}$-factorial. In this case, the Grothendieck-Lefschetz theorem states that every Weil (or Cartier) divisor on Y is the restriction of a Weil (or Cartier) divisor defined on $\mathbb{P}^4$. However, if the 3-fold Y is not $\mathbb{Q}$-factorial, very little is known about its topology. In this thesis, I study the topology of some mildly singular quartic 3-folds in $\mathbb{P}^4$.

Let $Y = Y_4^3 \subset \mathbb{P}^4$ be a quartic 3-fold. I assume that Y has *terminal* singularities. Notice in addition that Y is a locally complete intersection: it is therefore *Gorenstein*. These assumptions ensure, on the one hand, that the canonical class of Y is well defined as a Cartier divisor and, on the other, that pull-backs of the canonical class of Y to any resolution are well behaved. More precisely, let $\widetilde{Y} \to Y$ be a resolution of singularities. If Y is Gorenstein, the canonical sheaf ω_Y is locally free. Since $k(Y) = k(\widetilde{Y})$, a local generating section s of ω_Y near a singular point can be regarded as a rational differential on $\widetilde{Y}$. If Y has terminal singularities, the section s is regular as a rational differential on $\widetilde{Y}$ and vanishes along the exceptional locus of $\widetilde{Y} \to Y$. If Y has terminal Gorenstein singularities, Y is $\mathbb{Q}$-factorial if and only if it is factorial; that is, if its local rings are unique factorisation domains. In the case of quartic 3-folds, this means that Y is $\mathbb{Q}$-factorial if and only if $\dim H^2(Y,\mathbb{Z}) = \dim H_4(Y,\mathbb{Z})$. The Grothendieck-Lefschetz theorem on Picard groups states that $\operatorname{Pic} Y \simeq \operatorname{Pic}\mathbb{P}^4 \simeq \mathbb{Z}[\mathcal{O}_Y(1)]$. Yet, no such result holds for the group of Weil divisors $H_4(Y,\mathbb{Z})$. Let $\sigma(Y) = b_4(Y) - b^2(Y) = b_4(Y) - 1$ be the *defect* of Y. The defect of Y measures how far Y is from being $\mathbb{Q}$-factorial or, in other words, to what extent Poincaré duality fails on Y.

Such quartic 3-folds are a special case of Fano 3-folds with terminal Gorenstein singularities and Picard rank 1. A variety X is Fano if its anticanonical sheaf ω_X^{-1} is ample. The defect of a Fano 3-fold Y with terminal Gorenstein singularities can be defined as above. $\mathbb{Q}$-factorial Fano 3-folds with terminal singularities and Picard rank 1 play a crucial role in Mori theory: they arise as one of the possible end products of the Minimal Model Program for non-singular varieties. Moreover, it is known that terminal Gorenstein $\mathbb{Q}$-factorial Fano 3-folds are deformations of non-singular ones [Muk02]. Non-singular Fano 3-folds of Picard rank 1 have been classified [Isk77, Isk78]. Yet, very little is understood about the topology of non $\mathbb{Q}$-factorial Fano 3-folds.

The defect of some very simple quartic 3-folds with isolated singularities is already non-zero. For instance, if X is a sufficiently general quartic 3-fold containing a plane, then it has 9 ordinary double points and is not $\mathbb{Q}$-factorial. Similarly, a general determinantal quartic 3-fold has 20 ordinary double points and is not $\mathbb{Q}$-factorial. Finally, consider the linear system Σ of quartics spanned by the monomials $\{x_0^4, x_1^4, (x_4^2x_3 + x_2^3)x_0, x_3^3x_1, x_4^2x_1^2\}$ on $\mathbb{P}^4$. A general quartic $X \in \Sigma$ is not $\mathbb{Q}$-factorial and yet it has a unique singular point $P = (0:0:0:0:1)$, which is a cA_1 point [Mel04].

The study of quartics with ordinary double points suggests that any bound on the defect of terminal quartic 3-folds should be at least 15. Indeed, a quartic 3-fold $Y_4^3 \subset \mathbb{P}^4$ with no worse than ordinary double points is known to have at most 45 nodes [Fri86, Var83]. Up to projective equivalence, there is a unique quartic with 45 nodes: the Burkhardt quartic, given by the equation

$$\{x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\}.$$

It is easy to show that the defect of the Burkhardt quartic is at least 15 (Chapter 2). It is a natural question to ask how many topological types of quartic 3-folds with terminal Gorenstein singularities there are. I provide a partial answer to this question by bounding and studying the defect of such quartic 3-folds.

The first main result of this work is a bound on the defect of terminal quartic 3-folds.

Theorem 1.1.1 (Main Theorem 1)**.** *Let $Y_4^3 \subset \mathbb{P}^4$ be a quartic 3-fold with terminal singularities. The defect of Y is at most:*

1. 8 *if Y does not contain a plane or a quadric,*
2. 11 *if Y contains a quadric but no plane,*
3. 15 *if Y contains a plane.*

As Example 5.2.7 shows, this bound is attained by the Burkhardt quartic, a quartic with no worse than ordinary double points.

The second main result is a geometric "motivation" of the global topological property of $\mathbb{Q}$-factoriality. If Y is not $\mathbb{Q}$-factorial, by definition, Y contains a special surface. More precisely, Y contains a Weil non $\mathbb{Q}$-Cartier divisor. I show that this special surface belongs to a finite list. In particular, its degree is bounded.

Theorem 1.1.2 (Main Theorem 2)**.** *Let $Y_4^3 \subset \mathbb{P}^4$ be a terminal Gorenstein quartic 3-fold. Then one of the following holds:*

1. *Y is $\mathbb{Q}$-factorial.*
2. *Y contains a plane $\mathbb{P}^2$.*
3. *Y contains an irreducible reduced quadric Q.*
4. *Y contains an anticanonically embedded del Pezzo surface of degree 4.*
5. *Y has a structure of Conic Bundle over $\mathbb{P}^2$, $\mathbb{F}_0$ or $\mathbb{F}_2$.*
6. *Y contains a rational scroll $E \to C$ over a curve C whose genus and degree appear in the table on page 129.*

Typical examples of non $\mathbb{Q}$-factorial varieties are 3-folds that contain planes or quadrics, which are not Cartier. I show that, in accordance with geometric intuition, a quartic 3-fold has to contain a surface of low degree if it is not $\mathbb{Q}$-factorial.

1.2 Outline of the thesis

I have divided this thesis in six Chapters.

Chapter 2 reviews relevant material and results from Mixed Hodge Theory. The notion of defect of hypersurfaces was first introduced by Clemens [Cle83] in an attempt, based on Deligne's Mixed Hodge Theory [Del74], to extend Griffiths' results on the Hodge theory of hypersurfaces [Gri69] to mildly singular varieties. The traditional approach to the determination of the defect of hypersurfaces, or of Fano 3-folds, has focused on understanding their mixed Hodge structures, as in [Dim90, NS95]. Such an approach heavily relies on explicit computations of cohomology groups of specific varieties: it is impractical to determine a sharp bound on the defect of quartic 3-folds with terminal singularities.

In Chapter 3, I define the category of weak$*$ Fano 3-folds and I show that the Minimal Model Program can be run in this category. For any Fano 3-fold Y with terminal Gorenstein singularities, there is a small $\mathbb{Q}$-factorialisation $X \to Y$ [Kaw88]. If Y does not contain a plane, X belongs to the category of weak$*$ Fano 3-folds. Bounding the defect of Y is equivalent to determining the maximal number of divisorial contractions involved in a Minimal Model Program on X.

In Chapter 4, I bound the defect of any terminal Gorenstein Fano 3-fold that does not contain a plane. I show that if the anticanonical model Y of a weak$*$ Fano 3-fold X does not contain a quadric, this property is preserved when running a Minimal Model Program on X. This enables me to improve the bound on the defect when the 3-fold Y does not contain a quadric.

In Chapter 5, I study quartic 3-folds containing a plane. The 3-fold obtained by blowing up Y along this plane has a natural structure of del Pezzo fibration of degree 3. Using Corti's results [Cor96] to relate the defect of Y to the number of reducible fibres of this cubic fibration, I bound the number of reducible fibres: this completes the proof of Theorem 1.1.1.

Chapter 6 recalls several results on the deformation theory of so-called generalised Fano 3-folds. These results were mainly obtained by Namikawa, Kollár and Mori [Nam97, KM92]. Namikawa defines *generalised Fano 3-folds* and shows that any generalised Fano 3-fold is a one parameter flat deformation of a non-singular one. Any small (partial) $\mathbb{Q}$-factorialisation of a terminal Gorenstein Fano 3-fold and terminal Gorenstein Fano 3-folds themselves are generalised Fano 3-folds. Moreover, the degree and the Picard rank are constant in this deformation. The key observation is that each step of the Minimal Model Program on weak$*$ Fano 3-folds may be deformed, in a suitable sense, to an extremal contraction of a generalised Fano 3-fold of Picard rank 2.

In Chapter 7, I extend several ideas expressed by Takeuchi in [Tak89]. Any divisorial contraction of the Minimal Model Program on a small $\mathbb{Q}$-factorialisation X of Y induces an extremal divisorial contraction on Z, a Picard rank 2 partial $\mathbb{Q}$-factorialisation of Y. I study extremal divisorial contractions with Cartier exceptional divisor on partial $\mathbb{Q}$-factorialisations of terminal Gorenstein Fano 3-folds. Any such divisorial contraction can then be deformed to a divisorial contraction on a non-singular generalised Fano 3-fold of Picard rank 2. Following Takeuchi's approach, I then write systems of Diophantine equations associated to each extremal contraction of the Minimal Model Program on X. These equations translate numerically the properties of the extremal contraction. Such systems have very few solutions. The explicit study of the system associated to the first divisorial contraction on X yields a finite number of solutions. Each solution exhibits a possible surface that has to be contained in the quartic 3-fold Y. This establishes Theorem 1.1.2 and gives a geometric "motivation" of $\mathbb{Q}$-factoriality in the case of quartic 3-folds.

Similar methods may be used to bound the defect of any terminal Gorenstein Fano 3-fold with Picard rank 1 and degree greater than 4. I have bounded the defect of the Fanos with Picard rank 1 that contain no plane. An explicit study of terminal Gorenstein Fano 3-folds that contain a plane is also possible.

Conjecture 1.2.1. 1. *The defect of a non $\mathbb{Q}$-factorial terminal Gorenstein $Y_{2,3} \subset \mathbb{P}^5$ with Picard rank 1 is at most 8,*

2. *The defect of a non $\mathbb{Q}$-factorial terminal Gorenstein $Y_{2,2,2} \subset \mathbb{P}^6$ with Picard rank 1 is at most 8,*

3. *The defect of a Picard rank 1 non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold of genus $g \geq 6$ is at most $\left[\frac{12-g}{2}\right] + 5$.*

However, bounding the defect of the double cover of a sextic divisor in $\mathbb{P}^3$ is likely to be more complicated. The study of such double sextics, whether they contain planes or surfaces whose normalisations are planes, would be significantly more difficult than the quartic case. I conjecture that the defect of a terminal Gorenstein double sextic with Picard rank 1 is at most 18.

The methods I develop in this thesis can also be applied to terminal Gorenstein Fano 3-folds of larger Picard rank.

It would be natural to consider wider classes of singularities such as cDV singularities, canonical Gorenstein or even non-canonical singularities. It is likely that the analysis required will be much finer. Throughout this work, I made extensive use of the fact that crepant contractions are small and assumed that the anticanonical class was Cartier. I do not believe that the methods I have developped may be extended directly to these more general settings.

It is known that non-singular quartics are non-rational, while the Burkhardt quartic is rational. The defect could be used, in some cases, to determine whether a quartic 3-fold $Y_4^3 \subset \mathbb{P}^4$ is rational or not. Non $\mathbb{Q}$-factoriality does not necessarily imply rationality. Indeed, a general quartic containing a plane is non-rational. Yet, if Y does not contain a plane, the defect of Y can only be high when the end product of the Minimal Model Program on its $\mathbb{Q}$-factorialisation is rational. This could provide a rationality criterion for some values of the defect. Examples of rational non $\mathbb{Q}$-factorial quartics can be obtained by running a Minimal Model Program in reverse, using Takeuchi's numerical constraints.

Chapter 2

Mixed Hodge Theory

Clemens introduced the notion of defect of hypersurfaces [Cle83] in an attempt to generalise Griffiths' results [Gri69] on the Hodge theory of non-singular hypersurfaces to hypersurfaces with ordinary double points.

I recall in this chapter his definition of the defect of hypersurfaces and the subsequent generalisations of this notion to the framework of terminal Gorenstein Fano 3-folds. The defect is expressed in terms of local cohomology groups.

I show how the defect of a terminal Gorenstein Fano 3-fold Y is related to the mixed Hodge structure on the cohomology $H^\bullet(Y)$ and, in particular, to the weight filtration. The defect depends not only on the Hodge filtration, but also on the weight filtration: it is a global topological invariant. It cannot be determined by local analytic methods.

2.1 The notion of defect of a Fano 3-fold

Let Y be a non $\mathbb{Q}$-factorial Fano 3-fold with terminal Gorenstein singularities.

Kawamata shows [Kaw88] that a 3-fold Y with terminal singularities has a (not necessarily unique) small projective $\mathbb{Q}$-factorialisation. He proves:

Proposition 2.1.1 ($\mathbb{Q}$-factorialisation)**.** *Let Y be an algebraic threefold with only terminal singularities. Then, there is a birational morphism $f\colon X \to Y$ such that X is terminal and $\mathbb{Q}$-factorial, f is an isomorphism in codimension 1 and f is projective.*

Let X be a small crepant projective $\mathbb{Q}$-factorialisation of Y.

Definition 2.1.2. The *defect* of Y is

$$\begin{aligned}\sigma(Y) = \operatorname{rk}(\operatorname{Weil}(Y)/\operatorname{Pic}(Y)) &= \dim H_4(Y) - \dim H^2(Y)\\ &= \dim \operatorname{Pic}(X) - \dim \operatorname{Pic}(Y).\end{aligned}$$

Remark 2.1.3. In the case of hypersurfaces in projective space with isolated singularities, such as a quartic 3-fold $Y_4^3 = \{f = 0\} \subset \mathbb{P}^4$ with isolated singularities, it is known that the Euler characteristic depends only on the degree and on local invariants related to the singularities. More precisely, the Euler characteristic may in theory be computed once one understands the semi-simple part of the monodromy operator, i.e. the spectrum of f in the neighbourhood of the singularities. The defect, or rather $b_4(Y)$, is however a more subtle invariant: it also depends on the position of the singularities.

Lemma 2.1.4. *[NS95] Let Y be a terminal Gorenstein Fano 3-fold. Denote by Σ the singular locus $\Sigma = \operatorname{Sing}(Y)$ and by U its complement $U = Y \smallsetminus \Sigma$. The defect of Y is*

$$\sigma(Y) = \dim_{\mathbb{C}} \operatorname{Coker}[H^3(U, \mathbb{C}) \to H^4_\Sigma(Y, \mathbb{C})].$$

Proof. Denote by $\Sigma = \{P_1, \cdots, P_n\}$ the singular set of Y. Consider the local cohomology exact sequence of mixed Hodge structures:

$$H^3(U) \to H^4_\Sigma(Y) \to H^4(Y) \to H^4(U) \to H^5_\Sigma(Y) \tag{2.1}$$

Let $\{U_i\}_{1\leq i\leq n}$ be mutually disjoint open neighbourhoods of the singular points P_i. By excision, $H^5_\Sigma(Y) = \oplus H^5_\Sigma(U_i) = \oplus H^5_{\{P_i\}}(Y)$. In Section 2.3.3, I show that $H^5_{\{p_i\}}(Y) = 0$. Hence, $\sigma(Y) = b_4(Y) - b_4(U)$.

By definition U is non-singular, hence $H^4(U, \mathbb{C})$ is dual to the cohomology group with compact support $H^2_c(U)$. The group $H^2_c(U)$ is isomorphic to the relative cohomology $H^2(Y, \Sigma) \simeq H^2(Y)$ because the singularities of Y are isolated. □

The value of $b_n(Y)$ for a hypersurface Y of $\mathbb{P}^n$ is known to be related to the dimension of some linear systems of homogeneous polynomials that vanish at the singular points of Y [Cle83]. Dimca makes these relations explicit [Dim90] and shows that the defect depends on the mixed Hodge structure of the local cohomology groups at the singular points.

I recall his results in the case of $Y \subset \mathbb{P}^4$ a quartic 3-fold with isolated singularities.

Let $\Sigma = \{P_1, \dots, P_n\}$ be the singular set of Y and $U = Y \smallsetminus \Sigma$. Let i (resp. j) be the inclusion $Y \to \mathbb{P}^4$ (resp. $U \to \mathbb{P}^4$). The *primitive* part of the cohomology of Y (resp. U) is $H^\bullet_{\mathrm{prim}}(Y) = \mathrm{Coker}[i^*\colon H^\bullet(\mathbb{P}^4) \to H^\bullet(Y)]$ (resp. $H^\bullet_{\mathrm{prim}}(U) = \mathrm{Coker}(j^*)$). The natural inclusion $k\colon U \to Y$ induces a morphism $k_{\mathrm{prim}}\colon H^\bullet_{\mathrm{prim}}(Y) \to H^\bullet_{\mathrm{prim}}(U)$, and carries isomorphically the non-primitive part of $H^\bullet(Y)$ into the non-primitive part of $H^\bullet(U)$ except in the top dimension.

The Poincaré residue map

$$R\colon H^k((\mathbb{P}^4 \smallsetminus \Sigma) \smallsetminus (Y \smallsetminus \Sigma)) \to H^{k-1}(Y \smallsetminus \Sigma)$$

defines a $(-1,-1)$ isomorphism of Hodge structures from $H^k(\mathbb{P}^4 \smallsetminus Y) = H^k((\mathbb{P}^4 \smallsetminus \Sigma) \smallsetminus (Y \smallsetminus \Sigma))$ to the primitive part of the cohomology of $U = Y \smallsetminus \Sigma$, $H^{k-1}_{\mathrm{prim}}(U)$. From the exact sequence

$$H^3(U) \xrightarrow{\theta} H^4_\Sigma(Y) \to H^4(Y)$$

and the Poincaré isomorphism $R\colon H^4(\mathbb{P}^4 \smallsetminus \Sigma) \simeq H^3_{\mathrm{prim}}(U)$, we deduce the exact sequence:

$$H^4(\mathbb{P}^4 \smallsetminus Y) \xrightarrow{\delta} H^4_\Sigma(Y) \to H^4_{\mathrm{prim}}(Y) \to 0, \tag{2.2}$$

where $\delta = \theta \circ R$. The local cohomology $H^\bullet_\Sigma(Y)$ has a natural mixed Hodge structure inherited from that of the cohomology of the pair $(Y, Y - \Sigma)$ (Section 2.3). Let $\{U_i\}_{1 \le i \le n}$ be mutually disjoint open neighbourhoods of the singular points P_i; by excision, $H^k_\Sigma(Y) = \oplus H^k_{\{P_i\}}(U_i)$ and $H^4_\Sigma(V)$ is computable.

Denote by F the Hodge filtration. Recall that the Hodge filtration in the case of hypersurfaces with terminal singularities coincides with the filtration by the order of the pole on $\mathbb{P}^4 - Y$ [Gri69, DD90]. The polar filtration on the De Rham complex $A^\bullet = H^0(\mathbb{P}^4 - Y, \Omega^\bullet_{\mathbb{P}^4 - Y})$ is given by:

$$F^t A^j = \{\omega \in A^j \mid \omega \text{ has a pole along } Y \text{ of order at most } j - t\},$$

for $j - t \ge 0$ and $F^t A^j = 0$ for $j - t < 0$. Recall Griffiths' explicit description of A^4. Consider the differential 4-form $\Omega \in \Omega^4_{\mathbb{P}^4}$

$$\Omega = \sum_{0 \le i \le 4} (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_4.$$

If Y is the hypersurface $\{f = 0\} \subset \mathbb{P}^4$, any element $\omega \in A^4$ may be written

$$\omega = \frac{P\Omega}{f^t}$$

where P is a homogeneous polynomial of degree $4t - 5$. If f does not divide P, the order of the pole of ω along Y is precisely t.

Let t be the highest natural number such that $F^t H^4_\Sigma(Y) = H^4_\Sigma(Y)$ and call δ^t the linear map:

$$\delta^t \colon S_{4(4-t)-5} \to H^4_\Sigma(Y).$$

This map is, strictly speaking, a composition of the above δ with the natural map $S_{4(4-t)-5} \simeq F^t A^4 \to F^t H^4(Y \setminus \Sigma)$.

Definition-Lemma 2.1.5. [Dim90] Let $Y \subset \mathbb{P}^4$ be a quartic 3-fold. The defect of Y is:

$$\sigma(Y) = \dim H^4_\Sigma(Y) - \operatorname{Codim} \operatorname{Ker} \delta^t = \dim H^4_{\mathrm{prim}}(Y).$$

Remark 2.1.6. Note that if the quartic hypersurface $Y \subset \mathbb{P}^4$ has no worse than ordinary double points, the computations in Section 2.3.3 show that $t = 2$ and (2.2) gives a lower bound for the defect of Y [Dim92]:

$$\sigma(Y) \geq N - \frac{b_3(V)}{2} = N - 30,$$

where N is the number of double points and V is a non-singular quartic hypersurface in $\mathbb{P}^4$.

Proof. This expression agrees in the case of quartics with the expression of the defect given in Lemma 2.1.4. Indeed, the map δ^t factors through $H^3(U, \mathbb{C})$. More precisely, Dimca shows [Dim90] that the map δ factors as:

$$\delta \colon H^4(\mathbb{P}^4 \setminus Y) \xrightarrow{R} H^3(U) \to H^4_\Sigma(Y),$$

where the second map is the connecting morphism in the sequence of local cohomology. □

Remark 2.1.7. The defect of a quartic hypersurface with terminal singularities is expressed in terms of residuation from $A^\bullet = H^0(\mathbb{P}^4 \setminus Y, \Omega^\bullet)$. This extends Griffiths' results on the Hodge theory of non-singular hypersurfaces.

The equivalent definitions of the defect of a quartic hypersurface with terminal singularities show that the only singularities affecting the defect of a quartic 3-fold are *essential singularities*.

Definition 2.1.8. A singularity $P_i \in Y$ is *essential* if the local cohomology group $H^4_{\{P_i\}}(Y)$ is not trivial.

The local cohomology groups $H^k_{\{P_i\}}(Y)$ associated to isolated singularities are computed in Section 2.3.3.

Remark 2.1.9. [Dim90] Given a homogeneous polynomial $h \in S_{4(4-t)-5}$, $\delta^t(h) = 0$ means that h satisfies certain linear conditions $\mathcal{C}$ at the essential singular points. Denote by D the linear system in $S_{4(4-t)-5}$ defined by these conditions. The defect of Y is the difference between the number of linear conditions $\mathcal{C}$ and the codimension of D in $S_{4(4-t)-5}$. The defect does not depend only on the linear system D, but also on the number of conditions $\mathcal{C}$ used to define it. The conditions $\mathcal{C}$ are independent if and only if the defect is zero.

The expression of the defect given in 2.1.5 allows direct computations of the defect in some special cases, such as nodal hypersurfaces.

Lemma 2.1.10. *[Dim90] Let $Y \subset \mathbb{P}^4$ be a quartic 3-fold whose only essential singularities are nodes. Then the defect of Y is given by:*

$$\sigma(Y) = h^{2,2}(H^4_{prim}(Y))$$
$$= \dim H^3_\Sigma(Y) - \mathrm{Codim}\{h \in S_3 = S_{4(4-2)-5}, h(P) = 0 \textit{ for any node of } Y\}$$
$$= \sharp\{\textit{nodes}\} - \sharp\{\textit{conditions imposed on cubics by vanishing at the nodes of } Y\}.$$

2.2 Mixed Hodge structure of $H^\bullet(Y)$ and defect

I have explained in Section 2.1 that the defect is related to the cohomology groups $H^3(Y)$. I adapt ideas formulated by Namikawa and Steenbrink for Calabi-Yau 3-folds [NS95] to give a geometric description of the mixed Hodge structure on $H^3(Y)$.

The cohomology group $H^3(Y)$ is naturally endowed with a mixed Hodge structure supported in weights 2 and 3. There are at least two interpretations of this mixed Hodge structure.

First, the weight 3 part of $H^3(Y)$ is isomorphic to the pure Hodge structure $H^3(\widetilde{Y})$ on a good resolution $\widetilde{Y}$ of Y.

The weight 2 part of $H^3(Y)$ is non-zero if there exists a non trivial relation among the $H^2(E_i)$ in $H^2(\widetilde{Y})$ (where E_i denotes the exceptional divisor above the singular point P_i). This corresponds to the geometric intuition that the defect of a 3-fold is not only related to the analytic local type of the singularities, but also to their relative position.

Second, if there is a smoothing $\mathcal{Y} \to \Delta$ of Y, examining the relationship between the mixed Hodge structure of Y and the Hodge structure of a fibre $\mathcal{Y}_t$ for $t \neq 0$ expresses the defect in terms of the vanishing cohomology. The weight 3 part of $H^3(Y)$ is isomorphic to $H^3(\mathcal{Y}_t)$, while its weight 2 part depends on the cohomology of vanishing cycles.

2.2.1 Defect and cohomology of a good resolution of Y

Let Y be a terminal Gorenstein Fano 3-fold. Denote by $\Sigma = \{P_1, \cdots, P_n\}$ its singular set. Let $f\colon \widetilde{Y} \to Y$ be a good resolution of Y and $E = \sum E_i$ be the simple normal crossings exceptional divisor of f. For each i, $\{E_i^j\}_j$ are the irreducible components of E_i.

Definition-Lemma 2.2.1. [NS95] Let Y_i be a contractible Stein open neighbourhood of P_i and $\sigma^{\mathrm{an}}(P_i) = \mathrm{Weil}(Y_i)/\operatorname{Pic}(Y_i)$. The *analytic local defect* at P_i is:

$$\sigma^{\mathrm{an}}(P_i) = \dim(H^1(Y_i, \mathcal{O}^*_{Y_i})/\bigoplus \mathbb{Z}[E_i^j]) = \dim H^3_{\{P_i\}}(Y) = \dim H^4_{\{P_i\}}(Y).$$

Proof. The expressions for $H^3_{\{P_i\}}(Y)$ and $H^4_{\{P_i\}}(Y)$ are determined in Section 2.3.3. □

Remark 2.2.2. In particular, if P_i is an ordinary double point, $\sigma^{\mathrm{an}}(P_i) = 1$.

Proposition 2.2.3. *[NS95] Let Y be a terminal Gorenstein Fano 3-fold and $\Sigma = \{P_1, \cdots, P_n\}$ be the singular set of Y. The weight filtration on $H^3(Y)$ has the following description:*

$$Gr_k^W H^3(Y) = (0) \quad for \quad k \neq 2, 3$$
$$\dim W_2 H^3(Y) = \sum \sigma^{an}(P_i) - \sigma(Y)$$

Proof. As is explained in Section 2.3.1, the long exact sequence (2.4)

$$\cdots H^l(Y,\mathbb{Q}) \to H^l(\widetilde{Y},\mathbb{Q}) \oplus H^l(\Sigma,\mathbb{Q}) \to H^l(E,\mathbb{Q}) \to H^{l+1}(Y,\mathbb{Q}) \to \cdots$$

is compatible with the weight filtration. In particular, the following sequences are exact:

$$\begin{aligned} 0 \to Gr_3^W H^3(Y) \to H^3(\tilde{Y}) \to 0 \\ 0 \to Gr_2^W H^2(Y) \to H^2(\tilde{Y}) \to \oplus Gr_2^W H^2(E_i) \to Gr_2^W H^3(Y) \to 0 \\ 0 \to \oplus Gr_1^W H^2(E_i) \to Gr_1^W H^3(Y) \to 0 \\ 0 \to \oplus Gr_0^W H^2(E_i) \to Gr_0^W H^3(Y) \to 0 \end{aligned}$$

Section 2.3.2 shows that $H^2(E) = Gr_2^W H^2(E)$ is of pure weight 2. The exact sequence of local cohomology groups associated to Σ is:

$$\ldots \to H^2(Y) \to H^2(U) \xrightarrow{\alpha} H^3_\Sigma(Y) \to H^3(Y) \to H^3(U) \to \ldots.$$

The sequence

$$0 \to H^2_E(\widetilde{Y}) \to H^2(E) \to H^3_\Sigma(Y) \to 0$$

is exact and compatible with the weight filtration (Section 2.3.3).

The local cohomology group $H^3_\Sigma(Y)$ is purely of weight 2, while $H^3(U)$ is purely of weight 3. Moreover,

$$\operatorname{Im}(\alpha) = \operatorname{Im}[H^2(\tilde{Y}) \to H^2(U) \to H^3_\Sigma(Y)],$$

and $W_2H^3(Y) = H^3_\Sigma(Y)/\operatorname{Im}(\alpha) = \operatorname{Coker}[H^2(\tilde{Y}) \to H^3_\Sigma(Y)]$. The 3-fold Y is a Fano 3-fold and has rational singularities. By Kawamata-Viehweg vanishing (Theorem 3.1.5) and the Leray spectral sequence, $H^2(\tilde{Y},\mathcal{O}_{\tilde{Y}}) = (0)$ and $H^2(\tilde{Y}) = H^1(\tilde{Y},\mathcal{O}^*_{\tilde{Y}})\otimes\mathbb{C}$. The result then follows from the isomorphism:

$$\operatorname{Weil}(Y)/\operatorname{Pic}(Y) \simeq \operatorname{Im}\left[H^1(\tilde{Y},\mathcal{O}^*_{\tilde{Y}}) \to \bigoplus H^1(Y_i,\mathcal{O}^*_{Y_i})/\oplus_j \mathbb{Z}[E_i^j]\right].$$

This isomorphism is a direct application of the Kawamata-Viehweg vanishing theorem and of the Leray spectral sequence on Y_i [Kaw88, KM92]. □

Remark 2.2.4. Notice that this gives the following formula for the defect of a terminal Gorenstein Fano 3-fold:

$$\sigma(Y) = b_3(\widetilde{Y}) - b_3(Y) + \sum \sigma^{\mathrm{an}}(P_i).$$

Example 2.2.5 (Case of a nodal 3-fold)**.** Assume that the singularities of Y are ordinary double points (nodes). Denote by $\mu\colon \widetilde{Y} \to Y$ the blow up of Y at the nodes P_i and by $\nu\colon \widehat{Y} \to Y$ a small resolution of the nodes of Y.

1. The μ-exceptional locus is a disjoint union of non-singular quadrics $Q_i = \mathbb{P}^1 \times \mathbb{P}^1$. In that case, for each i, $\sigma^{\mathrm{an}}(P_i) = 1$. Hence, if the defect is strictly less than the number of nodes, $W_2H^3(Y) \neq (0)$. The sequence associated to the resolution $\widetilde{Y} \to Y$ (Section 2.3.1,(2.4)) reads

$$\cdots \to H^2(\widetilde{Y}) \to H^2(E) = \oplus H^2(Q_i) \to W_2H^3(Y) \to 0.$$

   Hence, if $W_2H^3(Y) \neq (0)$, there is a non trivial relation between the Q_i-s in $H^2(\tilde{Y}, \mathbb{C})$. In this case, the nodes fail to impose independent linear relations on the vanishing of cubics (Lemma 2.1.10).

2. Note for future reference that the second Betti numbers of $Y, \widehat{Y}$ and $\widetilde{Y}$ satisfy the following relations:

$$b_2(\widehat{Y}) = b_2(Y) + \sigma(Y) \quad \text{and} \quad b_2(\widetilde{Y}) = b_2(\widehat{Y}) + N, \tag{2.3}$$

   where N is the number of nodes.

2.2.2 Defect and cohomology of a smoothing of Y

Clemens compared the cohomology of a nodal 3-fold Y to the cohomology of a smoothing of Y [Cle83]. This lead him to introduce the notion of defect of Y: whereas the presence of nodes does not affect the second integral cohomology, the fourth homology is altered and Poincaré duality fails on Y. I recall how the defect relates to the cohomology of a smoothing of Y.

Definition 2.2.6. Let Y be a terminal Gorenstein 3-fold. A *smoothing* of Y is a proper flat map $f\colon \mathcal{Y} \to \Delta$ from an analytic space $\mathcal{Y}$ to a 1-dimensional complex disc Δ such that $f^{-1}(0) = Y$ and $f^{-1}(t) = \mathcal{Y}_t$ is a non-singular 3-fold for all $t \in \Delta \smallsetminus \{0\}$.

In Chapter 6, I recall several results on the deformation theory of Fano 3-folds. In particular, Namikawa shows [Nam97] that if Y is a nodal Fano 3-fold with Picard rank 1, there exists a smoothing $f\colon \mathcal{Y} \to \Delta$ of Y. The fibre $\mathcal{Y}_t$ is a non-singular Fano 3-fold for $t \neq 0$. If Y is a terminal Gorenstein Fano 3-fold with Picard rank 1, there exists a 1-parameter proper flat deformation

$f\colon \mathcal{Y} \to \Delta$ such that $\mathcal{Y}_{t_0}$ is non-singular for some $t_0 \in \Delta$ and $\mathcal{Y}_t$ is a terminal Gorenstein Fano 3-fold for all $t \in \Delta$. In such a deformation, the plurigenera are constant. The relation between the cohomology groups of Y and those of a fibre $\mathcal{Y}_t$ provides further information on the defect of Y.

Lemma 2.2.7. *[Mil68, Ste77] Let Y be a normal projective 3-fold with terminal Gorenstein singularities. Suppose that Y has a smoothing $f\colon \mathcal{Y} \to \Delta$. Then,* $\operatorname{Pic} Y \simeq \operatorname{Pic} \mathcal{Y}_t$.

Sketch Proof. Let $\{P_1, \cdots, P_n\}$ be the singular set of Y and denote by (U_i, P_i) a small Stein open neighbourhood of the singular point P_i. Terminal Gorenstein 3-fold singularities are isolated hypersurface singularities. Let $f_i\colon \mathcal{U}_i \to \Delta$ be a 1-parameter flat deformation of (U_i, p_i). Let B_i be a small ball of $\mathcal{U}_i$ with centre P_i and radius $\epsilon > 0$. For η sufficiently small, all fibres $(\mathcal{U}_i)_t$ for $|t| < \eta$ intersect B_i transversally and $(\mathcal{U}_i)_t \cap B_i = B_{i,t}$ is diffeomorphic to the Milnor fibre of P_i for $t \neq 0$. If the singularity at P_i is given in local analytic coordinates by $\{f(x, y, z, t) = 0\}$, for f a polynomial with an isolated critical point at the origin, the Milnor fibre is $\{f(x, y, z, t) = 1\}$. The Milnor fibre is homotopic to a bouquet of 3-spheres [Mil68]; its cohomology is supported in degrees 0 and 3. There is a homeomorphorphism between $U_i \setminus \{P_i\}$ and $(\mathcal{U}_i)_t \setminus B_{i,t}$, hence the exact sequence of relative cohomology shows that for $|t| < \eta$:

$$H^i(U_i, \mathbb{Z}) \simeq H^i((\mathcal{U}_i)_t, \mathbb{Z}) \quad \text{for } i \neq 3, 4,$$
$$0 \to H^3(U_i) \to H^3((\mathcal{U}_i)_t) \to H^3(B_{i,t}) \to H^4(U_i) \to H^4((\mathcal{U}_i)_t) \to 0.$$

There is a homeomorphism between $Y \setminus \{P_1, \cdots, P_n\}$ and $\mathcal{Y}_t \setminus \bigoplus_{1 \leq i \leq n} B_{i_t}$, so that for $|t|$ sufficiently small:

$$H^i(Y, \mathbb{Z}) \simeq H^i(\mathcal{Y}_t, \mathbb{Z}) \quad \text{for } i \neq 3, 4,$$
$$0 \to H^3(Y) \to H^3(\mathcal{Y}_t) \to \bigoplus_{1 \leq i \leq n} H^3(B_{i,t}) \to H^4(Y) \to H^4(\mathcal{Y}_t) \to 0.$$

□

Lemma 2.2.8. *If $\mathcal{Y} \to \Delta$ is a smoothing of Y, the defect of Y satisfies:*

$$\sigma(Y) = b_3(Y) - b_3(\mathcal{Y}_t) + \sum h^3(B_{i,t}).$$

Proof. The sequence of mixed Hodge structures

$$0 \to H^3(Y) \to H^3(\mathcal{Y}_t) \to \bigoplus H^3(B_i) \to H^4(Y) \to H^4(\mathcal{Y}_t) \to 0$$

is exact. It follows that

$$b_3(Y) + \sum h^3(B_{i,t}) - b_3(\mathcal{Y}_t) = b_4(Y) - b_4(\mathcal{Y}_t).$$

By Poincaré duality on the non-singular 3-fold $\mathcal{Y}_t$, $b_4(\mathcal{Y}_t) = b_2(\mathcal{Y}_t)$. As the Picard rank is constant in the smoothing, $b_2(\mathcal{Y}_t) = b_2(Y)$ and the result follows. □

Remark 2.2.9. Notice that $h^3(B_{i,t}) = \dim H^3_{\{P_i\}}(Y)$.

Remark 2.2.10. For an ordinary double point, $B_{i,t}$ is, by construction, a 3-sphere $\{x^2 + y^2 + z^2 + t^2 = |t|\}$, so that $h^3(B_{i,t}) = 1$. In particular, if Y is a quartic with no worse than ordinary double points and if there exists a smoothing of Y, the following holds:

$$\sigma(Y) = b_3(Y) + N - 60,$$

where N is the number of nodes. Indeed, the degree of a Fano 3-fold is constant in a flat family, and the third Betti number of a non-singular quartic 3-fold is 60.

Remark 2.2.11. As the third Betti numbers of non-singular Fano 3-folds with Picard rank 1 are known [IP99], Lemma 2.2.8 provides a bound on the number of essential singular points, once a bound on the defect of terminal Gorenstein Fano 3-folds is known.

Remark 2.2.12. This analysis could be carried out in the more general case of terminal or canonical singularities. However, in such cases, one would have to study the complex of vanishing cohomology (or vanishing cycles). This complex is not, in general, supported in degrees 0 and 3.

I have presented several definitions and formulae related to the defect of Gorenstein Fano 3-folds, based on their Mixed Hodge Theory. Lengthy computations are necessary in order to explicitly determine the defect of a particular terminal Gorenstein quartic or Fano 3-fold. These expressions rely on the analytic local type of the singularities and are too unwieldy to yield a bound on the defect of terminal quartic 3-folds.

2.3 Mixed Hodge structures on $H^\bullet(Y)$ and $H^\bullet_\Sigma(Y)$

In the previous sections, I used several results on the Mixed Hodge Theory of various cohomology groups associated to a Gorenstein terminal Fano 3-fold Y. In this Section, I indicate how these results are obtained. I do not provide complete proofs, but give an overview of the relevant aspects of the theory.

Let X be a complex algebraic variety. Deligne endows the cohomology groups $H^\bullet(X,\mathbb{Z})$ with a functorial mixed Hodge structure [Del74]. His formalism extends the Hodge theory of non-singular projective complex varieties.

Let X be a complex algebraic variety. Hironaka's resolution theorem states that there exists a resolution of singularities $f\colon \widetilde{X} \to X$. The map f is proper and birational and $\widetilde{X}$ is non-singular. The exceptional divisor of f has simple normal crossings. Deligne introduces the method of simplicial cohomological descent, which uses resolution of singularities to define a mixed Hodge structure on the cohomology of X. Whereas earlier applications of resolution theorems had implications on the hypercohomology of complexes of sheaves, cohomological descent yields results at the level of these complexes.

I use the techniques of cubic hyper-resolutions, developed in [GNAPGP88] to determine the mixed Hodge structure on a terminal Gorenstein Fano 3-fold Y.

2.3.1 Mixed Hodge structure on $H^\bullet(Y)$

Let Y be a terminal Gorenstein Fano 3-fold and let $\Sigma = \{P_1, \dots, P_n\}$ be the singular set of Y.

There is a good resolution $f\colon \widetilde{Y} \to Y$ of Y. The morphism f is proper and birational, and its exceptional set is a divisor $E = \sum E_i$ with simple normal crossings. As above, E_i denotes $f^{-1}(P_i)$ and $\{E_i^j\}_j$ are the irreducible components of E_i.

The diagram

$$\begin{tikzcd} \sum E_i \arrow[r, "\tilde{i}"] \arrow[d] & \tilde{Y} \arrow[d, "f"] \\ \Sigma \arrow[r, "i"] & Y \end{tikzcd}$$

induces the long exact sequence in cohomology:

$$\cdots \to H^l(Y,\mathbb{Q}) \to H^l(\widetilde{Y},\mathbb{Q}) \oplus H^l(\Sigma,\mathbb{Q}) \to H^l(E,\mathbb{Q}) \to H^{l+1}(Y,\mathbb{Q}) \to \cdots \tag{2.4}$$

This long exact sequence is compatible with the Hodge and weight filtrations.

