Second-Order Algebraic Theories

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بِسمِ اللهِ الرَّحمٰنِ الرَّحِيمِ

إهداء

إلى أمّي (رحمة اللّه عليها)
و إلى أبي
اللّذين علّموني .

Declaration

This dissertation is the result of my own work done under the guidance of my supervisor, Marcelo Fiore, and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

No parts of this dissertation have been, or will be, submitted for any other qualification at this or any other university.

This dissertation does not exceed the word limit of 60,000 words, including tables and footnotes, as set by the Degree Committee of the Computer Laboratory.

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Summary

Second-order universal algebra and second-order equational logic respectively provide a model theory and a formal deductive system for languages with variable binding and parameterised metavariables. This dissertation completes the algebraic foundations of second-order languages from the viewpoint of categorical algebra.

In particular, the dissertation introduces the notion of *second-order algebraic theory*. A main role in the definition is played by the second-order theory of equality $\mathbb{M}$, representing the most elementary operators and equations present in every second-order language. We show that $\mathbb{M}$ can be described abstractly via the universal property of being the free cartesian category on an exponentiable object. Thereby, in the tradition of categorical algebra, a second-order algebraic theory consists of a cartesian category $\mathscr{M}$ and a strict cartesian identity-on-objects functor $M : \mathbb{M} \to \mathscr{M}$ that preserves the universal exponentiable object of $\mathbb{M}$.

At the syntactic level, we establish the correctness of our definition by showing a categorical equivalence between second-order equational presentations and second-order algebraic theories. This equivalence, referred to as the Second-Order Syntactic Categorical Type Theory Correspondence, involves distilling a notion of syntactic translation between second-order equational presentations that corresponds to the canonical notion of morphism between second-order algebraic theories. Syntactic translations provide a mathematical formalisation of notions such as encodings and transforms for second-order languages.

On top of the aforementioned syntactic correspondence, we furthermore establish the Second-Order Semantic Categorical Type Theory Correspondence. This involves generalising Lawvere's notion of functorial model of algebraic theories to the second-order setting. By this semantic correspondence, second-order functorial semantics is shown to correspond to the model theory of second-order universal algebra.

We finally show that the core of the theory surrounding Lawvere theories generalises to the second order as well. Instances of this development are the existence of algebraic functors and monad morphisms in the second-order universe. Moreover, we define a notion of translation homomorphism that allows us to establish a 2-categorical type theory correspondence.

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Es nimmt der Augenblick, was Jahre geben.

- Johann Wolfgang von Goethe

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Chapter 1

INTRODUCTION

> *[...] The most effective illumination of algebraic practice by general algebra, both classical and categorical, has come from the explicit nature of the framework itself. The closure properties of certain algebraic sub-categories, the functoriality of semantics itself, the ubiquitous existence of functors adjoint to algebraic functors, the canonical method for extracting algebraic information from non-algebraic categories, have served (together with their many particular ramifications) as a partial guidance to mathematicians in dealing with the inevitably algebraic content of their subjects.*
>
> *William Lawvere [Adamek et al., 2009]*

Algebra is the study of operations on mathematical structures, and the constructions and relationships arising from them. These structures span the most basic algebraic entities, such as arithmetic, to the more abstract, such as groups, rings, lattices, etc. Based on these, Birkhoff [Birkhoff, 1935] laid out the foundations of a general unifying theory, now known as universal algebra. His formalisation of the notion of algebra starts with the introduction of equational presentations. These constitute the syntactic foundations of the subject. Algebras are then the semantics, or model theory, and play a crucial role in establishing the logical foundations. Indeed, Birkhoff introduced equational logic as a sound and complete formal deductive system for reasoning about algebraic structure.

The investigation of algebraic structure was further enriched by Lawvere's fundamental work on algebraic theories [Lawvere, 2004]. His approach gives an elegant categorical framework for providing a presentation-independent treatment of universal algebra, and it embodies the motivation for the present work.

As per Lawvere's own philosophy, we believe in the inevitability of algebraic content in mathematical subjects. We contend that it is only by looking at algebraic structure from all perspectives — syntactic, semantic, categorical — and the ways in which they interact, that the subject is properly understood.

In the context of computer science, for instance, consider that: (i) initial-algebra semantics provides canonical compositional interpretations [Goguen et al., 1978]; (ii) free constructions amount to abstract syntax [McCarthy, 1963] that is amenable to proofs by structural induction and definitions by structural recursion [Burstall, 1969]; (iii) equational presentations can be regarded as bidirectional rewriting theories and studied from a computational point of view [Knuth and Bendix, 1970]; (iv) algebraic theories come with an associated notion of algebraic translation [Lawvere, 2004], whose syntactic counterpart provides the right notion of syntactic translation between equational presentations [Fujiwara, 1959, Fujiwara, 1960]; (v) strong monads have an associated metalogic from which equational logics can be synthesised [Fiore and Hur, 2008b, Fiore and Hur, 2010].

The realm of categorical universal algebra has so far been restricted to first-order languages. This dissertation further extends it to include languages with variable-binding, such as the λ-calculus [Aczel, 1978] and predicate logic [Aczel, 1980]. We take the *explicit nature of the framework* introduced in Lawvere's seminal thesis as heuristic guidelines for applying the categorical algebra framework to second-order languages. In particular, emulating Lawvere's framework will enable us to:

- define second-order algebraic theories to be structure preserving functors from a suitable base category, the second-order theory of equality, to a category which abstractly *classifies* a given second-order presentation,
- extract syntactic information via internal languages from the categorical framework of second-order algebraic theories,
- synthesise a notion of syntactic translation from the canonical notion of morphism of algebraic theories, and vice versa,
- establish the functoriality of second-order semantics;

all in such a way that the expected categorical equivalences are respected. More precisely, we obtain:

1. the Second-Order *Syntactic* Categorical Type Theory Correspondence, by which second-order algebraic theories and their morphisms correspond to second-order equational presentations and syntactic translations; and
2. the Second-Order *Semantic* Categorical Type Theory Correspondence, by which algebras for second-order equational presentations correspond to second-order functorial models.

1.1 Background

We review the key developments of categorical algebra (Lawvere theories) and computer science (languages with variable binding) that are most relevant to us. Their combination forms the basis of this dissertation. Our approach towards syntactic notions of morphisms of equational presentations via syntactic translations is also briefly introduced and compared to existing such notions.

1.1.1 Algebraic theories

With the advent of category theory, the development of universal algebra was further advanced by Lawvere and his fundamental thesis on algebraic theories [Lawvere, 2004]. In it, Lawvere exhibited a presentation-independent category-theoretic formulation of finitary first-order theories; *finitary* in the sense that only operations of arity given by a finite cardinal are considered, and *first-order* in that the arguments of the operations do not allow variable-binding. We proceed to review Lawvere's categorical approach and its syntactic counterpart given by mono-sorted equational presentations.

The basic rough idea underlying Lawvere's abstraction is that an algebraic theory is a functor from a *base category* to a small category with strict finite products, whose morphishms can be thought of as tuples of abstract terms or derived operations. The base category intuitively represents the most fundamental equational theory, the theory of equality. It arises from the universal property of the categorical cartesian product.

Lawvere's axiomatisation of what is essentially the clone of an equational theory [Cohn, 1965] is along the following lines.

The first-order theory of equality. Let $\mathbb{F}$ be the category of finite cardinals and all functions between them. The objects of $\mathbb{F}$ are simply denoted by $n \in \mathbb{N}$; it comes equipped with a cocartesian structure given via cardinal sum $m + n$. Moreover, $\mathbb{F}$ can be universally characterised as the free cocartesian category generated by the object 1. By duality, the opposite of $\mathbb{F}$, which we shall denote by $\mathbb{L}$ for $\mathbb{L}$awvere, is equipped with finite products. This category, together with a suitable cartesian functor, form the main constituents of a Lawvere theory.

Definition 1.1 (Lawvere theory)**.** A *Lawvere theory* consists of a small category $\mathscr{L}$ with strictly associative finite products, together with a strict cartesian identity-on-objects functor $L\colon \mathbb{L} \to \mathscr{L}$. A morphism of Lawvere theories $L\colon \mathbb{L} \to \mathscr{L}$ and $L'\colon \mathbb{L} \to \mathscr{L}'$ is a cartesian functor $F\colon \mathscr{L} \to \mathscr{L}'$ which commutes with the theory functors L and L'. We write **LAW** for the category of Lawvere theories and their morphisms.

For a Lawvere theory $L\colon \mathbb{L} \to \mathscr{L}$, the objects of $\mathscr{L}$ are then precisely those of $\mathbb{L}$. For any $n \in \mathbb{N}$, morphisms in $\mathscr{L}(n,1)$ are referred to as the *operators* of the theory, and those arising from $\mathbb{L}(n,1)$ as the *elementary* such operators. For any $n, m \in \mathbb{N}$, morphisms in $\mathscr{L}(n,m)$ are m-tuples of operators, because $\mathscr{L}(n,m) \cong \mathscr{L}(n,1)^m$. Intuitively, a morphism of Lawvere theories encapsulates the idea of interpreting one theory in another.

Definition 1.2 (Functorial models of Lawvere theories)**.** A *functorial model* of a given Lawvere theory $L\colon \mathbb{L} \to \mathscr{L}$ in a cartesian category $\mathscr{C}$ is a cartesian functor $\mathscr{L} \to \mathscr{C}$.

Remark 1.3. Our reference to *algebraic theory* in this dissertation is solely in the categorical-algebra, functorial sense of Lawvere, with its syntactic counterpart given by equational presentations. Lawvere theories can be thought of as an abstract invariant notion different from the more concrete one of equational presentations. In fact, every equational presentation determines a Lawvere theory and every Lawvere theory is determined by an infinite class of equational presentations. As Hyland and Power point out [Hyland and Power, 2007], choosing good presentations for algebraic theories and, in the other direction, deriving an invariant, abstract, and universal description from a concrete presentation are important aspects of computer science. The transformation from one to the other is a main theme throughout this dissertation.

First-order equational presentations. An equational presentation consists of a signature defining its operations and a set of axioms describing the equations it should obey. Formally, a *mono-sorted first-order equational presentation* is specified as $\mathscr{E} = (\Sigma, E)$, where $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ is an indexed family of first-order operators. For a given $n \in \mathbb{N}$, we say that an operator $\omega \in \Sigma_n$ has *arity* n. The set of terms $T_\Sigma(V)$ on a set of variables V generated by the signature Σ is built up by the grammar

$$t \in T_\Sigma(V) \ := \ v \quad | \quad \omega(t_1, \ldots, t_k) \quad ,$$

where $v \in V$, $\omega \in \Sigma_k$, and for $i = 1, \ldots, k$, $t_i \in T_\Sigma(V)$. An equation is simply given by a pair of terms, and the set E of the equational presentation $\mathscr{E} = (\Sigma, E)$ contains equations, which we refer to as the axioms of $\mathscr{E}$.

The model-theoretic universe of first-order languages is classically taken to be the category **Set**. A (set-theoretic) algebra in this universe for a first-order signature Σ is a pair $(X, [\![-]\!]_X)$ consisting of a set X and interpretation functions $[\![\omega]\!]_X\colon X^{|\omega|} \to X$, where $|\omega|$ denotes the arity of ω. Algebras induce interpretations on terms (see for example [Fiore and Hur, 2008a] for details). An algebra for an equational presentation $\mathscr{E} = (\Sigma, E)$ is an algebra for Σ which satisfies all equations in E, in the sense that an equal pair of terms induces equal interpretation functions in **Set**.

We remind the reader that the passage from Lawvere theories and their functorial models to mono-sorted first-order equational presentations and their algebras is invertible (see Section 2.4.2 for a multi-sorted generalisation of this invertibility). This makes Lawvere theories an abstract, presentation-independent formalisation of equational presentations.

1.1.2 Rudiments of second-order languages

Variable-binding constructs are at the core of fundamental calculi and theories in computer science and logic [Church, 1936, Church, 1940]. Over the past two decades, many formal frameworks for languages with binding have been developed, including higher-order abstract syntax [Pfenning and Elliott, 1988] and Gabbay and Pitts' set-theoretic abstract syntax [Gabbay and Pitts, 2001]. The second-order framework we base this dissertation on is that of Fiore et al. [Fiore et al., 1999], as developed further by Hamana [Hamana, 2005], Fiore [Fiore, 2008], and Fiore and Hur [Fiore and Hur, 2010]. It provides a formal account of the principles of variable-binding and substitution.

Second-order languages. The passage from first to second order involves extending the language with both *variable-binding operators* and *parameterised metavariables*. Variable-binding operators bind a list of variables in each of their arguments, leading to syntax up to alpha equivalence [Aczel, 1978]. On top of variables, second-order languages come equipped with parameterised metavariables. These are essentially second-order variables for which substitution also involves instantiation.

We briefly review the mono-sorted version of the syntactic theory of second-order languages as developed by Fiore and Hur [Fiore and Hur, 2010]. Any simply-typed language with variable-binding fits their formalism. Examples of second-order languages spelled out in the literature include the λ-calculus [Aczel, 1978], the fixpoint operator [Klop et al., 1993], the primitive recursion operator [Aczel, 1978], the universal and existential quantifiers of predicate logic [Aczel, 1980], and the list iterator [van Raamsdonk, 2003].

Second-order signatures and their term calculus. A (mono-sorted) *second-order signature* $\Sigma = (\Omega, |-|)$ is specified by a set of operators Ω and an arity function $|-| \colon \Omega \to \mathbb{N}^*$. An operator $\omega \in \Omega$ of arity $|\omega| = (n_1, \dots, n_k)$ takes k arguments binding n_i variables in the i^{th} argument. Unlike the first-order universe, where terms are built up only from variables and (first-order) operators, second-order terms have *metavariables* as additional building blocks. A metavariable M of *meta-arity* m, denoted by $\mathrm{M} \colon [m]$, is to be parameterised by m terms. Therefore, second-order terms are considered in contexts $\mathrm{M}_1 \colon [m_1], \dots, \mathrm{M}_k \colon [m_k] \rhd x_1, \dots, x_n$ with two *zones*, each respectively declaring metavariables and variables. Second-order terms in context $\Theta \rhd \Gamma \vdash t$ are defined inductively as follows.

