Sequential Screening and Renegotiation

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Abstract

This paper considers a sequential screening problem. A seller sells an object to a buyer who is privately informed about the object's value. The value has two components. The buyer knows the first component when he contracts with the seller and learns the second component only later. The optimal contract when there is no commitment problem is a sequential mechanism in form of a menu of fee-price pairs. Paying the initial fee gives the buyer the right to purchase the good later at the corresponding price. High initial buyer types pay a high fee for a low price later. Each buyer chooses a different fee-price pair. If commitment is not feasible, the structure of the optimal contract is simpler. The optimal contract is either no contract, a simple forcing contract, or a contract in which high types buy for sure and low types pay an initial fee to buy the good at a price later. The difference to the setting with commitment is that all low buyer types obtain the same fee-price pair and all high buyer types buy for sure. There is no fine-tuning to specific buyer types. This might explain some simple real life sales agreements and why firms might find it optimal to group consumers into specific customer groups.

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1 Introduction

This paper studies a seller-buyer problem in which the buyer acquires private information over time. Specifically, we consider a seller's attempt to sell a good to a buyer with private information about his valuation for the good. This valuation has two components: One that the buyer knows at the time of initial contracting with the seller, and one that he observes only at a later point in time. Therefore, the buyer-seller relationship involves a sequential screening problem. We characterize the optimal contract under two different contracting environments: full-commitment and renegotiation.

As a benchmark, consider first the full-commitment case, in which the seller can commit not to renegotiate any of the contractual terms once the initial contract has been signed. The seller will optimally set up a sequential mechanism taking the form a menu of fee-price pairs. Each of these pairs is a different call option: By choosing a particular pair, the buyer pays the fee and gets the right to buy the good at the specified (strike) price later, once he has observed the second component of his valuation. Obviously, higher fees correspond to lower strike prices. The optimal mechanism discriminates between all buyer types: In equilibrium, buyers with different initial valuations (i.e. first components) select different fee-price pairs. Also, buyers with higher initial valuations choose higher-fee/lower-price pairs. Sequential price discrimination of this form is common practice in a variety of contexts including fidelity cards, book clubs or plane tickets.

Consider next the renegotiation case, in which after the initial contract is signed, the seller can make a new offer after the agent has observed his valuation's second component. Because no further information arrives, the revised offer is final and optimally takes the form of a single take-it-or-leave-it price offer.

The inability to commit to renegotiate the initial contract is a binding constraint on the buyer-seller relationship. Indeed, as is well known, a contract designed optimally to elicit hidden information usually becomes sub-optimal once this information is revealed. The uninformed party (here the seller) will therefore try to renegotiate the initial contract. This is anticipated by the informed party, making satisfying incentive compatibility more difficult. This so-called ratchet effect has been studied in a variety of circumstances.

We characterize the optimal contract under the simplifying assumption that the initial contract allocates the good non-stochastically. In this context, the optimal contract has a surprisingly simple structure and can take one of three possible simple forms. First, the seller might decide not to offer an initial contract at all and simply wait until the agent has observed the second component to make him a take-it-or-leave it price offer. Second, he might write a forcing contract in which he sells the good to every buyer type for the same fixed price. Finally, he might offer the buyer the choice between only two options: Buying the good immediately for a set price, or buying a call option. The option's strike price must be renegotiation-proof, i.e., optimal once the seller has received all the relevant reports from the buyer. Importantly, there is only one call option, not a whole menu as in the full-commitment case. In equilibrium, buyers divide into two groups: Those whose initial valuation is high enough simply buy the good, while all others buy the (same) call option.

Note that, as in the full-commitment case, the third type of contract involves call options. However, it implements much less fine-tuning to specific buyer types. Indeed, all low initial valuation buyers get the same call option and all high initial valuation buyers buy the good for sure. The coarseness of the optimal contract might explain the simplicity of some real life sales agreements and why firms often group consumers into specific customer categories rather than treat each customer on an individual basis.

The literature on screening with sequentially released information is small. It includes Courty and Li (2000) in static contracting environments, and Baron and Besanko (1984), in dynamic environments. These papers study the full-commitment case and make use of the Revelation Principle. Blume (1998), extending Hart and Tirole (1988), studies a buyer-seller relationship with time varying valuations and contract renegotiation. The author retains a persistent component for the buyer's valuation and in addition introduces a transient component. By assumption the seller does not want to screen this transient component. This is in contrast to the model in this paper, where the seller always has an incentive to ex-post screen the total valuation of the buyer. Kraehmer and Strausz (2007) contains a similar set-up as this paper but the buyer has to invest into acquiring information about the second component of his valuation. There is also no commitment problem.

The paper proceeds as follows. Section 2 presents the model. Sections 3 and 4 characterize the optimal contract under full commitment and renegotiation respectively. The last section concludes. Proofs are in the appendix.

2 The Model

A seller considers selling one unit of a good to a potential buyer. The buyer has a total valuation v for the good, which is made up of two components v_1 and v_2. For simplicity we assume an additive structure: $v = v_1 + v_2$.[1] The two variables v_1 and v_2 are both private information to the buyer and can be thought of as two different realizations of uncertainty or aspects of the state of nature that both influence the buyer's enjoyment of the good. Importantly, v_1 is realized at some point in time *before* v_2. We call v_1 the buyer's first stage type and v_2 his second stage type. His final type is v. We assume that both variables v_1 and v_2 are drawn independently on the intervals $[\underline{v}_i, \overline{v}_i]$ according to two commonly known cumulative distribution functions $F_i(\cdot)$ with strictly positive densities $f_i(\cdot)$, $i = 1, 2$. In addition, we assume that both distributions satisfy the monotone likelihood ratio property. The distributions on the v_i's induce a probability distribution of the final value v on the interval $[\underline{v}_1 + \underline{v}_2, \overline{v}_1 + \overline{v}_2]$, which can be written as

$$F(v) := \int_{\underline{v}_1}^{v-\underline{v}_2} \int_{\underline{v}_2}^{v-v_1} dF_2(v_2)\, dF_1(v_1). \tag{1}$$

The seller's production costs are fixed and normalized to 0. We assume that $\underline{v}_1 + \underline{v}_2 \geq 0$, which implies that the Pareto optimal solution involves selling to all final buyer types. The seller makes all contracting offers.

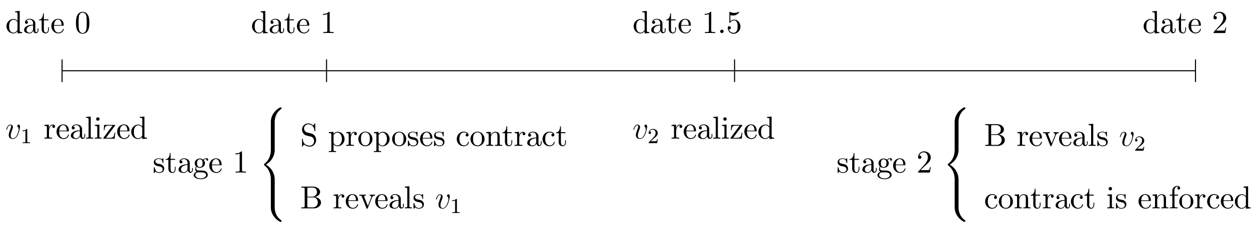
3 The Benchmark: Full Commitment

Consider first the set-up with full commitment. The revelation principle allows us to reduce the seller's problem to finding a two-stage direct revelation mechanism, in which the buyer announces v_1 at stage 1 and v_2 at stage 2. Conditional on his two announcements he receives the good with probability $\beta(v_1, v_2) \in \{0, 1\}$[2] and pays a price $p(v_1, v_2)$. Price and probability of trade are chosen to maximize the seller's

[1] To extend the analysis to a framework in which the valuation is a more general function of two consecutively realized states of nature is beyond the scope of this paper. It might be interesting material for future research.

[2] In general, we would like to allow for $\beta \in [0, 1]$. For the benchmark case with commitment the assumption that $\beta \in \{0, 1\}$ makes no difference because the optimal solution has either $\beta = 0$ or $\beta = 1$. For the case with renegotiation it constitutes a real constraint but it allows us to greatly simplify the analysis.

expected surplus subject to the constraint that the buyer tells the truth about both v_1 and v_2, and that, conditional on v_1, he receives a payoff of at least 0 in expectation. The timing is as follows:



For future reference, consider what happens if no contract is signed at date 1 and the seller tries selling his product only at date 2. At that date he faces a continuum of buyer types v distributed on $[\underline{v}_1 + \underline{v}_2, \overline{v}_1 + \overline{v}_2]$. Using a standard argument one can show that he optimally picks a price q to

$$\max_q q(1 - F(q)), \tag{2}$$

where $F(q)$ is given by (1). Call the solution to this problem v^* and assume that v^* is unique.