The above diagram is said to be of cohomological descent. It induces a resolution of the complex $\mathbb{Q}_Y$

$$0 \to \mathbb{Q}_Y \to \mathbb{R}f_*\mathbb{Q}_{\widetilde{Y}} \oplus \mathbb{R}i_*\mathbb{Q}_\Sigma \to \mathbb{R}(f\tilde{i})_*\mathbb{Q}_{\sum E_i} \to 0.$$

In the present case, this resolution reads :

$$0 \to \mathbb{Q}_Y \to \mathbb{R}f_*\mathbb{Q}_{\widetilde{Y}} \bigoplus \oplus\mathbb{Q}_{\{P_i\}} \to \mathbb{R}(f\tilde{i})_*\mathbb{Q}_{\sum E_i} \to 0.$$

The cohomology groups appearing in (2.4) satisfy the following set of properties. These properties follow either from definitions or from standard results in Mixed Hodge Theory [Del74, GNAPGP88] because $\widetilde{Y}$ is non-singular and projective and Y is projective with isolated singularities.

1. $H^l(\Sigma,\mathbb{Q}) = 0$ for $l \neq 0$ and $H^0(\Sigma,\mathbb{Q}) = \oplus\mathbb{Q}_{\{P_i\}}$.

2. For all l, $H^l(\widetilde{Y},\mathbb{Q})$ is a pure Hodge structure of weight l.

3. For all $j > l$, $Gr_j^W H^l(Y,\mathbb{Q}) = 0$.

4. For all l, $H^l(E,\mathbb{Q}) = \oplus H^l(E_i,\mathbb{Q})$.

Proposition 2.2.3 relates the defect of Y to the weight filtration $H^3(Y)$. The cohomology of $\widetilde{Y}$ has a pure Hodge structure, hence (2.4) shows that understanding the weight filtration on the cohomology of E suffices to determine the defect of Y.

In the next subsection, I recall how the weight filtration on $H^\bullet(E)$ is obtained.

2.3.2 Mixed Hodge structure on $H^\bullet(E,\mathbb{C})$

By definition, the irreducible components E_i^j of E_i are non-singular and intersect tranversally.

The mixed Hodge structure of $H^\bullet(E_i)$ is determined by a Mayer-Vietoris cubic hyper-resolution. From now onwards, I drop the index i and write E for E_i.

Let E^j, $1 \leq j \leq r$ be the irreducible components of E. For any $p \leq r$, let E_{p-1} denote the disjoint union of all p-fold intersections of components of E.

$$E_{p-1} = \amalg_{1 \leq j_0 < \cdots < j_p \leq r} (E^{j_0} \cap \cdots \cap E^{j_r}).$$

The projective varieties E_{p-1} and the maps

$$d_k^p \colon E_p \to E_{p-1}$$

for $k = 1, \cdots, p$ induced by the inclusions

$$E^{j_1} \cap \ldots \cap E^{j_{p+1}} \to E^{j_1} \cap \ldots \cap E_{j_{k-1}} \cap E_{j_{k+1}} \cap \ldots \cap E^{j_{p+1}},$$

and the natural augmentation maps $a_p \colon E_p \to E$ define a strict simplicial variety $E_\bullet$, that is a simplicial resolution of E [GNAPGP88].

The Mayer-Vietoris hyper-resolution of E is fairly simple. Very few terms are non-zero as E is a simple normal crossings divisor in a 3-fold. In particular, the p-fold intersections of components of E are empty for $p > 3$, so that E_p is non-empty only for $0 \leq p \leq 2$.

Each E_p is non-singular and projective, hence there exists a cohomological Hodge complex $((\mathbb{Q}_{E_p}, W), (\Omega_{E_p}, F, W))$ on E_p. These Hodge complexes yield a cohomological mixed Hodge complex $K = ((K_\mathbb{Q}, W), (K_\mathbb{C}, W, F))$ on E and, in particular, the following resolution of $\mathbb{Q}_E$:

$$0 \to \mathbb{Q}_E \to a_{0*}\mathbb{Q}_{E_0} \to a_{1*}\mathbb{Q}_{E_1} \to a_{2*}\mathbb{Q}_{E_2} \to 0.$$

I omit results related to the Hodge filtration. The weight spectral sequence reads:

$${}_wE_1^{p,q} = H^q(E_p, \mathbb{C}).$$

The spectral sequence $({}_wE_r, d_r)$ abuts to $H^{p+q}(E)$. It degenerates at ${}_wE_2$ and $d_1^{p,q}$ is induced by $\sum_{k=0}^{p-1}(-1)^{p+k}(d_k^{p-1})^*$ [GNAPGP88]. In particular,

$$Gr_p^W H^{p+q}(Y) = {}_wE_2^{p,q} = H({}_wE_1^{p-1,q} \to {}_wE_1^{p,q} \to {}_wE_1^{p+1,q}).$$

In the present case, $E_p = \emptyset$ for $p > 2$, E_0 has dimension 2, E_1 has dimension 1 and E_2 is a set of points. The only non-zero terms in the spectral sequence associated to the weight filtration are ${}_wE_1^{0,q}$ for $0 \leq q \leq 4$, ${}_wE_1^{1,r}$ for $0 \leq r \leq 2$ and ${}_wE_1^{2,0}$.

The definition of the map $d_1^* : H^\bullet(E_0) \to H^\bullet(E_1)$ implies that $H^2(E)$ is of pure weight 2.

I give the result of the computation of $H^4(E)$, which is used in the next subsection:

$$H^4(E, \mathbb{C}) = Gr_4^W H^4(E) \simeq \oplus \mathbb{C}[E^j].$$

2.3.3 Mixed Hodge structure on the local cohomology $H^\bullet_\Sigma(X, \mathbb{C})$

I use the same notation as above. The cohomology of the pair $H^\bullet(Y, Y \smallsetminus \Sigma)$, and therefore the local cohomology $H^\bullet_\Sigma(Y)$, carries a mixed Hodge structure. These mixed Hodge structures are independent of the choice of resolution [Ste83].

I recall some results on local cohomology groups associated to isolated singularities.

Theorem 2.3.1 (Goresky-MacPherson vanishing)**.** *[Ste83] Let Y_i be a contractible Stein neighbourhood of the isolated singular point P_i and $\widetilde{Y}_i$ a good resolution with exceptional divisor E_i. The restriction map*

$$H^k(\widetilde{Y}_i, \mathbb{Q}) \to H^k(\widetilde{Y}_i \smallsetminus E_i, \mathbb{Q}) \simeq H^k(Y_i \smallsetminus \{P_i\}, \mathbb{Q})$$

is surjective for $k \leq 2$ and the zero map for $k \geq 3$.

Proposition 2.3.2 (Fujiki duality)**.** *[Fuj80] There is a duality isomorphism:*

$$H^k_E(\widetilde{X}) \simeq \mathrm{Hom}(H^{6-k}(E), \mathbb{Q}(-3)), \tag{2.5}$$

where $\mathbb{Q}(-3)$ is the mixed Hodge structure on $\mathbb{C}$ with rational lattice $\frac{1}{(2\pi i)^3}\mathbb{Q}$ purely of weight $(3,3)$.

Proposition 2.3.3. *[Ste83] The local cohomology groups $H^k_{\{P_i\}}(Y_i)$ fit in the following exact sequences:*

$$0 \to H^k_{E_i}(\widetilde{Y}_i) \to H^k(E_i) \to H^{k+1}_{\{P_i\}}(Y_i) \to 0 \quad \textit{for} \quad k < 3 \tag{2.6}$$

$$0 \to H^3_{E_i}(\widetilde{Y}_i) \to H^3(E_i) \to 0 \tag{2.7}$$

$$0 \to H^k_{\{P_i\}}(Y_i) \to H^k_{E_i}(\widetilde{Y}_i) \to H^k(E_i) \to 0 \quad \textit{for} \quad k > 3. \tag{2.8}$$

Remark 2.3.4. The exact sequences in Proposition 2.3.3 are obtained by applying Goresky-MacPherson vanishing to the local cohomology exact sequence associated to $H^\bullet_{E_i}(\widetilde{Y}_i)$, noting that as Y_i is contractible, $H^k(\widetilde{Y}_i) = H^k(E_i)$.

Remark 2.3.5. Fujiki shows [Fuj80] that the maps $\alpha_k \colon H^k_{E_i}(\widetilde{Y}_i) \to H^k(E_i)$ satisfy $\alpha_k =^t \alpha_{6-k}$ under the duality (2.5). In particular, this shows that for $k < 3$, the exact sequences:

$$0 \to H^k_{E_i}(\widetilde{Y}_i) \to H^k(E_i) \to H^{k+1}_{\{P_i\}}(Y_i) \to 0$$

and

$$0 \to H^{6-k}_{\{P_i\}}(Y_i) \to H^{6-k}_{E_i}(\widetilde{Y}_i) \to H^{6-k}(E_i) \to 0$$

are dual. The local cohomology groups $H^{k+1}_{\{P_i\}}(Y_i)$ and $H^{6-k}_{\{P_i\}}(Y_i)$ are dual to each other.

Lemma 2.3.6. *The local cohomology of Y at P_i is:*

$$H^k_{\{P_i\}}(Y) = (0), \quad \textit{for} \quad k \neq 1,3,4,6$$
$$H^1_{\{P_i\}}(Y) \simeq H^6_{\{P_i\}}(Y) \simeq \mathbb{C}[E_i]$$
$$H^3_{\{P_i\}}(Y) \simeq H^4_{\{P_i\}}(Y) \simeq H^2(E_i)/\sum \mathbb{C}[E_i^j]$$

Proof. First, notice that, by excision, $H^k_{\{P_i\}}(Y_i) = H^k_{\{P_i\}}(Y)$. The local cohomology exact sequence shows that $H^0_{\{P_i\}}(Y_i) = (0)$. As Remark 2.3.5 shows, the result need only be checked for $k = 1, 2$.

The result is clear for $k = 1$ by the exact sequence (2.6), because $H^0_{E_i}(\widetilde{Y}_i)$ is dual to $H^6(E_i) = (0)$.

For $k = 2$, as the singularities are assumed to be rational $H^1(\widetilde{Y}_i, \mathcal{O}_{\widetilde{Y}_i}) = (0)$ and $H^1(E_i) \simeq H^1(\widetilde{Y}_i) = (0)$, the result follows from the exact sequence (2.6).

The exact sequence (2.6) gives:

$$0 \to H^2_{E_i}(\widetilde{Y}_i) \to H^2(E_i) \to H^3_{\{P_i\}}(Y) \to 0.$$

The $H^k(E_i)$ were determined in Section 2.3.2 and, by Fujiki duality, $H^2_{E_i}(Y_i)$ is dual to $H^4(E_i)$, so that:

$$H^3_{\{P_i\}}(Y) = H^2(E_i)/\bigoplus \mathbb{C}[E_i^j].$$

The singularity $P_i \in Y$ is rational and Y_i is a contractible neighbourhood:

$$H^1(Y_i, \mathcal{O}^*_{Y_i}) = H^2(Y_i, \mathbb{Z}) = H^2(\widetilde{Y}_i, \mathbb{Z}) = H^2(E_i, \mathbb{Z}).$$

The local cohomology of Y at P_i is of the form:

$$H^3_{\{P_i\}}(Y) = H^1(Y_i, \mathcal{O}^*_{Y_i})/\bigoplus \mathbb{C}[E_i^j].$$

$\square$

Chapter 3

The category of weak$*$ Fano 3-folds

Let Y be a terminal Gorentein non $\mathbb{Q}$-factorial Fano 3-fold and denote by X a small $\mathbb{Q}$-factorialisation of Y. The Picard rank of X is equal to the rank of the Weil group of Y. If Y has Picard rank 1 and defect 1, the Picard rank of X is 2. One of three things occurs: either X has a structure of del Pezzo fibration over $\mathbb{P}^1$, or of conic bundle over $\mathbb{P}^2$, or there exists an extremal contraction $\phi\colon X \to X'$, where X' is a $\mathbb{Q}$-factorial terminal Fano 3-fold with Picard rank 1. As X has isolated hypersurface singularities, a direct geometric analysis of the contraction ϕ is possible (Theorem 3.2.1) and this describes the Weil non-$\mathbb{Q}$-Cartier divisors that can lie on Y.

One could hope to bound the defect and to obtain some information on the Weil group of Y by running a Minimal Model Program (MMP) on X. The category of terminal $\mathbb{Q}$-factorial 3-folds is stable under the operations of the MMP. In general, however, this approach is too naive: if $\phi\colon X \to X'$ is an extremal contraction, X' does not necessarily have hypersurface singularities and nef anticanonical divisor. I show that if Y does not contain a plane, the MMP can be run on X, and that the 3-folds encountered when doing so are terminal Gorenstein and have nef and big anticanonical divisor.

In this chapter, I define the category of *weak$*$ Fano 3-folds*. If X is a weak$*$ Fano 3-fold, its anticanonical model Y is a terminal Gorenstein Fano 3-fold that does not contain a plane. The 3-fold Y is in general not $\mathbb{Q}$-factorial. Conversely, any small $\mathbb{Q}$-factorialisation X of a terminal Gorenstein Fano 3 fold Y whose anticanonical ring is generated in degree 1 is a weak$*$ Fano unless Y contains a plane.

This chapter shows that the category of weak$*$ Fano 3-folds is stable under the operations of the MMP. If X is a weak$*$ Fano, the end product of the MMP on X is described.

3.1 Definitions and basic results

Definition 3.1.1. A 3-fold X is *weak Fano* if X is Gorenstein, terminal and if its anticanonical divisor $-K_X$ is nef and big.

The *anticanonical ring* of X is $R(X, -K_X) = \bigoplus_{n\in\mathbb{N}} H^0(X, -nK_X)$.

The *anticanonical model* of X is $Y = \operatorname{Proj} R(X, -K_X)$.

Definition 3.1.2. X is *weak$*$* if moreover:

1. X is $\mathbb{Q}$-factorial,
2. the morphism $X \to Y$ is small,
3. Y contains no plane $\mathbb{P}^2$ with $-K_Y|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$,
4. the anticanonical ring $R = R(X, -K_X)$ is generated by R_1.

Remark 3.1.3. The category of weak$*$ Fano 3-folds is stable under flops.

Definition-Lemma 3.1.4. If X is a weak Fano 3-fold, then:

$$h^0(X, -mK_X) = 2m + 1 + \frac{1}{12}m(m+1)(2m+1)(-K_X)^3.$$

Denote by $g = \frac{1}{2}(-K_X)^3 + 1$ the *genus* of X; in particular:

$$h^0(X, -K_X) = g + 2.$$

I recall here the statement of the Kawamata-Viehweg vanishing theorem.

Theorem 3.1.5 (Kawamata-Viehweg vanishing,[KM98])**.** *Let (X, Δ) be a proper Kawamata log terminal (klt) pair. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathcal{O}_X(-N)) = (0)$ for $i < \dim X$.*

Proof of 3.1.4. The Kawamata-Viehweg vanishing theorem applied to the pair $(X, 0)$ and to the divisor $-(m+1)K_X$ shows that:

$$H^i\big(X, \mathcal{O}_X((m+1)K_X)\big) = (0)$$

for $i < 3$ and $m \geq 0$. By Serre duality, this implies

$$H^j(X, \mathcal{O}_X(-mK_X)) = (0)$$

for $j > 0$ and $m \geq 0$ and

$$\chi(\mathcal{O}_X(-mK_X)) = h^0(X, -mK_X).$$

The Riemann-Roch theorem for Gorenstein 3-folds [Rei87] reads:

$$\chi(\mathcal{O}_X(-mK_X)) = (1+2m)\chi(\mathcal{O}_X) + \frac{1}{12}m(m+1)(2m+1)(-K_X)^3.$$

□

Lemma 3.1.6 (Cone theorem for weak Fano 3-folds)**.** *If X is a weak Fano 3-fold then $NE(X)$ is a finite rational polyhedron (in particular $NE(X) = \overline{NE}(X)$). If $R \subset NE(X)$ is an extremal ray, then either $K_X \cdot R < 0$, or $K_X \cdot R = 0$ and there exists an effective divisor D such that $D \cdot R < 0$. There is a contraction morphism ϕ_R associated to each extremal ray R. If ϕ_R is small, it is a flopping contraction.*

Proof. The 3-fold X has terminal singularities. By the standard cone theorem [KMM87, Theorem 4-2-1]:

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum C_j$$

where the extremal rays C_j are discrete in the half space $\{K_X < 0\}$ and can be contracted. Since $-K_X$ is nef, for any $z \in N^1(X)$, $K_X \cdot z \geq 0$ if and only if $K_X \cdot z = 0$. The anticanonical divisor $-K_X$ is big: for some integer $m > 0$, $m(-K_X) \sim A + D$, where A is an ample divisor and D is effective. By the Nakai-Moishezon criterion for ampleness, if $K_X \cdot z = 0$ then $D \cdot z < 0$. In particular, for $0 < \epsilon \ll 1$,

$$\overline{NE}(X) \subset K_{X<0} \cup (K_X + \epsilon D)_{<0}.$$

By the usual compactness argument, NE(X) is a finitely generated rational polyhedron. Extremal rays can be contracted by the contraction theorem [KMM87, Theorem 3-2-1]. Finally, if ϕ_R is small, R flips or flops. A flipping curve γ on a terminal 3-fold X satisfies $-K_X \cdot \gamma < 1$ [Ben85]. The anticanonical divisor is Cartier and nef: ϕ_R is a flopping contraction. □

Theorem 3.1.7. *If X is a weak Fano 3-fold then one of the following holds:*

1. *The linear system $|-K_X|$ has a non-singular section.*

2. *The anticanonical model of X, $Y = \operatorname{Proj} R(X, -K_X)$, is birational to a special complete intersection $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$ with a node. More precisely, Y is defined by equations of the form:*

$$\begin{cases} a_2 = 0 \\ w^2 + v^3 + va_4 + a_6 = 0 \end{cases}$$

where the coordinates x_i have degree 1, v and w have degree 2 and 3 respectively, and where each a_j is a homogeneous form of degree j in the variables x_i.

Remark 3.1.8. If X is a weak∗ Fano 3-fold, the linear system $|-K_X|$ is basepoint free. The anticanonical divisor of X is nef and big, hence by the basepoint free theorem [KM98, Theorem 3.3], the linear system $|-nK_X|$ has no basepoint for n sufficiently large. If the anticanonical ring of X is generated in degree 1, the map

$$H^0(X, -K_X)^{\otimes n} \to H^0(X, -nK_X)$$

is surjective for any $n \in \mathbb{N}$. The linear system $|-K_X|$ is itself basepoint free.

I first prove the following lemma:

Lemma 3.1.9. *Let X be a weak Fano 3-fold. The general member S of $|-K_X|$ is a $K3$ surface with no worse than Du Val singularities.*

Definition 3.1.10. Let X be a weak Fano 3-fold. The *Fano index* of X, $i(X)$, is the maximal integer such that $-K_X = i(X)H$ with H a nef and big Cartier divisor.

Remark 3.1.11. The Fano index of a weak Fano 3-fold X with small anticanonical map is the same as that of its anticanonical model Y. Indeed, the anticanonical map $f\colon X \to Y$ is birational and small. As the divisor $-K_X$ is relatively trivial, by the basepoint free/contraction theorem [Kaw88, Corollary 1.5], $-K_X = f^*D$, with D ample. The map f is crepant, so that $D = -K_Y$ and the Fano indices of Y and X are equal.

For a weak Fano 3-fold X, by Kawamata-Viehweg vanishing and the Riemann-Roch theorem:

$$h^0(X,H) = 1 + \frac{2}{i(X)} + \frac{1}{12}H^3(1+i(X))(2+i(X)) \geq 2.$$

Lemma 3.1.12. $\mathrm{Pic}(X)$ *has no torsion.*

Proof. Let D be a torsion divisor on X. By definition, there exists a smallest integer $m > 1$ such that $mD \sim 0$. The divisor $-K_X + D$ is nef and big and therefore, by Kawamata-Viehweg vanishing,

$$H^i(X,D) = H^i(X, K_X + (-K_X + D)) = (0) \quad \text{for} \quad i > 0.$$

Moreover, $H^0(X,D) = (0)$, as otherwise D itself would be linearly equivalent to 0. The divisor D has Euler characteristic $\chi(\mathcal{O}_X(D)) = (0)$.

As D is numerically trivial, by the Riemann-Roch theorem,

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X).$$

This contradicts $\chi(\mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$ on a weak Fano 3-fold. $\square$

Remark 3.1.13. The divisor class H of Definition 3.1.10 is uniquely determined.

Definition 3.1.14. Let X be a weak$*$ Fano 3-fold of Fano index $i(X)$. The *degree* of X is H^3. The *anticanonical degree* of X is $-K_X^3$.

I use Kawamata's basepoint free technique to show that $|-K_X|$ has a section with canonical singularities. I recall the notion and properties of non-Kawamata log-terminal (non-klt) centres introduced in [Kaw97a].

Definition 3.1.15. Let X be a normal variety and let $D = \sum d_i D_i$ be an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier.

1. The *non-klt locus* of (X,D) is the set of points where (X,D) is not klt, that is:
$$\mathrm{nklt}(X,D) = \{x \in X : (X,D) \text{ is not klt at } x\}.$$

2. If the pair (X,D) is log canonical (lc), a subvariety $W \subset X$ is a *log canonical centre* or *lc centre* for the pair (X,D) if there is a log resolution $\mu\colon \widetilde{X} \to X$ and a prime divisor E on $\widetilde{X}$ with discrepancy coefficient $e = -1$ such that $\mu(E) = W$.

Remark 3.1.16. Let (X, D) be a pair and $\mu\colon \widetilde{X} \to X$ a log resolution. Denote by $E_1, \cdots, E_m$ the prime divisors with discrepancy less than or equal to -1. The non-klt locus of (X, D) is:

$$\mathrm{nklt}(X, D) = \mu(E_1 + \cdots E_m).$$

Remark 3.1.17. Let (X, D) be a lc pair. A divisor W is a codimension 1 lc centre if $D = W + D'$, where the support of D' does not contain W.

Proposition 3.1.18. *[Kaw97a] Let X be a klt variety and (X, D) a lc pair.*

1. *Let W_1 and W_2 be lc centres of (X, D). If W is an irreducible component of $W_1 \cap W_2$, then W is a lc centre as well. In particular, if $x \in \mathrm{nklt}(X, D)$, there is a well defined minimal lc centre containing x.*

2. *If W is a minimal lc centre, then it is normal.*

Proposition 3.1.19 (Subadjunction,[Kaw97b, Kol07])**.** *Let X be a normal variety with klt singularities and let D be an effective $\mathbb{Q}$-Cartier divisor such that the pair (X, D) is log canonical. Let $W \subset \mathrm{nklt}(X, D)$ be an isolated minimal lc centre. There is an effective $\mathbb{Q}$-divisor D_W on W such that:*

1. *The pair (W, D_W) is klt,*

2. *$(K_X + D)_{|W} \sim_{\mathbb{Q}} K_W + D_W$.*

The choice of the divisor D_W is not canonical: D_W is the sum of a boundary divisor B_W and of a general divisor M_W in a nef $\mathbb{Q}$-linear equivalence class $J(W, D)$. Write $D = D' + D''$, where D' (resp. D'') is the sum of components of D that contain (resp. do not contain) W. The boundary part B_W is uniquely determined and is supported on $D''_{|W}$. The moduli part $J(W, D)$ is determined only by the pair (X, D'); the choice of $M_W \in J(W, D)$ is not canonical.

Proof of 3.1.9. I follow ideas exposed in [Rei83, Ale91, Mel99]. As is recalled in Definition-Lemma 3.1.4, $h^0(X, -K_X) = g + 2$, where g denotes the genus of X. Let S be a general member of the linear system $|-K_X|$.

Step 1. *The surface S is connected.*

The sequence

$$0 \to H^0(X, \mathcal{O}_X(-S)) \to H^0(X, \mathcal{O}_X) \to H^0(S, \mathcal{O}_S) \to H^1(X, \mathcal{O}_X(-S)) \to \cdots$$

is exact. The surface S is linearly equivalent to $-K_X$. Kawamata-Viehweg vanishing shows that:

$$H^0(X, \mathcal{O}_X) \simeq H^0(S, \mathcal{O}_S)$$

and S is connected.

Set up 3.1.20. The anticanonical divisor $-K_X$ is Cartier and nef and big and X has terminal singularities, hence by [KM98, Proposition 2.61], there is a resolution $\mu\colon \widetilde{X} \to X$ and a divisor with normal crossings $\sum E_i$ such that:

1. $K_{\widetilde{X}} = \mu^*(K_X) + \sum a_i E_i$, with $a_i \in \mathbb{N}$ and $a_i > 0$ if and only if E_i is exceptional;
2. $\mu^*(-K_X) - \sum p_i E_i$ is an ample $\mathbb{Q}$-Cartier divisor on $\widetilde{X}$, for suitable $p_i \in \mathbb{Q}, 0 \leq p_i \ll 1$.

A small perturbation of the rational numbers p_i does not affect the ampleness of $\mu^*(-K_X) - \sum p_i E_i$. We can blow up further to obtain a new resolution $\mu\colon \widetilde{X} \to X$, that still satisfies the conditions above, and that is a log resolution of the linear system $|-K_X|$. Then:

$$\mu^*(|-K_X|) = |L| + \sum r_i E_i, \tag{3.1}$$

where $|L|$ is a free linear system, $r_i \in \mathbb{N}$ and $r_i > 0$ if $\mu(E_i)$ is in the base locus of $|-K_X|$. The divisor $\sum E_i$ has simple normal crossings.

Note that μ determines a log resolution of the general member S of $|-K_X|$: the divisor $\mu^*(S) + \mathrm{Exc}(\mu)$ has simple normal crossing support. Indeed, if the divisor $\sum E_i$ has simple normal crossings on the non-singular variety $\widetilde{X}$ and if L is a general section of the free linear system $|L|$, then $L + \sum E_i$ has simple normal crossings as well. Then:

$$\mu^*(S) = L + \sum r_i E_i, \tag{3.2}$$

where L is a member of the free linear system $|L|$. Reid proves that the free linear system $|L|$ is not composed with a pencil [Rei83].

Step 2. *The surface S is irreducible, reduced and has no worse than du Val singularities if no component of the non-klt locus of (X, S) is contained in the base locus of $|-K_X|$.*

If the free linear system $|L|$ is not composed with a pencil, by Bertini's theorem, its general member is reduced and irreducible; the surface S is irreducible and reduced if the linear system $|-K_X|$ has no base component. Recall that a base component of $|-K_X|$ is the image by μ of a divisor E_i with $a_i = 0$ and $r_i \geq 1$.

The surface S has canonical singularities if the discrepancy of every exceptional divisor appearing in a resolution of S is non-negative. The restriction of μ to $\mu_*^{-1}(S) = L$ is a resolution of S. By Bertini, S is non-singular away from the base locus of $|-K_X|$, so that S has canonical singularities if $a_i - r_i \geq 0$ (that is: $a_i - r_i > -1$, as a_i and r_i are natural numbers) for every exceptional divisor with centre contained in the base locus of $|-K_X|$.

In the formalism introduced above, the surface S is reduced, irreducible and has no worse than canonical singularities, if no component of the non-klt locus of (X, S) is contained in the base locus of $|-K_X|$.

Remark 3.1.21. The rational numbers a_i are non-negative because X is terminal. If $W_i = \mu(E_i)$ is a lc centre, the coefficient r_i is positive and W_i is contained in the base locus of $|-K_X|$.

Step 3. *Assume that the pair (X, S) is not purely log terminal. For some $0 < b \leq 1$, the pair (X, bS) is log canonical but not klt.*

The pair (X, S) is not log terminal, therefore there exists an index $i_0 = 0$ such that $a_0 - r_0 \leq -1$, that is $a_0 + 1 \leq r_0$. Define:

$$b = \min\{\frac{a_i + 1}{r_i}\} \leq 1, \tag{3.3}$$

where the minimum is taken over the indices i such that r_i is non-zero. The rational number b is positive because $a_i \geq 0$ for all i. The pair (X, bS) is strictly log canonical, as the prime divisors E_i for the indices where the minimum is attained have discrepancy -1.

Step 4. *There is a divisor $D \in |-K_X|$ and a rational number $k \in \mathbb{Q}$, such that the non-klt locus of $(X, b(1-\epsilon)S + \epsilon kD)$ is irreducible for all $0 < \epsilon \ll 1$. The lc centre $W = \mathrm{nklt}(X, b(1-\epsilon)S + \epsilon kD)$ is a minimal lc centre for (X, D).*

Let E_i for $0 \leq i \leq d$ be the prime divisors such that:

$$\mathrm{nklt}(X, bS) = \mu(E_0) \cup \cdots \cup \mu(E_d),$$

so that:

$$K_{\widetilde{X}} = \mu^*(K_X + bS) - bL + \sum_{i \geq d+1} (a_i - br_i)E_i - \sum_{0 \leq i \leq d} E_i \tag{3.4}$$

with $a_i - br_i > -1$ for all $i \geq d+1$.

As a slight increase of the p_i does not affect the ampleness of the $\mathbb{Q}$-Cartier divisor $\mu^*(-K_X) - \sum p_i E_i$, we may assume that the minimum

$$k = \min_{0 \leq i \leq d} \{\frac{br_i}{p_i}\}$$

is attained only for one index $i_0 = 0$.

Let D be an element of the linear system $|-K_X|$ such that $\mu^*(D) - \sum p_i E_i$ is an ample divisor; from (3.4), we get, for $0 < \epsilon \ll 1$:

$$\begin{aligned} K_{\widetilde{X}} &= \mu^*(K_X + b(1-\epsilon)S + \epsilon k D) - b(1-\epsilon)L - \epsilon k(\mu^*(D) - \sum p_i E_i) \\ &+ \sum_{i \geq d+1} (a_i - br_i + \epsilon(br_i - kp_i))E_i + \sum_{1 \leq i \leq d} (-1 + \epsilon(br_i - kp_i))E_i - E_0. \end{aligned} \tag{3.5}$$

By construction, for all $1 \leq i \leq d$ and all $\epsilon > 0$,

$$-1 + \epsilon(br_i - kp_i) > -1.$$

For ϵ sufficiently small, $a_i - br_i + \epsilon(br_i - kp_i) > -1$ for $i \geq d+1$ and $\epsilon k(\mu*(-K_X) - \sum p_i E_i)$ is a boundary divisor.

From now on, I include L and a Cartier multiple of the ample divisor $\mu*(-K_X) - \sum p_i E_i$ among the divisors E_i. Denote by $b_{i,\epsilon}$ the discrepancy coefficient of E_i and set:

$$E = E_0, \quad F = \sum_{i \neq 0, b_{i,\epsilon} < 0} -b_{i,\epsilon}E_i, \quad \text{and } A = \sum_{b_{i,\epsilon} \geq 0} b_{i,\epsilon}E_i.$$

The divisor A is effective and exceptional, since $b_{i,\epsilon} > 0$ for $0 < \epsilon \ll 1$ is only possible for $a_i > 0$, i.e. for E_i exceptional. The divisor F is a boundary, and (3.5) reads:

$$K_{\widetilde{X}} = \mu^*(K_X + b(1-\epsilon)S + \epsilon k D) + A - F - E. \tag{3.6}$$

The non-klt locus of $(X, b(1-\epsilon)S + \epsilon k D)$ is the irreducible component $\mu(E)$.

Step 5. *If W is the unique lc centre for $(X, b(1-\epsilon)S+\epsilon kD)$, then W is not contained in* $\mathrm{Bs}\,|-K_X|$.

Consider the divisor:

$$N(t) = \mu^*(-tK_X) - (K_{\widetilde{X}} + F) + A - E. \tag{3.7}$$

From equation (3.6) we see that:

$$N(t) = \mu^*(-tK_X) - \mu^*(K_X + b(1-\epsilon)S + \epsilon kD)$$

and $N(t)$ is numerically equivalent to $\mu^*\big(-(t+1-b+\epsilon(b-k))K_X\big)$; $N(t)$ is nef and big for $t+1-b+\epsilon(b-k) > 0$.

By the Kawamata-Viehweg vanishing theorem,

$$H^1(\widetilde{X}, N(t) + K_{\widetilde{X}} + F) = (0)$$

for $t+1-b+\epsilon(b-k) > 0$. The constant b is less than or equal to 1; for $t = 1$ and ϵ sufficiently small,

$$H^1(\widetilde{X}, \mu^*(-K_X) + A - E) = 0,$$

and the map

$$H^0(\widetilde{X}, \mu^*(-K_X) + A) \to H^0(E, (\mu^*(-K_X) + A)_{|E})$$

is surjective.

By the Leray spectral sequence, $H^0(\widetilde{X}, \mu^*(-K_X) + A) \simeq H^0(X, -K_X)$, and any section of $H^0(\widetilde{X}, \mu^*(-K_X) + A)$ that does not vanish on E pushes forward to a section of $H^0(X, -K_X)$ not vanishing identically on W; if $H^0(E, (\mu^*(-K_X)+A)_{|E})$ is non-trivial, W is not contained in the base locus of $|-K_X|$.

The following cases need to be considered:

1. If $\mathrm{Codim}(W) = 3$, W is a closed point $\{x\}$. Then:

   $$H^0(\widetilde{X}, \mu^*(-K_X) + A) \to H^0(E, (\mu^*(-K_X) + A)_{|E}) \simeq H^0(E, A_{|E}) \to 0,$$

   and the Leray spectral sequence shows that $h^0(\widetilde{X}, A) = h^0(X, \mathcal{O}_X) = 1$. The divisor $A_{|E}$ is effective, therefore $h^0(E, A_{|E}) = 1$. Hence, $\{x\}$ does not belong to the base locus of $|-K_X|$.

2. If $\mathrm{Codim}(W) = 2$, W is a curve Γ. By Propositions 3.1.18 and 3.1.19, Γ is non-singular and there is an effective divisor M_Γ on Γ such that:

$$(K_X + b(1-\epsilon)S + \epsilon k D)_{|\Gamma} = (1-b)K_X = K_\Gamma + M_\Gamma.$$

The divisor K_X has non-positive degree on Γ and $K_X + b(1-\epsilon)S + \epsilon k D$ is numerically equivalent to $(1 - b - \epsilon(k-b))K_X$, hence, as $0 < \epsilon \ll 1$ can be taken arbitrarily small, Γ has arithmetic genus $p_a(\Gamma) = 0$ or 1. A flopping curve on X is rational [Kol89]: if Γ is elliptic, $(-K_X)_{|\Gamma}$ has positive degree. In both cases, $h^0(\Gamma, -K_{X|\Gamma}) > 0$ and

$$H^0(E, (\mu^*(-K_X)+A)_{|E}) \supset H^0(E, (\mu^*(-K_X))_{|E}) \simeq H^0(\Gamma, -K_{X|\Gamma}) \neq (0)$$

shows that Γ is not contained in $\mathrm{Bs}\,|-K_X|$.

3. If $\mathrm{Codim}(W) = 1$, $f_*(E) = W$ is a base component of the linear system $|-K_X|$.

Denote by $P(t)$ the polynomial of degree 2 defined as the Euler characteristic $\chi((t\mu^*(-K_X) + A)_{|E})$ of the linear system $|t\mu^*(-K_X) + A|_{|E}$ on the divisor E.

By definition of the divisor $N(t)$ (3.7),

$$(t\mu^*(-K_X) + A)_{|E} = (N(t) + K_{\widetilde{X}} + E + F)_{|E}.$$

The divisor E is Cartier, hence $(K_{\widetilde{X}} + E)_{|E} = K_E$; the support of $F \cup E$ has simple normal crossings because μ is a log resolution, the divisor $F_{|E}$ therefore is a boundary divisor and $P(t) = h^0((t\mu^*(-K_X) + A)_{|E}))$ by the Kawamata-Viehweg vanishing for $t + 1 - b + \epsilon(b-k) > 0$. Recall that $W = \mu(E_0)$ is a minimal lc centre of codimension 1 of the pair (X, bS), hence W is a fixed component of $|-K_X|$. From (3.3), the multiplicity of the fixed component W in $|-K_X|$ is $\frac{1}{r_0}$.

If $b = 1$, $r_0 = 1$ and W is a fixed component of $|-K_X|$ with multiplicity 1. Since the divisor $-K_X$ is connected and movable, $S \in |-K_X|$ is singular along a codimension 2 set $\Gamma \subset W$. This contradicts W being a minimal lc centre of (X, S).

The multiplicity r_0 of the fixed component W is therefore strictly greater than 1 and $b < 1$. For ϵ sufficiently small, $1 - b + \epsilon(b-k) > 0$ and $P(t) = h^0((t\mu^*(-K_X) + A))$ for all $t \geq 0$.

The divisor W is a base component of the linear system $|-K_X|$, therefore $P(1) = h^0(E, (\mu^*(-K_X) + A)_{|E}) = h^0(W, -K_X) = 0$. As $h^0(\widetilde{X}, A) = h^0(X, \mathcal{O}_X) = 1$ and $A_{|E}$ is effective, $P(0) = 1$. The leading coefficient of $P(t)$ is $\frac{((\mu^*(-K_X))_{|E})^2}{2!} = \frac{(-K_X)^2 \cdot W}{2!}$ and $P(t)$ can be written:

$$P(t) = \frac{1}{2}\big((-K_X)^2 \cdot W \cdot t^2 + (-(-K_X)^2 \cdot W - 2) \cdot t + 2\big).$$

By Riemann-Roch, however,

$$\begin{aligned} P(t) &= \frac{1}{2}((t\mu^*(-K_X) + A)_{|E}) \cdot ((t\mu^*(-K_X) + A)_{|E} - K_E) + 1 \\ &= \frac{1}{2}(t\mu^*(-K_X) + A)_{|E})^2 - \frac{1}{2}K_E \cdot ((t\mu^*(-K_X) + A)_{|E})) + 1. \end{aligned}$$

The leading coefficients in both expressions agree, equating the coefficients of the terms of degree 1 in t shows that:

$$-(-K_X)^2 \cdot W - 2 = (A_{|E} - \frac{1}{2}K_E)\mu^*(-K_X)_{|E}. \tag{3.8}$$

The anticanonical divisor $-K_X$ is nef and big, so that the expression on the left hand side is strictly negative. The divisor E is Cartier, therefore, by adjunction:

$$\begin{aligned} K_E = (K_{\widetilde{X}} + E)_{|E} &= (\mu^*(-K_X + b(1-\epsilon)S + \epsilon k D) + A - F)_{|E} \\ &\sim \big(\mu^*(-(1 - b + \epsilon(b-k))K_X) + A - F\big)_{|E} \end{aligned}$$

where A is an effective exceptional divisor and F is effective and supported on $\cup_{1 \leq i \leq d} E_i$.

The left hand side of (3.8) becomes (up to a factor of 2):

$$\begin{aligned} &(A + F)_{|E} \cdot \mu^*(-K_X)_{|E} + \mu^*(-(1 - b + \epsilon(b-k))K_X)_{|E} \cdot \mu^*(-K_X)_{|E} \\ &\qquad \sim (A + F)_{|E} \cdot \mu^*(-K_X)_{|E} + (1 - b + \epsilon(b-k))(\mu^*(-K_X)_{|E})^2. \end{aligned}$$

The divisor $(A + F)_{|E}$ is effective and the first term is non-negative because the divisor $\mu^*(-K_X)_{|E}$ is nef. Recall that ϵ has be chosen so that $(1 - b + \epsilon(b-k)) > 0$.

The left hand side of (3.8) is positive; this yields a contradiction. The linear system $|-K_W|$ has no base component.

I have shown that the linear system $|-K_X|$ has no base component. The free linear system $|L|$ is not composed with a pencil [Rei83] and hence, by Bertini, the general member $S \in |-K_X|$ is irreducible and reduced. For all indices i of exceptional divisors E_i whose centre is contained in the base locus of $|-K_X|$, $a_i - r_i \geq 0$. The surface S is irreducible, reduced and has canonical singularities.

Step 6. *The general member $S \in |-K_X|$ is a $K3$ surface with rational double points.*

I first recall the definition of $K3$ surfaces:

Definition 3.1.22. A $K3$ surface is a surface S with no worse than Du Val singularities such that:

1. $H^1(S, \mathcal{O}_S) = (0)$,
2. The canonical divisor class K_S is trivial.

Consider the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0. \tag{3.9}$$

By Kawamata-Viehweg vanishing,

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X(-S)) = (0),$$

and $H^1(S, \mathcal{O}_S) = (0)$. By adjunction, $K_S = (K_X + S)_{|S} \sim 0$ and S is a $K3$ surface with rational double points. □

Tensor the exact sequence (3.9) with $\mathcal{O}_X(-K_X)$. The surface S is a section of $|-K_X|$. By the Kawamata-Viehweg vanishing theorem, the map

$$H^0(X, -K_X) \to H^0(S, -K_{X|S})$$

is surjective. It follows that the base points of $|-K_X|$ are precisely the base points of $|-K_{X|S}|$.

I study the linear system $|-K_{X|S}|$ on the surface S to complete the proof of Theorem 3.1.7. Linear systems on non-singular K3 surfaces were studied by Saint-Donat [SD74]. I recall his — now classical — results below.