- For $x \in \Gamma$, $$\frac{}{\Theta \rhd \Gamma \vdash x}$$
- For $(\mathrm{M} \colon [m]) \in \Theta$, $$\frac{\Theta \rhd \Gamma \vdash t_i \quad (1 \le i \le m)}{\Theta \rhd \Gamma \vdash \mathrm{M}[t_1, \dots, t_m]}$$
- For $\omega \colon (n_1, \dots, n_k)$, $$\frac{\Theta \rhd \Gamma, \overrightarrow{x}_i \vdash t_i \quad (1 \le i \le k)}{\Theta \rhd \Gamma \vdash \omega\big((\overrightarrow{x}_1)t_1, \dots, (\overrightarrow{x}_k)t_k\big)} \quad (\overrightarrow{x}_i = x_1^{(i)}, \dots, x_{n_i}^{(i)})$$

The second-order nature of the syntax requires a two-level substitution calculus, as formalised in [Aczel, 1978] and [Fiore, 2008]. Each level respectively accounts for the substitution of variables and metavariables, with the latter operation depending on the former. See Section 4.1.3 for a detailed account of both substitution and *metasubstitution*.

Second-order equational logic. A *Second-order equational presentation* $\mathscr{E} = (\Sigma, E)$ is obtained by adding equations on top of the above constructions. It is specified by a second-order signature Σ together with a set of equations E, where a second-order equation $\Theta \rhd \Gamma \vdash s \equiv t$ is given by a pair of second-order terms $\Theta \rhd \Gamma \vdash s$ and $\Theta \rhd \Gamma \vdash t$. The rules of *Second-Order Equational Logic* are given in Figure 4.1 in Section 4.2.1. They provide a sound and complete formal deductive system for reasoning about second-order equational presentations [Fiore and Hur, 2010].

Second-order semantic universe. In the framework developed by Fiore et al. in [Fiore et al., 1999], instead of working within the objects of the category **Set**, one takes the category $\mathbf{Set}^{\mathbb{F}}$ of covariant presheaves (or *variable sets*). In the model theory, algebras over sets are replaced by so-called binding algebras over variable sets. Binding algebras are essentially presheaves endowed with both an algebra structure and a compatible substitution structure. The suitability of taking $\mathbf{Set}^{\mathbb{F}}$ as the mathematical universe in which to deal with variable binding can be seen as follows. The index category $\mathbb{F}$ provides a notion of cartesian context and allows for the familiar operations on contexts, such as exchange, weakening, and contraction. The *presheaf of variables* $V : \mathbb{F} \to \mathbf{Set}$ is simply the inclusion of $\mathbb{F}$ in **Set**, and for any $n \in \mathbb{F}$ and presheaf $X : \mathbb{F} \to \mathbf{Set}$, the set $X(n)$ can be seen as giving the terms with at most n free variables. It is well known that the category $\mathbf{Set}^{\mathbb{F}}$ is cartesian closed. In particular, exponentiating any X with respect to the presheaf of variables V yields an abstract view of variable binding via the resulting exponential X^V. Indeed, one has the equality $X^V(n) = X(n+1)$ for any $n \in \mathbb{F}$.

1.1.3 Theories of translations

One of the principle dogmas of category theory is that for every mathematical structure, no matter how general or specific, there exists a category whose objects have that structure and, more importantly, whose morphisms preserve it [Goguen, 1991]. The significance of the latter lies in the categorical convention that morphisms are in fact more fundamental than objects, as they reveal what the structure really is.

It is for this reason rather surprising that there is no *generic* agreed-upon syntactic notion of morphism between equational presentations. In the context of Lawvere theories, the canonical notion of morphism is given by a cartesian functor (Definition 1.1). In the categorical algebra framework, we expect this to be the presentation-independent formalisation of some syntactically defined notion of morphism of equational presentations. We will show that this notion is precisely given by *syntactic translations*.

Notions of mappings of signatures and presentations have been developed in the first-order setting by Fujiwara [Fujiwara, 1959, Fujiwara, 1960], Goguen et al. [Goguen et al., 1978], and Vidal and Tur [Vidal and Tur, 2008], all of which use the common definition that a syntactic notion of morphism maps *operators to terms*. We briefly review these approaches.

Fujiwara mappings. A formalisation for mappings of mono-sorted first-order finite product theories was constructed by Fujiwara in [Fujiwara, 1959, Fujiwara, 1960]. His general theory of such mappings between algebraic systems is defined via a so-called *system of P-mappings*. For first-order mono-sorted signatures $\Sigma = \{(\Sigma)_n\}_{n\in\mathbb{N}}$ and $\Sigma' = \{(\Sigma')_n\}_{n\in\mathbb{N}}$, a morphism $\Sigma \to \Sigma'$ is given by a pair (Φ, P), where Φ is a set of *mapping variables* and $P = \{P_n\}_{n\in\mathbb{N}}$ is a family of mappings, where for $n \in \mathbb{N}$, P_n sends a pair $(\varphi, \omega) \in \Phi \times \Sigma_n$ of a mapping variable φ and an operator ω of arity n to a term $t_{n,\omega}$ of Σ' formed on top of the set of variables $\Phi \times (v_1, \ldots, v_n)$. The mapping variables here are to be replaced by mappings from a Σ-algebra to another Σ-algebra derived from a Σ'-algebra. What is important here is to note that the set of variables $\Phi \times (v_1, \ldots, v_n)$ used to form the term $t_{n,\omega}$ is determined by the arity of the operator ω. We will see in Chapter 3 and Chapter 6 that syntactic translations impose a similar condition on the context of the terms which operators are mapped to.

Polyderivors. Polyderivors were introduced by Goguen et al. in [Goguen et al., 1978]. They provide a formal notion of syntactic morphism similar to that of Fujiwara, but in a multi-sorted framework. A *polyderivor* thereby consists of two mappings. One mapping relates the sets of sorts of the signatures. It assigns to each sort in the first signature a *derived sort* in the second signature, which is a word on the set of sorts in the second signature. The other mapping assigns to each operator in the first signature a family of terms in the second. The context of each of these terms is again specified by the arity of the operator being mapped.

Syntactic translations. There are three constituents defining the notion of morphism of (generic) mono-sorted equational presentations $\mathscr{E} = (\Sigma, E) \to \mathscr{E}' = (\Sigma', E')$:

1. An operator ω of Σ is mapped to a term $\Gamma \vdash t$ of Σ', with its context Γ given by the arity of ω.

2. The above mapping induces a mapping between the terms of Σ and Σ' in such a way that the axioms of E are respected.

3. The generalisation to include sorts yields a mapping of sorts of Σ to *tuples* of sorts of Σ', and operators to *tuples* of terms.

We will show in Section 3.2 and Section 6.2 that a syntactic morphism with these properties mirrors the behaviour of morphisms of first- and second-order algebraic theories, respectively. Indeed, we define syntactic translations to be exactly those maps specified by the above three components. Both polyderivors and Fujiwara mappings satisfy the above and therefore coincide with our notion of (first-order) syntactic translation.

1.2 Contributions

Motivated by Lawvere's observation that algebraic structure is inevitable in mathematics, this work illustrates the imminence of his abstract categorical treatment of syntactic equational presentations in the setting of second-order languages. We develop the main ingredients in such a development *à-la*-Lawvere, which encompasses: (i) the definition of second-order algebraic theories and their morphisms; (ii) the formalisation of a syntactic notion of morphism of second-order presentations; and (iii) the functorial semantics for second-order algebraic theories.

With the second-order syntactic theory reviewed in Section 1.1.2 in mind, we now give an overview of the above three main contributions of this dissertation. A more detailed chapter-by-chapter synopsis is provided in Section 1.3.

1.2.1 Second-order algebraic theories

The second-order theory of equality. In the notion of categorical algebraic theory, the elementary theory of equality represents the most fundamental theory and plays a pivotal role. The second-order algebraic theory of equality $\mathbb{M}$ has objects given by $\mathbb{N}^*$ and morphisms $(m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$ given by tuples

$$\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_i} \vdash t_i \rangle_{1 \le i \le l}$$

of so-called *elementary* second-order terms. These are built from variables and metavariables only. Just as composition in the first-order theory of equality $\mathbb{L}$ is given by substitution, composition in $\mathbb{M}$ is defined via metasubstitution. Another similarity lies in the cartesian structure of $\mathbb{M}$, which is given by the concatenation of tuples. Its universal structure goes beyond that of the categorical product though. In fact, every object $(n) \in \mathbb{M}$ is the exponential $(0)^n \Rightarrow (0)$. This exponential structure provides a universal semantic characterisation of $\mathbb{M}$. Loosely speaking, $\mathbb{M}$ is the free cartesian category on the exponentiable object (0).

Second-order algebraic theories. The core contribution of this dissertation is the introduction of second-order algebraic theories and their morphisms.

> A *second-order algebraic theory* consists of a cartesian category $\mathscr{M}$ and a strict cartesian identity-on-objects functor $M : \mathbb{M} \to \mathscr{M}$ that preserves the exponentiable object (0). A *second-order algebraic translation* between second-order algebraic theories $M : \mathbb{M} \to \mathscr{M}$ and $M' : \mathbb{M} \to \mathscr{M}'$ is a cartesian functor $F : \mathscr{M} \to \mathscr{M}'$ satisfying $M' \circ F = M$.

We obtain the category **SOAT** of second-order algebraic theories and their algebraic translations, with the evident functorial identity and functorial composition.

Classifying algebraic theories and internal languages. Second-order equational presentations induce second-order algebraic theories, and vice versa. For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, one can start by constructing the *classifying category* $\mathbb{M}(\mathscr{E})$, which has the same set of objects as the elementary theory $\mathbb{M}$ and morphisms $(m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$ given by tuples of equivalence classes of terms generated from Σ under the equivalence relation identifying two terms if and only if they are provably equal from $\mathscr{E}$ in Second-Order Equational Logic (Figure 4.1). This canonical methodology for constructing classifying categories is borrowed from traditional categorical type theory. For a second-order equational presentation $\mathscr{E}$, the category $\mathbb{M}(\mathscr{E})$ together with the canonical functor $M_{\mathscr{E}} \colon \mathbb{M} \to \mathbb{M}(\mathscr{E})$ is a second-order algebraic theory, referred to as the classifying theory. Going in the other direction, the *internal language* $\mathfrak{L}(M)$ of a second-order algebraic theory $M \colon \mathbb{M} \to \mathscr{M}$ has operators specified by the morphisms of $\mathscr{M}$, and equations specified by the morphism equalities of $\mathbb{M}$ and $\mathscr{M}$. Again, this mirrors the classical way of extracting syntactic theories from categorical ones.

Second-order theory/presentation correspondence. The correctness of our definition of second-order algebraic theory is verified by establishing its correspondence to the notion of second-order equational presentation. Indeed, every second-order algebraic theory $M \colon \mathbb{M} \to \mathscr{M}$ is isomorphic to the second-order algebraic theory of its associated second-order equational presentation $\mathbb{M} \to \mathbb{M}(\mathfrak{L}(M))$.

1.2.2 Second-order syntactic translations

Morphisms of second-order equational presentations. Algebraic theories come with an associated notion of algebraic translation, their morphisms. While the syntactic counterpart of these morphisms has been developed in one form or another in the first-order setting, in the second-order universe, a notion of syntactic morphism has yet to be formalised. Our main contribution in this regard is the generalisation of the notion of syntactic translation as introduced in Section 1.1.3 above to second-order languages.

> A *second-order syntactic translation* $\tau \colon \Sigma \to \Sigma'$ between second-order signatures is given by a mapping from the operators of Σ to the terms of Σ' as follows:
>
> $$\omega \colon (m_1, \ldots, m_k) \quad \mapsto \quad \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd - \vdash \tau_\omega$$
>
> We will show that a translation $\tau \colon \Sigma \to \Sigma'$ extends to a mapping from the terms of Σ to the terms of Σ'. When translating between equational presentations, we take syntactic translations $\mathscr{E} = (\Sigma, E) \to \mathscr{E}' = (\Sigma', E')$ to be those signature translations $\tau \colon \Sigma \to \Sigma'$ which preserve the equational theory of $\mathscr{E}$ in the sense that axioms are mapped to theorems.

Note that this definition satisfies properties 1 and 2 in our proposed development of syntactic translations of Section 1.1.3. This shows that our framework for a general theory of morphisms between algebraic systems is easily generalisable to second-order algebraic systems. We will also define a canonical identity translation and translation composition, which leads us to construct the category **SOEP** of second-order equational presentations and their syntactic translations.

Second-order presentation/theory correspondence. By considering syntactic translation isomorphisms, we are able to establish at the syntactic level whether two equational presentations are essentially the same without having to revert to their categorical counterparts. This explicit machinery of syntactic comparison is used to prove that every second-order equational presentation $\mathscr{E}$ is isomorphic to the second-order equational presentation $\mathfrak{E}(M_{\mathscr{E}})$ of its associated algebraic theory $M_{\mathscr{E}} : \mathbb{M} \to \mathbb{M}(\mathscr{E})$.

Second-order syntactic categorical type theory correspondence. This correspondence constitutes another core contribution of the dissertation, as it precisely and completely establishes the correctness of (i) the definition of second-order algebraic theories, and (ii) the definition of second-order syntactic translations. This is done by establishing the categorical equivalence of **SOAT** and **SOEP**. Note that not only does this categorical equivalence demonstrate the strong similarities of second-order algebraic theories and presentations, but it makes the notion of algebraic theory even more powerful: it creates the opportunity to translate theorems between abstract and concrete second-order algebraic systems, knowing that the essential meaning of those theorems is preserved under this equivalence.

1.2.3 Second-order functorial semantics

Second-order functorial models. We show that Lawvere's functorial semantics for algebraic theories admits generalisation to the second-order universe, in which a *second-order functorial model* of a second-order algebraic theory is given in terms of a suitable functor from the algebraic theory to **Set**, as follows:

> A *second-order functorial model* of a second-order algebraic theory $M : \mathbb{M} \to \mathscr{M}$ is given by a cartesian functor $\mathscr{M} \to \mathscr{C}$, for $\mathscr{C}$ a cartesian category. We obtain the category $\mathbb{M}\mathrm{od}(M, \mathscr{C})$ of functorial models of M in $\mathscr{C}$, with morphisms (necessarily monoidal) natural transformations between them. A second-order *set-theoretic functorial model* of a second-order algebraic theory $M : \mathbb{M} \to \mathscr{M}$ is simply a cartesian functor from $\mathscr{M}$ to **Set**. We obtain the category $\mathbb{M}\mathrm{od}(M)$ of set-theoretic functorial models of M in **Set**.

