Optimal Asset Allocation with Factor Models for Large Portfolios

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Abstract

This paper characterizes the asymptotic behaviour, as the number of assets gets arbitrarily large, of the portfolio weights for the class of tangency portfolios belonging to the Markowitz paradigm. It is assumed that the joint distribution of asset returns is characterized by a general factor model, with possibly heteroskedastic components. Under these conditions, we establish that a set of appealing properties, so far unnoticed, characterize traditional Markowitz portfolio trading strategies. First, we show that the tangency portfolios fully diversify the risk associated with the factor component of asset return innovations. Second, with respect to determination of the portfolio weights, the conditional distribution of the factors is of second-order importance as compared to the distribution of the factor loadings and that of the idiosyncratic components. Third, although of crucial importance in forecasting asset returns, current and lagged factors do not enter the limit portfolio returns. Our theoretical results also shed light on a number of issues discussed in the literature regarding the limiting properties of portfolio weights such as the diversifiability property and the number of dominant factors.

JEL Classifications: C32, C52, C53, G11

Key Words: Asset allocation, Large Porftolios, Factor models, Diversification.

1 Introduction

Factor models represent a parsimonious, yet flexible way of modelling the conditional joint probability distribution of asset returns when there are a large number of assets under consideration. Prominent use of factor models initially focused on parameterizing the conditional mean, following the highly influential capital asset pricing model of Sharpe (1964) and Lintner (1965), and the arbitrage pricing theory of Ross (1976). In fact parsimony plays an even a more important role when modeling conditional covariance matrix of a large number of asset returns.

Given that the main 'rationale' for using factor models is to deal with portfolios with a large number of assets, this paper characterizes the distribution of portfolio weights, as the number of assets, N, increases without bounds, in the case of the commonly used mean-variance efficient portfolios (hereafter MV). Our analysis is confined to 'myopic' asset allocation rules, all particular cases of Markowitz (1952) theory, which are optimal only for a constant investment opportunity set. Focusing on myopic trading strategies is justified from a practical perspective in the case of large portfolios where application of dynamic asset allocation strategies can be prohibitive and is rarely tried in practice. The literature on dynamic asset allocation is often confined to a few broad asset classes, such as Treasury Bills, long term bonds and equities (see, for example, Campbell and Viceira (2002)).

A number of papers have already examined the limiting behavior of MV efficient portfolios when there are a countably infinite number of primitive assets under consideration. Chamberlain (1983) and Chamberlain and Rothschild (1983) studied the implications of no arbitrage for the MV efficient frontier as N tends to infinity. They then considered factor models and extended the arbitrage pricing theory (APT) result of Ross (1976) to the case where asset returns follow an approximate factor structure. The latter extends the exact factor model by permitting certain (limited) degree of correlation across the idiosyncratic component of asset returns. Hansen and Richard (1987) extended the static framework of Chamberlain (1983) and Chamberlain and Rothschild (1983), but did not focus on factor structures. Subsequently, Green and Hollifield (1992) clarified the relationships that exist between diversification and MV efficiency in a general setting. Employing a factor structure, these authors provided a further generalization showing that even the approximate factor structure is too stringent for the APT to hold. Whereas Chamberlain (1983) characterize diversifiability by looking at the rate at which the square norm of the portfolio weights converge to zero as N tends to infinity, Green and Hollifield (1992) characterize diversifiability in terms of sup-norm criteria. Sentana (2004) compares the statistical properties of static and dynamic factor representing portfolios, using a dynamic version of the APT.

By and large all of the above papers focused on various aspects and generalizations of the APT, under the maintained assumption of an underlying factor structure as $N \to \infty$. However, once one abstracts from the APT, a number of other interesting issues arise that have hitherto been neglected in the literature. For instance, the precise behavior of the MV portfolio weights as $N \to \infty$ has been surprisingly overlooked. Likewise, to our knowledge the statistical properties of the limit portfolio return have not been spelled out. It turns out that interesting, and in fact, somewhat counter-intuitive results arises from these investigations, in particular regarding the role played by the conditional distribution of the factors. In this paper we do not make use of no-arbitrage assumption and, therefore, do not investigate any implications for the APT, unlike Hubermann (1982), Chamberlain (1983), Chamberlain and Rothschild (1983), Stambaugh (1983), Connor (1984), Ingersoll (1984), Grinblatt and Titman (1987), Green and Hollifield (1992), Sentana (2004) among others.

We make the maintained hypothesis that the vector of asset returns is distributed according to a dynamic factor model, with a specification of the conditional variance matrix of the idiosyncratic components which is more general than the approximate factor structure of Chamberlain and Rothschild (1983). Under this assumption, the paper establishes three main results:

(a) In the limit the MV portfolios fully diversify the innovations in the common factor components of asset returns. It is well known that MV portfolios do fully diversify the idiosyncratic component of asset returns innovations, but to our knowledge it is not recognized that the same applies to the factor innovations. This is an important feature of MV portfolios with practical implications which we discuss below.

(b) The limit MV portfolio weights (the first-order limit approximation for large N of the MV portfolio weights) are functionally independent of the conditional distribution of the factors. Notice that this does not imply that the factors themselves are not important but only that their (conditional) moments are not relevant insofar the calculation of the MV portfolio weights is concerned. For example, estimation of the factors and their loadings are required for a consistent estimation of the idiosyncratic components.

(c) In the limit the MV portfolio returns are functionally independent of the current and lagged values of the common factors. The factors could play a central role for forecasting asset returns but, as N gets larger, their role vanishes in terms of their contribution to the limit portfolio returns. In other words, at any point in time in the limit as $N \to \infty$, the conditional distribution of the limit returns on MV portfolios are functionally independent of the conditional distribution of the common factor component.

Neither of the above findings strictly implies the other and are all of independent interest. Initially these results seem rather counter intuitive since it is generally believed that the factor components, being dominant, are likely to be more important in the determination of asset returns. But MV portfolios are functions of the *inverse* of the variance matrix of asset returns, and the common factor part of asset returns that generate strong cross return dependence will become turn into weak cross dependence when the inverse of the variance matrix is considered. By comparison, the idiosyncratic components of asset returns that exhibit weak cross section dependence will begin to play a central role in determination of MV portfolios as N starts to become sufficiently large. Concepts of weak and strong cross section dependence are developed in Pesaran and Tosetti (2007). In particular the concept of weak cross section dependence allows the maximum eigenvalue of the covariance matrix of the idiosyncratic component of asset returns to rise like $o(N)$. Formally, our results follows from a form of asymptotic orthogonality between the inverse of the conditional covariance matrix of asset returns and the matrix of factor loadings, newly established in this paper.

The above findings also have a number of further implications of interest that we summarize below:

(d-i) The limit MV portfolios are time-invariant unless, depending on the trading strategies, the risk free rate is time-varying and the idiosyncratic component features time-varying conditional heteroskedasticity.

(d-ii) The limit MV portfolio weights are invariant to any orthogonal rotation of the factors.

(d-iii) Primitive conditions required for full-diversification in the sup-norm sense of Green and Hollifield (1992) are established.

(d-iv) Analytical characterizations of the occurrence of negative portfolio weights and of the related issue of factor dominance, in the sense of Green and Hollifield (1992) and Jagannathan and Ma (2003), are provided.

The remainder of the paper is organized as follows. Section 2 introduces the concepts, sets out the dynamic factor model, and discusses its properties. Section 3 presents the main results with respect to the commonly used trading strategies: the global minimum-variance and the maximum expected quadratic utility portfolios. Section 4 elaborates and discusses the implications of the theoretical results. The main findings are illustrated, as an example, with respect to a single factor model in Section 5. Section 6 extends the results to two other tangency portfolios, namely the minimum-variance and the maximum expected return portfolios. Section 7 concludes. Mathematical proofs are collected in an appendix.

2 Factor model: definitions and assumptions

We assume the N-dimensional vector $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{Nt})'$ of asset returns can be characterized by the following linear dynamic factor model

$$\mathbf{r}_t = \alpha_{t-1} + \mathbf{B}\mathbf{f}_t + \varepsilon_t, \tag{1}$$

where $\mathbf{f}_t$ is the $k \times 1$ vector of possibly latent common factors, $\mathbf{B} = (\beta_1, ..., \beta_N)'$ is an $N \times k$ matrix of factor loadings, ε_t is an $N \times 1$ vector of idiosyncratic components, and the $N \times 1$ vector α_{t-1} represents the part of the conditional mean of the $\mathbf{r}_t$ that does not depend on the common factors. Throughout it will be assumed that k remains fixed as $N \to \infty$. We identify the factor model by means of the following assumptions:

Assumption 1 *(conditional mean returns)* *The vector of latent factors* $\mathbf{f}_t$ *can be decomposed into its predictable component,* $\mu_{f,t-1}$*, and the remainder* $\mathbf{u}_t$ *as*

$$\mathbf{f}_t = \mu_{f,t-1} + \mathbf{u}_t, \tag{2}$$

where $\mu_{f,t-1} = E(\mathbf{f}_t \mid \mathcal{Z}_{t-1})$*, with* $\mathcal{Z}_{t-1}$ *being the sigma-algebra induced by a* $N \times g$ *matrix of observed variates* $\{\mathbf{Z}_{t-s}, s > 0\}$*.*

$$\alpha_{t-1} = E(\mathbf{r}_t - \mathbf{B}\mathbf{f}_t \mid \mathcal{Z}_{t-1}), \tag{3}$$

$$\alpha_t,\ \mathbf{u}_s \text{ are independently distributed for all } t, s. \tag{4}$$

Under this assumption the conditional mean of asset returns is given by

$$E(\mathbf{r}_t \mid \mathcal{Z}_{t-1}, \mathbf{B}) \equiv \mu_{t-1} = \alpha_{t-1} + \mathbf{B}\mu_{f,t-1}, \tag{5}$$

and the innovations in the common components, $\mathbf{u}_t$ is a martingale difference process with respect to $\mathcal{Z}_{t-1}$.

It is worth noting that the decomposition in Assumption 1 can also be defined with respect to the sigma-algebra spanned by the unobserved information set $f_{t-s}, s > 0$. This will not affect our main conclusions, but will raise a number of additional difficulties with respect to the empirical implementation of the model.

Also Assumption 1 rules out a dynamic factor representation of asset returns (see Forni, Hallin, Lippi, and Reichlin (2000)) such as $r_{it} = \alpha_{i,t-1} + \beta_i(1 - c_i L)^{-1} u_t + \varepsilon_{it}$, where L is a lag operator and c_i differs across i. This does not seem a particularly important limitation in the case of asset pricing models where the returns are only tenuously serially correlated.

In practice, specification and estimation of μ_{t-1} could be a major empirical undertaking, particularly in the case of large portfolios. But given the focus of our analysis, in what follows we take the specification of μ_{t-1}, especially its α_{t-1} component, as given. It is important, nevertheless, to separate α_{t-1} from $\mu_{f,t}$ since the latter, as we shall see, does not enter the limit MV portfolios as N gets large.