Theorem 3.1.23. *[SD74] Let X be a non-singular $K3$ surface and D a nef and big effective divisor on X. One of the following is true:*

1. *The linear system $|D|$ is basepoint free, and:*
   (a) *Either: the morphism $\phi_{|D|}$ induced by $|D|$ on X is birational on its image $\overline{X}$, $\overline{X}$ has rational double points, and $\phi_{|D|}$ is the minimal resolution.*
   (b) *Or: D is hyperelliptic and $\phi_{|D|}$ is generically 2-to-1.*
2. *D is monogonal, that is $D \sim kE + \Gamma$, with $|E|$ a free elliptic pencil and Γ a (-2)-curve, which is the fixed component of $|D|$. In that case, $\Gamma^2 = -2$, $\Gamma \cdot E = 1$.*

I prove the following easy extension of Saint-Donat's result to linear systems on K3 surfaces with rational double points:

Theorem 3.1.24. *Let S be a $K3$ surface with rational double points and let D be a nef and big Cartier divisor on S. Then one of the following holds:*

1. *$|D|$ is basepoint free.*
2. *D is monogonal. In that case:*
$$D \sim kE + \Gamma,$$
   where $|E|$ is a free elliptic pencil, $E \cdot \Gamma = 1$ and Γ is a (-2)-curve contained in the non-singular locus of S,
3. *S is birational to a (special) complete intersection $X_{2,6} \subset \mathbb{P}(1^3, 2, 3)$ with an ordinary double point given by:*
$$\begin{cases} a_2 = 0 \\ z^2 + y^3 + ya_4 + a_6 = 0 \end{cases}$$
   where a_i is a homogeneous form of degree i in the coordinates x_1, x_2, x_3.

Proof. Let $f\colon T \to S$ be the minimal resolution of S. The surface T is a non-singular K3 surface. Indeed, the exceptional locus of f consists of a set of (-2)-curves G_i. By adjunction, $K_T \cdot G_i = 0$. Moreover, K_T may be written as
$$K_T = f^*(K_S) + \sum_i a_i G_i = \sum_i a_i G_i,$$

with $a_i \geq 0$. The matrix $(G_i \cdot G_j)$ is negative definite, hence $a_i = 0$ for all i and K_T is trivial.

The singularities of S are rational and, in particular, $R^1 f_* \mathcal{O}_T = 0$. By the Leray spectral sequence, $H^1(S, \mathcal{O}_S) = H^1(T, \mathcal{O}_T) = (0)$.

Consider the complete linear system $|D'| = f^*|D| = f^*|-K_{X|S}|$ on T. The divisor D' is nef and big: the results of [SD74] show that $|D'|$ is basepoint free or D' is monogonal.

In the first case, $\operatorname{Bs}|D'| = \operatorname{Bs}|D| = \emptyset$.

Now assume that $D' = kE + \Gamma$, the base locus of $|D'|$ is a (-2)-curve Γ and $|E|$ is a free elliptic pencil.

Let G be an exceptional irreducible curve of the resolution f. Then,

$$G \cdot D' = 0 = kG \cdot E + G \cdot \Gamma$$

and E is nef. There are two possible configurations:

1. Either G is disjoint from both E and Γ, and $\operatorname{Bs}|-K_{X|S}| = f_*(\Gamma)$ is a (-2)-curve contained in the non-singular locus of S, so we are in the second case of the theorem.

2. Or $G = \Gamma$ and, as $-kE \cdot G = k = G^2$, this implies $k = 2$. Thus, $D' \sim 2E + \Gamma$, and the base locus $\operatorname{Bs}|-K_{X|S}| = \{p\}$ is an ordinary double point on the surface S.

Assume that $G = \Gamma$. By the observation above, any section C of the linear system $|-K_{X|S}|$ is of the form $C = E_1 + E_2$ with $E_1 \cdot E_2 = 1$. The intersection of E_1 and E_2 is a reduced point $\{p\}$.

I determine explicitly the graded ring

$$R(S, -K_{X|S}) = \bigoplus_{n \in \mathbb{N}} H^0(S, -nK_{X|S})$$

and show that the linear system $|-nK_{X|S}|$ defines a birational morphism from S to $\operatorname{Proj} R(S, -K_{X|S})$.

Let C be a section of the linear system $|-K_{X|S}|$ on the K3 surface S. By Kawamata-Viehweg vanishing on S, the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_S((n-1)(-K_{X|S})) \to \mathcal{O}_S(n(-K_{X|S})) \to \mathcal{O}_C(-nK_{X|S}) \to 0$$

shows that

$$R(C, -K_{X|S}) = R(S, -K_{X|S})/sR(S, -K_{X|S}),$$

where s is a section of $D = -K_{X|S}$. The exact sequence on S

$$0 \to \mathcal{O}_{E_1 \cup E_2} \to \mathcal{O}_{E_1} \oplus \mathcal{O}_{E_2} \to \mathcal{O}_{E_1 \cap E_2} \to 0$$

gives rise to the long exact sequence in cohomology:

$$\ldots \to H^i(C, \mathcal{O}_C(nD)) \to H^i(E_1, \mathcal{O}_{E_1}(nD)) \oplus H^i(E_2, \mathcal{O}_{E_2}(nD)) \to \ldots$$
$$\to H^i(C, \mathcal{O}_{\{p\}}(nD)) \to H^{i+1}(C, \mathcal{O}_C(nD)) \to \ldots.$$

By Kawamata-Viehweg vanishing, $H^1(C, \mathcal{O}_C(nD)) = (0)$, and therefore:

$$0 \to H^0(C, \mathcal{O}_C(nD)) \to \ldots$$
$$\ldots \to H^0(C, \mathcal{O}_{E_1}(nD)) \oplus H^0(C, \mathcal{O}_{E_2}(nD)) \xrightarrow{\alpha} H^0(C, \mathcal{O}_{\{p\}}(nD)) \to 0,$$

in particular:

$$R(C, D) = (R(E_1, D) \oplus R(E_2, D))/(\alpha R(p, D)).$$

By the above,

$$H^0(C, \mathcal{O}_{E_i}(nD)) = H^0(E_i, \mathcal{O}_{E_i}(nD)) = H^0(E_i, \mathcal{O}_{E_i}(np))$$

and the map α is a relation of degree 2.

Applying Riemann-Roch's theorem to the linear system $|np|$ on the nonsingular elliptic curve E_i yields:

$$R(E_i, p) = R(E_i, D_{|E_i}) \simeq \mathbb{C}[x_i, v, w]/(\phi),$$

where x_i, v, w have degrees 1, 2 and 3 respectively and ϕ is a relation of degree 6. Finally, the map α is a relation of degree 2 of the form $\{b_2(x_1, x_2) = 0\}$, where b_2 is a homogeneous map of degree 2. This determines $R(C, D)$ and $R(S, D)$.

Hence $\operatorname{Proj} R(S, D_{|S})$ is a special complete intersection

$$X_{2,6} \subset \mathbb{P}(1, 1, 1, 2, 3) = \mathbb{P}(x_1, x_2, x_3, v, w)$$

with an ordinary double point.

The map $S \to \operatorname{Proj} R(S, D_{|S})$ is birational, induced by $|2D_{|S}|$ because $R(S, D_{|S})$ is generated in degrees less or equal to 2. The proof of Theorem 3.1.24 is complete. □

Remark 3.1.25. In the third case of Theorem 3.1.24, the map induced by $|-K_X|$ is generically 2-to-1. The map

$$S \to \{a_2(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2 \subset \mathbb{P}(1,1,1,2,3)$$

is 2-to-1.

End of proof of 3.1.7. By 3.1.9, the general section S of $|-K_X|$ is a K3 surface with rational double points. Consider the long exact sequence in cohomology associated to:

$$0 \to \mathcal{O}_X(-nK_X - S) \to \mathcal{O}_X(-nK_X) \to \mathcal{O}_S(-nK_{X|S}) \to 0. \tag{3.10}$$

By Kawamata-Viehweg vanishing, since $-nK_X - S \sim -(n-1)K_X$ with $-K_X$ nef and big, $H^1(X, \mathcal{O}_X(-nK_X - S)) = (0)$ for $n \geq 0$. In particular, $H^0(X, -K_X) \to H^0(S, -K_{X|S})$ is surjective and $\operatorname{Bs}|-K_X| = \operatorname{Bs}|-K_{X|S}|$. By Bertini's theorem, a general member S can only have singularities at base points of $|-K_X|$.

According to 3.1.23, one of the following holds:

1. $|-K_X|$ is basepoint free,
2. $\operatorname{Bs}|-K_X| = \Gamma$, where Γ is a (-2)-curve contained in the non singular locus of X, or
3. $\operatorname{Bs}|-K_X| = \{p\}$ an ordinary double point.

In the first case, $|-K_X|$ has a non-singular member. As is noted in Remark 3.1.8, if X is a weak$*$ Fano 3-fold, this is the only possible case.

In the second case, let $S \in |-K_X|$ be a general $K3$ section. Theorem 3.1.24 shows that S is non-singular along $\operatorname{Bs}|-K_X|$, hence (by Bertini) it is non-singular everywhere.

In the third case, the anticanonical map is a birational map (determined by $|-2K_X|$) to $\operatorname{Proj} R(X, -K_X)$. The exact sequence (3.10) implies the "hyperplane section principle":

$$R(S, -K_{X|S}) \simeq R(X, -K_X)/(sR(X, -K_X)),$$

where s is a variable of degree 1. The anticanonical model of X is therefore birational to a special $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$ with a node, given by equations of the following form (the variables x_i have degree 1, v and w have degrees 2 and 3 respectively):

$$\begin{cases} a_2 = 0 \\ z^2 + y^3 + ya_4 + a_6 = 0 \end{cases}$$

where each a_j is a homogeneous form of degree j in the variables x_i, $i = 0, \cdots, 4$.

To conclude the proof of Theorem 3.1.7, notice that, as is mentioned in Remark 3.1.8, if X is weak∗, the linear system $|-K_X|$ is basepoint free. $\square$

Remark 3.1.26. Weak Fano 3-folds such that $\operatorname{Bs}|-K_X| \simeq \mathbb{P}^1$ is a (-2)-curve are called monogonal. One can show that the only occurences are:

1. $X = \mathbb{P}^1 \times S$, where S is the weighted hypersurface $S_6 \subset \mathbb{P}(1, 1, 2, 3)$; X has Picard rank 10,

2. $X = \operatorname{Bl}_\Gamma V$, the blow up of V, the weighted hypersurface $V_6 \subset \mathbb{P}(1^3, 2, 3)$ along the plane $\Pi = \{x_0 = x_1 = 0\}$; X has Picard rank 2.

If X is monogonal, the linear system $|-K_X|$ determines a rational map from X to a surface.

3.2 Minimal Model Program for weak∗ Fano 3-folds

In [Cut88], Cutkosky studies the contractions of extremal rays on projective 3-folds with normal Gorenstein $\mathbb{Q}$-factorial terminal singularities. He proves the following results:

Theorem 3.2.1 (Birational operations of the minimal model program)**.** *Let $f\colon X \to Y$ be the birational contraction of an extremal ray. Assume that f is not an isomorphism in codimension 1. Then Y is factorial and:*

E1: *Suppose that $f\colon X \to Y$ contracts a surface E to a curve Γ. Then Y is non-singular near Γ, Γ is locally a complete intersection and has planar singularities: in the local ring $\mathcal{O}_{Y,p}$ of any point $p \in \Gamma$, one of the local equations of Γ is a smooth hypersurface near p. The contraction f is the blow up of the ideal sheaf I_Γ. The 3-fold X has only cA_n singularities on E.*

Suppose that $f\colon X \to Y$ contracts a surface E to a point p. Then f is one of the following:

E2: *The 3-fold Y is non-singular, $E \simeq \mathbb{P}^2$ and $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$: f is the inverse of the blow up of a non-singular 3-fold at a point.*

E3: *The local ring of Y at p is of the form $\mathcal{O}_{Y,p} = k[x,y,z,w]/(x^2+y^2+z^2+w^2)$, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_E(E) = \mathcal{O}(-1,-1)$: f is the inverse of the blow up of an ordinary double point.*

E4: *The local ring of Y at p is of the form $\mathcal{O}_{Y,p} = k[x,y,z,w]/(x^2+y^2+z^2+w^n)$, with $n \geq 3$ and $E \simeq Q$, an irreducible reduced singular quadric surface in $\mathbb{P}^3$, with normal bundle $\mathcal{O}_E(E) \simeq \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$: f is the inverse of the blow up of a cA_{n-1} singular point.*

E5: *The local ring of Y at p is of the form $\mathcal{O}_{Y,p} = k[[x,y,z]]^{(2)}$, the ring of invariants for the $\mathbb{Z}_2$-action. $E \simeq \mathbb{P}^2$ with $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$: f is the inverse of the blow up of a non Gorenstein point of index 2.*

Theorem 3.2.2 (Conic bundles)**.** *Suppose that $f\colon X \to Y$ is the contraction of an extremal ray to a surface Y. Then Y is non-singular and X is a possibly singular conic bundle over Y.*

I want to determine whether I can run a Minimal Model Program in the categories of 3-folds I have introduced in Section 3.1. The category of weak Fano 3-folds is not suitable, since a birational contraction of type E5 would take us out of the category. I prove that the category of weak$*$ Fano 3-folds, by contrast, is stable under the operations of the Minimal Model Program.

Theorem 3.2.3. *The category of weak$*$ Fano 3-folds is stable under the birational operations of the Minimal Model Program.*

Proof. Let X be a weak$*$ Fano 3-fold and let $\phi : X \to X'$ be a divisorial contraction. By Theorem 3.2.1, the variety X' is $\mathbb{Q}$-factorial and has terminal singularities.

Step 1. *The extremal contraction ϕ is not of type E5.*

By the description of extremal contractions given in Theorem 3.2.1, if ϕ is of type E5, X contains a Cartier divisor $E \simeq \mathbb{P}^2$ with normal bundle $\mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$. By adjunction, the restriction of the anticanonical divisor to E is $-K_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(3-2) = \mathcal{O}_{\mathbb{P}^2}(1)$. The rational map $\Phi_{|-K_X|_{|E}}$ coincides with a projection from a possibly empty linear subspace

$$\nu\colon \mathbb{P}(H^0(E, \mathcal{O}_{\mathbb{P}^2}(1))) \simeq \mathbb{P}^2 \dashrightarrow \mathbb{P}(H^0(E, |-K_X|_{|E})$$

associated to the inclusion $|-K_X|_{|E} \subset |\mathcal{O}_{\mathbb{P}^2}(1)|$. By Theorem 3.1.7, the linear system $|-K_X|_{|E}$ is basepoint free, so that $\Phi_{|-K_X|_{|E}}$ is a morphism, and $|-K_X|_{|E} = |\mathcal{O}_{\mathbb{P}^2}(1)|$. The anticanonical model Y contains a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$: this contradicts X being weak∗.

Step 2. *The anticanonical divisor $-K_{X'}$ is nef and big.*

I first establish the following lemma:

Lemma 3.2.4. *Let X be a weak Fano 3-fold and let $\phi\colon X \to X'$ be a divisorial extremal contraction. Then:*

1. *The anticanonical divisor $-K_{X'}$ is big.*
2. *One of the following holds:*
   - *(a) The anticanonical divisor $-K_{X'}$ is nef,*
   - *(b) The contraction ϕ is of type E1; its exceptional divisor E is isomorphic to $\mathbb{F}_1$. The negative section σ of E is contracted by the anticanonical map, i.e. $-K_X \cdot \sigma = 0$.*

Proof of 3.2.4. The morphism ϕ is an extremal divisorial contraction and

$$-K_X = \phi^*(-K_{X'}) - aE \tag{3.11}$$

where E is the effective prime exceptional divisor of ϕ and a is a positive natural number. More precisely, $a = 2$ when ϕ is of type E2 and 1 otherwise (recall that ϕ is not of type E5).

In particular, the anticanonical rings of X and X' satisfy

$$R(X, -K_X) \subset R(X', -K_{X'}),$$

so that the anticanonical model of X' has dimension at least 3: $-K_{X'}$ is big.

Assume first that ϕ contracts a divisor E to a point. Let Z' be an effective irreducible curve lying on X'. Denote by $Z = \phi_*^{-1}(Z')$ the proper transform of Z'. The curve Z is effective and irreducible and it either instersects E properly or not at all. In particular, by the projection formula:

$$-K_{X'} \cdot Z' = \phi^*(-K_{X'}) \cdot Z = -K_X \cdot Z + aE \cdot Z \geq -K_X \cdot Z \geq 0.$$

The anticanonical divisor $-K_{X'}$ is nef.

Assume now that ϕ contracts a divisor E to Γ, an irreducible reduced curve. The anticanonical divisors of X and X' satisfy:

$$\phi^*(-K_{X'}) = -K_X + E.$$

Let Z' be an irreducible effective curve lying on X' and denote by Z any effective irreducible curve on X that maps 1-to-1 to Z'. There are two cases to consider: either Z' and Γ intersect in a 0-dimensional (or empty) set or $Z' = \Gamma$.

If Z' and Γ intersect in a 0-dimensional or empty set, $E \cdot Z \geq 0$ and by the projection formula:

$$-K_{X'} \cdot Z' = -K_X \cdot Z + aE \cdot Z \geq -K_X \cdot Z \geq 0.$$

If $Z' = \Gamma$, by contrast,

$$-K_{X'} \cdot \Gamma = -K_X \cdot Z + E \cdot Z.$$

As above $-K_X \cdot Z \geq 0$ but $E \cdot Z$ can be negative and $-K_{X'}$ can fail to be nef.

If $-K_{X'}$ is not nef, $-E_{|E}$ is ample. The divisor E is Cartier and, by adjunction,

$$-K_E = (-K_X - E)_{|E}$$

is ample as the sum of a nef and an ample divisor. The surface E is a possibly nonnormal del Pezzo surface.

Claim 3.2.5. *The surface E and the curve Γ are normal.*

Theorem 3.2.1 states that the curve Γ has planar singularities and that at any point $P \in \Gamma$, one of the local equations of Γ is a smooth hypersurface near P. Locally, the equation of Γ is of the form $\{x = f(y,z)\}$. Near a singular point $P \in \Gamma$, the blow up of Γ (and hence E) is given by the equation:

$$\{t_0 x - t_1 f(y,z) = 0\} \subset \mathbb{P}^1_{t_0,t_1} \times \mathbb{C}^3.$$

Writing down the equations of E on the affine pieces of the blow up shows that E is a $\mathbb{P}^1$-bundle over Γ.

Consider the normalisation maps $\Gamma^\nu \to \Gamma$ and $E^\nu \to E$. The surface E^ν has a structure of $\mathbb{P}^1$-bundle over Γ^ν which makes the following diagram commutative:

$$\begin{tikzcd} E^\nu \arrow[r, "n"] \arrow[d, "g^\nu"'] & E \arrow[d, "g"] \\ \Gamma^\nu \arrow[r, "n"] & \Gamma \end{tikzcd}.$$

Denote by Δ (resp. δ) the divisor of E^ν (resp. Γ^ν) defined by the conductor ideal of the normalisation map [Rei94], so that

$$n^*(K_\Gamma) = K_{\Gamma^\nu} + \delta$$
$$n^*(K_E) = K_{E^\nu} + \Delta$$

where $\Delta = g^{\nu *}(\delta)$ is an effective sum of fibres of $E^\nu \to \Gamma^\nu$. The Weil divisor Δ is defined scheme theoretically by the conductor ideal $\mathcal{I}_{\Delta, E^\nu}$, i.e. the inverse image by n of the ideal $\mathrm{Ann}(n_*(\mathcal{O}_{E^\nu})/\mathcal{O}_E)$. As a set, Δ is the codimension 1 double locus of n.

The anticanonical divisor $-K_{E^\nu}$ is ample. Indeed, $n^*(-K_E)$ is ample as the pullback of an ample divisor by a finite morphism and Δ is nef as an effective sum of fibres of the $\mathbb{P}^1$-bundle E^ν over a non-singular curve Γ^ν. The surface E^ν is a normal del Pezzo and has a structure of $\mathbb{P}^1$-bundle over a non-singular curve Γ^ν. According to the classification of normal del Pezzo surfaces, E^ν is isomorphic to $\mathbb{F}^1$.

Let σ be a negative section of E^ν. Since $n^*(-K_E)$ is ample,

$$(-K_{E^\nu} - \Delta) \cdot \sigma = 1 - \Delta \cdot \sigma > 0.$$

Consequently, Δ is empty and E is normal. Moreover, by adjunction:

$$-K_E \cdot \sigma = (-K_X - E)_{|E} \cdot \sigma = 1$$

and as $-E_{|E}$ is ample and $-K_X$ is nef, $-E_{|E} \cdot \sigma = 1$ and $-K_X \cdot \sigma = 0$. □

Claim 3.2.6. *The anticanonical divisor $-K_{X'}$ is nef and big: X' is a weak Fano 3-fold.*

Lemma 3.2.4 shows that $-K_{X'}$ is always big and that it fails to be nef only if ϕ is an E1 contraction and if the exceptional divisor E is a rational scroll $\mathbb{F}_1$ with negative section σ such that $-K_X \cdot \sigma = 0$.

I show that this case does not occur when X is a weak∗ Fano 3-fold.

The computations in the proof of Lemma 3.2.4 show that $|-K_X|_{|E}$ is contained in the linear system $|\sigma + f|$ on the scroll $\mathbb{F}_1 = E$ (the linear system that contracts the negative section). Following the usual notation conventions, I denote by σ the negative section on the scroll, by f a fibre and the intersection numbers are $\sigma^2 = -1$, $\sigma f = 1$ and $f^2 = 0$.

By Theorem 3.1.7, the anticanonical linear system is basepoint free. The basepoint free linear system $|-K_X|_{|E}$ determines a morphism onto a surface $\overline{E}$. More precisely, the morphism $\Phi_{|-K_X|_{|E}}$ has a factorisation $\nu \circ \Phi_{|\sigma+f|}$ where ν is the projection from a (possibly empty) linear subspace

$$\mathbb{P}(H^0(E, |\sigma + f|)) \simeq \mathbb{P}^2 \dashrightarrow \mathbb{P}(H^0(E, |-K_X|_{|E}\,)$$

associated to the inclusion $|-K_X|_{|E} \subset |\sigma+f|$. The surface $\overline{E}$ is the projection of $\mathbb{P}^2 = \Phi_{|\sigma+f|}(E)$ from a possibly empty linear subspace: it is equal to $\mathbb{P}^2$. This shows that $|-K_X|_E$ is the complete linear system $|\sigma + f|$. The anticanonical model Y then contains a plane $\mathbb{P}^2 = \overline{E}$, which is the image of the scroll $E = \mathbb{F}_1$ by the anticanonical map. Moreover, $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ and this contradicts X being weak∗.

Step 3. *The anticanonical map $X' \to Y'$ contracts finitely many curves.*

Let Z' be an irreducible effective curve of X' that is contracted by the anticanonical map of X'. Denote by E the exceptional divisor of the contraction ϕ and recall that

$$\phi^*(-K_{X'}) = -K_X + aE$$

for some positive integer a. Either Z' meets the centre of the contraction $\phi(E)$ in a 0-dimensional or empty set or ϕ is an E1 contraction with centre Z'. If Z' meets $\phi(E)$ in a 0-dimensional or empty set, denote by Z its proper transform. Then, by the projection formula, $-K_{X'} \cdot Z' = 0$ if and only if $E \cdot Z = 0$ and $-K_X \cdot Z = 0$, that is if Z is itself contracted by the anticanonical map of X and does not meet the centre of the contraction ϕ. The 3-fold X is weak∗: there are finitely many such curves. The anticanonical map $X' \to Y'$ is small.

Step 4. *The anticanonical ring $R(X', -K_{X'})$ is generated in degree 1.*

The proof of Theorem 3.1.7 shows that the anticanonical ring $R(X', -K_{X'})$ of X' is generated in degree 1 if and only if the rational map $\Phi_{|-K_{X'}|}$ determined by $|-K_{X'}|$ is birational onto its image. More precisely, it shows that if $\Phi_{|-K_{X'}|}$ is not birational onto its image, either $\Phi_{|-K_{X'}|}$ is generically 2-to-1 and Y' is birational to a special complete intersection $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$ with a node, or $\Phi_{|-K_{X'}|}$ maps X' to a surface (X' is then monogonal).

Recall that

$$\phi^*(-K_{X'}) = -K_X + aE,$$

for some positive integer a, and hence:

$$0 \to H^0(X, -K_X) \to H^0(X, \phi^*(-K_{X'})) \simeq H^0(X', -K_{X'}).$$

The linear system $|-K_X|$ is naturally a subsystem of $|\phi^*(-K_{X'})|$. This embedding defines a natural projection from a linear subspace:

$$\nu\colon \mathbb{P}(H^0(X, \phi^*(-K_{X'}))) \dashrightarrow \mathbb{P}((H^0(X, -K_X))).$$

The rational map $\Phi_{|-K_X|}$ determined by $|-K_X|$ is birational onto its image because X is a weak∗ Fano 3-fold. The rational map $\Phi_{|-K_X|}$ factorises as $\nu \circ \Phi_{|\phi^*(-K_{X'})|}$, hence the dimension of $\Phi_{|-K_X|}(X)$ is less than or equal to that of $\Phi_{|\phi^*(-K_{X'})|}(X) = \Phi_{|-K_{X'}|}(X')$, and X' is not monogonal. The map $\Phi_{|-K_X|}$ cannot be birational onto its image if $\Phi_{|-K_{X'}|}$ is generically 2-to-1. Hence, the map $\Phi_{|-K_{X'}|}$ is birational onto its image.

The anticanonical ring of X' is generated in degree 1. This implies that $|-K_{X'}|$ is basepoint free (Remark 3.1.8).

Step 5. *The anticanonical model Y' of X' does not contain a plane $\mathbb{P}^2$ with $-K_{Y'|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$.*

Denote by g (resp. g') the anticanonical map of X (resp. X'). Assume that X' contains a surface S that is mapped by g' to a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. The linear system $|-K_{X'}|$ is basepoint free and determines the anticanonical map of X', therefore, as above, $|-K_{X'}|_{|S} = |(g')^*\mathcal{O}_{\mathbb{P}^2}(1)|$. Denote by $\widetilde{S}$ the proper transform of S. The linear system

$$|-K_X|_{|\widetilde{S}} = |\phi^*(-K_{X'}) - aE|_{|\widetilde{S}}$$

is a subsystem of $|\phi^*(-K_{X'})|_{|\widetilde{S}} = |(g' \circ \phi)^*\mathcal{O}_{\mathbb{P}^2}(1)|_{|\widetilde{S}}$. The linear system $|-K_X|_{\widetilde{S}}$ is strictly contained in $|\phi^*(-K_{X'})|_{|\widetilde{S}}$ if the surface S intersects the centre of the contraction ϕ. By assumption on X, $|-K_X|_{|\widetilde{S}}$ is basepoint free and determines a morphism $\Phi_{|-K_X|_{|\widetilde{S}}}$ from $\widetilde{S}$ to a surface. This morphism factorises as $\Phi_{|-K_X|_{|\widetilde{S}}} = \nu \circ \Phi_{|(g'\circ\phi)^*\mathcal{O}_{\mathbb{P}^2}(1)|_{|\widetilde{S}}}$ where ν is the projection from a possibly empty linear subspace

$$\mathbb{P}(H^0(\widetilde{S}, (g'\circ\phi)^*\mathcal{O}_{\mathbb{P}^2}(1)_{|\widetilde{S}})) \simeq \mathbb{P}^2 \dashrightarrow \mathbb{P}(H^0(\widetilde{S}, |-K_X|_{\widetilde{S}}))$$

associated to the inclusion $|-K_X|_{\widetilde{S}} \subset |(g'\circ\phi)^*\mathcal{O}_{\mathbb{P}^2}(1)|_{|\widetilde{S}}$. The image of $\widetilde{S}$ by $\Phi_{|-K_X|_{|\widetilde{S}}}$ is therefore $\nu(\mathbb{P}^2) = \nu(\Phi_{|(g'\circ\phi)^*\mathcal{O}_{\mathbb{P}^2}(1)|_{|\widetilde{S}}}(\widetilde{S}))$. The linear system $|-K_X|_{|\widetilde{S}}$ determines a morphism onto a surface: ν is the identity, $|-K_X|_{|\widetilde{S}}$ is complete and equal to $|\mathcal{O}_{\mathbb{P}^2}(1)|$. This contradicts X being a weak∗ Fano 3-fold. □

The Minimal Model Program can therefore be run in the category of weak∗ Fano 3-folds.

Lemma 3.2.7. *If $X := X_0$ is a weak∗ Fano 3-fold whose anticanonical divisor Y_0 has Picard rank 1, there is a sequence of extremal contractions:*

$$\begin{tikzcd}
X_0 \arrow[r, "\phi_1"] \arrow[d] & X_1 \arrow[r, "\phi_2"] \arrow[d] & \cdots & X_{n-1} \arrow[r, "\phi_n"] \arrow[d] & X_n \arrow[d] \\
Y_0 & Y_1 & \cdots & Y_{n-1} & Y_n
\end{tikzcd}$$

where for each i, X_i is a weak∗ Fano 3-fold, Y_i is its anticanonical model, and ϕ_i is a birational contraction of an extremal ray. Moreover, for each i, the Picard rank of Y_i, $\rho(Y_i)$ is equal to 1. The 3-fold X_n either has Picard rank 1 or is an extremal Mori fibre space.

Proof. Assume that $\rho(X_i) > 1$. By Lemma 3.1.6, there is an extremal ray R on X_i and R can be contracted. If the contraction associated to R is not birational, $X_i = X_n$ is an extremal Mori fibre space and there is nothing to prove.

If the extremal contraction

$$\phi_i \colon X_i \to X_{i+1}$$

is birational, Theorem 3.2.3 shows that X_{i+1} is a weak∗ Fano 3-fold. I prove that the anticanonical model Y_{i+1} of X_{i+1} has Picard rank 1.

If ϕ_i is a flopping contraction, there is nothing to prove as $Y_i = Y_{i+1}$. Let us assume that ϕ_i is a divisorial contraction of type E1-E4. Recall that the proof of Theorem 3.2.3 shows that extremal contractions of weak∗ Fano 3-folds are not of type E5.

Let E be the exceptional divisor of the contraction ϕ_i and let f_i be the anticanonical map of X_i.

Step 1. *The image of E by the anticanonical map $\overline{E} = f_i(E)$ is a Weil non $\mathbb{Q}$-Cartier divisor.*

The divisor E is covered by K_{X_i}-negative rational curves Γ that have strictly negative intersection with E, $E \cdot \Gamma < 0$. If the divisor $\overline{E}$ is Cartier, $\overline{E}$ is covered by curves $\overline{\Gamma} = (f_i)_*(\Gamma)$ and by the projection formula:

$$\overline{E} \cdot \overline{\Gamma} = \overline{E} \cdot (f_i)_*(\Gamma) = (f_i)^*(\overline{E}) \cdot \Gamma < 0.$$

The 3-fold Y_i has Picard rank 1: $\overline{E}$ is not $\mathbb{Q}$-Cartier because it is not ample.

Denote by $Z_i = \underline{\mathrm{Proj}} \bigoplus_{n \geq 0} f_{i*}\mathcal{O}_{X_i}(nE)$ a small partial $\mathbb{Q}$-factorialisation of Y_i; Z_i is the symbolic blow up of Y_i along the Weil non $\mathbb{Q}$-Cartier divisor $f_i(E)$.

$$\begin{tikzcd} E \subset X_i \arrow[r, "\phi_i"] \arrow[d, "g_i"'] & X_{i+1} \\ E' \subset Z_i \arrow[d, "h_i"'] & \\ \overline{E} \subset Y_i & \end{tikzcd}$$

In the above diagram, $h_i \circ g_i = f_i$, $E = f_i^*(\overline{E})$. By construction, the divisor E' is Cartier on Z_i.

Step 2. *The 3-fold Z_i has Picard rank 2. There is an extremal contraction on Z_i that contracts E'.*

The 3-fold Z_i is the crepant blow up of Y_i along a single Weil non $\mathbb{Q}$-Cartier divisor $\overline{E}$: $\rho(Z_i/Y_i) = 1$. The ($\mathbb{Q}$-Cartier) divisor E' is covered by curves Γ' such that $E' \cdot \Gamma' < 0$ and $-K_{Z_i} \cdot \Gamma' > 0$. Indeed, the map g_i is small and $E = g_i^*(E'), -K_{X_i} = g_i^*(-K_{Z_i})$. The divisor E is covered by curves Γ such that $E \cdot \Gamma < 0$ and $-K_{X_i} \cdot \Gamma < 0$; by the projection formula, so is E'. The cone $\overline{NE}(Z_i)$ is 2-dimensional and rational polyhedral; by the contraction theorem [KMM87, Theorem 3-2-1], there exists an extremal ray R on which E' is negative and this extremal ray may be contracted.

Denote by ψ_i the contraction of the extremal face E'; ψ_i fits in the following diagram:

$$\begin{tikzcd} E \subset X_i \arrow[r, "\phi_i"] \arrow[d, "g_i"'] & X_{i+1} \\ E' \subset Z_i \arrow[r, "\psi_i"] \arrow[d, "h_i"'] & Z_{i+1} \\ Y_i & \end{tikzcd}$$

and $\rho(Z_i/Z_{i+1}) = \rho(X_i/X_{i+1}) = 1$.

Step 3. *There is a small map $g_{i+1}\colon X_{i+1} \to Z_{i+1}$ such that $\psi_i \circ g_i = g_{i+1} \circ \phi_i$.*

Consider the projective and surjective morphism

$$\psi_i \circ g_i\colon X_i \to Z_{i+1}$$

and run a relative Minimal Model Program on X_i over Z_{i+1}. The divisor $E \in \overline{NE}(X_i/Z_{i+1})$ is covered by curves $\Gamma \in N_1(X_i/Z_{i+1})$ with $E \cdot \Gamma < 0$ and $-K_{X_i} \cdot \Gamma > 0$, because E is covered by such curves $\Gamma \in N_1(X_i)$ and these are contracted by $\psi_i \circ g_i$ by definition of ψ_i. The contraction theorem shows that the contraction $\phi_i\colon X_i \to X_{i+1}$ of the extremal face E factorises $\psi_i \circ g_i$ and makes the diagram

$$\begin{tikzcd} E \subset X_i \arrow[r, "\phi_i"] \arrow[d, "g_i"'] & X_{i+1} \arrow[d, "g_{i+1}"] \\ E' \subset Z_i \arrow[r, "\psi_i"] \arrow[d, "h_i"'] & Z_{i+1} \\ Y_i & \end{tikzcd}$$

commutative.

The map g_{i+1} is crepant because $\rho(X_i/X_{i+1}) = \rho(Z_i/Z_{i+1})$ and g_i maps the exceptional divisor of ϕ_i to that of ψ_i.

Step 4. *The anticanonical model of Z_{i+1} is Y_{i+1}. The Picard rank of Y_{i+1} is 1.*

Since $K_{X_{i+1}} = g_{i+1}^*(K_{Z_{i+1}})$, X_{i+1} and Z_{i+1} have the same anticanonical model. The Picard rank of Z_{i+1} is $\rho(Z_i) - 1 = 1$ because $\overline{E}$ is $\mathbb{Q}$-Cartier, hence the anticanonical model of Z_{i+1} has Picard rank 1. □

I now study strict Mori fibrations. Let ϕ_{n+1} be an extremal contraction that is a strict Mori fibration, i.e. a del Pezzo fibration over a curve Γ or a conic bundle over a surface S.

If ϕ_{n+1} is a del Pezzo fibration over a curve Γ, Γ has arithmetic genus 0 by the Leray spectral sequence because $h^1(X_n, \mathcal{O}_{X_n}) = 0$. The curve Γ is isomorphic to $\mathbb{P}^1$.

Lemma 3.2.8. *If $\phi\colon X \to \mathbb{P}^1$ is a weak* Fano 3-fold that is an extremal del Pezzo fibration of degree k, k is not equal to 1 or 2.*

Proof. A general fibre F of ψ is a non-singular del Pezzo surface of degree k. As $-K_{X|F} = -K_F$, the linear system $|-K_X|_{|F}$ is naturally a subsystem of $|-K_F|$. Theorem 3.1.7 shows that $|-K_X|$ is basepoint free, hence the degree k cannot be equal to 1 as the anticanonical linear system of a non-singular del Pezzo surface of degree 1 has base points.

Since X is a weak* Fano, the anticanonical map is birational, contracts finitely many curves, and is determined by the linear system $|-K_X|$. The restriction of the anticanonical map to F factorises as $\Phi_{|-K_X|_{|F}} = \nu \circ \Phi_{|-K_F|}$, where ν is the projection from a possibly empty linear subspace

$$\nu\colon \mathbb{P}(H^0(F, -K_F)) \dashrightarrow \mathbb{P}(H^0(F, -K_{X|F}))$$

naturally associated to the inclusion of linear systems $|-K_X|_{|F} \subset |-K_F|$.

This is impossible if $k = 2$, as $\Phi_{|-K_F|}$ is generically 2-to-1. □

Assume that ϕ_{n+1} is a conic bundle, then Cutkosky shows (Theorem 3.2.2) that the surface S is non-singular.

Definition 3.2.9. A conic bundle $\phi\colon X \to S$ is a *weak Fano conic bundle* if X is a weak Fano 3-fold and $\rho(X/S) = 1$.

Lemma 3.2.10. *If $\phi\colon X \to S$ is a weak Fano conic bundle, $-K_S$ is nef.*

Proof. By definition of the discriminant curve Δ of a conic bundle,

$$-4K_S = \phi_*(-K_X)^2 + \Delta.$$

Assume that $-K_S$ is not nef and let $C \subset S$ be an irreducible curve such that $-K_S \cdot C < 0$. The curve C is necessarily contained in Δ, as $\phi_*(-K_X)^2$ is nef and $C^2 \leq C \cdot \Delta < 0$. By adjunction:

$$\begin{aligned} -2 \leq 2p_a(C) - 2 \leq (K_S + C) \cdot C \\ \leq (K_S + \Delta) \cdot C \leq (-3K_S - \phi_*(-K_X)^2) \cdot C \leq -3K_S \cdot C. \end{aligned}$$

This is impossible because $-K_S \cdot C$ is an integer. $\square$

If the surface S is not minimal, one can run a Minimal Model Program on S. There is a chain of contractions of (-1)-curves $S \to S_1 \to \ldots \to S_N$ such that either K_{S_N} is nef, or S_N is $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a curve Γ.

I show the following basic lemma:

Lemma 3.2.11. *Let $\phi\colon X \to S$ be a weak Fano conic bundle. Then, there exists a weak Fano conic bundle $\phi'\colon X' \to S'$, with $S' = \mathbb{P}^2, \mathbb{F}_0$ or $\mathbb{F}_2$, such that the following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' . \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

Proof. I show that steps of the Minimal Model Program on S are dominated by extremal contractions of X.

Step 1. *Let $X \to S$ be a weak Fano conic bundle and $S \to S'$ the contraction of a (-1)-curve. Then, there exists a weak Fano conic bundle $X' \to S'$ and an extremal contraction $X \to X'$ making the following diagram commutative:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' . \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

The relative Picard rank $\rho(X/S')$ is equal to 2, so that by Lemma 3.1.6, $\overline{NE}(X/S')$ has exactly 2 extremal rays that can be contracted. A 2-ray game gives the following two possible situations:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d] & X' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & T \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

where ψ is birational and f is a conic bundle.

In the first case, denote by Γ the (-1)-curve on S contracted to a point $P \in S'$. Then, by definition $X_{|S \smallsetminus \Gamma} \simeq X'_{|S' \smallsetminus \{P\}}$, and $X' \to S'$ naturally has a structure of weak Fano conic bundle.

In the second case, all maps are proper. Considering analytic neighbourhoods of the contracted (-1)-curves in S and in T shows that S and T are isomorphic. The two maps $X \to S$ and $X \to T$ correspond to the contraction of the same extremal ray.

There is a sequence of contractions of (-1)-curves $S \to S_1 \to \ldots \to S_N$ such that K_{S_N} is nef or S_N is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 2. *The minimal surface S_N is birational to $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a rational curve.*

As $-K_{S_N}$ is nef, the surface S_N is isomorphic to $\mathbb{P}^2$ or is a $\mathbb{P}^1$-bundle over a curve Γ.

If S_N is a $\mathbb{P}^1$-bundle over a curve Γ, as $h^1(S_N, \mathcal{O}_{S_N}) = h^1(X_N, \mathcal{O}_{X_N}) = 0$, Γ has arithmetic genus $h^1(\Gamma, \mathcal{O}_\Gamma) = 0$ and Γ is isomorphic to $\mathbb{P}^1$.

Suppose that S_N is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The surface S_N is a Hirzebruch surface of the form $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$. Since $-K_{S_N}$ is nef, S_N is either $\mathbb{F}_0$ (i.e. $\mathbb{P}^1 \times \mathbb{P}^1$) or $\mathbb{F}_2$. $\square$

Theorem 3.2.12 (End product of the MMP)**.** *By the above, the end product of the Minimal Model Program X_n is a weak$*$ Fano 3-fold. More precisely, X_n is one of the following:*

1. *X_n is a terminal Gorenstein $\mathbb{Q}$-factorial Fano 3-fold of rank 1. X_n is the deformation of a non-singular Fano 3-fold of rank 1.*

2. *$X_n \to \mathbb{P}^1$ is a del Pezzo fibration of degree k, with $3 \leq k \leq 9$. X_n has Picard rank 2.*

3. *$X_n \to S$ is a conic bundle and S is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. X_n has Picard rank 2 or 3.*

Proof. Theorem 3.2.3 shows that the category of weak$*$ Fano 3-folds is stable under the birational operations of the Minimal Model Program. Running the Minimal Model Program on a weak$*$ Fano 3-fold X_0 yields a sequence of extremal contractions and weak$*$ Fano 3-folds:

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_n} X_n.$$

The Program terminates if $\rho(X_n) = 1$ or if X_n is a Mori fibre space over a minimal base.