Let us now turn to the full problem. It turns out that the following relatively simple contract implements the optimal direct revelation mechanism: At date 1, the buyer pays an initial fee that is dependent on his type v_1, call it $A(v_1)$. This gives him the right to purchase the good at date 2 for a price $p(v_1)$. Obviously, he will only exercise this option if his final valuation $v_1 + v_2$ lies above the price $p(v_1)$.

Proposition 1 *With full commitment the seller offers a menu of contracts* $\{A(v_1), p(v_1)\}$, $v_1 \in [\underline{v}_1, \overline{v}_1]$. *The buyer selects a pair* $(A(v_1), p(v_1))$ *at date* 1 *and pays* $A(v_1)$. *At date* 2 *he decides whether or not to buy the good at the price* $p(v_1)$.

The seller's optimization problem is formalized below. For this we need the following notation. Set $U(v_1, v_2) = \beta(v_1, v_2)(v_1 + v_2) - p(v_1, v_2)$. This is the utility of a final $v_1 + v_2$-type buyer who announces both v_1 and v_2 truthfully at the relevant screening stages. In what follows it will be also useful to consider two other levels of utility. The first, $U(v_1, v_2, \hat{v}_1) = \beta(\hat{v}_1, \hat{v}_2)(v_1 + v_2) - p(\hat{v}_1, \hat{v}_2)$, is the utility of a final $v_1 + v_2$-type buyer who announces $\hat{v}_1$ at the first revelation stage and $\hat{v}_2$ at the second stage, where $\hat{v}_2$ is given by

$$\hat{v}_2(v_1, v_2, \hat{v}_1) = \arg\max_x \beta(\hat{v}_1, x)(v_1 + v_2) - p(\hat{v}_1, x). \tag{3}$$

The second, $U(v_1, v_2, \hat{v}_2) = \beta(v_1, \hat{v}_2)(v_1 + v_2) - p(v_1, \hat{v}_2)$, is the utility of a final $v_1 + v_2$-type buyer who announces v_1 truthfully at the first stage and then chooses some announcement $\hat{v}_2$ at the second revelation stage. Finally, $\bar{U}(v_1, \hat{v}_1) := \int_{\underline{v}_2}^{\overline{v}_2} U(v_1, v_2, \hat{v}_1)\, dF_2(v_2)$ is the expected utility of an initial v_1-type buyer who announces $\hat{v}_1$ and then chooses his second announcement according to 3. Set $\bar{U}(v_1) := \bar{U}(v_1, v_1)$.

With these notations in place, the seller's program can be written as

$$\max_{\beta, U} \int_{\underline{v}_1}^{\overline{v}_1} \int_{\underline{v}_2}^{\overline{v}_2} [\beta(v_1, v_2)(v_1 + v_2) - U(v_1, v_2)]\, dF_2(v_2)\, dF_1(v_1) \tag{4}$$

s.t.

$$(IC_2) \qquad U(v_1, v_2) \geq U(v_1, v_2, \hat{v}_2), \qquad \forall\, v_1, v_2, \hat{v}_2 \tag{5}$$

$$(IC_1) \qquad \int_{\underline{v}_2}^{\overline{v}_2} U(v_1, v_2)\, dF_2(v_2) \geq \bar{U}(v_1, \hat{v}_1), \qquad \forall\, v_1, \hat{v}_1 \tag{6}$$

$$(IR) \qquad \int_{\underline{v}_2}^{\overline{v}_2} U(v_1, v_2)\, dF_2(v_2) \geq 0 \qquad \forall\, v_1 \tag{7}$$

Line (5) formalizes the incentive constraint at the final screening stage. Given that v_1 is revealed truthfully at the first screening stage (IC_2) ensures that v_2 is also revealed truthfully. Line (6) contains the incentive constraint at the initial screening stage. The left-hand side is a v_1-type buyer's expected utility when he announces both his types truthfully in the two consecutive revelation stages. The right-hand side is his expected utility when he announces some other type $\hat{v}_1$ in the first stage and then chooses his second stage announcement $\hat{v}_2$ according to (3). Line (7) formalizes a v_1-type buyer's individual rationality constraint. He must receive at least his reservation utility in expectation if he tells the truth in both stages.

The proof of Proposition 1 can be found in the appendix. It uses some variations of the techniques developed by Mirlees (1971) to replace both (IC_1) and (IC_2) with the relevant local first-order and monotonicity conditions. The solution to this contracting problem has some interesting features. It is optimal for the seller to trade with all types v, such that

$$v \geq \frac{1 - F_1(v)}{f_1(v)}$$

First, remark that the solution is "bang-bang" although there is some non-linearity introduced through the expectation operator in the incentive constraint (IC_I). The solution is similar to a static problem in which the good is sold if and only if $v \geq \frac{1-F(v)}{f(v)}$. In the sequential model total valuation must lie above $\frac{1-F_1(v_1)}{f_1(v_1)}$, the hazard rate of the first variable's distribution function, because only the first variable is known when the contract offer is made. Also, the allocation depends on the realization of the second variable but the price at which the good is sold only depends on the first variable. This is in contrast to what happens without commitment as will be seen in the next section.

The model is a special case of Courty and Li (2000), a version of which they discuss as an example. The authors study general sequential screening problems with commitment when the buyer has some private information with respect to the distribution of his total valuation. Here, he has no better information about the distribution but he knows the support. In Courty and Li (2000) the support is fixed and therefore a slightly different proof must be employed.

4 Sequential Screening with No Commitment

We now consider the situation in which the seller is not committed by his date 1 contract and can instead try and renegotiate the contractual terms with the buyer once new information has arrived, that is, once v_2 is realized. Renegotiation could in principle happen anytime at or after date 2. The implicit assumption here is, that although a long-term commitment (i.e. from date 1 to date 2) is not possible for the seller, he can commit in the short-term: A contract that is signed and enforced within the same time period is binding. Due to the Revelation Principle, a new contract offer can without loss of generality be taken to be a direct mechanism, in which the buyer truthfully reveals his final type. This is not so with the date 1 contract. The Revelation Principle breaks down due to the lack of commitment and it is not immediately obvious what type of contract is optimal at that date. Nevertheless, a version of the Revelation Principle that is due to Bester and Strausz (2000) applies in this setting. Their paper shows that in a private information setting with a single agent (buyer) and direct communication (no transmission/garbling devices) any mechanism without commitment can without loss of generality be taken to be a direct revelation mechanism in which the agent sends messages about his type to the principle (seller). He does not necessarily tell the truth:

he can mix between announcements. So, in our setting we take a date 1 contract to be composed of two stages. It is described by two message spaces M_1 and M_2[3] with typical elements m_1 and m_2, and for each message pair (m_1, m_2) a probability of trade $\beta(m_1, m_2)$ and a price $p(m_1, m_2)$. Message m_1 is sent by the buyer after the contract offer at stage 1, message m_2 is sent by him at stage 2 after he has learnt v_2. To simplify notation, we write $M := M_1 \times M_2$.

There are two points in time at which renegotiation could enter into this setting, namely, either *before* or *after* the prescriptions of the date 1 contract have been carried out. To distinguish the two possible timings we will call the former *interim renegotiation* and the latter *ex-post renegotiation.* Since the date 1 contract has its own time line (stage 1, stage 2) interim renegotiation can also happen at different points in time. More precisely, the seller could offer a new contract before stage 2, that is, before the buyer sends message m_2, or after stage 2 but before the contract is enforced. Whether one or the other timing is more reasonable and whether they deliver different results is not clear. To cut through this complication and to gain a first insight into what happens with renegotiation in a sequential setting, we concentrate instead on ex-post renegotiation. Ex-post renegotiation happens right at the end of contract 1, when all the prescriptions of the contract have been carried out. At that point, the seller either faces a buyer who has already bought the object and, since this is the efficient outcome, there is no more to renegotiate. Alternatively, he faces a buyer who has not bought the object and so the seller makes him a final offer. This offer is just a take-it-or-leave-it offer with a new price, which we call the renegotiation price. Of course, this new price depends on the information the buyer has revealed during the date 1 contract.

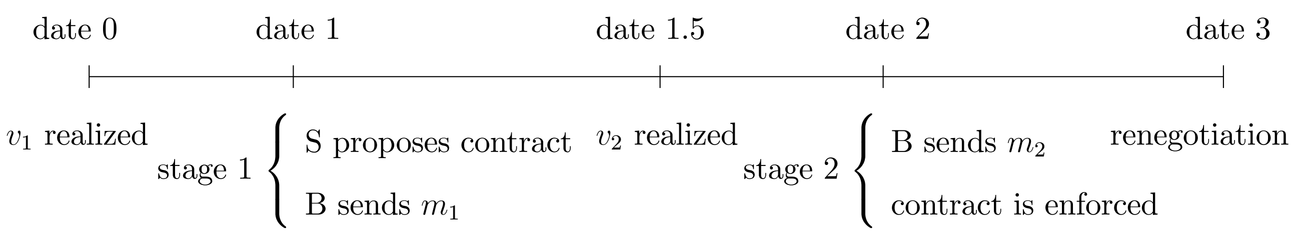
Note, that the different types of renegotiation imply slightly different assumptions on the parties' commitment abilities. With interim renegotiation, there is little commitment between the dates. A date 1 contract can be written but there is no guarantee that its prescriptions will be carried out. With ex-post renegotiation, there is commitment to the terms of the original contract in the sense that it will be carried out and enforced by a court. But a court cannot prevent the seller to reopen trade in case the original contract led to an inefficient outcome.