Conditions (2), (3) and (4) together imply

$$\mathbf{r}_t = \mu_{t-1} + \mathbf{B}\mathbf{u}_t + \varepsilon_t. \tag{6}$$

Hereafter, we shall refer to (6) as the factor model with $\mathbf{u}_t$ being the $k \times 1$ vector of factor innovations without further reference to $\mathbf{f}_t$.

Regarding the factor loadings, we consider the case where the elements of $\mathbf{B}$ are random variates satisfying the following limit condition:

Assumption 2 *(factor loadings) As* $N \to \infty$

$$\frac{\mathbf{B}'\mathbf{e}}{N} \to_p \bar{\beta} \neq \mathbf{0}, \tag{7}$$

where $\mathbf{e} = (1, ..., 1)'$ *is an* $N \times 1$ *vector of ones, and* $\rightarrow_p$ *denotes convergence in probability.*

From (7) it follows that $\overline{\beta}$ represents the mean vector $E(\beta_i)$. Assumption 2 is an ergodicity assumption over the cross section. It is much weaker than the *i.i.d.* assumption typically made when considering random factor loadings. For instance, a strong sufficient condition for (7) to hold is when the factor loadings have finite second-order moments and absolutely summable cross covariances but, in fact, Assumption 2 is compatible with a much more substantial degree of (cross-sectional) dependence among the elements of β_i. The results presented in this paper can be generalized further to the case of heterogeneous yet non-random β_i.

Assumption 3 *(innovations) At any given point in time* t

$$\mathbf{u}_t \mid \mathcal{Z}_{t-1} \sim (0, \mathbf{\Omega}_{t-1}), \;\; \varepsilon_t \mid \mathcal{Z}_{t-1} \sim (0, \mathbf{H}_{t-1}), \tag{8}$$

$$\varepsilon_t \textit{ and } \mathbf{u}_t \textit{ are mutually independent}, \tag{9}$$

where $\mathbf{\Omega}_{t-1}$ *and* $\mathbf{H}_{t-1}$ *are positive definite matrices, respectively, of dimension* $k \times k$ *and* $N \times N$ *for a fixed* k *and any finite* N.

The results that follow do not depend on a particular specification of the volatility model characterizing the asset returns. Moreover, the factors can either be observable or non-observable. As a consequence, Ω_{t-1} and $\mathbf{H}_{t-1}$ could belong to the multivariate stochastic volatility class as well as to the generalized autoregressive conditional heteroskedasticity class of volatility models. Particular examples, to which Assumption 3 applies, are discussed below.

To derive the limiting behavior (as $N \rightarrow \infty$) of the various tangency portfolio weights to be considered below, we further require the following assumption:

Assumption 4 *(mixed limit conditions) At any given point in time* t *as* $N \rightarrow \infty$

$$\frac{(\mathbf{B} - \mathbf{e}\overline{\beta}')'\mathbf{H}_t^{-1}(\mathbf{B} - \mathbf{e}\overline{\beta}')}{N} \rightarrow_p \mathbf{A}_t > 0, \tag{10}$$

$$\frac{\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}}{N} \rightarrow_p \mathbf{C}_t \geq 0, \tag{11}$$

and

$$\frac{\mathbf{e}'\mathbf{H}_t^{-1}\mathbf{e}}{N} \to_p a_t > 0, \tag{12}$$

$$\frac{\mathbf{e}'\mathbf{H}_t^{-1}\alpha_t}{N} \to_p c_t, \tag{13}$$

$$\frac{\alpha_t'\mathbf{H}_t^{-1}\alpha_t}{N} \to_p d_t > 0, \tag{14}$$

where hereafter > 0 *and* ≥ 0 *means, respectively, positive definitive and positive semi-definite.*

Moreover $a_t, c_t, d_t, \mathbf{A}_t, \mathbf{C}_t$ *are* $O_p(1)$ *(element by element) such that*

$$d_t a_t - c_t^2 > 0 \text{ almost surely} \tag{15}$$

and

$$\mathbf{B} \text{ is independently distributed from both } \mathbf{H}_t,\ \alpha_t. \tag{16}$$

The common feature of the limits presented in Assumption 4 is that they involve, possibly weighted, averages of the elements of $\mathbf{H}_t^{-1}$. In particular, they impose implicitly an upper bound on the speed with which the maximum eigenvalue of $\mathbf{H}_t^{-1}$ could diverge to infinity. (Recall that the largest eigenvalue of $\mathbf{H}_t^{-1}$ coincide with the smallest eigenvalue of $\mathbf{H}_t$, by construction.) This is clearly seen from condition (12): assuming for illustrative purposes that $\mathbf{H}_t^{-1}$ is diagonal, with $h_{ii,t}^{-1}$ in the (i,i)th entry, then (12) allows $max_{1\leq i\leq N} h_{ii,t}^{-1} = o_p(N)$. Condition (11) requires a further constraint on the speed of divergence of $max_{1\leq i\leq N} h_{ii,t}^{-1}$ which can now be at most $o_p(N^{\frac{1}{2}})$. Even this case is much weaker than $max_{1\leq i\leq N} h_{ii,t}^{-1} \leq C < \infty$, for some constant C, implied by the definition of approximate factor structure (see Chamberlain and Rothschild (1983)). Green and Hollyfield (1992) were the first to note that, insofar as optimal asset allocation is concerned, a degree of cross-sectional dependence stronger than the one implied by the approximate factor structure is permitted. When $\mathbf{H}_t^{-1}$ is non-diagonal, the previous discussion applies to its largest eigenvalues.

Conditions (10) and (11) require the existence of the second-order moments of the factor loadings and impose certain constraints on the degree of cross-sectional dependence of the β_i. Note that when (7), (12) and (16) hold, then (10) is equivalent to saying that $N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B}$ has a positive definite limit. When β_i are *i.i.d.* and $\mathbf{H}_t$ is diagonal then $\mathbf{A}_t = a_t\mathbf{cov}(\beta_i)$. Concerning (16), note that $\mathbf{H}_t$ and α_t need not be, and in general will not be, mutually independent. Conditions (13) and (14) also require the elements of α_t not to grow, if any, too fast as compared with N. The limit c_t in condition (13) is bounded, in absolute value, by $(a_t\, d_t)^{\frac{1}{2}}$. The limit d_t in condition (14) is finite whenever (12) holds and $\alpha_t'\alpha_t/N$ has a finite limit. Condition (15) is not needed in the case where α_t is a non-degenerate random variable.

For some results, in particular to derive the limit distribution of the MV portfolio weights, a stronger version of Assumption 4 is needed as set out below:

Assumption 5 *(further mixed limit conditions)*
For any i and at any given point in time t, as $N \to \infty$

$$\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e}_i^{(N)} \to_p \xi_{1it}, \tag{17}$$

$$\mathbf{e}'\mathbf{H}_t^{-1}\mathbf{e}_i^{(N)} \to_p \xi_{2it}, \tag{18}$$

$$\alpha_t'\mathbf{H}_t^{-1}\mathbf{e}_i^{(N)} \to_p \xi_{3it}, \tag{19}$$

with $\| \xi_{jit} \| = O_p(1)$, for j=1,2,3, where $\mathbf{e}_i^{(N)}$ is the i^{th} column of the identity matrix $\mathbf{I}_N$ and $\| \cdot \|$ denotes the Euclidean norm.

$$N^{\frac{1}{2}} \begin{pmatrix} N^{-1}vech(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B}) & - & vech(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}') \\ N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} & - & a_t\bar{\beta} \\ N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\alpha_t & - & c_t\bar{\beta} \\ N^{-1}\mathbf{e}'\mathbf{H}_t^{-1}\mathbf{e} & - & a_t \end{pmatrix} \to_d \mathbf{N}(\mathbf{0}, \mathbf{V}_t), \tag{20}$$

for some positive semi-definite matrix $\mathbf{V}_t$, where $\to_d$ denotes convergence in distribution and $vech(\mathbf{A})$ stacks the diverse elements of a symmetric matrix $\boldsymbol{A}$ into a column vector.

Conditions (17)-(18)-(19), impose a finite upper bound to each of the columns of $\mathbf{H}_t^{-1}$ and are therefore much stronger than (10)-(12)-(13) that are expressed in terms of averages. In particular, (18) is satisfied by an approximate factor structure. Condition (20) is somewhat weaker than the other parts of Assumption 5 although, again, it allows for a smaller degree of cross-sectional dependence than the one permitted by Assumption 4. In particular, note that (20) rules out that α_t contains a common factor structure. This can be relaxed without a substantial impact on our results.

In view of (8), the factor structure (6) implies the well-known form of the asset return conditional variance-covariance matrix:

$$E\left((\mathbf{r}_t - \mu_{t-1})(\mathbf{r}_t - \mu_{t-1})' \mid \mathcal{Z}_{t-1}, \mathbf{B}\right) = \mathbf{\Sigma}_{t-1} = \mathbf{B}\mathbf{\Omega}_{t-1}\mathbf{B}' + \mathbf{H}_{t-1}. \qquad (21)$$

Thus model (6) nests the various factor models with time-varying conditional second moment proposed in the econometrics literature (see among many others Diebold and Nerlove (1989), King, Sentana, and Wadhwani (1994), Chib, Nardari, and Shephard (2002), Fiorentini, Sentana, and Shephard (2004), Connor, Korajczyk, and Linton (2006)). These papers, which focus on estimation of volatility factor models, in particular when $\mathbf{u}_t$ is not observable, all assume constant conditional first-order moments. On the other hand, the finance literature dealing with factor models-based asset allocation assumes homoskedastic factors whereby $\mathbf{\Omega}_{t-1} = \mathbf{\Omega}$, often normalized to be equal to the identity matrix (see among many others Pesaran and Timmermann (1995) and Kandel and Stambaugh (1996)). A few contributions analyze asset allocation problems allowing for volatility dynamics but impose constant conditional means (see for instance Aguilar and West (2000) and Fleming, Kirby, and Ostdiek (2001)). Only recently, a limited number of asset allocation exercises have considered time variations in both the first and second conditional moments of asset returns (see for instance Johannes, Polson, and Stroud (2002) and Han (2006)). Model (6) nests all of the above specifications although with some abuse of notation, as particular cases of our set-up we have also referred to stochastic volatility models whereby the conditional moments of $\mathbf{u}_t$ are not functions of observed information.

