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Analytic Solutions for Supply Function Equilibria: Uniqueness and Stability

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Supply Function Equilibria (SFE) offer an attractive equilibrium concept for an electricity Pool in which all suppliers receive the market clearing price and are an important tool for examining market power. It is helpful to have analytical solutions available for simple models to explore market behaviour and to check computational solutions. This note derives analytic solutions for the symmetric case of linear and quadratic costs, and where each firm has an identical set of constant but different marginal cost technologies, as in most practical applications to data. Such stepped marginal cost schedules can replicate general marginal cost schedules to any desired degree of accuracy and hence symmetric SFEs can be solved analytically by piecing together recursively defined supply functions for general cost functions. The paper discusses the question of the uniqueness and stability of these symmetric solutions, but notes that finding asymmetric analytic solutions is generally difficult. It collects together and extends results scattered in earlier working papers to make them more accessible.

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Analytic Solutions for Supply Function Equilibria: Uniqueness and Stability

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September 18, 2008

Abstract

Supply Function Equilibria (SFE) offer an attractive equilibrium concept for an electricity Pool in which all suppliers receive the market clearing price and are an important tool for examining market power. It is helpful to have analytical solutions available for simple models to explore market behaviour and to check computational solutions. This note derives analytic solutions for the symmetric case of linear and quadratic costs, and where each firm has an identical set of constant but different marginal cost technologies, as in most practical applications to data. Such stepped marginal cost schedules can replicate general marginal cost schedules to any desired degree of accuracy and hence symmetric SFEs can be solved analytically by piecing together recursively defined supply functions for general cost functions. The paper discusses the question of the uniqueness and stability of these symmetric solutions, but notes that finding asymmetric analytic solutions is generally difficult. It collects together and extends results scattered in earlier working papers to make them more accessible.

1 Introduction

A Supply Function Equilibrium (SFE) is a set of supply functions $S_i(p)$ (made up of offers to supply by firms i, $i = 1, 2, ...n$, when the price is p) such that each firm maximizes its profit taking the supply functions of the other firms as given, and is thus a Nash equilibrium in supply

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functions. It can be found from the profit maximizing behaviour of firm i facing residual demand $R_i(p) = D(p) - \sum_{j \neq i} S_j(p)$, where $D(p)$ is demand facing the n firms at price p.

Klemperer and Meyer (1989) were interested in the case in which demand was uncertain, so that offers to supply had to be valid for a range of possible outcomes. Thus if $D(p, t)$ varies with t, which they took to be states of the world, then p will be a function of that variable, and the SF is chosen given some probability distribution over t before the state of the world is realized. Green and Newbery (1992) argued that t could be taken as time during the course of a day for which offers to supply at varying prices into an electricity Pool were valid. In practice both electricity demand and supply are subject to random shocks at each moment, so that t describes both predictable and unpredictable variations over the duration of the offers. This additional uncertainty, notably the Loss of Load Probability that is the reason for the reserve margin, contributes a critical determinant of uniqueness (Holmberg, 2005a; 2008).

Supply Function Equilibria offer an attractive equilibrium concept for an electricity Pool in which all suppliers receive the market clearing price, MCP (sometimes termed the system marginal price, or SMP). They appear to capture aspects of reality, in that the price-cost margin is low when demand is low relative to available capacity (Bertrand-like behaviour) and high when demand is high relative to available capacity (Cournot-like behaviour). An SFE is most simply characterized when supply and demand can be represented as differentiable functions of price. The profit function for firm i offering a supply function $S_i(p)$ and facing a residual demand function $R_i(p) = D(p) - \sum_{j \neq i} S_j(p)$ when the price is p is $\pi_i(p, t) = pR_i(p) - C_i(R_i(p))$ where $C_i(q_i)$ is the cost function when production is q_i, or

$$\pi_i(p, t) = p \left(D(p) - \sum_{j \neq i} S_j(p) \right) - C_i \left(D(p) - \sum_{j \neq i} S_j(p) \right). \tag{1}$$

The first order condition for firm i maximizing profit can be written

$$S_i(p) = (p - C_i'\{S_i(p)\})(\sum_{j \neq i} S_j'(p) - D_p(p, t)), \quad i = 1, 2, ...n. \tag{2}$$

where derivatives with respect to the dependent variable are marked with a dash, or, in the case of demand, with subscript p. Klemperer and Meyer (1989) showed that the solutions to this set of n interdependent differential equations are potentially supply function equilibria.

In addition to these various market relations, there are further conditions needed for feasibility: the solutions must be (weakly) monotonically increasing in price, outputs cannot be negative, and output must be feasible given capacity constraints. Anderson and Hu (2008) use the term *strongly optimal* if the supply function gives the highest achievable profit subject to constraints on price, capacity, and the other supplies offered, for every realization of demand, ruling out cases where the best supply response has a negative slope (and must therefore be constrained to a constant over an interval whose choice depends on the probability distribution of demand shocks). They characterise a *strong* SFE as one in which supply functions are piecewise smooth and strongly optimal. Weakly monotonically increasing piecewise smooth supply functions satisfy these criteria.

The effect of requiring weak monotonicity is most readily appreciated in the case of a symmetric n-firm oligopoly with separable demand, $D(p,t) = D(p) + \varepsilon(t)$, for which (2) can be written

$$q(p) = (p - C'(q))((n-1)\frac{dq}{dp} - D_p),$$

$$(n-1)\frac{dq}{dp} = \frac{q}{p - C'(q)} + D_p, \tag{3}$$

where $D_p \leq 0$. For valid solutions $\infty \geq dS_i^*(p)/dp = dq/dp \geq 0$. If $p = C'(q)$, then $dq/dp = \infty$ (i.e. $dp/dq = 0$) and the solution is competitive, while if $dq/dp = 0$, then $q + (p - C'(q))D_p = 0$, which is the Cournot solution. Valid solutions are therefore bounded between the competitive solution and the Cournot solution, between which there may be a continuum of possible SFE. This multiplicity of possible equilibria is troubling, and much of the study of SFE has been concerned to narrow down the set of possible SFEs, preferably to a unique case.

If we consider the more conventional representation with output on the x-axis, then the most competitive SFE meets the marginal cost schedule (assuming it increases) at its intersection with maximum demand, where its slope is flat: $dp/dq = 0$, and the least competitive SFE meets the intersection of maximum demand with the Cournot line at $\widehat{p}$ with slope there vertical: $dp/dq = \infty$. All valid SFEs lie between these two extreme solutions: $C'(q(p) \leq p \leq C'(q(p)) + q(p)/(-D_p)$ Figure 1 illustrates this for the case of England and Wales in 1990, as fitted in Green and Newbery (1992). it shows that the range of feasible SFE becomes far smaller as the number of generating companies increases from two (the chosen structure) to five (as recommended by Green and Newbery, and implemented in e.g. Victoria, Australia shortly after).

Newbery (1992a, 1998) shows how capacity constraints can narrow down the range of valid SFEs, in some cases to a unique solution, as shown in section 2.3. If generators are subject to a potentially binding price cap $\overline{p} < \widehat{p}$, assumed to be above the maximum value of marginal cost, then the range of solutions will also be restricted. In short-hand, then, a solution to (1), S^*, is a set of supply functions $S_i^*(p) \geq C_i'(S_i^*(p)$, $i = 1, 2, ..n$, satisfying (2) for $\underline{p} \leq p \leq \overline{p}$, with $\infty \geq dS_i^*(p)/dp \geq 0$, where, if there is no formal price cap, $\overline{p} = \widehat{p}$.