[3] In principle, the second message space could depend on the first message and should be denoted by $M_2(m_1)$. To simplify notation we suppress this dependency and simply write M_2. As it turns out the optimal solution does not make use of this possibility.

We concentrate on ex-post renegotiation although this is not how renegotiation is commonly modelled in the literature. Nevertheless, we believe that it provides interesting results in its own right, and that it constitutes an important first step in analyzing renegotiation in a sequential setting. Future research should be carried out on interim renegotiation and on how the two renegotiation regimes compare. Finally, this can lead to a discussion about the timing of renegotiation and how different limits to commitment affect welfare.

4.1 Ex-Post Renegotiation

With ex-post renegotiation the time line is as follows:



The overall game consists of the contract offer at date 1, which determines the message games played at stages 1 and 2, and the final renegotiation offer at date 3. The solution concept is that of Perfect Bayesian Nash Equilibrium.

The seller's strategy in the game is as follows. First, he designs the date 1 contract. That is, he chooses the two message spaces M_1 and M_2 and the probability of trade $\beta(m_1, m_2)$ and the price $p(m_1, m_2)$. Second, the seller sets a renegotiation price at date 3. This is denoted by $q(m_1, m_2)$. It is dependent on the message pair (m_1, m_2) because the seller updates his belief about the buyer's type after receiving the two messages.

The buyer's strategy in the game consists of a pair of probability distributions (μ_1, μ_2), μ_1 is a probability distribution over message space M_1, and μ_2 is a probability distribution over message space M_2. More precisely, $\mu_1(m_1 \mid v_1)$ is the probability with which a v_1-type buyer sends message m_1 at stage 1, and $\mu_2(m_2 \mid v_1, v_2, m_1)$ is the probability with which a $v_1 + v_2$-type buyer who has sent message m_1 at stage 1 sends message m_2 at stage 2. There is no need to explicitly account for the buyer's acceptance decision of the date 1 contract, because this is done via the individual rationality constraint in the seller's optimization problem. Neither do we need to model the buyer's acceptance decision of the renegotiation price at date 3: He accepts any price offer q that is lower than his final valuation.

It is convenient to write the problem as in section 3. We define the utility that a final v-type buyer receives under the date 1 contract, that is, *before* the final renegotiation offer as:

$$U(m_1, m_2, v) := \beta(m_1, m_2)v - p(m_1, m_2).$$

We now turn to the analysis of this game. As stated above, at date 3 the seller offers a renegotiation price $q(m_1, m_2)$ that maximizes $q(1 - F(q \mid m_1, m_2))$, where $F(\cdot \mid m_1, m_2)$ is the seller's updated belief about the buyer's final type after having received the message pair (m_1, m_2), arrived at via Baye's Rule. A buyer type v accepts this offer if and only if $v \geq q(m_1, m_2)$. We now analyze the message sending stages.

4.2 Message Sending Stages

Fix a contract $[M_1, M_2, \beta(\cdot, \cdot), p(\cdot, \cdot)]$ with induced renegotiation price $q(\cdot, \cdot)$. The first lemma states that, if the seller's belief (i.e. his renegotiation offer) is independent of the buyer's messages then the contract itself cannot be made dependent on those messages.

Lemma 1 *Assume that* $q(m_1, m_2) \equiv q \ \forall (m_1, m_2)$. *Then,* $q = v^*$, *where* v^* *is the solution to (2), and for almost all message pairs* (m_1, m_2), *the contract has* $\beta(m_1, m_2) \equiv \beta$ *and* $p(m_1, m_2) = p$.

It remains to investigate the situation in which the seller, depending on the buyer's messages, sets different prices at the renegotiation stage. For this we proceed in two steps. We first investigate how the contract depends on the message at stage 2 and then on how it depends on the message at stage 1. For this, fix the first message m_1. The next Lemma shows that even if the second stage message affects the seller's belief and his final renegotiation offer, the contract depends in only a very limited way on the second message.

Lemma 2 *For a fixed* m_1, *assume that the seller sets different renegotiation prices* $\{q_1, q_2, ...\}$ *depending on the buyer's message at stage* 2. *Without affecting final payoffs, the contract can be assumed to contain only two messages at stage* 2, m_2 *and* $\hat{m}_2$, *with*

$q(m_1, m_2) = q := \min q_i$ and $q(m_1, \hat{m}_2) = \hat{q} := \min_{q_i \neq q} q_i$. Also, $\beta(m_1, m_2) \leq 1$, $\beta(m_1, \hat{m}_2) = 1$, and $p(m_1, \hat{m}_2) = p(m_1, m_2) + (1 - \beta(m_1, m_2))q$. Final buyer types $v \geq q$ send both messages m_2 and $\hat{m}_2$ and obtain a level of ex-post utility equal to $U(m_1, v) = v - p(m_1, \hat{m}_2)$. Final buyer types $v < q$ only send message m_2 and obtain a level of ex-post utility equal to $U(m_1, v) = \beta(m_1, m_2)v - p(m_1, m_2)$.

Lemma 2 allows us to simplify the prescriptions of the contract. Set $\beta := \beta(m_1, m_2)$ and $p := p(m_1, m_2)$. Then, Lemma 2 tells us that the buyer sends a message $m_1 \in M_1$ at stage 1 and either message m_2 or message $\hat{m}_2$ at stage 2. If he sends message $\hat{m}_2$ he obtains the good under the date 1 contract for a price $p + (1 - \beta)q$. If he sends message m_2 he obtains the good with probability β at the price p. After message $\hat{m}_2$ there is nothing more to renegotiate, after message m_2, if the good is not sold under the contract the seller offers to sell it at q. Thus, all final buyer types above q (independent of the message they send) obtain $v - p - (1 - \beta)q$ and final buyer types below q obtain $\beta v - p$.

We now turn to the incentive constraints imposed by the first message sending stage. For this, it is convenient to derive expressions for a first stage buyer type's expected utility from sending message m_1. Following the above result and maintaining the same notation, a first stage buyer type v_1 expects the following payoff from a message m_1:

$$\begin{aligned} \bar{U}(m_1, v_1) &= \int_{\underline{v}_2}^{q-v_1} \beta(v_1 + v_2) - p \, dF_2(v_2) + \int_{q-v_1}^{\overline{v}_2} v_1 + v_2 - p - (1 - \beta)q \, dF_2(v_2) \\ &= v_1 + E[v_2] - p - (1 - \beta) \int_{\underline{v}_2}^{\overline{v}_2} \min(v_1 + v_2, q) \, dF_2(v_2) \\ &: = v_1 + E[v_2] - p - (1 - \beta)\Phi(v_1, q). \end{aligned}$$

Note, that $\Phi(v_1, q)$ is positive and strictly increasing in v_1 and q. The next Lemma details how the contract can depend on the first message. To simplify, we first assume that there are only two possible first stage messages.

Lemma 3 *Take two different first stage messages m_1 and m_1' with $q < q'$. Then, the date 1 contract can have two possible forms. Either $\beta = \beta' = 1$ and $p = p'$, that is, the contract is a simple forcing contract. Alternatively, the buyer's equilibrium strategy at the first message sending must be a partition equilibrium strategy, that is, there exists a first stage buyer type $v_1^{\lim}$, such that all first stage buyer types in $V := [\underline{v}_1, v_1^{\lim}]$ send* *message* m_1 *and all first stage buyer types in* $V' := [v_1^{\lim}, \overline{v}_1]$ *send message* m_1'*. In this case, the additional constraints on the contract are*

$$p' - p = (1-\beta)\Phi(v_1^{\lim}, q) - (1-\beta')\Phi(v_1^{\lim}, q') \quad (8)$$

$$(1-\beta)\int_{q-v_1'}^{q-v_1} F_2(v_2)\,dv_2 \geq (1-\beta')\int_{q'-v_1'}^{q'-v_1} F_2(v_2)\,dv_2. \qquad \forall v_1 \in V,\ v_1' \in V' \quad (9)$$

The incentive constraint in (9) implies that we only need to consider two possible solution pairs: $(\beta, \beta') = (0, 1)$ and $(\beta, \beta') = (1, 1)$.