3 Efficient portfolios

We start with characterizing the limiting behavior of the global minimum-variance (gmv) portfolio weights, $\mathbf{w}_t^{gmv} = (w_{1t}^{gmv}, ..., w_{Nt}^{gmv})'$, defined by the optimization problem

$$\mathbf{w}_{t-1}^{gmv} = \text{argmax}_{\mathbf{w}}\left(\mathbf{w}'\mathbf{\Sigma}_{t-1}\mathbf{w}\right), \text{ such that } \mathbf{w}'\mathbf{e} = 1,$$

that yields

$$\mathbf{w}_{t-1}^{gmv} = \frac{\mathbf{\Sigma}_{t-1}^{-1}\mathbf{e}}{\mathbf{e}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{e}}. \qquad (22)$$

We refer to $\mathbf{w}_{t-1}^{gmv}$ as the `gmv` portfolio. It is well known that this portfolio does not belong to the efficient frontier, except when the conditional expected returns $\mu_{i,t-1}$ are the same across i but, with some abuse of notation, we will view it as belonging to the set of MV trading strategies. Nevertheless, this portfolio is still of interest since its implementation does not require the estimation of expected returns. Jagannathan and Ma (2003) show that, in terms of asset allocation, its out-of-sample performance is comparable with the performance of other tangency portfolios.

In the theorems that follows we suppose that $\mathbf{r}_t$ is generated according to the factor model (6), Assumptions 1,2 and 3 hold, and all the limits are taken for each t and as $N \to \infty$.

Theorem 1 *(global minimum-variance portfolio)*
(i) Let

$$\mathring{w}_{it}^{gmv} = N^{-1} \frac{\mathbf{e}_i^{(N)\prime} \mathbf{H}_t^{-1}}{a_t} \left[\mathbf{e} + a_t (\mathbf{e} \overline{\beta}' - \mathbf{B}) \mathbf{A}_t^{-1} \overline{\beta} \right], \tag{23}$$

and recall that $\mathbf{e}_i^{(N)\prime}$ *is a* $N \times 1$ *row vector of zeros except for its* i^{th} *element which is unity. When Assumptions (7), (10), (12) and (16)*

$$N \left(w_{it}^{gmv} - \mathring{w}_{it}^{gmv} \right) \to_p 0. \tag{24}$$

(ii) When, in addition to the assumptions made in (i), (17)-(18)-(20) hold

$$\begin{aligned} w_{it}^{gmv} &= \mathring{w}_{it}^{gmv} + N^{-3/2} z_{it}^{gmv} + \\ & N^{-2} b_t \left[\mathbf{e}_i^{(N)\prime} \mathbf{H}_t^{-1} \mathbf{B} \left(\mathbf{A}_t + a_t \overline{\beta} \overline{\beta}' \right)^{-1} \mathbf{\Omega}_t^{-1} \left(\mathbf{A}_t + a_t \overline{\beta} \overline{\beta}' \right)^{-1} \overline{\beta} \right] + o_p(N^{-2}), \end{aligned} \tag{25}$$

in which

$$b_t = (1 + a_t \overline{\beta}' \mathbf{A}_t^{-1} \overline{\beta}), \tag{26}$$

and z_{it}^{gmv} *is a mixture of normally distributed random variables that are only functions of* $\mathbf{B}$ *and* $\mathbf{H}_t$.
(iii) When, in addition to the assumptions made under (i), relations (11) and (13) hold:

$$\rho_t^{gmv} = \mathbf{r}_t' \mathbf{w}_{t-1}^{gmv} \quad \to_p \quad \frac{c_{t-1}}{a_{t-1}}, \tag{27}$$

$$N^{-\frac{1}{2}} \left(\frac{\mu_{\rho,t-1}^{gmv}}{\sigma_{\rho,t-1}^{gmv}} \right) \quad \to_p \quad \frac{c_{t-1}}{\sqrt{a_{t-1}}}, \tag{28}$$

where $\mu_{\rho,t-1}^{gmv} = E(\rho_t^{gmv} \mid \mathcal{Z}_{t-1})$, *and* $\sigma_{\rho,t-1}^{gmv} = \sqrt{\mathrm{var}(\rho_t^{gmv} \mid \mathcal{Z}_{t-1})}$.

Remark 1(a) The gmv portfolio weight of the i^{th} asset is, asymptotically in N, equivalent to $\mathring{w}_{it}^{gmv}$. Inspecting (23) it emerges that $\mathring{w}_{it}^{gmv}$ is functionally independent from the factors covariance matrix, $\mathbf{\Omega}_t$. Instead, it is a function of the factor loadings $\mathbf{B}$, of their first moments $\bar{\beta}$, of the mixed moment $\mathbf{A}_t$ and of the (inverse of the) idiosyncratic component covariance matrix, $\mathbf{H}_t$.
Remark 1(b) From (25) it is also easily seen that the effect of $\mathbf{\Omega}_t$ on the dispersion of the w_{it}^{gmv} around $\mathring{w}_{it}^{gmv}$ vanishes at a sufficiently rapid rate such that even the asymptotic distribution of w_{it}^{gmv} does not depend on $\mathbf{\Omega}_t$ as N tends to infinity.
Remark 1(c) The gmv portfolio becomes fully diversified with respect to the idiosyncratic as well as the factor components of asset return innovations as $N \to \infty$. Moreover, the limit portfolio return is $\mathcal{Z}_{t-1}$-adapted as well as independent of the factor component of asset returns conditional mean $\mu_{f,t-1}$.
Remark 1(d) The *ex ante* Sharpe ratio, defined by $\mu_{\rho,t-1}^{gmv}/\sigma_{\rho,t-1}^{gmv}$, diverges at the rate of $N^{\frac{1}{2}}$, unless $c_{t-1} = 0$. But it is not guaranteed that the *ex ante* Sharpe ratio in the case of gmv will diverge to plus infinity. The outcome depends on sign of c_{t-1} which is not guaranteed to be positive. This arises since gmv portfolio does not make use of expected mean returns.

Suppose now that besides the N risky assets, investors can also allocate their funds to a risk free asset with a time-varying rate of return, r_{0t}, which is known at the start of trading day t. We now consider a tangency portfolio, namely the maximum expected utility (henceforth meu) portfolio based on a mean-variance utility function.

The meu portfolio weights $\mathbf{w}_t^{meu} = (w_{1t}^{meu}, ..., w_{Nt}^{meu})'$ are defined by

$$\mathbf{w}_{t-1}^{meu} = \text{argmax}_{\mathbf{w}} \left(\mathbf{w}'\mu_{t-1} + (1 - \mathbf{w}'\mathbf{e})r_{0,t-1} - \frac{\gamma_{t-1}}{2}\mathbf{w}'\mathbf{\Sigma}_{t-1}\mathbf{w} \right),$$

where $0 < \gamma_{t-1} < \infty$ is the parameter of risk aversion (possibly time-varying), implying

$$\mathbf{w}_{t-1}^{meu} = \frac{1}{\gamma_{t-1}}\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - \mathbf{e}r_{0,t-1}). \tag{29}$$

Theorem 2 *(maximum expected utility portfolio)*
(i) Let

$$\mathring{w}_{it}^{meu} = \frac{\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}}{\gamma_t b_t} \left\{ (\alpha_t - \mathbf{e}r_{0t}) + [a_t(\alpha_t - \mathbf{e}r_{0t})\bar{\beta}' - (c_t - a_t r_{0t})\mathbf{B}]\mathbf{A}_t^{-1}\bar{\beta} \right\}. \tag{30}$$

When conditions (7), (10), (11), (12), (13), (16), (17) and (19) hold:

$$w_{it}^{meu} - \mathring{w}_{it}^{meu} \to_p 0. \tag{31}$$

(ii) When, in addition to the conditions in (i), (20) also holds:

$$w_{it}^{meu} = \mathring{w}_{it}^{meu} + N^{-1/2} z_{it}^{meu} + \tag{32}$$
$$N^{-1} \left\{ \gamma_t^{-1} \mathbf{e}_i^{(N)\prime} \mathbf{H}_t^{-1} \mathbf{B} \left(\mathbf{A}_t + a_t \bar{\beta}\bar{\beta}' \right)^{-1} \mathbf{\Omega}_{t-1}^{-1} \left[\mu_{ft} + \left(\mathbf{A}_t + a_t \bar{\beta}\bar{\beta}' \right)^{-1} \bar{\beta}(c_t - a_t r_{0t}) \right] \right\}$$
$$+ o_p(N^{-1}),$$

where z_{it}^{meu} *is a mixture of normally distributed random variables that are only functions of* γ_t, r_{0t}, α_t, $\mathbf{B}$, *and* $\mathbf{H}_t$.
(iii) When, in addition to the conditions in (i), (14)also holds

$$\rho_t^{meu} = \mathbf{r}_t' \mathbf{w}_{t-1}^{meu} + (1 - \mathbf{e}' \mathbf{w}_{t-1}^{meu}) r_{0,t-1},$$

satisfies

$$N^{-1} \rho_t^{meu} \to_p \frac{e_{t-1}}{\gamma_{t-1} b_{t-1}}, \tag{33}$$

$$N^{-\frac{1}{2}} \left(\frac{\mu_{\rho,t-1}^{meu} - r_{0,t-1}}{\sigma_{\rho,t-1}^{meu}} \right) \to_p \sqrt{e_{t-1}}, \tag{34}$$

where $\mu_{\rho,t-1}^{meu} = E(\rho_t^{meu} \mid \mathcal{Z}_{t-1})$, $\sigma_{\rho,t-1}^{meu} = \sqrt{\text{var}(\rho_t^{meu} \mid \mathcal{Z}_{t-1})}$,

$$e_t = d_t - 2 r_{0,t} c_t + a_t r_{0,t}^2 + (a_t d_t - c_t^2) \bar{\beta}' \mathbf{A}_t^{-1} \bar{\beta}, \tag{35}$$

and $e_{t-1} > 0$ *almost surely.*

Remark 2(a) At a given point in time t, the `meu` portfolio weight of the i^{th} asset is asymptotically equivalent to $\mathring{w}_{it}^{meu}$ and does not converge to zero. Moreover, $\mathring{w}_{it}^{meu}$ is functionally independent from the factors covariance matrix, $\mathbf{\Omega}_t$, as well as from the factors conditional mean, μ_{ft}.
Remark 2(b) There is no contribution from either $\mathbf{\Omega}_t$ and μ_{ft} to the asymptotic distribution of w_{it}^{meu} around $\mathring{w}_{it}^{meu}$.
Remark 2(c) The `meu` portfolio does not achieve complete diversification of the idiosyncratic and the factors component of asset return innovations. Moreover, the part of the portfolio return involving the factors component is of the same order of magnitude, in N, as the part involving the idiosyncratic

component. Diversification of both components is achieved if one considers $N^{-1}w_{it}^{meu}$. For the same reasons, convergence of the portfolio return ρ_t^{meu} is achieved when normalizing by N and its limit is $\mathcal{Z}_{t-1}$-adapted. In particular, the limiting value of $N^{-1}\rho_t^{meu}$ will be a function of α_{t-1}, but not of $\mu_{f,t-1}$.
Remark 2(d) The *ex ante* Sharpe ratio diverges at the rate $N^{\frac{1}{2}}$, and the limit is always positive. Note that limit of the normalized Sharpe ratio is independent of the coefficient of risk aversion, γ_{t-1}.

Analog results can be derived for the minimum-variance (`mv`) and the mean expected (`me`) portfolios, as discussed in Section 6.