Second-order semantic categorical type theory correspondence. Second-order functorial models are proven to correspond to second-order algebras, as developed by Fiore in [Fiore, 2008]. More precisely, for every second-order equational presentation $\mathscr{E}$, the category of $\mathscr{E}$-models $\mathbf{Mod}(\mathscr{E})$ and the category of second-order functorial models $\mathbb{M}\mathrm{od}(M_{\mathscr{E}})$ are equivalent.

Second-order translational semantics. Second-order functorial semantics enables us to take a model of an algebraic theory in any cartesian category $\mathscr{C}$. Moreover, the notion of algebraic translation between second-order algebraic theories encapsulates the idea of a simple interpretation of one theory in another. We observe that a second-order syntactic translation is the equivalent syntactic such idea. We have thus introduced a less abstract, more concrete way of giving semantics to equational presentations. We refer to it as (second-order) *Translational Semantics*.

1.3 Synopsis

This dissertation begins with three chapters (Chapters 2-4) dedicated to setting the background of first-order algebraic theories and translations, and of second-order syntax. The following three chapters (Chapters 5-7) develop the three main contributions discussed above. We conclude in Chapter 8 by showing that many of the developments surrounding Lawvere theories still hold in the second-order universe. We also propose two concrete research directions based on the work introduced here.

Chapter 2: First-Order Algebraic Theories. In this chapter, we review the syntactic framework of first-order equational presentations, and the categorical counterpart given by first-order algebraic theories. Our exposition lies in the multi-sorted universe and can be viewed as a generalisation of Lawvere theories. We review the classical set-theoretic semantics, and the multi-sorted version of functorial semantics. We conclude this chapter by recalling the *first-order categorical type theory correspondence* - the *syntactic* correspondence being the equivalence of algebraic theories and equational presentations, and the *semantic* correspondence being that of set-theoretic and functorial semantics.

Chapter 3: First-Order Syntactic Translations. This chapter introduces the notion of syntactic translation in the multi-sorted first-order universe. We show that syntactic translations can be defined as Kleisli *maps*. The correctness of our syntactic definition is established by proving its correspondence to that of a canonical morphism of algebraic theories.

Chapter 4: Second-Order Syntax and Semantics. This chapter gives an introduction to the work of Fiore and Hur [Fiore and Hur, 2010] on second-order universal algebra. It lays the syntactic foundations of the second-order universe, whose categorical counterpart is developed in the following chapters. Our summary recalls: (i) the notion of second-order equational presentation, that allows the specification of equational theories by means of schematic identities over signatures of variable-binding operators; (ii) the model theory of second-order equational presentations by means of second-order algebras; and (iii) the deductive system underlying formal reasoning about second-order algebraic structure.

Chapter 5: Second-Order Algebraic Theories. In this chapter, we present the main contribution of this dissertation. We define second-order algebraic theories, their algebraic translations, and establish the correctness of our definition by showing a categorical equivalence between second-order equational presentations and second-order algebraic theories.

Chapter 6: Second-Order Syntactic Translations. The notion of syntactic translation is generalised to the second-order setting, and we show that it corresponds to the notion of second-order algebraic translation. This completes the *Second-Order Syntactic Categorical Type Theory Correspondence* by which second-order algebraic theories and their translations are categorically equivalent to second-order equational presentations and their syntactic translations.

Chapter 7: Second-Order Functorial Semantics. In this chapter, we show that Lawvere's functorial semantics for algebraic theories is generalisable to the second-order universe. Second-order functorial models are shown to correspond to second-order algebras as defined in Chapter 4. This completes the *Second-Order Semantic Categorical Type Theory Correspondence*.

Chapter 8: Concluding Remarks. We conclude the dissertation by generalising the notions of algebraic functors and monad morphisms to the second-order setting. We also define a notion of *translation homomorphism*, which allows us to establish a 2-categorical equivalence between syntactic and categorical presentations of equational theories. Finally, we propose two concrete directions for future research.

1.3.1 Published work

The work presented here is largely based on [Fiore and Mahmoud, 2010] written by the author together with Marcelo Fiore, but has been significantly expanded in this dissertation.

Chapter 2

First-Order Algebraic Theories

Equational presentations provide a syntactic formalisation of the notion of algebraic equational theory by specifying a set of operations, the *signature*, and the laws that these operations must satisfy, the *axioms*. Lawvere theories abstract away from particular syntactic descriptions by giving a syntax-independent formulation of presentations. With these fundamental counterparts in mind, the purpose of the following two chapters is twofold. First, we recall these two approaches in the first-order setting and prove their mutual correspondence. While this is a classic result, we nevertheless review the details to motivate our analogous development for second-order algebraic theories. Second, recalling that a cartesian functor defines a morphism between Lawvere theories, we introduce in Chapter 3 the notion of *syntactic translation* between first-order equational presentations and validate our definition by establishing its equivalence with cartesian functors.

Our exposition lies in the multi-sorted universe and is presented as follows. We start by reviewing the syntactic definition of first-order equational presentations (Section 2.1) and its set-theoretic and categorical semantics (Section 2.2). We then move on to the categorical counterparts given by first-order algebraic theories and functorial semantics (Section 2.3). We conclude by recalling the *categorical type theory correspondence* in Section 2.4 - the *syntactic* correspondence being the equivalence of algebraic theories and their corresponding equational presentations, and the *semantic* equivalence being that of set-theoretic and functorial models. This chapter together with the following one serve as a motivational review and are not a prerequisite to understanding the core contributions of this dissertation. We therefore skip or sketch proofs of classical results and refer the reader to literature for existing proofs.

2.1 First-Order Syntactic Theory

The purely abstract approach to algebraic theories, as developed by Lawvere, is often not sufficient for the needs of the computer scientist. We review the concrete structures of first-order multi-sorted algebraic signatures and equational presentations and the syntactic machinery surrounding them.

2.1.1 Signatures and their term calculus

A multi-sorted (first-order) algebraic signature, or just *signature*, $\Sigma = (S, \Omega, |-|)$ is given by a set of sorts S, a set of operators Ω, and a function $|-|\colon \Omega \to S^* \times S$ specifying the operator arity. We typically write $\omega\colon \sigma_1, \ldots, \sigma_n \to \tau$ to indicate an operator $\omega \in \Omega$ with arity $|\omega| = (\sigma_1, \ldots, \sigma_n), \tau$. Note that a signature is an object of the indexed category $\mathbf{Set}^{S^* \times S}$.

Example 2.1. *A typical example of a (mono-sorted) first-order algebraic signature is the signature $\Sigma_{\mathscr{G}}$ of the theory of* groups *specifying the algebraic structure of groups. Recalling that for mono-sorted signatures operator arities are equivalently given by natural numbers, $\Sigma_{\mathscr{G}}$ consists of the following three operators:*

- $\mathtt{e}\colon 0$ (identity)
- $\mathtt{i}\colon 1$ (inverse)
- $\mathtt{m}\colon 2$ (multiplication)

Remark 2.2 (Notational convention). Throughout this dissertation, we will, for any $n \in \mathbb{N}$, denote by $\|n\|$ the set $\{1, \ldots, n\}$.

Contexts. Given a countable set V of variables, a *context* is a finite sequence of variable declarations of the form $\Gamma = (x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n)$, where $\sigma_i \in S$ for all $i \in \|n\|$, and all variables are assumed to be distinct. Concatenation of contexts $\Gamma = (x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n)$ and $\Gamma' = (y_1\colon \tau_1, \ldots, y_k\colon \tau_k)$ is defined as $\Gamma, \Gamma' := (x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n, x_{n+1}\colon \tau_1, \ldots, x_{n+k}\colon \tau_k)$, noting that the variables, which are merely placeholders, remain distinct.

Terms. We associate to a signature Σ its term calculus, which specifies the rules for term generation. The set of *raw terms* $T_\Sigma(V)$ generated by the signature Σ over the set of variables V is given by the grammar

$$t \in T_\Sigma(V) \quad := \quad v \quad | \quad \omega(t_1, \ldots, t_k) \quad ,$$

where $v \in V$, $\omega \in \Omega$, and $t_1, \ldots, t_k \in T_\Sigma(V)$. *Terms-in-context*, or simply *terms*, denoted by $\Gamma \vdash t\colon \sigma$, are described with respect to a finite set of variables receiving their type assignments in the contexts. They are generated via the following rules.

$$\frac{}{\Gamma, x\colon \sigma \vdash x\colon \sigma} \qquad\qquad \frac{\Gamma \vdash t_i\colon \sigma_i \quad (1 \leq i \leq k)}{\Gamma \vdash \omega(t_1, \ldots, t_k)\colon \sigma} \; (\omega\colon \sigma_1, \ldots, \sigma_k \to \sigma)$$

The terms of every first-order signature come equipped with *structural rules*, which are often not listed explicitly, as they are derivable. They respectively allow adding an extra variable declaration in the context, replacing two variables of the same sort by a single variable, and permuting contexts.

Substitution. The operation of *simultaneous substitution* maps terms $\Gamma, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \sigma$ and $\Gamma \vdash s_i : \sigma_i$, for $i \in \|n\|$, to the term

$$\Gamma \vdash t\{x_i := s_i\}_{i \in \|n\|} : \sigma \quad ,$$

which is defined by induction on the structure of t as follows:

- $x_j\{x_i := s_i\}_{i \in \|n\|} = s_j$
- $\omega(t_1, \ldots, t_k)\{x_i := s_i\}_{i \in \|n\|} = \omega\big(t_1\{x_i := s_i\}_{i \in \|n\|}, \ldots, t_1\{x_i := s_i\}_{i \in \|n\|}\big)$

It is easy to verify that the operation of substitution is well-defined and well-typed (see e.g. [Jacobs, 1999]) and moreover associative, as expressed formally by the following fundamental lemma.

Lemma 2.3 (First-Order Substitution Lemma). *Given terms*

$$\Gamma, x_1 : \sigma_1, x_n : \sigma_n, y_1 : \tau_1, \ldots, y_k : \tau_k \vdash t : \sigma \quad ,$$

$$\Gamma \vdash s_i : \sigma_i \quad (1 \leq i \leq n) \qquad \text{and} \qquad \Gamma \vdash r_j : \tau_j \quad (1 \leq j \leq k) \quad ,$$

we have the following syntactic equality:

$$\Gamma \vdash t\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \;=\; t\big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\big\}_{i \in \|n\|} \quad .$$

The definition of substitution together with the First-Order Substitution Lemma play a principal role in the categorical formulation of first-order signatures and presentations, as composition in their *classifying categories* is defined via term substitution (see Section 2.4).

2.1.2 Equational presentations

Adding equations to signatures yields equational presentations. An equation-in-context, or simply *equation*, written $\Gamma \vdash t_1 \equiv t_2 : \sigma$, is given by a pair of terms $\Gamma \vdash t_1 : \sigma$ and $\Gamma \vdash t_2 : \sigma$. A (multi-sorted first-order) *equational presentation* $\mathscr{E} = (\Sigma, E)$ is specified by a first-order algebraic signature Σ and a set of equations E. Elements of E are the *axioms* of the equational presentation $\mathscr{E}$ and are denoted $\Gamma \vdash_E t_1 \equiv t_2 : \sigma$.

Example 2.4. *The equational presentation $\mathscr{E}_{\mathscr{G}} = (\Sigma_{\mathscr{G}}, E_{\mathscr{G}})$ associated to the theory of groups has a set of equations $E_{\mathscr{G}}$ expressing the usual group axioms:*

- (Associativity) $\quad \Gamma, x, y, z \vdash \mathtt{m}\big(\mathtt{m}(\mathtt{x}, \mathtt{y}), \mathtt{z}\big) \equiv \big(\mathtt{x}, \mathtt{m}(\mathtt{y}, \mathtt{z})\big)$
- (Identity) $\quad \Gamma, x \vdash \mathtt{m}\big(\mathtt{x}, \mathtt{e}()\big) \equiv \mathtt{x}$ *and* $\Gamma, x \vdash \mathtt{m}\big(\mathtt{e}(), \mathtt{x}\big) \equiv \mathtt{x}$
- (Inverse) $\quad \Gamma, x \vdash \mathtt{m}\big(\mathtt{i}(\mathtt{x}), \mathtt{x}\big) \equiv \mathtt{e}()$ *and* $\Gamma, x \vdash \mathtt{m}\big(\mathtt{x}, \mathtt{i}(\mathtt{x})\big) \equiv \mathtt{e}()$

2.1.3 First-order equational logic

First-order equational presentations $\mathscr{E}$ have the following derivability rules:

Axioms

$$(\textsc{ax})\quad \frac{\Gamma \vdash_E t_1 \equiv t_2 : \sigma}{\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_2 : \sigma}$$

Equality rules

$$(\textsc{refl})\quad \frac{\Gamma \vdash t : \sigma}{\Gamma \vdash_{\mathscr{E}} t \equiv t : \sigma} \qquad (\textsc{sym})\quad \frac{\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_2 : \sigma}{\Gamma \vdash_{\mathscr{E}} t_2 \equiv t_1 : \sigma} \qquad (\textsc{trans})\quad \frac{\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_2 : \sigma \quad \Gamma \vdash_{\mathscr{E}} t_2 \equiv t_3 : \sigma}{\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_3 : \sigma}$$

Substitution

$$(\textsc{sub})\quad \frac{\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_2 : \sigma \quad \Gamma, x : \sigma \vdash_{\mathscr{E}} s : \tau}{\Gamma \vdash_{\mathscr{E}} s\{x := t_1\} \equiv s\{x := t_2\} : \tau}$$

An equation $\Gamma \vdash_{\mathscr{E}} t_1 \equiv t_2 : \sigma$ derivable from first-order equational logic is called a *theorem* of the equational presentation $\mathscr{E}$.

It is well-known that First-Order Equational Logic is sound and complete for first-order equational presentations (Birkhoff 1935, Goguen and Mesenguer 1985), in the sense that an equation is derivable if and only if it is satisfied by all *algebras* for the presentation (see Section 2.2).

2.2 First-Order Model Theory

We recall the fundamental development of set-theoretic and categorical semantics for multi-sorted first-order languages.

2.2.1 Categorical semantics

The power of the categorical language as an organisational tool allows us to consider interpretations of syntactically defined theories in the abstract setting of a category. Intuitively, terms are morphisms, term substitution is interpreted by composition of morphisms, and model soundness is established by looking at morphism equality. To illustrate this approach, we recall the notion of algebra for a first-order signature and equational presentation in a cartesian category. As a special case, we obtain the traditional set-theoretic model theory, which we generalise to include multiple sorts.