5 Contract Offer Stage

Last, we need to compute the contract that maximizes the seller's expected payoff. We first consider the constraints imposed on a contract if the renegotiation offer does not vary with the buyer's messages. Then, Lemma 1 tells us that independent of the buyer's messages the contract prescribes a probability of trade β and the buyer pays a price p. The seller's expected revenue is

$$p + (1-\beta)(1 - F(v^*))v^*,$$

he obtains the price p and the expected renegotiation revenue $(1 - F(v^*))v^*$ if the good is not sold under the date 1 contract.

To maximize this revenue, the lowest first stage buyer type is kept at his reservation utility. The reservation utility of a type v_1 buyer is the expected payoff he receives from no contract at date 1 followed by the seller's final offer at date 3, which consists of the take-it-or-leave-it price offer v^*. Formally, it is

$$R(v_1, v^*) = \int_{v^*-v_1}^{\overline{v}_2} v_1 + v_2 - v^*\, dF_2(v_2).$$

Then, the lowest first stage buyer type's binding participation constraint can be written as

$$\beta(\underline{v}_1 + E[v_2]) - p + (1-\beta)R(\underline{v}_1, v^*) = R(\underline{v}_1, v^*).$$

Substituting for p in the seller's objective function we obtain

$$\beta\Phi(\underline{v}_1, v^*) + (1-\beta)(1 - F(v^*))v^*.$$

The first part of this expression is the price that he receives under a contract, the second part is the expected price he receives at renegotiation. The optimal contract in this case is either a forcing contract ($\beta = 1$, $p = \Phi(\underline{v}_1, v^*)$) or no contract ($\beta = p = 0$). The latter generates a revenue of $(1 - F(v^*))v^*$.

Now, let us consider a contract as detailed in Lemma 2.

The seller's objective function is

$$\int_{\underline{v}_1}^{v_1^{\lim}} \left(p + (1-\beta) \int_{q-v_1}^{\overline{v}_2} q \, dF_2(v_2) \right) dF_1(v_1) \tag{10}$$
$$+ \int_{v_1^{\lim}}^{\overline{v}_1} \left(p' + (1-\beta') \int_{q'-v_1}^{\overline{v}_2} q' \, dF_2(v_2) \right) dF_1(v_1).$$

The first part of (10) is the expected payment he receives from first stage buyer types below $v_1^{\lim}$. They send message m_1 and pay a price p under the contract. With probability $1-\beta$ trade does not occur before the final offer and all final buyer types $v_1 + v_2 > q$ accept the renegotiation offer q. The second part of (10) is the expected payment that the seller receives from first stage buyer types above $v_1^{\lim}$. Using (8) we can replace p' in (10):

$$p + [(1-\beta)\Phi(v_1^{\lim}, q) - (1-\beta')\Phi(v_1^{\lim}, q')](1 - F_1(v_1^{\lim}))$$
$$+(1-\beta) \int_{\underline{v}_1}^{v_1^{\lim}} \int_{q-v_1}^{\overline{v}_2} q \, dF_2(v_2) \, dF_1(v_1) + (1-\beta') \int_{v_1^{\lim}}^{\overline{v}_1} \int_{q'-v_1}^{\overline{v}_2} q' \, dF_2(v_2) \, dF_1(v_1).$$

From the lowest first stage buyer type's binding participation constraint we obtain

$$\underline{v}_1 + E[v_2] - p - (1-\beta)\Phi(\underline{v}_1, q) = R(\underline{v}_1 v^*)$$

and substituting for p in the seller's objective function we derive the following linear expression in $1-\beta$ and $1-\beta'$

$$\Phi(\underline{v}_1, v^*)$$
$$+(1-\beta)\left(\Phi(v_1^{\lim}, q)(1 - F_1(v_1^{\lim})) - \Phi(\underline{v}_1, q) + \int_{\underline{v}_1}^{v_1^{\lim}} \int_{q-v_1}^{\overline{v}_2} q \, dF_2(v_2) \, dF_1(v_1) \right)$$
$$+(1-\beta')\left(-\Phi(v_1^{\lim}, q')(1 - F_1(v_1^{\lim})) + \int_{v_1^{\lim}}^{\overline{v}_1} \int_{q'-v_1}^{\overline{v}_2} q' \, dF_2(v_2) \, dF_1(v_1) \right)$$

If the solution is $(\beta, \beta') = (1, 1)$ the contract is a forcing contract as discussed above.

If the solution is $(\beta, \beta') = (0, 1)$, this expression reduces to

$$\Phi(\underline{v}_1, v^*) - \Phi(\underline{v}_1, q) + \int_{v_1^{\text{lim}}}^{\overline{v}_1} \Phi(v_1^{\text{lim}}, q)\, dF_1(v_1) + \int_{\underline{v}_1}^{v_1^{\text{lim}}} \int_{q-v_1}^{\overline{v}_2} q\, dF_2(v_2)\, dF_1(v_1) \qquad (11)$$

Here, one can see two possible benefits of contracting for the seller. Because the contract allows the seller to credibly commit to a lower final price offer q after message m_1, he can ex-ante extract the possibly positive rent $\Phi(\underline{v}_1, v^*) - \Phi(\underline{v}_1, q)$ that the lowest first stage buyer type obtains from this decrease. Next, a contract might ensure trade with some high first stage buyer types who would not have traded for sure without a contract. First stage buyer types above v_1^{lim} pay a lower price than without a contract because $\min(\hat{v}_1 + v_2, q) < v^*$, but more buyer types accept trade.

From the above discussion it should be clear, that we do not need to consider more than two messages at the first stage: Take any other renegotiation offer q'' with associated message m_1''. If $q'' > q$, a similar proof as the one for Lemma 4 shows that $(\beta, \beta', \beta'') \in \{(0, 1, 1), (1, 1, 1)\}$ and the seller's revenue is as above. If $q'' < q$ the same argument shows that now $(\beta, \beta', \beta'') \in \{(1, 1, 0), (1, 1, 1)\}$ and again the same payoffs are achieved, where q'' takes the place of q.

The above discussion is summarized in the following proposition:

Proposition 2 *The seller has three contract choices: He can write no contract and generate a revenue of $(1 - F(v^*))v^*$. He can write a forcing contract and generate a revenue of $\Phi(\underline{v}_1, v^*)$. Finally, he can design a contract that splits the first stage buyer types into two groups. The 'eager' group ($V' = [v_1^{\text{lim}}, \overline{v}_1]$) obtains the good for sure at the price $p' = \Phi(\underline{v}_1, v^*) - \Phi(\underline{v}_1, q) + \Phi(v_1^{\text{lim}}, q)$. The 'hesitant' group ($V = [\underline{v}_1, v_1^{\text{lim}}]$) buys the right to buy the good at a price q for an initial fee $p = \Phi(\underline{v}_1, v^*) - \Phi(\underline{v}_1, q)$. The seller generates the revenue given in (11). In this case he chooses v_1^{lim} and q to maximize (11), under the constraint that q is a consistent renegotiation offer given v_1^{lim}.*

6 Conclusion

In this paper we discussed a contracting problem in which information is released sequentially over time. We studied two contracting scenarios, the first with full commitment and the second without commitment. We have shown that with full commitment, the contracting solution is a menu of options with different strike prices that distinguishes between every initial informational type. Without commitment the optimal contract is much coarser. It is either a non-discriminatory contract that either sells to nobody, or sells to everybody at the same price, or it distinguishes only between low valuation and high valuation types.

There are some questions that remain open and are interesting for further research. First, we only considered ex-post renegotiation. Another timing that is interesting to look at is interim renegotiation. Interim renegotiation could happen either after the realization of v_2 but before message m_2 is sent, or after message m_2 is sent but before the prescriptions of the contract are carried out. It is important to determine whether the results we obtained in this paper are sensitive to the timing of renegotiation. Second, we considered non-stochastic contracts and an extension to $\beta \in [0, 1]$ should be discussed in future work.