4 Discussion of results

4.1 Contribution of factors to portfolio return

Part (iii) of the above theorems establish the limit portfolio return, normalized with a suitable scaling factor, for various MV trading strategies. In particular, ρ_t^{gmv} has a well defined limit whereas ρ_t^{meu} requires the scaling factor N^{-1}. The scaling factor is necessary since the `meu` portfolio weights do not converge to zero but are in fact $O_p(1)$.

Inspecting the results, it is evident that the limit MV portfolio returns are $\mathcal{Z}_{t-1}$-adapted, that is they are neither functions of the idiosyncratic innovations, ε_t, nor the common innovations, $\mathbf{u}_t$. The first result is well known, namely that the contribution of the idiosyncratic innovations to the portfolio return vanishes in mean square as $N \to \infty$. One of the novel results of this paper is to show that MV trading strategies also succeed in diversifying the effects of the common innovations, $\mathbf{u}_t$. This result is driven by the fact that the MV trading strategies make use of the inverse of the conditional covariance matrix $\mathbf{\Sigma}_{t-1}$ in a convenient way. In particular, the MV portfolio weights have the form $\mathbf{\Sigma}_{t-1}^{-1}\mathbf{p}_{t-1}$, for some $N \times 1$ vector $\mathbf{p}_{t-1}$ function of $\mathcal{Z}_{t-1}$, the exact form of which depends on the type of trading strategy under consideration. As a consequence, the portfolio return can be decomposed as:

$$\mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{r}_t = \mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\alpha_{t-1} + \mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{B}\mu_{f,t-1} + \mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{B}\mathbf{u}_t + \mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\varepsilon_t.$$

Lemma A in the appendix establishes that $\| \mathbf{\Sigma}_{t-1}^{-1}\mathbf{B} \|^2 = O_p(N^{-1})$, so that $\mathbf{\Sigma}_{t-1}^{-1}$ and $\mathbf{B}$ are asymptotically orthogonal, and therefore the contribution of the common factor innovation, $\mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{B}\mathbf{u}_t$, to the return portfolio $\mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{r}_t$ is of smaller order than the mean term $\mathbf{p}_{t-1}'\mathbf{\Sigma}_{t-1}^{-1}\alpha_{t-1}$, as N gets large. Obviously, The contribution of the idiosyncratic term, $\mathbf{p}'_{t-1}\mathbf{\Sigma}^{-1}_{t-1}\varepsilon_t$, is also of smaller order. Therefore

$$\mathbf{p}'_{t-1}\mathbf{\Sigma}^{-1}_{t-1}\mathbf{r}_t = \mathbf{p}'_{t-1}\mathbf{\Sigma}^{-1}_{t-1}\alpha_{t-1}(1 + o_p(1)) \text{ as } N \to \infty.$$

This implies that, subject to a suitable normalization, the contributions of $\mathbf{u}_t$ and ε_t to the limit portfolio return converges to zero, the only difference between the two being that convergence occurs in first mean in the case of the terms involving $\mathbf{u}_t$, and in mean square in the case of the terms in ε_t.

Given the asymptotic orthogonality of $\mathbf{\Sigma}^{-1}_{t-1}$ and $\mathbf{B}$ it also happens that the contribution of the factors to the returns conditional mean, namely $\mathbf{p}'_{t-1}\mathbf{\Sigma}^{-1}_{t-1}\mathbf{B}\ \mu_{f,t-1}$, typically involving lagged factors f_s, $s < t$, is also of smaller order. Therefore the limit portfolio return will be given simply by the limit of $\mathbf{p}'_{t-1}\mathbf{\Sigma}^{-1}_{t-1}\alpha_{t-1}$, where this limit is $\mathcal{Z}_{t-1}$-adapted.

We have seen that different MV trading strategies implies different rates at which the corresponding portfolio weights converge, if any, to zero. However, the defined for a given MV strategy s by

$$\frac{\mu^s_{\rho,t-1} - r_{0,t-1}}{\sigma^s_{\rho,t-1}}$$

all diverge as N tends to infinity, and at the same rate $N^{\frac{1}{2}}$. However, this is not true of the *ex ante* Sharpe ratio of the `gmv` strategy, which could divergence towards minus infinity! This partly reflects the sub-optimal nature of the `gmv` strategy that does not make use of the expected means, μ_{t-1}.

4.2 Contribution of factors to portfolio weights

The conditional distribution of the factors, $\mathbf{f}_t$, is irrelevant, as far as the form of the limiting MV portfolio weights $\mathbf{w}^s_t$ is concerned. In fact, the factors conditional mean $\mu_{f,t-1}$ and conditional covariance matrix $\mathbf{\Omega}_{t-1}$ do not appear in the first-order limit approximations set out in (23) and (30). This outcome is a direct consequence of lemma A proved in the Appendix. An immediate implication is that when evaluating the MV portfolio weights empirically one can avoid specifying, let alone estimating, the conditional mean and the conditional covariance matrix of the common factors. For a finite N, this clearly would involve an approximation error since the finite-N expression of the MV weights will necessarily be a function of $\mathbf{\Omega}_{t-1}, \mu_{f,t-1}$. However, such approximation error decreases to zero as N increases and, at the same time, using the limit portfolio formulae permits avoiding modeling and estimation risk related to the common factors - namely the consequences of incorrectly specifying or poorly estimating $\mathbf{\Omega}_{t-1}$ and $\mu_{f,t-1}$.

Part (i) of Theorems 1 and 2 can be interpreted as a consistency result, showing the form of the limit approximations, as $N \to \infty$, of the MV portfolio weights. Part (ii) of these theorems considers if the conditional distribution of $\mathbf{f}_t$ plays a role with respect to the dispersion of the finite-N portfolios around their limit approximation. Under suitable regularity conditions, the MV portfolio weights have an asymptotic distribution, centered around the limit portfolio weights, which is distributed independently of the conditional moments of $\mathbf{f}_t$. In other words, the contribution of these moments to the (finite-N) MV portfolio weights vanishes at a suitably fast rate, faster than the rate required to obtain the asymptotic distribution of the portfolio weights.

The result in part (i) of the above theorems hold not only point-wise for each $i = 1, 2, ..., N$ but also jointly for the entire vector of portfolio weights. In fact, it can be shown that $\| \mathbf{w}_t^{gmv} - \mathring{\mathbf{w}}_t^{gmv} \| = o_p(N^{-1})$ and $\| \mathbf{w}_t^{meu} - \mathring{\mathbf{w}}_t^{meu} \| = o_p(1)$.

Another important consequence of part (i) of these theorems is that the limiting portfolio weights will not be time-varying unless $\mathbf{H}_t$ is, that is only if the idiosyncratic component ε_t features dynamic conditional heteroskedasticity. The mean-variance portfolio `meu` will be time-varying both due to possible time variation in $\mathbf{H}_t$, and in the risk-free rate r_{0t}. If we relax our assumptions, say allowing $\mathbf{B}$ to be time-varying $\mathbf{B}_t$, then for instance the `gmv` portfolio weights (23) become, under regularity conditions similar to the ones spelled out in Theorem 1,

$$N\, w_{it}^{gmv} - \frac{1}{a_t}\mathbf{e}_i^{(N)\prime}\mathbf{H}_t \left[\mathbf{e} + a_t(\mathbf{e}\bar{\beta}_t' - \mathbf{B}_t)\mathbf{A}_t^{-1}\bar{\beta}_t\right] \to_p 0 \text{ as } N \to \infty.$$

For this case to be genuinely interesting, $\mathbf{B}_t$ needs to be independent from the factors $\mathbf{f}_t$ though. This rules out the case $\mathbf{B}_t = \mathbf{B}\mathbf{\Omega}_t^{\frac{1}{2}}$, which, as far as the dynamics of $\mathbf{r}_t$ is concerned, is observationally equivalent to (1). If instead one alternatively assumes the parameter-free form $\mathbf{\Omega}_t = \mathbf{I}_k$, our result continues to apply since the limit portfolios continue to be functionally independent of any parametric aspect of $\mathbf{\Omega}_t$.

Factor models are inherently undetermined since (6) yields the same vector $\mathbf{r}_t$ given a non-singular $k \times k$ matrix $\mathbf{C}$ and replacing $\mathbf{B}$ and $\mathbf{u}_t$ by $\mathbf{BC}$ and $\mathbf{C}^{-1}\mathbf{u}_t$, respectively. Determination of $\mathbf{C}$ is crucial for identification and estimation of model (6). This is particularly relevant in our context since besides the factor loadings, the matrix $\mathbf{C}$ induces also a rotation of $\mathbf{\Omega}_t$ and $\mu_{f,t}$ and, due to their time-variation, the risk of possible lack of identification is even more pronounced. However, this issue is of second-order importance since the limit portfolio weights do not dependent on the conditional mean and covariance matrix of $\mathbf{f}_t$. One can easily verify this by replacing $\mathbf{B}$, $\mathbf{A}_t^{-1}$ and $\bar{\beta}$ with $\mathbf{BC}$, $\mathbf{C}^{-1}\mathbf{A}_t^{-1}\mathbf{C}'^{-1}$, and $\mathbf{C}'\bar{\beta}$, respectively into (23) and (30).

4.3 Portfolio diversification

Under our assumptions

$$w_{it}^{gmv} - \mathring{w}_{it}^{gmv} \rightarrow_p 0 \text{ as } N \rightarrow \infty,$$

where $N\mathring{w}_{it}^{gmv} = O_p(1)$, for given i and t, and are different from zero almost surely. Therefore, the `gmv` portfolio is diversified in the sense that each coefficient w_{it}^{gmv} becomes arbitrarily small as N grows.

More formally, if $\sup_{1\leq i\leq N} \mid w_{it}^{gmv} \mid = o_p(1)$ for each t, then we achieve full diversification in the sup-norm sense of Green and Hollifield (1992). Using the limit approximation $\mathring{w}_{it}^{gmv}$ it turns out to be much easier to find sufficient conditions for full diversification. For instance, using results of Theorem 1, one obtains

$$\sup_{1\leq i\leq N}\left(\mid \mathbf{h}_{(i)t}'\mathbf{e} \mid + \sum_{j=1}^{k} \mid \mathbf{h}_{(i)t}'\beta^{(j)} \mid \right) = o_p(N) \tag{36}$$

where $\beta^{(j)} = \mathbf{B}\mathbf{e}_j^{(k)}$ and $\mathbf{h}_{(i)t} = \mathbf{H}_t^{-1}\mathbf{e}_i^{(N)}$. If full diversification at rate N^{-1} is required, the left hand side of the previous expression must be $O_p(1)$. In turn this is satisfied whenever $\sup_{1\leq i\leq N}\sup_{1\leq j\leq k} \mid \beta_{ij} \mid = O_p(1)$ and $\mid \mathbf{h}_{(i)t}'\mathbf{e} \mid = O_p(1)$.