However, there are a number of problems in solving for SFEs. Analytical solutions exist (and provide useful numerical checks) for the simple case of linear demand, if all firms have the same constant marginal costs at each level of output, even with varying capacity constraints. These

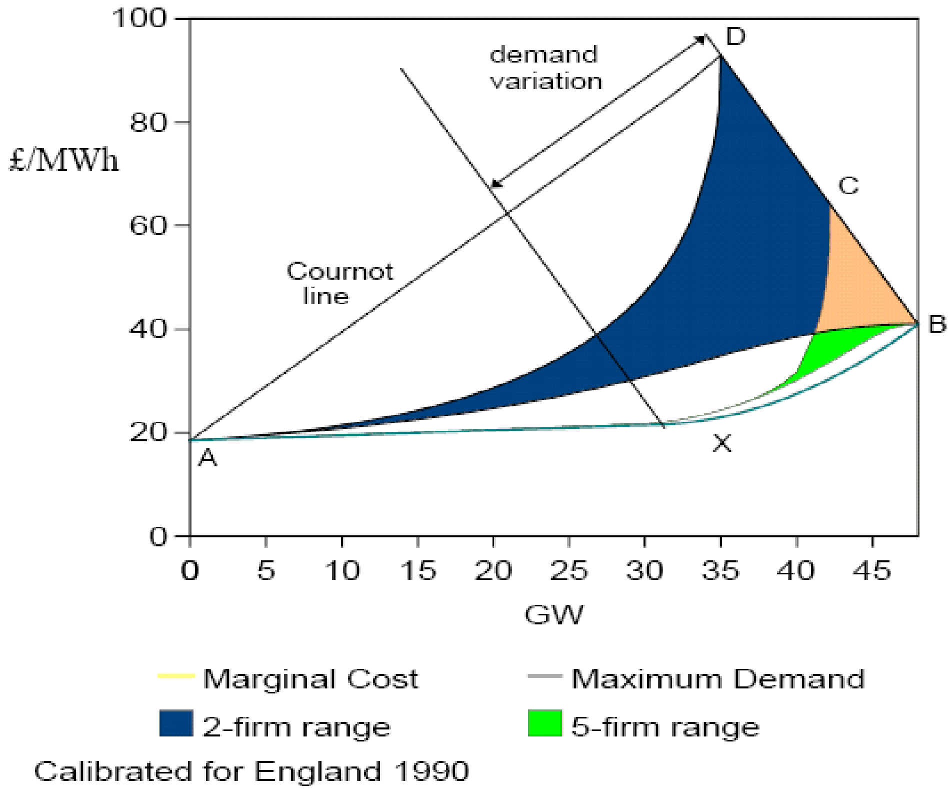

Figure 1: Supply functions for a duopoly and quintopoly

have the property that there is normally a continuum of equilibria bounded by a most and least competitive outcome, though the range can shrink to a unique equilibrium under certain demand and capacity conditions (Newbery, 1998). Specific affine solutions (i.e. solutions where offers are linear in price) can be derived for the case of linear demand and a quadratic cost function, giving linear marginal costs (Green, 1999; Baldick, Grant and Kahn, 2004). Analytic solutions can also be derived for the case of symmetrical firms and general cost functions provided the residual demand elasticity is zero (Rudkevich, 1998). This can be seen by rewriting (3) with $D_p = 0$ and multiplying by the integrating factor q^{1-n} to give

$$\begin{aligned} \frac{dp}{dq} &= \frac{(n-1)(p-C'(q))}{q}, \\ \frac{d}{dq}(q^{1-n}p) &= (1-n)q^{-n}C'(q), \\ p &= q^{n-1}\int(1-n)q^{-n}C'(q)dq. \end{aligned} \tag{4}$$

Thus for the case of affine (i.e. linear with an intercept) marginal costs, $C'(q) = a + bq$, (4) gives

$$p = q^{n-1} \int (1-n)(aq^{-n} + bq^{1-n})dq,$$

$$= a + \frac{n-1}{n-2} bq + Kq^{n-1}. \tag{5}$$

The case of inelastic but uncertain demand, with a positive probability that capacity constraints will bind, yields unique SFE, as demonstrated by Holmberg (2004) and discussed in 2.3 below.

Newbery (2002) shows that analytical solutions (but in implicit function form) can also be derived for the symmetric n-firm oligopoly case with linear marginal costs and linear demand, and these have a similar graphical appearance to the constant marginal cost case. Their derivation is set out in section 3 below.

2 SFEs with constant marginal costs

Consider the case of n symmetric core strategic firms with constant marginal costs (set at zero by a choice of the price intercept), facing an demand schedule $Q(t)$ =Max$(A(t) - \gamma p, 0)$.[1] The differential equation is given from (3):

$$(n-1)\frac{dq}{dp} = \frac{q}{p} - \gamma. \tag{6}$$

This is readily integrated to give

$$q = Kp - \gamma p \ln p, \qquad n = 2, \tag{7}$$

$$q = Kp^{1/(n-1)} - \frac{\gamma p}{(n-2)}, \qquad n > 2, \tag{8}$$

where K is a constant of integration. The constant will depend on how competitive the industry is, and may be determined by entry conditions that drive average prices down to the average cost of new entrants (Newbery, 1998). The least competitive solution is found by the joint intersection of the SF with the Cournot line $Q^c = n\gamma p$ and the maximum demand schedule, $\overline{D} = \overline{A} - \gamma p$ at $\widehat{p}$. For example, in the case of $n = 3$, $\widehat{p} = \frac{1}{4}\overline{A}/\gamma$ and $K = 2\gamma\sqrt{\widehat{p}}$, so that $q = 2\gamma\sqrt{\widehat{p}p} - \gamma p$.

2.1 Symmetric stepped marginal cost functions

The next step in complexity is to consider a set of n identical firms with a series of generating sets, each of the same capacity but different variable costs. These can be numbered in order of increasing cost, with marginal costs given by $C'(q) = m_j$, $k_j \leq q < k_{j+1}$, $j = 1, ...N$, $k_1 = 0 = m_1$

[1] This could be written in the form $A(t) - p$ by a suitable choice of quantity units, but it will be convenient later to be able to vary the slope parameter γ.

(by choice of the origin of prices), facing demand $A - p$, (i.e. setting $\gamma = 1$ by a suitable choice of units). Suppose for the moment that each firm has an unlimited amount of the final generating set, numbered N. Suppose that there are $n > 2$ firms (the case of $n = 2$ has a different functional form (7) but can be solved by the same techniques). The differential equations are given by (3). Define $y = p - m_j$, and consider a range of supply offers $k_j \leq q < k_{j+1}$, for which the differential equation can be written

$$\begin{aligned} (n-1)\frac{dq}{dy} &= \frac{q}{y} - 1, \quad \text{for which the solution is} \\ q &= K_j(p - m_j)^{1/(n-1)} - \frac{(p - m_j)}{(n-2)}, \quad k_j \leq q < k_{j+1}. \end{aligned} \tag{9}$$

It remains to select a set of constants K_j such that the supply functions (SFs) that solve (9) are both feasible and continuous (i.e. are strong equilibria, see Proposition 2 below) at values $(q, p) = (k_{j+1}, p_{j+1})$. Feasibility requires that the SFs lie between the Cournot line and the marginal cost schedule, but now the Cournot line is discontinuous (as in figure 2) and defined in segments as $Q_j^c = n(p - m_j)$, $j = 1, ...N$. The constants must be such that there is a continuous solution threading back between these two boundaries from a suitable end point boundary condition.[2]

For example, suppose $n = 3$, $N = 2$, so there are two different types of generation set with variable costs zero and m, with $k_2 < \frac{1}{4}(\overline{A} - m)$ (so that both types will be used) The Cournot lines are $Q_1^c = 3p$, $0 \leq Q_1^c \leq 3k_2$, $Q_2^c = 3(p - m)$, $3k_2 < Q_2^c$. For existence, the Cournot line cannot intersect the next step in the marginal cost schedule, so $m < k_2$. One can then check to see if the least competitive solution for the second segment is feasible for the first segment. That is found by the joint intersection of the SF with the Cournot line, and for this to occur when the marginal cost is m, $Q^c = 3(p - m)$, meets the maximum demand schedule, $\overline{D} = \overline{A} - p$ at $\widehat{p} = \frac{1}{4}\overline{A} + \frac{3}{4}m$. $\widehat{q} = \frac{1}{4}(\overline{A} - m)$, (at B in figure 2) so that

$$\begin{aligned} \widehat{q} &= \frac{1}{2}K_2\sqrt{\overline{A} - m} - \frac{1}{4}(\overline{A} - m) = \frac{1}{4}(\overline{A} - m), \\ K_2 &= \sqrt{\overline{A} - m}. \end{aligned}$$

Then at (k_2, p_2) $q(p_2 - \varepsilon) \to q(p_2 + \varepsilon)$ as $\varepsilon \downarrow 0$, or

$$\begin{aligned} k_2 &= K_1\sqrt{p_2} - p_2 = K_2\sqrt{p_2 - m} - (p_2 - m), \\ K_1 &= \sqrt{p_2} + k_2/\sqrt{p_2}, \end{aligned}$$

[2] It is natural to start with the highest possible value and work back as that choice indexes all solutions to the differential equation, while there are an infinite number of solutions that pass through the origin. One could, however, choose any point bounded away from the local marginal cost to index solutions.