7 Appendix

Proof. (Proposition 1)

First, we consider the (IC_2) constraint. Following a standard argument it can be replaced by a local first-order condition with respect to v_2 and a monotonicity condition, that is,

$$\left.\frac{\partial U(v_1, v_2, \hat{v}_2)}{\partial \hat{v}_2}\right|_{\hat{v}_2 = v_2} = 0$$

$$\frac{\partial \beta(v_1, v_2)}{\partial v_2} \geq 0.$$

The monotonicity condition is neglected in what follows and we check ex-post that the derived solution satisfies this condition. Using the local first-order condition, a $v_1 + v_2$-type buyer's utility can be written as

$$U(v_1, v_2) = U(v_1, \underline{v}_2) + \int_{\underline{v}_2}^{v_2} \beta(v_1, y)\, dy. \tag{12}$$

The utility level $U(v_1, \underline{v}_2)$ for each v_1 needs to be determined which is done by looking at (IC_1). We show that (IC_1) can be replaced by the following two conditions

$$\left.\frac{\partial \bar{U}(v_1, \hat{v}_1)}{\partial \hat{v}_1}\right|_{\hat{v}_1 = v_1} = 0 \tag{13}$$

$$\frac{\partial \bar{\beta}(v_1)}{\partial v_1} \geq 0, \tag{14}$$

where $\bar{\beta}(v_1) := \int_{\underline{v}_2}^{\overline{v}_2} \beta(v_1, v_2)\, dF_2(v_2)$. The first condition (13) is a local first-order condition which is necessary for $\hat{v}_1 = v_1$ to be the optimal announcement at the first stage. It implies that

$$\frac{d\bar{U}(v_1)}{dv_1} = \left.\frac{\partial \bar{U}(v_1, \hat{v}_1)}{\partial v_1}\right|_{\hat{v}_1 = v_1} = \left.\int_{\underline{v}_2}^{\overline{v}_2} \beta(\hat{v}_1, \hat{v}_2)\, dF_2(v_2)\right|_{\hat{v}_1 = v_1} = \int_{\underline{v}_2}^{\overline{v}_2} \beta(v_1, v_2)\, dF_2(v_2), \tag{15}$$

where the last equality follows because $\hat{v}_2(v_1, v_2, v_1) = v_2$.

The second condition (14) is a monotonicity condition that replaces the second-order condition of the buyer's maximization problem with respect to $\hat{v}_1$. To see that (14) is necessary, consider two possible types v_1 and $\hat{v}_1$. For the (IC_I) constraint to hold for both types, we need

$$\bar{U}(v_1) \geq \bar{U}(v_1, \hat{v}_1) = \bar{U}(\hat{v}_1) + \bar{\beta}(\hat{v}_1)(v_1 - \hat{v}_1)$$

$$\bar{U}(\hat{v}_1) \geq \bar{U}(\hat{v}_1, v_1) = \bar{U}(v_1) + \bar{\beta}(v_1)(\hat{v}_1 - v_1).$$

The two together imply

$$\bar{\beta}(v_1)(v_1 - \hat{v}_1) \geq \bar{\beta}(\hat{v}_1)(v_1 - \hat{v}_1),$$

which implies (14).

For sufficiency, assume that (13) and (14) hold but that (IC_1) is violated for two types v_1 and $\hat{v}_1$. Assume, w.l.o.g., that $v_1 > \hat{v}_1$. Then,

$$\bar{\beta}(\hat{v}_1)(v_1 - \hat{v}_1) > \bar{U}(v_1) - \bar{U}(\hat{v}_1),$$

or equivalently

$$\int_{\hat{v}_1}^{v_1} \bar{\beta}(\hat{v}_1)dx > \int_{\hat{v}_1}^{v_1} \bar{\beta}(x)dx,$$

where the right-hand side follows from (13). But this is in contradiction with (14) since $\bar{\beta}(\hat{v}_1) \leq \bar{\beta}(x)$ for all $x \in [\hat{v}_1, v_1]$. If $v_1 < \hat{v}_1$, a similar argument works.

Again we neglect the monotonicity condition and use (15) to obtain

$$\bar{U}(v_1) = \bar{U}(\underline{v}_1) + \int_{\underline{v}_1}^{v_1} \int_{\underline{v}_2}^{\overline{v}_2} \beta(x, v_2)\, dF_2(v_2)dx. \tag{16}$$

The lowest first stage buyer type is optimally kept at his reservation level and therefore $\bar{U}(\underline{v}_1) = 0$. Combining (12) and (16), using $\bar{U}(v_1) = \int_{\underline{v}_2}^{\overline{v}_2} U(v_1, v_2)\, dF_2(v_2)$ we obtain

$$\int_{\underline{v}_1}^{v_1} \int_{\underline{v}_2}^{\overline{v}_2} \beta(x, v_2)\, dF_2(v_2)dx = U(v_1, \underline{v}_2) + \int_{\underline{v}_2}^{\overline{v}_2} \int_{\underline{v}_2}^{v_2} \beta(v_1, y)\, dy\, dF_2(v_2),$$

and by changing order of integration on the left-hand side

$$U(v_1, \underline{v}_2) = \int_{\underline{v}_2}^{\overline{v}_2} \left[\int_{\underline{v}_1}^{v_1} \beta(x, v_2)dx - \int_{\underline{v}_2}^{v_2} \beta(v_1, y)\, dy \right] dF_2(v_2).$$

Replacing this expression in (12) we obtain

$$\begin{aligned} U(v_1, v_2) \quad = \quad & \int_{\underline{v}_2}^{\overline{v}_2} \left[\int_{\underline{v}_1}^{v_1} \beta(x, z)\, dx - \int_{\underline{v}_2}^{z} \beta(v_1, y)\, dy \right] dF_2(z) \\ & + \int_{\underline{v}_2}^{v_2} \beta(v_1, y)\, dy. \end{aligned}$$

This expression can be substituted into the seller's objective function in (4) and the seller's optimization problem becomes

$$\begin{aligned} \max_{\beta} \int_{\underline{v}_1}^{\overline{v}_1} \int_{\underline{v}_2}^{\overline{v}_2} & \left[\beta(v_1, v_2)(v_1 + v_2) - \int_{\underline{v}_2}^{\overline{v}_2} \int_{\underline{v}_1}^{v_1} \beta(x, z)\, dx\, dF_2(z) \right. \\ & \left. + \int_{\underline{v}_2}^{\overline{v}_2} \int_{\underline{v}_2}^{z} \beta(v_1, y)\, dy\, dF_2(z) - \int_{\underline{v}_2}^{v_2} \beta(v_1, y)\, dy \right] dF_2(v_2)\, dF_1(v_1). \end{aligned}$$

Because the first term in the second line is independent of v_2 the line can be written equivalently as

$$\int_{\underline{v}_1}^{\overline{v}_1}\int_{\underline{v}_2}^{\overline{v}_2}\int_{\underline{v}_2}^{z}\beta(v_1,y)\,dy\,dF_2(z)\,dF_1(v_1)-\int_{\underline{v}_1}^{\overline{v}_1}\int_{\underline{v}_2}^{\overline{v}_2}\int_{\underline{v}_2}^{v_2}\beta(v_1,y)\,dy\,dF_2(v_2)\,dF_1(v_1),$$

so it is equal to 0.

The second term in the first line is also independent of v_2 and can therefore be written as

$$\int_{\underline{v}_1}^{\overline{v}_1}\int_{\underline{v}_2}^{\overline{v}_2}\int_{\underline{v}_1}^{v_1}\beta(x,z)\,dx\,dF_2(z)\,dF_1(v_1).$$

Thus, by changing the order of integration the seller's overall problem can be equivalently written as

$$\max_{\beta}\int_{\underline{v}_2}^{\overline{v}_2}\int_{\underline{v}_1}^{\overline{v}_1}\left[\beta(v_1,v_2)(v_1+v_2)-\int_{\underline{v}_1}^{v_1}\beta(x,v_2)\,dx\right]dF_1(v_1)\,dF_2(v_2).$$

Using integration by parts for the second part of this expression we obtain

$$\begin{aligned}\int_{\underline{v}_1}^{\overline{v}_1}\int_{\underline{v}_1}^{v_1}\beta(x,v_2)\,dx\,dF_1(v_1) &= \int_{\underline{v}_1}^{v_1}\beta(x,v_2)\,dx\cdot F_1(v_1)\Big|_{\underline{v}_1}^{\overline{v}_1}-\int_{\underline{v}_1}^{\overline{v}_1}\beta(v_1,v_2)F_1(v_1)\,dv_1\\ &= \int_{\underline{v}_1}^{\overline{v}_1}\beta(v_1,v_2)(1-F_1(v_1))\,dv_1.\end{aligned}$$

So the problem becomes:

$$\max_{\beta}\int_{\underline{v}_2}^{\overline{v}_2}\int_{\underline{v}_1}^{\overline{v}_1}\beta(v_1,v_2)\left(v_1+v_2-\frac{1-F_1(v_1)}{f_1(v_1)}\right)dF_1(v_1)\,dF_2(v_2).$$

Pointwise maximization yields the cut-off rule

$$\beta^*(v_1,v_2)=\begin{cases}1 & \text{if} \quad v_1+v_2\geq\frac{1-F_1(v_1)}{f_1(v_1)}\\ 0 & \text{otherwise.}\end{cases}.$$

The probability of trade is monotonically increasing in v_2 without any further assumption and so the second order-condition of IC_2 is satisfied. Given the MLRP of $F_1(\cdot)$, the probability of trade is also monotonically increasing in v_1, which implies that it is increasing in expectation. So also the second order condition of IC_2 is satisfied. An initial v_1-type buyer's expected utility is given by (16):

$$\bar{U}(v_1)=\int_{\underline{v}_1}^{v_1}\int_{\min\left(\underline{v}_2,\frac{1-F_1(x)}{f_1(x)}\right)}^{\overline{v}_2}dF_2(v_2)dx$$

