

## Remark on pion scattering lengths

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First it is shown that the tree amplitude for pion pion scattering in the minimal linear sigma model has an exact expression which is proportional to a geometric series in the quantity  $(s-m_\pi^2)/(m_B^2-m_\pi^2)$ , where  $m_B$  is the sigma mass which appears in the Lagrangian and is the only a priori unknown parameter in the model. This induces an infinite series for every predicted scattering length in which each term corresponds to a given order in the chiral perturbation theory counting. It is noted that, perhaps surprisingly, the pattern, though not the exact values, of chiral perturbation theory predictions for both the isotopic spin 0 and isotopic spin 2 s-wave pion-pion scattering lengths to orders  $p^2$ ,  $p^4$  and  $p^6$  seems to agree with this induced pattern. The values of the  $p^8$  terms are also given for comparison with a possible future chiral perturbation theory calculation. Further aspects of this approach and future directions are briefly discussed.

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### I. INTRODUCTION

The chiral perturbation theory approach [1]-[4] provides a systematic method for improving the "current algebra" or tree level "non-linear chiral Lagrangian" results for low energy QCD in powers of a characteristic squared momentum,  $p^2$  (or number of derivatives). Intuitive understanding of the resulting physics in some cases has been obtained by computing the amplitudes of interest based on pole-dominance. For example vector meson dominance is known to be good at low energies; a typical well known immediate prediction gives the squared charge radius of the pion simply as  $r_\pi^2 = 6/m_\rho^2$ . This kind of approach may be theoretically justified to some extent by invoking the  $1/N$  expansion of QCD [5], [6] which yields tree level dominance.

In the case of the pion s-wave scattering lengths, the long controversial, but now apparently accepted, sigma particle would appear to play the role of the rho meson. However, a simple sigma dominance approximation is not viable because it would not guarantee the nearly spontaneous breakdown of chiral symmetry mechanism which is crucial for QCD. Such a mechanism is guaranteed by the use of a linear sigma model of some type. The properties of the sigma suggest that it may be a four quark (i.e.  $qq\bar{q}\bar{q}$ ) state of some kind or a mixture of four quark and two quark components [7]. In such an instance the  $1/N$  expansion would not hold in its usual form [8] and those models have a lot of subtleties. Here, we will not discuss such questions but will just point out that the minimal SU(2) linear sigma model [9] provides a useful approximation to the lightest sigma of a model which may contain a number of them. A crucial effect is that the linear sigma model has an important contact term. The actual low energy scattering is known to result from an enormous cancellation between the sigma pole and the contact contributions. This unpleasant feature is mitigated in the non-linear sigma model (which forms the basis of the chiral perturbation scheme). Another way to mitigate this feature is at the amplitude level. Then the amplitude is expanded [10], [11] in a Taylor series about  $s = m_\pi^2$  and the cancellation may be explicitly made. The result is proportional to a simple geometric series in the variable  $(s - m_\pi^2)/(m_B^2 - m_\pi^2)$ . Then in order to compare it with something, it is natural to compare it with another power series in squared momentum- chiral perturbation theory.

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## II. EXPANDED SCATTERING LENGTHS

With the Mandelstam notation, the invariant pion scattering amplitude computed at tree level in the minimal SU(2) linear sigma model reads:

$$A(s, t, u) = \frac{2(m_B^2 - m_\pi^2)}{F_\pi^2} \left[ \left(1 - \frac{s - m_\pi^2}{m_B^2 - m_\pi^2}\right)^{-1} - 1 \right], \quad (1)$$

where  $F_\pi = 131$  MeV and  $m_B$  denotes the ‘‘bare’’ sigma mass which appears in the Lagrangian.

This equation is seen to contain a contact term as well as a pole term which has been rewritten for convenience. In this form it is apparent that there is a geometric series expansion in powers of  $(s - m_\pi^2)/(m_B^2 - m_\pi^2)$ , which should be rapidly convergent for  $s$  close to the pion-pion threshold:

$$A(s, t, u) = \frac{2(s - m_\pi^2)}{F_\pi^2} \left[ 1 + \frac{s - m_\pi^2}{m_B^2 - m_\pi^2} + \frac{(s - m_\pi^2)^2}{(m_B^2 - m_\pi^2)^2} + \frac{(s - m_\pi^2)^3}{(m_B^2 - m_\pi^2)^3} + \dots \right]. \quad (2)$$

Actually, a similar expansion may be derived when a number of different scalar mesons are present [11]. In that instance the lowest lying scalar meson is expected to dominate near threshold.