In contrast, the `meu` portfolio is not fully diversifiable in the sense that its weights do not converge to zero and instead $\mathring{w}_{it}^{meu} = O_p(1)$. Therefore, as a consequence, the limit portfolio ρ_t^{meu} requires the normalization N^{-1} in order to obtain a well-defined limit.

Thus, the common practice of building (optimal) portfolios imposing the restriction that the portfolio weights are smaller than a given predetermined quantity, appears justified for the `meu` portfolio. In fact, there is no guarantee that the weights will be smaller the larger the number of assets under consideration. On the other hand, under conditions such as (36) or variations of, the `gmv` portfolio weights gets arbitrarily small, for a sufficiently large N.

The definition of complete diversifiability of Chamberlain and Rothschild (1983) instead requires, for the s trading strategy, $\sum_{i=1}^{N}(\mathring{w}_{it}^{s})^2 = o_p(N^2)$ for each t, and sufficient conditions can be easily derived. For instance, for the gmv portfolio it is required

$$\mathbf{e}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{e} = o_p(N^2), \;\; \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B} = o_p(N^2).$$

Notice that the second condition is implied by (11). This definition of complete diversifiability requires stronger conditions than the notion based on the sup-norm discussed earlier.

4.4 Short-selling and factor dominance

When $\mathbf{H}_t$ is diagonal, it easily follows that $\mathbf{A}_t = a_t\mathbf{\Sigma}_\beta$, where $\mathbf{\Sigma}_\beta$ is the covariance matrix of the β_i, yielding for the gmv portfolio weights

$$Nw_{it}^{gmv} \rightarrow_p \frac{h_{ii,t}^{-1}}{a_t}\left[1 - \bar{\beta}'\mathbf{\Sigma}_\beta^{-1}(\beta_i - \bar{\beta})\right]. \tag{37}$$

Moreover, if $\mathbf{\Sigma}_\beta$ is diagonal, with $\sigma_{\beta j}$ being its $(j,j)^{th}$ entry, (37) simplifies further to

$$Nw_{it}^{gmv} \rightarrow_p \frac{h_{ii,t}^{-1}}{a_t}\left[1 - \left(\frac{\bar{\beta}_1}{\sigma_{\beta 1}}\right)^2\left(\frac{\beta_{i1} - \bar{\beta}_1}{\bar{\beta}_1}\right) - .... - \left(\frac{\bar{\beta}_k}{\sigma_{\beta k}}\right)^2\left(\frac{\beta_{ik} - \bar{\beta}_k}{\bar{\beta}_k}\right)\right], \tag{38}$$

where $\bar{\beta}_j$ and β_{ij} are the j^{th} element of $\bar{\beta} = (\bar{\beta}_1, ..., \bar{\beta}_k)'$ and $\beta_i = (\beta_{i1}, ..., \beta_{ik})'$, respectively.

Green and Hollifield (1992) argue that the possibility of short-selling, in the sense of a repeated finding of negative optimal portfolio weights, is related to the presence of one dominant factor. Our result sheds some light on this. One can see from (38) that the limit portfolio weights only depend on factor loadings if the mean of these laodings is non-zero (i.e. if $\bar{\beta}_i \neq 0$). Such factors are regarded as 'dominant' by Jagannathan and Ma (2003)

More generally, a negative weight can arise whenever the factor loading assumes values smaller than their cross-sectional average. This effect is magnified, the larger is the 'Sharpe ratio' of the factor loading, defined by $\bar{\beta}_j/\sigma_{\beta j}$. A large dispersion implies a smaller chance of finding negative weights, corroborating the findings based on simulations reported by Jagannathan and Ma (2003). On the other hand, note also that the larger the number of dominant factors under consideration (in the sense of Jagannathan and Ma (2003)), the less likely it is that a negative weight would be encountered. Similar outcomes obtain for non-diagonal $\mathbf{H}_t$. This reinforces Green and Hollifield (1992)'s conjecture about the presence of a single dominant factor whenever large negative weights are observed.

Under the same conditions as above, for the `meu` portfolio weights one obtains

$$w_{it}^{meu} \to_p \frac{h_{ii,t}^{-1}}{\gamma_t} \left[\alpha_{it} - r_{0t} + \left(\frac{\bar{\beta}_1}{\sigma_{\beta 1}} \right)^2 \left((\alpha_{it} - r_{0t}) - (c_t - a_t r_{0t}) \frac{\beta_{i1}}{a_t \bar{\beta}_1} \right) \dots \right. \\ \left. + \left(\frac{\bar{\beta}_k}{\sigma_{\beta k}} \right)^2 \left((\alpha_{it} - r_{0t}) - (c_t - a_t r_{0t}) \frac{\beta_{ik}}{a_t \bar{\beta}_k} \right) \right] \tag{39}$$

Therefore, as with the `gmv` portfolio weights one can see that a negative weight is more likely for the asset for which $\alpha_{it} - r_{0t} < 0$.

Assuming $c_t > a_t r_{0t}$, a negative weight is more likely to arise whenever the factor loading assumes values smaller than their cross-sectional average and this effect is magnified, the larger is the 'Sharpe ratio' of the factor loading. Finally, the larger the number of dominant factors under consideration, the less likely that a negative weight would be encountered.

5 An illustrative example: a single factor model

Here we illustrate our results using a single factor model ($k = 1$) where (6) becomes

$$\mathbf{r}_t = \mu_{t-1} + \beta u_t + \varepsilon_t, \tag{40}$$

where Assumptions 1, 2 and 3 hold. Therefore now u_t is a scalar martingale difference process with conditional variance $\omega_{t-1} > 0$, and β is a $N \times 1$ vector of factor loadings with mean $\bar{\beta}\mathbf{e} \neq \mathbf{0}$, and the variance matrix $\sigma_\beta^2 \mathbf{I}_N > 0$, where $\bar{\beta}$ is a scalar. For simplicity, let us also assume that the idiosyncratic errors ε_{it} are cross-sectionally uncorrelated, implying a diagonal $\mathbf{H}_t$, with conditional variances $h_{ii,t-1}$. The conditional covariance matrix of $\mathbf{r}_t$ will then be

$$\mathbf{\Sigma}_{t-1} = \omega_{t-1} \, \beta\beta' + \mathbf{H}_{t-1}.$$

By part (i) of Theorem 1 we have

$$N\, w_{it}^{gmv} - \frac{h_{ii,t}^{-1}}{a_t}\left(1 - \frac{\bar{\beta}}{\sigma_\beta^2}(\beta_i - \bar{\beta})\right) \to_p 0 \text{ as } N \to \infty, \tag{41}$$

where $N^{-1}\sum_{i=1}^{N} h_{ii,t}^{-1} \to_p a_t$. Result (41) shows that the limit `gmv` portfolio weights are functionally independent from the factor conditional variance ω_t. The limit `gmv` portfolio weights will be time-varying only if $h_{ii,t-1}$ are time-varying. Moreover, it is well known that any factor model is undetermined only up to a non-singular transformation. This implies that (40) is unchanged if we substitute f_t by cf_t, for a non-zero constant c, and β by $c^{-1}\beta$. However, the limit `gmv` portfolio weights (41) are identified and do not depend on c, since

$$\left(1 - \frac{c^{-1}\bar{\beta}}{c^{-2}\sigma_\beta^2}(c^{-1}\beta_i - c^{-1}\bar{\beta})\right) = \left(1 - \frac{\bar{\beta}}{\sigma_\beta^2}(\beta_i - \bar{\beta})\right),$$

for any $c \neq 0$. Part (ii) of Theorem 1 now yields

$$w_{it}^{gmv} = \frac{1}{N}\frac{h_{ii,t}^{-1}}{a_t}\left[1 - \frac{\bar{\beta}}{\sigma_\beta^2}(\beta_i - \bar{\beta})\right] + \frac{z_{it}^{gmv}}{N^{\frac{3}{2}}} + \frac{1}{N^2}\frac{\omega_t^{-1}h_{ii,t}^{-1}}{a_t^2\sigma_\beta^2} + o_p(\frac{1}{N^2}).$$

This shows how the `gmv` portfolio weights are a function of ω_t, for a finite N, but that this term is of a smaller order, decreasing to zero at rate N^{-2}. Concerning the limit portfolio return $\rho_t^{gmv} = \mathbf{r}_t'\mathbf{w}_{t-1}^{gmv}$, part (iii) of Theorem 1 yields

$$\rho_t^{gmv} = \mathbf{r}_t'\mathbf{w}_{t-1}^{gmv} \to_p \frac{c_{t-1}}{a_{t-1}} \text{ as } N \to \infty,$$

where $N^{-1}\sum_{i=1}^{N} \alpha_{i,t}h_{ii,t}^{-1} \to_p c_t$.

A similar discussion applies to the `meu` trading strategy. For instance, part (i) of Theorem 2 yields

$$w_{it}^{meu} - \frac{h_{ii,t}^{-1}}{\gamma_t b_t}\left\{(\alpha_{it} - r_{0t}) + \frac{\bar{\beta}^2}{\sigma_\beta^2}[(\alpha_{it} - r_{0t}) - (\frac{c_t}{a_t} - r_{0t})\frac{\beta_i}{\bar{\beta}}]\right\}.$$

Consider now the issue of diversification. (41) shows clearly that for each i the weight w_{it}^{gmv} decays to zero at rate of N^{-1}. However, full diversification in the sup-norm sense of Green and Hollifield (1992) requires the stronger condition $\sup_{1\le i\le N} \mid \beta_i/h_{ii,t} \mid = o_p(N)$ for each t. Finally, if the even stronger requirement of diversification at the exact rate $1/N$ is desired, then $\sup_{1\leq i\leq N}|\beta_i/h_{ii,t}| = O_p(1)$ is needed. A sufficient condition for this is boundedness of the factor loadings, $\sup_{1\leq i\leq N}|\beta_i| = O(1)$ and $h_{ii,t} \geq \delta > 0$ almost surely for any i and t.

Regarding short-selling and factor dominance, from (41), one can see that short-selling for the i^{th} asset ($w_{it}^{gmv} < 0$) arises, whenever the i^{th} factor loading, β_i, is greater than its (cross-sectional) mean, $\bar{\beta}$, by a certain amount which is a function of β and σ_{β}^2. This holds assuming a positive factor loading mean $\bar{\beta}$. Short-selling is more likely to occur, the smaller is the factor loading variance, σ_{β}^2, and the larger is $\bar{\beta}$.

We have derive (41) assuming $\bar{\beta} \neq 0$, but this is not required. When $\bar{\beta} = 0$ we have

$$N\, w_{it}^{gmv} - \frac{h_{ii,t}^{-1}}{a_t} \to_p 0, \text{ as } N \to \infty. \tag{42}$$

It turns out that the same result also holds irrespective of whether $\sigma_{\beta}^2 = 0$ or not. This result does not follow directly from (41), but one needs to start from the definition of the gmv portfolio for a given N, set $\sigma_{\beta}^2 = 0$, and then take the limit.