Definition 2.5. An *algebra* $(X_S, [\![-]\!]_{X_S})$ for the signature $\Sigma = (S, \Omega, |-|)$ in a cartesian category $\mathscr{C}$ is given by an S-indexed collection $X_S = \{X_\sigma\}_{\sigma \in S}$ of objects of $\mathscr{C}$ together with, for every operator

$\omega\colon \sigma_1,\ldots,\sigma_n \to \sigma$, a morphism $[\![\omega]\!]_{X_S}\colon X_{\sigma_1} \times \ldots X_{\sigma_n} \to X_\sigma$ of $\mathscr{C}$. A *homomorphism* of Σ-algebras $(X_S, [\![-]\!]_{X_S}) \to (Y_S, [\![-]\!]_{Y_S})$ is specified by giving a collection of morphisms $f : X_\sigma \to Y_\sigma$ of $\mathscr{C}$ for each sort $\sigma \in S$ such that for every operator $\omega\colon \sigma_1,\ldots,\sigma_n \to \sigma$, we have

$$[\![\omega]\!]_{Y_S} \circ (f_{\sigma_1} \times \cdots \times f_{\sigma_n}) = f_\sigma \circ [\![\omega]\!]_{X_S} \quad .$$

Σ-algebras in $\mathscr{C}$ and their homomorphisms form the category Σ-**Alg**($\mathscr{C}$).

Such an algebra $(X_S, [\![-]\!]_{X_S})$ induces the interpretation

$$[\![t]\!]_{X_S}\colon X_{\sigma_1} \times \cdots \times X_{\sigma_n} \to X_\sigma$$

for a term $x_1\colon \sigma_1,\ldots,x_n\colon \sigma_n \vdash t\colon \sigma$ as follows:

- $[\![x_i]\!]_{X_S} := \pi_i$, where $\pi_i\colon X_{\sigma_1} \times \cdots \times X_{\sigma_n} \to X_{\sigma_i}$ is the i-th projection in $\mathscr{C}$.
- $[\![\omega(t_1,\ldots,t_k)]\!]_{X_S} := [\![\omega]\!]_{X_S} \circ \langle [\![t_1]\!]_{X_S},\ldots,[\![t_k]\!]_{X_S}\rangle$.

A Σ-algebra $(X_S, [\![-]\!]_{X_S})$ in a cartesian category $\mathscr{C}$ is said to *satisfy* an equation $\Gamma \vdash t_1 \equiv t_2\colon \sigma$ if $[\![t_1]\!]_{X_S}$ and $[\![t_2]\!]_{X_S}$ are equal morphisms in $\mathscr{C}$.

Satisfiability of the axioms of equational presentations determines their algebras, which we define as follows.

Definition 2.6. An *algebra for an equational presentation* $\mathscr{E} = (\Sigma, E)$ in a cartesian category $\mathscr{C}$ is a Σ-algebra in $\mathscr{C}$ that satisfies all equations in E. $\mathscr{E}$-algebra homomorphisms are simply Σ-algebra homomorphisms. We write $\mathscr{E}$-**Alg**($\mathscr{C}$) for the category of $\mathscr{E}$-algebras and their homomorphisms.

Theorem 2.7 (Soundness)**.** *An algebra for an equational presentation $\mathscr{E} = (\Sigma, E)$ in a cartesian category $\mathscr{C}$ satisfies all theorems derivable from E.*

2.2.2 Set-theoretic semantics

We review folklore results from first-order set-theoretic model theory. To generalise the universe of discourse to include sorts, one takes a sort-indexed collection of sets as the base category rather than just **Set**. Thus, for a set of sorts S, we consider the category $\mathbf{Set}^S$ whose objects are S-indexed sets $X_S := \{X_\sigma\}_{\sigma\in S}$ and whose morphisms $f_S\colon X_S \to X'_S$ are S-indexed functions $\{f_\sigma\}_{\sigma\in S}$. More precisely, $\mathbf{Set}^S$ is a fibre within the category **MSSet** of multi-sorted sets and functions, whose objects are sets indexed over arbitrary sets of sorts. Furthermore, $\mathbf{Set}^S$ is bicomplete; we refer the reader to [Tarlecki et al., 1991] for a straightforward proof involving machinery from indexed category theory.

First-order signature algebras. An *algebra* $(X_S, [\![-]\!]_{X_S})$ for a multi-sorted first-order signature $\Sigma = (S, \Omega, |-|)$ in **Set** is given by an object $X_S \in \mathbf{Set}^S$ together with interpretation functions

$$[\![\omega]\!]_{X_S} \colon X_{\sigma_1} \times \cdots \times X_{\sigma_n} \to X_\sigma$$

for every operator $\omega \colon \sigma_1, \ldots, \sigma_n \to \sigma$ in Ω. We write Σ-**Alg** for the category of set-theoretic Σ-algebras and their homomorphisms, with the evident composition and identity.

Note that by simply referring to Σ-*algebras* without specifying the cartesian category they are considered in, we mean by default Σ-algebras in $\mathbf{Set}^S$ and use the widely used notation Σ-**Alg** rather than the more precise Σ-$\mathbf{Alg}(\mathbf{Set}^S)$.

It is generally known that a signature induces an algebra-preserving endofunctor on its model-theoretic base category. The signature endofunctor $\underline{\Sigma} \colon \mathbf{Set}^S \to \mathbf{Set}^S$ corresponding to the signature Σ is defined by

$$\{X_\sigma\}_{\sigma \in S} \mapsto \Big\{ \coprod_{(\sigma_1,\ldots,\sigma_n) \in S^*} \Omega_{(\sigma_1,\ldots,\sigma_n),\tau} \times \prod_{1 \le i \le n} X_{\sigma_i} \Big\}_{\tau \in S} \quad .$$

A $\underline{\Sigma}$-*algebra* is then an algebra for the endofunctor $\underline{\Sigma}$.

Definition 2.8. Given the endofunctor $F \colon \mathscr{C} \to \mathscr{C}$, an F-*algebra* (X, φ) is given by a *carrier* object X and a *structure map* $\varphi \colon FC \to C$. A *homomorphism* of F-algebras $(X, \varphi) \to (Y, \psi)$ is a morphism $f \colon X \to Y$ in $\mathscr{C}$ such that $f \circ \varphi = \psi \circ Ff$. F-algebras and their homomorphisms form the category F-**Alg**.

Algebras for signature endofunctors are an abstract formulation of signature models. Indeed, there is an isomorphism between the category Σ-**Alg** and $\underline{\Sigma}$-**Alg**. We also obtain the well-known left adjoint to the canonical forgetful functor $U \colon \underline{\Sigma}\text{-}\mathbf{Alg} \to \mathbf{Set}^S$, which maps X_S to the free $\underline{\Sigma}$-algebra generated by X_S. The underlying endofunctor $\mathbf{T}_\Sigma \colon \mathbf{Set}^S \to \mathbf{Set}^S$ of this adjunction maps an S-indexed set X_S to the initial $(X_S + \underline{\Sigma})$-algebra. Moreover, this adjunction is monadic, making the categories $\mathbf{T}_\Sigma$-**Alg** and $\underline{\Sigma}$-**Alg** equivalent.

First-order presentation algebras. An algebra for an equational presentation $\mathscr{E} = (\Sigma, E)$ is simply a Σ-algebra satisfying the equations of E in **Set**. We again simplify notation and write $\mathscr{E}$-**Alg** for the category of $\mathscr{E}$-algebras and their homomorphisms, noting that it is a full subcategory of Σ-**Alg**.

The existence of free algebras for an equational presentation $\mathscr{E} = (\Sigma, E)$ is one of the most fundamental developments in universal algebra. Define the equivalence relation $\sim_E$ on the set of terms $T_\Sigma(X_S)$ generated over the S-indexed set X_S by identifying two terms if and only if they are derivably equal using equations of E and first-order equational logic. Then the free $\mathscr{E}$-algebra generated by X_S is given by $(T_\Sigma(X_S)/\sim_E, [\![-]\!]_{T_\Sigma(X_S)/\sim_E})$, where $T_\Sigma(X_S)/\sim_E$ is the set of equivalence classes $[-]_\mathscr{E}$ of terms of

$T_\Sigma(X_S)$ under $\sim_E$, and the interpretation function is given by defining $[\![\omega]\!]_{T_\Sigma(X_S)/\sim_E}([t_1]_{\mathscr{E}}, \ldots, [t_n]_{\mathscr{E}})$ for each operator ω as $[\omega(t_1, \ldots, t_n)]_{\mathscr{E}}$. This development yields the monadic forgetful functor $\mathscr{E}$-**Alg** $\to$ **Set**^{S}, hence the category of $\mathscr{E}$-algebras is isomorphic to the category of Eilenberg-Moore algebras for the monad induced by the free $\mathscr{E}$-algebras. Finally, as another well-known property we have that the category $\mathscr{E}$-**Alg** is complete and cocomplete.

2.3 First-Order Algebraic Theories

First-order equational presentations are abstractly formalised as algebraic theories. The details surrounding the strong connection to finitary monads, that is monads preserving filtered colimits, will be omitted here.

2.3.1 Algebraic theories and their translations

We generalise first-order algebraic theories of Lawvere (Definition 1.1) to the multi-sorted universe. This involves a generalisation of the first-order theory of equality to include sorts.

The multi-sorted first-order theory of equality. For S a set of sorts, let $\mathbb{L}_S$ be the opposite of the category whose objects are pairs $(n, \sigma_{(-)})$, with $\sigma_{(-)} \colon \|n\| \to S$ a function mapping $i \in \|n\|$ to σ_i, and with morphisms $f : (n, \sigma_{(-)}) \to (n', \sigma'_{(-)})$ given by functions $f : \|n\| \to \|n'\|$ such that $\sigma_{(-)} = \sigma'_{f(-)}$. Composition is simply function composition, and the identity on $(n, \sigma_{(-)})$ is just the identity on $\|n\|$. Informally, we think of objects of $\mathbb{L}_S$ as given by tuples $(\sigma_1, \ldots, \sigma_n)$ of $(S')^*$.

The opposite of $\mathbb{L}_S$ comes equipped with a cocartesian structure given by the concatenation

$$\sigma_1, \ldots, \sigma_n, \sigma'_{n+1}, \ldots, \sigma'_{n+k}$$

of tuples $(n, \sigma_{(-)})$ and $(k, \sigma'_{(-)})$, with injections $(n^{(i)}, \sigma^{(i)}_{(-)}) \to \coprod_j \left(n^{(j)}, \sigma^{(j)}_{(-)}\right)$ given by

$$\|n^{(i)}\| \to \|\sum_j n^{(j)}\|, \qquad k \mapsto k + \sum_{j=1}^{i-1} n^{(j)} \quad .$$

The initial object is the empty tuple (), equivalently represented as $\phi \to S$. This of course means that $\mathbb{L}_S$ is cartesian.

We refer to $\mathbb{L}_S$ as the *multi-sorted first-order theory of equality*. Like its mono-sorted version, $\mathbb{L}$, we can characterise $\mathbb{L}_S$ abstractly via its universal cartesian structure.

Lemma 2.9 (Universal property)**.** *For a set of sorts S, the first-order theory of equality $\mathbb{L}_S$ is the free* *cartesian category generated by S.*

Definition 2.10 (Multi-sorted first-order algebraic theories)**.** A *multi-sorted first-order algebraic theory* consists of a set of sorts S, a small cartesian category $\mathscr{L}$, and a strict identity-on-objects cartesian functor $L \colon \mathbb{L}_S \to \mathscr{L}$.

Remark 2.11. Although, strictly, the combination of a strict cartesian functor L and a cartesian category $\mathscr{L}$ defines an algebraic theory, we informally refer to both L and $\mathscr{L}$ separately as algebraic theories.

Since for the one-element set $\{*\}$ we evidently have $\mathbb{L} = \mathbb{L}_{\{*\}}$, our first example of an algebraic theory is a Lawvere theory, that is a mono-sorted first-order algebraic theory $L \colon \mathbb{L}_{\{*\}} \to \mathscr{L}$.

First-order algebraic translations. For multi-sorted first-order algebraic theories $L \colon \mathbb{L}_S \to \mathscr{L}$ and $L' \colon \mathbb{L}_{S'} \to \mathscr{L}'$, a (multi-sorted) *first-order algebraic translation* is given by a cartesian functor $F \colon \mathscr{L} \to \mathscr{L}'$, together with a function $\varphi \colon S \to (S')^*$, making the following commute

$$\begin{array}{ccc} \mathbb{L}_S & \xrightarrow{\mathbb{L}_\varphi} & \mathbb{L}_{S'} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L'} \\ \mathscr{L} & \xrightarrow{F} & \mathscr{L}' \end{array}$$

noting that $\mathbb{L}_\varphi \colon \mathbb{L}_S \to \mathbb{L}_{S'}$ is the functor induced by φ mapping the tuple $(\sigma_1, \ldots, \sigma_n)$ to the concatenation of the tuples $\varphi(\sigma_i)$, for $1 \leq i \leq n$.

The category of first-order algebraic theories. We denote by **FOAT** the category of multi-sorted first-order algebraic theories and algebraic translations, with the evident identity and composition. We furthermore obtain, for a fixed set of sorts S, the category $\mathbf{FOAT}_S$ of S-sorted first-order theories, whose algebraic translations all have component maps $\varphi \colon S \to S^*$; $\sigma \mapsto (\sigma)$, together with the resulting identity functor $\mathbb{L}_S \to \mathbb{L}_S$. Note that this results in the categorical equivalence $\mathbf{FOAT}_{\{*\}} \cong \mathbf{LAW}$.

The category **LAW** of Lawvere theories is known to be bicomplete [Lawvere, 2004], and this result has been extended to include many-sorted algebraic theories, see for example [Goguen and Burstall, 1984a, Goguen and Burstall, 1984b].

Theorem 2.12. *The category* **FOAT** *of multi-sorted first-order algebraic theories and algebraic translations is bicomplete.*

We use completeness and cocompleteness to provide examples of some basic algebraic theories via universal properties arising from (co)limiting constructions.

- The most elementary algebraic theory is the identity $\mathbb{L} \to \mathbb{L}$, which is initial in **FOAT**. It is mono-sorted and has no operators or axioms. A model of it is just a set, and it is therefore often referred to as 'the theory of sets'.