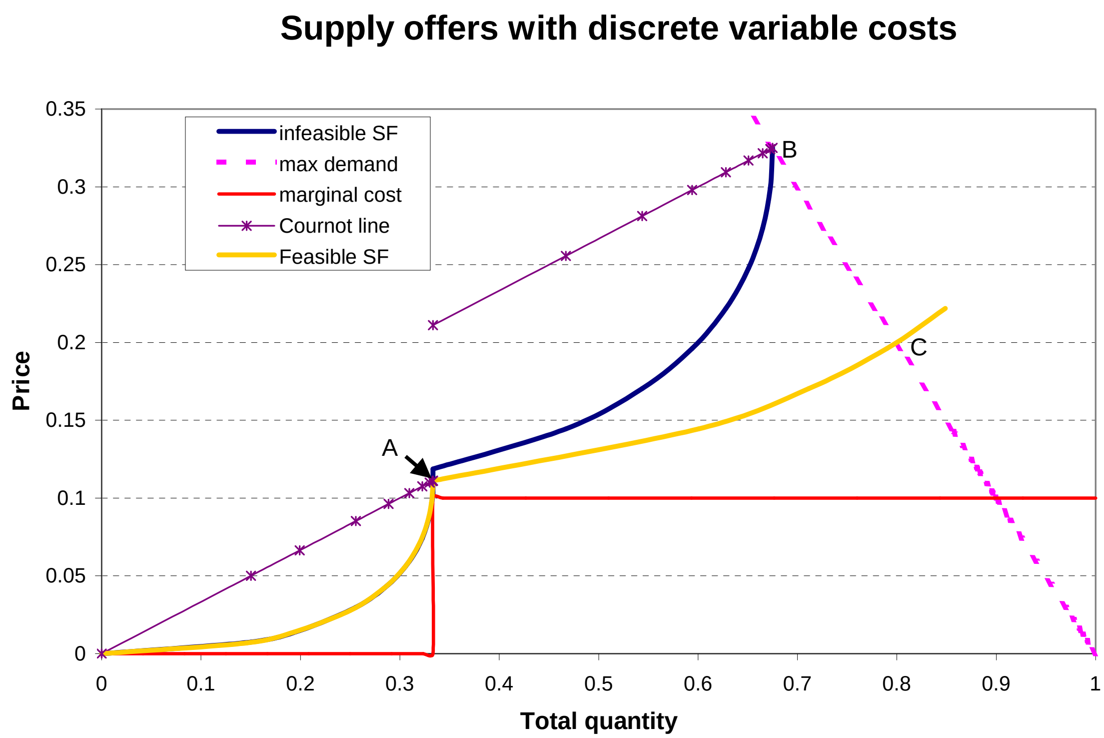

Figure 2: Supply functions for $n = 3, \gamma = 1 = \overline{A}$, $m = 0.1$, $k_2 = \frac{1}{9}$.

where p_2 can be solved in terms of K_2 and k_2. For example, if $\overline{A} = 1$, $m = 0.1 < k_2 = \frac{1}{9}$, $\widehat{p} = 0.325$, then $K_2 = 0.95$, and $y = p_2 - 0.1$ solves a quadratic giving solution $p_2 = 0.12$, $K_1 = 0.67$. This is readily seen to be infeasible, as it lies above the first Cournot line, whose value at $k_2 = 0.111 < p_2$. Instead the least competitive solution meets the first Cournot line at $(k_2, p_2) = (k_2, k_2)$, (at point A in figure 2) so $K_1 = \frac{2}{3}$. Continuity at (k_2, k_2) determines $K_2 = 1.16$, and as figure 2 shows, leads to a SF that is considerably below the infeasible least competitive second segment SF, reaching maximum demand at point C.

Whilst it is generally difficult to solve for a smoothly continuous general marginal cost function $C(q)$, except for inelastic demand, in practice most electricity market models assume constant marginal costs for each type of generation unit (nuclear, coal, gas, gas turbine, etc.), and so this extension is of considerably practical use, provided that all price-setting generating companies have the same amount of each type of generating set. Figure 3 gives one of several marginal cost schedules for the six countries studied in the EU Sector Inquiry, and it demonstrates that a stepped marginal cost schedule is a better approximation that a smooth curve.

It is possible to solve (2) numerically for the asymmetric case, although with difficulty as the solutions are very sensitive to initial conditions. Fortunately, the solutions for asymmetric

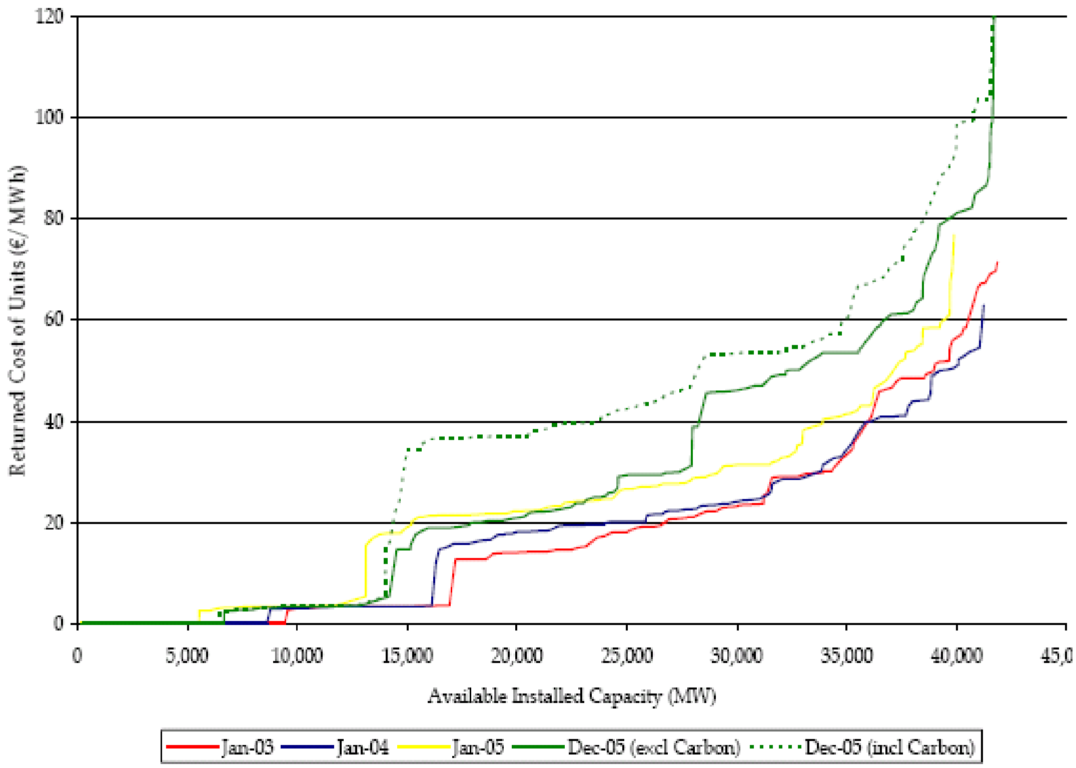

Figure 3: Merit Order Curve (incl. Carbon) - Spain (London Economics, 2007, p403)

configurations seem to be perturbations from symmetric configurations, so the latter may provide a reasonable approximation (as Rudkevich, 1998, assumed in his empirical illustration).

2.2 The effect of price caps

If the market is subject to a price cap of $\overline{p} < \widehat{p}$, and if the demand function is again written as $A(t) - \gamma p$, then the least competitive SFE for the case $n = 3$ in (8) has $K = \frac{\overline{A}-2\gamma\overline{p}}{3\sqrt{\overline{p}}}$ and so aggregate supply $Q = 3q$:

$$Q(p, \gamma) = (\overline{A} - 2\gamma\overline{p})\sqrt{p/\overline{p}} - 3\gamma p.$$

For any value of p, Q is clearly decreasing in γ, so there is a well-defined sense in which the least competitive solution is that for which $\gamma = 0$, and so solving for the case of inelastic demand provides an upper bound (for prices, given supply) on the range of feasible SFEs. Setting $\gamma = 0$ in (8) gives the same solution as (5) for the case $a = b = 0$.

2.3 The effect of capacity constraints

Newbery (1992a) solves for the zero marginal cost duopoly with potentially different capacity constraints, and that method is readily extended to an n-firm symmetric constant marginal cost oligopoly. Holmberg (2005a; 2007) provides an exhaustive analysis of the identical constant marginal cost case with differing capacities, and provides convenient proofs of the more important propositions. Genc and Reynolds (2004) show that the existence of a pivotal supplier reduces the set of possible SFE, and Holmberg (2005a) establishes uniqueness with inelastic but uncertain demand, a positive probability that the capacity constraint will bind, and a price cap, $\overline{p}$ (to constrain prices with monopoly and inelastic demand). He also establishes useful propositions for this special case that justify the way in which solutions to the differential equations are pieced together, which it is useful to reproduce here. They apply for the case of firms with identical constant marginal costs, facing inelastic but uncertain demand, a positive probability that the capacity constraint will bind, and a price cap, $\overline{p}$, but apply under a wider range of conditions. Thus Anderson and Hu (2008) give similar propositions for more general cost and demand functions for strong supply function equilibria but under the assumption that demand is sufficiently elastic to prevent any price caps binding.