Finally, it is easy to see that this allocation can be achieved by the menu of contracts $\{A(v_1), p(v_1)\}_{v_1 \in [\underline{v}_1, \overline{v}_1]}$ described in the statement of the proposition. The final price is set $p(v_1) = \frac{1-F_1(v_1)}{f_1(v_1)}$ and $A(v_1)$ is given by

$$A(v_1) = \int_{\min\left(\underline{v}_2, \frac{1-F_1(v_1)}{f_1(v_1)}\right)}^{\overline{v}_2} \left(v_1 + v_2 - \frac{1-F_1(v_1)}{f_1(v_1)} \right) dF_2(v_2) - \bar{U}(v_1).$$

■

Proof. (Lemma 1)

We first introduce some notation. Call $\mu(m_1, m_2)$ the expected probability that message pair (m_1, m_2) is sent by the buyer, where expectation is taken over all first and second stage buyer types. So,

$$\mu(m_1, m_2) = \int_{\underline{v}_1}^{\overline{v}_1} \int_{\underline{v}_2}^{\overline{v}_2} \mu_1(m_1 \mid v_1)\, \mu_2(m_2 \mid v_1, v_2, m_1)\, dF_2(v_2)\, dF_1(v_1).$$

Call $\mu(m_1)$ the expected probability that message m_1 is sent by the buyer, where expectation is taken over all first stage buyer types. So,

$$\mu(m_1) = \int_{\underline{v}_1}^{\overline{v}_1} \mu_1(m_1 \mid v_1)\, dF_1(v_1).$$

The seller's updated beliefs about buyer types v_1 and v_2 after message pair (m_1, m_2) are calculated via Baye's Rule:

$$F_1(v_1 \mid m_1, m_2) = \frac{1}{\mu(m_1, m_2)} \int_{\underline{v}_1}^{v_1} \int_{\underline{v}_2}^{\overline{v}_2} \mu_1(m_1 \mid x)\, \mu_2(m_2 \mid x, v_2, m_1)\, dF_2(v_2)\, dF_1(x),$$

$$F_2(v_2 \mid m_1, m_2) = \frac{1}{\mu(m_1, m_2)} \int_{\underline{v}_1}^{\overline{v}_1} \int_{\underline{v}_2}^{v_2} \mu_1(m_1 \mid v_1)\, \mu_2(m_2 \mid v_1, y, m_1)\, dF_2(y)\, dF_1(v_1).$$

Similarly, the seller's updated belief about buyer type v_1 after message m_1 is given by

$$F_1(v_1 \mid m_1) = \frac{1}{\mu(m_1)} \int_{\underline{v}_1}^{v_1} \mu_1(m_1 \mid x)\, dF_1(x).$$

Finally, the seller's updated belief about the buyer's final buyer type v after the message pair (m_1, m_2) is given by

$$F(v \mid m_1, m_2) = \int_{\underline{v}_1}^{v-\underline{v}_2} \int_{\underline{v}_2}^{v-v_1} dF_2(v_2 \mid m_1, m_2)\, dF_1(v_1 \mid m_1, m_2)$$

and his belief after message m_1 is given by

$$F(v \mid m_1) = \int_{\underline{v}_1}^{v-\underline{v}_2} \int_{\underline{v}_2}^{v-v_1} dF_2(v_2)\, dF_1(v_1 \mid m_1).$$

We first show that $q = v^*$. Because q is the seller's renegotiation offer after every message pair (m_1, m_2), we have

$$(1 - F(q \mid m_1, m_2))q \geq (1 - F(v^* \mid m_1, m_2))v^* \qquad \forall (m_1, m_2).$$

Taking expectations over all message pairs $(m_1, m_2) \in M := M_1 \times M_2$:

$$\int_M (1 - F(q \mid m_1, m_2))q \, d\mu(m_1, m_2) \geq \int_M (1 - F(v^* \mid m_1, m_2))v^* \, d\mu(m_1, m_2),$$

which, because of Bayes' Rule, is equivalent to

$$(1 - F(q))q \geq (1 - F(v^*))v^*.$$

Since v^* is the unique maximizer of $(1 - F(v))v$, this implies $q = v^*$.

Next, we show that for any two message pairs (m_1, m_2) and (m_1, m_2') in M, except for a set of message pairs with zero measure, $\beta(m_1, m_2) = \beta(m_1, m_2')$ and $p(m_1, m_2) = p(m_1, m_2')$, so the contract does not depend on the second message.

First, we argue that for every message pair (m_1, m_2) there must exist a final buyer type v above v^*, to which the seller attaches a positive probability after receiving the message pair (m_1, m_2). That is, there must exist a first stage buyer type v_1 and a second stage buyer type v_2 with $v_1 + v_2 = v \geq v^*$, such that $F(v \mid m_1, m_2) > 0$. To see this, note that if there is no such $v \geq v^*$, the seller obtains an expected payoff of 0 by offering the renegotiation price v^*. But he obtains a strictly positive expected payoff if he chooses a lower renegotiation price and captures some of the final buyer types below v^*.

Second, it is shown that for every message pair (m_1, m_2), except for a set of messages with zero measure, there must also exist a final buyer type v strictly below v^*, to which the seller attaches a positive probability after receiving the message pair (m_1, m_2). That is, there must exist a first stage buyer type v_1 and a second stage buyer type v_2 with $v_1 + v_2 = v < v^*$, such that $F(v \mid m_1, m_2) > 0$.

To show this we first argue that because v^* maximizes $(1 - F(v \mid m_1, m_2))v$ for all message pairs (m_1, m_2), v^* must also maximize $(1 - F(v \mid m_1))v$ for all m_1. For this, fix m_1. We have for all m_2 and all v:

$$(1 - F(v^* \mid m_1, m_2))v^* \geq (1 - F(v \mid m_1, m_2))v.$$

Taking expectations over all messages m_2 we have for all v

$$\int_{M_2} (1 - F(v^* \quad | \quad m_1, m_2))v^* \, d\mu(m_2 \mid m_1)$$

$$\geq \int_{M_2} (1 - F(v \mid m_1, m_2))v \, d\mu(m_2 \mid m_1),$$

where $\mu(m_2 \mid m_1) = \mu(m_1, m_2)/\mu(m_1)$ is the expected probability that m_2 is sent at the second stage given that m_1 was sent at the first stage. Baye's Rule implies that this is equivalent to

$$(1 - F(v^* \mid m_1)v^* \geq (1 - F(v \mid m_1))v.$$

Now, divide $M = M_1 \times M_2$ into two disjoint subsets $\underline{M}$ and $\overline{M}$. $\underline{M}$ is defined such that for all $(m_1, m_2) \in \underline{M}$, the seller, after receiving the message pair (m_1, m_2), attaches a positive probability to some final buyer types below and some final buyer types above v^*. $\overline{M}$ is defined such that for all $(m_1, m_2) \in \overline{M}$, the seller, after receiving the message pair (m_1, m_2), attaches positive probability only to final buyer types above v^*, that is, for all $v < v^*$, $F(v \mid m_1, m_2) = 0$. The aim is to show that $\overline{M}$ has measure 0. Assume the contrary.

Because v^* is the seller's optimal renegotiation offer after every message pair (m_1, m_2), it must be that

$$v^* \int_{v^*}^{\overline{v}} dF(v \mid m_1, m_2) \geq (v^* - \alpha) \int_{v^*-\alpha}^{\overline{v}} dF(v \mid m_1, m_2) \qquad \forall (m_1, m_2) \in M, \ \forall \alpha > 0. \tag{17}$$

Using Bayes Rule and the definition of $\overline{M}$ we can write

$$\begin{aligned} 1 - F(v^*) &= \int_{\overline{M}} \int_{v^*}^{\overline{v}} dF(v \mid m_1, m_2) \, d\mu(m_1, m_2) + \int_{\underline{M}} \int_{v^*}^{\overline{v}} dF(v \mid m_1, m_2) \, d\mu(m_1, m_2) \\ &= \int_{\overline{M}} d\mu(m_1, m_2) + \int_{\underline{M}} \int_{v^*}^{\overline{v}} dF(v \mid m_1, m_2) \, d\mu(m_1, m_2). \end{aligned}$$

Given the assumption that $\overline{M}$ has a positive mass in M, this implies that $x := \int_{\underline{M}} \int_{v^*}^{\overline{v}} dF(v \mid m_1, m_2) \, d\mu(m_1, m_2) < 1 - F(v^*)$. By taking expectation over all (m_1, m_2) in $\underline{M}$ in (17) we obtain

$$v^* x \geq (v^* - \alpha) \left(x + \int_{\underline{M}} \int_{v^*-\alpha}^{v^*} dF(v \mid m_1, m_2) \, d\mu(m_1, m_2) \right) \qquad \forall \alpha > 0, \tag{18}$$

Again, using the definition of $\overline{M}$ and Bayes Rule we can write

$$\begin{aligned}\int_{\underline{M}} \int_{v^*-\alpha}^{v^*} dF(v &\mid m_1, m_2)\, d\mu(m_1, m_2) \\ &= \int_{\overline{M}} \int_{v^*-\alpha}^{v^*} dF(v \mid m_1, m_2)\, d\mu(m_1, m_2) + \int_{\underline{M}} \int_{v^*-\alpha}^{v^*} dF(v \mid m_1, m_2)\, d\mu(m_1, m_2) \\ &= F(v^*) - F(v^* - \alpha)\end{aligned}$$

So (18) is equivalent to

$$v^* x \geq (v^* - \alpha)\,(x + F(v^*) - F(v^* - \alpha)) \qquad \forall \alpha > 0,$$

which is equivalent to

$$x \geq v^* G(\alpha) - (F(v^*) - F(v^* - \alpha)),$$

where $G(\alpha) := \frac{F(v^*)-F(v^*-\alpha)}{\alpha}$. Now, as α approaches 0, this inequality is violated. The first part of the right-hand side approaches $v^* f(v^*)$, which is, substituting for the definition of v^*, equal to $1 - F(v^*)$, whereas the second part approaches 0. Therefore, the right-hand side approaches $1 - F(v^*) > x$.