- The most elementary S-sorted algebraic theory is given by the identity functor $\mathbb{L}_S \to \mathbb{L}_S$. It is again free of operators and axioms, and is the initial object in the category $\mathbf{FOAT}_S$ of S-sorted algebraic theories.

- The terminal object of **FOAT** is a mono-sorted so-called *trivial* algebraic theory and defined as follows. Let $\mathscr{L}_T$ be the category with objects those of $\mathbb{L}$ and exactly one morphism from any object to another, making it equivalent to the unit category $\mathbf{1}$. The trivial theory $L_T \colon \mathbb{L} \to \mathscr{L}_T$ is the identity on objects but trivial on morphisms, and in that sense it identifies all morphisms in a given hom-set $\mathbb{L}(m,n)$.

- One of the most interesting constructions in universal algebra is the *tensor product* of algebraic theories, which we illustrate here in the mono-sorted setting for simplicity. Given Lawvere theories $L\colon \mathbb{L} \to \mathscr{L}$ and $L'\colon \mathbb{L} \to \mathscr{L}'$, the tensor product theory $(L \otimes L')\colon \mathbb{L} \to (\mathscr{L} \otimes \mathscr{L}')$ is constructed by taking the coproduct of $\mathscr{L}$ and $\mathscr{L}'$ and imposing the following equality in the category $\mathscr{L} \otimes \mathscr{L}'$: for every morphism $f : m \to 1$ in $\mathscr{L}$ and $g : n \to 1$ in $\mathscr{L}'$, $f \circ g^m = g \circ f^n$. Intuitively, this requirement enforces the operators of both Lawvere theories to commute in their tensor product theory. The tensor product operation is associative, commutative, and admits the 'theory of sets' as a unit. Moreover, it can be combined with a coequaliser to construct the tensor product of two algebraic theories over a third one. The importance of this universal construction lies in the fact that the following categories of mono-sorted functorial models (Definition 1.2) are equivalent:

$$\mathbf{FMod}(L, \mathscr{L}') \cong \mathbf{FMod}(L', \mathscr{L}) \cong \mathbf{FMod}(L \otimes L', \mathbf{Set})$$

2.3.2 Functorial Semantics

The mono-sorted functorial model theory of Lawvere presented in Section 1.1.1 generalises easily to the multi-sorted universe. Functorial models are again defined to be cartesian functors.

Definition 2.13. A *functorial model* of an algebraic theory $L \colon \mathbb{L}_S \to \mathscr{L}$ in a cartesian category $\mathscr{C}$ is given by a cartesian functor $F : \mathscr{L} \to \mathscr{C}$. For any cartesian category $\mathscr{C}$, $\mathbf{FMod}(L, \mathscr{C})$ denotes the category of functorial models of $L \colon \mathbb{L}_S \to \mathscr{L}$ in $\mathscr{C}$ and natural transformations between them. We denote by $\mathbf{FMod}(L)$ the category of *set-theoretic functorial models* $\mathscr{L} \to \mathbf{Set}^S$ of the theory $L \colon \mathbb{L}_S \to \mathscr{L}$ in the category $\mathbf{Set}^S$.

Note that functorial models are defined to be cartesian rather than strict cartesian, which is a fundamental difference pointed out by Lawvere in [Lawvere, 2004]. With **Set** and $\mathbf{Set}^S$ being the primary semantic universes of interest, note that their finite products are not strictly associative, whereas they are associative in any algebraic theory. The importance of this can be seen in Lawvere's example of the category of functorial models of the algebraic theory of monoids, which would be empty under strict cartesian models rather than the category of monoids as one would expect.

Remark 2.14. In defining the category of functorial models, the correctness of taking all natural transformations as morphisms rather than monoidal ones can be easily verified. A natural transformation $\alpha\colon F \to G$ between cartesian functors $F, G\colon \mathbb{L}_S \rightrightarrows \mathscr{C}$ is *monoidal* if it respects the cartesian structure, in the sense that

$$\begin{array}{ccc} F(\sigma_1,\ldots,\sigma_n) & \xrightarrow{\alpha_{(\sigma_1,\ldots,\sigma_n)}} & G(\sigma_1,\ldots,\sigma_n) \\ \cong\downarrow & & \downarrow\cong \\ F(\sigma_1)\times\cdots\times F(\sigma_n) & \xrightarrow{\langle\alpha_{\sigma_1},\ldots,\alpha_{\sigma_n}\rangle} & G(\sigma_1)\times\cdots\times G(\sigma_n) \end{array}$$

commutes for all $(\sigma_1,\ldots,\sigma_n)$ of $\mathbb{L}_S$. The subtlety here is in that natural transformations between cartesian functors are necessarily monoidal, as for $(\sigma_1,\ldots,\sigma_n)\in\mathbb{L}_S$, the morphism

$$\alpha_{(\sigma_1,\ldots,\sigma_n)}\colon F(\sigma_1,\ldots,\sigma_n)\to G(\sigma_1,\ldots,\sigma_n)$$

is simply the n-ary product of α_{σ_i}, for $i\in \|n\|$.

2.4 First-Order Categorical Type Theory Correspondence

A main theme throughout this dissertation is the formulation of an abstract view of syntactic universal algebra, and, vice versa, the extraction of syntactic presentations, their morphisms and models from categorical constructions inspired by Lawvere. Having presented the two developments in the multi-sorted first-order setting independently, we now proceed to review what we refer to as the *syntactic* and *semantic* categorical type theory correspondences, which respectively establish

- the equivalence between first-order algebraic theories and first-order equational presentations, making algebraic theories a syntax-independent presentation of equational theories; and

- the equivalence between the corresponding first-order algebras and functorial models.

2.4.1 Classifying algebraic theories and internal languages

We start by illustrating that a first-order equational presentation induces an algebraic theory, and, vice versa, that any algebraic theory has an underlying equational presentation.

Classifying categories. A *classifying category* for a syntactic specification is the 'smallest', up-to-equivalence unique category in which it can be soundly modelled. Given an S-sorted equational presentation $\mathscr{E} = (\Sigma, E)$, its classifying category is a cartesian category $\mathbb{L}(\mathscr{E})$ equipped with a *generic* $\mathscr{E}$-algebra

$(G_S, [\![-]\!]_{G_S})$, which is generic in the sense that for any $\mathscr{E}$-algebra $(D_S, [\![-]\!]_{D_S})$ in a cartesian category $\mathscr{D}$, there is a unique functor $F\colon \mathbb{L}(\mathscr{E}) \to \mathscr{D}$ such that for all operators ω of $\mathscr{E}$, $F([\![\omega]\!]_{G_S}) = [\![\omega]\!]_{D_S}$.

The universal category $\mathbb{L}(\mathscr{E})$ arises through a formal construction from the syntactic definition of $\mathscr{E}$ as follows. Objects are tuples of sorts, and morphisms $(\sigma_1, \ldots, \sigma_k) \to (\sigma'_1, \ldots, \sigma'_n)$ are tuples $\langle [t_1]_{\mathscr{E}}, \ldots, [t_n]_{\mathscr{E}} \rangle$ of equivalence classes of terms $x_1\colon \sigma_1, \ldots, x_k\colon \sigma_k \vdash t_i\colon \sigma'_i$ under the equivalence $\sim_{\mathscr{E}}$ identifying two terms if and only if their equational congruence $\equiv$ is derivable from E.

This construction does indeed yield a category. The identity on the tuple $(\sigma_1, \ldots, \sigma_k)$ is the tuple $\langle [x_1\colon \sigma_1, \ldots, x_k\colon \sigma_k \vdash x_i\colon \sigma_i]_{\mathscr{E}} \rangle_{i \in \|n\|}$, and the composition of $\langle [\Gamma \vdash t_i\colon \sigma'_i]_{\mathscr{E}} \rangle_{i \in \|n\|}$ with $\langle [\Gamma' \vdash s_j\colon \sigma''_j]_{\mathscr{E}} \rangle_{j \in \|k\|}$ is given via substitution by $\langle [s_j\{x_i := t_i\}_{i \in \|n\|}]_{\mathscr{E}} \rangle_{j \in \|k\|}$, where x_i are the variables appearing in Γ'. Associativity of composition is a consequence of the First-Order Substitution Lemma (Lemma 2.3), and its well-definedness is an immediate consequence of the (Substitution) derivability rule of First-Order Equational Logic (Section 2.1.3). Finally, we note that the classifying category $\mathbb{L}(\mathscr{E})$ is cartesian, with products given by tuple concatenation, the terminal object being the empty tuple and the terminal map the empty tuple. The projection $\pi_i\colon \sigma_1 \times \cdots \times \sigma_n \to \sigma_i$ is given by $\langle [x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n \vdash x_i\colon \sigma_i]_{\mathscr{E}} \rangle$.

Note that we can construct the classifying category $\mathbb{L}(\Sigma)$ of a signature by taking the set of axioms to be empty. Morphisms are then simply tuples of equivalence classes of terms under the empty set, or equivalently, tuples of terms rather than equivalence classes of terms.

Lemma 2.15. *Let $\mathscr{E}_0 = (\Sigma_0, \{\})$ be the 'elementary' S-sorted equational presentation with underlying empty signature $\Sigma_0 = (S, \{\})$ and no axioms. Its classifying category $\mathbb{L}(\mathscr{E}_0)$ is (isomorphic to) the first-order theory of equality $\mathbb{L}_S$.*

Classifying algebraic theories. An S-sorted equational presentation $\mathscr{E} = (\Sigma, E)$ induces the algebraic theory $L_{\mathscr{E}}\colon \mathbb{L}_S \to \mathbb{L}(\mathscr{E})$, where $L_{\mathscr{E}}$ is the canonical cartesian functor mapping $\langle t \rangle$ to $\langle [t]_{\mathscr{E}} \rangle$. In consistency with the terminology above, we refer to $L_{\mathscr{E}}$ as the *classifying algebraic theory* of $\mathscr{E}$.

Remark 2.16. Generally, the notion of classifying algebraic theory for an equational presentation $\mathscr{E}$ is formalised as the theory $L^*\colon \mathbb{L} \to \mathbb{L}^*$ resulting in the categorical equivalence

$$\mathscr{E}\text{-}\mathbf{Alg}(\mathscr{C}) \cong \mathbf{FMod}(L^*, \mathscr{C})$$

for any cartesian category $\mathscr{C}$. We show in Section 2.4.3 that our reference to the algebraic theory $L_{\mathscr{E}}\colon \mathbb{L}_S \to \mathbb{L}(\mathscr{E})$ as being classifying is justified, as we do indeed obtain the equivalence $\mathscr{E}\text{-}\mathbf{Alg}(\mathscr{C}) \cong \mathbf{FMod}(L_{\mathscr{E}}, \mathscr{C})$.

Internal languages. In the other direction, the *internal language* $\mathfrak{L}(L) = (\Sigma(L), E(L))$ of an algebraic theory $L\colon \mathbb{L}_S \to \mathscr{L}$ is the equational presentation defined by taking the objects of $\mathscr{L}$, or equivalently of $\mathbb{L}_S$, as its set of sorts and a morphism $f : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ of $\mathscr{L}$ to be an operator $\omega_f : \sigma_1, \ldots, \sigma_n \to \sigma$. One canonically obtains the algebra $(\mathrm{ob}(\mathscr{L}), [\![-]\!]_*)$ of Σ_L in $\mathscr{L}$ by defining $[\![\omega_f]\!]_* := f$. The equations $E(L)$ of $\mathfrak{L}(L)$ are obtained by setting $\Gamma \vdash_{E(L)} t_1 \equiv t_2 \colon \sigma$ if and only if $[\![t_1]\!]_*$ and $[\![t_2]\!]_*$ are equal morphisms in $\mathscr{L}$. The algebra $[\![-]\!]_*$ is referred to as the *generic algebra* of $\mathscr{L}$ induced by its own internal language.

The semantic definition of the equations $E(L)$ associated with an internal language can be given more explicitly, but equivalently, by the following axioms:

($\mathscr{E}$1) For any projection $\pi_i \colon \sigma_1 \times \cdots \times \sigma_n \to \sigma_i$ in $\mathscr{L}$, we set

$$x_1 \colon \sigma_1 \ldots, x_n \colon \sigma_n \vdash_{E(L)} x_i \equiv \omega_{\pi_i}(x_1, \ldots, x_n) \colon \sigma_i \quad .$$

($\mathscr{E}$2) For morphisms $h \colon \sigma_1 \times \cdots \times \sigma_n \to \sigma$, $g : \tau_1 \times \cdots \times \tau_l \to \sigma$, and $f_i \colon \sigma_1 \times \cdots \times \sigma_n \to \tau_i$ $(1 \leq i \leq l)$ of $\mathscr{L}$ with $h = g \circ \langle f_1, \ldots, f_l \rangle$, we set

$$x_1 \colon \sigma_1, \ldots, x_n \colon \sigma_n \vdash_{E(L)} \omega_h(x_1, \ldots, x_n) \equiv \omega_g(y_1, \ldots, y_l)\{y_i := \omega_{f_i}(x_1, \ldots, x_n)\}_{i \in \|l\|} \colon \sigma \quad .$$

2.4.2 Towards first-order syntactic categorical type theory correspondence

We prove the first part of the syntactic categorical type theory correspondence, namely that an algebraic theory is essentially the same as the classifying theory of its internal language.

Theorem 2.17 (First-order theory/presentation correspondence)**.** *A multi-sorted first-order algebraic theory $L\colon \mathbb{L}_S \to \mathscr{L}$ is isomorphic to the classifying algebraic theory $L_{\mathfrak{L}(L)}\colon \mathbb{L}_S \to \mathbb{L}(\mathfrak{L}(L))$ of its own internal language $\mathfrak{L}(L)$.*

Proof sketch. The isomorphism is trivial on objects, as a tuple $(\sigma_1, \ldots, \sigma_n)$ of $\mathbb{L}(\mathfrak{L}(L))$ is just the cartesian product $\sigma_1 \times \cdots \times \sigma_n$ as in $\mathscr{L}$. A morphism $f : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ of $\mathscr{L}$ is mapped under the isomorphism to $\langle [\omega_f(x_1, \ldots, x_n)]_{\mathfrak{L}(L)} \rangle \colon (\sigma_1, \ldots, \sigma_n) \to \sigma$ of $\mathbb{L}(\mathfrak{L}(L))$. In the other direction, a morphism $\langle [t]_{\mathfrak{L}(L)} \rangle \colon (\sigma_1, \ldots, \sigma_n) \to \sigma$ of $\mathbb{L}(\mathfrak{L}(L))$ is mapped to the term interpretation $[\![t]\!]_*$ induced by the generic algebra of $\mathfrak{L}(L)$ in $\mathscr{L}$. Note that this mapping respects the equivalence relation $\sim_{\mathfrak{L}(L)}$ as by definition the generic algebra satisfies all equations of $\mathfrak{L}(L)$. These mappings are indeed mutual inverses: a morphism $f : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ of $\mathscr{L}$ is trivially equal to the composite $f \circ \langle \pi_1, \ldots, \pi_n \rangle$. The other direction is given by the equational theory of $\mathfrak{L}(L)$ and established by induction on term structure:

- $\langle [x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i]_{\mathfrak{L}(L)} \rangle$ corresponds to the morphism $\langle [\omega_{\pi_i}(x_1, \ldots, x_n)]_{\mathfrak{L}(L)} \rangle$, which are indeed equal by $(\mathscr{E}1)$.