Proposition 1 *In equilibrium no capacity is offered below marginal cost or withheld.*

Proposition 2 *There are no discontinuities in the equilibrium price.*

Proposition 3 *In equilibrium, no firm can have a perfectly elastic supply below the price cap, and at most one firm can have perfectly elastic supply at the price cap.*

Proposition 4 *In equilibrium, all firms for whom capacity is not constrained in an interval must have identical SFs on that interval.*

Proposition 5 *There is no equilibrium in which the SF of a firm is inelastic in an interval* $[p_L, p_U]$ *where marginal cost* $c \leq p_L < p_U \leq \overline{p}$ *unless its capacity constraint is binding.*

As a result every firm offers its first (infinitesimal) increment at marginal cost and has price responsive supply up to full capacity. If demand is price responsive, i.e. $\gamma > 0$, then it is no longer necessary to impose a price cap to ensure finite solutions and so, in the absence of a price cap, there should be no perfectly elastic segments of the SFE.

Consider the deterministic demand case in which there is no risk (zero probability) of loss of load. If firm i has capacity k_i, so that $q_i \leq k_i$, then the solutions (7,8) must be modified to reflect the constraints. As an example, consider the asymmetric capacity case of $n = 3$, $k_1 < k_2 < k_3$, with demand schedule $Q(t) = \text{Max}(A(t) - \gamma p, 0)$. Define $K_j = \sum_{i=1}^{j} k_i$ as cumulative capacity of the first j firms, and consider solutions for respectively 3, 2, and then only 1 price setting firm(s):

$$\begin{aligned}
q_i &= B_1\sqrt{p} - \gamma p, \qquad p \le p_1 = q_1^{-1}(k_1), \\
q_i &= B_2 p - \gamma p \ln p, \quad p_1 \le p \le p_2 = k_2/\gamma, \quad i = 2, 3, \\
q_1 &= k_1, \quad p \ge p_1, \\
q_3 &= \gamma p, \quad q_2 = k_2, \quad p \ge p_2.
\end{aligned}$$

The constants for the least competitive solution are determined by working back from the equilibrium at maximum demand.

If the intersection of maximum demand with the single firm Cournot line $q^c = \gamma p$ is at demand $\overline{D} = \overline{A} - \gamma p$ greater than K_2, then the largest firm will be pivotal, i.e. the single price-setter, setting the Cournot price, over some range of prices. A sufficient condition for a terminal monopoly is that $\overline{A} \ge A(t) > k_1 + 3k_2 = A(t_2)$, for a range of values of t. If that condition is satisfied, then the second largest firm must reach full capacity on its Cournot line $q^c = \gamma p$ at $p_2 = k_2/\gamma$, giving the unique value of $B_2 = \gamma(1 + \ln(k_2/\gamma))$. Output of firm 2 will be $q_2 = \gamma p(1 - \ln(\gamma p/k_2))$, which will decrease with price to k_1 at price p_1, where $k_1 = \gamma p_1(1 - \ln(\gamma p_1/k_2)) > \gamma p_1$, which has a valid solution $p_1 < p_2$, for which $B_1 = (k_1 + \gamma p_1)/\sqrt{p_1}$ The result will be a unique piecewise continuous SFE. Multiple SFE require that the largest firm is never pivotal (i.e. it is possible to meet maximum demand with capacity K_2). This can be summarized in

Proposition 6 *In a deterministic market with linear price responsive demand and all firms having identical constant marginal costs, the SFE is unique if a single firm is pivotal (but never capacity constrained) for a range of demand levels.*

Figure 4 (this time graphing quantities against price rather than the more usual presentation of price against quantity) illustrates the case for capacities $k_1 = 0.5,\ k_2 = 0.7,\ k_3 = 0.8,\ \gamma = 1,\ \overline{A} = 2.8,\ \underline{A} = 1.5$, showing that at minimum demand none of the firms is capacity constrained, but at peak demand both the smaller firms are capacity constrained, so that the largest firm is pivotal for a short period, but never reaches full capacity. The restriction that supplies must be non-decreasing in price binds for firms 1 and 2.

2.4 The effect of contracts and entry

Electricity spot markets are volatile, and both suppliers and consumers wish to hedge risk with contracts, typically for base-load, peak, or possibly for other sets of hours. The effect of contracting is to add an extra term $(f - p)x_i$ to profit, where x_i is the size of the (base-load) contract sold

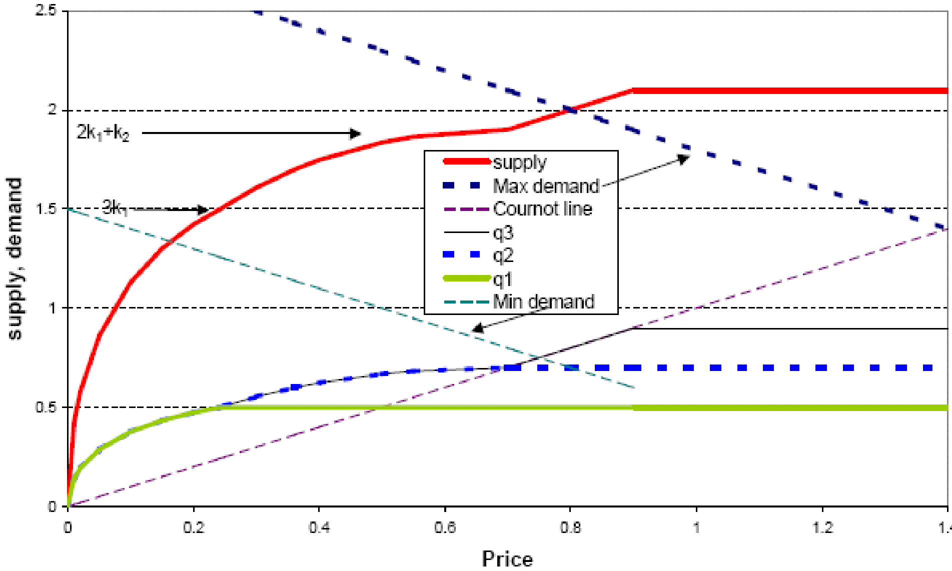

Figure 4: Asymmetrically capacity constrained triopoly

at strike price f. Newbery (1998) shows that the effect of contracting is to reduce the spot market exposure from q_i to $q_i - x_i$, and the differential equations remain as before if $y_i = q_i - x_i$ replaces q_i. Thus all SFEs can be considered as spot market solutions for uncontracted quantities.

If there are fixed capacity costs in addition to variable costs, then equilibrium in an electric supply industry under free entry will determine the maximum sustainable average price, which will set the price of the base-load contracts, f, while arbitrage will force the average of the spot prices in a risk-neutral industry down to f. Contracts allow incumbents to commit to delivering an average price f, possibly after accepting entry or new incumbent investment, and reducing the range of possible SFE to a unique SFE.

2.5 Piece-wise linear demands and fringe suppliers

Baldick, Grant and Kahn (2004) argue for using piecewise linear supply functions (PWLSFs) as providing simpler solutions for more complex cases where there is a fringe of competitive generators and a core of strategic generators. If the fringe has a linear supply function, and total demand is linear, as before, their presence can be represented by a kinked residual demand function facing the core producers. They also consider the case in which each firm has quadratic costs and hence affine marginal costs, $a_i + b_i q_i$, where each firm may have a different minimum cost, a_i. Again, the residual demand facing firms can be represented as piecewise linear.

Their argument for using PWLSFs is one of tractability, but it leads to solutions that are unlikely to be optimal. It seems preferable to attempt to understand the optimal SF given piecewise linear continuous demand schedules. This section therefore starts with the simple case of constant marginal costs, and then shows how to extend this to the case of a competitive fringe of suppliers whose collective supply is linear in price up to full capacity. This can be extended to the case in which each member of the fringe has a possibly different capacity-constrained linear supply function leading to a piece-wise linear aggregate fringe supply function.