If $\rho(X_n) = 1$, the anticanonical divisor is ample. Indeed, assume $-K_X$ is nef but not ample. There exists an effective curve Z such that $-K_X \cdot Z = 0$. Since $\rho(X_n) = 1$, $-K_{X_n}$ is trivial and therefore not big: this contadicts X_n being weak$*$. In this case, Mukai shows that X_n can be deformed to a non-singular Fano 3-fold with Picard rank 1 [Muk02]. Such Fano 3-folds have been classified [Isk77, Isk78].

If $X_n \to \mathbb{P}^1$ is a del Pezzo fibration, then $\rho(X_n) = \rho(\mathbb{P}^1) + 1 = 2$.

If $X_n \to S$ is a conic bundle, by the discussion above, either S is $\mathbb{P}^2$ and $\rho(X_n) = 2$, or S is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ and $\rho(X_n) = 3$. $\square$

Chapter 4

A bound on the defect of some Fano 3-folds

The results obtained in Chapter 3 show that it is possible to run a Minimal Model Program (MMP) in the category of weak$*$ Fano 3-folds. If X is a weak$*$ Fano 3-fold and Y is its anticanonical model, the end product of the MMP on X is either a Fano 3-fold, a del Pezzo fibration over $\mathbb{P}^1$, or a conic bundle over $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. Since the defect of Y is equal to $\operatorname{rk}\operatorname{Pic}(X) - \operatorname{rk}\operatorname{Pic}(Y)$, the number of divisorial contractions needed to reach an end product of the Minimal Model Program on a weak$*$ Fano 3-fold X determines the defect of its anticanonical model Y.

In this Chapter, I show that if the anticanonical model Y of a weak$*$ Fano 3-fold X does not contain a quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$, a divisorial contraction increases the anticanonical degree by at least 4. I then prove that this condition on Y is preserved by the operations of the MMP. Let the weak$*$ Fano 3-folds X_i, for $1 \leq i \leq n$, be the intermediate steps of the MMP on $X = X_0$ and let Y_i be the anticanonical model of X_i. If $Y = Y_0$ does not contain an irreducible reduced quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$, then no Y_i contains an irreducible reduced quadric Q with $-K_{Y_i|Q} = \mathcal{O}_Q(1)$. If Y does contain an irreducible reduced quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$ however, there exists an extremal divisorial contraction $\phi\colon X \to X_1$ whose exceptional divisor is $\widetilde{Q}$, the pull back of Q by the anticanonical map. In this case, I show that there are at most $10 - g$ contractions of quadrics when running the MMP on X.

Following these observations, I determine a bound on the defect of terminal Gorenstein Fano 3-folds Y that contain neither a plane, nor a quadric. I then give a general bound on the defect of terminal Gorenstein Fano 3-folds that do not contain a plane.

4.1 Further study of the Minimal Model Program of weak* Fano 3-folds

Lemma 4.1.1. *Let X be a weak* Fano 3-fold. Assume that Y, the anticanonical model of X, does not contain an irreducible reduced quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ or $Q \subset \mathbb{P}^3$ with $-K_{Y|Q} = \mathcal{O}_Q(1)$. Every divisorial extremal contraction ϕ increases the anticanonical degree $(-K)^3$ by at least 4.*

Proof. Let $\phi\colon X \to X'$ be a divisorial contraction.

Step 1. *The contraction ϕ is of type E1 or E2.*

Recall from the proof of Theorem 3.2.3 that the contraction ϕ is not of type E5.

If ϕ is of type E3 (resp. E4), X contains a quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ (resp. $Q \subset \mathbb{P}^3$) with normal bundle $\mathcal{O}_Q(Q) = \mathcal{O}_Q(-1) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ (resp. $\mathcal{O}_Q(Q) = \mathcal{O}_Q(-1)$). The divisor Q is Cartier, hence, by adjunction $-K_{X|Q} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) = \mathcal{O}_Q(1)$ (resp. $-K_{X|Q} = \mathcal{O}_Q(1)$).

Since X is a weak* Fano 3-fold, the linear system $|-K_X|_{|Q}$ is basepoint free (Theorem 3.1.7) and determines a morphism $\Phi_{|-K_X|_{|Q}}$ onto a surface.

As $|-K_X|_{|Q} \subset |\mathcal{O}_Q(1)|$, the morphism $\Phi_{|-K_X|_{|Q}}$ factorises as $\nu \circ \Phi_{|\mathcal{O}_Q(1)|}$, where ν is the projection from a possibly empty linear subspace

$$\mathbb{P}(H^0(Q, \mathcal{O}_Q(1))) \simeq \mathbb{P}^3 \dashrightarrow \mathbb{P}(H^0(Q, -K_{X|Q}))$$

naturally associated to the inclusion of linear systems. The image of Q by $\Phi_{|-K_X|_{|Q}}$ is $\overline{Q} = \nu(Q)$. The map ν is the identity or a projection from a point, line or plane. As $\overline{Q}$ is a surface, ν can only be the identity or a projection from a point and $\overline{Q}$ either is the quadric Q or is a plane $\mathbb{P}^2$ and, by construction, $-K_{Y|\overline{Q}} = \mathcal{O}_{\overline{Q}}(1)$.

In both cases, this contradicts the assumption on X since Y contains neither a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ nor a quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$. The contraction ϕ cannot therefore be of type E3 or E4; ϕ is of type E1 or E2.

Step 2. *The anticanonical degree increases by at least 4: $-K_{X'}^3 \geq -K_X^3 + 4$.*

If ϕ is of type E2, then $-K_X = \phi^*(-K_{X'}) - 2E$ and the anticanonical degrees of X and X' satisfy $-K_X^3 = -K_{X'}^3 - 8E^3 = -K_{X'}^3 - 8$. The degree increases by precisely 8.

Assume that $\phi\colon X \to X'$ is an E1-contraction and denote by Γ the centre of ϕ. Lemma 7.1.5 shows that the following intersection table holds:

$$(-K_X)^3 = (-K_{X'})^3 - 2(-K_{X'} \cdot \Gamma - p_a(\Gamma) + 1) \tag{4.1}$$

$$(-K_X)^2 \cdot E = -K_{X'} \cdot \Gamma + 2 - 2p_a(\Gamma) \tag{4.2}$$

$$(-K_X) \cdot E^2 = 2p_a(\Gamma) - 2 \tag{4.3}$$

$$E^3 = -\deg \mathcal{N}_{\Gamma/X'} = 2 - 2p_a(\Gamma) - (-K_{X'} \cdot \Gamma). \tag{4.4}$$

The anticanonical divisor $-K_X$ is Cartier and nef. The anticanonical map is small and determined by $|-K_X|$. The anticanonical map $\Phi_{|-K_X|}$ contracts no divisor, hence in particular, $(-K_X)^2 \cdot E > 0$. Equation (4.2) implies that:

$$\begin{aligned} -K_{X'} \cdot \Gamma &\geq 2p_a(\Gamma) - 1 \quad \text{that is:} \\ -K_{X'} \cdot \Gamma - p_a(\Gamma) + 1 &\geq p_a(\Gamma). \end{aligned}$$

Equation (4.1) shows that the required result holds for $p_a(\Gamma) \geq 2$.

Assume that the centre of ϕ is a curve Γ with arithmetic genus $p_a(\Gamma) \leq 1$. Since Γ has planar singularities (Theorem 3.2.1), if its arithmetic genus is 1, its degree $-K_{X'} \cdot \Gamma$ is at least 3 and the desired inequality holds.

Finally, assume that the curve Γ has arithmetic genus $p_a(\Gamma) = 0$, that is that Γ is rational and non-singular. Equation (4.1) shows that the inequality holds for $-K_{X'} \cdot \Gamma \geq 1$.

Claim 4.1.2. *The centre of ϕ is not a flopping curve.*

Suppose that the centre of ϕ is a curve Γ with $-K_{X'} \cdot \Gamma = 0$. From (4.2), $-K_X^2 \cdot E > 0$, and the curve Γ is rational and non-singular.

The degree of $\mathcal{N}_{\Gamma/X'}$, the normal bundle of Γ in X', is -2. The curve $\Gamma \simeq \mathbb{P}^1$ is locally a complete intersection, so that its normal bundle is

$$\mathcal{N}_{\Gamma/X'} = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-2-n)$$

for some integer $n \geq -1$.

Let s be a section of $\mathcal{N}_{\Gamma/X'}$ corresponding to the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2+n) \to \mathcal{N}_{\Gamma/X'} \to \mathcal{O}_{\mathbb{P}^1}(-n) \to 0.$$

The intersection of s with the anticanonical divisor is $-K_{X'} \cdot s = -n$. Since $-K_{X'}$ is nef, n is either 0 or -1. The exceptional divisor E of the contraction ϕ is $\mathbb{P}(\mathcal{N}_{\Gamma/X'})$, that is either $E \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq \mathbb{F}_2$ or $E \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

I show that these cases are impossible unless Y contains a quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$.

If $E \simeq \mathbb{F}_2$, the linear system $|-K_X|_{|E}$ is basepoint free and determines a morphism $\Phi_{|-K_X|_{|E}}$ onto a surface. The morphism $\Phi_{|-K_X|_{|E}}$ is determined by a linear system $|-K_X|_{|E} \subset |af + b\sigma|$ on the scroll. By convention, f denotes the fibre of the scroll, σ its negative section, and the intersection numbers are $f^2 = 0$, $\sigma^2 = -2$, and $\sigma \cdot f = 0$. Since f is a fibre of the contraction ϕ, $-K_X \cdot f = 1$ and $a = 1$. From (4.2), the degree is $(-K_X)^2 \cdot E = (\sigma + bf)^2 = 2$, so that $b = 2$. Since $(\sigma + 2f) \cdot \sigma = 0$ and $(\sigma + 2f) \cdot f = 1$, $|\sigma + 2f|$ determines a morphism that contracts the negative section and that maps the scroll onto an irreducible reduced singular quadric $\overline{Q} = \mathrm{Proj}(\mathbb{F}_2, |\sigma + 2f|) \subset \mathbb{P}^3 = \mathbb{P}(H^0(\mathbb{F}_2, |\sigma + 2f|))$. The image of the morphism $\Phi_{|-K_X|_{|E}}$ is $\nu(\overline{Q})$, where ν is the projection from a (possibly empty) linear subspace

$$\mathbb{P}^3 = \mathbb{P}(H^0(\mathbb{F}_2, |\sigma + 2f|)) \dashrightarrow \mathbb{P}(H^0(E, -K_{X|E}))$$

associated to the inclusion $|-K_X|_{|E} \subset |\sigma + 2f|$.

Similarly, if $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$, the linear system $|-K_X|_{|E}$ is contained in the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$, because $-K_X \cdot l = 1$ for any ruling l of E. The linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$ determines a morphism of E onto an irreducible reduced non-singular quadric $\overline{Q} = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = \mathbb{P}(H^0(E, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)))$. The image of the morphism $\Phi_{|-K_X|_{|E}}$ is $\nu(\overline{Q})$, where ν is the projection from a (possibly empty) linear subspace

$$\mathbb{P}^3 = \mathbb{P}(H^0(E, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))) \dashrightarrow \mathbb{P}(H^0(E, -K_{X|E})$$

associated to the inclusion $|-K_X|_{|E} \subset |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)|$.

In both cases, the image of $\Phi_{|-K_X|_{|E}}$ is a surface that is the image of an irreducible reduced quadric $\overline{Q} \subset \mathbb{P}^3$ under a projection of $\mathbb{P}^3$ from a linear subspace. As above, $\Phi_{|-K_X|_{|E}}(E)$ is a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ or the quadric $\overline{Q}$ with $-K_{Y|\overline{Q}} = \mathcal{O}_{\overline{Q}}(1)$. . This yields a contradiction if X is weak$*$ and Y does not contain a quadric $\overline{Q}$ with $-K_{Y|\overline{Q}} = \mathcal{O}_{\overline{Q}}(1)$. □

I show that the hypotheses of Lemma 4.1.1 are preserved by the birational operations of the Minimal Model Program.

Lemma 4.1.3. *Let X be a weak$*$ Fano 3-fold and $\phi\colon X \to X'$ a birational extremal contraction. Denote by Y and Y' the anticanonical models of X and X'. If Y' contains an irreducible reduced quadric Q' with $-K_{Y'|Q'} = \mathcal{O}_{Q'}(1)$, Y also contains an irreducible reduced quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$.*

Proof. Denote by g (resp. g') the anticanonical map of X (resp. X'). Assume that Y' contains an irreducible quadric Q' with $-K_{Y'|Q'} = \mathcal{O}_{Q'}(1)$. Denote by $\widetilde{Q'}$ the proper transform of Q' by g' and by $\widetilde{Q}$ the proper transform of $\widetilde{Q'}$ on X.

The linear system $|-K_X|_{|\widetilde{Q}} = |\phi^*(-K_{X'}) - E|_{|\widetilde{Q}}$ is naturally a subsystem of $|(\phi)^*(-K_{X'})|_{|\widetilde{Q'}} = |(g' \circ \phi)^*\mathcal{O}_{Q'}(1)|$. The inclusion is strict if the centre of ϕ intersects $\widetilde{Q}$. The 3-fold X is a weak$*$ Fano hence, by Theorem 3.1.7, $|-K_X|_{|\widetilde{Q}}$ is basepoint free and determines a morphism $\Phi_{|-K_X|_{|\widetilde{Q}}}$ that maps $\widetilde{Q}$ onto a surface.

Let Q be the image of $\Phi_{|-K_X|_{|\widetilde{Q}}}$. The morphism $\Phi_{|-K_X|_{|\widetilde{Q}}}$ is the composition $\nu \circ \Phi_{|(g'\circ\phi)^*\mathcal{O}_{Q'}(1)|}$ where ν is the projection from a (possibly empty) linear subspace

$$\mathbb{P}(H^0(\widetilde{Q}, (g' \circ \phi)^*\mathcal{O}_{Q'}(1))) \simeq \mathbb{P}^3 \dashrightarrow \mathbb{P}(H^0(\widetilde{Q}, -K_{X_{|\widetilde{Q}}}))$$

associated to the inclusion $|-K_{X_{|\widetilde{Q}}}| \subset |(g' \circ \phi)^*\mathcal{O}_{Q'}(1)|$.

The surface Q therefore is the image of $Q' \subset \mathbb{P}^3$ under the projection ν of $\mathbb{P}^3$ from a linear subspace. As Q is a surface, the rational map ν is either the identity, or the projection of $\mathbb{P}^3$ from a point. The surface Q is either Q' or a plane $\mathbb{P}^2$ and $-K_{Y|Q} = \mathcal{O}_Q(1)$ by construction. The surface Q is not a plane because X is a weak$*$ Fano 3-fold, hence Q is an irreducible reduced quadric $Q \simeq Q'$. The proof shows in addition that $\widetilde{Q'} \subset X'$ does not intersect the centre of ϕ. □

4.2 A bound on the defect of some Fano 3-folds

Let Y be a non $\mathbb{Q}$-factorial normal Gorenstein terminal Fano 3-fold of genus $g \geq 3$. The 3-fold Y is not $\mathbb{Q}$-factorial, hence:

$$\operatorname{rk}(\operatorname{Weil}(Y)) = \dim H_4(Y, \mathbb{Z}) \neq \dim H^2(Y, \mathbb{Z}) = \operatorname{rk}(\operatorname{Pic}(Y)).$$

If Y is a quartic 3-fold with terminal (Gorenstein) singularities, the Grothendieck-Lefschetz hyperplane theorem shows that $\rho(Y) = \operatorname{rk}\operatorname{Pic}(Y) = 1$. In this section, I always assume that the Picard rank of Y is 1.

Recall the statement of Kawamata's $\mathbb{Q}$-factorialisation theorem (Proposition 2.1.1).

Proposition 4.2.1 ($\mathbb{Q}$-factorialisation). *Let Y be an algebraic threefold with only terminal singularities. Then, there is a birational morphism $f\colon X \to Y$ such that X is terminal and $\mathbb{Q}$-factorial, f is an isomorphism in codimension 1 and f is projective.*

Remark 4.2.2. The map $f\colon X \to Y$ is a chain of 'symbolic blow-ups' of Weil non $\mathbb{Q}$-Cartier divisors on Y. The anticanonical map introduced in Chapter 3 is a $\mathbb{Q}$-factorialisation map.

Remark 4.2.3. As mentioned in Remark 3.1.11, the Fano indices of X and Y are equal. The map f is crepant, and so the 3-fold X is also Gorenstein.

The morphism f is small and X is $\mathbb{Q}$-factorial, hence:

$$\operatorname{rk}\operatorname{Weil}(Y) = \operatorname{rk}\operatorname{Weil}(X) = \operatorname{rk}\operatorname{Pic}(X),$$

and the defect of Y is:

$$\sigma(Y) = \operatorname{rk}\operatorname{Pic}(X) - 1.$$

The $\mathbb{Q}$-factorialisation X of Y is Gorenstein, terminal, $\mathbb{Q}$-factorial and its anticanonical divisor $-K_X = f^*(-K_Y)$ is nef and big. The 3-fold X is a weak Fano 3-fold.

The proof of Theorem 3.1.7 shows that if $|-K_X|$ is not basepoint free, then either X is birational to a special complete intersection $X_{2,6} \subset \mathbb{P}(1,1,1,2,3)$ and X has genus 2, or X is monogonal and X has Picard rank at least 2. If Y has Picard rank 1 and genus $g \geq 3$, $R(Y, -K_Y) = R(X, -K_X)$ is generated in degree 1.

Unless Y contains a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$, its $\mathbb{Q}$-factorialisation X is a weak$*$ Fano 3-fold.

I state here a weak version of Theorem 6.2.8 [Nam97], which is established in Chapter 6. This theorem is used in the rest of this section.

Theorem 4.2.4. *Let Y be a Fano 3-fold with terminal Gorenstein singularities. There is a 1-parameter flat deformation of Y $f\colon \mathcal{Y} \to \Delta$, where $\mathcal{Y}_t$ is a terminal Gorenstein Fano 3-fold for all $t \in \Delta$ and $\mathcal{Y}_{t_0}$ is non-singular for some $t_0 \in \Delta \smallsetminus \{0\}$.*

Remark 4.2.5. If $\mathcal{Y} \to \Delta$ is such a 1-parameter flat deformation, $\operatorname{rk}\operatorname{Pic}(Y) = \operatorname{rk}\operatorname{Pic}(\mathcal{Y}_t)$ for all $t \in \Delta$ (Lemma 2.2.7). The Fano index and the plurigenera are constant in a 1-parameter flat deformation, so that $-K_Y^3$, the degree of Y, is equal to the degree of $\mathcal{Y}_{t_0}$ (see Chapter 6).

Corollary 4.2.6. *(of all the above): Let Y be a terminal Gorenstein Fano 3-fold with Fano index 1, Picard rank 1 and genus $g \geq 3$. If Y does not contain a quadric or a plane, the defect of Y is bounded by $\left[\frac{12-g}{2}\right]+4$, where g is the genus of Y.*

Proof. Denote by X a small $\mathbb{Q}$-factorialisation of Y, X is a weak$*$ Fano 3-fold. I prove that the Picard rank of X is at most $\left[\frac{12-g}{2}\right]+5$.

The anticanonical model Y of X is a terminal Gorenstein Fano 3-fold. The anticanonical degree of X is of the form $-K_Y^3 = -K_X^3 = 2g-2$ with $2 \leq g \leq 10$ or $g = 12$ by Iskovskih's classification [Isk77, Isk78] and Theorem 4.2.4.

Lemma 4.1.1 states that, when running the Minimal Model Program on X, each divisorial contraction increases the anticanonical degree by at least 4. By Theorem 4.2.4, at each step of the MMP, the anticanonical model Y_i of X_i is a terminal Gorenstein Fano 3-fold and it can be smoothed to a non-singular Fano 3-fold with Picard rank 1.

Iskovskih's classification of non-singular Fano 3-folds shows that at each step, $-K_{X_i}^3$ can only take one of finitely many values. If the Fano index of X_i is 1, it is of the form $2g_i - 2$, with $g + 2 \leq g_i \leq 10$ or $g_i = 12$. If X_i has Fano index 2, it is of the form $8d$, with $1 \leq d \leq 5$. If X_i is a possibly singular quadric, it is equal to 54, and if $X_i \simeq \mathbb{P}^3$, it is equal to 64 (see Theorem 5.1.1).

In each case, the Picard rank of the weak$*$ Fano 3-fold X is equal to the sum of the number of divisorial contractions and the Picard rank of the end product of the Minimal Model Program. The end product of the MMP is one of: a terminal Gorenstein Fano 3-fold, a del Pezzo fibration over $\mathbb{P}^1$, or a conic bundle over $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$.

If, at any intermediate step, X_i is a weak$*$ Fano 3-fold of Fano index 2, ϕ_i can only be an E2 contraction, and X_{i+1} also has Fano index 2 (Lemma 5.1.2). If the end product X_n has index 2, it is one of: a Fano index 2 Fano 3-fold, an étale conic bundle, or it admits a fibration by del Pezzo surfaces of degree 4. More precisely, depending on the end product of the Minimal Model Program, one of the folllowing cases occurs:

1. Y_n is a Fano 3-fold of Fano index 1. The genus decreases by at least

2 with each divisorial contraction. The Picard rank of X is bounded above by $\left[\frac{12-g}{2}\right]+1$.

2. Y_n is a Fano 3-fold of Fano index 2. Similarly, by inspection, the Picard rank of X is bounded by $\left[\frac{12-g}{2}\right]+3$.

3. Y_n is a possibly singular quadric. In this case, no intermediate 3-fold can have index 2 and the Picard rank is bounded by $\left[\frac{12-g}{2}\right]+2$.

4. $Y_n \simeq \mathbb{P}^3$. No intermediate 3-fold can have Fano index 2, hence the Picard rank of X is bounded by $\left[\frac{12-g}{2}\right]+3$.

5. Y_n is a strict Mori fibre space. In this case, as Y_n is a terminal Gorenstein Fano 3-fold with Picard rank 1, I can apply the same results to bound the degree of Y_n (refining the argument to take account of possible indices of Mori fibre spaces does not improve the bound). This yields an upper bound on the Picard rank equal to $\left[\frac{12-g}{2}\right]+5$.

□

Corollary 4.2.7. *Let Y be a terminal Gorenstein Fano 3-fold with Fano index 1, Picard rank 1 and genus $g \geq 3$. If Y contains a quadric but does not contain a plane, the defect of Y is at most $14-g$.*

Proof. Denote by $g\colon X \to Y$ a small $\mathbb{Q}$-factorialisation of Y; X is a weak$*$ Fano 3-fold. I prove that the Picard rank of X is at most $15-g$.

I can run a Minimal Model Program on X. If Q is a quadric lying on Y with $-K_{Y|Q} = \mathcal{O}_Q(1)$, denote by $\widetilde{Q}$ its proper transform on X. We claim that there is a K-negative extremal ray of X on which $\widetilde{Q}$ is negative, implying that there is an extremal contraction that contracts $\widetilde{Q}$. Indeed, let $\overline{\Gamma}$ be any ruling of Q and let Γ be the proper transform of Γ. As $\widetilde{Q}$ is Cartier,

$$\widetilde{Q} \cdot \Gamma = -K_X \cdot \Gamma + K_\Gamma - (\Gamma \cdot \Gamma)_{\widetilde{Q}}.$$

In addition, $-K_X \cdot \Gamma = -K_Y \cdot \overline{\Gamma} = 1$ and $-K_\Gamma = -K_{\overline{\Gamma}} = -2$ by the Leray spectral sequence. The divisor $\overline{\Gamma} \subset Q$ is nef: $(\Gamma \cdot \Gamma)_{\widetilde{Q}} \geq 0$. Hence, $\widetilde{Q} \cdot \Gamma < 0$ and there exists a K-negative extremal ray R, on which $\widetilde{Q}$ is negative. The contraction theorem [KMM87] shows that R can be contracted. Denote by $\phi\colon X \to X'$ the extremal divisorial contraction associated to R.

Claim 4.2.8. *The degree increases by at least 2: $-K_X^3 \leq -K_{X'}^3 + 2$.*

If ϕ contracts a quadric $\widetilde{Q}$ to a point, ϕ is of type E3 or E4 and the degree increases by 2. If $\widetilde{Q}$ is contracted to a curve Γ, (4.2) shows that $-K_{X'}\cdot\Gamma = 2p_a(\Gamma)$ and therefore by (4.1), $(-K_X)^3 = (-K_{X'})^3 - 2(p_a(\Gamma)+1)$. The degree increases by at least 2.

The classification of non-singular Fano 3-folds of Picard rank 1 shows that the number of quadrics lying on Y is bounded by $11 - g = 10 - g + 1$. As long as quadrics are contracted, the indices of the intermediate steps of the Minimal Model Program are equal to 1. The bound on the Picard rank then follows from Corollary 4.2.7. □

Remark 4.2.9. I have not managed to construct a terminal Gorenstein quartic 3-fold $Y = Y_4^3 \subset \mathbb{P}^4$ that contains a quadric but no plane for which the bound on the defect would be attained. My guess is that this bound is not optimal. First, if Y contains two quadrics $Q_0 = \{x_0 = q_0 = 0\}$ and $Q_1 = \{x_1 = q_1 = 0\}$, Y naturally has a structure of del Pezzo fibration of degree 4 over $\mathbb{P}^1$. One could try to build an example by considering such del Pezzo fibrations with 3 reducible fibres. Second, if Y does not contain two quadrics lying in distinct hyperplane sections, Y contains two conjugate quadrics of the form $Q = \{x_0 = q = 0\}$ and $Q' = \{x_0 = q' = 0\}$ and Y can be "unprojected" to a terminal Gorenstein $Y_{2;3} \subset \mathbb{P}^5$ with a node at $P = (0:0:0:0:0:1)$. One then needs to consider quadrics in $Y_{2,3}$ that do not contain the point P (Lemma 4.1.3).

Remark 4.2.10. This analysis need not be limited to Gorenstein terminal Fano 3-folds of Picard rank 1. Let Y be a non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold of Picard rank $\rho(Y)$ and denote by X a small $\mathbb{Q}$-factorialisation of Y. It is possible to run a Minimal Model Program on X. Whereas Lemma 3.2.7 shows that the Picard rank of the anticanonical models of the intermediate steps of the MMP is always 1 when $\rho(Y) = 1$, in the general case, at each step of the MMP, the Picard rank of the anticanonical model Y_i is at most $\rho(Y)$. Following the strategy of Corollaries 4.2.6 and 4.2.7, it is possible to bound the number of divisorial contractions. This time, the end product is either a terminal Gorenstein Fano 3-fold of Picard rank at most $\rho(Y)$ or a strict Mori fibre space. The anticanonical model of each intermediate step can be deformed to a non-singular Fano 3-fold; these have been classified by Mori and Mukai [MM86, MM82].

Chapter 5

Fano 3-folds containing a plane

In chapters 3 and 4, I have determined an upper bound on the defect of terminal Gorenstein Fano 3-folds with Picard rank 1 that do not contain a plane. The simplest examples of non $\mathbb{Q}$-factorial Fano 3-folds are, however, singular Fano 3-folds Y with Picard rank 1 that contain a plane. In this chapter, I study terminal Gorenstein Fano 3-folds with Picard rank 1 that contain a plane.

Let Y be a non $\mathbb{Q}$-factorial Fano 3-fold with terminal Gorenstein singularities and Picard rank 1. Let $E \simeq \mathbb{P}^2 \subset Y$ be a plane contained in Y. The plane E is a Weil non $\mathbb{Q}$-Cartier divisor. The anticanonical divisor of Y is ample and Y has Picard rank 1, hence $-K_Y = \mathcal{O}_Y(i(Y))$, where $i(Y)$ is the Fano index of Y. In particular, if E is any Weil non $\mathbb{Q}$-Cartier plane lying on Y, $-K_{Y|E} = \mathcal{O}_E(i(Y))$. If Y is a non $\mathbb{Q}$-factorial Fano 3-fold with terminal Gorenstein singularities and Picard rank 1, then the restriction of the anticanonical divisor to any plane lying on Y is determined by the Fano index of Y.

In Section 5.1, I study terminal Gorenstein Fano 3-folds of Fano index strictly greater than 1 that contain a plane. The methods used in chapters 3 and 4 can be applied and yield a bound on the defect.

In Section 5.2, I focus on the case of terminal quartic 3-folds $Y = Y_4^3 \subset \mathbb{P}^4$ that contain a plane. Notice that such a quartic 3-fold $Y = Y_4^3 \subset \mathbb{P}^4$ has Fano index 1 and Y is Gorenstein because it is a hypersurface in $\mathbb{P}^4$. Chapter 2 explains that the defect of a terminal Gorenstein Fano 3-fold depends on the number of singular points and on the position of its singularities. If $Y = Y_4^3 \subset \mathbb{P}^4$ is nodal, that is if Y has no worse than ordinary double points, Y has at most 45 nodes [Var83, Fri86]. Further, it is known that, up to projective equivalence, there is a unique quartic hypersurface with 45 nodes [dJSBVdV90]: the Burkhardt quartic (Example 5.2.7). This quartic contains forty planes and has defect 15. I show in Section 5.2 that this is an upper bound for the defect of terminal quartic 3-folds. Finally, I make a conjecture (5.2.14) on the defect of terminal Gorenstein Fano 3-folds with Picard rank 1 and Fano index 1 that contain a plane.

5.1 Non $\mathbb{Q}$-factorial Fano 3-folds of higher index

Higher Fano index canonical Gorenstein Fano 3-folds are classified by Fujita and Shin. Shin proves the following:

Theorem 5.1.1 ([Shi89])**.** *Let Y be a Gorenstein Fano 3-fold with at most canonical singularities. Denote by $i(Y)$ the Fano index of Y, then:*

1. $i(Y) \leq 4$ *with equality if and only if* $Y \simeq \mathbb{P}^3$.
2. $i(Y) = 3$ *if and only if Y is a possibly singular quadric in* $\mathbb{P}^4$.

Let Y be a terminal Gorenstein Fano 3 fold with Fano index $i(Y) > 1$ and Picard rank 1, and let X be a small $\mathbb{Q}$-factorialisation of Y. Recall that $i(X) = i(Y)$ by Remark 3.1.11. The 3-fold X is a weak$*$ Fano 3-fold. Indeed, Y cannot contain a plane with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$, the anticanonical ring of Y is generated in degree 1 (Theorem 3.1.7) and the anticanonical map is small. The results obtained in Chapter 3 show that it is possible to run a Minimal Model Program on X.

Let Y be a terminal Gorenstein Fano 3-fold of Fano index 2 and let X be a small $\mathbb{Q}$-factorialisation of Y. I state in the following Lemma some straightforward results on contractions of extremal rays on X.

Lemma 5.1.2. *Let X be a Fano index 2 weak$*$ Fano 3-fold and $f\colon X \to X'$ the contraction of an extremal ray. Then, one of the following holds:*

1. *f is birational, and f is either a flop or an E2 contraction.*
2. *$f\colon X \to S$ is a conic bundle, in fact f is an étale $\mathbb{P}^1$-bundle.*
3. *$f\colon X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 8, and f is a quadric bundle.*

Remark 5.1.3. Similarly, if X has Fano index 3 and $\rho(X) \geq 2$, then $\rho(X) = 2$. Any contraction of an extremal ray is either a flop or a del Pezzo fibration of degree 9, that is, X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$.

Proof. Let $f\colon X \to X'$ be the birational contraction of an extremal ray. Since X is weak$*$, f cannot be of type E5. If f is of type E1, E3 or E4, an irreducible reduced curve C contracted by f has anticanonical degree $-K_X \cdot C = 1$: this contradicts $i(X) = 2$.

If f is a birational contraction, it is either a flopping contraction or the inverse of the blow up of a smooth point. The exceptional divisor of f is then a plane $E \simeq \mathbb{P}^2$ with $\mathcal{O}_E(-E) = \mathcal{O}_E(1)$, that is, by adjunction, with $-K_{X|E} = \mathcal{O}_{\mathbb{P}^2}(2)$.

Let f be a fibering contraction. Unless f is an étale $\mathbb{P}^1$-bundle over $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, or a del Pezzo fibration of degree 8, there exists a curve C contracted by f with anticanonical degree $-K_X \cdot C = 1$ or 3. $\square$

Remark 5.1.4. In particular, if $\operatorname{rk}\operatorname{Pic}(X) > 2$, X has to contain a plane. The problem of bounding the defect of a terminal Gorenstein Fano 3-fold Y of Fano index 2 is equivalent to that of determining the maximal number of disjoint planes that can lie on X (see the proof of Corollary 5.1.7).

Lemma 5.1.5. *Let $f\colon X \to X'$ be an extremal contraction of type E2. Assume that X is a weak$*$ Fano 3-fold of index 2 and degree $h^3 = \frac{-K_X^3}{8}$. Then X' is a weak$*$ Fano 3-fold of Fano index 2 and degree $h'^3 = h^3 - 1$.*

Proof. By Theorem 3.2.3, X' is a weak$*$ Fano 3-fold. The anticanonical divisors of X and X' satisfy the relation:

$$-K_X = f^*(-K_{X'}) - 2E,$$

and $-K_X^3 = -K_{X'}^3 - 8$. This also shows that 2 divides the Fano index of X', so that $i(X') = 2$ or $i(X') = 4$.

The 3-fold X' cannot have Fano index 4. If it did, its anticanonical model Y' would be $\mathbb{P}^3$ (Theorem 5.1.1). By Theorem 4.2.4, Y is a one-parameter deformation of a non-singular Fano 3-fold Y_t with Fano index 2 and Picard rank 1. The degree is invariant in a one-parameter flat deformation, hence $-K_Y^3 = -K_X^3 \leq 40$. In particular, X' cannot have degree $64 = -K_{\mathbb{P}^3}^3$; thus X' has Fano index 2. $\square$

Let Y be a non $\mathbb{Q}$-factorial Gorenstein Fano 3-fold with Picard rank 1 and Fano index 2. Let X be a $\mathbb{Q}$-factorialisation of Y. The 3-fold X is a weak$*$ Fano 3-fold. I run a Minimal Model Program on X.

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \dots \xrightarrow{f_n} X_n$$

By Lemma 5.1.2, each birational contraction is either a flop or of type E2. Theorem 3.2.12 gives the following possible end products of the Minimal Model Program on X:

1. X_n is a terminal Gorenstein $\mathbb{Q}$-factorial Fano 3-fold of rank 1 and Fano index 2.

2. $X_n \to \mathbb{P}^1$ has a structure of del Pezzo fibration of degree 8 (the general fibre is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$).

3. $X_n \to S$ has a structure of étale $\mathbb{P}^1$-bundle over $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{F}_2$.

The Picard number of X is determined by the number of planes contracted while running the Minimal Model Program, that is by the number of E2 contractions.

Corollary 5.1.6. *Let Y be a non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold of Picard rank 1 and Fano index 2, and let X be a small $\mathbb{Q}$-factorialisation of Y. Denote by $h^3 = \frac{-K_X^3}{8}$ the degree of X and Y. The Picard rank of X is at most $8 - h^3$.*

Proof. By Lemma 5.1.5, each divisorial contraction increases the degree by 1. Denote by l the number of divisorial contractions encountered in running the Minimal Model Program on X.

1. X_n is a terminal Gorenstein $\mathbb{Q}$-factorial 3-fold of Fano index 2. By Theorem 4.2.4, X_n has degree $1 \leq h_n^3 \leq 5$. As $h_n^3 = h^3 + l$ and $\rho(X) = l + 1$, $\rho(X) \leq 6 - h^3$.

2. $X_n \to \mathbb{P}^1$ is a del Pezzo fibration of degree 8. By Theorem 4.2.4, Y_n can be deformed to a non-singular Fano 3-fold of Fano index 2 and Picard rank 1. In particular, the degree $h_n^3 \leq 5$. Since $h_n^3 = h^3 + l$ and $\rho(X) = l + 2$, $\rho(X) \leq 7 - h^3$.

3. $X_n \to \mathbb{P}^2$ (resp. $X_n \to \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$) is an étale $\mathbb{P}^1$-bundle, and as above X_n has degree $h_n^3 \leq 5$. Since $h_n^3 = h^3 + l$ and $\rho(X) = l + 2$ (resp. $\rho(X) = l + 3$), $\rho(X) \leq 7 - h^3$ (resp. $\rho(X) \leq 8 - h^3$).

□

Corollary 5.1.7. *Let Y be a non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold of Picard rank 1 and Fano index 2 and let X be a small $\mathbb{Q}$-factorialisation of Y. Then X contains at most $7 - h^3$ disjoint planes.*

Proof. Let $\overline{E} = \mathbb{P}^2$ be a plane contained in X, then $-K_{X|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(2)$.

By the contraction theorem, there is an extremal contraction which contracts E to a point. Indeed, E is covered by curves Γ with $-K_X \cdot \Gamma = 2$ and $E \cdot \Gamma = -1$.

Claim 5.1.8. *The planes contracted when running the Minimal Model Program on X are disjoint non $\mathbb{Q}$-Cartier planes on Y.*

I show that I may assume that the flopping contractions are performed first, that is, that the Minimal Model Program on X is of the form:

$$X \to X_1 \to \cdots \to X_{n-l-1} \to \cdots \to X_n$$

where all the extremal contractions f_i for $i \leq n-l-1$ are flopping contractions and the extremal contractions f_i for $i \geq n-l$ are divisorial contractions of type E2.

A flopping contraction and a divisorial contraction of type E2 always commute. Indeed, the exceptional locus of a flopping contraction is K-trivial, while that of an E2 contraction is K-negative. More precisely, assume that an E2 contraction f_{i-1} is followed by a flopping contraction f_i:

$$X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1}$$

Then the centre P of f_{i-1} does not belong to a flopping curve C. Otherwise, let $\widetilde{C}$ be a curve, which maps 1-to-1 to C; $-K_{X_{i-1}} \cdot \widetilde{C} < 0$ and this contradicts X_{i-1} being weak$*$.

Let E_i be the exceptional divisor of f_i for $i \geq n-l$. Without loss of generality, I want to prove that for $i \geq n-l+1$, the centre P_i of f_i on X_i does not belong to E_{i+1}. The proper transform of any line through P_i is a flopping curve in X_{i-1}. The morphism determined by $|-K_{X_{i-1}}|$ contracts finitely many curves, in particular, there are finitely many curves C through P_i such that $-K_{X_i} \cdot C = 2$. The point P_i cannot lie on E_{i+1}.

In particular, the proper transforms on X of the divisors contracted when running the Minimal Model Program on X are disjoint planes. There is no other plane disjoint from these on X, as this would give rise to another extremal contraction. $\square$

Remark 5.1.9. The image of a plane $E \subset X$ on Y is $\nu(S)$, where $S \simeq \mathbb{P}^2 \subset \mathbb{P}^5 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ is the Veronese surface and ν is the projection of $\mathbb{P}^5$ from a possibly empty linear subspace

$$\mathbb{P}^5 = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))) \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^2, -K_{X|\mathbb{P}^2})$$

associated to the inclusion of linear subspaces $|-K_{X|\mathbb{P}^2}| \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$. As X is weak$*$, $|-K_X|_{|E}$ is basepoint free and ν is the identity or the projection of S from a point $P \in \mathbb{P}^5 \smallsetminus S$.

5.2 Quartic 3-fold containing a plane

Assume that $Y = Y_4^3 \subset \mathbb{P}^4$ is a quartic 3-fold with terminal singularities that contains a plane $\Pi = \{x_0 = x_1 = 0\}$. As Y is a complete intersection in $\mathbb{P}^4$, Y is Gorenstein. The Grothendieck-Lefschetz hyperplane theorem shows that $\rho(Y) = 1$. I aim at determining a bound on the defect of Y. The equation of Y is of the form:

$$Y = \{x_0 \cdot a_3(x_0, \cdots, x_4) + x_1 \cdot b_3(x_0, \cdots, x_4) = 0\}. \tag{5.1}$$

Let X be the blow up of Y along the plane Π. Locally, the equation of X can be written as:

$$\begin{multline}\{t_0 \cdot a_3(t_0x, t_1x, x_2, x_3, x_4) + t_1 \cdot b_3(t_0x, t_1x, x_2, x_3, x_4) = 0\} \\ \subset \mathbb{P}(t_0, t_1) \times \mathbb{P}(x, x_2, x_3, x_4), \end{multline} \tag{5.2}$$

where the variable x is defined by $x_0 = t_0 \cdot x$, $x_1 = t_1 \cdot x$.

I fix some notation for rational scrolls over $\mathbb{P}^1$. Let t_0, t_1 be coordinates on $\mathbb{P}^1$ and consider $\mathbb{C}^{n+1}$ with coordinates $x_0, \dots, x_n$. Fix a set of non negative integers $a_0, \dots, a_n$ in increasing order.