- For $f : \tau_1 \times \cdots \times \tau_k \to \sigma$, the morphism $\langle [\omega_f(t_1, \ldots, t_k)]_{\mathfrak{L}(L)} \rangle$ corresponds to the

$$\left\langle \left[\omega_f(y_1, \ldots, y_k)\{y_i := \omega_{[\![t_i]\!]_*}(\overrightarrow{z_i})\}_{i \in \|k\|} \right]_{\mathfrak{L}(L)} \right\rangle \quad ,$$

  which are similarly equal by $(\mathscr{E}2)$.

□

2.4.3 Semantic categorical type theory correspondence

We conclude by recalling the semantic component of the Categorical Type Theory Correspondence given by the correspondence between functorial models for first-order algebraic theories, algebras for first-order equational presentations, and Eilenberg-Moore algebras for finitary monads. We refer the reader to [Borceux, 1994] for detailed proofs.

Theorem 2.18 (First-Order Semantic Categorical Type Theory Correspondence)**.** *For every S-sorted first-order equational presentation $\mathscr{E}$ with classifying algebraic theory $L_{\mathscr{E}} : \mathbb{L}_S \to \mathbb{L}(\mathscr{E})$, the category $\mathscr{E}$-**Alg** of $\mathscr{E}$-algebras and the category of functorial models $\mathbf{FMod}(L_{\mathscr{E}}, \mathbf{Set}^S)$ are equivalent. Similarly, for every first-order algebraic theory $L : \mathbb{L}_S \to \mathscr{L}$, the category of functorial models $\mathbf{FMod}(L, \mathbf{Set}^{\mathrm{ob}(\mathscr{L})})$ is equivalent to the category $\mathfrak{L}(L)$-**Alg** of algebras for the internal language $\mathfrak{L}(L)$.*

Proposition 2.19. *For every S-sorted first-order equational presentation $\mathscr{E}$, there exists a finitary monad **T** on $\mathbf{Set}^S$ such that the category of $\mathscr{E}$-algebras is isomorphic to that of Eilenberg-Moore algebras for **T**. Also, for a set S and every finitary monad **T** on $\mathbf{Set}^S$, there exists a first-order algebraic theory $L : \mathbb{L}_S \to \mathscr{L}$ such that the category of Eilenberg-Moore algebras for **T** is isomorphic to the category of functorial models $\mathbf{FMod}(L, \mathbf{Set}^S)$.*

Chapter 3

First-Order Syntactic Translations

Formal comparison of equational presentations is traditionally obtained by comparing their categories of models. If those are categorically equivalent, we say that the presentations are *Morita equivalent*. Alternatively, if known, one may look at the respective classifying categories, in which case an equivalence of presentations would be established if the classifying categories are isomorphic. We seek to develop a syntactic mathematical formalism for notions such as equivalence and conservative extension, amongst others, enabling us to compare equational presentations at the syntactic level. To this end, we introduce our notion of (first-order) *syntactic translation* between equational presentations. We justify the correctness of our definition by establishing its correspondence with that of algebraic translations, the canonical notion of morphism between first-order algebraic theories.

Despite that our definition coincides in principle with that of Fujiwara [Fujiwara, 1959, Fujiwara, 1960] and with the concept of *polyderivor* [Vidal and Tur, 2008], the notion of syntactic translation carries its advantages. Its syntactic formulation enables an explicit description of the mapping of the components (sorts, operators, equations) defining equational presentations, and is, as a result, easily generalisable to the second-order setting. Moreover, as it is the syntactic counterpart of algebraic translation, it encapsulates the idea of a *syntactic interpretation* of one presentation in another.

Our development begins with the notion of a *syntactic map* of equational presentations (Section 3.1), which are simply functions between the corresponding sets of sorts and operators. Maps are what one may initially believe to be the correct definition of morphism of equational presentations; however, we show that syntactic translations, defined in Section 3.2, arise as *Kleisli syntactic maps* (Section 3.3). We establish the correctness of our definition of syntactic translations in Section 3.4 by proving that they correspond to algebraic translations. Syntactic translations moreover enable an explicit description of the notion of isomorphism of equational presentation. We use this to establish the syntactic counterpart of the Theory/Presentation Correspondence, by syntactically verifying that an equational presentation is isomorphic to the internal language of its classifying algebraic theory (Theorem 3.8).

3.1 Syntactic Maps

Signature maps. A (first-order) *syntactic map* $\mu\colon \Sigma \to \Sigma'$ between multi-sorted first-order signatures $\Sigma = (S, \Omega, |-|)$ and $\Sigma' = (S', \Omega, |-|)$ is given by functions between the corresponding sets of sorts and operators as follows:

$$\begin{array}{rcl} S & \to & S' \\ \sigma & \mapsto & \mu(\sigma) \end{array}$$

$$\begin{array}{rcl} \Omega & \to & \Omega' \\ \omega\colon \sigma_1, \ldots, \sigma_k \to \sigma & \mapsto & \mu(\omega)\colon \mu(\sigma_1), \ldots, \mu(\sigma_k) \to \mu(\sigma) \end{array}$$

A signature map $\mu\colon \Sigma \to \Sigma'$ induces an evident mapping on contexts and terms, by mapping a context $\Gamma = (x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n)$ of Σ to $\mu(\Gamma) = (x_1\colon \mu(\sigma_1), \ldots, x_n\colon \mu(\sigma_n))$ of Σ', and a term $\Gamma \vdash t\colon \sigma$ to $\mu(\Gamma) \vdash \mu(t)\colon \mu(\sigma)$, which is defined by induction on term structure as follows:

- $x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n \vdash x_i\colon \sigma_i$ is mapped to $x_1\colon \mu(\sigma_1), \ldots, x_n\colon \mu(\sigma_n) \vdash x_i\colon \mu(\sigma_i)$.

- $\Gamma \vdash \omega(t_1, \ldots, t_k)\colon \sigma$ is mapped to $\mu(\Gamma) \vdash \mu(\omega)\big(\mu(t_1), \ldots, \mu(t_k)\big)\colon \mu(\sigma)$.

Syntactic maps. A first-order *syntactic map* $\mu\colon \mathscr{E} \to \mathscr{E}'$ between equational presentations $\mathscr{E} = (\Sigma, E)$ and $\mathscr{E}' = (\Sigma', E')$ is a signature map $\mu\colon \Sigma \to \Sigma'$ such that for every axiom $\Gamma \vdash_E t \equiv t'\colon \sigma$ of $\mathscr{E}$, the judgement $\mu(\Gamma) \vdash_{\mathscr{E}'} \mu(t) \equiv \mu(t')\colon \mu(\sigma)$ is a theorem of $\mathscr{E}'$.

The *identity syntactic map* $\mu^{\mathscr{E}}\colon \mathscr{E} \to \mathscr{E}$ is simply the identity function on the sets of sorts, operators and equations, and composition of syntactic maps is given by composition of the underlying functions. We write $\mathbf{Sig}_\mu$ for the category of multi-sorted first-order signatures and syntactic maps, and $\mathbf{FOEP}_\mu$ for the category of first-order equational presentations and syntactic maps.

3.2 Syntactic Translations

A syntactic map is not the appropriate notion of a morphism for equational presentations. However, it plays a subtle role in the definition of *syntactic translation*, which we explicitly define next.

Signature translations. A first-order *syntactic translation* $\tau\colon \Sigma \to \Sigma'$ between multi-sorted signatures $\Sigma = (S, \Omega, |-|)$ and $\Sigma' = (S', \Omega', |-|)$ maps sorts to tuples of sorts and operators to tuples of terms in a context determined by the operator arity. Formally, τ is given by mappings

$$
\begin{array}{rcl}
\tau\colon S & \to & (S')^* \\
\sigma & \mapsto & \left(\tau(\sigma)_1, \ldots, \tau(\sigma)_{|\tau(\sigma)|}\right)
\end{array}
$$

$$
\begin{array}{rcl}
\tau\colon \Omega & \to & T_{\Sigma'} \\
\omega\colon \sigma_1, \ldots, \sigma_k \to \sigma & \mapsto & \left\langle \Gamma'_{\tau(\sigma_1)}, \ldots, \Gamma'_{\tau(\sigma_n)} \vdash \tau(\omega)_i\colon \tau(\sigma)_i \right\rangle_{1 \le i \le |\tau(\sigma)|}
\end{array} ,
$$

where $\Gamma'_{\tau(\sigma_j)}$ denotes the context declaring the sorts $\tau(\sigma_j)_i$ for $1 \le i \le |\tau(\sigma_j)|$.

Remark 3.1 (Notational Convention). In the above definition, we write $|-|$ for the length of any tuple, and we moreover denote the i-th element of a tuple $(-)$ by $(-)_i$.

Translation of contexts. A signature translation $\tau\colon \Sigma \to \Sigma'$ induces an evident mapping from the contexts of Σ to the contexts of Σ', defined for a context $\Gamma = (x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n)$ of Σ by $\tau(\Gamma) := \Gamma'_{\tau(\sigma_1)}, \ldots, \Gamma'_{\tau(\sigma_n)}$.

Translation of terms. A signature translation $\tau\colon \Sigma \to \Sigma'$ further extends to a mapping $\tau\colon T_\Sigma \to (T_{\Sigma'})^*$ from the terms of Σ to tuples of terms of Σ' according to the following definition by structural induction.

- The variable term $x_1\colon \sigma_1, \ldots, x_n\colon \sigma_n \vdash x_i\colon \sigma_i$ is mapped to the tuple

$$
\left\langle \Gamma'_{\tau(\sigma_1)}, \ldots, \Gamma'_{\tau(\sigma_n)} \vdash x_{i,j}\colon \tau(\sigma_i)_j \right\rangle_{1 \le j \le |\tau(\sigma_i)|} .
$$

- For an operator $\omega\colon \sigma_1, \ldots, \sigma_k \to \sigma$ of Σ with image under τ given by

$$
\left\langle \Gamma'_{\tau(\sigma_1)}, \ldots, \Gamma'_{\tau(\sigma_k)} \vdash \tau(\omega)_i\colon \tau(\sigma)_i \right\rangle_{1 \le i \le |\tau(\sigma)|}
$$

  and for terms $\Gamma \vdash t_i\colon \sigma_i$ $(1 \le i \le k)$ with images under τ given by

$$
\left\langle \tau(\Gamma) \vdash \tau(t_i)_j\colon \tau(\sigma_i)_j \right\rangle_{1 \le j \le |\tau(\sigma_i)|} ,
$$

  the term $\Gamma \vdash \omega(t_1, \ldots, t_k)\colon \sigma$ is mapped under the translation τ to the tuple

$$
\left\langle \tau(\Gamma) \vdash \tau(\omega)_h \left\{ \{y_{i,j} := \tau(t_i)_j\}_{1 \le j \le |\tau(\sigma_i)|} \right\}_{i \in \|k\|} \right\rangle_{1 \le h \le |\tau(\sigma)|} .
$$

Lemma 3.2 (Compositionality)**.** *The extension of a first-order syntactic translation on terms commutes with substitution. Formally, for a translation $\tau\colon \Sigma \to \Sigma'$ and term $\Gamma \vdash t\{x_k := s_k\}_{k \in \|n\|}\colon \sigma$, where $k \in \|n\|$*

and $\Gamma \vdash s_k : \sigma_k$, *we have for all* $1 \leq i \leq |\tau(\sigma)|$,

$$\tau(\Gamma) \vdash \tau\big(t\{x_k := s_k\}_{k\in\|n\|}\big)_i \;=\; \tau(t)_i\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(\sigma_k)|} : \tau(\sigma)_i \quad .$$

Proof. We proceed by induction on the structure of the term $\Gamma, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \sigma$.

$$\begin{aligned}
- \quad & \tau\big(x_l\{x_k := s_k\}_{k\in\|n\|}\big)_i \\
= \; & \tau(s_l)_i \\
= \; & x_{l,i}\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(s_k)|} \\
= \; & \tau(x_l)\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(s_k)|}
\end{aligned}$$

$$\begin{aligned}
- \quad & \tau\big(\omega(\ldots,t,\ldots)\{x_k := s_k\}_{k\in\|n\|}\big)_i \\
= \; & \tau\big(\omega(\ldots,t\{x_k := s_k\}_{k\in\|n\|},\ldots)\big)_i \\
= \; & \tau(\omega)_i\big\{y_l := \tau\big(t\{x_k := s_k\}_{k\in\|n\|}\big)_l\big\}_{1\leq l\leq|\tau(t)|} \\
= \; & \tau(\omega)_i\big\{y_l := \tau(t)_l\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(s_k)|}\big\}_{1\leq l\leq|\tau(t)|} \\
= \; & \tau(\omega)_i\{y_l := \tau(t)_l\}_{1\leq l\leq|\tau(t)|}\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(s_k)|} \\
= \; & \tau\big(\omega(\ldots,t,\ldots)\big)_i\{x_{k,j} := \tau(s_k)_j\}_{k\in\|n\|,1\leq j\leq|\tau(s_k)|}
\end{aligned}$$

□

Syntactic translations. A *syntactic translation* $\tau : \mathscr{E} \to \mathscr{E}'$ between first-order equational presentations $\mathscr{E} = (\Sigma, E)$ and $\mathscr{E}' = (\Sigma', E')$ is a signature translation $\tau : \Sigma \to \Sigma'$, such that for every axiom $\Gamma \vdash_E t_1 \equiv t_2 : \sigma$ of $\mathscr{E}$, the judgements $\tau(\Gamma) \vdash \tau(t_1)_i \equiv \tau(t_2)_i : \tau(\sigma)_i$ (for all $1 \leq i \leq |\tau(\sigma)|$) are derivable from E'.