If aggregate fringe supply is $Q^f = (\gamma - \mu)p,\ \ p \leq p^*$, $\gamma > \mu$, where full capacity $(\gamma - \mu)p^*$ is reached at p^*, so that above that price fringe supply is constant at $(\gamma - \mu)p^*$, and if total demand is the non-negative part of $A(t) + \gamma p^* - \mu p$, then the net demand facing the core oligopoly takes the quasi-concave piecewise linear form

$$\begin{aligned} D(p,t) &= \mathrm{Max}(A(t) - \gamma(p - p^*), 0), \quad p \leq p^*, \\ &= \mathrm{Max}(A(t) - \mu(p - p^*), 0), \quad p > p^*, \end{aligned} \tag{10}$$

where $A(t) \in [\underline{A}, \overline{A}]$, $0 < \underline{A} < \overline{A}$. The solutions from (7,8) to each segment of demand can be stitched together by a suitable choice of the constant. For example, consider the case of $n = 3$:

$$\begin{aligned} q &= K\sqrt{p} - \gamma p, \quad p \leq p^*, \\ q &= M\sqrt{p} - \mu p, \quad p > p^*, \quad M = K - (\gamma - \mu)\sqrt{p^*}. \end{aligned}$$

These solutions are piecewise continuous, but the Cournot line $Q^c(p)$ is discontinuous at p^*, reflecting the discontinuity of the marginal revenue schedule, and is given by

$$q = \theta p, \quad Q^c = 3\theta p, \quad \text{where } \theta = \gamma, \quad p \leq p^*; \quad \theta = \mu, \quad p > p^*.$$

The Cournot line thus may have either one or two intersections with realizations of the demand schedule, depending on the value of $A(t)$. If there is only ever a single intersection with demand for $p < p^*$, then all SFEs have $p < p^*$, corresponding to a single effective demand schedule $D(p,t)$ =Max$(A(t) - \gamma(p - p^*), 0)$ from (10). If there is only ever a single intersection with demand for $p > p^*$, then the least competitive SFE will be piecewise continuous with a discontinuous slope at p^*, meeting the intersection of maximum demand and the Cournot line for $p > p^*$ with $dq/dp = 0$. If there are two intersections for some values of $A(t)$, then the candidate for the least competitive solution is the SF that meets the intersection of maximum demand and the Cournot line for $p > p^*$, provided it does not first cross the Cournot line for $p < p^*$, for beyond that point it would have negative slope and thus not be a valid SFE (at least for an electricity market). A necessary condition for this is that $\gamma < \frac{\overline{A}}{4\mu p^*} + \frac{1}{4}$, the intersection of the maximum demand and the Cournot line for $p > p^*$.

Figure 5 shows an example where the Cournot line has two intersections for low levels of demand, but only one at higher levels of demand, and the least competitive SF meets the

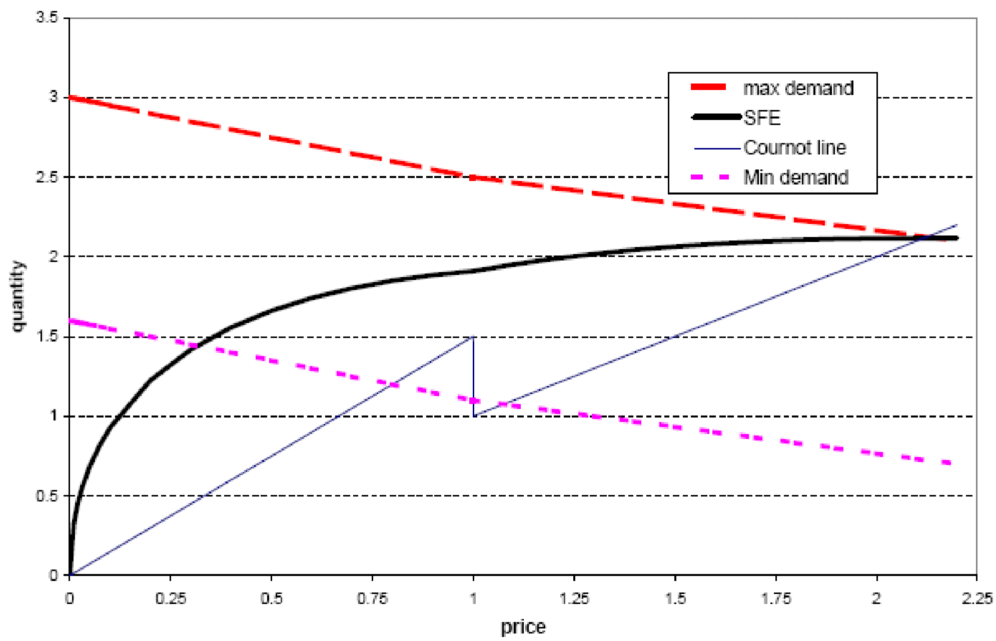

Figure 5: Triopoly with kinked demand showing unconstrained least competitive SFE

intersection of maximum demand and the Cournot line for $p > p^*$ without intersecting the Cournot line for $p < p^*$. All other SFEs lie between that and the y-axis (which, with zero marginal cost, is the line of marginal cost and the most competitive SFE).

If, as in Figure 6, the prospective SF that meets the intersection of maximum demand and the Cournot line for $p > p^*$ also crosses the Cournot line at $p < p^*$, then it has a section of (invalid) negative slope and cannot be an SFE. The least competitive SFE meets the intersection of maximum demand and the Cournot line for $p < p^*$, and all SFEs lie between that and the line of marginal cost (the y-axis). In both cases the value of the constant K for the least competitive SFE is determined by the intersection of the SFE with the Cournot line at maximum demand. Lower values of K then trace out more competitive solutions.

Capacity constraints (which differ by firm) can also be included (Newbery, 1992a). As each firm reaches capacity output along its SFE, so the remaining firms compete in an oligopoly of one fewer firms, until the final (largest) firm can play the Cournot strategy, all other firms producing an inflexible maximum amount. Again the aggregate SFE will be piecewise continuous with slope changes at each capacity constraint.

It is straightforward to extend this analysis to the case in which each fringe firm has supply

$$\begin{aligned} q_i^f &= \text{Max}(\text{Min}(\beta_i(p - a_i), k_i), 0), \\ S^f(p) &= \sum_i q_i^f. \end{aligned}$$

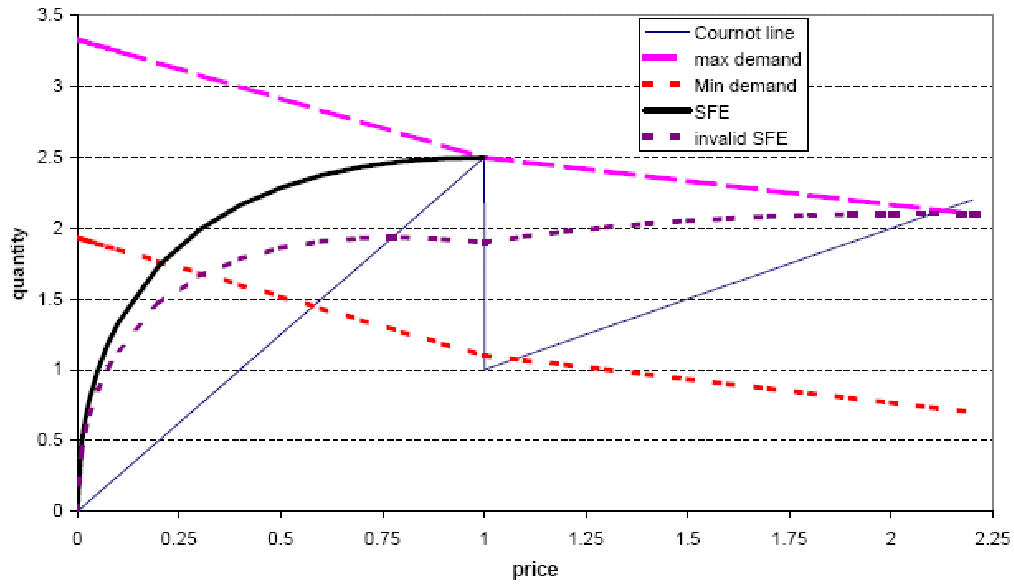

Figure 6: Triopoly with kinked demand: least competitive SFE constrained by Cournot line

The resulting net demand facing the oligopoly is piecewise continuous with kinks depending on a_i, k_i, and the solutions to (6) for each segment can again be pieced together to give piecewise continuous SFs. As the slopes of successive segments of demand approach each other (in the example as $\gamma - \mu \to 0$, so $M \to K$, and the SF's become closer to a continuous curve, suggesting that quasi-convex demand schedules that are not linear can be approximated by piece-wise linear demand schedules.