I consider actions of the group $G = \mathbb{C}^\times \times \mathbb{C}^\times$ on the affine space $\mathbb{C}^2 \times \mathbb{C}^{n+1}$, where the two factors of G act by:

$$\lambda : (t_0, t_1, x_0, \ldots, x_n) \mapsto (\lambda t_0, \lambda t_1, \lambda^{-a_0} x_0, \ldots, \lambda^{-a_n} x_n)$$
$$\mu : (t_0, t_1, x_0, \ldots, x_n) \mapsto (t_0, t_1, \mu x_0, \ldots, \mu x_n)$$

I use the following matrix notation to summarise this action:

$$\begin{pmatrix} 1 & 1 & a_0 & \ldots & a_n \\ 0 & 0 & 1 & \ldots & 1 \end{pmatrix}$$

The scroll $\mathbb{F}(a_0, \ldots, a_n)$ is the quotient $(\mathbb{C}^2 \smallsetminus \{0\}) \times (\mathbb{C}^{n+1} \smallsetminus \{0\})/G$. The scroll $\mathbb{F}$ is a $\mathbb{P}^n$-bundle over $\mathbb{P}^1$. In the case of $\mathbb{P}^3$-bundles over $\mathbb{P}^1$, I denote by $\mathbb{F}_{a_1,a_2,a_3}$ the scroll $\mathbb{F}(0, a_1, a_2, a_3)$. Line bundles on $\mathbb{F}$ are in 1-to-1 correspondence with characters $\chi \colon G \to \mathbb{C}^*$ of the group G. Indeed, associate to a character χ the line bundle L_χ such that the space of sections of L_χ is:

$$H^0(\mathbb{F}, L_\chi) = \{f : (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^{n+1} - \{0\}) \to \mathbb{C} \mid f(gx) = \chi(g) f(x)\}.$$

Let L and M be the line bundles associated to the characters $(\lambda, \mu) \mapsto \lambda$ and $(\lambda, \mu) \mapsto \mu$ respectively. Denoting by $\pi \colon \mathbb{F} \to \mathbb{P}^1$ the natural projection, L is the pull back $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$.

The 3-fold X is naturally embedded in the scroll $\mathbb{F}_{0,0,1}$ over $\mathbb{P}^1$. It is an element of the linear system $|3M + L|$ on $\mathbb{F}$, and it has a natural structure of cubic del Pezzo fibration over $\mathbb{P}^1$. The generic fibre X_η is reduced and irreducible because Y is. However, special fibres might be reducible. Moreover, X is a weak Fano 3-fold, $\rho(X/Y) = 1$ and the map $X \to Y$ is small (X is a small partial $\mathbb{Q}$-factorialisation of Y).

Bounding the rank of the group of Weil divisors of X suffices to determine the defect of Y.

Lemma 5.2.1. *[Cor96] Let X_i, $0 \leq i \leq n$, be the reducible fibres of the cubic fibration X. Let $D_{\eta,k}$ be irreducible generators of $NE(X_\eta)$, D_k the closure of $D_{\eta,k}$ in X (each D_k is irreducible) and let the $E_{i,j}$ be the irreducible components of X_i. The cone of effective divisors $N^1(X)$ is generated by the $E_{i,j}$ and by the divisors D_k.*

Remark 5.2.2. A similar description holds for any del Pezzo fibration.

Corti proves in [Cor96] that, when projecting a terminal weak Fano del Pezzo fibration away from a plane contained in a reducible fibre, the number of irreducible components of this reducible fibre decreases. Let me recall his results more precisely.

Let $\mathcal{O}$ be a discrete valuation ring, with fraction field K, parameter t and residue field k. In the cases I consider, $\mathcal{O}$ is the local ring of $\mathbb{P}^1$ at a point P that corresponds to a reducible fibre and k is $\mathbb{C}$. Let $S = \operatorname{Spec}(\mathcal{O})$, $\eta = \operatorname{Spec} K$ be the generic point and $0 = \operatorname{Spec} k$. Let $\mathbb{P} = \mathbb{P}^n_{\mathcal{O}}$ be an n-dimensional projective space over S and $L = L_d$ a d-dimensional projective subspace, defined over k. If $d \leq n-1$, there is a birational transformation $\Phi = \Phi_L \colon \mathbb{P} \to \mathbb{P}$, centred at L. The map Φ is a projection from L. In homogeneous coordinates, it is given by:

$$\Phi\colon (x_0 : \ldots : x_n) \mapsto (tx_0 : \ldots : tx_d : x_{d+1} : \ldots : x_n).$$

Now, let $X \subset \mathbb{P}$ be a cubic surface defined over S, such that $X_\eta = X_K$ is smooth. I assume that X is a weak Fano 3-fold in the sense of Definition 3.1.1 and that the anticanonical map of X is small. Let $n(X)$ denote the number of k-irreducible components of the central fibre X_0.

Lemma 5.2.3. *[Cor96, Lemma 2.17] Assume that X_0 contains a 2-plane $\Pi \subset \mathbb{P}_0$, defined over k. Let $\Phi\colon \mathbb{P} \dashrightarrow \mathbb{P}^+$ be the projection away from Π and $X^+ = \Phi_* X$. Then:*

1. *X^+ has terminal singularities, $-K_{X^+}$ is nef and big over* $\operatorname{Spec}(\mathcal{O})$.
2. *$n(X^+) < n(X)$.*

Remark 5.2.4. Notice that in such an operation of projection away from a plane, the Weil rank decreases by at most two. Indeed, this follows from Lemma 5.2.1, as such a transformation only affects X in the central fibre.

Corollary 5.2.5 ([Cor96], Flowchart 2.13 and Corollary 2.20)**.** *If $X \subset \mathbb{P}^3_{\mathcal{O}}$ is a weak Fano cubic fibration with small anticanonical map, then projecting away from planes contained in the central fibre yields a standard integral model X' for X_η, that is a flat subscheme $X' \subset \mathbb{P}^3_{\mathcal{O}}$ with isolated cDV singularities and reduced and irreducible central fibre.*

Lemma 5.2.6. *The rank of the Weil group of X is bounded by $8+2\cdot N+M$, where N is the number of reducible fibres with 3 irreducible components and M the number of reducible fibres with 2 irreducible components. The defect of Y is hence bounded above by $7+2\cdot N+M$.*

Proof. If a fibre of $X \to \mathbb{P}^1$ is reducible, it has at most 3 irreducible components (one of which has to be a plane). Project away from these reducible components in order to obtain a standard integral model. The result follows from the following exact sequence, valid when $\pi\colon X' \to \mathbb{P}^1$ has reduced and irreducible fibres:

$$0 \to \mathbb{Z}[\pi^*\mathcal{O}_{\mathbb{P}^1}(1)] \to \mathrm{Weil}(X') \to \mathrm{Pic}(X'_\eta) \to 0.$$

□

The problem is now to bound the possible number of reducible fibres of X.

Example 5.2.7 (The Burkhardt quartic)**.** It is known that a nodal quartic hypersurface in $\mathbb{P}^4$ has at most 45 nodes [Var83, Fri86]. Moreover, up to projective equivalence, there is only one such $Y = Y_4^3 \subset \mathbb{P}^4$ [dJSBVdV90] and it has the following equation:

$$\{x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4 = 0\}.$$

Let $\widetilde{Y}$ be the blow up of Y at the 45 nodes. The Hodge theoretic approach (Chapter 2,(2.3)) leads to $\sigma(Y) = b_2(\widetilde{Y}) - 45 - b_2(Y) = b_2(\widetilde{Y}) - 46$. The cohomology of $\widetilde{Y}$ is determined in [HW01] and $b_2(\widetilde{Y}) = 61$, so that the defect of the Burkhardt quartic is 15.

Alternatively, the plane $\Pi = \{x_0 = x_1 = 0\}$ is contained in Y. Let X be the 3-fold obtained by blowing up Y along the plane Π; X has the following expression:

$$t_0(t_0^3 - t_1^3)x^3 - t_0(x_2^3 + x_3^3 + x_4^3) + 3t_1x_2x_3x_4 = 0,$$

which on the affine piece $t_1 \neq 0$ reads

$$t_0(t_0^3 - 1)x^3 - t_0(x_2^3 + x_3^3 + x_4^3) + 3x_2x_3x_4 = 0.$$

The central fibre, which corresponds to $(t_0 : t_1) = (0 : 1)$, has three irreducible components: the planes $\{t_0 = x_i = 0\}$. Each fibre reduced and irreducible for $(t_0 : t_1) \neq (0 : 1)$ and $(t_0 : t_1) \neq (1 : \omega^i)$ for $0 \leq i \leq 2$, where ω is a cube root of unity (notice that considering the affine piece $t \neq 1$ yields the same results). The generic fibre X_η is a non-singular cubic surface in $\mathbb{P}^3$.

Consider for instance the fibre X_1 over $(1 : 1)$. The fibre X_1 is the union of three planes in $\mathbb{P}(x, x_2, x_3, x_4)$, namely $\Pi_1 = \{x_2 + x_3 + x_4 = 0\}$,

$\Pi_2 = \{\omega x_2+x_3+\omega^2 x_4 = 0\}$ and $\Pi^3 = \{\omega^2 x_2+x_3+\omega x_4 = 0\}$. The situation is analogous for the other two reducible fibres. There are 27 closed subschemes in $X \smallsetminus \bigcup_{i=0,1,2} X_i$ isomorphic to $\mathbb{P}^1_{\mathbb{P}^1 \smallsetminus (\cup_i \{(\omega^i:1)\})}$. In other words, the 27 lines on the generic fibre may be completed to divisors on X, they are rational over $\mathbb{P}^1 \smallsetminus (\cup_i \{(\omega^i : 1)\})$. The Picard rank of X_η is 7 and the generators of $\mathrm{Pic}(X_\eta)$ complete to independent Cartier divisors on X.

This shows that the rank of the Weil group of X is $1+7+8$. The rank of the Weil group of the Burkhardt quartic is 16 and its defect is 15.

I show that if X is a cubic fibration obtained by blowing up a plane lying on a quartic 3-fold $Y = Y_4^3 \subset \mathbb{P}^4$ with terminal singularities, then X has at most 4 reducible fibres. By construction, Y is the anticanonical model of X.

Lemma 5.2.8. *Let $Y = Y_4^3 \subset \mathbb{P}^4$ be a terminal quartic 3-fold that contains a plane Π and let X be the cubic del Pezzo fibration obtained by blowing up Y along Π. The cubic fibration X has at most 4 reducible fibres.*

Corollary 5.2.9. *The defect of a terminal quartic 3-fold $Y = Y_4^3 \subset \mathbb{P}^4$ that contains a plane is at most 15.*

Proof of Lemma 5.2.8. Recall that X is the blow up of a terminal quartic 3-fold $Y \subset \mathbb{P}^4$ of the form

$$Y = \{x_0 a_3(x_0, \cdots, x_4) + x_1 b_3(x_0, \cdots, x_4) = 0\} \subset \mathbb{P}^4 \tag{5.3}$$

along the plane $\Pi = \{x_0 = x_1 = 0\}$.

From (5.2), the equation of X reads:

$$X = \{t_0 a_3(t_0 x, t_1 x, x_2, x_3, x_4) + t_1 b_3(t_0 x, t_1 x, x_2, x_3, x_4) = 0\},$$

where a_3 and b_3 are sections of the linear system $|3M|$ on the scroll $\mathbb{F}_{0,0,1}$. Denote by f the morphism $f\colon X \to Y$.

The terminal Gorenstein 3-fold X is a section of the linear system $|3M+L|$ on the scroll $\mathbb{F}_{0,0,1}$.

A change of coordinates on the scroll $\mathbb{F}_{0,0,1}$ is given by a usual change of coordinates on $\mathbb{P}^1$ and by the action of a matrix of the following form on $(x_2 : x_3 : x_4 : x)$:

$$\begin{pmatrix} a & b & c & l_1(t_0,t_1) \\ d & e & f & l_2(t_0,t_1) \\ g & h & i & l_3(t_0,t_1) \\ 0 & 0 & 0 & j \end{pmatrix},$$

where the upper 3 by 3 minor is an invertible complex matrix, l_1, l_2 and l_3 are linear forms and j is a non-zero complex number.

The fibre of the scroll $\mathbb{F}_{0,0,1}$ over $(\lambda : \mu) \in \mathbb{P}^1$ is isomorphic to the hyperplane $H_{(\lambda:\mu)} = \{\mu x_0 - \lambda x_1 = 0\} \subset \mathbb{P}^4$. There is a 1-to-1 correspondence between fibres of the scroll $\mathbb{F}_{0,0,1}$ and hyperplanes of $\mathbb{P}^4$ that contain the plane Π.

The hyperplane section of $Y \subset \mathbb{P}^4$ corresponding to $H_{(\lambda:\mu)}$ is a reducible quartic surface that contains the plane Π. The intersection $Y \cap H_{(\lambda:\mu)}$ is $\Pi \cup Y'_{(\lambda:\mu)}$, where $Y'_{(\lambda:\mu)}$ is a possibly reducible cubic surface, naturally isomorphic to the fibre $X_{(\lambda:\mu)}$.

If X has a reducible fibre over $(\lambda : \mu) \in \mathbb{P}^1$, X contains a plane either of the form:

$$\Pi_{(\lambda:\mu)} = \{\lambda t_0 + \mu t_1 = l(x_2, x_3, x_4) + l'(t_0 x, t_1 x) = 0\},$$

where l and l' are linear, or of the form:

$$\Pi_{(\lambda:\mu)} = \{\lambda t_0 + \mu t_1 = x = 0\}.$$

The plane $\Pi_{(\lambda:\mu)} \subset X_{(\lambda:\mu)}$ is isomorphic to a plane $\Pi'_{(\lambda:\mu)}$ lying on $Y \cap H_{(\lambda:\mu)}$. In the first case, $\Pi'_{(\lambda:\mu)}$ is of the form:

$$\Pi'_{(\lambda:\mu)} = \{\lambda x_0 + \mu x_1 = l(x_2, x_3, x_4) + l'(x_0, x_1) = 0\}$$

and meets Π in a line $L_{(\lambda:\mu)} = \{x_0 = x_1 = l(x_2, x_3, x_4) = 0\}$, while in the second case $\Pi'_{(\lambda:\mu)} = \Pi$.

Step 1. *The cubic fibration X does not contain two planes of the form:*

$$\{\alpha t_0 + \beta t_1 = x = 0\}.$$

I show that if X contains two distinct planes of this form, every fibre of X is reducible. This contradicts Y having terminal singularities.

Indeed, we may change coordinates on $\mathbb{P}^1$ so that the two reducible fibres lie over 0 and ∞. Let us assume that X contains the planes $\{t_0 = x = 0\}$ and $\{t_1 = x = 0\}$. Then the equation of X reads:

$$t_0 t_1 A_{3M-L} + x B_{2M+2L} = 0,$$

where A_{3M-L} is a section of $|3M - L|$ and B_{2M+2L} is a section of $|2M + 2L|$. Every monomial in $H^0(\mathbb{F}_{0,0,1}, 3M - L)$ is divisible by x, hence X is reducible.

From the point of view of $Y \subset \mathbb{P}^4$, if X contains two such planes, the hyperplane sections of $Y \cap H_{(0:1)}$ and $Y \cap H_{(1:0)}$ both contain the plane $\Pi = \{x_0 = x_1 = 0\}$ with multiplicity 2, so that in the equation (5.3), both a_3 and b_3 lie in the ideal of Π. The quartic Y has generic multiplicity at least 2 along Π.

Step 2. *Assume that X has at least two reducible fibres. Let $\Pi_1 = \Pi_{(\lambda_1:\mu_1)}$ and $\Pi_2 = \Pi_{(\lambda_2:\mu_2)}$ be two planes lying in distinct fibres of X and denote by $\Pi'_1 = \Pi'_{(\lambda_1:\mu_1)}$ and $\Pi'_2 = \Pi'_{(\lambda_2:\mu_2)}$ the planes of $\mathbb{P}^4$ to which they are isomorphic. If Π'_1 and Π'_2 are both distinct from Π, they meet at a point.*

I show that if X contains two such planes Π_1 and Π_2, associated to planes Π'_1 and Π'_2, that intersect Π along the same line $L_1 = L_2 = L$, Y has multiplicity 2 along the line L. This contradicts Y having isolated singularities.

The planes Π'_1 and Π'_2 of $\mathbb{P}^4$ lie in distinct hyperplanes of $\mathbb{P}^4$ because they correspond to distinct fibres of the cubic; Π'_1 and Π'_2 intersect Π in lines L_1 and L_2 respectively. If $L_1 = L_2$, without loss of generality, I may assume that $\Pi'_1 = \{x_0 = x_2 = 0\}$ and $\Pi'_2 = \{x_1 = x_2 = 0\}$ after coordinate change on $\mathbb{P}^4$. The planes Π'_1 and Π'_2 both intersect the plane Π along the line $L_1 = L_2 = L = \{x_0 = x_1 = x_2 = 0\}$.

The plane Π'_1 (resp. Π'_2) lies on Y if and only if, in (5.3), the homogeneous form b_3 is in the ideal $\langle x_0, x_2\rangle$ of Π'_1 (resp. the homogeneous form a_3 is in the ideal $\langle x_1, x_2\rangle$ of Π'_2). The quartic Y then has multiplicity 2 along the line L.

Step 3. *If X contains three planes Π_1, Π_2 and Π_3 which lie in distinct fibres and correspond to planes Π'_1, Π'_2 and Π'_3 lying on Y that are all distinct from Π, the three planes do not meet at a single point.*

I show that if the planes Π'_1, Π'_2 and Π'_3 meet at a point P, the quartic Y does not have isolated cDV singularities. More precisely, I show that X, a small partial $\mathbb{Q}$-factorialisation of Y, does not have isolated singularities. The cubic fibration X is hence not terminal: $P \in Y$ is not a cDV point.

Up to coordinate change on $\mathbb{P}^4$, we may assume that

$$\begin{aligned} \Pi'_1 &= \{x_0 = x_2 = 0\} \\ \Pi'_2 &= \{x_0 = x_2 = 0\} \\ \Pi'_3 &= \{x_0 + x_1 = x_2 + x_3 + l(x_0, x_1) = 0\}, \end{aligned}$$

where l is a linear form.

Equivalently, up to global coordinate change on the scroll, X contains the following 3 planes:

$$\begin{aligned}\Pi_1 &= \{t_0 = x_2 = 0\}\\ \Pi_2 &= \{t_1 = x_3 = 0\}\\ \Pi_3 &= \{t_0 + t_1 = x_2 + x_3 + l(t_0, t_1)x = 0\}.\end{aligned}$$

The equation of X therefore has to be in the ideal spanned by the monomials:

$$\begin{aligned}I = \{&(t_0 + t_1)t_0t_1, t_0t_1(x_2 + x_3 + l(t_0, t_1)x), (t_0 + t_1)t_1x_2, (t_0 + t_1)t_0x_3,\\ &t_1x_2(x_2 + x_3 + l(t_0, t_1)x), t_0x_3(x_2 + x_3 + l(t_0, t_1)x), (t_0 + t_1)x_2x_3,\\ &x_2x_3(x_2 + x_3 + l(t_0, t_1)x)\},\end{aligned}$$

The cubic fibration X has multiplicity 2 along $\Gamma = \{x = x_2 = x_3 = 0\}$.

Indeed, the only monomials in I which do not have multiplicity 2 along Γ are $(t_0 + t_1)t_0t_1$, $t_0t_1(x_2 + x_3 + l(t_0, t_1)x)$, $(t_0 + t_1)t_1x_2$ and $(t_0 + t_1)t_0x_3$. However, $(t_0 + t_1)t_0t_1$ is a section of $|3L|$, so that if it appears in the equation of $X \in |3M + L|$, it is multiplied by A_{3M-2L}, a section of $|3M - 2L|$. Any monomial in $|3M-2L|$ is divisible by x^2, hence the term $(t_0+t_1)t_0t_1 \times A_{3M-2L}$ which appears in the equation of X (if non-zero) has multiplicity 2 along Γ.

Similarly, $t_0t_1(x_2+x_3+l(t_0, t_1)x)$, $(t_0+t_1)t_1x_2$ and $(t_0+t_1)t_0x_3$ are sections of $|M+2L|$ and have multiplicity 1 along Γ. If they appear in the equation of X, they are multiplied by a section of $|2M - L|$. Any monomial of $|2M - L|$ is divisible by x, so that these terms, if non-zero, also have multiplicity 2 along Γ.

As Y has terminal Gorenstein singularities, X, a small partial resolution of Y, also has isolated singularities: this yields a contradiction.

In terms of Y, if X contains such a combination of planes, the singular point $(0 : 0 : 0 : 0 : 1) \in Y$ is not a cDV point.

If X contains three planes Π_1, Π_2 and Π_3 lying in distinct fibres, which correspond to planes Π'_1, Π'_2 and $\Pi'_3 \subset Y$ all distinct from Π, without loss of generality, up to coordinate change on $\mathbb{P}^4$, we may assume that:

$$\begin{aligned}\Pi'_1 &= \{x_0 = x_2 = 0\} \subset H_{(0:1)}\\ \Pi'_2 &= \{x_1 = x_3 = 0\} \subset H_{(1:0)}\\ \Pi'_3 &= \{x_0 + x_1 = x_4 = 0\} \subset H_{(1:1)}.\end{aligned}$$

Equivalently, up to global change of coordinates on the scroll, we may assume that X contains the following 3 planes:

$$\begin{aligned}\Pi_1 &= \{t_0 = x_2 = 0\} \subset X_{(0:1)}\\ \Pi_2 &= \{t_1 = x_3 = 0\} \subset X_{(1:0)}\\ \Pi_3 &= \{t_0 + t_1 = x_4 = 0\} \subset X_{(1:1)}\end{aligned}$$

Step 4. *The cubic fibration X has at most 4 reducible fibres.*

I argue by contradiction. Assume that X has at least 5 reducible fibres and denote by $\Pi_1, \cdots, \Pi_5$ planes that lie in distinct reducible fibres. Denote by $\Pi_1', \cdots, \Pi_5' \subset \mathbb{P}^4$ the planes lying on Y naturally associated to $\Pi_1, \cdots, \Pi_5$. Recall that Step 1 shows that at most 1 of the planes $\Pi_1', \cdots, \Pi_5'$ is isomorphic to Π. Without loss of generality, I may assume that the four planes Π_1', Π_2', Π_3' and Π_4' are distinct from Π and denote by $L_i = \Pi \cap \Pi_i'$ for $1 \leq i \leq 4$. Steps 2 and 3 show that the lines $L_i \subset \Pi$ are distinct and that no three of these lines meet at a point. Up to change of coordinates, we may therefore assume that:

$$\begin{aligned}\Pi_1 &= \{t_0 = x_2 = 0\} \subset X_{(0:1)}\\ \Pi_2 &= \{t_1 = x_3 = 0\} \subset X_{(1:0)}\\ \Pi_3 &= \{t_0 + t_1 = x_4 = 0\} \subset X_{(1:1)}\end{aligned}$$

and that

$$\Pi_4 = \{t_0 + \lambda t_1 = a x_2 + b x_3 + c x_4 + l(t_0, t_1) x = 0\} \subset X_{(1:\lambda)},$$

where a, b and c are constants and l is a linear form. The constant $\lambda \neq 0, 1$ or ∞ because no two of the planes Π_i lie in the same fibre of X. The constants a, b and c are all non-zero, as otherwise either two of the lines L_i coincide, or three of the lines L_i meet at a point. Up to rescaling, we may assume that:

$$\Pi_4 = \{t_0 + \lambda t_1 = x_2 + x_3 + x_4 + l(t_0, t_1) x = 0\} \subset X_{(1:\lambda)}.$$

Claim 5.2.10. *The plane $\Pi_5' \subset Y$ is not Π.*

If X has a fifth reducible fibre containing a plane Π_5 such that $\Pi_5' = \Pi$, X contains a plane of the form:

$$\Pi_5 = \{t_0 + \mu t_1 = x = 0\}.$$

with $\mu \neq 1, 0, \infty, \lambda$.

The equation of X may be written uniquely in the form:

$$A(t_0,t_1)x^3+B(t_0,t_1,x_2,x_3,x_4)x^2+C(t_0,t_1,x_2,x_3,x_4)x+D(t_0,t_1,x_2,x_3,x_4) = 0, \tag{5.4}$$

with D in the ideal

$$\{t_0x_3x_4(x_2+x_3+x_4), t_1x_2x_4(x_2+x_3+x_4), \\ (t_0+t_1)x_2x_3(x_2+x_3+x_4), (t_0+\lambda t_1)x_2x_3x_4\}, \tag{5.5}$$

i.e. D is a linear combination of the above monomials.

In the expression (5.4), D has to be of the form

$$D(t_0,t_1,x_2,x_3,x_4) = (t_0+\mu t_1)P(x_2,x_3,x_4),$$

with P a polynomial of degree 3. Writing D as a linear combination of the 4 monomials in (5.5) shows that the only possibility is for D to be equal to a scalar multiple of one of these monomials. In this case, μ is equal to one of $0, 1, \infty$ or λ, so that Π_5 lies in one of the fibres above $(0:1), (1:0), (1:1)$ or $(1, \lambda)$. This is a contradiction.

Claim 5.2.11. *X cannot have a fifth reducible fibre that contains a plane Π_5 such that Π_5' is distinct from Π.*

If X contains such a plane Π_5, the equation of Π_5 is of the form:

$$\Pi_5 = \{t_0+\mu t_1 = \alpha x_2 + \beta x_3 + \gamma x_4 + l'(t_0,t_1)x = 0\},$$

where l' is a linear form and α, β and γ are constants. As any three lines of $L_1, \cdots, L_5$ have to satisfy the conditions of Steps 2 and 3, α, β and γ are all non-zero. I may assume that $\alpha = 1$. Considering triples of lines (L_4, L_5, L_i) for $1 \leq i \leq 3$ shows that $\beta \neq 1$, $\gamma \neq 1$, and $(\beta : \gamma) \neq (1:1)$.

In the expression (5.4), $D(t_0,t_1)$ may be written as a linear combination of the monomials

$$\{t_0x_3x_4(x_2+x_3+x_4), t_1x_2x_4(x_2+x_3+x_4), \\ (t_0+t_1)x_2x_3(x_2+x_3+x_4), (t_0+\lambda t_1)x_2x_3x_4\}$$

or as a linear combination of the monomials

$$\{t_0x_3x_4(x_2+\beta x_3+\gamma x_4), t_1x_2x_4(x_2+\beta x_3+\gamma x_4), \\ (t_0+t_1)x_2x_3(x_2+\beta x_3+\gamma x_4), (t_0+\mu t_1)x_2x_3x_4\}.$$

Equating these two expressions, one finds that either $D = 0$ or $\lambda = \mu$. If $D = 0$, every fibre of X is reducible and this yields a contradiction. If $\lambda = \mu$, Π_5 is contained in the reducible fibre over $(1 : \lambda)$.

This show that X has at most 4 reducible fibres. $\square$

I have proved the following:

Theorem 5.2.12 (Main Theorem 1)**.** *Let $X_4^3 \subset \mathbb{P}^4$ be a quartic 3-fold. Then the defect of X is at most:*

1. 8 *if X does not contain a plane or a quadric,*
2. 11 *if X contains a quadric but no plane,*
3. 15 *if X contains a plane.*

Remark 5.2.13. Let Y be a terminal Gorenstein Fano 3-fold of Fano index 1. Denote by g its genus. It is known that if $g \geq 6$, Y does not contain a plane. If g is equal to 4 or 5 and if Y contains a plane Π, a similar analysis can be carried out. Let X be the crepant blow up of Π. Projecting Y away from Π shows that X is a conic bundle over $\mathbb{P}^2$ if $g = 4$, and is birational to $\mathbb{P}^3$ if $g = 5$. I make the following conjecture:

Conjecture 5.2.14.

1. *The defect of a non $\mathbb{Q}$-factorial terminal Gorenstein $Y_{2,3} \subset \mathbb{P}^5$ with Picard rank* 1 *is at most* 8*,*
2. *The defect of a non $\mathbb{Q}$-factorial terminal Gorenstein $Y_{2,2,2} \subset \mathbb{P}^6$ with Picard rank* 1 *is at most* 8*,*
3. *The defect of a non $\mathbb{Q}$-factorial terminal Gorenstein Fano* 3*-fold with Picard rank* 1 *of genus $g \geq 6$ is at most $\left[\frac{12-g}{2}\right] + 5$.*

Remark 5.2.15. I would like to conjecture that the defect of a terminal Gorenstein double sextic with Picard rank 1 ($g = 2$) is at most 18. I unfortunately have no evidence to support that conjecture.

Chapter 6

Deformation Theory

Let X be a weak$*$ Fano 3-fold and Y its anticanonical model. In this chapter, I show that deformation theory can be used to determine invariants of the intermediate weak$*$ Fano 3-folds X_i and to describe the extremal contractions encountered when running the Minimal Model Program on X. It follows from Namikawa's results (Theorem 6.2.8) and from Lemma 2.2.7 that the degrees of the intermediate weak$*$ Fano 3-folds X_i can only take values among the list of degrees of non-singular Fano 3-folds with Picard rank less than or equal to $\rho(Y)$, the Picard rank of Y. This result has been used in Chapters 4 and 5 to bound the Picard rank of X, and hence the defect of Y. I show that each step $\phi_i \colon X_i \to X_{i+1}$ of the MMP on X can be deformed— in a suitable sense— to the contraction of an extremal ray on a non-singular weak Fano 3-fold with Picard rank 2.

First, I recall the set-up of the theory of deformation functors of Schlessinger and Lichtenbaum [LS67]. Second, I survey results obtained by Friedman and Namikawa on the deformation theory of *generalised Fano 3-folds* [Nam97, Fri86]. Generalised Fano 3-folds are algebraic 3-folds that admit a small birational proper map to a Fano 3-fold Y with terminal Gorenstein singularities. Weak$*$ Fano 3-folds are generalised Fano 3-folds, but the category of generalised Fano 3-folds is larger, as it does not require $\mathbb{Q}$-factoriality. For instance, any small partial $\mathbb{Q}$-factorialisation of a terminal Gorenstein Fano 3-fold Y is a generalised Fano 3-fold. Third, I recall and extend some results of Kollár and Mori [KM92] on deformations of extremal contractions.

6.1 Theoretical setup

Let k be a field and let Λ be a complete local noetherian k-algebra with residue field k and maximal ideal $\mathfrak{m}_\Lambda$. Consider $\mathcal{C}_\Lambda$, the category of local Λ-algebras with residue field k and local homomorphisms.

Definition 6.1.1. Consider a covariant functor $F\colon \mathcal{C}_\Lambda \to Sets$.

1. Let $k[\epsilon]$ with $\epsilon^2 = 0$ be the ring of dual numbers over k; the *tangent space* to F is $t_F = F(k[\epsilon])$.

2. The functor F *has a hull* if there exists a complete local Λ-algebra R and a morphism

$$\phi\colon \operatorname{Hom}_{\text{local }\Lambda\text{-algebras}}(R, \cdot) \to F,$$

   such that for all $A \in \mathcal{C}_\Lambda$, $\phi(A)$ is surjective, and such that ϕ induces an isomorphism on tangent spaces.

3. The functor F is *pro-representable* if it is isomorphic to a functor of the form:

$$F \simeq \operatorname{Hom}_{\text{local }\Lambda\text{-algebras}}(R, \cdot),$$

   with R a complete local Λ-algebra such that $R/\mathfrak{m}_R^n \in \mathcal{C}_\Lambda$ for all n.

4. Let $A, A' \in \mathcal{C}_\Lambda$ and $p\colon A' \to A$ be a morphism. The morphism p is a *small extension* if $\ker p$ is a principal ideal annihilated by the maximal ideal of A'.

Schlessinger states the following conditions for a functor of Artin rings to have a hull or to be pro-representable.

Theorem 6.1.2 ([Sch68]). *Let $F\colon \mathcal{C}_\Lambda \to Sets$ be a covariant functor such that $F(k)$ is restricted to one element (e). Let A, A' and A'' be objects of $\mathcal{C}_\Lambda$, and let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}_\Lambda$. Consider the natural map:*

$$\psi_{A,A',A''}\colon F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

1. *F has a hull if and only if the following conditions $H1$ to $H3$ hold:*

   [H1] $\psi_{A,A',A''}$ is a surjection whenever $A'' \to A$ is a small extension,

   [H2] $\psi_{k,A',k[\epsilon]}$ is a bijection, so that $F(A' \times_k k[\epsilon]) \simeq F(A') \times_{(e)} t_F$,

   [H3] $\dim_k t_F$ is finite.

2. *F is pro-representable if moreover:*

   [H4] For any small extension $A' \to A$, $\psi_{A,A',A'}$ is a bijection.

Remark 6.1.3. 1. The condition H2 applied to $A' = k[W]$ for W a k-vector space endows t_F with a canonical k-vector space structure.

2. Let A be an element of $\mathcal{C}_\Lambda$ and let I be an ideal annihilated by the maximal ideal of A; $A \times_{A/I} A$ is naturally isomorphic to $A \times_k k[I]$ by the map sending (x, y) to $(x, x_0 - x + y)$, where x_0 denotes the k-residue of x.

   Let $p\colon A' \to A$ be a small extension with kernel I. If condition H2 holds, $\psi_{A,A',A'}$ may be written:

   $$\psi_{A,A',A'}\colon F(A') \times_{F(A)} (t_F \otimes I) \xrightarrow{\sim} F(A') \times_{F(A)} F(A').$$

   For any $\eta \in F(A)$, ψ defines a group action of $t_F \otimes I$ on $F(p^{-1}(\eta)) \subset F(A')$. Condition H1 ensures that this action is transitive, while condition H4 makes $F(p^{-1}(\eta))$ a formally principal homogeneous space (i.e. a torsor) under $t_F \otimes I$.

   If the first three conditions are satisfied, pro-representability is obstructed by fixed points of the action of $t_F \otimes I$. In the geometric setting, a point $\eta' \in F(p^{-1}(\eta))$ is fixed under the action of $t_F \otimes I$, if there is an automorphism of an object $y \in [\eta]$ that cannot be extended to an automorphism of some object of $[\eta']$.

Let $\hat{\mathcal{C}}$ be the category of complete local Λ-algebras A such that $A/\mathfrak{m}_A^n \in \mathcal{C}_\Lambda$ for all n. The functor F can be extended to $\hat{\mathcal{C}}$ by defining $\hat{F}(A)$ as $\operatorname{projLim} F(A/\mathfrak{m}_A^n)$. For any such ring $R \in \hat{\mathcal{C}}$, $h_R = \operatorname{Hom}(R, \cdot)$ defines a functor on $\mathcal{C}$. For any functor on $\mathcal{C}$, there is a canonical isomorphism $\hat{F}(R) \simeq \operatorname{Hom}(h_R, F)$.

Definition 6.1.4. 1. (A, ξ) is a *pro-couple* for F if $A \in \mathcal{C}$ and $\xi \in F(A)$. A morphism of pro-couples $u\colon (A, \xi) \to (A', \xi')$ is a morphism in $\mathcal{C}$ such that $F(u)(\xi) = \xi'$.

2. A pro-couple (R, ξ) for $\hat{F}$ is *pro-representing* if $h_R \to F$ is an isomorphism induced by ξ in the following sense. Since by definition $\xi = \operatorname{projLim} \xi_n \in \hat{F}(R)$, it is possible to associate to ξ a morphism of

functors Φ_ξ defined as follows: for any morphism $u\colon R \to A$ factoring through $u_n\colon R/\mathfrak{m}_R^n \to A$ for some n, let

$$\Phi_\xi(A) : u \in h_R(A) \to F(u_n)(\xi_n) \in F(A).$$

Let X be a pre-scheme over k. An *infinitesimal deformation* of X/k to a local Λ-algebra A is a diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A, \end{tikzcd}$$

with $X \simeq Y \times_{\operatorname{Spec} A} \operatorname{Spec} k$. The pre-scheme Y is required to be flat over $\operatorname{Spec} A$ and i is necessarily a closed immersion.

If Y'/A is another infinitesimal deformation of X/k, it is equivalent to Y/A if there exists a morphism of pre-schemes $f\colon Y \to Y'$ defined over A that induces the identity on the closed (or central) fibre X.

Given a deformation Y/A and a morphism $A \to B$, there is an induced deformation $(Y \otimes_A B)/B$. The notion of morphism of deformations is thus well defined.

Define a deformation functor $\mathcal{D}$ of X/k by associating to each local Λ-algebra A the equivalence class of infinitesimal deformations of X/k to A. Notice that $\mathcal{D}(k) = \{X/k\}$. Let the obstruction space T of $\mathcal{D}$ be the space making, for any $A, A' \in \mathcal{C}$ and $p\colon A' \to A$ small, the following sequence exact:

$$\mathcal{D}(A') \to \mathcal{D}(A) \to T \otimes \ker p \to 0.$$

The definition of the obstruction space is functorial and is consistent with Remark 6.1.3.

Theorem 6.1.5 ([Sch68])**.** *If X is proper over k, then $\mathcal{D}$ has a hull (R, ξ). The pro-couple (R, ξ) pro-represents $\mathcal{D}$ if and only if, for each small extension $A' \to A$ and for each deformation Y'/A' of X/k, every automorphism of the deformation $Y' \otimes_{A'} A$ is induced by an automorphism of A.*

Definition 6.1.6. Let X be a projective variety. The *Kuranishi space* $\operatorname{Def}(X)$ of X is the hull of the functor $\mathcal{D}$, that is, the semi-universal (or miniversal) space of flat deformations of X/k. If $\mathcal{D}$ is pro-representable, the pro-representing couple $(\operatorname{Def}(X), \mathcal{X})$ is a universal deformation object and $\mathcal{X}$ is called the *Kuranishi family* of X.

Lichtenbaum, Schlessinger [LS67] and Illusie [Ill71] make explicit the relation between the problem of infinitesimal deformations of X/k and that of extensions of local Λ-algebras. In that spirit, they relate infinitesimal deformations of X/k to the cotangent complex of X.

Proposition 6.1.7 ([LS67, Ill71]). *Let X be a proper locally complete intersection (lci) variety. Let $A \in \mathcal{C}_\Lambda$. The obstruction to the existence of a flat deformation of X/k to A is a class ω lying in $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$. If ω is zero, the isomorphism classes of deformations of X/k to A is a torsor under $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ and the group of automorphisms of a given deformation is canonically identified with $\mathrm{Ext}^0(\Omega^1_X, \mathcal{O}_X) = \mathrm{Hom}(\Omega^1_X, \mathcal{O}_X)$.*

Remark 6.1.8. In particular, if $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X) = (0)$, the Kuranishi space $\mathrm{Def}(X)$ of X is a smooth analytic complex space. The tangent space of the deformation functor (and of $\mathrm{Def}(X)$ at the origin) is $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$. If $\mathrm{Hom}(\Omega^1_X, \mathcal{O}_X) = (0)$, the deformation functor of X/k is pro-representable (or, equivalently, the Kuranishi space is universal). The functor of first order *local deformations* is defined similarly, and is controlled by the sheaves $\mathcal{E}\mathrm{xt}^i(\Omega^1_X, \mathcal{O}_X)$.

Let (X, D) be a pair of a normal 3-fold over k and an effective Cartier divisor. An *infinitesimal deformation of the pair* $(X, D)/k$ to a local Λ-algebra A is a proper flat morphism $X_A \to \operatorname{Spec} A$ together with an effective divisor D_A on X_A, such that $X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k = X$ and $D_A \times_{\operatorname{Spec} A} \operatorname{Spec} k = D$.

Define, as above, a deformation functor $\mathcal{D}'$ of (X, D) by associating to each local Λ-algebra A the equivalence class of infinitesimal deformations of the pair (X, D) to A; in particular $\mathcal{D}'(k) = \{(X, D)/k\}$.

I recall Kawamata's construction of the sheaf $\Omega^1_X(\log D)$ of logarithmic differential forms [Kaw85] when D is a not necessarily non-singular effective Cartier divisor.

Let X be a complete algebraic variety, $(D_j)_{j\in J}$ a finite number of effective divisors, and let D be the Cartier divisor $D = \sum_{j\in J} D_j$. The Cartier divisor D does not necessarily have normal crossings.

Let $U = \operatorname{Spec} A \subset X$ be an affine open subset with a closed embedding $U \to R$ into a non-singular affine algebraic variety $R = \operatorname{Spec} B$ such that:

1. For each $j \in J$, $D_j \cap U = (h_j)$ with $h_j \in A$ is extended to a non-singular prime divisor $D'_j = (h'_j)$ with $h'_j \in B$ on R;

2. the divisors D'_j intersect transversally.

Denote by $D' = (h')$ the normal crossings divisor $\sum_{j\in J} D'_j$ on R.