The isospin 0 scattering length is proportional to  $3A(s, t, u) + A(t, s, u) + A(u, t, s)$  evaluated at  $s = 4m_\pi^2, t = u = 0$  while the isospin 2 scattering length is obtained by evaluating  $A(t, s, u) + A(u, t, s)$  instead. Then we find for the ‘‘dimensionless’’ s-wave scattering lengths:

$$m_\pi a_0^0 = \frac{m_\pi^2}{16\pi F_\pi^2} \left[ 7 + 29 \frac{m_\pi^2}{m_B^2 - m_\pi^2} + 79 \frac{m_\pi^4}{(m_B^2 - m_\pi^2)^2} + 245 \frac{m_\pi^6}{(m_B^2 - m_\pi^2)^3} + \dots \right], \quad (3)$$

and,

$$m_\pi a_0^2 = -\frac{m_\pi^2}{8\pi F_\pi^2} \left[ 1 - \frac{m_\pi^2}{m_B^2 - m_\pi^2} + \frac{m_\pi^4}{(m_B^2 - m_\pi^2)^2} - \frac{m_\pi^6}{(m_B^2 - m_\pi^2)^3} + \dots \right]. \quad (4)$$

Evidently, these terms may be consecutively interpreted as  $p^2, p^4, p^6$ , and  $p^8$  etc. contributions.

## III. NUMERICAL COMPARISON

The chiral perturbation theory results to the first three orders [12], [13],[14] as well as the comparison with experiment may be conveniently read from Fig. 10 of [15]. We have subtracted the values presented there to get the incremental corrections for comparison with Eqs.(3) and (4). The order  $p^2$  entries [12] in Table I are of course the same and we made the choice  $m_\pi = 140$  MeV to enforce this feature. The only unfixed parameter in the linear sigma model is the bare sigma mass,  $m_B$  which we chose to be 550 MeV to give a  $p^4$  contribution to the resonant partial wave scattering length which approximately agrees with chiral perturbation theory at that order. (Alternatively, a similar value can be found on an a priori basis by fitting the near threshold I=0, s-wave scattering data).

Order:	$p^2$	$p^4$	$p^6$	$p^8$
$m_\pi a_0^0$ in ChPT:	0.16	0.04	$0.02 \pm 0.005$	-
$m_\pi a_0^0$ in LSM:	0.159	0.046	0.009	0.0019
$m_\pi a_0^2$ in ChPT:	-0.046	0.004	$-0.002 \pm 0.001$	-
$m_\pi a_0^2$ in LSM:	-0.0454	0.0031	-0.0002	0.000015

TABLE I: Comparison of scattering length increments

We notice that the increments to  $m_\pi a_0^2$  are predicted to alternate in sign with increasing order. This pattern manifestly agrees with what was found in the first three orders of chiral perturbation theory.

If the  $p^4$  increment of  $m_\pi a_0^0$  is taken as approximately a common input, the magnitude of the  $p^4$  increment to  $m_\pi a_0^2$  is predicted to be about 75 percent of the chiral perturbation theory one. Also the magnitude of the  $p^6$  increment to  $m_\pi a_0^0$  is predicted to be about 50 percent of the chiral perturbation theory one. Finally, the magnitude of the  $p^6$

increment to  $m_\pi a_0^0$  could be about 20 percent of the chiral perturbation theory one (which contains a large uncertainty however). Thus it seems fair to say that the tree level linear sigma model result exactly reproduces the signs of the chiral perturbation amplitudes and tracks well the magnitudes. It will be interesting to compare the predicted  $p^8$  increments given above when the chiral perturbation theory calculation is carried to that order.

Differences between the chiral perturbation results for the s-wave scattering lengths and the present ones may be evidently interpreted physically as due to contributions from effects other than the existence of the sigma meson. It is likely that the next most important effects should arise from including the rho meson and a higher mass scalar meson like the  $f_0(980)$  in the formulation of the chiral invariant linear sigma model. Work in this direction is under way.

#### IV. DISCUSSION

Different recent discussions of the pion scattering lengths in the linear sigma model are given in [16], [17], [18].

The main new feature in the present approach seems to be the realization that the use of the simplest linear sigma model at tree level does not give just one number (a scattering length) but gives an infinite series of numbers which can be conveniently compared with the series resulting from chiral perturbation theory.

Another amusing feature is that this approach provides a specific model for the expansion parameter of this series; namely  $m_\pi^2/(m_B^2 - m_\pi^2)$ .

Of course, in comparison with chiral perturbation theory, there is an obvious difference in that the latter approach includes the effect of loop integrals. The loop integrals enforce that chiral perturbation theory carried to all orders should result in fully unitarized scattering amplitudes. In the present approach it is possible to obtain exactly unitary partial wave amplitudes without introducing any new parameters by means of the K-matrix technique. For example, in a 3 flavor linear sigma model [19], the resonant partial wave amplitude up to about 1.2 GeV was simply fit by including just the sigma and  $f_0(980)$  scalar mesons together with K-matrix unitarization. An approach equivalent to K matrix unitarization for the 2 flavor linear sigma model was described in [20]. It is easy to see [21] that K matrix unitarization actually does not change the predicted value of the scattering length from the one determined at tree level.

It is well known that to accurately model low energy pion physics it is necessary to take the rho meson into account in addition to the sigma. So certainly, the next step in the present approach is to investigate the power series expansion of the pion scattering amplitude computed from a linear sigma model in which the rho meson as well as an axial vector meson (for chiral symmetry) are included. This can be expected to improve the agreement with the chiral perturbation theory expansion. (Of course the pion scattering has been treated from other points of view in *non-linear* models containing both scalars and vectors; see for examples, [22]-[25]).

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