2.6 Solutions with differing marginal costs

The more challenging task is to derive solutions where each firm has differing (but constant) marginal costs, where the simple normalization of setting marginal costs to zero no longer works. Thus if we consider a duopoly in which firm i has constant marginal costs of c_i, $c = c_2 - c_1 > 0$, and if p is defined to be market price less c_1, then the SFE is defined by the pair of differential equations (where demand has negative slope γ that can be set either to unity for price-responsive demand or to zero for the inelastic case):

$$\begin{aligned} Dy - \frac{z}{p-c} + \gamma &= 0, \\ Dz - \frac{y}{p} + \gamma &= 0, \end{aligned} \tag{11}$$

where Dy is the first derivative of y w.r.t. p and where supply of firm 1 is y and supply of firm 2 is z. Differentiate each again to give two second-order linear ordinary differential equations

(ODEs):

$$\begin{aligned}
D^2y - \frac{Dz}{p-c} + \frac{z}{(p-c)^2} &= 0, \\
(p-c)D^2y + Dy - \frac{y}{p} &= -2\gamma, \\
pD^2z - Dy - \frac{y}{p} &= 0, \\
pD^2z - Dz - \frac{z}{p-c} &= 0.
\end{aligned}$$

If demand is inelastic, $\gamma = 0$, then the ODEs are homogenous, and in particular have linear solutions:

$$\begin{aligned}
y &= \text{Max}(\beta p, c), \\
z &= \text{Max}(\beta(p-c), 0).
\end{aligned}$$

Baldick and Hogan (2002, section 4.4) show that there are no affine solutions except for this special case in which $\gamma = 0$ and there are only two firms.

It is tempting to see if one can find simple solutions for the inelastic demand case in which firms i differ in the capacities they have, k_{ij}, of a set of different technologies, such that technology j has a constant marginal cost m_j, ranked such that $m_{j+1} > m_j$. The SFs now satisfy

$$Dy = \frac{z}{p-c_i}, \tag{12}$$

$$Dz = \frac{y}{p-m_i}, \tag{13}$$

where $c_i = m_j$ for some j. Differentiate (12) and (13) again

$$\begin{aligned}
(p-c_i)(p-m_i)D^2y + (p-m_i)Dy - y &= 0, \\
(p-c_i)(p-m_i)D^2z + (p-c_i)Dz - z &= 0.
\end{aligned}$$

Consider the piecewise linear solutions:

$$y = \beta_i(p-m_i), \tag{14}$$

$$z = \beta_i(p-c_i). \tag{15}$$

Suppose again that $m_1 = 0$, $m_2 = c$, and that firm 1 is the larger firm, with $k_{11} > k_{21}$, and that both firms have unbounded capacities of technology 2. Let $a_1 = k_{21}$ define the end of the first (zero marginal cost) segment, $a_2 = k_{11} > a_1$ the end of the second segment (with differing costs), and the third (unbounded) segment has both firms with the same costs, c. It is then straightforward to show that there are no continuous piecewise linear solutions to (14) and (15) (even if we allow for flat segments). Consider possible SFs with $y = z = \beta_1 p$, $y, z < a_1$, $y = \beta_1 p$,

$z = \beta_2(p - m)$, $a_1 < y, z < a_2$, $y = z = \beta_3(p - m)$, $a_2 < y, z$. Continuity requires $(\beta_2 - \beta_1)a_1 = \beta_1\beta_2 m$, $(\beta_3 - \beta_1)a_2 = \beta_1\beta_3 m$, and $\beta_3 = \beta_2$, which is impossible if $a_2 > a_1$. It therefore appears that it is not possible to find piecewise linear SFs for the asymmetric case in which generators have different capacities of each technology. It may be that (12) and (13) have non-linear solutions that could be pieced together to deal with this multiple technology asymmetric case, which would be very useful in extending the earlier symmetric multiple technology case.

Otherwise the ODEs in (11) may be attacked by Sturm-Liouville theory by writing them in the form

$$-D[f(p)Ds(p)] + g(p)s(p) = \lambda w(p)s(p),$$

where $s(p) = y(p)$ or $z(p)$, as above.

3 Quadratic cost functions

Consider the case of a symmetric n-firm oligopoly, each member of which has total costs $C(q) = aq + \frac{1}{2}bq^2$, and thus affine marginal costs $C'(q) = a + bq$, facing a demand schedule $A - \gamma p$. Renormalise and define a new variable y:

$$y \equiv \frac{p - a}{b}, \text{ or } \; p = a + by, \tag{16}$$

then (3) becomes (when $D_p = -\gamma$)

$$(n - 1)\frac{dq}{dy} = \frac{q}{y - q} - b\gamma. \tag{17}$$

This has as a linear solution $q = \beta y$, where

$$\beta = \frac{n - 2 - b\gamma + \sqrt{(n - 2)^2 + 2nb\gamma + b^2\gamma^2}}{2(n - 1)} < 1. \tag{18}$$

If $\gamma = 0$, this takes the simple form $q = \frac{n-2}{n-1}p$, as in (8), and for $\beta\gamma$ small, β is approximately

$$\beta = \left(\frac{n - 2}{n - 1}\right)\left(1 + \frac{b\gamma}{(n - 2)^2}\right) < 1.$$

More generally, one can search for linear supply function solutions for the asymmetrical case with affine marginal costs (that is, firm i has cost function $C_i(q_i) = a_i q_i + \frac{1}{2}c_i q_i^2$, with marginal costs $C_i' = a_i + c_i q_i$) and search for supply function solutions $S_i(p) =$Max$(\beta_i(p - \alpha_i), 0)$, as shown by Baldick, Grant and Kahn (2004).

3.1 General analytic solutions

In general we seek analytic solutions which pass through $(0, 0)$ and lie between the p-axis and the Cournot line $Q^c = nq = \frac{nb\gamma}{1+b\gamma}y$ where it meets maximum demand $\overline{A} - \gamma p = \overline{A} - a\gamma - b\gamma y = D(p, \overline{t})$

at $\bar{q}$, where y is defined by (16). The intersection is where $(n+1+b\gamma)\bar{q} = \overline{A} - a\gamma$ (at which point $dq/dy = \infty$). To make progress, first substitute $u = y - q$ in (17) to give the differential equation

$$\frac{du}{dy} = K - \frac{y}{(n-1)u}, \quad \text{where } K = \frac{n+b\gamma}{n-1}.$$

Substitute $u = y/x$ to give

$$\int \frac{dy}{y} = \int \frac{n-1}{x(n-1-(n+b\gamma)x+x^2)} dx = \int \left(\frac{1}{x} + \frac{\alpha}{x-B} - \frac{1+\alpha}{x-C} \right) dx$$

where A and B are the roots of the quadratic equation:

$$\begin{aligned} B &= \frac{n+b\gamma - \sqrt{(n+b\gamma)^2 - 4(n-1)}}{2} < 1; \\ C &= \frac{n+b\gamma + \sqrt{(n+b\gamma)^2 - 4(n-1)}}{2} > 1; \quad \alpha = \frac{C}{B-C}. \end{aligned}$$

Note that the term under the square root $(n+b\gamma)^2 - 4(n-1) = (n-2)^2 + 2nb\gamma + b^2\gamma^2$, the term under the root in (18), so that $B = (n-1)(1-\beta)$, with β given by (18). As the product of the roots, $BC = n-1$, it follows that

$$C = \frac{1}{1-\beta}, \quad \frac{B}{C} = (n-1)(1-\beta)^2 \equiv v < 1. \tag{19}$$

Equation (17) can now be integrated:

$$y = K_1 \frac{x(x-B)^\alpha}{(x-C)^{1+\alpha}}.$$

Substituting x and u and simplifying gives

$$((1-C)y + Cq)^{1+\alpha} = K_1((1-B)y + Bq)^\alpha,$$

$$y = \frac{C}{C-1}q + K_2(1-B)y + Bq)^{C/B} = \frac{1}{\beta}q + K_3 \left(q + \frac{1-B}{B}y \right)^{1/v}, \tag{20}$$

(substituting for $C/(C-1) = 1/\beta$ from (19)), and a constant of integration K_2 or K_3 to be determined by the boundary conditions. This is unfortunately in implicit form. In terms of the original variables solution (20) can be written as

$$p - a = bq/\beta + K_4 \left(bq + M(p-a) \right)^{1/v}, \quad M = \frac{2-(\beta+n)+\beta n}{(1-(\beta+n)+\beta n)}. \tag{21}$$