6 Other optimization strategies

We now present results for two other MV tangency portfolios considered in the literature, namely the minimum variance and the maximum expected return portfolios. We present two corresponding theorems without proof, and comment on the results afterwards.

The `mv` portfolio weights $\mathbf{w}_t^{mv} = (w_{1t}^{mv}, ..., w_{Nt}^{mv})'$ are defined by

$$\mathbf{w}_{t-1}^{mv} = \text{argmin}_{\mathbf{w}}\left(\mathbf{w}'\mathbf{\Sigma}_{t-1}\mathbf{w}\right), \text{ such that } \mathbf{w}'\mu_{t-1} + (1 - \mathbf{w}'\mathbf{e})r_{0,t-1} = \mu_{\rho},$$

where μ_{ρ} is the targeted expected portfolio return assumed to exceed $r_{0,t-1}$ ($\mu_{\rho} > r_{0,t-1}$), yielding

$$\mathbf{w}_{t-1}^{mv} = \frac{\mu_{\rho} - r_{0,t-1}}{(\mu_{t-1} - \mathbf{e}r_{0,t-1})'\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - \mathbf{e}r_{0,t-1})}\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - \mathbf{e}r_{0,t-1}). \tag{43}$$

Theorem 3 *(minimum variance portfolio)* *(i) Let*

$$\mathring{w}_{it}^{mv} = N^{-1}\frac{(\mu_\rho - r_{0t})}{e_t}\mathbf{e}_i'\mathbf{H}_t^{-1}\left\{(\alpha_t - \mathbf{e}r_{0t}) + [a_t(\alpha_t - \mathbf{e}r_{0t})\bar{\beta}' - (c_t - a_t r_{0t})\mathbf{B}]\mathbf{A}_t^{-1}\bar{\beta}\right\}. \tag{44}$$

When conditions (7), (10), (11), (12), (13), (14), (16), (17) and (19) hold:

$$N(w_{it}^{mv} - \mathring{w}_{it}^{mv}) \to_p 0.$$

(ii) When, in addition to the conditions in (i), (20) holds:

$$\begin{aligned} w_{it}^{mv} &= \mathring{w}_{it}^{mv} + N^{-3/2} z_{it}^{mv} \\ &+ N^{-2}\left\{\left(\frac{\mu_\rho - r_{0t}}{e_t}\right)\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{B}\left(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}'\right)^{-1}\mathbf{\Omega}_t^{-1} \times \right.\\ &\left.\left[\mu_{ft} + \left(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}'\right)^{-1}\bar{\beta}(c_t - a_t r_{0t})\right]\right\} \\ &+ o_p(N^{-2}) \end{aligned}$$

where z_{it}^{mv} is a mixture of normally distributed random variables that are only functions of μ_ρ, r_{0t}, α_t, $\mathbf{B}$, and $\mathbf{H}_t$.
(iii) When the conditions in (i) hold

$$\rho_t^{mv} = \mathbf{r}_t'\mathbf{w}_{t-1}^{mv} + (1 - \mathbf{e}'\mathbf{w}_{t-1}^{mv})r_{0,t-1},$$

satisfies

$$\rho_t^{mv} \to_p \quad \mu_\rho, \tag{45}$$

$$N^{-\frac{1}{2}}\left(\frac{\mu_{\rho,t-1}^{mv}}{\sigma_{\rho,t-1}^{mv} - r_{0,t-1}}\right) \to_p \quad \sqrt{e_{t-1}}, \tag{46}$$

setting $\mu_{\rho,t-1}^{mv} = E(\rho_t^{mv} \mid \mathcal{Z}_{t-1})$ and $\sigma_{\rho,t-1}^{mv} = \sqrt{\mathrm{var}(\rho_t^{mv} \mid \mathcal{Z}_{t-1})}$.

For the maximum expected return portfolio, $\mathbf{w}_t^{me} = (w_{1t}^{me}, ..., w_{Nt}^{me})'$, we have

$$\mathbf{w}_{t-1}^{me} = \mathrm{argmax}_{\mathbf{w}}\mathbf{w}'\mu_{t-1} + (1 - \mathbf{w}'\mathbf{e})r_{0,t-1}, \text{ such that } \mathbf{w}'\mathbf{\Sigma}_{t-1}\mathbf{w} = \sigma_\rho^2,$$

where σ_ρ^2 is the targeted portfolio variance, yielding the maximum expected return portfolio

$$\mathbf{w}_{t-1}^{me} = \left[\frac{\sigma_\rho^2}{(\mu_{t-1} - \mathbf{e}r_{0,t-1})'\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - \mathbf{e}r_{0,t-1})}\right]^{\frac{1}{2}}\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - \mathbf{e}r_{0,t-1}).$$

Theorem 4 *(maximum expected return portfolio)*
(i) Let

$$\mathring{w}_{it}^{me} = N^{-\frac{1}{2}} \frac{\sigma_\rho}{\sqrt{e_t}} \mathbf{e}_i^{(N)\prime} \mathbf{H}_t^{-1} \left\{ (\alpha_t - \mathbf{e} r_{0t}) + [a_t(\alpha_t - \mathbf{e} r_{0t})\bar{\beta}' - (c_t - a_t r_{0t})\mathbf{B}]\mathbf{A}_t^{-1}\bar{\beta} \right\}. \tag{47}$$

When conditions (7), (10), (11), (12), (13), (14), (16), (17) and (5.2due) hold:

$$N^{\frac{1}{2}}(w_{it}^{me} - \mathring{w}_{it}^{me}) \to_p 0.$$

(ii) When, in addition to the conditions in (i), (20) hold:

$$\begin{aligned} w_{it}^{me} &= \mathring{w}_{it}^{me} + N^{-1} z_{it}^{me} \\ &+ N^{-3/2} \frac{\sigma_\rho}{\sqrt{e_t}} \mathbf{e}_i^{(N)\prime} \mathbf{H}_t^{-1} \mathbf{B} \left(\mathbf{A}_t + a_t \bar{\beta}\bar{\beta}' \right)^{-1} \mathbf{\Omega}_t^{-1} \left[\mu_{f,t} + \left(\mathbf{A}_t + a_t \bar{\beta}\bar{\beta}' \right) \bar{\beta}(c_t - a_t r_{0t}) \right] \\ &+ o_p(N^{-\frac{3}{2}}), \end{aligned} \tag{48}$$

where z_{it}^{me} is a mixture of normally distributed random variables, function of σ_ρ^2, r_{0t}, α_t, $\mathbf{B}$, and $\mathbf{H}_t$, only.
(iii) When the conditions in (i) hold:

$$\rho_t^{me} = \mathbf{r}_t' \mathbf{w}_{t-1}^{me} + (1 - \mathbf{e}' \mathbf{w}_{t-1}^{me}) r_{0,t-1},$$

satisfies

$$N^{-\frac{1}{2}} \rho_t^{me} \to_p \quad \sigma_\rho \sqrt{e_{t-1}}, \tag{49}$$

$$N^{-\frac{1}{2}} \left(\frac{\mu_{\rho,t-1}^{me} - r_{0,t-1}}{\sigma_{\rho,t-1}^{me}} \right) \to_p \quad \sqrt{e_{t-1}}, \tag{50}$$

where $\mu_{\rho,t-1}^{me} = E(\rho_t^{me} \mid \mathcal{Z}_{t-1})$, and $\sigma_{\rho,t-1}^{me} = \sqrt{\mathrm{var}(\rho_t^{me} \mid \mathcal{Z}_{t-1})}$.

Remark 3-4(a) The `mv` and `me` portfolio weights of the i^{th} asset are, asymptotically in N, equivalent to $\mathring{w}_{it}^{mv}$ and $\mathring{w}_{it}^{me}$, respectively. Moreover, the latter are functionally independent of $\mathbf{\Omega}_t$ and $\mu_{f,t}$.
Remark 3-4(b) The asymptotic distributions of w_{it}^{mv} and w_{it}^{me}, respectively around $\mathring{w}_{it}^{mv}$and $\mathring{w}_{it}^{mv}$, do not depend on $\mathbf{\Omega}_t$ and/or μ_{ft}.
Remark 3-4(c) The `mv` and `me` portfolios achieve full diversification of both the idiosyncratic and common factor components of asset return innovations, the latter when normalized by $N^{\frac{1}{2}}$. Under the same conditions, the corresponding limit portfolio returns are $\mathcal{Z}_{t-1}$-adapted.

Remark 3-4(d) The *ex ante* Sharpe ratio diverges to plus infinity at rate $N^{\frac{1}{2}}$. The limit of the normalized Sharpe ratio is independent of μ_ρ and of σ_ρ^2 for `mv` and `me` trading strategies, respectively. In particular, once normalized by $N^{-\frac{1}{2}}$, the limit is the same and coincide with the one obtained for the `meu` portfolio return. This follows since all the three MV tangency portfolio weights are proportional to one another. This important propert is not shared by the the `gmv` portfolio.
Remark 3-4(e) Part (i) of the above theorems hold also jointly for the entire vector of portfolio weights, that is $\| \mathbf{w}_t^{mv} - \mathring{\mathbf{w}}_t^{mv} \| = o_p(N^{-1})$ and $\| \mathbf{w}_t^{me} - \mathring{\mathbf{w}}_t^{me} \| = o_p(N^{-\frac{1}{2}})$.
Remark 3-4(f) As for the other optimization strategies, both (44) and (47) do not depend on any particular rotation of the factors and factor loadings.
Remark 3-4(g) Both the `mv` and the `me` portfolios are fully diversifiable, although at different rates of N^{-1} and $N^{-1/2}$, respectively, achieved whenever $\sup_{1 \leq i \leq N} \mid \mathbf{h}'_{(i)t}\alpha_t \mid = O_p(1)$ and the left hand side of (36) is $O_p(1)$.

7 Final remarks

In this paper we have provided a number of theoretical results for the MV tangency portfolios as the number of assets in the portfolio gets large. Under fairly general conditions we have shown that to a first order approximation the portfolio weights and the associated *ex ante* Sharpe ratios do not depend on the means and the variance-covariances of the common factors. This result has a number of important practical implications. It is well know that under the assumption of correct model specification, factor model-based optimal portfolios weights leads to more efficient estimates of the corresponding portfolio variance, as compared to the familiar sample moment estimates (see Fan, Fan, and Lv (2007)). However, the asymptotic independence of optimal portfolio weights from the common factors, established in this paper, suggests that in the case of large portfolios it might be prudent to side-step the tasks of specification and estimation of the conditional distribution of the factors and instead use the formulae for the limit portfolio weights advanced in this paper. In this way it might be possible to avoid the adverse effects of model and parameter uncertainties that surround the specification of the unobserved common factor models. But before this issue can be examined one also needs to consider the extent to which the properties of the limit portfolios are still valid when the remaining unknown parameters are replaced by their estimates. An extensive Monte-Carlo exercise might be required to complement the asymptotic results provided in this paper. These issues will be addressed in a subsequent work by the authors.