Now, fix m_1 and consider four final buyer types x, y, x' and y' who have all sent message m_1 at the first revelation stage.[4] Types x and y send message m_2 at the second revelation stage and are such that $x \leq v^* < y$. Types x' and y' send another message m_2' and are such that $x' \leq v^* < y'$. We consider these four types' incentive constraints. Type x must prefer message m_2 over message m_2':

$$\beta(m_1, m_2)x - p(m_1, m_2) \geq \beta(m_1, m_2')x - p(m_1, m_2'). \qquad (19)$$

Type y' must prefer message m_2' over message m_2:

$$\begin{aligned}\beta(m_1, m_2')y' - p(m_1, m_2') + (1 - \beta(m_1, m_2'))(y' - v^*) \;\;&\geq \qquad (20) \\ \beta(m_1, m_2)y' - p(m_1, m_2) + (1 - \beta(m_1, m_2))(y' - v^*).&\end{aligned}$$

(19) and (20) together imply the following:

$$(\beta(m_1, m_2) - \beta(m_1, m_2'))x \geq (\beta(m_1, m_2) - \beta(m_1, m_2'))v^*,$$

[4]More precisely, there are first and second stage buyer types x_i, y_i, x_i' and y_i', $i = 1, 2$, such that $x_1 + x_2 = x$, $y_1 + y_2 = y$, $x_1' + x_2' = x'$ and $y_1' + y_2' = y'$ and x_1, y_1, x_1' and y_1' have all sent m_1.

that is, $\beta(m_1, m_2) \leq \beta(m_1, m_2')$. The same two inequalities can be written for the buyer types x' and y, where the roles of m_2 and m_2' are reversed, which imply similar constraints as (19) and (20) and thus

$$(\beta(m_1, m_2) - \beta(m_1, m_2'))v^* \geq (\beta(m_1, m_2) - \beta(m_1, m_2'))x'.$$

This implies $\beta(m_1, m_2) \geq \beta(m_1, m_2')$, and therefore $\beta(m_1, m_2) = \beta(m_1, m_2{}')$ and also $p(m_1, m_2) = p(m_1, m_2')$. We consequently simplify notation by suppressing the dependency of $\beta(m_1, m_2)$, $p(m_1, m_2)$ and $U(m_1, m_2, v)$ on m_2.

Finally, we show that for any two messages m_1 and m_1' in M_1, $\beta(m_1) = \beta(m_1')$ and $p(m_1) = p(m_1')$, so the contract does not depend on the first message either.

To see this, we turn to the incentive constraints of a first stage buyer type who sends message m_1. We compute the expected utility that he receives by sending this message. A second stage buyer type $v \geq v^*$ who has sent message m_1 obtains a payoff of

$$\beta(m_1)v + (1 - \beta(m_1))(v - v^*) - p(m_1) = (v - v^*) + U(m_1, v^*),$$

A second stage buyer type $v < v^*$ who has sent message m_1 obtains a payoff of

$$\beta(m_1)v - p(m_1) = \beta(m_1)(v - v^*) + U(m_1, v^*).$$

Therefore, a first stage buyer type v_1 expects the following payoff from message m_1:

$$\begin{aligned} \bar{U}(m_1, v_1) &= \beta(m_1) \int_{\underline{v}_2}^{v^* - v_1} (v_1 + v_2 - v^*) \, dF_2(v_2) \\ &\qquad + \int_{v^* - v_1}^{\overline{v}_2} (v_1 + v_2 - v^*) \, dF_2(v_2) + U(m_1, v^*) \\ : &= \beta(m_1)\xi(v_1) + \zeta(v_1) + U(m_1, v^*). \end{aligned}$$

Note, that $\xi(v_1) < 0$ and $\zeta(v_1) > 0$ and that both are monotonically increasing in v_1 and decreasing in v^*.

Take first the situation in which two different first stage buyer types v_1', v_1 *both* send the two messages m_1' and m_1 in equilibrium. This implies that

$$\begin{aligned} \beta(m_1')\xi(v_1') + \zeta(v_1') + U(m_1', v^*) &= \beta(m_1)\xi(v_1') + \zeta(v_1') + U(m_1, v^*), \\ \beta(m_1')\xi(v_1) + \zeta(v_1) + U(m_1', v^*) &= \beta(m_1)\xi(v_1) + \zeta(v_1) + U(m_1, v^*), \end{aligned}$$

and consequently that

$$(\beta(m_1') - \beta(m_1))\xi(v_1') = (\beta(m_1') - \beta(m_1))\xi(v_1).$$

Because $\xi(v_1') \neq \xi(v_1)$ if $v_1' \neq v_1$, this is only possible if $\beta(m_1') = \beta(m_1)$.

Next, consider the situation in which no two buyer types send the same two messages m_1' and m_1. So, there are disjoint sets $V(m_1')$, $V(m_1) \subseteq [\underline{v}_1, \overline{v}_1]$, s.t. all buyer types in $V(m_1')$ send only message m_1' and all buyer types in $V(m_1)$ send only message m_1. Then, either $\beta(m_1') = \beta(m_1)$ or $V(m_1')$ and $V(m_1)$ must be connected. Assume for example that $V(m_1')$ is not connected. Then there must exist three types $v_1' < v_1 < v_1''$ with $v_1', v_1'' \in V(m_1')$ and $v_1 \in V(m_1)$. The incentive constraints of v_1' and v_1 yield

$$(\beta(m_1') - \beta(m_1))\xi(v_1') \geq (\beta(m_1') - \beta(m_1))\xi(v_1)$$

and $\beta(m_1') \leq \beta(m_1)$. The incentive constraints of types v_1 and v_1'' together imply that

$$(\beta(m_1') - \beta(m_1))\xi(v_1'') \geq (\beta(m_1') - \beta(m_1))\xi(v_1),$$

which is only satisfied if $\beta(m_1') \geq \beta(m_1)$. Therefore, $\beta(m_1') = \beta(m_1)$ if $V(m_1')$ is not connected.

Finally, consider the situation in which $V(m_1')$ and $V(m_1)$ are two disjoint intervals with $V(m_1')$ 'below' $V(m_1)$. But then it is impossible that v^* simultaneously maximizes $(1 - F(v \mid m_1'))v$ and $(1 - F(v \mid m_1))v$. The v that maximizes $(1 - F(v \mid m_1'))v$ should in fact lie strictly below the one that maximizes $(1 - F(v \mid m_1))v$. Therefore, the fact that $V(m_1')$ and $V(m_1)$ are two disjoint intervals is inconsistent with our initial assumption that the renegotiation offer is the same after every message. This proves that $\beta(m_1') = \beta(m_1)$ from which it follows that $U(m_1', v^*) = U(m_1, v^*)$ and also $p(m_1') = p(m_1)$. ■

Proof. Lemma 2

Fix m_1. Consider q and $\hat{q}$, defined as in the statement of the Lemma, and any of the higher renegotiation prices $q_i > \hat{q}$. We consider five possible messages at the second revelation stage: m_2, m_2', which both lead to renegotiation price q, $\hat{m}_2$, $\hat{m}_2'$, which both lead to $\hat{q}$, and m_2^i, which leads to q_i. This is without loss of generality, because the naming of messages is arbitrary. Following an identical argument as in the proof of Lemma 2 one can show that there must exist final buyer types $v, v' \geq q$, such that v sends m_2 and v' sends m_2' with positive probability in equilibrium. Similarly, there must exist final buyer types $\hat{v}, \hat{v}' \geq \hat{q}$, such that $\hat{v}$ sends $\hat{m}_2$ and $\hat{v}'$ sends $\hat{m}_2'$ with positive probability in equilibrium. Finally, there must exist a final buyer type $v^i \geq q^i$, such that v^i sends m_2^i with positive probability in equilibrium

Next, we want to show that we can find v, v', such that $\hat{q} > v, v'$. Assume to the contrary that all final buyer types that send message m_2 or m_2' either lie strictly below q or above $\hat{q}$. But then, instead of setting the renegotiation price q after message m_2 or m_2' the seller can set a renegotiation price at or above $\hat{q}$. This raises his payoff because he obtains a higher price from the buyer types with whom he trades and he does not loose any buyer types by charging the higher price.