The value of K_4 can be determined, e.g. for the least competitive solution, which is the intersection of the Cournot line $q^c = b\gamma(y - q^c)$ or $Q^c = nq^c = n\gamma(p-a)/(1+b\gamma)$ with maximum demand $\overline{A} - \gamma p$ at $q^* = (\overline{A} - \gamma a)/(1+n-b\gamma)$; $p^* - a = (1+b\gamma)\, q^*/\gamma$. If these values of q, p are substituted into (21) then K_4 can be determined as:

$$K_4 = \frac{p^* - a - bq^*/\beta}{(bq^* + M(p^*-a))^{1/v}}.$$

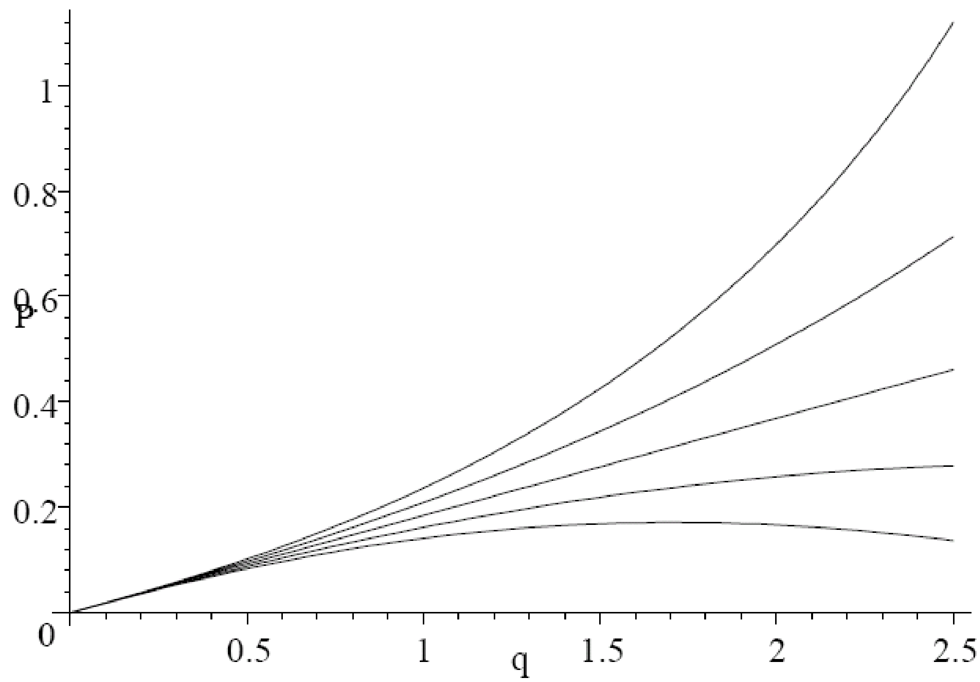

Figure 7: SFEs for the case $b = 0.2, \gamma = 1, n = 3, \overline{A} = 3$

Figure 7 plots the solutions to (21) for specific values of the parameters, where the linear solution is the middle of the five solutions graphed. Note that the lowest SF becomes infeasible as it decreases before reaching marginal cost (not shown), and hence would be invalid.

For the special case of inelastic demand, $\gamma = 0$, $B = 1$, and again we have $p = a + \frac{n-1}{n-2}bq + Kbq^{n-1}$, as before in (5). Moreover, if $K_3 = 0$, this allows the solution $q = \beta(p-a)/b$ as in (5). Equation (21) can be written as

$$q(p) = \beta(p-a)/b + K_5(q(p) + M(p-a)/b)^{1/v}. \tag{22}$$

showing that the SFE can be considered as an amplification of the linear solution under inelastic demand. Again the value of K_5 in (22) can be determined for e.g. the least competitive solution as above.

4 Stepped approximations to general cost functions

It is relatively easy to solve analytically for the stepped marginal cost function, where marginal costs are constant at m_j for step j. As an example, consider approximating the linear marginal cost function $C'(q) = bq$ over the range $[0, (N + \frac{1}{2})\delta]$, (where as the number of steps increases, so $\delta \to 0$, and $N\delta \to \widehat{Q}$). The marginal cost is $m_j = jb\delta$ over $((j - \frac{1}{2})\delta, (j + \frac{1}{2})\delta]$, $j = 1, 2, ...N$, and

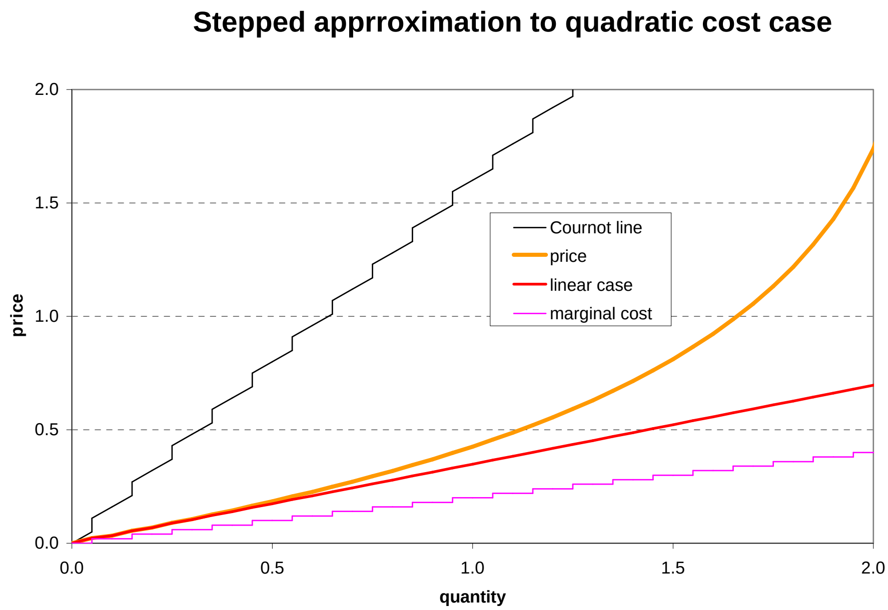

Figure 8: SFEs for the case $b = 0.2$, $\gamma = 1$, $n = 3$, $\overline{A} = 3$

$m_0 = 0$ over $[0, \frac{1}{2}\delta]$. The Cournot line is then $Q_j^c = n(p - m_j)$ over this interval and the solution is given by (9). The general solution must be feasible (i.e. lie between the Cournot line and the marginal cost schedule) and continuous, and over the j-th segment satisfy

$$q = K_j(p - m_j)^{1/(n-1)} - \frac{(p - m_j)}{(n-2)}, \quad (j - \frac{1}{2})\delta < q \leq (j + \frac{1}{2})\delta,$$

for suitable constants K_j.

Consider the case $n = 3$ and suppose at $Q = (N + \frac{1}{2})\delta$, the solution is p_N where

$$\begin{aligned} q &= (N + \frac{1}{2})\delta = K_N\sqrt{(p_N - Nb\delta)} - (p_N - Nb\delta), \\ K_N &= \sqrt{(p_N - Nb\delta)} + \frac{(N + \frac{1}{2})\delta}{\sqrt{(p_N - Nb\delta)}}. \end{aligned}$$

Once K_N is determined, the values of K_j can be determined recursively, by noting that at $q_{j-1} = (j - \frac{1}{2})\delta$ the SF on the $j-1$-th and j-th intervals are continuous at p_{j-1}, so, letting $p_{j-1} - m_j = y$,

$$\begin{aligned} q_{j-1} &= K_j\sqrt{y} - y = K_{j-1}\sqrt{y + b\delta} - (y + b\delta), \\ 0 &= y^2 + (2q_{j-1} - K_j^2)y + q_{j-1}^2, \\ K_{j-1} &= \frac{K_j\sqrt{y} + b\delta}{\sqrt{y + b\delta}} < K_j, \quad j = 2, ..., N. \end{aligned}$$

Figure 8 shows the results for $\delta = 0.1$, $b = 0.2$, for two cases, one setting $p_N = 2$, the linear case demonstrating that indeed if $p_N = bq_N/\beta$, where β is defined by (18), then the resulting SF is linear, replicating the analytical result for this quadratic cost case. With sufficiently small step lengths the symmetric SFE for any cost function can be solved by similar recursive techniques.

5 Stability analysis

Numerical solutions of the differential equations can in principle be found for the general case of equation (2), though for asymmetric firms the solutions are notoriously sensitive to the exact starting solution (Green and Newbery, 1992; Baldick and Hogan, 2001). Recently, Baldick and Hogan (2006) have raised a number of additional problems that they claim have the effect of considerably reducing the range of solutions to (2) that are eligible equilibria. Specifically, they cast doubt on whether the full range from least to most competitive solutions meeting all the required boundary and monotonicity conditions are stable. although this concept of stability has more to do with out-of-equilibrium behaviour and whether such behaviour will converge to an equilibrium (and if so to which one) than it has to existence. There are further problems that arise if there are price caps or discontinuities in the slope of the residual demand schedule already noted by Baldick, Grant and Kahn (2000) that are ignored in the following discussion.