Appendix A: mathematical proofs

We start with a Lemma where we show that for a given t and as $N \to \infty$, $\mathbf{\Sigma}_{t-1}^{-1}$ and $\mathbf{B}$ are asymptotically orthogonal. This result turns out to be critical for characterizing the behavior of optimal portfolios as N gets large.

Lemma A Let $\mathbf{P}_t$ be a sequence of random positive definitive matrices such that

$$\frac{\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B}}{N} \to_p \mathbf{P}_t > 0 \text{ as } N \to \infty. \tag{51}$$

Recalling that $\mathbf{e}_i^{(N)}$ is the i^{th} column of the identity matrix $\mathbf{I}_N$, then for any t,i and j

$$\mathbf{e}_i^{(N)\prime}\mathbf{\Sigma}_t^{-1}\beta^{(j)} \to_p 0 \text{ as } N \to \infty, \;\; 1 \leq j \leq k, \tag{52}$$

where $\beta^{(j)}$ denotes the j^{th} column of $\mathbf{B} = (\beta^{(1)} \dots \beta^{(k)})$.
Under (51) and

$$\frac{\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}}{N} \to_p \mathbf{Q}_t \geq 0, \tag{53}$$

where $\mathbf{Q}_t$ denotes a sequence of random positive semi-definitive matrices, for any t

$$\| \mathbf{\Sigma}_t^{-1}\beta^{(j)} \|^2 = O_p(N^{-1}), \;\; 1 \leq j \leq k, \quad \text{as } N \to \infty. \tag{54}$$

Proof of Lemma A. The results follow from the identity

$$\mathbf{\Sigma}_t^{-1} = \mathbf{H}_t^{-1} - \mathbf{H}_t^{-1}\mathbf{B}(N^{-1}\mathbf{\Omega}_t^{-1} + N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}. \tag{55}$$

Pre-multiplying both sides by $\mathbf{e}_i^{(N)\prime}$ and post-multiplying both sides by $\beta^{(j)}$ yields (52).

We deal with (54) more explicitly. First note that ($\mathbf{e}_j^{(k)}$ denotes the j^{th} column of the identity $\mathbf{I}_k$ matrix)

$$\begin{aligned}
& (N^{-1}\mathbf{\Omega}_t^{-1} + N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\beta^{(j)} - \mathbf{e}_j^{(k)} \\
= \; & (N^{-1}\mathbf{\Omega}_t^{-1} + N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\beta^{(j)} - (N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\beta^{(j)} \\
= \; & N^{-1}\left[-(N^{-1}\mathbf{\Omega}_t^{-1} + N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{\Omega}_t^{-1}(N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\beta^{(j)}\right] \\
\equiv \; & N^{-1}\mathbf{g}_t^{(j)},
\end{aligned}$$

where notice that $\mathbf{g}_t^{(j)}$ is a $k \times 1$ vector with a finite norm.

Therefore, substituting the latter expression into (55) and recalling that $\mathbf{B}\mathbf{e}_j^{(k)} = \beta^{(j)}$ it follows that

$$\boldsymbol{\Sigma}_t^{-1}\beta^{(j)} = \mathbf{H}_t^{-1}\beta^{(j)} - \mathbf{H}_t^{-1}\mathbf{B}(\mathbf{e}_j^{(k)} + N^{-1}\mathbf{g}_t^{(j)}),$$

yielding

$$\begin{aligned}\| \quad & \boldsymbol{\Sigma}_t^{-1}\beta^{(j)} \|^2 = \beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} + N^{-1}\mathbf{g}_t^{(j)\prime}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} - \beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} \\ & -N^{-1}\mathbf{g}_t^{(j)\prime}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} - N^{-1}\beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}\mathbf{g}_t^{(j)} - N^{-2}\mathbf{g}_t^{(j)\prime}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}\mathbf{g}_t^{(j)} \\ & -\beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} + \beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\beta^{(j)} + N^{-1}\beta^{(j)\prime}\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}\mathbf{g}_t^{(j)} \\ = \quad & -N^{-1}\mathbf{g}_t^{(j)\prime}\left(N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{H}_t^{-1}\mathbf{B}\right)\mathbf{g}_t^{(j)} = O_p(N^{-1}\mathbf{g}_t^{(j)\prime}\mathbf{Q}_t\mathbf{g}_t^{(j)}). \ \square\end{aligned}$$

Proof of Theorem 1 All the limits below are based on $N \to \infty$.

(i) For $N < \infty$, set $\mathbf{w}_t^{gmv} = \mathbf{C}_{t,N}/D_{t,N}$ where

$$\mathbf{C}_{t,N} = \mathbf{H}_t^{-1}\mathbf{e} - \mathbf{H}_t^{-1}\mathbf{B}(\boldsymbol{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e},$$

and

$$D_{t,N} = \mathbf{e}'\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{e}'\mathbf{H}_t^{-1}\mathbf{B}(\boldsymbol{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e},$$

which easily follow from the identity (55).

For $\tilde{\mathbf{B}} = \mathbf{B} - \mathbf{e}\overline{\beta}'$

$$\begin{aligned} N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B} &= N^{-1}\overline{\beta}\overline{\beta}'\mathbf{e}'\mathbf{H}_t^{-1}\mathbf{e} + N^{-1}\tilde{\mathbf{B}}'\mathbf{H}_t^{-1}\tilde{\mathbf{B}} + \\ & \quad N^{-1}\overline{\beta}\mathbf{e}'\mathbf{H}_t^{-1}\tilde{\mathbf{B}} + N^{-1}\tilde{\mathbf{B}}'\mathbf{H}_t^{-1}\mathbf{e}\overline{\beta}', \end{aligned}$$

so that collecting terms

$$N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B} \to_p \mathbf{A}_t + a_t\overline{\beta}\overline{\beta}',$$

since $N^{-1}\tilde{\mathbf{B}}'\mathbf{H}_t^{-1}\mathbf{e} \to_p \mathbf{0}$ by $E(\tilde{\beta}_i) = \mathbf{0}'$. Similarly

$$N^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} \to_p a_t\overline{\beta}.$$

Hence, using the identity

$$(\mathbf{A}_t + a_t\overline{\beta}\overline{\beta}')^{-1} = \left(\mathbf{A}_t^{-1} - \frac{a_t}{(1 + a_t\overline{\beta}'\mathbf{A}_t^{-1}\overline{\beta})}\mathbf{A}_t^{-1}\overline{\beta}\overline{\beta}'\mathbf{A}_t^{-1}\right),$$

which yields $\bar{\beta}'(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\bar{\beta} = \bar{\beta}'\mathbf{A}_t^{-1}\bar{\beta}/b_t$, by Slutsky's theorem,

$$N^{-1}D_{t,N} \to_p a_t \left(1 - a_t\bar{\beta}'(\mathbf{A}_t^{-1} - \frac{a_t\mathbf{A}_t^{-1}\bar{\beta}\bar{\beta}'\mathbf{A}_t^{-1}}{(1 + a_t\bar{\beta}'\mathbf{A}_t^{-1}\bar{\beta})})\bar{\beta}\right)$$

but the right hand side simplifies yielding

$$N^{-1}D_{t,N} \to_p a_t b_t^{-1}$$

By the same arguments, since $(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\bar{\beta}a_t = a_t b_t^{-1}\mathbf{A}_t^{-1}\bar{\beta}$, then

$$\mathbf{e}_i^{(N)\prime}\mathbf{C}_{t,N} = b_t^{-1}\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\left(\mathbf{e} + a_t(\mathbf{e}\bar{\beta}' - \mathbf{B})\mathbf{A}_t^{-1}\bar{\beta}\right) + o_p(1).$$

(ii) For (25)

$$\begin{aligned} &\mathbf{C}_{t,N} = \mathbf{H}_t^{-1}\mathbf{e} - \mathbf{H}_t^{-1}\mathbf{B}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} = \\ &\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{H}_t^{-1}\mathbf{B}(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} \\ &+\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{H}_t^{-1}\mathbf{B}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} \\ = \; &\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{H}_t^{-1}\mathbf{B}(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e} \qquad (56) \\ &+\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{\Omega}_t^{-1}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e}. \qquad (57) \end{aligned}$$

In relation to (56), we seek the asymptotic distribution of

$$N^{\frac{1}{2}}\left(Nw_{it}^{gmv} - N\mathring{w}_{it}^{gmv}\right)$$

where

$$w_{it}^{gmv} = \frac{1}{N}\left(\frac{C_{i,t,N}}{N^{-1}D_{t,N}}\right) = \frac{1}{N}\left(\frac{\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e}}{N^{-1}D_{t,N}}\right),$$

$$\mathring{w}_{it}^{gmv} = \frac{1}{N}\frac{b_t^{-1}\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\left(\mathbf{e} + a_t(\mathbf{e}\bar{\beta}' - \mathbf{B})\mathbf{A}_t^{-1}\bar{\beta}\right)}{a_t b_t^{-1}}$$

$$= \frac{1}{N}\left(\frac{\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{e} - \mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}a_t\bar{\beta}}{a_t b_t^{-1}}\right).$$

By the continuous mapping theorem

$$N^{\frac{1}{2}}\left(\frac{(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{e}}{N^{-1}D_{t,N}} - \frac{(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta})'^{-1}a_t\bar{\beta}}{a_t b_t^{-1}}\right) \to_d \zeta_{1t}^{gmv},$$

$$N^{\frac{1}{2}}\left(\frac{1}{N^{-1}D_{t,N}} - \frac{1}{a_t b_t^{-1}}\right) \to_d \zeta_{2t}^{gmv},$$

where $\zeta_{1t}^{gmv}, \zeta_{2t}^{gmv}$ are a $k \times 1$ and a scalar normally distributed random variable, respectively, with zero mean. Therefore by standard results

$$N^{\frac{1}{2}} \left(N w_{it}^{gmv} - N \mathring{w}_{it}^{gmv} \right) \rightarrow_d \xi_{1i,t}' \zeta_{1t}^{gmv} - \xi_{2i,t} \zeta_{2t}^{gmv} = z_{i,t}^{gmv},$$

which is a mixture of normal random variables, unless $\xi_{1i,t}$, $\xi_{2i,t}$ are both non-random.