Lemma 3.3. *The extension of a first-order syntactic translation on terms preserves equational derivability.*

Proof. We verify for each rule of First-Order Equational Logic (Section 2.1.3) that the hypothesis is mapped under a syntactic translation to a finite collection of derivable equations. One needs to only check the Substitution derivability rule. For $\tau : \mathscr{E} \to \mathscr{E}'$ a syntactic translation of equational presentations, let

$$\tau(\Gamma), \tau(x : \sigma) \vdash \tau(s)_j : \tau(\sigma')_j$$

be a term and

$$\tau(\Gamma) \vdash \tau(t_1)_i \equiv \tau(t_2)_i : \tau(\sigma)_i$$

be an equation of $\mathscr{E}'$. Then we indeed have from the substitution rule of the First-Order Equational Logic of $\mathscr{E}'$

$$\tau(\Gamma) \vdash \tau(s)_j\{y_j := \tau(t_1)_i\}_{1\leq i\leq|\sigma|} \equiv \tau(s)_j\{y_j := \tau(t_2)_i\}_{1\leq i\leq|\sigma|} : \tau(\sigma')_j \quad ,$$

for all $1 \leq j \leq |\tau(s)|$, which further implies

$$\tau(\Gamma) \vdash \tau\big(s\{x := t_1\}\big) \equiv \tau\big(s\{x := t_2\}\big) : \tau(\sigma')_j$$

by the Compositionality Lemma (Lemma 3.2). □

For the detailed syntactic definitions to yield some intuition, we provide examples of (mono-sorted) syntactic translations from classical universal algebra.

Example 3.4.

(1) One may define a syntactic translation from the presentation $\mathscr{E}_{\mathscr{G}}$ of the theory of groups to itself, according to the following mappings of operators to terms:

$$\begin{array}{lcl} \mathtt{e} & \mapsto & - \vdash \mathtt{e}() \\ \mathtt{i} & \mapsto & x \vdash \mathtt{i}(\mathtt{x}) \\ \mathtt{m} & \mapsto & x_1, x_2 \vdash \mathtt{m}(\mathtt{x}_1, \mathtt{x}_2) \end{array}$$

The axioms of group theory are just mapped to themselves. In fact, we will see below that this is an example of an identity *syntactic translation.*

(2) We can also translate the presentation $\mathscr{E}_{\mathscr{G}}$ of the theory of groups into that of the theory of rings, $\mathscr{E}_{\mathscr{R}}$, which has operators $+ : 2$, $\mathbf{0} : 0$, $- : 1$, $\bullet : 2$, and $\mathbf{1} : 1$. Recall that the axioms of $\mathscr{E}_{\mathscr{R}}$ are given by associativity of $+$ and $\bullet$, identity with respect to both $+$ and $\bullet$, existence of an inverse $-$, commutativity of $+$, and finally distributivity of $\bullet$ over $+$. We define the syntactic translation $\tau_{\mathscr{G}\to\mathscr{R}} : \mathscr{E}_{\mathscr{G}} \to \mathscr{E}_{\mathscr{R}}$ by the following mapping (where infix notation is used for the operators of $\mathscr{E}_{\mathscr{R}}$):

$$\begin{array}{lcl} \mathtt{e} & \mapsto & - \vdash \mathbf{0}() \\ \mathtt{i} & \mapsto & x \vdash -(x) \\ \mathtt{m} & \mapsto & x_1, x_2 \vdash x_1 + x_2 \end{array}$$

Axioms of $\mathscr{E}_{\mathscr{G}}$ translate to axioms of $\mathscr{E}_{\mathscr{R}}$ representing associativity, identity, and the existence of an inverse for the operator $+$.

Translation composition and identity. We define the *composition* $\tau' \circ \tau : \mathscr{E}_1 \to \mathscr{E}_3$ of translations $\tau : \mathscr{E}_1 \to \mathscr{E}_2$ and $\tau' : \mathscr{E}_2 \to \mathscr{E}_3$ to be the composition of the underlying mappings, more formally given by

$$\begin{array}{rcl} \sigma & \mapsto & \tau'(\tau(\sigma)_1), \ldots, \tau'(\tau(\sigma)_{|\tau(\sigma)|}) \\ \omega : \sigma_1, \ldots, \sigma_k \to \sigma & \mapsto & \big\langle \Gamma_{\tau'(\tau(\sigma_1))}, \ldots, \Gamma_{\tau'(\tau(\sigma_k))} \vdash \tau'(\tau(\omega)_i)_{j_i} : \tau'(\tau(\sigma)_i)_{j_i} \big\rangle_{1 \leq i \leq |\tau(\sigma)|,\ 1 \leq j_i \leq |\tau'(\tau(\sigma)_i)|} \end{array}$$

Note that this definition immediately implies that the extension of the composite $\tau' \circ \tau$ on a term t is simply given by $\tau'(\tau(t))$, and that composition of translations is an associative operation. Translation composition moreover preserves equational derivability. More precisely, an axiom $\Gamma \vdash s \equiv t : \sigma$ of $\mathscr{E}_1$ has its image under the composite $\tau' \circ \tau$ given by the tuple

$$\left\langle \tau'(\tau(\Gamma)) \vdash \tau'(\tau(s))_i \equiv \tau'(\tau(t))_i : \tau'(\tau(\sigma))_i \right\rangle_{1 \leq i \leq |(\tau' \circ \tau)(\sigma)|} \quad ,$$

whose component equations are indeed derivable in $\mathscr{E}_3$ because each of τ and τ' preserves equational derivability.

Furthermore, we define the *identity syntactic translation* $\tau^{\mathscr{E}} : \mathscr{E} \to \mathscr{E}$ on an equational presentation $\mathscr{E} = (\Sigma, E)$ by the following mappings

$$\begin{array}{rcl} \sigma & \mapsto & (\sigma) \\ \omega : \sigma_1, \ldots, \sigma_n \to \sigma & \mapsto & x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \omega(x_1, \ldots, x_n) : \sigma \end{array}$$

The extension of the identity translation evidently acts as the identity on contexts and terms; hence axioms are just mapped to themselves. Note that $\tau^{\mathscr{E}}$ behaves indeed as the identity with respect to syntactic translation composition. Given a translation $\tau : \mathscr{E} \to \mathscr{E}'$, the fact that $\tau \circ \tau^{\mathscr{E}} = \tau = \tau^{\mathscr{E}'} \circ \tau$ is clear for sorts. For an operator $\omega : \sigma_1, \ldots, \sigma_n \to \sigma$ of $\mathscr{E}$, $(\tau \circ \tau^{\mathscr{E}})(\omega)$ is the image of the term

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \omega(x_1, \ldots, x_n) : \sigma$$

under τ, which is just $\tau(\omega)$. On the other hand, the image of $\tau(\omega)$ under $\tau^{\mathscr{E}'}$ is also simply $\tau(\omega)$, as the extension of the identity syntactic translation on terms is the identity mapping.

The category of first-order equational presentations. Using the previous development, we define the category **FOEP** to have objects first-order equational presentations and morphisms given by syntactic translations.

3.3 The Signature/Theory Adjunction

A signature generates a free algebraic theory, and every algebraic theory is given by a quotient of a free algebraic theory. We take a little diversion from the categorical type theory correspondence to recall this fundamental adjunction presented by Lawvere [Lawvere, 2004]. Although it is an integral element of the development of algebraic theories, its aim here is to illustrate that syntactic translations are syntactically constructed Kleisli maps under the signature/theory adjunction.

Recall that $\mathbf{Sig}_\mu$ is the category of multi-sorted first-order signatures and maps. Define the functor

$U\colon \mathbf{FOAT} \to \mathbf{Sig}_\mu$ by mapping an algebraic theory $L\colon \mathbb{L}_S \to \mathscr{L}$ to the signature $\Sigma(L)$ of its internal language $\mathfrak{L}(L)$. An algebraic translation $F\colon \mathscr{L} \to \mathscr{L}'$ $(\varphi\colon S \to (S')^*)$ between algebraic theories $L\colon \mathbb{L}_S \to \mathscr{L}$ and $L'\colon \mathbb{L}_{S'} \to \mathscr{L}'$ is mapped to the signature map

$$\begin{aligned} \hat{\mu}_F\colon \Sigma(L) &\to \Sigma(L') \\ \sigma &\mapsto F\sigma \\ \omega_f\colon \sigma_1,\ldots,\sigma_n \to \sigma &\mapsto F\omega \circ \cong_{\sigma_1,\ldots,\sigma_n} \end{aligned}$$

where $\cong_{\sigma_1,\ldots,\sigma_n}\colon F\sigma_1 \times \cdots \times F\sigma_n \to F(\sigma_1 \times \cdots \times \sigma_n)$ is the canonical isomorphism.

Theorem 3.5 (Signature/theory adjunction)**.** *In the above setting, the functor $U\colon \mathbf{FOAT} \to \mathbf{Sig}_\mu$ has a left adjoint $F\colon \mathbf{Sig}_\mu \to \mathbf{FOAT}$.*

Proof sketch. The left adjoint maps an S-sorted signature Σ to its classifying algebraic theory $L_\Sigma\colon \mathbb{L}_S \to \mathbb{L}(\Sigma)$, and a signature map $\mu\colon \Sigma \to \Sigma'$, for $\Sigma' = (S', \Omega', |-|)$, to (φ^*_μ, F^*_μ), where

$$\begin{aligned} \varphi^*_\mu\colon S \to (S')^*, &\qquad \sigma \mapsto \mu(\sigma) \\ F^*_\mu\colon \mathbb{L}(\Sigma) \to \mathbb{L}(\Sigma'), &\qquad \sigma_1,\ldots,\sigma_n \mapsto \mu(\sigma_1),\ldots,\mu(\sigma_n), \quad \langle t_1,\ldots,t_k\rangle \mapsto \langle \mu(t_1),\ldots,\mu(t_n)\rangle \end{aligned}$$

It is clear that this satisfies the definition of algebraic translation. The counit ϵ of this adjunction has component at an algebraic theory $L\colon \mathbb{L}_S \to \mathscr{L}$ given by the algebraic translation (id_S, ϵ_L), where id_S is just the identity on the set of sorts S, and $\epsilon_L\colon \mathbb{L}(\Sigma(L)) \to \mathscr{L}$ is the functor mapping $(\sigma_1,\ldots,\sigma_n)$ to $\sigma_1 \times \cdots \times \sigma_n$. On morphisms, ϵ_L is defined by induction on term structure as follows. The variable term $x_1\colon \sigma_1,\ldots,x_n\colon \sigma_n \vdash x_i\colon \sigma_i$ is mapped to the projection $\pi_i\colon \sigma_1 \times \cdots \times \sigma_n \to \sigma_i$, and for an operator $\omega_f\colon \tau_1,\ldots,\tau_k \to \tau$, the term $x_1\colon \sigma_1,\ldots,x_n\colon \sigma_n \vdash \omega_f(t_1,\ldots,t_k)\colon \tau$ is mapped to the composite $f \circ \langle \epsilon_L(t_1),\ldots,\epsilon_L(t_k)\rangle$. □

We use the free theory construction of Theorem 3.5 to provide an equivalent definition of the notion of syntactic signature translation.

Proposition 3.6. *Let $F \dashv U$ be the signature/theory adjunction. The Kleisli category $\mathbf{Sig}_\mu(\mathbf{T})$ for the monad $\mathbf{T} = UF$ is isomorphic to the category $\mathbf{Sig}_\tau$ of multi-sorted first-order signatures and their syntactic translations.*

Indeed, a Kleisli map $\Sigma \to \Sigma'$ maps sorts to tuples of sorts and operators to tuples of terms, which defines a signature translation. The identity translation on Σ is given by the component $\eta_\Sigma\colon \Sigma \to \mathbf{T}\Sigma$ at Σ of the unit η of this adjunction.

3.4 First-Order Syntactic Categorical Type Theory Correspondence

We have only shown one direction of the First-Order Syntactic Categorical Type Theory Correspondence, namely the equivalence of an algebraic theory and the classifying algebraic theory of its own internal language (Theorem 2.17). We now complete this correspondence by proving the other direction, which states that an equational presentation is isomorphic to the internal language of its own classifying theory (Theorem 3.8) and finally establishing that syntactic translations are the correct syntactic counterpart of algebraic translations (Theorem 3.10).

3.4.1 Presentation/theory correspondence

Although Theorem 2.17 is now an integral result of the development surrounding categorical universal algebra, its syntactic counterpart has not been spelled out elsewhere. This, we believe, is due to the non-existence of an explicit, syntactically specified notion of isomorphism of equational presentations, which, given our definition of syntactic translation, is now trivial to formalise.

Definition 3.7. A syntactic translation $\tau\colon \mathscr{E} \to \mathscr{E}'$ of equational presentations is an *isomorphism* if there exists a syntactic translation $\tau^{-1}\colon \mathscr{E}' \to \mathscr{E}$ such that $\tau \circ \tau^{-1}$ is naturally isomorphic to $\tau^{\mathscr{E}'}$ and $\tau^{-1} \circ \tau$ is naturally isomorphic $\tau^{\mathscr{E}}$, where $\tau^{\mathscr{E}}$ and $\tau^{\mathscr{E}'}$ are the identity syntactic translations on $\mathscr{E}$ and $\mathscr{E}'$, respectively.