We now use the incentive constraints of the buyer types v, v', $\hat{v}$, $\hat{v}'$ and v^i to formulate restrictions on the contract. To simplify notation, we set $\beta(m_1, m_2) = \beta$, $\beta(m_1, m_2') = \beta'$, $\beta(m_1, \hat{m}_2) = \hat{\beta}$, $\beta(m_1, \hat{m}_2') = \hat{\beta}'$ and $\beta(m_1, m_2^i) = \beta^i$. Define p, p', $\hat{p}$, $\hat{p}'$ and p^i similarly.

For v to be sending message m_2 rather than $\hat{m}_2$ and for $\hat{v}$ to be sending message $\hat{m}_2$ rather than m_2 it must be that

$$\begin{aligned} \beta v + (1-\beta)(v-q) - p &\geq \hat{\beta} v - \hat{p}, \\ \hat{\beta}\hat{v} + (1-\hat{\beta}))(\hat{v}-\hat{q}) - \hat{p} &\geq \beta\hat{v} + (1-\beta)(\hat{v}-q) - p, \end{aligned}$$

which together imply

$$(1-\hat{\beta})v - (1-\beta)q \geq p - \hat{p} \geq (1-\hat{\beta})\hat{q} - (1-\beta)q.$$

So, $\hat{\beta} = 1$ and $\hat{p} = p + (1-\beta)q$. Using v and $\hat{v}'$'s incentive constraints, the same argument shows that also $\hat{\beta}' = 1$ and $\hat{p}' = p + (1-\beta)q$. Similarly, v and v^i's incentive constraints together imply $\beta^i = 1$ and $p^i = p + (1-\beta)q$. Finally, using the incentive constraints of v and v', one can show that $p + (1-\beta)q = p' + (1-\beta')q$.

Summarizing, the contract prescribes the same outcome after the message pairs $(m_1, \hat{m}_2)$, $(m_1, \hat{m}_2')$ and (m_1, m_2^i). Therefore, both messages $\hat{m}_2'$ and m_2^i can be deleted. Since q_i was chosen arbitrarily, this also implies that any other message leading to a different renegotiation price q_j can be deleted. In what follows we therefore only consider the remaining three messages m_2, m_2' and $\hat{m}_2$ and the two renegotiation prices q and $\hat{q}$.

Take any final buyer type $v \geq q$. This buyer type obtains the same payoff whether he sends message m_2, m_2' or $\hat{m}_2$:

$$\begin{array}{ccc} m_2 & m_2' & \hat{m}_2 \\ v-(1-\beta)q-p & v-(1-\beta')q-p' & v-\hat{p}. \end{array}$$

Also, the seller obtains the same payoff from this buyer type regardless of which message the latter sends:

$$\begin{array}{ccc} m_2 & m_2' & \hat{m}_2 \\ p(1-\beta)q & p' + (1-\beta')q & \hat{p}. \end{array}$$

Take any final buyer type $v < q$. This buyer type obtains the following payoffs depending on the message he sends

$$\begin{array}{ccc} m_2 & m_2' & \hat{m}_2 \\ \beta v - p & \beta' v - p' & v - \hat{p} \end{array}$$

From the expressions for $\hat{p}$ and p it follows that this buyer type gets a lower payoff from sending $\hat{m}_2$ than from sending either m_2 or m_2'. Furthermore, he sends m_2 if and only if $\beta \leq \beta'$. Because the naming of messages is arbitrary, we can assume without loss of generality that $\beta \leq \beta'$. Therefore, the message pair (m_1, m_2') generates the same final payoffs as the message pair (m_1, m_2) for the seller and for every buyer type that sends it with positive probability in equilibrium, and m_2' can consequently be deleted.

This leaves us with messages m_2 and $\hat{m}_2$ and the following ex-post utility levels. Any buyer type $v \geq q$ receives a payoff of

$$v - p - (1-\beta)q,$$

Any final buyer type $v < q$ receives a payoff of

$$\beta v - p.$$

■

Proof. Lemma 3

First, assume that there are two first stage buyer types $v_1 \neq v_1'$ who both send messages m_1 and m_1'. It will be shown that then $\beta = \beta' = 1$. Both types must be indifferent between sending either message and therefore

$$(1-\beta)\Phi(v_1', q) - (1-\beta')\Phi(v_1', q') = p' - p = (1-\beta)\Phi(v_1, q) - (1-\beta')\Phi(v_1, q')$$

Therefore, the constraint is

$$\beta(\Phi(v_1, q) - \Phi(v_1', q)) + \Phi(v_1', q) - \Phi(v_1', q') = \beta'(\Phi(v_1, q') - \Phi(v_1', q')) + \Phi(v_1, q) - \Phi(v_1, q'). \tag{21}$$

Note, that

$$\begin{aligned}
\Phi(v_1, q) &= \int_{\underline{v}_2}^{\overline{v}_2} \min(v_1 + v_2, q)\, dF_2(v_2) \\
&= \int_{\underline{v}_2}^{q-v_1} v_1 + v_2\, dF_2(v_2) + \int_{q-v_1}^{\overline{v}_2} q\, dF_2(v_2) \\
&= qF_2(q - v_1) - \int_{\underline{v}_2}^{q-v_1} F_2(v_2) dv_2 + q(1 - F_2(q - v_1)) \\
&= q - \int_{\underline{v}_2}^{q-v_1} F_2(v_2) dv_2
\end{aligned}$$

With this we obtain for (21)

$$\beta \left(\int_{q-v_1}^{q-v_1'} F_2(v_2)\, dv_2 \right) + q - q' + \int_{q-v_1'}^{q'-v_1'} F_2(v_2)\, dv_2 = \beta' \left(\int_{q'-v_1}^{q'-v_1'} F_2(v_2)\, dv_2 \right) + q - q' + \int_{q-v_1}^{q'-v_1} F_2(v_2)\, dv_2$$

and so

$$-(1-\beta) \int_{q-v_1}^{q-v_1'} F_2(v_2)\, dv_2 = -(1-\beta') \int_{q'-v_1}^{q'-v_1'} F_2(v_2)\, dv_2 \tag{22}$$

Because $F_2(\cdot)$ is strictly increasing, it follows that $\int_{q-v_1}^{q-v_1'} F_2(v_2)\, dv_2 \neq \int_{q'-v_1}^{q'-v_1'} F_2(v_2)\, dv_2$ for $q \neq q'$ and $v_1 \neq v_1'$ and therefore $\beta = \beta' = 1$.

Next, assume that there are two disjoint subsets V and V' of first stage buyer types who send messages m_1 and m_1' respectively. We want to show that *unless* V and V' are two connected intervals, $\beta = \beta' = 1$. Assume w.l.o.g. that $V = [\underline{v}_1, v_1^1] \cup [v_1^2, \overline{v}_1]$ and $V' = [v_1^1, v_1^2]$ and that $V \neq \varnothing$ and that $V' \neq \varnothing$, so that $v_1^1 \neq v_1^2$. Then, the two types v_1^1 and v_1^2 must be indifferent between sending messages m_1 and m_1' and the same argument as above with can be applied to show that $\beta = \beta' = 1$.

Finally, it is possible that V and V' are two connected, disjoint intervals. Without loss of generality: $V = [\underline{v}_1, v_1^{\lim}]$ and $V' = [v_1^{\lim}, \overline{v}_1]$. Therefore, the incentive constraint of a first stage buyer type $v_1 \in V$ together with the incentive constraint of a type $v_1' \in V'$ imply that

$$(1-\beta)\Phi(v_1', q) - (1-\beta')\Phi(v_1', q') \geq p' - p \geq (1-\beta)\Phi(v_1, q) - (1-\beta')\Phi(v_1, q')$$

So,

$$-(1-\beta)\int_{q-v_1}^{q-v_1'} F_2(v_2)\,dv_2 \geq -(1-\beta')\int_{q'-v_1}^{q'-v_1'} F_2(v_2)\,dv_2$$

Or, since $v_1 < v_1'$,

$$(1-\beta)\int_{q-v_1'}^{q-v_1} F_2(v_2)\,dv_2 \geq (1-\beta')\int_{q'-v_1'}^{q'-v_1} F_2(v_2)\,dv_2.$$

■

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