The usual residue exact sequence for a normal crossing divisor D' on a non-singular affine algebraic variety $R = \operatorname{Spec} B$ is:

$$0 \to \Omega^1_B \to \Omega^1_B(\log(h')) \to \bigoplus_{j\in J} B/(h'_j) \to 0.$$

The sequence

$$0 \to \Omega^1_B \otimes_B A \xrightarrow{\alpha} \Omega^1_B(\log(h')) \otimes_B A \to \bigoplus_{j\in J} B/(h'_j) \otimes_B A = \bigoplus_{j\in J} A/(h_j) \to 0 \tag{6.1}$$

is also exact, because $\operatorname{Tor}^B_1(B/(h'_j), A)$ is trivial. Let I be the ideal of B such that $A = B/I$, so the conormal sequence is:

$$I/I^2 \to \Omega^1_B \otimes_B A \xrightarrow{\beta} \Omega^1_A \to 0. \tag{6.2}$$

The exact sequence (6.1) pushed forward along β yields an extension of modules:

$$0 \to \Omega^1_A \to \mathcal{M} \to \bigoplus_{j\in J} A/(h_j) \to 0, \tag{6.3}$$

where the module $\mathcal{M}$ is

$$\mathcal{M} = \frac{(\Omega^1_B(\log(h')) \otimes_B A) \oplus \Omega^1_A}{(\alpha \oplus -\beta)(\Omega^1_B \otimes_B A)}.$$

If $e_j, j \in J$, are the images in $\mathcal{M}$ of the elements $\frac{dh'_j}{h'_j} \in \Omega^1_B(\log(h'))$, $\mathcal{M}$ is generated by Ω^1_A and the elements e_j.

Definition 6.1.9. The sheaf $\Omega^1(\log D)$ is defined locally by setting

$$\Omega^1_A(\log D) = \mathcal{M}.$$

Remark 6.1.10. If X is non-singular and D has normal crossings, the definition of $\Omega^1_X(\log D)$ agrees with Deligne's definition. Note however that the sheaf $\Omega^1_X(\log D)$ may have torsion, even when X is non-singular.

Remark 6.1.11. The residue sequence (6.3) is exact.

Lemma 6.1.12. *[Kaw85, Gro62] Let (X, D) be a pair comprising a normal complete 3-fold and an effective Cartier divisor. Assume that X and D have no worse than isolated l.c.i. singularities. The infinitesimal deformations of the pair $(X, D)/k$ are controlled by $\Omega^1_X(\log D)$. Let $A \in \mathcal{C}_\Lambda$. The obstruction to the existence of a flat deformation of $(X, D)/k$ to A is a class ω lying in $\mathrm{Ext}^2(\Omega^1_X(\log D), \mathcal{O}_X)$. If ω is zero, the isomorphism classes of deformations of $(X, D)/k$ to A is a torsor under $\mathrm{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X)$ and the group of automorphisms of a given deformation is canonically identified with $\mathrm{Ext}^0(\Omega^1_X(\log D), \mathcal{O}_X) = \mathrm{Hom}(\Omega^1_X(\log D), \mathcal{O}_X)$.*

Remark 6.1.13. Let $\mu\colon \widetilde{X} \to X$ be a good resolution of the pair (X, D). By definition of μ, the divisor $\mathrm{Exc}(\mu) \cup \widetilde{D}$, where $\widetilde{D}$ is the proper transform of D, has simple normal crossings. Denote by E the union of $\mu^*(D)$ and of some irreducible components of the exceptional locus of μ. Kawamata proves [Kaw85] that there is a natural transformation of functors $\mathcal{D}'' \to \mathcal{D}'$, where $\mathcal{D}'$ is the functor of deformations of (X, D) and $\mathcal{D}''$ the functor of deformations of $(\widetilde{X}, E)$. This is consistent with the construction of the sheaf $\Omega^1_X(\log D)$, which can be interpreted as relating $\Omega^1_X(\log D)$ to $\Omega^1_{\widetilde{X}}(\log \widetilde{D})$ for a good resolution μ of the pair (X, D).

6.2 Deformations of generalised Fano 3-folds

Definition 6.2.1. A 3-fold X is a *generalised Fano 3-fold* if there exists a small birational proper morphism $X \to Y$ to a Fano 3-fold with terminal Gorenstein singularities.

Remark 6.2.2.
1. Note that X is a weak Fano 3-fold; in particular X has terminal Gorenstein singularities.
2. A weak$*$ Fano 3-fold is a generalised Fano 3-fold, but the notion of generalised Fano 3-fold does not require $\mathbb{Q}$-factoriality. Any small partial $\mathbb{Q}$-factorialisation of a terminal Gorenstein Fano 3-fold is a generalised Fano 3-fold.

Let X be a generalised Fano 3-fold and let D be a general section of the anticanonical linear system $|-K_X|$. Let Y be the anticanonical model of X.

Recall that Theorem 3.1.7 shows that a general member D of $|-K_X|$ is a K3-surface with no worse than Du Val singularities. The pair (X, D) is therefore log canonical. Let $\mathrm{Sing}(X) \cup \mathrm{Sing}(D) = \Sigma = \{P_1, \cdots, P_n\}$.

If Y is not birational to a special complete intersection with a node $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$, there always exists a non-singular section of $|-K_X|$. If Y is, in addition, not monogonal, the general section $D \in |-K_X|$ is non-singular. In particular, if Y has Picard rank 1 and genus g greater than or equal to 3, the general section D is non-singular.

Remark 6.2.3. I use in this section some results on the Hodge theory of surfaces with rational double points. If Z has isolated quotient singularities and if $j\colon Z \smallsetminus \mathrm{Sing}(Z) \to Z$ is the natural inclusion, the complex of coherent sheaves $\widehat{\Omega}^\bullet_Z = j_*(\Omega^\bullet_{Z\smallsetminus \mathrm{Sing}(Z)})$ is a resolution of the constant sheaf $\mathbb{C}$ [Ste77]. For each p, $\widehat{\Omega}^p_Z$ coincides with the double dual of Ω^p_Z. The spectral sequence of hypercohomology $E_1^{p,q} = H^q(Z, \widehat{\Omega}^p_Z)$ abuts to $H^{p+q}(Z, \mathbb{C})$, degenerates at E_1 and the induced filtration on $H^{p+q}(Z, \mathbb{C})$ coincides with the canonical Hodge filtration. If $\nu\colon \widetilde{Z} \to Z$ is a resolution of singularities, the complexes $\widehat{\Omega}^\bullet_Z$ and $\nu_*\Omega^\bullet_{\widetilde{Z}}$ are equal.

Lemma 6.2.4. *[Nam97, Kaw92] Let X be a generalised Fano 3-fold and D a general anticanonical section. Then:*

1. $\mathrm{Def}(X)$ *and* $\mathrm{Def}(X, D)$ *are smooth.*
2. $\mathrm{Def}(X, D)$ *is universal.*
3. *The natural map* $\phi\colon \mathrm{Def}(X, D) \to \mathrm{Def}(X)$ *is smooth.*

Proof. Denote by $\Omega^1_X(\log D)$ the sheaf of logarithmic differential forms as constructed in Definition 6.1.9; $\Omega^1_X(\log D)$ is not, in general, locally free at points $P_i \in \Sigma = \mathrm{Sing}(X) \cup \mathrm{Sing}(D)$.

By Proposition 6.1.7, the Kuranishi spaces $\mathrm{Def}(X)$ and $\mathrm{Def}(X, D)$ are smooth analytic spaces if:

$$\mathrm{Ext}^2(\Omega^1_X(\log D), \mathcal{O}_X) = \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X) = (0).$$

The residue exact sequence (6.3)

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{O}_D \to 0$$

yields the exact sequence of Ext groups:

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{O}_D, \mathcal{O}_X) \to \mathrm{Hom}(\Omega^1_X(\log D), \mathcal{O}_X) \to \mathrm{Hom}(\Omega^1_X, \mathcal{O}_X) \to \\ \to \mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}_X) \to \mathrm{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \\ \to \mathrm{Ext}^2(\mathcal{O}_D, \mathcal{O}_X) \to \mathrm{Ext}^2(\Omega^1_X(\log D), \mathcal{O}_X) \to \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X) \to \cdots \end{aligned} \quad (6.4)$$

Claim 6.2.5. *The* Ext *groups* $\operatorname{Ext}^i(\mathcal{O}_D, \mathcal{O}_X)$ *for* $i = 0, 1, 2$ *are as follows:*

$$\begin{aligned} \operatorname{Ext}^0(\mathcal{O}_D, \mathcal{O}_X) = \operatorname{Ext}^2(\mathcal{O}_D, \mathcal{O}_X) &= (0) \\ \operatorname{Ext}^1(\mathcal{O}_D, \mathcal{O}_X) &= H^0(X, \mathcal{O}_D(D)). \end{aligned}$$

The standard resolution of the sheaf $\mathcal{O}_D$

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

yields the long exact sequence of $\mathcal{E}\mathrm{xt}$ sheaves:

$$\begin{aligned} 0 \to \mathcal{H}\mathrm{om}(\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{H}\mathrm{om}(\mathcal{O}_X(-D), \mathcal{O}_X) \to \\ \to \mathcal{E}\mathrm{xt}^1(\mathcal{O}_D, \mathcal{O}_X) \to \mathcal{E}\mathrm{xt}^1(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{E}\mathrm{xt}^1(\mathcal{O}_X(-D), \mathcal{O}_X) \to \cdots \end{aligned}$$

As $\mathcal{O}_X$ and $\mathcal{O}_X(-D)$ are locally free,

$$\mathcal{E}\mathrm{xt}^i(\mathcal{O}_X(-D), \mathcal{O}_X) = \mathcal{E}\mathrm{xt}^i(\mathcal{O}_X, \mathcal{O}_X) = 0$$

for all $i > 0$. The sheaf $\mathcal{H}\mathrm{om}(\mathcal{O}_D, \mathcal{O}_X)$ is trivial and the long exact sequence reduces to

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{E}\mathrm{xt}^1(\mathcal{O}_D, \mathcal{O}_X) \to 0.$$

This shows that $\mathcal{E}\mathrm{xt}^i(\mathcal{O}_D, \mathcal{O}_X) = 0$ for $i \neq 1$ and $\mathcal{E}\mathrm{xt}^1(\mathcal{O}_D, \mathcal{O}_X) = \mathcal{O}_D(D)$. The local-to-global spectral sequence of Ext sheaves and groups has E_1 term

$$E_1^{p,q} = H^q(X, \mathcal{E}\mathrm{xt}^p(\mathcal{O}_D, \mathcal{O}_X)),$$

and abuts to $\operatorname{Ext}^{p+q}(\mathcal{O}_D, \mathcal{O}_X)$. This shows that $\operatorname{Ext}^0(\mathcal{O}_D, \mathcal{O}_X) = (0)$ and that the following sequence is exact [God73]:

$$\begin{aligned} 0 \to H^1(X, \mathcal{H}\mathrm{om}(\mathcal{O}_D, \mathcal{O}_X)) \to \operatorname{Ext}^1(\mathcal{O}_D, \mathcal{O}_X) \to H^0(X, \mathcal{E}\mathrm{xt}^1(\mathcal{O}_D, \mathcal{O}_X)) \\ \to H^2(X, \mathcal{H}\mathrm{om}(\mathcal{O}_D, \mathcal{O}_X))) \to \operatorname{Ext}^2(\mathcal{O}_D, \mathcal{O}_X) \to 0 \end{aligned}$$

In particular, $\operatorname{Ext}^2(\mathcal{O}_D, \mathcal{O}_X) = (0)$ and $\operatorname{Ext}^1(\mathcal{O}_D, \mathcal{O}_X) = H^0(X, \mathcal{O}_D(D))$.

Step 1. *The Kuranishi spaces* $\operatorname{Def}(X)$ *and* $\operatorname{Def}(X, D)$ *are smooth.*

As $\operatorname{Ext}^2(\mathcal{O}_D, \mathcal{O}_X) = (0)$, if $\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X) = (0)$, both $\operatorname{Def}(X)$ and $\operatorname{Def}(X, D)$ are smooth by (6.4).

The group $\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ is Serre dual to $H^1(X, \Omega^1_X \otimes \omega_X)$. The exact sequence

$$0 \to \Omega^1_X \otimes \omega_X \to \Omega^1_X \to \Omega^1_{X|D} \to 0$$

yields the long exact sequence of cohomology groups:

$$\begin{aligned}0 \to H^0(X, \Omega^1_X \otimes \omega_X) \to H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_{X|D}) \to \\ \to H^1(X, \Omega^1_X \otimes \omega_X) \to H^1(X, \Omega^1_X) \xrightarrow{\beta} H^1(X, \Omega^1_{X|D}) \to \cdots \quad (6.5)\end{aligned}$$

The conormal sequence is exact because D is a Cartier divisor, hence:

$$0 \to \mathcal{O}_D(-D) \to \Omega^1_{X|D} \to \Omega^1_D \to 0.$$

This shows that $H^0(X, \Omega^1_{X|D}) = (0)$. Indeed, D is a K3 surface with Du Val singularities; by Remark 6.2.3, $H^0(D, \widehat{\Omega}^1_D) = H^1(D, \mathcal{O}_D) = (0)$. The divisor D is nef and big, hence by the Kawamata-Viehweg vanishing theorem $H^0(D, \mathcal{O}_D(-D)) = (0)$. The map $\Omega^1_D \to \widehat{\Omega}^1_D$ is an injection because Ω^1_D is torsion free; thus $H^0(X, \widehat{\Omega}^1_D) = H^0(X, \Omega^1_D) = (0)$ and $H^0(X, \Omega^1_{X|D}) = (0)$.

Consequently, it is sufficient to show that the map β in (6.5) is injective in order to complete the proof.

The surface D is a K3 with no worse than Du Val singularities, hence $H^1(D, \mathcal{O}_D)$ is trivial. The map

$$\frac{1}{2i\pi} dlog \colon H^1(D, \mathcal{O}^*_D) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(D, \widehat{\Omega}^1_D)$$

is injective and factors through $H^1(D, \Omega^1_D)$:

$$\frac{1}{2i\pi} dlog \colon H^1(D, \mathcal{O}^*_D) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(D, \Omega^1_D)$$

is also an injection.

The following diagram is commutative:

$$\begin{tikzcd} H^1(X, \mathcal{O}^*_X) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "j"] \arrow[d, "\frac{1}{2i\pi} dlog"', "\simeq"] & H^1(D, \mathcal{O}^*_D) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[d, "\frac{1}{2i\pi} dlog"'] \\ H^1(X, \Omega^1_X) \arrow[r, "k"] & H^1(D, \Omega^1_D). \end{tikzcd}$$

The conormal sequence shows that k factors through β, hence the morphism $H^1(X, \Omega^1_X) \to H^1(D, \Omega^1_{X|D})$ is injective.

Claim 6.2.6. *The natural restriction map* $\operatorname{Pic} X \to \operatorname{Pic} D$ *is injective.*

Following [Har70], one shows that the pair (X, D) satisfies the Lefschetz condition, i.e. that for any coherent sheaf $\mathcal{F}$, the cohomology groups $H^i(X - D, \mathcal{F})$ are trivial for $i > 1$. Denoting by $\widehat{X}$ a formal completion of X along D, $\operatorname{Pic} X \to \operatorname{Pic} \widehat{X}$ is then an injection. The natural map $\operatorname{Pic} \widehat{X} \to \operatorname{Pic} D$ is also an injection because $\mathcal{O}_D(-D)$ is nef and big. In the above diagram, j is injective.

Claim 6.2.7. *If X is a generalised Fano 3-fold, the map*

$$\frac{1}{2i\pi} dlog \colon H^1(X, \mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(X, \Omega_X^1)$$

is an isomorphism.

Let $\mu \colon \widetilde{X} \to X$ be a good resolution of X. The Leray spectral sequence shows that the cohomology groups $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(X, \mathcal{O}_X)$ because the singularities of X are rational. As X is a generalised Fano 3-fold, by the Kawamata-Viehweg vanishing theorem, $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = (0)$ for $i > 0$. The map

$$\frac{1}{2i\pi} dlog \colon H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^*) \otimes_{\mathbb{Z}} \mathbb{C} \to H^1(\widetilde{X}, \Omega_{\widetilde{X}}^1) \simeq H^2(\widetilde{X}, \mathbb{C})$$

is an isomorphism. Note first that when X has isolated hypersuface singularities and $P \in \operatorname{Sing}(X)$, the local cohomology groups $H^i_{\{P\}}(X, \Omega_X^1) = (0)$ for $i = 0, 1$ [Nam94]. This implies that the natural map $\Omega_X^1 \to \mu_* \Omega_{\widetilde{X}}^1$ is an isomorphism. The diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mu_* \Omega_{\widetilde{X}}^1) = H^1(X, \Omega_X^1) \arrow[r] & H^1(\widetilde{X}, \Omega_{\widetilde{X}}^1) \arrow[r] & H^0(X, R^1 \mu_* \Omega_{\widetilde{X}}^1) \\
0 \arrow[r] & H^1(X, \mu_* \mathcal{O}_{\widetilde{X}}^*) = H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[u] & H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^*) \arrow[r] \arrow[u] & H^0(X, R^1 \mu_* \mathcal{O}_{\widetilde{X}}^*) \arrow[u]
\end{tikzcd}$$

is commutative. The second vertical map is surjective by the Hodge decomposition of the non-singular 3-fold $\widetilde{X}$. It is sufficient to prove that $H^0(X, R^1 \mu_* \mathcal{O}_{\widetilde{X}}^*) \to H^0(X, R^1 \mu_* \Omega_{\widetilde{X}}^1)$ is injective. This can be done exactly as in [Nam94]. Denote by E the exceptional divisor of the resolution μ, by V a neighbourhood of E in $\widetilde{X}$, and by L a line bundle in V. If $\frac{1}{2i\pi} dlog(L) = 0$ in $H^0(X, R^1 \mu_* \Omega_{\widetilde{X}}^1)$, then L is μ-numerically trivial. The 3-fold X has rational singularities: it follows that L is a torsion line bundle on V.

The map k, and hence β, is an injection; thus $\operatorname{Def}(X)$ and $\operatorname{Def}(X, D)$ are smooth analytic spaces.

Step 2. *The group* $\operatorname{Ext}^0(\Omega^1_X(\log D))$ *is trivial; the Kuranishi space* $\operatorname{Def}(X)$ *is universal.*

By Serre duality, it is sufficient to prove that:

$$H^3(X, \Omega^1_X(\log D) \otimes \omega_X) = H^3(X, \Omega^1_X(\log D)(-D)) = (0).$$

The sequence

$$0 \to \Omega^1_X(\log D)(-D) \to \Omega^1_X \to \Omega^1_D \to 0$$

is exact. As D is a K3 surface with rational double points, Ω^1_D is torsion free and the natural map $0 \to \Omega^1_D \to \widehat{\Omega}^1_D$ has cokernel supported at the singular points. In particular, $H^2(D, \Omega^1_D) = H^2(D, \widehat{\Omega}^1_D) = (0)$ by Hodge symmetry. It is sufficient to prove that $H^3(X, \Omega^1_X) = (0)$.

The group $H^3(X, \Omega^1_X)$ is Serre dual to $H^0(X, \Theta_X(-D))$, where Θ_X denotes $\mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)$. Let $\widetilde{X} \to X$ be a good resolution of X. Since X has terminal singularities, $f_*(\Theta_{\widetilde{X}} \otimes K_{\widetilde{X}}) = \Theta_X \otimes K_X$. The cohomology group $H^3(\widetilde{X}, \Omega^1_{\widetilde{X}})$ is trivial by Hodge symmetry on the non-singular generalised Fano $\widetilde{X}$; the result follows.

Step 3. *The natural map* $\phi\colon \operatorname{Def}(X, D) \to \operatorname{Def}(X)$ *is surjective.*

The long exact sequence (6.4) shows that this follows from triviality of the group $\operatorname{Ext}^2(\mathcal{O}_D, \mathcal{O}_X)$. □

The canonical local-to-global spectral sequence of Ext groups relates global to local first order deformation functors and reads:

$$0 \to H^1(X, \mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)) \to \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\alpha} H^0(X, \mathcal{E}\mathrm{xt}^1(\Omega^1_X, \mathcal{O}_X)) \to \\ \to H^2(X, \mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)).$$

The map α can be regarded as the homomorphism between the space of first order global deformations and the space of first order local deformations. If α is surjective, X can be deformed to a non-singular 3-fold, which can be shown to be a generalised Fano (Remark 6.2.21).

Define similarly the homomorphism:

$$\alpha_{\log}\colon \operatorname{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X) \to H^0(X, \mathcal{E}\mathrm{xt}^1(\Omega^1_X(\log D), \mathcal{O}_X)),$$

which relates the spaces of global and local first order deformations of (X, D).

Theorem 6.2.8. *[Nam97] Let X be a generalised Fano 3-fold with terminal Gorenstein singularities. Then X can be deformed to a non-singular generalised Fano 3-fold. In particular, any Fano 3-fold with Gorenstein terminal singularities is smoothable by a flat deformation.*

More precisely, if X has no worse than ordinary double points, α is surjective. In the general case, there is a "good" direction $\eta \in \mathrm{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X)$ such that deforming X along η improves the singularities. After finitely many infinitesimal deformations along "good" directions, X becomes a generalised Fano with no worse than ordinary double points. I give an overview of Namikawa's proof.

Sketch Proof. Let $\Theta_X = \mathcal{H}\mathrm{om}(\Omega^1_X, \mathcal{O}_X)$ be the dual of the cotangent sheaf and $T^i_X = \mathcal{E}\mathrm{xt}^i(\Omega^1_X, \mathcal{O}_X)$.

Step 1. *Assume that X has no worse than ordinary double points; then there is a smoothing of X.*

Lemma 6.2.9. *[Nam97, Fri86] If X has no worse than ordinary double points, the cohomology group $H^2(X, \Theta_X)$ is trivial. In particular, α is surjective and X is smoothable by a flat deformation.*

Proof of Lemma 6.2.9. Let $\pi\colon \widetilde{X} \to X$ be a small resolution of X. By definition, the exceptional locus of π above an ordinary double point P_i is a rational curve C_i with normal bundle $\mathcal{N}_{C_i/\widetilde{X}} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Friedman shows that if the only singularities are ordinary double points, $H^0(X, R^1\pi_*\Theta_{\widetilde{X}}) = (0)$ [Fri86].

The spectral sequence for $R\pi_*\Theta_{\widetilde{X}}$ yields the exact sequence [God73]:

$$H^1(\widetilde{X}, \Theta_{\widetilde{X}}) \to H^0(X, R^1\pi_*\Theta_{\widetilde{X}}) \to H^2(X, \Theta_X) \to H^2(\widetilde{X}, \Theta_{\widetilde{X}}).$$

It is therefore sufficient to prove that $H^2(\widetilde{X}, \Theta_{\widetilde{X}}) = (0)$. By Serre Duality on the non-singular 3-fold $\widetilde{X}$, $H^2(\widetilde{X}, \Theta_{\widetilde{X}})$ is dual to $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}} \otimes \omega_{\widetilde{X}})$. Denote by $\widetilde{D}$ the proper transform of D by π. The map π is small, hence $\widetilde{D}$ is a section of $-K_{\widetilde{X}}$ and is a K3 surface with no worse than Du Val singularities. Tensoring the residue exact sequence of $(\widetilde{X}, \widetilde{D})$ by $K_{\widetilde{X}}$ shows that the following sequence is exact:

$$0 \to \Omega^1_{\widetilde{X}} \otimes \omega_{\widetilde{X}} \to \Omega^1_{\widetilde{X}}(\log \widetilde{D}) \otimes \omega_{\widetilde{X}} \to \mathcal{O}_{\widetilde{D}} \otimes \omega_{\widetilde{X}} \to 0. \tag{6.6}$$

The line bundle $-K_{\widetilde{X}|\widetilde{D}}$ is nef and big, so that $H^0(\widetilde{D}, \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{D}}) = (0)$ by the Kawamata-Viehweg vanishing theorem. The long exact sequence in cohomology associated to (6.6) shows that if

$$H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D}) \otimes \omega_{\widetilde{X}}) = H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D})(-\widetilde{D})) = (0),$$

then $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}} \otimes \omega_{\widetilde{X}}) = (0)$. The sequence

$$0 \to \Omega^1_{\widetilde{X}}(\log \widetilde{D})(-\widetilde{D}) \to \Omega^1_{\widetilde{X}} \to \Omega^1_{\widetilde{D}} \to 0 \tag{6.7}$$

is exact. The surface $\widetilde{D}$ has no worse than Du Val singularities and $\Omega^1_{\widetilde{D}}$ is torsion free, so that $\Omega^1_{\widetilde{D}} \to \widehat{\Omega}^1_{\widetilde{D}}$ is an injection. The cohomology group $H^0(\widetilde{D}, \widehat{\Omega}^1_{\widetilde{D}})$ is trivial because $\widetilde{D}$ is a K3 surface and $H^0(\widetilde{D}, \widehat{\Omega}^1_{\widetilde{D}}) = (0)$. As $\operatorname{Pic} \widetilde{X} \to \operatorname{Pic} \widetilde{D}$ is an injection, $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}) \to H^1(\widetilde{D}, \widehat{\Omega}^1_{\widetilde{D}})$ is an injection. In addition, since it factors through $H^1(\widetilde{D}, \Omega^1_{\widetilde{D}})$, the map $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}) \to H^1(\widetilde{D}, \Omega^1_{\widetilde{D}})$ is also an injection.

The long exact sequence in cohomology associated to (6.7) shows that $H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log \widetilde{D})(-\widetilde{D})) = (0)$; this completes the proof. □

Step 2. *We define an invariant $\mu(P_i)$ for all $P_i \in \operatorname{Sing} X$, which is strictly positive unless P_i is an ordinary double point.*

Let X be a generalised Fano 3-fold and let $P_i \in \Sigma$ be a singular point. As will be made clear in Lemma 6.2.16, if $\mu > 0$, there is a small deformation η of X which "improves" the singularity at P_i in the following sense: for any resolution $\widetilde{X} \to X$ of the singularity at P_i, η is not in the image of the map $\operatorname{Def}(\widetilde{X}) \to \operatorname{Def}(X)$.

I first state some results on the local cohomology groups at the singular set of the sheaves $\Theta_X(\log D)$. Recall that $\Sigma = \operatorname{Sing}(X) \cup \operatorname{Sing}(D) = \{P_1, \cdots, P_n\}$, that $U = X \smallsetminus \Sigma$, and denote by U_i a Stein open neighbourhood of P_i in X.

Lemma 6.2.10. *[Nam97]*

1. $H^2_{\{P_i\}}(U_i, \Theta_X(-\log D)) = H^1(U_i \smallsetminus \{P_i\}, \Theta_X(-\log D))$ *and both are equal to* $H^0(U_i, \mathcal{E}\mathrm{xt}(\Omega^1_X(\log D), \mathcal{O}_X))$.
2. $H^2_\Sigma(X, \Theta_X(-\log D)) = H^1(U, \Theta_X(-\log D)) = \operatorname{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X))$.

Sketch of proof. Let $D_i = D \cap U_i$ and denote by U_i^* (resp. D_i^*) the punctured neighbourhood $U_i \setminus \{P_i\}$ (resp. $D_i \setminus \{P_i\}$). If D_i is empty, the result is proved by Schlessinger [Sch71]. The first equality is obtained by comparing the long exact sequences of Ext groups associated to the residue exact sequences of $(U_i, D \cap U_i)$ and (U_i^*, D_i^*).

The second equality is obtained by comparing the local-to-global spectral sequence of $\mathcal{E}$xt sheaves and Ext groups with the long exact sequence of local cohomology at the place Σ of the sheaf $\Theta_X(-\log D)$. $\square$

Let V be the germ of an isolated rational singularity and let $\widetilde{V}$ be a resolution of V.

Definition 6.2.11. [Nam97]

$$\mu(V) = \dim_{\mathbb{C}} \operatorname{Coker}[\frac{1}{2i\pi} d\log\colon H^1(\widetilde{V}, \mathcal{O}^*_{\widetilde{V}}) \otimes \mathbb{C} \to H^1(\widetilde{V}, \Omega^1_{\widetilde{V}})]$$

Namikawa shows [NS95] that $\mu(V)$ is independent of the chosen resolution.

Lemma 6.2.12. *[NS95] Let V be a terminal Gorenstein singularity of dimension 3. Then $\mu(V) = 0$ if and only if V is non-singular or if V is an ordinary double point.*

Let $f\colon \widetilde{V} \to V$ be a good resolution of the germ of a terminal Gorenstein singularity and let $D \subset V$ be a Cartier divisor with no worse than a rational double point at the singular point. Let $F = \widetilde{D} \cup E$, where $\widetilde{D}$ is the proper transform of D and E is the exceptional divisor of f. The divisor F has simple normal crossings and $\widetilde{D}$ is a resolution of a sufficiently small open neighbourhood of a rational double point on D.

Lemma 6.2.13. *[Nam97] Using the above notation, the following hold:*

1. $H^1(F, \mathcal{O}^*_F) \simeq H^1(E, \mathcal{O}^*_E)$.
2. $H^1(F, \widehat{\Omega}^1_F) \simeq H^1(E, \widehat{\Omega}^1_E)$.
3. *In the natural commutative diagram*

$$\begin{tikzcd}
H^1(\widetilde{V}, \Omega^1_{\widetilde{V}}) \arrow[r, "\sigma"] & H^1(F, \widehat{\Omega}^1_F) \\
H^1(\widetilde{V}, \mathcal{O}^*_{\widetilde{V}}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[u, "\phi"] \arrow[r, "\sigma'"] & H^1(F, \mathcal{O}^*_F) \otimes_{\mathbb{Z}} \mathbb{C}, \arrow[u, "\frac{1}{2i\pi} d\log"']
\end{tikzcd}$$

σ' is an isomorphism and $\frac{1}{2i\pi} d\log$ is a surjection.

Sketch proof of Lemma 6.2.13. This Lemma follows from computations similar to the ones detailed in the Appendix 2.3. A base change diagram in the fashion of [GNAPGP88] gives a resolution of $\mathbb{Q}_V$ in terms of $\mathbb{Q}_{\widetilde{V}}$ and $\mathbb{Q}_E$. The Mixed Hodge theory of E and F is then determined by a Mayer-Vietoris resolution. The computations of the cohomology groups and properties of Hodge numbers of generalised Fano 3-folds yield the results. □

Step 3. *The invariants $\mu(P_i)$ can be used to identify a "good" direction of deformations of the pair (X, D).*

Let $f \colon \widetilde{X} \to X$ be a good resolution of X and let F be the simple normal crossing divisor $E \cup \widetilde{D}$.

Claim 6.2.14. *There is an injection*

$$\Theta_{\widetilde{X}}(-\log F) = \mathcal{H}\mathrm{om}(\Omega^1_{\widetilde{X}}(\log F), \mathcal{O}_X) \to \Omega^2_{\widetilde{X}}(\log F).$$

Since the pair (X, D) has log canonical singularities, $K_{\widetilde{X}}+F$ is an effective divisor. The exact sequence

$$0 \to \Omega^3_{\widetilde{X}}(\log F)(-F) \to \Omega^3_X \to \Omega^3_F \to 0$$

shows that $\Omega^3_{\widetilde{X}}(\log F) \simeq \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}+F)$ because Ω^3_F is trivial. The cup product map $\Omega^1_{\widetilde{X}}(\log F) \otimes \Omega^2_{\widetilde{X}}(\log F) \to \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + F)$ gives the desired injection.

Recall that $\alpha_{\log}$ is defined as

$$\alpha_{\log} \colon \operatorname{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X) \to H^0(X, \mathcal{E}\mathrm{xt}^1(\Omega^1_X(\log D), \mathcal{O}_X)).$$

Lemma 6.2.10 identifies the morphism $\alpha_{\log}$ with the coboundary map of local cohomology

$$H^1(U, \Theta_X(-\log D)) \to H^2_{\operatorname{Sing} X}(X, \Theta_X(-\log D)).$$

As $\Omega^2_X(\log D)_{|U} \simeq \Theta_X(-\log D)_{|U}$ and

$$H^2_{\{P_i\}}(X, f_*\Omega^2_{\widetilde{X}}(\log F)) \simeq H^2_{\{P_i\}}(X, \Theta_X(-\log D)),$$

the diagram

$$\begin{tikzcd}
H^1(f^{-1}(U), \Omega^2_{\widetilde{X}}(\log F)) \arrow[r] & \oplus H^2_{E_i}(\widetilde{X}, \Omega^2_{\widetilde{X}}(\log F)) \arrow[r, "\oplus\gamma_i"] & H^2(\widetilde{X}, \Omega^2_{\widetilde{X}}(\log F)) \\
H^1(U, \Theta_X(-\log D)) \arrow[u] \arrow[r, "\alpha_{\log}"] & H^2_{\{p_i\}}(X, \Theta_X(-\log D)) \arrow[u, "\oplus\tau_i"'] &
\end{tikzcd}$$

is commutative.

Lemma 6.2.15. *[Nam97] Assume that $P_i \in X$ is neither non-singular nor an ordinary double point; then γ_i is not an injection. Moreover,* $\dim\ker\gamma_i \geq \mu(U_i)$.

Sketch proof of Lemma 6.2.15. Set $V_i = f^{-1}(U_i)$ for U_i a contractible Stein open neighbourhood of $\{P_i\}$. Consider the dual map

$$\gamma_i^*\colon H^1(\widetilde{X}, \Omega^1_{\widetilde{X}}(\log F)\otimes \mathcal{O}_{\widetilde{X}}(-F)) \to H^1(V_i, \Omega^1_{V_i}(\log F)\otimes \mathcal{O}_{V_i}(-F)).$$

Namikawa shows [Nam97] that $\dim \operatorname{Coker}\gamma_i^* \geq \mu(U_i)$ by studying the long exact sequences in cohomology on $\widetilde{X}$ and V_i associated to

$$0 \to \Omega^1_{\widetilde{X}}(\log F)(-F) \to \Omega^1_{\widetilde{X}} \to \hat{\Omega}^1_F \to 0,$$
$$0 \to \Omega^1_{V_i}(\log F)(-F) \to \Omega^1_{V_i} \to \hat{\Omega}^1_{F\cap V_i} \to 0$$

and the commutative diagram:

$$\begin{tikzcd}
H^1(V_i, \Omega^1_{V_i}) \arrow[r, "\sigma"] & H^1(F_i, \hat{\Omega}^1_{F_i}) \\
H^1(V_i, \mathcal{O}^*_{V_i})\otimes_{\mathbb{Z}}\mathbb{C} \arrow[u, "\phi"] \arrow[r, "\sigma'"] & H^1(F, \mathcal{O}^*_{F_i})\otimes_{\mathbb{Z}}\mathbb{C}. \arrow[u, "\frac{1}{2i\pi}d\log"']
\end{tikzcd}$$

□

Let β_i be the natural morphism from the space of global first order deformations of (V_i, F_i) to the space of local first order deformations of (U_i, D_i) defined as follows:

$$\begin{aligned}\beta_i\colon H^1(V_i, \Theta_{V_i}(-\log F)) &\to H^1(V_i \smallsetminus E_i, \Theta_{V_i}(-\log F))\\ &\simeq H^1(U_i \smallsetminus P_i, \Theta_{U_i}(-\log D)) \simeq H^0(U_i, \mathcal{E}\mathrm{xt}^1(\Omega^1_X(\log D), \mathcal{O}_X)).\end{aligned}$$

Lemma 6.2.16. *[Nam97] There is an element $\eta \in \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ such that $\alpha_{\log}(\eta)_i$ is not in the image of β_i, for all i such that $P_i \in X$ is neither non-singular, nor an ordinary double point.*

Proof of Lemma 6.2.16. By Lemma 6.2.15, if P_i is neither non-singular nor an ordinary double point, γ_i is not injective.

There is an element $\eta \in \operatorname{Ext}^1(\Omega^1_X(\log D), \mathcal{O}_X)$ such that $\tau_i(\alpha_{\log}(\eta)_i) \neq 0$. The map τ_i factors as:

$$H^2_{\{P_i\}}(X, \Theta_X(-\log D)) \xrightarrow{\tau_i'} H^2_{E_i}(\widetilde{X}, \Theta_{\widetilde{X}}(-\log F)) \to H^2_{E_i}(\widetilde{X}, \Omega^2_{\widetilde{X}}(\log F)).$$

In particular, $\tau_i'(\alpha_{\log}(\eta)_i) \neq 0$, and as the spectral sequence of local cohomology reads

$$H^0(X, R^1 f_* \Theta_{\widetilde{X}}(-\log F)) \xrightarrow{\beta_i} H^2_{\{P_i\}}(X, \Theta_X(-\log D)) \xrightarrow{\tau_i'} H^2_{E_i}(\widetilde{X}, \Theta_{\widetilde{X}}(-\log F)),$$

$\alpha_{\log}(\eta)_i$ is not in $\operatorname{Im}\beta_i$. □

Step 4. *After a finite number of deformations along distinguished directions, X becomes a generalised Fano 3-fold with no worse than ordinary double points.*

Let $g_i\colon (\mathcal{U}_i, \mathcal{D}_i) \to \operatorname{Def}(U_i, D_i)$ be the universal family. Let $g\colon \mathcal{X} \to S$ be a smooth morphism and let $\mathcal{D} = \sum \mathcal{D}_j$ be a divisor of $\mathcal{X}$ with simple normal crossings.

Definition 6.2.17. The morphism $g\colon (\mathcal{X}, \mathcal{D}) \to S$ is *log smooth* if

1. $\mathcal{D}_t = \sum \mathcal{D}_{j,t}$ is a divisor of $\mathcal{X}_t$ with simple normal crossings for each $t \in S$ and
2. for any point $p \in \mathcal{X}$, g is locally a trivial deformation of $(\mathcal{X}_t, \mathcal{D}_t)$ around $g(p) = t$.

Remark 6.2.18. Notice that g_i is log smooth over a non-empty Zariski open subset $S_i^0 \subset \operatorname{Def}(U_i, D_i)$ by Sard's theorem.

Namikawa constructs iteratively a stratification of $\operatorname{Def}(U_i, D_i)$ into locally closed non-singular subsets with the following properties:

1. $S_0^i \subset \operatorname{Def}(U_i, D_i)$ is a Zariski open subset and g_i is log smooth over S_0^i.
2. S_k^i is a locally closed non-singular subset of pure codimension, and $\operatorname{Codim}(S_i^k, \operatorname{Def}(U_i, D_i))$ is strictly increasing with k.
3. If $k > l$, $\overline{S_k^i} \cap S_l^i = \emptyset$.
4. $(\mathcal{U}_i, \mathcal{D}_i)$ has a simultaneous resolution over each S_k^i, that is, there exists $\nu_k^i\colon (\mathcal{V}_i, \mathcal{F}_i) \to (\mathcal{U}_i, \mathcal{D}_i)$ a resolution over S_k^i, such that $g_i^k \circ \nu_i^k\colon (\mathcal{V}_i, \mathcal{F}_i) \to S_i^k$ is log smooth, where $g_i^k\colon (\mathcal{U}_i, \mathcal{D}_i) \to S_i^k$ is the base change of g_i to S_i^k.

Step 5. *Let X be a generalised Fano 3-fold with terminal Gorenstein singularities. Then X can be deformed to a non-singular generalised Fano 3-fold.*

Fix a stratification as above for each $P_i \in \operatorname{Sing}(X)$.

Let $q_i \in \operatorname{Def}(U_i, D_i)$ be the point corresponding to $(U_i, D_i)/k$ and let S_i^k be the stratum containing q_i. Let $\nu_i \colon V_i \to U_i$ be the resolution induced by the log smooth simultaneous resolution of $(\mathcal{U}_i, \mathcal{D}_i)$ over S_k^i. Since ν_i is an isomorphism above $U_i \smallsetminus P_i$, the resolutions ν_i can be patched to a global resolution $\nu \colon \widetilde{X} \to X$. Consider the divisor $F = E \cup \widetilde{D}$, where E is the exceptional divisor of ν and $\widetilde{D}$ the proper transform of D.

Pick an $\eta \in \operatorname{Ext}^1(\Omega_X^1(\log D), \mathcal{O}_X)$ as in Lemma 6.2.16: since $\operatorname{Def}(X, D)$ is non-singular and universal, η determines a small deformation $g \colon (\mathcal{X}, \mathcal{D}) \to \Delta_\epsilon^1$. There is a holomorphic map $\phi_i \colon \Delta_\epsilon^1 \to \operatorname{Def}(U_i, D_i)$ for each i, with $\phi_i(0) = q_i$. By definition of η, if P_i is neither non-singular, nor an ordinary double point, $\operatorname{Im} \phi_i$ is not contained in S_i^k.

Pick a suitable $t \in \Delta_\epsilon^1 \smallsetminus \{0\}$ such that $\phi_i(t) \in S_i^{k'}$ for some $k' < k$. Apply the same procedure to $(\mathcal{X}_t, \mathcal{D}_t)$. This is possible because $\operatorname{Def}(U_i, D_i)$ is versal at every point near q_i. After a finite number of iterations, X becomes a generalised Fano 3-fold with no worse than ordinary double points and the result follows from Lemma 6.2.9. □

Remark 6.2.19. Namikawa's proof shows that if X is a generalised Fano 3-fold with no worse than ordinary double points, there is a flat (global) smoothing $f \colon \mathcal{X} \to \Delta$. If X is a generalised Fano 3-fold, there is a 1-parameter flat deformation $f \colon \mathcal{X} \to \Delta$, such that $\mathcal{X}_t$ is a non-singular Fano 3-fold for some $t \in \Delta$. The construction of the small deformation shows that $\mathcal{X}_t$ has terminal Gorenstein singularities for all $t \in \Delta$.