Baldick and Hogan (2002, section 6) claim that even for the symmetric case with linear marginal costs and linear residual demand and with no capacity constraints, the range of stable equilibria is limited to a considerably lower set than that delimited by the most and least competitive solutions to (2). This is surprising, as Green and Newbery examined this case and showed that solutions to (2) that lie within this range also satisfy the second-order condition for profits to be maximized, not at a local minimum or point of inflection. At equilibrium, then, each firm is maximising its profits given the actions of others and has no reason to deviate. The reason that Baldick and Hogan claim many solutions are unstable has to do with the nature of perturbations. The normal interpretation of stability of a solution would seem to mean that if other firms continue to supply along the solution path S^*_{-i}, then any deviation by i from $S^*_i(p)$ would be unprofitable and hence would be corrected as soon as possible. Nevertheless, out of equilibrium behaviour and its stability depends sensitively on speeds of response and the decision variable (e.g. price or quantity) that is adjusted - there are cobweb models that are stable or unstable depending on the slopes of demand and marginal cost and whether responses are lagged one or more periods (see e.g. Newbery and Stiglitz, 1981).

Baldick and Hogan adopt a specific concept of, and test for, stability. They consider a particular form of deviation from a solution S^* in which all other firms follow S^* up to price p^ε, and thereafter each firm j follows a linear extrapolation (or affine supply function, ASF) from $S^*_j(p^\varepsilon)$, whose slope β_j at p^ε is the slope of $dS^*_j(p)/dp$ at p^ε. This has the attraction that the best response to an ASF of arbitrary slope is an ASF with a slope that is a function of the slopes

β_j. They then show that if the solution S^* is concave (and hence above the unique ASF that is an SFE, that we may therefore term the ASFE) then the best response to the ASF deviation is an ASF that is more deviant than the original deviant ASFs. In other words, any deviation to an ASF sets in train a response of increasing deviations to ASFs, that will, under reasonably conditions, converge on the ASFE (assuming also that the point of deviation can gradually move back towards the zero output point).

This result is not surprising, given two already known results. The first is that there is a unique ASFE, so any ASF that is not the ASFE will not be an equilibrium. The second is that the best ASF response to an arbitrary set of ASFs is closer to the ASFE. Together these imply that the best ASF response to an ASF lies between that ASF and the ASFE, and in this case outside the range of the ASF and the original solution set S^*.

I would argue that this does not demonstrate the general instability of SF solutions, S^*, except for a particular form of disequilibrium dynamics. There are several objections, but consider the following. The worst case is a duopoly, for then the rest of the industry consists of just one firm, so one can reasonably accept that all other firms make the same deviation. Much will then depend on the nature of the deviation, of which there are uncountably many (as there are many ways to specify cobweb behaviour, for example). Suppose the other firm deviates to a candidate SF $\in S^*$ (which makes some sense) and our original firm is already on an SF in S^*. Given that the second-order conditions are satisfied, we have an apparent puzzle. Each firm would earn higher profits moving to the SF chosen by his rival, but one of these SFs has higher profits than the other (at least in the short run if there is no threat of entry and no existing contracts). Logically, they should coordinate on the higher joint profit SF (which will involve higher offer prices for each level of supply), so the deviation might be a signal to coordinate on a more collusive equilibrium. In that case they would typically move in the opposite direction to that suggested by Baldick and Hogan's analysis. If, on the other hand, there are compelling reasons to select a less collusive equilibrium SF (because of the threat of entry, or of damaging regulatory scrutiny), then a determined firm adhering to the lower SF (i.e. lower prices offered for each output) could make it more profitable for the one offering a higher SF to deviate downwards, once it became clear that the firm offering the lower SF were determined to stick with that.

If there are more than two firms, and they are currently selecting the same SF that lies above the (equilibrium) ASFE, and one firm deviates to (and remains with) a new SF, then matters are more complex, as for each other firm, their SF is no longer an equilibrium given the new residual demand facing them. One can imagine an iterative process in which each of the other firms selects a new SF from the set S^* that is a weighted average of the original SF and that of the deviant. (An argument, which it may be possible to make rigorous, is that the slope of the SF offered is a function of the residual demand slope, which will in turn involve sums over the slopes of the other SFs offered, and these will be affected monotonically.) It is possible that if the original deviant continues to offer the same (new) SF, that eventually all other firms will converge on that SF, and a new SFE will have been reached. That will presumably depend on the beliefs of the firms about the desirability of coordinating on a higher or lower SF, as in the duopoly case.

It is also possible to imagine that firms will adopt punishment strategies for deviations that reduce their profits (and that are not justified by entry deterrence or other intelligent responses to changed circumstances). One such would be to select the lowest profit SF from the set S^* for a given number of periods. The problem with this argument is that punishments are normally selected to deter deviations that are individually rational but collectively profit reducing, and the second-order conditions mean that any individual deviation while all other firms stick to their original SFs would be individually profit reducing. There is a sense in which the concept of an SFE already contains its own punishment strategy.

What might go wrong with these arguments? We know from earlier cobweb models that iteratively responding to earlier price signals may or may not lead to convergence to the full equilibrium, depending on the relative slopes of supply and demand. We also know that learning models in which agents respond to price information that partially reveals information about cost and demand conditions may or may not converge to the full information equilibrium. Clearly in the electricity spot market although cost and (the distribution of residual) demand conditions may be common knowledge, the contracting position and hence optimal bidding strategy of other players will not be known, so learning models are relevant. They are somewhat pessimistic about convergence.

Baldick and Hogan have, however, noted the considerable difficulty in integrating the differential equations defining SFEs, and are anxious to propose an implementable method for selecting stable candidate SFEs. Their method would seem to restrict choices to the set of SFEs that are more competitive than the ASFE, for the function space over which they iterate is made up of piece-wise linear non-decreasing functions with uniformly distributed break points (Baldick and Hogan, 2001, section 8.2). If their argument that any SF above the ASFE is unstable to deviations to linear SFs applies, then they should not find any stable piece-wise linear SFs (PWLSF) above the ASFE. Apparently they do find some apparently stable PWLSF equilibria, perhaps because the deviations considered are by a single firm, with all other firms remaining on the previous candidate PWLSF. Note that the previous stability argument assumed that all other firms deviated (in a PWL fashion) from a candidate continuously differentiable SFE. Whether or not their algorithm will identify all the continuously differentiable SFs must be doubtful, and it would therefore be dangerous to assume that, for example, the SFE is actually more competitive than earlier work suggested, as less competitive candidate SFEs are unstable. All they will have demonstrated is that in some class of deviations and approximated PWLSFs, the set of feasible solutions is more limited than may have been predicted.

Recently several alternative solution strategies have been proposed to overcome the computational difficulties of finding numerical solutions. Holmberg (2005b) and Anderson and Hu (2008) propose different numerical algorithms to solve for SFE of markets with asymmetric firms and general cost functions, removing one argument for concentrating on linear solutions. More recently, Holmberg, Newbery and Ralph (2008) show that stepped supply function equilibria converge to continuous SFE as the number of steps increases, suggesting a possible discrete approximation method to finding solutions to continuous SFE.

6 Conclusions

Continuous supply functions are appealing representations of bidding behaviour in electricity pools and power exchanges, even where bids and offers are required to take the form of steps or price ladders (as Holmberg, Newbery and Ralph, 2008, demonstrate). Just as in standard Industrial Organization theory it is useful to have simple analytical models of imperfect competition (e.g. linear or quadratic cost functions under Cournot oligopoly) to explore market equilibria, so it is useful to have analytically tractable counterparts for the more complex supply function models of electricity markets. The aim of this paper was to derive solutions for the cases of symmetric linear or quadratic cost functions, and also to show that the quadratic cost case is qualitatively quite similar in general form to the simpler constant marginal cost case. It is also straightforward to solve the case of linear demand and identical oligopolists, each of which has the same portfolio of plants with differing but constant marginal costs. Such stepped marginal cost schedules are standard in modelling the electricity supply industry. In that sense if one is willing to accept symmetry, it is not necessary to use more complex smooth marginal cost schedules to approximate the underlying set of different technologies, and it is still possible to find analytical solutions, extending Rudkevich's (1998) approach to the case of price-sensitive demand.

However, whereas I-O economists can readily solve for static Cournot equilibria with differing marginal costs, it is in general very difficult to derive analytical solutions for SFE with differing marginal costs, unless all the asymmetries are confined to the competitive fringe or price-unresponsive suppliers (such as hydro or nuclear plant), which can be subtracted from aggregate demand to give a net demand facing the (symmetric) oligopolists.

Whether or not these equilibria are stable depends on out-of-equilibrium behaviour, and, contra Baldick and Hogan (2006), there are no good reasons for thinking that all but affine solutions are unstable.

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