Concerning term (57) that involves $\mathbf{\Omega}_t$

$$\begin{aligned} & \mathbf{e}_i' \mathbf{H}_t^{-1} \mathbf{B} (\mathbf{B}' \mathbf{H}_t^{-1} \mathbf{B})^{-1} \mathbf{\Omega}_t^{-1} (\mathbf{\Omega}_t^{-1} + \mathbf{B}' \mathbf{H}_t^{-1} \mathbf{B})^{-1} \mathbf{B}' \mathbf{H}_t^{-1} \mathbf{e} \\ = \ & a_t \mathbf{e}_i' \mathbf{H}_t^{-1} \mathbf{B} (\mathbf{A}_t + a_t \overline{\beta} \overline{\beta}')^{-1} \frac{\mathbf{\Omega}_t^{-1}}{N} (\mathbf{A}_t + a_t \overline{\beta} \overline{\beta}')^{-1} \overline{\beta} (1 + o_p(1)). \end{aligned}$$

(iii) To establish (27), since:

$$\mathbf{w}_{t-1}^{gmv\prime} \mathbf{r}_t = \frac{1}{\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e}} \left[\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{B} (\mu_{f,t-1} + \mathbf{u}_t) + \mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} (\alpha_{t-1} + \varepsilon_t) \right],$$

then by Lemma A the first term on the right hand side satisfies

$$\frac{1}{\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e}} \mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{B} (\mu_{f,t-1} + \mathbf{u}_t) = O_p(N^{-1}).$$

The covariance matrix of the term involving ε_t is

$$(\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e})^{-2} \mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{H}_{t-1} \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e} = O_p(N^{-1}),$$

since by easy calculations

$$\frac{\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{H}_{t-1} \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e}}{N} - \frac{\mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e}}{N} \rightarrow_p 0,$$

yielding $\varepsilon_t' \mathbf{w}_{t-1}^{gmv} = O_p(N^{-\frac{1}{2}})$. Therefore, collecting terms

$$\rho_t^{gmv} = \frac{N^{-1} \mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \alpha_{t-1}}{N^{-1} \mathbf{e}' \mathbf{\Sigma}_{t-1}^{-1} \mathbf{e}} + O_p(N^{-\frac{1}{2}}),$$

which is asymptotically equivalent, by part (i) of this proof, to

$$\frac{1}{N a_{t-1}} \left(\mathbf{e}' \mathbf{H}_{t-1}^{-1} \alpha_{t-1} + a_{t-1} \overline{\beta}' \mathbf{A}_{t-1}^{-1} (\mathbf{B}' - \overline{\beta} \mathbf{e}') \mathbf{H}_{t-1}^{-1} \alpha_{t-1} \right) \rightarrow_p \frac{c_{t-1}}{a_{t-1}}.$$

(28) follows easily since $\sigma^{gmv\,2}_{\rho,t-1} = (\mathbf{e}'\mathbf{\Sigma}^{-1}_{t-1}\mathbf{e})^{-1}$ and where the limit of ρ^{gmv}_t, just established, coincide with the limit of its conditional mean $\mu^{gmv}_{\rho,t-1}$.
□

Proof of Theorem 2. All the limits below are based on $N \to \infty$.

(i) By identity (55)

$$\begin{aligned}\mathbf{w}^{meu}_t &= \frac{1}{\gamma_t}\mathbf{\Sigma}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) \\ &= \frac{1}{\gamma_t}\left(\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) - \mathbf{H}^{-1}_t\mathbf{B}(\mathbf{\Omega}^{-1}_t + \mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t})\right),\end{aligned}$$

for $N < \infty$. Since

$$\mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) = \mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\alpha_t - \mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\mathbf{e}r_{0t} + \mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\mathbf{B}\mu_{f,t},$$

we just need to determine the behavior of the first term on the right hand side. In fact, the second term can be written as $-\mathbf{e}^{(N)\prime}_i\mathbf{C}_{t,N}r_{0t}$ with $\mathbf{C}_{t,N}$ defined in the proof of Theorem 1 and the third term, $\mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\mathbf{B}\mu_{f,t}$, goes to $\mathbf{0}'$ by Lemma A. Thus, using the same arguments used in proof of Theorem 1,

$$\mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\alpha_t = \mathbf{e}^{(N)\prime}_i\mathbf{H}^{-1}_t\alpha_t - \mathbf{e}^{(N)\prime}_i\mathbf{H}^{-1}_t\mathbf{B}(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\bar{\beta}c_t + o_p(1),$$

since $(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\bar{\beta}c_t = c_t b^{-1}_t\mathbf{A}^{-1}_t\bar{\beta}$, straightforward manipulation yields

$$\mathbf{e}^{(N)\prime}_i\mathbf{\Sigma}^{-1}_t\alpha_t = \frac{1}{b_t}\mathbf{e}^{(N)\prime}_i\mathbf{H}^{-1}_t\left(\alpha_t + (a_t\alpha_t\bar{\beta}' - \mathbf{B}c_t)\mathbf{A}^{-1}_t\bar{\beta}\right) + o_p(1). \quad \square$$

(ii) For (32)

$$\begin{aligned}&\mathbf{w}^{meu}_t = \mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) - \mathbf{H}^{-1}_t\mathbf{B}(\mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) \\ &+\mathbf{H}^{-1}_t\mathbf{B}(\mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) - \mathbf{H}^{-1}_t\mathbf{B}(\mathbf{\Omega}^{-1}_t + \mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}) \\ = \; &\mathbf{H}^{-1}_t(\mathbf{I}_N - \mathbf{B}(\mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t)(\mu_t - \mathbf{e}r_{0t}) \quad (58) \\ &+\mathbf{H}^{-1}_t\mathbf{B}(\mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{\Omega}^{-1}_t(\mathbf{\Omega}^{-1}_t + \mathbf{B}'\mathbf{H}^{-1}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}^{-1}_t(\mu_t - \mathbf{e}r_{0t}). \quad (59)\end{aligned}$$

In relation to (58), we seek the asymptotic distribution of

$$N^{\frac{1}{2}}\left(w^{meu}_{it} - \mathring{w}^{meu}_{it}\right)$$

where

$$w_{it}^{meu} = \frac{1}{\gamma_t}\left(\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}(\mu_t - \mathbf{e}r_{0,t}) - \mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}(\mu_t - \mathbf{e}r_{0,t})\right),$$

$$\mathring{w}_{it}^{meu} = \frac{1}{\gamma_t}\left(\mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}(\alpha_t - \mathbf{e}r_{0,t}) - \mathbf{e}_i^{(N)\prime}\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}(c_t - a_t r_{0,t})\bar{\beta}\right).$$

By the continuous mapping theorem

$$N^{\frac{1}{2}}\left((\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}(\alpha_t - \mathbf{e}r_{0,t}) - (\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}(c_t - a_t r_{0,t})\bar{\beta}\right) \to_d \zeta_t^{meu},$$

where ζ_t^{meu} is a $k \times 1$ normally distributed random vector with zero mean yielding

$$N^{\frac{1}{2}}\left(w_{it}^{meu} - \mathring{w}_{it}^{meu}\right) \to_d \gamma_t^{-1}\xi_{1i,t}'\zeta_t^{meu} = z_{i,t}^{meu},$$

which is a mixture of normal random variables, unless $\xi_{1i,t}$ is non-random.

Concerning term (59) that involves $\mathbf{\Omega}_t$ and $\mu_{f,t}$, by Lemma A,

$$\begin{aligned} & \mathbf{e}_i'\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{\Omega}_t^{-1}(\mathbf{\Omega}_t^{-1} + \mathbf{B}'\mathbf{H}_t^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t^{-1}(\mu_t - \mathbf{e}r_{0t}) \\ = \; & \mathbf{e}_i'\mathbf{H}_t^{-1}\mathbf{B}(\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\left(N^{-1}\mathbf{\Omega}_t^{-1}\right)\left[\mu_{f,t} + (\mathbf{A}_t + a_t\bar{\beta}\bar{\beta}')^{-1}\bar{\beta}(c_t - a_t r_{0,t})\right](1 + o_p(1)). \end{aligned}$$

(iii) By Lemma A

$$\mathbf{w}_{t-1}^{meu\,\prime}\mathbf{r}_t = \gamma_t^{-1}\left[(\alpha_{t-1} - \mathbf{e}r_{0t-1})'\mathbf{\Sigma}_{t-1}^{-1}(\alpha_{t-1} + \varepsilon_t)\right](1 + O_p(1)).$$

By part (i)

$$N^{-1}(\alpha_{t-1} - \mathbf{e}r_{0t})'\mathbf{\Sigma}_{t-1}^{-1}\alpha_{t-1} = \frac{1}{b_{t-1}}\left[d_{t-1} + (a_{t-1}d_{t-1} - c_{t-1}^2)\bar{\beta}'\mathbf{A}_{t-1}^{-1}\bar{\beta} - r_{0t-1}c_{t-1}\right] + o_p(1),$$

and since

$$N^{-1}(\alpha_{t-1} - \mathbf{e}r_t)'\mathbf{\Sigma}_{t-1}^{-1}(\alpha_{t-1} - \mathbf{e}r_t) = e_{t-1}/b_{t-1} + o_p(1),$$

where e_t is defined in (35), one gets

$$\begin{aligned} & N^{-1}(\alpha_{t-1} - \mathbf{e}r_{0t-1})'\mathbf{\Sigma}_{t-1}^{-1}\varepsilon_t \\ = \; & O_p\left(N^{-1}((\alpha_{t-1} - \mathbf{e}r_{0t-1})'\mathbf{\Sigma}_{t-1}^{-1}\mathbf{H}_{t-1}\mathbf{\Sigma}_{t-1}^{-1}(\alpha_{t-1} - \mathbf{e}r_{0t-1}))^{\frac{1}{2}}\right) = O_p(N^{-\frac{1}{2}}(e_{t-1}/b_{t-1})^{\frac{1}{2}}), \end{aligned}$$

the last equality being obtained by Lemma A since $\mathbf{H}_t = \mathbf{\Sigma}_t - \mathbf{B}\mathbf{\Omega}_t\mathbf{B}'$. Similarly,

$$N^{-1}\mathbf{w}_{t-1}^{meu\,\prime}\mathbf{e} = \gamma_t^{-1}(c_{t-1} - a_{t-1}r_{0,t-1})(1 + o_p(1)).$$

It is easy to see that $e_t > 0$ almost surely. In fact e_t is given by the sum of $d_t - 2r_{0,t}c_t + a_t r_{0,t}^2$ and $(a_t d_t - c_t^2)\bar{\beta}'\mathbf{A}_t^{-1}\bar{\beta}$. The latter term is positive since $a_t d_t > c_t^2$ by our assumption and $\mathbf{A}_t$ is positive definite. The first term $d_t - 2r_{0,t}c_t + a_t r_{0,t}^2$ is certainly non-negative since it equals the probability limit of the quadratic form $N^{-1}(\alpha_t - r_{0,t}\mathbf{e})'\mathbf{H}_t^{-1}(\alpha_t - r_{0,t}\mathbf{e})$. Finally, (34) follows since $\mu_{\rho,t-1}^{meu} = (r_{0,t-1} + \gamma_{t-1}^{-1}(\mu_{t-1} - r_{0,t-1}\,\mathbf{e})'\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - r_{0,t-1}\mathbf{e}))$ and $\sigma_{\rho,t-1}^{meu\,2} = \gamma_{t-1}^{-2}(\mu_{t-1} - r_{0,t-1}\mathbf{e})'\mathbf{\Sigma}_{t-1}^{-1}(\mu_{t-1} - r_{0,t-1}\mathbf{e})$.
□

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