The total space of the 1-parameter flat deformation satisfies the following additional properties.

Lemma 6.2.20. *[JR06] Let $\mathcal{X} \to \Delta$ be a one-parameter smoothing of X. The total space $\mathcal{X}$ is normal, parafactorial and has at most isolated Gorenstein singularities.*

Proof. A point $P \in \mathcal{X}$ belongs to a fibre $\mathcal{X}_t$, that is some reduced Cartier divisor $\mathcal{X}_t$ with no worse than terminal Gorenstein singularities. Denote by $\mathcal{I}_t$ the ideal defining $\mathcal{X}_t$ in $\mathcal{X}$; the conormal sequence

$$\mathcal{I}_t/\mathcal{I}_t^2 \to \Omega^1_{\mathcal{X}|\mathcal{X}_t} \to \Omega^1_{\mathcal{X}_t} \to 0$$

shows that the local embedding dimension of $\mathcal{X}$ at P is 4 or 5. In particular, P is either an isolated analytic hypersurface singularity or a non-singular point and $\mathcal{X}$ is Cohen Macaulay, normal and Gorenstein. By inversion of adjunction ([KM98, Theorem 5.50]), the pair $(\mathcal{X}, X)$ is purely log terminal because $(X, 0)$ is klt. The variety $\mathcal{X}$ is therefore terminal.

As $\mathcal{X}$ is a locally complete intersection, $\mathcal{X}$ is parafactorial by [Gro05, XI, Théorème 3.13]. $\square$

Remark 6.2.21. Let $\pi\colon \mathcal{X} \to \Delta$ be a one parameter flat deformation of a generalised Fano 3-fold $X = \pi^{-1}(0)$. Then $\mathcal{X}_t$ is a generalised Fano 3-fold for all t and has the same Picard rank as X.

Proof of Remark 6.2.21. This result is well known, I include a sketch of proof for convenience. As the total space $\mathcal{X}$ is Goresntein, $-K_\pi$ is Cartier and for all $t \in \Delta$, $\mathcal{X}_t$ is Gorenstein, so that the relative anticanonical divisor $-K_{\pi|\mathcal{X}_t} = -K_{\mathcal{X}_t}$ is Cartier.

The divisor $-K_X = -K_{\pi|\mathcal{X}_0}$ is nef, and so is $-K_{\pi|\mathcal{X}_t}$ for all nearby t. Indeed, Proposition 6.3.1 shows that π induces a 1-parameter deformation $\mathcal{Y} \to \Delta$ of Y. The Kuranishi spaces $\mathrm{Def}(X)$ and $\mathrm{Def}(Y)$ are smooth complex analytic spaces; the proper morphism $X \to Y$ is small, hence $\mathrm{Def}(X) \to \mathrm{Def}(Y)$ is a closed immersion. For all $t \in \Delta$, the map $\mathcal{X}_t \to \mathcal{Y}_t$ is small and contracts K-trivial curves by Theorem 6.3.7. For all $t \in \Delta$, $\mathcal{Y}_t$ is a terminal Gorenstein Fano 3-fold, because ampleness is an open condition on flat families; $\mathcal{X}_t$ is a generalised Fano 3-fold. Lemma 2.2.7 establishes that the Picard rank is constant in a smoothing of a terminal Gorenstein generalised Fano 3-fold. The exact same arguments can be used in the case of the explicitly constructed smoothing $f\colon \mathcal{X} \to \Delta$. Recall that each fibre of f has isolated hypersurface singularities. One can define vanishing cycles $B_{i,t}$ whose cohomology is supported in degrees 0 and 3 and construct a homeomorphism between $X \smallsetminus \{P_1, \cdots, P_n\}$ and $X_t \smallsetminus \cup B_{i,t}$.

The Picard rank is also constant in the 1-parameter flat deformation of the anticanonical model $\mathcal{Y} \to \Delta$ induced by π and for all $t \in \Delta$, $\mathcal{Y}_t$ is a terminal Gorenstein Fano 3-fold. The degree $-K^3_{\mathcal{Y}_t}$ is invariant in a flat deformation, because the plurigenera are. $\square$

6.3 Deformations of extremal contractions

Let X be a generalised Fano 3-fold and Y its anticanonical model. Denote by f the anticanonical map of X. If E is a $\mathbb{Q}$-Cartier divisor of X such that

$\overline{E} = f(E)$ is not $\mathbb{Q}$-Cartier on Y, denote by Z the symbolic blow-up of $\overline{E}$ on Y. The map f factors through Z in the following way:

$$f\colon X \xrightarrow{h} Z \xrightarrow{g} Y.$$

In particular, when $\operatorname{Pic}(X/Y) = 1$, h is the identity. I recall some known results about the relations between deformations of X, Z and Y.

Proposition 6.3.1. *[KM92] Let X be a projective 3-fold and let $f\colon X \to X'$ be a proper map with connected fibres. Assume that $R^1 f_* \mathcal{O}_X = 0$. Then, there exist natural morphisms F and $\mathcal{F}$ that make the following diagram commutative:*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{X}' \arrow[d] \\ \operatorname{Def}(X) \arrow[r, "F"] & \operatorname{Def}(X') \end{tikzcd}$$

where $\mathcal{X}$ (resp. $\mathcal{X}'$) is the Kuranishi family of X (resp. of X') and $\operatorname{Def}(X)$ (resp. $\operatorname{Def}(X')$) is the Kuranishi space of X (resp. of X'); $\mathcal{F}_{|X}$ coincides with f.

In particular, there exist maps $\mathcal{G}$ and $\mathcal{H}$ that restrict to g and h on the central fibre and make the following diagram commutative:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathcal{H}"] \arrow[d] & \mathcal{Z} \arrow[r, "\mathcal{G}"] \arrow[d] & \mathcal{Y} \arrow[d] \\ \operatorname{Def}(X) \arrow[r, "H"] & \operatorname{Def}(Z) \arrow[r, "G"] & \operatorname{Def}(Y). \end{tikzcd}$$

Theorem 6.3.2. *[KM92] Let Y be a projective 3-fold with canonical singularities and let $f\colon X \to Y$ be a projective, $\mathbb{Q}$-factorial terminal and crepant partial resolution. Then $F\colon \operatorname{Def}(X) \to \operatorname{Def}(Y)$ is a finite map.*

Sketch of proof. Let $T = \operatorname{Spec} \mathbb{C}[[t]]$ be the formal disc. The proof relies on showing that no non-trivial deformation of X over T can induce a trivial deformation of Y over T. Assume that F is not finite. There exists a deformation $\mathcal{X}/T$ such that $F\colon \mathcal{X} \to Y \times T$. In particular, there is a birational map $g\colon \mathcal{X} \dashrightarrow X \times T$. The map g is induced by a map g_0 on X and $\mathcal{X}$ is also a trivial deformation. $\square$

Proposition 6.3.3. *[KM92] Let Y be a terminal Gorenstein projective 3-fold and $X \to Y$ a small $\mathbb{Q}$-factorialisation. The subspace* $\mathrm{Im}[\mathrm{Def}(X) \to \mathrm{Def}(Y)]$ *is closed and independent of the choice of small $\mathbb{Q}$-factorialisation.*

Remark 6.3.4. Let Θ_Y (resp. Θ_Y) be the dual of Ω^1_Y (resp. of Ω^1_X) and let C be the exceptional locus of $X \to Y$. There is a natural map of functors $\mathrm{Def}(X) \to \mathrm{Def}(Y)$ because Y has terminal singularities and $R^1\pi_*\mathcal{O}_Y = (0)$. At the level of tangent spaces, the kernel is given by the local cohomology group $H^1_C(\Theta_X)$. This group vanishes when C is a curve [Fri86].

The following result shows that $\mathbb{Q}$-factoriality is an open condition on the base of flat deformations of algebraic 3-folds.

Theorem 6.3.5 (Factoriality and deformations, [KM92])**.** *Let $g\colon X \to S$ be a flat family of algebraic varieties. Assume that the fibres have rational singularities and that for any $s \in S$,* $\mathrm{Codim}(\mathrm{Sing}(X_s), X_s) \geq 3$*. For instance, this is the case of a flat family of 3-folds with isolated singularities. Then*

$$X_{\mathbb{Q}-fact} = \{x \in X | g^{-1}(g(x)) \textit{ is } \mathbb{Q}\textit{-factorial}\}$$

is open in X. In particular, if a fibre is $\mathbb{Q}$-factorial, so are nearby fibres.

Remark 6.3.6. Let $f\colon \mathcal{Y} \to \Delta$ be a 1-parameter flat and proper deformation of a generalised Fano 3-fold Y with $\mathcal{Y}_t$ non-singular for some $t \in \Delta$ as contructed by Namikawa (see Section 6.2). The set $\Delta_{\mathbb{Q}-\mathrm{fact}} = \{t \in \Delta | \mathcal{X}_t$ is $\mathbb{Q}$-factorial $\}$ is open.

I now recall some results on deformations of extremal rays.

Theorem 6.3.7 (Flops in families, [KM92])**.** *Let $f_0\colon X_0 \to Y_0$ be a proper morphism between normal 3-folds. Assume that X_0 has only terminal singularities, and that f_0 contracts a curve $C_0 \subset X_0$ to a point $Q_0 \in Y_0$. Assume moreover that every component of C_0 is K_{X_0}-trivial. Let $X_S \to S$ be a flat deformation of X_0 over the germ of a complex space $0 \in S$. Then,*

1. *f_0 extends to a contraction morphism $F_S\colon X_S \to Y_S$.*

2. *The flop $F_S^+\colon X_S^+ \to Y_S$ exists and commutes with any base change.*

Sketch proof. This follows from Theorem 6.3.2. For each $s \in S$, the singularities of Y_s are terminal. Using a local analytic description of Y_s near a singular point, the flop $F_S\colon X_S^+ \to Y_S$ can be explicitly constructed as in [Kol91]. □

Theorem 6.3.8 (Deformation of extremal rays, [KM92])**.** *Let $g\colon X \to S$ be a proper flat morphism of projective varieties. Assume that for some $0 \in S$ the fibre X_0 is a 3-fold with no worse than $\mathbb{Q}$-factorial canonical singularities. Let $f\colon X_0 \to X_0'$ be the contraction of an extremal ray $C_0 \subset X_0$. There then exist a proper flat morphism $h\colon X' \to S$ (by Proposition 6.3.1) and a factorisation $g\colon X \xrightarrow{f} X' \xrightarrow{h} S$. Moreover, there exists an open neighbourhood U of $0 \in S$ such that:*

(i) *if f_0 contracts a subset of* Codim ≥ 2 *(resp. a divisor, resp. is a fibre space of generic relative dimension k), f_s contracts a subset of* Codim $\geq$ 2 *(resp. a divisor, resp. is a fibre space of generic relative dimension k) for all $s \in U$,*

(ii) *f_s is the contraction of an extremal ray.*

Remark 6.3.9. Lemma 7.1.3 describes contractions of extremal rays with Cartier exceptional divisor on small partial $\mathbb{Q}$-factorialisations of terminal Gorenstein Fano 3-folds. This Lemma is a mild generalisation of Cutkosky's results (Theorem 3.2.1), and shows that the same classification of extremal rays essentially applies as in the $\mathbb{Q}$-factorial terminal Gorenstein case.

Lemma 6.3.10. *Let Z be a small partial $\mathbb{Q}$-factorialisation of a terminal Gorenstein Fano 3-fold Y. Denote by $\mathcal{Z} \to S$ a proper flat 1-parameter deformation of X over the spectrum of a complete discrete valuation ring $S = \operatorname{Spec} \mathcal{O}_S$ with residue field $\mathbb{C}$. Let 0 be the closed point of S and η the generic point. Assume that each fibre has terminal Gorenstein singularities. If $f_0\colon Z \to Z'$ is a divisorial extremal contraction with $\mathbb{Q}$-Cartier exceptional divisor E, there is an S-morphism $f\colon \mathcal{Z} \xrightarrow{f} \mathcal{Z}'$ to a projective 1-parameter flat deformation of Z'. The restrictions f_η and f_0 are extremal divisorial contractions. In the notation of Theorem 3.2.1, either the contraction f_η is of the same type as f_0, or f_η and f_0 are of types E3 and E4.*

Proof. I give an outline of the argument, as the assumption that E is Cartier ensures that it can be proved as in [Mor82, Proposition 3.47]. If $\mathcal{D}_\eta$ is a Cartier divisor of $\mathcal{Z}_\eta$, the completion $\mathcal{D}_S$ to S is an irreducible divisor which is $\mathbb{Q}$-Cartier because $\mathcal{Z}$ is parafactorial: any proper Weil divisor on S that is Cartier outside of finitely many fibres is $\mathbb{Q}$-Cartier. The specialisation map associates to $\mathcal{D}_\eta$ the $\mathbb{Q}$-Cartier divisor $\operatorname{Red}(\mathcal{D}_\eta) = \mathcal{D}_S \times_S \{0\}$ of Z. As Z is Gorenstein, $\operatorname{Red}(\mathcal{D}_\eta)$ is in fact Cartier [Kaw88, Lemma 6.3];

Red is an injective homomorphism $\mathrm{Red}\colon NS(\mathcal{Z}_\eta) \to NS(Z)$. This homomorphism is bijective because the Picard rank is constant in a 1-parameter flat deformation of a 3-fold with isolated hypersurface singularities. Similarly, if $\mathcal{C}_\eta$ is an effective curve, denote by $\mathrm{Red}(\mathcal{C}_\eta) = \mathcal{C}_S \times_S \{0\}$ the specialisation of $\mathcal{C}_\eta$. The 1-cycle $\mathrm{Red}(\mathcal{C}_\eta)$ has non-negative coefficients and Red defines an injective homomorphism $\mathrm{Red}\colon \overline{NE}(\mathcal{Z}_\eta) \to \overline{NE}(Z)$ on the cone of effective curves. In addition, if $\mathcal{D}_\eta$ is a Cartier divisor and $\mathcal{C}_\eta$ is a 1-cycle, $\mathcal{D}_\eta \cdot \mathcal{C}_\eta = \mathrm{Red}(\mathcal{D}_\eta \cdot \mathcal{C}_\eta) = \mathrm{Red}(\mathcal{D}_\eta) \cdot \mathrm{Red}(\mathcal{C}_\eta)$ [Ful98].

Recall from Remark 6.2.21 that for all $t \in S$, $\mathcal{Z}_t$ is a generalised Fano 3-fold, in particular, $NE(\mathcal{Z}_\eta)$ is rational polyhedral (Lemma 3.1.6).

Case 1. *The contraction $f_0\colon Z \to Y$ is of type E1, i.e. f_0 contracts a surface E to a curve $\Gamma \subset Y$.*

Denote by l a general fibre of the contraction $E \to \Gamma$. Lemma 7.1.3 shows that E is a $\mathbb{P}^1$-bundle over Γ, $l \simeq \mathbb{P}^1$ and that $-K_Z \cdot l = -E \cdot l = 1$. The curve l lies on a Cartier divisor, it is a l.c.i. variety and by adjunction on E, $\mathcal{N}_{\Gamma/Z} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Let $\mathcal{H}$ be the connected component of the Hilbert prescheme $\mathrm{Hilb}_{Z/S}$ containing the class $[l]$ of l. The Zariski tangent space to $\mathrm{Hilb}_{Z/S}$ at $[l]$ is canonically isomorphic to $H^0(Z, \mathcal{N}_{l/Z})$ and if $H^1(Z, \mathcal{N}_{l/Z}) = (0)$, $\mathcal{H}_0$ is non-singular over S at $[l]$ [Gro62, IV, Corollaire 5.4]. The component $\mathcal{H}_0$ is non-singular at $[l]$ ($h^1(Z, \mathcal{N}_{l/Z}) = 0$) and has relative dimension 1 at 0 ($h^0(Z, \mathcal{N}_{l/Z}) = 1$). The base S is complete, and hence the component $\mathcal{H}_0$ is connected. As this is true for all but a finite number of fibres, the component $\mathcal{H}_0$ is isomorphic to Γ. Denote by $\mathcal{D} \subset \mathcal{Z} \times_S \mathcal{H}$ the universal closed subscheme. The projection of $\mathcal{D}$ to $\mathcal{Z}$ induces an embedding $\mathcal{D}_0 \simeq E$. The component $\mathcal{H}$ is non-singular over S at any point of $\mathcal{H}_0$, hence it is smooth over S and $\mathcal{H}_\eta$ is a non-singular projective curve.

The universal subscheme $\mathcal{D}$ is by definition flat over $\mathcal{H}$. As $\mathcal{D}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{H}_0 \simeq \Gamma$, $\mathcal{D}_\eta$ is also a $\mathbb{P}^1$-bundle over $\mathcal{H}_\eta$. Denote by $\mathcal{C}_\eta$ any fibre of $\mathcal{D}_\eta \to \mathcal{C}_\eta$; $\mathrm{Red}(\mathcal{C}_\eta)$ is a fibre l of $E \to \Gamma$; it therefore generates an extremal ray R_0 of $NE(Z) \supset \mathrm{Red}(NE(\mathcal{Z}_\eta))$. The curve $\mathcal{C}_\eta$ generates an extremal ray R_η of $\mathcal{Z}_\eta$. This extremal ray may be contracted because $\mathcal{Z}_\eta$ is a weak Fano 3-fold, and since $\mathcal{D}_\eta \cdot \mathcal{C}_\eta = D \cdot l = -1$, the divisor $\mathcal{D}_\eta$ is exceptional for the contraction $cont_{R_\eta}$. If $\mathcal{G}_\eta \subset \mathcal{D}_\eta$ denotes a curve that dominates $\mathcal{H}_\eta$ such that $\mathrm{Red}(\mathcal{G}_\eta)$ dominates $\mathcal{H}_0$, $[\mathcal{G}_\eta]$ does not belong to the class R_η because $\mathrm{Red}(\mathcal{G}_\eta)$ does not belong to R_0. This proves that the contraction of R_η is an extremal contraction of type E1.

Case 2. *The contraction $f_0 \colon Z \to Y$ is of type E2-E5, i.e. f_0 contracts a surface E to a point $P \in Y_0$.*

The divisor E is a l.c.i. subscheme because E is Cartier. The cohomology groups $H^0(E, \mathcal{O}_E(E))$ and $H^1(E, \mathcal{O}_E(E))$ are both trivial (Lemma 7.1.3), hence the connected component $\mathcal{H}$ of the Hilbert scheme $\mathrm{Hilb}_{Z/S}$ containing $[E]$ has relative dimension 0 at $[E]$ and $\mathcal{H}_0$ is non-singular at $[E]$. The component $\mathcal{H}$ therefore is isomorphic to S. Denote by $\mathcal{D} \subset \mathcal{Z} \times_S S$ the universal subscheme; the projection to $\mathcal{Z}$ determines an embedding $\mathcal{D}_0 \simeq E$. As in case 1, any effective curve lying on $\mathcal{D}_\eta$ belongs to the class of an extremal ray R_η, that may be contracted. The contraction $cont_{R_\eta}$ contracts a divisor to a point. As $\mathcal{D}_\eta^3 = \mathrm{Red}(\mathcal{D}_\eta)^3 = E^3$, the contractions f_0 and $cont_{R_\eta}$ are of the same type, except possibly if one of them is of type E3 and the other is of type E4.

Claim 6.3.11. *There is an S-morphism $f \colon \mathcal{Z} \to \mathcal{Z}'$ that restricts to f_0 on the central fibre and to $cont_{R_\eta}$ on the generic fibre.*

The Picard rank is constant in the 1-parameter flat deformation and $\mathrm{Pic}(\mathcal{Z}) \simeq \mathrm{Pic}(Z) \simeq \mathrm{Pic}(\mathcal{Z}_\eta)$, since each fiber has terminal Gorenstein singularities. Denote by L a nef invertible sheaf on Z associated to f_0, i.e. such that $L^\perp \cap NE(Z) = R_0$ and f_0 is the morphism determined by $|mL|$ for $m \gg 0$. The invertible sheaf L comes from an invertible sheaf $\mathcal{L}$ on $\mathcal{Z}$ and $\mathcal{L}_\eta^\perp \cap NE(\mathcal{Z}_\eta) = R_\eta$ by construction. The Cartier divisor $|m\mathcal{L}|$ determines the desired S-morphism $\mathcal{Z} \to \mathcal{Z}'$, because by the base change theorem, $H^0(\mathcal{Z}, m\mathcal{L}) \otimes \mathbb{C} \simeq H^0(Z, mL)$. $\square$

Every divisorial step of the Minimal Model Program on a weak$*$ Fano 3-fold can be deformed to an extremal divisorial contraction on a non-singular weak$*$ Fano 3-fold with Picard rank 2.

Assume that the extremal contraction $\phi_i \colon X_i \to X_{i+1}$ is divisorial and denote by E_i its exceptional divisor. I use the notation introduced in the proof of Lemma 3.2.7. The generalised Fano 3-fold Z_i is a crepant blow up of Y_i along the image of the exceptional divisor E_i of ϕ_i by the anticanonical map. More precisely, Z_i is defined as $\underline{\mathrm{Proj}} \bigoplus_n f_{i*}\mathcal{O}(nE_i)$ and fits in the diagram:

$$\begin{tikzcd} X_i \arrow[r, "\phi_i"] \arrow[d] & X_{i+1} \arrow[d] \\ Z_i \arrow[r, "\psi_i"] \arrow[d] & Z_{i+1} \\ Y_i & \end{tikzcd}$$

where $Z_{i+1} = Y_{i+1}$ is the anticanonical model of X_{i+1}. The following corollary is a direct consequence of Theorem 6.2.8 and of Lemma 6.3.10.

Corollary 6.3.12. *There is a proper* 1*-parameter flat deformation* $\mathcal{Z}_i$ *of* Z_i *over* Δ *such that* $\mathcal{Z}_{i,t}$ *is a terminal Gorenstein generalised Fano with Picard rank* 2 *for all* $t \in \Delta$ *and* $\mathcal{Z}_{i,t_0}$ *is non-singular for some* $t_0 \in \Delta$*. There is a morphism* $\Psi_i \colon \mathcal{Z}_i \to \mathcal{Z}_{i+1}$ *which restricts to* ψ_i *on the central fibre, and such that* $\mathcal{Z}_{i+1}$ *is a* 1*-parameter flat deformation of* Z_{i+1}*. In a neighbourhood of* $0 \in \Delta$*,* $(\mathcal{Z}_i)_t$ *is a Picard rank* 2 *terminal Gorenstein weak Fano* 3*-fold and* $(\mathcal{Z}_{i+1})_t$ *is a Picard rank* 1 *terminal Gorenstein Fano* 3*-fold. The morphism* Ψ_t *restricts to the contraction of an extremal ray of the same type as* Ψ_{i0}*, except if* Ψ_{i0} *is of type E*3 *and* $\Psi_{i\eta}$ *is of type E*4*.*

Remark 6.3.13. The same holds for an extremal contraction of fibering type. Note that, in a neighbourhood of 0, $\mathcal{Z}_{i+1,t}$ is projective because Z_{i+1} is.

Chapter 7

Takeuchi game

In this Chapter, I generalise some constructions studied by Takeuchi [Tak89] to determine explicitly the extremal divisorial contractions encountered when running the Minimal Model Program on a weak$*$ Fano 3-fold X. Iskovskikh obtained a classification of Fano 3-folds with Picard rank 1 by studying the double projection of Fano 3-folds from lines lying on them [Isk77, Isk78]. Takeuchi investigated projections of Fano 3-folds from a point or from a general conic [Tak89]. Both constructions are special cases of Sarkisov elementary links [Cor95]. Simple numerical calculations based on the theory of extremal rays led to a considerable refinement of Iskovskikh's methods. I generalise Takeuchi's construction in order to study some projections of a terminal Gorenstein Fano 3-fold Y from curves lying on it.

Let Y be a terminal Gorenstein Fano 3-fold with Picard rank 1 that does not contain a plane and let Z be a small partial $\mathbb{Q}$-factorialisation of Y. I assume that the Picard rank of Z is 2, and that there exists on Z an extremal contraction ϕ with Cartier exceptional divisor. In Sections 7.1 and 7.2, I study a generalised Takeuchi construction on Z. This analysis yields systems of Diophantine equations, whose solutions describe the possible divisorial contractions encountered when the MMP is run on X, a small $\mathbb{Q}$-factorialisation of Y. This approach provides a theoretical method to construct explicit examples of non $\mathbb{Q}$-factorial Gorenstein terminal Fano 3-folds.

In Section 7.3, I use these techniques to give a "geometric motivation" of $\mathbb{Q}$-factoriality for terminal Gorenstein quartic 3-folds. If $Y_4^3 \subset \mathbb{P}^4$ fails to be $\mathbb{Q}$-factorial, by definition, Y contains a surface $\overline{E}$, which is a Weil, non $\mathbb{Q}$-Cartier divisor. I show that the surface $\overline{E}$ is a plane, a quadric or is one of the surfaces listed in the table on page 129.

7.1 Elementary contractions of terminal Gorenstein Fano 3-folds

Let X be a projective non-singular 3-fold such that K_X is not nef. Mori shows [Mor82] that X admits an elementary contraction morphism ϕ that corresponds to an extremal ray of $\overline{NE}(X)$; he classifies all such contractions and shows that either ϕ is of fibering type, or ϕ contracts a Cartier divisor E to a point or a curve. In the case of non-singular surfaces, the exceptional divisor of an extremal contraction is a (-1)-curve; Mori shows that in the 3-fold case, E either is a plane or a quadric with anti-ample normal bundle or E is a $\mathbb{P}^1$-bundle over a non-singular curve. Up to minor generalisations, the classification of divisorial extremal contractions still holds when X is assumed to have terminal Gorenstein $\mathbb{Q}$-factorial singularities [Cut88]. I study extremal contractions with Cartier exceptional divisor on small partial $\mathbb{Q}$-factorialisations of terminal Gorenstein Fano 3-folds.

Set up 7.1.1. Let Z be a normal, terminal Gorenstein weak Fano 3-fold and let Y be its anticanonical model. Assume that the anticanonical map $h\colon Z \to Y$ is small, that the anticanonical ring of Z is generated in degree 1 and that Y has Picard rank 1.

Recall from Theorem 3.1.7 that this is the case unless Z is monogonal or Y is birational to a special complete intersection $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$ with a node. This is always true if Y has Picard rank 1 and genus at least 3.

Let $\psi\colon Z \to Z_1$ be a K_Z-negative extremal contraction. Since a flipping curve γ on a terminal variety Z satisfies $-K_Z \cdot \gamma < 1$ [Ben85], ψ is not an isomorphism in codimension 1. Assume that the exceptional divisor E of the contraction ψ is $\mathbb{Q}$-Cartier. Note that the Weil divisor $\overline{E} = h(E)$ is not $\mathbb{Q}$-Cartier. If it were, as Y has Picard rank 1, E would have to be ample. Note that as Z has terminal Gorenstein singularities, a divisor is $\mathbb{Q}$-Cartier if and only if it is Cartier [Kaw88, Lemma 6.3].

Let $g\colon X \to Z$ be a small $\mathbb{Q}$-factorialisation and let $\widetilde{E}$ be the pull back of E on X. There is a K_X-negative extremal ray on which $\widetilde{E}$ is negative. Indeed, g is small, hence $K_X = g^*(K_Z)$ and $\widetilde{E} = g^*(E)$. The divisor $\widetilde{E}$ is covered by curves Γ such that $K_X \cdot \Gamma < 0$ and $\widetilde{E} \cdot \Gamma < 0$, because E has this property. The 3-fold X is a weak Fano, by Lemma 3.1.6, there is a K_X-negative extremal contraction ϕ on X with exceptional divisor $\widetilde{E}$.

Denote by $f\colon X \to Y$ the anticanonical map of X. As in the proof of Lemma 3.2.7, the following diagram is commutative because $\widetilde{E}$ is negative on an extremal ray of $NE(X/Z_1)$:

$$\begin{tikzcd} \widetilde{E} \subset X \arrow[r, "\phi"] \arrow[d, "g"'] & X_1 \arrow[d, "g_1"] \\ E \subset Z \arrow[r, "\psi"] \arrow[d, "h"'] & Z_1 \\ \overline{E} \subset Y & \end{tikzcd} \tag{7.1}$$

In the diagram, $g \circ h = f$, $\widetilde{E} = f^*(\overline{E})$ and g_1 is an isomorphism in codimension 1.

Remark 7.1.2. In the terminology introduced in Chapter 3, if Y does not contain a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$, then X is a weak$*$ Fano 3-fold.

Lemma 7.1.3. *E1: If ψ contracts E to a curve Γ, Γ is locally a complete intersection and has planar singularities. The contraction ψ is locally the blow up of the ideal sheaf I_Γ. In the local ring $\mathcal{O}_{Z_1,P}$ of any point $P \in \Gamma$, one of the local equations of Γ is a smooth hypersurface near P.*

If ψ contracts E to a point P, then one of the following holds:

E2: $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ and P is a non-singular point.

E3: $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$.

E4: $(E, \mathcal{O}_E(-E)) \simeq (Q, \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1))$, with Q an irreducible reduced singular quadric surface in $\mathbb{P}^3$.

E5: $(E, \mathcal{O}_E(E)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)))$, and P is a non-Gorenstein point of index 2.

Proof. The diagram (7.1) is commutative, therefore g_1 maps the centre of the contraction ϕ to the centre of the contraction ψ. If ϕ contracts a divisor to a point, so does ψ. The map $g_1 \colon X_1 \to Z_1$ is a small $\mathbb{Q}$-factorialisation of Z_1, so that if the centre of ϕ is a curve Γ and that of ψ is a point $\{P\}$, $-K_{X_1} \cdot \Gamma = 0$, the curve Γ is non-singular, $-K_{X_1}$ is nef and big and defines a small map by the proof of Theorem 3.1.7. As in the proof of Lemma 4.1.3, $\widetilde{E} \simeq \mathbb{F}_2$ or $\widetilde{E} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The Cartier divisor $E = g_*(\widetilde{E})$ is an irreducible reduced quadric in both cases, and $-K_{Z|E} = \mathcal{O}_E(1)$. The contraction ψ is of type E3 or E4.

I assume that the contractions ϕ and ψ either both contract a divisor to a curve or both contract a divisor to a point.

Case 1. *The contraction $\psi\colon Z \to Z_1$ contracts the divisor E to a curve Γ.*

The 3-fold Z has terminal Gorenstein singularities, in particular its singularities are isolated. Let l be a fibre of the contraction ψ that contains no singularities of Z. The map g is small and by construction of ψ, there exists a fibre $\tilde{l}$ of ϕ mapping 1-to-1 to l. Then, $-K_Z \cdot l = g^*(-K_Z)\cdot \tilde{l} = -K_X \cdot \tilde{l} = 1$ and $E \cdot l = g^*(E) \cdot \tilde{l} = \widetilde{E} \cdot \tilde{l} = 1$. The contraction ψ is the blow up of a non-singular curve away from finitely many points.

The anticanonical model Y of Z has Picard rank 1 and genus $g \geq 3$, so the weak Fano 3-fold Z itself has basepoint free anticanonical system $|-K_Z|$. Let S be a general section of $|-K_Z|$; S is non-singular by Theorem 3.1.7. As $|-K_Z| = |\psi^*(-K_{Z_1}) - E|$ (since $-K_X = \phi^*(-K_{X_1}) - \widetilde{E}$), S is mapped to S_1, a section of $|-K_{Z_1}|$ that contains the curve Γ. The contraction ϕ is not of type E5, therefore X_1 and Z_1 are Gorenstein and S_1 is Cartier. The morphism $\psi_{|S}\colon S \to S_1$ is an isomorphism away from Γ. For any fibre l of ψ, l is not contained in S because S is general in $|-K_Z|$, and l intersects S in a finite number of points. Over an affine neighbourhood of $P = \psi(l) \in \Gamma$, the morphism $\psi_{|S}\colon S \to S_1$ is a finite birational morphism, which is an isomorphism away from l. In particular, the singularities of S_1 are isolated. Further, S_1 is normal because S_1 is a Cartier divisor in Z, which is Cohen Macaulay, and S_1 is regular in codimension 1. The morphism $\psi_{|S}$ is finite birational, hence, by Zariski's main theorem, it is an isomorphism. The curve Γ lies on a non-singular Cartier divisor: it is locally a complete intersection. Hence, the curve Γ has planar singularities.

Case 2. *The contraction $\psi\colon Z \to Z_1$ contracts the divisor E to a point P.*

The morphism ψ is the elementary contraction of a K_Z (and E) negative extremal ray. The divisor $-K_Z - E$ is Cartier, and its restriction to E, $-K_E = (-K_Z - E)_{|E}$, is anti-ample. The exceptional divisor E therefore is a Gorenstein, possibly nonnormal, del Pezzo surface. The map $g\colon X \to Z$ is small since the anticanonical map is an isomorphism in codimension 1; it induces a morphism $g_{|\widetilde{E}}\colon \widetilde{E} \to E$. The morphism $g_{|\widetilde{E}}$ uniquely induces a morphism between the normalisations of E and $\widetilde{E}$: $\tilde{g}_{|\widetilde{E}}\colon \widetilde{E}^\nu \to E^\nu$. Since the map g is small, E^ν has the same anticanonical degree as $\widetilde{E}^\nu = \widetilde{E}$. Theorem 3.2.1 shows that this degree is $1, 2$ or 4. The normalisation of E is either a plane or a quadric.

By the Serre criterion, E is nonnormal if and only if it is not regular in codimension 1. Indeed, Z is Cohen Macaulay and E is Cartier: E satisfies the S_2-condition. Any curve C lying on $\widetilde{E}$ has $-K_X \cdot C < 0$ so that no g-exceptional curve lies on $\widetilde{E}$ and $g_{|\widetilde{E}}$ is an isomorphism outside of a finite number of points: if E is not regular in codimension 1, neither is $\widetilde{E}$. Cutkosky's classification (Theorem 3.2.1) shows that $\widetilde{E}$ is normal, hence E is also normal and the result follows. □

Remark 7.1.4. The assumption that Z is a small partial $\mathbb{Q}$-factorialisation of a terminal Gorenstein Fano 3-fold is not necessary here: the proof works when Z is assumed to be a small partial $\mathbb{Q}$-factorialisation of a terminal Gorenstein 3-fold with Picard rank 1. The surface S in Case 1 can be replaced by a general member of $|nH - K_Z|$, where H is the linear system that determines the contraction ψ.

Lemma 7.1.5. *Let Y be a non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold with Picard rank 1 and genus $g \geq 3$. Let Z be a small partial $\mathbb{Q}$-factorialisation of Y. Assume that there exists an extremal divisorial contraction $\psi \colon Z \to Z_1$ that contracts a Cartier divisor E to a curve Γ. The following relations hold:*

$$\begin{aligned}(-K_Z)^3 = (-K_Y)^3 = 2g-2 = -K_{Z_1}^3 - 2((-K_{Z_1}\cdot\Gamma) + 1 - p_a(\Gamma))\\ (-K_Z)^2\cdot E = -K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma)\\ -K_Z\cdot E^2 = -2 + 2p_a(\Gamma)\\ E^3 = -(-K_{Z_1}\cdot\Gamma - 2 + 2p_a(\Gamma))\end{aligned}$$

Proof. By Lemma 7.1.3, the contraction ψ is the blow up of the curve Γ and Γ is locally a complete intersection. The anticanonical divisors of Z and Z_1 satisfy:

$$-K_Z = \psi^*(-K_{Z_1}) - E. \tag{7.2}$$

The proof of Lemma 7.1.3 shows that the curve Γ lies on a non-singular section S_1 of the anticanonical linear system $|-K_{Z_1}|$, such that the proper transform S of S_1 on Z is a non-singular section of $|-K_Z|$. The surfaces S and S_1 are moreover isomorphic. As $\psi^*S_1 = S + E$, E^3 satisfies:

$$E^3 = -S\cdot E^2 + \psi^*S_1\cdot E^2 = -(E_S\cdot E_S)_S - S_1\cdot\Gamma.$$

The curve Γ is locally a complete intersection, and by construction of S, $E_S \simeq \Gamma$ and $(E_S \cdot E_S)_S = (\Gamma \cdot \Gamma)_{S_1}$. As S_1 is Cartier, $K_{S_1} = (K_{Z_1} + S_1)_{|S_1}$ and since S_1 is non-singular, the adjunction formula for Γ gives $K_\Gamma = (K_{S_1} + \Gamma)_{|\Gamma}$. No correction term is needed at any point. In particular,

$$(\Gamma \cdot \Gamma)_{S_1} = -K_{S_1} \cdot \Gamma + K_\Gamma = -K_{Z_1} \cdot \Gamma - S_1 \cdot \Gamma + 2p_a(\Gamma) - 2$$

and therefore:

$$E^3 = K_{Z_1} \cdot \Gamma + S_1 \cdot \Gamma + 2 - 2p_a(\Gamma) - S_1 \cdot \Gamma = K_{Z_1} \cdot \Gamma + 2 - 2p_a(\Gamma).$$

The relations then follow from (7.2) and from the projection formula:

$$\psi^*(-K_{Z_1}) \cdot E^2 = -(-K_{Z_1} \cdot \Gamma).$$

□

Lemma 7.1.6. *Let Y be a non $\mathbb{Q}$-factorial terminal quartic 3-fold. Assume that Y does not contain a plane $\mathbb{P}^2$ with $|-K_Y|_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. Let Z be a small partial $\mathbb{Q}$-factorialisation of Y with $\rho(Z/Y) = 1$. Assume that there exists an extremal divisorial contraction $\psi \colon Z \to Z_1$ that contracts a Cartier divisor E to a curve Γ. Denote by $i(Z_1)$ the Fano index of Z_1. The following bounds on the degree and arithmetic genus of Γ hold:*

1. *If $i(Z_1) = 1$ and Z_1 has genus g_1, then $\deg(\Gamma) = g_1 - 4 + p_a(\Gamma)$ and $p_a(\Gamma) \le g_1 - 3$,*
2. *If $i(Z_1) = 2$ and $(-K_{Z_1})^3 = 8d$ for some $1 \le d \le 5$, then $2\deg(\Gamma) = 4d - 3 + p_a(\Gamma)$ and $p_a(\Gamma) = 2k - 1$, where $1 \le k \le d + 1$,*
3. *If Z_1 is a possibly singular quadric in $\mathbb{P}^4$, then $3\deg(\Gamma) = 24 + p_a(\Gamma)$ and $p_a(\Gamma) = 3k$, for some $0 \le k \le 7$,*
4. *If $Z_1 = \mathbb{P}^3$, then $4\deg(\Gamma) = 29 + p_a(\Gamma)$ and $p_a(\Gamma) = 3 + 4k$, for some $0 \le k \le 6$.*

Proof. Lemmas 7.1.3 and 7.1.5 show that ψ is the inverse of the blow up of the curve Γ and that the following relations hold:

$$(-K_Z)^3 = (-K_Y)^3 = 4 = -K_{Z_1}^3 - 2((-K_{Z_1} \cdot \Gamma) + 1 - p_a(\Gamma)) \tag{7.3}$$
$$(-K_Z)^2 \cdot E = -K_{Z_1} \cdot \Gamma + 2 - 2p_a(\Gamma). \tag{7.4}$$

The anticanonical linear system $|-K_Z|$ is basepoint free and it is equal to $|\psi^*(-K_{Z_1}) - E| = |\psi^*(-K_{Z_1}(-\Gamma)|$: the curve Γ is a scheme-theoretic intersection of members of $|-K_{Z_1}|$. In particular,

$$-K_{Z_1} \cdot \Gamma = i(Z_1) \deg(\Gamma) \leq (-K_{Z_1})^3.$$

The following relations and bounds on the degree of Γ therefore hold:

1. If $i(Z_1) = 1$ and if Z_1 has genus g_1, with $4 \leq g_1 \leq 10$ or $g_1 = 12$, then

$$\deg(\Gamma) = g_1 - 4 + p_a(\Gamma) \quad \text{and} \quad \deg(\Gamma) \leq 2g_1 - 2;$$

2. If $i(Z_1) = 2$ and if $(-K_{Z_1})^3 = 8d$ for some $1 \leq d \leq 5$, then

$$2\deg(\Gamma) = 4d - 3 + p_a(\Gamma) \quad \text{and} \quad \deg(\Gamma) \leq 4d;$$

3. If $i(Z_1)) = 3$, Z_1 is a possibly singular quadric in $\mathbb{P}^4$, then

$$3\deg(\Gamma) = 24 + p_a(\Gamma) \quad \text{and} \quad \deg(\Gamma) \leq 18;$$

4. If $Z_1 = \mathbb{P}^3$, then

$$4\deg(\Gamma) = 29 + p_a(\Gamma) \quad \text{and} \quad \deg(\Gamma) \leq 16.$$

Now that the degree of Γ has been bounded, the genus of Γ can be bounded using (7.3).

By assumption, $-K_Z$ is nef and big and induces a small map, therefore $(-K_Z)^2 \cdot E > 0$. □

Remark 7.1.7. The bound on the genus of Γ obtained using (7.3) is sharper than the Castelnuovo bound when Γ is a non-singular curve.

7.2 A generalised Takeuchi construction

Takeuchi formulates numerical constraints associated to contractions of extremal rays [Tak89]. This approach simplifies considerably the methods of birational classification using projection of varieties from points or lines, which lie at the heart of Iskovskikh's classification of Fano 3-folds [Isk77, Isk78].

This application of the theory of extremal rays makes for a unified treatment of projections of Fano 3-folds from any centre. Takeuchi's work focuses on non-singular weak Fano 3-folds with Picard rank 2 that are small $\mathbb{Q}$-factorialisations of terminal Fano 3-folds with Picard rank 1 and defect 1. Given Z, a non-singular weak Fano 3-fold with Picard rank 2, there is an elementary Sarkisov link on Z involving two extremal contractions that are not isomorphisms in codimension 1. The numerical constraints associated to each type of extremal contraction yield systems of Diophantine equations, whose solutions correspond to the only possible Sarkisov elementary links. I generalise Takeuchi's construction, in order to classify non $\mathbb{Q}$-factorial terminal quartic 3-folds with arbitrary defect.

Set up 7.2.1 (A generalised Takeuchi construction)**.** Let Y be a non $\mathbb{Q}$-factorial terminal Gorenstein Fano 3-fold with Picard rank 1 and Fano index 1. Assume that Y contains neither a plane with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ nor an irreducible reduced quadric Q with $-K_{Y|Q} = \mathcal{O}_Q(1)$.

Let $f\colon X \to Y$ be a small $\mathbb{Q}$-factorialisation of Y. The 3-fold X is a weak$*$ Fano 3-fold and has Picard rank $\rho(X) > 1$. Lemma 3.1.6 shows that there is an extremal ray $R \in NE(X)$ and that R can be contracted. A Minimal Model Program (MMP) can be run on X (Chapter 3).

Let ϕ be the first extremal contraction of the MMP that is not an isomorphism in codimension 1. Assume that ϕ is divisorial; ϕ necessarily contracts a divisor E to a curve, since Y is of index 1 and contains neither a plane nor a quadric. Taking a different small $\mathbb{Q}$-factorialisation X of Y if necessary, we may assume that the first extremal contraction of the MMP on X is $\phi\colon X \to X_1$.

Denote by $\overline{E} \subset Y$ the image of E by the anticanonical map. Recall from the proof of Lemma 3.2.7 that $\overline{E}$ is a Weil non $\mathbb{Q}$-Cartier divisor on Y. Let Z be the small partial $\mathbb{Q}$-factorialisation of Y defined by:

$$Z = \underline{\mathrm{Proj}} \bigoplus_{n\geq 0} f_*\mathcal{O}_X(n\widetilde{E}).$$

The 3-fold Z is a weak Fano and has Picard rank 2; denote by $h\colon Z \to Y$ its anticanonical map. If E' is the image of E on Z, there is an extremal contraction $\psi\colon Z \to Z_1$ that contracts E' (Lemma 3.2.7). The proof of Lemma 7.1.5 shows that since Y does not contain an irreducible quadric, ψ contracts the divisor E' to a curve Γ.

Theorem 6.2.8 shows that there is a 1-parameter flat deformation $\mathcal{Z} \to \Delta$ of Z such that for each $t \neq 0$, $\mathcal{Z}_t$ is a generalised Fano 3-fold of Picard rank 2. Proposition 6.3.1 and Theorem 6.3.7 show that $\mathcal{Z}/\Delta$ induces a 1-parameter flat deformation $\mathcal{Y}/\Delta$ of Y and that $\mathcal{Z} \to \Delta$ factors through a morphism $H\colon \mathcal{Z} \to \mathcal{Y}$. The morphism H restricts to the anticanonical map on each fibre. For each $t \neq 0$, $\mathcal{Y}_t$ is a defect 1, Picard rank 1, terminal Gorenstein Fano 3-fold.

Corollary 6.3.12 shows that there exists a 1-parameter flat deformation of Z_1 and a morphism Ψ that fit in a diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "\Psi"] \arrow[d] & \mathcal{Z}_1 \arrow[d] \\ \Delta \arrow[r, equal] & \Delta \end{tikzcd}$$

such that Ψ restricts to ψ on the central fibre, and such that for all $t \in \Delta$, Ψ_t is an E1-contraction.

Pick $t \in \Delta \smallsetminus \{0\}$; we may assume that $\mathcal{Z}_t$ is a $\mathbb{Q}$-factorial weak Fano 3-fold with small anticanonical map $h_t\colon \mathcal{Z}_t \to \mathcal{Y}_t$. Indeed, for some $t \in \Delta$, $\mathcal{Z}_t$ is non-singular and $\mathbb{Q}$-factoriality is an open condition on the base. The Cone theorem 3.1.6 shows that $\mathcal{Z}_t$ has exactly 2 extremal rays and that they may be contracted. One of the extremal contractions contracts E'_t (the divisor on $\mathcal{Z}_t$ mapped to E' on the central fibre) to a curve Γ_t. Denote this divisorial contraction by Ψ_t. On $\mathcal{Z}_t$, a 2-ray game yields a diagram of the form:

$$\begin{tikzcd} & \mathcal{Z}_t \arrow[dl, "\Psi_t"'] \arrow[dr, "h_t"] \arrow[rr, dashed, leftrightarrow, "\Phi_t"] & & \widetilde{\mathcal{Z}_t} \arrow[dl, "\widetilde{h}_t"'] \arrow[dr, "\alpha_t"] & \\ (\mathcal{Z}_1)_t & & \mathcal{Y}_t & & (\widetilde{\mathcal{Z}_1})_t. \end{tikzcd}$$

where

1. $\mathcal{Z}_t$ and $\widetilde{\mathcal{Z}_t}$ are $\mathbb{Q}$-factorial weak Fano 3-folds with Picard rank 2; the anticanonical maps h_t and $\widetilde{h_t}$ of $\mathcal{Z}_t$ and $\widetilde{\mathcal{Z}_t}$ are small,

2. $(\mathcal{Z}_1)_t$ and $\mathcal{Y}_t$ are terminal Gorenstein Fano 3-folds with Picard rank 1,

3. Φ_t is a composition of flops,

4. α_t is an extremal contraction that is not an isomorphism in codimension 1,

5. $(\widetilde{\mathcal{Z}_1})_t$ is one of:

   (a) a terminal Gorenstein Fano 3-fold with Picard rank 1 if α_t is birational,

   (b) $\mathbb{P}^2$ if α is a conic bundle,

   (c) $\mathbb{P}^1$ if α is a del Pezzo fibration.

Claim 7.2.2. *The elementary Sarkisov link on $\mathcal{Z}_t$, $t \neq 0$, induces an elementary Sarkisov link on the central fibre of $\mathcal{Z} \to \Delta$.*

As in the proof of Lemma 6.3.10, we may assume that $\mathcal{Z}$ is a proper flat deformation over the spectrum of a complete ring S with residue field k, closed point 0 and generic point η. It is enough to show that if $\mathcal{Z}/S$ is a 1-parameter proper flat deformation of a generalised Fano 3-fold Z with Picard rank 2 such that the generic fibre is terminal Gorenstein and $\mathbb{Q}$-factorial, an extremal contraction on the generic fibre $\mathcal{Z}_\eta$ induces an S-morphism that restricts to the contraction of an extremal ray on the central fibre.

Recall that the specialisation map induces an isomorphism of the Neron Severi groups of $\mathcal{Z}_\eta$ and Z, and that $\mathrm{Pic}(\mathcal{Z}_\eta) \simeq \mathrm{Pic}(Z) \simeq \mathrm{Pic}(\mathcal{Z})$ because the singularities of each fibre are terminal and Gorenstein. As the cone $NE(Z)$ is rational polyhedral and generated by extremal rays, the specialisation of every extremal ray R_η on Z_η, belongs to the class of an extremal ray R_0 on Z, and R_0 may be contracted. If R_0 is a flopping contraction, Theorem 6.3.7 ensures that the contraction of R_η also is a flopping contraction. If R_0 is a K-negative extremal ray, so is R_η (Lemma 6.3.10). If $cont_{R_\eta}$ is divisorial, it has Cartier exceptional divisor E_η. The specialisation $\mathrm{Red}(E_\eta)$ is ($\mathbb{Q}$-)Cartier and is the exceptional divisor of $cont_{R_0}$. The contraction of R_η on $\mathcal{Z}_\eta$ induces a projective S-morphism $\mathcal{Z} \to \mathcal{Z}'$ that restricts to the contraction of R_η and R_0.

The flop Φ_t induces a deformation $\widetilde{\mathcal{Z}}/\Delta$ and a morphism $\Phi\colon \mathcal{Z} \dashrightarrow \widetilde{\mathcal{Z}}$ that restricts to a flop on each fibre; in particular each fibre of $\widetilde{\mathcal{Z}}/\Delta$ is a terminal Gorenstein generalised Fano 3-fold with Picard rank 2. The contraction α_t induces a deformation $\widetilde{\mathcal{Z}_1}/\Delta$ and a morphism $\alpha\colon \widetilde{\mathcal{Z}} \to \widetilde{\mathcal{Z}_1}$ that restricts to an extremal contraction. The contraction α has Cartier exceptional divisor if it is divisorial.

I denote by Φ and α the restriction to the central fibre of Φ and α. I presume this will not lead to any confusion. On the central fibre, there is a diagram:

$$\begin{tikzcd}
& Z \arrow[dl, "\psi"'] \arrow[dr, "h"] \arrow[rr, dashed, leftrightarrow, "\Phi"] & & \widetilde{Z} \arrow[dl, "\tilde{h}"'] \arrow[dr, "\alpha"] & \\
Z_1 & & Y & & \widetilde{Z_1}
\end{tikzcd} . \tag{7.5}$$

where

1. Z and $\widetilde{Z}$ are terminal Gorenstein weak Fano 3-folds with Picard rank 2; the anticanonical maps h and $\widetilde{h}$ of Z and $\widetilde{Z}$ are small,
2. Z_1 and Y are terminal Gorenstein Fano 3-folds with Picard rank 1,
3. Φ is a composition of flops,
4. α is an extremal contraction that is not an isomorphism in codimension 1,
5. $\widetilde{\mathcal{Z}}_1$ is one of:
   (a) a terminal Gorenstein Fano 3-fold with Picard rank 1 if α is birational,
   (b) $\mathbb{P}^2$ if α is a conic bundle,
   (c) $\mathbb{P}^1$ if α is a del Pezzo fibration.

Fix the following notation:

$\widetilde{E}$ is the strict transform of E on $\widetilde{Z}$,

$e = E^3 - \widetilde{E}^3$.

Remark 7.2.3. The purpose of this construction via deformation theory is that if α is a birational contraction, its exceptional divisor is $\mathbb{Q}$-Cartier. Direct constructions on Z would have been possible, but these could have involved an extremal contraction α with Weil non $\mathbb{Q}$-Cartier exceptional divisor. Notice also that the 3-fold Y is not $\mathbb{Q}$-factorial and the map h is not the identity by construction of Z. In particular, the birational map Φ is not an isomorphism.

The birational map Φ is a sequence of flops, therefore:

$$\begin{aligned}(-K_Z)^2 \cdot E &= (-K_{\widetilde{Z}})^2 \cdot \widetilde{E} \\ -K_Z \cdot E^2 &= -K_{\widetilde{Z}} \cdot \widetilde{E}^2 \\ \widetilde{E}^3 &= E^3 - e.\end{aligned} \tag{7.6}$$

Lemma 7.2.4. *[Tak02] In the construction* (7.5)*, e is a strictly positive integer.*

Proof. The Cartier divisor E is effective and ψ-negative. For any exceptional curve γ of Φ, since γ is a flopping curve, $E \cdot \gamma$ is strictly positive. The map Φ is an E-flopping contraction and by the construction of [Kol89], $\widetilde{E}$ is Cartier: e is an integer.

Consider a common resolution of Z and $\widetilde{Z}$:

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Z \arrow[rr, dashed, leftrightarrow, "\Phi"] & & \widetilde{Z}. \end{tikzcd}$$

The 3-folds Z and $\widetilde{Z}$ are terminal, so that

$$\begin{aligned}K_W &= p^*(K_Z) + E_1 + F \\ K_W &= q^*(K_{\widetilde{Z}}) + E_2 + G\end{aligned}$$

where E_1, E_2 are effective p and q-exceptional divisors and F (resp. G) is a (possibly empty) effective p but not q (resp. q but not p) exceptional divisor.

Then:

$$p^*(K_Z) = q^*(K_{\widetilde{Z}}) + G + H,$$

where G is effective and contains no p-exceptional component and H is p-exceptional. Since $p_*(q^*(K_{\widetilde{Z}})+G) = K_Z$, $q^*(K_{\widetilde{Z}})$ is p-nef, and by the standard negativity lemma [Cor95, (2.5)], H is effective, that is $F = 0$ and $E_2 - E_1 \geq 0$.

Reversing the roles of Z and $\widetilde{Z}$ shows that, as Z and $\widetilde{Z}$ are terminal, every exceptional divisor is p and q-exceptional. Moreover, $p^*(K_Z) = q^*(K_{\widetilde{Z}}) + E_1 - E_2$ and $q^*(K_{\widetilde{Z}}) = p^*(K_Z) + E_2 - E_1$, so that, by the negativity lemma, $p^*(K_Z) = q^*(K_{\widetilde{Z}})$.

One may write

$$p_*^{-1}E = p^*E - R = q^*(\widetilde{E}) - R',$$

where R and R' are effective exceptional divisors for p and q. In particular:

$$-p^*(E) = -q^*(\widetilde{E}) + R' - R.$$

By the construction of the E-flop, $-q^*(\widetilde{E})$ is p-nef. By the negativity lemma, the divisor $R' - R$ is strictly effective because Φ is not an isomorphism, and its push forward $p_*(R' - R)$ is effective. Then

$$\begin{aligned} \widetilde{E}^3 &= (q^*\widetilde{E} - (R' - R))(q^*\widetilde{E})^2 = p^*E(q^*\widetilde{E})^2 \\ &= p^*E(p^*E + (R' - R))^2 = E^3 + Ep_*(R' - R)^2. \end{aligned}$$

This concludes the proof: $-p_*(R' - R)^2$ is a non-zero effective 1-cycle contained in the indeterminacy locus of Φ: it has strictly positive intersection with E. □

The Cartier divisors $-K_Z$ and E are linearly independent. The divisor $\widetilde{E}$ is a prime divisor, because E is prime and Φ is an isomorphism in codimension 1. Let $i(\widetilde{Z})$ be the Fano index of $\widetilde{Z}$ and $\widetilde{H}$ the uniquely determined Cartier divisor such that $-K_{\widetilde{Z}} = i(\widetilde{Z})\widetilde{H}$. The divisors $\widetilde{H}$ and $\widetilde{E}$ form a $\mathbb{Z}$-basis of $\operatorname{Pic} \widetilde{Z}$.

If α is birational, denote by D its exceptional divisor. The divisor D is Cartier, so that there exist integers x, y such that:

$$D = \frac{x}{i(\widetilde{Z})}(-K_{\widetilde{Z}}) - y\widetilde{E} \tag{7.7}$$

If α is of fibering type, denote by L the pull back of an ample generator of $\operatorname{Pic} Z_1$. The divisor L is Cartier, so that there exist integers x, y such that:

$$L = \frac{x}{i(\widetilde{Z})}(-K_{\widetilde{Z}}) - y\widetilde{E}. \tag{7.8}$$

Remark 7.2.5. As is noted in Remark 3.1.11, the indices of Y, Z and $\widetilde{Z}$ are equal. By Theorem 5.1.1, if the Fano index of Y is 4, Y is isomorphic to $\mathbb{P}^3$. If the Fano index of Y is 2, by Lemma 5.1.2 both α and ψ are either E2 contractions, étale conic bundles or quadric bundles. By Remark 5.1.3, if the Fano index of Y is 3, then α and ψ are $\mathbb{P}^2$-bundles over $\mathbb{P}^1$.

I now assume that $i(\widetilde{Z}) = i(Z) = 1$. I am mainly interested in the case of terminal quartic 3-folds $Y \subset \mathbb{P}^4$, which have Fano index 1; other cases can be treated similarly.

The morphism α is the contraction of an extremal ray on a Picard rank 2 weak Fano 3-fold $\widetilde{Z}$. If the contraction α is divisorial, its exceptional divisor is Cartier. Note that α is induced by an extremal contraction on a weak$*$ Fano 3-fold $\widetilde{X}$, which is a small $\mathbb{Q}$-factorialisation of Y. The initial small $\mathbb{Q}$-factorialisation of Y, X, and $\widetilde{X}$ are related by a sequence of flops.

The results of Lemma 7.1.3 apply to α: one can associate to α numerical constraints on the intersection numbers of powers of D (resp. L) with $-K_{\widetilde{Z}}$. When considered together, the constraints associated to α and ψ yield systems of Diophantine equations on x, y.

1. α is a conic bundle.

   Claim 7.2.6. *The surface $\widetilde{Z}_1$ is $\mathbb{P}^2$.*

   As in the proof of Lemma 3.2.11, $-K_{\widetilde{Z}_1}$, the anticanonical divisor of the surface $\widetilde{Z}_1$, is nef. The 3-fold $\widetilde{Z}$ has Picard rank 2, hence $\widetilde{Z}_1$ is $\mathbb{P}^2$. The divisor L is $\alpha^*\mathcal{O}_{\mathbb{P}^2}(1)$.

   Claim 7.2.7. *The integers x and y are positive and coprime; y can only be equal to* 1 *or* 2.

   The integers x, y are such that $L \simeq x(-K_{\widetilde{Z}}) - y\widetilde{E}$ and the divisor $\widetilde{E}$ is fixed because it is the image by Φ of a fixed divisor on Z. Assume that x is not positive, then $|L| \subset |y\widetilde{E}|$. Some positive multiple of L defines a map to $\mathbb{P}^2$, yet the linear system $|y\widetilde{E}|$ is 0-dimensional: this is impossible. Now assume that y is not positive. The linear system $|L|$ contains $|x(-K_{\widetilde{Z}})|$, so that L is big. This contradicts α being of fibering type. The integers x, y are coprime because $-K_{\widetilde{Z}}$ and $\widetilde{E}$ form a $\mathbb{Z}$-basis of $\operatorname{Pic}\widetilde{Z}$, L is prime and L is not an integer multiple of either of them.

   Remark 7.2.8. This proof shows that x, y are positive coprime integers when α is an extremal contraction of fibering type.

   Denote by l an effective non-singular curve that is contracted to a point by α. Then $-K_{\widetilde{Z}} \cdot l \leq 2$, and $x(-K_{\widetilde{Z}} \cdot l) = L \cdot l + y\widetilde{E} \cdot l = y\widetilde{E} \cdot l$. The divisor $\widetilde{E}$ is Cartier and therefore y divides $(-K_{\widetilde{Z}} \cdot l)x$. As x and y are coprime, this shows that y can only be equal to 1 or 2.

Let Δ be the discriminant curve of the conic bundle α. Recall that the curve Δ is linearly equivalent to $-\alpha_*(-K_{\widetilde{Z}/\widetilde{Z}_1})^2$.

$$\begin{cases} L^3 = 0 \\ L^2 \cdot (-K_{\widetilde{Z}}) = 2 \\ L \cdot (-K_{\widetilde{Z}})^2 = 12 - \deg(\Delta) \end{cases}$$

Recall that g is the genus of Y, Z and $\widetilde{Z}$. These numerical constraints, together with the intersection table (7.6) and the numerical constraints associated to ψ in Lemma 7.1.5, yield the system of equations:

$$\begin{aligned} (2g-2)x^3 - 3(-K_{Z_1} \cdot \Gamma + 2 - 2p_a(\Gamma))x^2y + 3(2p_a(\Gamma) - 2)xy^2 \\ +(-K_{Z_1} \cdot \Gamma - 2 + 2p_a(\Gamma) + e)y^3 = 0 \\ (2g-2)x^2 - 2(-K_{Z_1} \cdot \Gamma + 2 - 2p_a(\Gamma))xy \\ +(2p_a(\Gamma) - 2)y^2 = 2 \\ (2g-2)x - (-K_{Z_1} \cdot \Gamma + 2 - 2p_a(\Gamma))y = 12 - \deg(\Delta) \end{aligned}$$

2. α is a del Pezzo fibration.

Claim 7.2.9. *The curve $\widetilde{Z}_1$ is $\mathbb{P}^1$.*

If α is a del Pezzo fibration, by the Leray spectral sequence and the Kawamata-Viehweg vanishing theorem, $\widetilde{Z}_1$ is $\mathbb{P}^1$. The divisor L is $\alpha^*\mathcal{O}_{\mathbb{P}^1}(1)$. Let d be the degree of the generic fibre.

Claim 7.2.10. *The integers x and y are positive and coprime; y can only be equal to $1, 2$ or 3.*

As mentioned in Remark 7.2.8, x and y are coprime. Denote by l an effective curve of $\widetilde{Z}$ that is mapped to a point by α. Depending on the degree of the generic fibre, $-K_{\widetilde{Z}} \cdot l$ can only be $1, 2$ or 3. Moreover, $x(-K_{\widetilde{Z}} \cdot l) = L \cdot l + y\widetilde{E} \cdot l = y\widetilde{E} \cdot l$. The divisor $\widetilde{E}$ is Cartier and therefore y divides $(-K_{\widetilde{Z}} \cdot l)x$. As x and y are coprime, this shows that y can only be equal to $1, 2$ or 3.

$$\begin{cases} L^2 \cdot (-K_{\widetilde{Z}}) = 0 \\ L^2 \cdot \widetilde{E} = 0 \\ L \cdot (-K_{\widetilde{Z}})^2 = d \end{cases}$$

The following sytem of equations is associated to the configuration (ψ, α):

$$\begin{aligned}(2g-2)x^2 - 2(-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))xy +& \\ (2p_a(\Gamma)-2)y^2 &= 0\\ (-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma)) - 2(2p_a(\Gamma)-2)xy& \\ -(-K_{Z_1}\cdot\Gamma - 2 + 2p_a(\Gamma) + e)y^2 &= 0\\ (2g-2)x - (-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))y &= d\end{aligned}$$

3. α is a divisorial contraction. Recall that Y contains neither a plane with $|-K_Y|_{|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ nor an irreducible quadric with normal bundle (-1). In particular, Z_1 and $\widetilde{Z}_1$ are Gorenstein and Lemma 7.1.3 shows that α is of type E1 or E2. The 3-fold Y has Fano index 1 and Picard rank 1, so it cannot contain a plane $\mathbb{P}^2$ with $-K_{Y|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(2)$. The morphism α contracts a divisor D to a curve C.

   The contraction α is naturally induced by the contraction $\tilde{\alpha}$ of an extremal ray on a weak$*$ Fano 3-fold $\widetilde{X}$ that is a small $\mathbb{Q}$-factorialisation of $\widetilde{Z}$. The 3-folds X and $\widetilde{X}$ are related by a sequence of flops. The exceptional divisor D of the contraction α is by construction Cartier on $\widetilde{Z}$ and
$$-K_{\widetilde{Z}} = \alpha^*(-K_{\widetilde{Z}_1}) - D. \tag{7.9}$$

   The divisor $\alpha(\widetilde{E})$ is Cartier at the centre of α, except possibly at finitely many points. The formulae (7.7) and (7.9) show that y divides $x+1$. As above, I assume that the indices of Y, Z and $\widetilde{Z}$ are equal to 1. Define k by $x+1 = yk$.

   Note that the integer y is equal to the Fano index of $\widetilde{Z}_1$.

   Let C be the centre of the contraction α. The curve C is locally a complete intersection and has planar singularities. Lemma 7.1.3 shows that the Cartier divisor D satisfies the following equations.

$$\begin{cases} (-K_{\widetilde{Z}}+D)^3 = (-K_{\widetilde{Z}}+D)^2(-K_{\widetilde{Z}}) = (-K_{\widetilde{Z}_1})^3\\ (-K_{\widetilde{Z}}+D)^2D = 0\\ (-K_{\widetilde{Z}}+D)D(-K_{\widetilde{Z}}) = -K_{\widetilde{Z}_1}\cdot C = i(\widetilde{Z}_1)\deg(C)\\ (-K_{\widetilde{Z}})D^2 = 2p_a(C)-2\end{cases}$$

The following system of equations can be associated to the configuration (ψ, α).

$$\begin{aligned}
y^2[(2g-2)k^2 - 2(-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))k + 2p_a(\Gamma) - 2]& \\
= -K^3_{\widetilde{Z}_1}& \\
(2g-2)k^2(yk-1) + (-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))(2k - 3k^2y)& \\
+(2p_a(\Gamma)-2)(3ky-1) + (-K_{Z_1}\cdot\Gamma - 2 + 2p_a(\Gamma) + e)y = 0& \\
(2g-2)k(yk-1) - (-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))(2yk-1)& \\
+(2p_a(\Gamma)-2)y = \frac{i(\widetilde{Z}_1)}{y}\deg(C)& \\
(2g-2)(yk-1)^2 - 2(-K_{Z_1}\cdot\Gamma + 2 - 2p_a(\Gamma))y(yk-1)& \\
+(2p_a(\Gamma)-2)y^2 = 2p_a(C) - 2&
\end{aligned}$$

These systems of Diophantine equations have very few solutions, once a value is chosen for the genus of Y, Z and $\widetilde{Z}$. The solutions of such systems for a given value of the genus of Y exhibit all possible Sarkisov links with midpoint along Y.

7.3 Geometric Motivation of non $\mathbb{Q}$-factoriality

Let $Y_4^3 \subset \mathbb{P}^4$ be a terminal Gorenstein non $\mathbb{Q}$-factorial quartic 3-fold. There exists a Weil non-Cartier divisor on Y. On the one hand, well-known examples of non $\mathbb{Q}$-factorial quartic 3-folds contain planes or quadrics. On the other hand, a very general determinantal quartic hypersurface is known to be non $\mathbb{Q}$-factorial but to contain neither a plane nor a quadric. Yet, it does contain a Bordigo surface of degree 6. I show that Y has to contain some surface of relatively low degree. In other words, the degree of the surface lying on Y that breaks $\mathbb{Q}$-factoriality cannot be arbitrarily large.

Assume that Y does not contain a quadric or a plane. Let X be a small $\mathbb{Q}$-factorialisation of X. The 3-fold X is a weak$*$ Fano 3-fold on which a Minimal Model Program can be run.

$$\begin{tikzcd}
X = X_0 \arrow[r, "\phi_0"] \arrow[d] & X_1 \arrow[r, "\phi_1"] \arrow[d] & \cdots & X_{n-1} \arrow[r, "\phi_n"] \arrow[d] & X_n \arrow[d] \\
Y_0 & Y_1 & \cdots & Y_{n-1} & Y_n
\end{tikzcd}$$

At each step, Y_i, the anticanonical model of X_i, has Picard rank 1. Assume that the first extremal contraction $\phi\colon X \to X_1$ is not an isomorphism in codimension 1. The small $\mathbb{Q}$-factorialisation $X \to Y$ can always be chosen for this to be the case. Assume that ϕ is divisorial. As mentioned above, ϕ can only be of type E1 because Y contains neither a plane nor a quadric and Y has Fano index 1.

Let E be the exceptional divisor of ϕ and let $\overline{E}$ be the image of E by the anticanonical map. Denote by Z the Picard rank 2 symbolic blow up of Y along $\overline{E}$. Recall from Lemma 3.2.7 that there exists an extremal contraction ψ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X_1 \arrow[d] \\ Z \arrow[r, "\psi"] \arrow[d] & Z_1 \\ Y & \end{tikzcd}$$

commutative. The contraction ψ is studied in Lemmas 7.1.3, 7.1.5 and 7.1.6. Section 7.2 shows that there is a diagram of the form:

$$\begin{tikzcd} & Z \arrow[dl, "\psi"'] \arrow[dr, "h"] \arrow[rr, dashed, leftrightarrow, "\Phi"] & & \widetilde{Z} \arrow[dl, "\tilde{h}"'] \arrow[dr, "\alpha"] & \\ Z_1 & & Y & & \widetilde{Z_1} \end{tikzcd},$$

where α is an extremal contraction that is not an isomorphism in codimension 1. If α is divisorial, moreover, its exceptional divisor is Cartier. Each configuration of type (ψ, α) correspond to a solution of a system of Diophantine equations determined by the types of ψ and α. The configurations listed in the table on page 129 are the only solutions of these systems.

Remark 7.3.1. I have not listed solutions with e negative (Lemma 7.2.4).

Remark 7.3.2. We can further refine the list of solutions by eliminating the solutions such that $h^{2,1}(Z) \neq h^{2,1}(\widetilde{Z})$. The Hodge numbers themselves are not known, but as Z and $\widetilde{Z}$ have equal defect and the same analytic type of singularities, from (2.1), it is sufficient to determine that $h^{2,1}(\mathcal{Z}_t) = h^{2,1}(\widetilde{\mathcal{Z}}_t)$ for the elementary Sarkisov link on the non-singular fibre.

On the non-singular fibre, $h^{2,1}(\mathcal{Z}_t) = h^{2,1}((\mathcal{Z}_1)_t) + p_a(\Gamma_t)$ (resp. $h^{2,1}(\widetilde{\mathcal{Z}}_t) = h^{2,1}((\widetilde{\mathcal{Z}}_1)_t) + p_a(C_t))$). The terminal Gorenstein Fano 3-fold $(\mathcal{Z}_1)_t$ (resp. $(\widetilde{\mathcal{Z}}_1)_t$) is a flat degeneration of a non-singular Fano 3-fold Z_1' (resp. $\widetilde{Z}_1'$) with Picard rank 1 and the same genus. The Hodge numbers $h^{2,1}((\mathcal{Z}_1)_t)$ and $h^{2,1}((\widetilde{\mathcal{Z}}_1)_t)$ might not be easily computable, but they are bounded above by $h^{2,1}(Z_1')$ and $h^{2,1}(Z_1')$ respectively. This is sufficient to rule out cases $16, 25$ and 32 in the table on page 129.

Notation 7.3.3. In the table on page 129, I write $X_{2g-2} \subset \mathbb{P}^{g+1}$ for Fano 3-folds of Fano index 1, V_d for Fano 3-folds of Fano index 2 and Q for the unique terminal Gorenstein Fano 3-fold of Fano index 1.

The surface F is defined as the image by the anticanonical map of Z of the exceptional divisor of ψ. More precisely, the exceptional divisor E is mapped by the anticanonical map to a surface F of degree at most $-K_Z^2 \cdot E$. Lemma 7.1.5 shows that $-K_Z^2 \cdot E = -K_{Z_1} \cdot \Gamma' + 2 - 2p_a(\Gamma) = i(Z_1)\deg(\Gamma) + 2 - 2p_a(\Gamma)$. If α is also an E1 contraction with exceptional divisor D, F is the image of D or of E, depending on which one has smallest degree.

Theorem 7.3.4 (Main Theorem 2)**.** *Let $Y_4^3 \subset \mathbb{P}^4$ be a terminal Gorenstein quartic 3-fold. Then one of the following holds:*

1. *Y is $\mathbb{Q}$-factorial.*
2. *Y contains a plane $\mathbb{P}^2$.*
3. *Y contains an irreducible reduced quadric Q.*
4. *Y contains an anticanonically embedded del Pezzo surface of degree 4.*
5. *Y has a structure of Conic Bundle over $\mathbb{P}^2$, $\mathbb{F}_0$ or $\mathbb{F}_2$.*
6. *Y contains a rational scroll $E \to C$ over a curve C whose genus and degree appear in the table on page 129.*

Proof. The only thing there is to prove is that if Y does not contain a plane and if X, a small $\mathbb{Q}$-factorialisation of Y, admits no divisorial contraction and does not have a structure of Conic bundle, then Y contains an anticanonically embedded del Pezzo surface of degree 4.

Step 1. *If Y has defect 1 and if Y is the midpoint of a link between two del Pezzo fibrations, both these fibrations have degree 4.*

Vologodsky shows in [Vol01] that if Y has defect 1 and if a two ray game with midpoint along Y involves two del Pezzo fibrations, then they have the same degree d and d is either 2 or 4. Recall from Lemma 3.2.8 that a del Pezzo fibration of degree 2 is impossible when the anticanonical ring of X is generated in degree 1.

Step 2. *If X has a structure of del Pezzo fibration of degree 4, then either Y contains a plane, or Y contains an anticanonically embedded del Pezzo surface of degree 4, and the equation of Y can be written:*

$$Y = \{a_2 q + b_2 q' = 0\} \subset \mathbb{P}^4$$

where a_2, b_2, q and q' are homogeneous forms of degree 2 on $\mathbb{P}^4$.

Let F be a general fibre of $X \to \mathbb{P}^1$. The fibre F is a non-singular del Pezzo surface of degree 4 and $-K_F = -K_{X|F}$. The linear system $|-K_X|_{|F}$ is a subsystem of $|-K_F|$. The restriction of the anticanonical map to F factorises as $\Phi_{|-K_X|_{|F}} = \nu \circ \Phi_{|-K_F|}$ where ν is the projection from a (possibly empty) linear subspace

$$\mathbb{P}(H^0(F, -K_F)) \simeq \mathbb{P}^4 \dashrightarrow \mathbb{P}(H^0(F, |-K_X|_{|F}))$$

associated to the inclusion of linear systems $|-K_X|_{|F} \subset |-K_F|$. As X is a weak$*$ Fano 3-fold, $|-K_X|$ is basepoint free and the image of F is a surface. The morphism ν can only be the identity or the projection either from a line not meeting $\Phi_{|-K_F|}(F)$ or from a point not lying on $\Phi_{|-K_F|}(F)$.

Note that if ν is not the identity, as $h^0(-K_F) = h^0(-K_X) = 5$, the map i appearing in the long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(X, -K_X - F) \to H^0(X, -K_X) \xrightarrow{i} H^0(F, -K_F) \to \\ \to H^1(X, -K_X - F) \to 0 \end{aligned}$$

is not surjective. In particular, $H^0(X, -K_X - F)$ is not trivial. There is a hyperplane section of Y that contains $\Phi_{|-K_X|}(F)$. As this holds for the general fibre F, the fibration $X \to \mathbb{P}^1$ is induced by a pencil of hyperplanes on Y.

Case 3. *If $H^0(X, -K_X - F) \neq (0)$ for the general fibre F, Y contains a plane.*

The rational map $Y \dashrightarrow \mathbb{P}^1$ is determined by a pencil of hyperplanes $\mathcal{H}$. Without loss of generality, we may assume that $\mathcal{H}$ is the pencil $\mathcal{H}_{(\lambda:\mu)}) = \{\lambda x_0 + \mu x_1 = 0\}$ for $(\lambda : \mu) \in \mathbb{P}^1$. The anticanonical map $X \to Y$ is small and $X \to \mathbb{P}^1$ is a del Pezzo fibration; the map $X \to Y$ is a resolution of the base locus of the pencil $\mathcal{H}$ on Y. As the map $X \to Y$ is small, the pencil $\mathcal{H}$ has a base component on Y; This is only possible if Y contains the plane $\Pi = \{x_0 = x_1 = 0\} = \operatorname{Bs}\mathcal{H}$ itself.

Case 4. *Assume that $H^0(X, -K_X - F)$ is trivial, then Y contains an anticanonically embedded non-singular del Pezzo surface S of degree 4, that is the intersection of two quadric hypersurfaces in $\mathbb{P}^4$.*

The equation of S is $\{q(x_0, \dots, x_4) = q'(x_0, \dots, x_4) = 0\}$, where q and q' are homogeneous forms of degree 2. The equation of Y is:

$$Y = \{a_2 q + b_2 q' = 0\} \subset \mathbb{P}^4$$

with a_2 and b_2 homogeneous forms of degree 2. Geometrically, the structures of del Pezzo fibrations on small $\mathbb{Q}$-factorialisations of Y arise as the maps induced by pencils of quadrics (eg $\mathcal{L} = \{q, q'\}$ amd $\mathcal{M} = \{a_2, b_2\}$) after blowing up their base locus on Y, which are anticanonically embedded del Pezzo surfaces of degree 4. Considering for example the unprojection of Y with variables of weight 0:

$$t = \frac{q}{q'} = -\frac{b_2}{a_2}$$
$$t' = \frac{q}{b_2} = -\frac{q'}{a_2}$$

and X (resp. X') the blow-up of X along S (resp. along $S' = \{q = b_2 = 0\}$), there is a diagram

$$\begin{tikzcd}
& X \arrow[rr, dashed] \arrow[dl] \arrow[dr] & & X' \arrow[dl] \arrow[dr] & \\
\mathbb{P}^1 & & Y & & \mathbb{P}^1
\end{tikzcd}$$

The 3-fold X (resp. X') lies on $Q \times \mathbb{P}^1$ (resp. $Q' \times \mathbb{P}^1$) for $Q \subset \mathbb{P}^4$ (resp. Q') a quadric that is the proper transform of $\{a_2 = 0\}$ under the blow-up of $\mathbb{P}^4$ along S (resp. S'). The 3-fold X (resp. X') is the section of a linear system $|2M + 2F|$ on $Q \times \mathbb{P}^1$ (resp. $Q' \times \mathbb{P}^1$), where $M = p_1^*\mathcal{O}_Q(1)$ (resp. $M = p_1^*(\mathcal{O}_{Q'}(1)))$ and $F = p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$.

These have natural structures of del Pezzo surfaces of degree 4 that correspond to $t = \frac{t_0}{t_1}$ and $t' = \frac{t_0}{t_1}$. The map $X \dashrightarrow X'$ is a flop in the lines that are the preimages of the points defined by $\{q = q' = a_2 = b_2 = 0\}$. $\square$

	Z_1	$\widetilde{Z_1}$	$p_a(\Gamma)$	$\deg(\Gamma)$	α	max deg F
1	X_{22}	X_{22}	0	8	E1, $p_a(C) = 0$, $\deg(C) = 8$	10
2	X_{22}	V_5	1	9	E1, $p_a(C) = 1$, $\deg(C) = 9$	9
3	X_{22}	X_{22}	2	10	E1, $p_a(C) = 2$, $\deg(C) = 10$	8
4	X_{22}	$\mathbb{P}^2$	2	10	Conic Bundle,deg $\Delta = 4$	8
5	X_{22}	X_{12}	3	11	E1, $p_a(C) = 0$, $\deg(C) = 3$	5
6	X_{18}	X_{18}	0	6	E1, $p_a(C) = 0$, $\deg(C) = 6$	8
7	X_{18}	V_4	1	7	E1, $p_a(C) = 1$, $\deg(C) = 7$	7
8	X_{18}	X_{18}	2	8	E1, $p_a(C) = 2$, $\deg(C) = 8$	6
9	X_{18}	$\mathbb{P}^2$	2	8	Conic Bundle, deg $\Delta = 6$	6
10	X_{16}	Q	0	5	E1, $p_a(C) = 3$, $\deg(C) = 9$	7
11	X_{16}	X_{16}	1	6	E1, $p_a(C) = 1$, $\deg(C) = 6$	6
12	X_{16}	$\mathbb{P}^1$	1	6	Del Pezzo fibration of degree 6	6
13	X_{16}	X_8	2	7	E1, $p_a(C) = 0$, $\deg(C) = 1$	3
14	X_{16}	V_4	2	7	E1, $p_a(C) = 5$, $\deg(C) = 9$	5
15	X_{14}	X_{14}	0	4	E1, $p_a(C) = 0$, $\deg(C) = 4$	6
16	X_{14}	Q	1	5	E1, $p_a(C) = 9$, $\deg(C) = 11$	5
17	X_{14}	V_3	1	5	E1, $p_a(C) = 1$, $\deg(C) = 5$	5
18	X_{12}	X_{22}	0	3	E1, $p_a(C) = 3$, $\deg(C) = 11$	5
19	X_{12}	$\mathbb{P}^3$	0	3	E1, $p_a(C) = 7$, $\deg(C) = 9$	5
20	X_{12}	X_{12}	1	4	E1, $p_a(C) = 1$, $\deg(C) = 4$	4
21	X_{12}	$\mathbb{P}^1$	1	4	Del Pezzo fibration of degree 4	4
22	X_{10}	X_{10}	0	2	E1, $p_a(C) = 0$, $\deg(C) = 2$	4
23	X_{10}	V_5	0	2	E1, $p_a(C) = 7$, $\deg(C) = 12$	4
24	X_{10}	V_2	1	3	E1, $p_a(C) = 1$, $\deg(C) = 3$	3
25	X_{10}	$\mathbb{P}^3$	1	3	E1, $p_a(C) = 15$, $\deg(C) = 11$	3
26	X_8	$\mathbb{P}^2$	0	1	Conic Bundle, deg $\Delta = 7$	3
27	X_8	X_{16}	0	1	E1, $p_a(C) = 2$, $\deg(C) = 7$	3
28	V_2	$\mathbb{P}^1$	1	3	Del Pezzo fibration of degree 6	6
29	V_2	X_{16}	1	3	E1, $p_a(C) = 1$, $\deg(C) = 6$	6
30	V_3	$\mathbb{P}^3$	3	12	E1, $p_a(C) = 3$, $\deg(C) = 8$	20
31	V_5	$\mathbb{P}^1$	9	13	Del Pezzo fibration of degree 6	6
32	Q	X_{22}	12	12	E1, $p_a(C) = 0$, $\deg(C) = 8$	10

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