Theorem 3.8 (First-order presentation/theory correspondence)**.** *Every multi-sorted first-order equational presentation $\mathscr{E} = (S, \Sigma, E)$ is isomorphic to the internal language $\mathfrak{L}(L_{\mathscr{E}})$ of its own classifying first-order algebraic theory $L_{\mathscr{E}}\colon \mathbb{L}_S \to \mathbb{L}(\mathscr{E})$.*

Proof. Let $\mathscr{E} = (S, \Sigma, E)$ be an equational presentation, and define the syntactic translation

$$\xi_{\mathscr{E}}\colon \mathscr{E} \to \mathfrak{L}(L_{\mathscr{E}})$$

by mapping a sort σ to itself (more correctly to the single tuple (σ) of itself, but for ease of readability, and without compromising our proof, we will not make this slight distinction in what follows) and an operator $\hat{\omega}\colon \sigma_1, \ldots, \sigma_k \to \sigma$ to the term

$$x_1\colon \sigma_1, \ldots, x_k\colon \sigma_k \vdash \omega_{f(\hat{\omega})}(x_1, \ldots, x_k)\colon \sigma \quad ,$$

where we write $f(\hat{\omega})$ for the morphism

$$\langle [x_1\colon \sigma_1, \ldots, x_k\colon \sigma_k \vdash \hat{\omega}(x_1, \ldots, x_k)\colon \sigma]_{\mathscr{E}} \rangle\colon \sigma_1 \times \cdots \times \sigma_k \to \sigma$$

of $\mathbb{L}(\mathscr{E})$, which induces the operator $\omega_{f(\hat{\omega})}$ of $\mathfrak{L}(L_{\mathscr{E}})$. Note that $\xi_{\mathscr{E}}$ acts as the identity on contexts, and its extension on a term $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash t : \sigma$ of $\mathscr{E}$ is given by

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \omega_{\langle [t]_{\mathscr{E}} \rangle}(x_1, \ldots, x_n) : \sigma$$

of $\mathfrak{L}(L_{\mathscr{E}})$. The correctness of this extension mapping can be seen by structural induction on t:

- The image of $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i$ under $\xi_{\mathscr{E}}$ is given by

$$\begin{array}{rcl} x_1 : \sigma_1, \ldots, x_n : \sigma_n & \vdash & x_i \\ & \stackrel{(\mathscr{E}1)}{\equiv} & \omega_{\pi_i}(x_1, \ldots, x_n) \\ & = & \omega_{\langle [x_i]_{\mathscr{E}} \rangle}(x_1, \ldots, x_n) \ : \sigma_i \quad , \end{array}$$

  where $\langle [x_i]_{\mathscr{E}} \rangle : \sigma_1 \times \cdots \times \sigma_n \to \sigma_i$ is the i-th projection π_i in $\mathbb{L}(\mathscr{E})$.

- For an operator $\hat{\omega} : \tau_1, \ldots, \tau_k \to \sigma$, the image of $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \hat{\omega}(t_1, \ldots, t_k) : \sigma$ under $\xi_{\mathscr{E}}$ is

$$\begin{array}{rcl} x_1 : \sigma_1, \ldots, x_n : \sigma_n & \vdash & \xi_{\mathscr{E}}(\hat{\omega})\{y_i := \xi_{\mathscr{E}}(t_i)\}_{i \in \|k\|} \\ & = & \omega_{f(\hat{\omega})}(y_1, \ldots, y_k)\{y_i := \omega_{\langle [t_i]_{\mathscr{E}}(x_1, \ldots, x_n) \rangle}\}_{i \in \|k\|} \\ & \stackrel{(\mathscr{E}2)}{\equiv} & \omega_{\langle [\hat{\omega}(t_1, \ldots, t_k)]_{\mathscr{E}} \rangle}(x_1, \ldots, x_n) \ : \sigma \quad , \end{array}$$

  where, recall, $f(\hat{\omega})$ is the morphism $\langle [\hat{\omega}(y_1, \ldots, y_k)]_{\mathscr{E}} \rangle : \tau_1 \times \cdots \times \tau_k \to \sigma$ of $\mathbb{L}(\mathscr{E})$, and the validity of applying $(\mathscr{E}2)$ above follows from the morphism equality

$$\langle [\hat{\omega}(t_1, \ldots, t_k)]_{\mathscr{E}} \rangle = f(\hat{\omega}) \circ \langle [t_1]_{\mathscr{E}}, \ldots, [t_k]_{\mathscr{E}} \rangle$$

  in $\mathbb{L}(\mathscr{E})$.

Moreover, the translation $\xi_{\mathscr{E}}$ maps axioms of $\mathscr{E}$ to derivable equations of $\mathfrak{L}(L_{\mathscr{E}})$ and is therefore justifiably a translation of equational presentations. Indeed, given an axiom

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash_E t \equiv s : \sigma$$

of $\mathscr{E}$, we know that $\langle [t]_{\mathscr{E}} \rangle$ and $\langle [s]_{\mathscr{E}} \rangle$ are the same morphism in $\mathbb{L}(\mathscr{E})$ and therefore induce the same operator $\omega_{\langle [t]_{\mathscr{E}} \rangle} = \omega_{\langle [s]_{\mathscr{E}} \rangle}$ of $\mathfrak{L}(L_{\mathscr{E}})$. This means that we have the equality

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \omega_{\langle [t]_{\mathscr{E}} \rangle}(x_1, \ldots, x_n) \equiv \omega_{\langle [s]_{\mathscr{E}} \rangle}(x_1, \ldots, x_n) : \sigma$$

in $\mathfrak{L}(L_{\mathscr{E}})$, which further gives

$$x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash \xi_{\mathscr{E}}(t) \equiv \xi_{\mathscr{E}}(s) : \sigma$$

from the definition of the extension of $\xi_{\mathscr{E}}$ on terms.

In the other direction, define the syntactic translation

$$\bar{\xi}_{\mathscr{E}} : \mathfrak{L}(L_{\mathscr{E}}) \to \mathscr{E}$$

by mapping a sort $\sigma_1 \times \cdots \times \sigma_n$ of $\mathfrak{L}(L_{\mathscr{E}})$ to the tuple $(\sigma_1, \dots, \sigma_n)$ of sorts σ_i of $\mathscr{E}$. For a morphism $\langle [t]_{\mathscr{E}} \rangle : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ of $\mathbb{L}(\mathscr{E})$, the operator $\omega_{\langle [t]_{\mathscr{E}} \rangle} : \sigma_1, \dots, \sigma_n \to \sigma$ is mapped under $\bar{\xi}_{\mathscr{E}}$ to the term $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash t : \sigma$ of $\mathscr{E}$. A few requirements need to be satisfied for $\bar{\xi}_{\mathscr{E}}$ to be a well-defined syntactic translation. First, note that it has been defined on representatives of equivalence classes $[-]_{\mathscr{E}}$. However, these are well-respected, as given congruent terms t and s, the morphisms $\langle [t]_{\mathscr{E}} \rangle$ and $\langle [s]_{\mathscr{E}} \rangle$ are equal in $\mathbb{L}(\mathscr{E})$, and therefore they induce the same operator $\omega_{\langle [t]_{\mathscr{E}} \rangle} = \omega_{\langle [s]_{\mathscr{E}} \rangle}$ of $\mathfrak{L}(L_{\mathscr{E}})$, whose images under $\bar{\xi}_{\mathscr{E}}$ must therefore be equal. Moreover, the definition of $\bar{\xi}_{\mathscr{E}}$ ensures that axioms of $\mathfrak{L}(L_{\mathscr{E}})$ are mapped to theorems of $\mathscr{E}$.

We finally show that the syntactic translations $\xi_{\mathscr{E}}$ and $\bar{\xi}_{\mathscr{E}}$ are mutual inverses in the sense of Definition 3.7, thereby proving the *syntactic isomorphism* $\mathscr{E} \cong \mathfrak{L}(L_{\mathscr{E}})$. The isomorphism is evident on sorts – we have $(\bar{\xi}_{\mathscr{E}} \circ \xi_{\mathscr{E}})(\sigma) = \bar{\xi}(\sigma) = \sigma$, and

$$(\xi_{\mathscr{E}} \circ \bar{\xi}_{\mathscr{E}})(\sigma_1 \times \cdots \times \sigma_n) = \xi_{\mathscr{E}}(\sigma_1, \dots, \sigma_n) = (\sigma_1, \dots, \sigma_n) = \sigma_1 \times \cdots \times \sigma_n \quad .$$

For an operator $\hat{\omega} : \sigma_1, \dots, \sigma_n \to \sigma$ of $\mathscr{E}$, we have

$$(\bar{\xi}_{\mathscr{E}} \circ \xi_{\mathscr{E}})(\hat{\omega}) \;=\; \bar{\xi}_{\mathscr{E}}\big(\omega_{\langle [\hat{\omega}(x_1, \dots, x_n)]_{\mathscr{E}} \rangle}\big) \;=\; \hat{\omega}(x_1, \dots, x_n) = \tau^{\mathscr{E}}(\hat{\omega}) \quad .$$

On the other hand, given a term $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash t : \sigma$ of $\mathscr{E}$, we have

$$(\xi_{\mathscr{E}} \circ \bar{\xi}_{\mathscr{E}})(\omega_{\langle [t]_{\mathscr{E}} \rangle}) \;=\; \xi_{\mathscr{E}}(t) \;=\; \omega_{\langle [t]_{\mathscr{E}} \rangle}(x_1, \dots, x_n) = \tau^{\mathfrak{L}(L_{\mathscr{E}})}(\omega_{\langle [t]_{\mathscr{E}} \rangle}) \quad .$$

□

3.4.2 Induced syntactic and algebraic translations

To ensure the correctness of our development of syntactic translations, we verify that the notion of algebraic translation is equivalent to that of syntactic translation in the context of the Syntactic Categorical Type Theory Correspondence. We start by illustrating how to construct an algebraic translation from a syntactic one, and, vice versa, a syntactic translation from an algebraic translation.

Induced algebraic translations. Let $\tau\colon \mathscr{E} \to \mathscr{E}'$ be a first-order syntactic translation between equational presentations $\mathscr{E} = (S, \Sigma, E)$ and $\mathscr{E}' = (S', \Sigma', E')$ equational presentations with respective classifying algebraic theories $L_{\mathscr{E}}\colon \mathbb{L}_S \to \mathbb{L}(\mathscr{E})$ and $L_{\mathscr{E}'}\colon \mathbb{L}_{S'} \to \mathbb{L}(\mathscr{E}')$. Define the functor

$$\begin{array}{rcrcl} \mathbb{L}(\tau) & : & \mathbb{L}(\mathscr{E}) & \to & \mathbb{L}(\mathscr{E}') \\ & & (\sigma_1, \ldots, \sigma_n) & \mapsto & \tau(\sigma_1), \ldots, \tau(\sigma_n) \\ & & \langle [t]_{\mathscr{E}} \rangle & \mapsto & \langle [\tau(t)_1]_{\mathscr{E}'}, \ldots, [\tau(t)_{|\tau(t)|}]_{\mathscr{E}'} \rangle \quad . \end{array}$$

$\mathbb{L}(\tau)$ respects the equivalence classes $[-]_{\mathscr{E}}$ because the translation τ preserves equational derivability. Note also that $\mathbb{L}(\tau)$ is functorial: evidently, $\mathbb{L}(\tau)(id_{(\sigma_1,\ldots,\sigma_n)}) = id_{\tau(\sigma_1),\ldots,\tau(\sigma_n)}$, and compositionality is implied by the fact that the extension of τ on terms of $\mathscr{E}$ commutes with substitution (Lemma 3.2). Moreover, the functor $\mathbb{L}(\tau)$, together with the mapping $\varphi(\tau)\colon S \to (S')^*$ $(\sigma \mapsto \tau(\sigma))$, is indeed an algebraic translation. By definition, it is cartesian, and the following diagram commutes.

$$\begin{array}{ccc} \mathbb{L}_S & \xrightarrow{\mathbb{L}_{\varphi(\tau)}} & \mathbb{L}_{S'} \\ {\scriptstyle L_{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle L_{\mathscr{E}'}} \\ \mathbb{L}(\mathscr{E}) & \xrightarrow{\mathbb{L}(\tau)} & \mathbb{L}(\mathscr{E}') \end{array}$$

We have this way defined a functor

$$\begin{array}{rcrcl} \mathbb{L}(-) & : & \mathbf{FOEP} & \to & \mathbf{FOAT} \\ & & \mathscr{E} & \mapsto & L_{\mathscr{E}}\colon \mathbb{L}_S \to \mathbb{L}(\mathscr{E}) \\ & & \tau & \mapsto & \mathbb{L}(\tau) \end{array}$$

mapping a first-order equational presentation to its classifying first-order algebraic theory, and a first-order syntactic translation to its induced algebraic translation.

Induced syntactic translations. Let $L\colon \mathbb{L}_S \to \mathscr{L}$ and $L'\colon \mathbb{L}_{S'} \to \mathscr{L}'$ be first-order algebraic theories, and $F\colon \mathscr{L} \to \mathscr{L}'$, together with $\varphi\colon S \to (S')^*$, be a first-order algebraic translation. Define the syntactic translation

$$\mathfrak{E}(F)\colon \mathfrak{E}(L) \to \mathfrak{E}(L')$$

by mapping σ to $F(\sigma)$, and an operator $\omega_f : \sigma_1, \ldots, \sigma_n \to \sigma$ of $\mathfrak{E}(L)$ to the tuple

$$\langle \Gamma_{F\sigma_1}, \ldots, \Gamma_{F\sigma_n} \vdash \omega_{Ff_i}(\overrightarrow{y}) \rangle_{1 \le i \le |F(\sigma)|}$$

of terms of $\mathfrak{E}(L')$, where ω_f is the operator induced by the morphism $f : \sigma_1 \times \cdots \times \sigma_n \to \sigma$ of $\mathscr{L}$, the notation $\overrightarrow{y}$ denotes the list of variables y_j with length given by the arity of ω_{Ff_i}, and Ff of $\mathscr{L}'$ is the morphism

$$\langle (Ff)_1, \ldots, (Ff)_{|F\sigma|} \rangle : F(\sigma_1) \times F(\sigma_n) \to F(\sigma)_1 \times \cdots \times F(\sigma)_{|F\sigma|} \quad .$$

Lemma 3.9. *The induced syntactic translation $\mathfrak{E}(F)(t)$ on a term t of sort σ of $\mathfrak{E}(L)$ is given by the tuple*

$$\langle \omega_{F[\![t]\!]_{*j}}(\overrightarrow{y}) \rangle_{1 \le j \le |F\sigma|} \quad ,$$

where $[\![-]\!]_*$ is the canonical algebra of $\mathfrak{E}(L)$ in $\mathscr{L}$, and $F[\![t]\!]_{*j}$ is the j-th component morphism of the tuple $F[\![t]\!]_*$.

Proof. By structural induction on t.

- The image of $\Gamma \vdash x_i : \sigma_i$ under $\mathfrak{E}(F)$ is given by $\langle y_1, \ldots, y_{|F(\sigma_i)|} \rangle$, where for each y_j we have

$$y_j \overset{(\mathscr{E}1)}{\equiv} \omega_{\pi_j^{\mathscr{L}'}}(\overrightarrow{y}) \;=\; \omega_{F(\pi_j^{\mathscr{L}})}(\overrightarrow{y}) \;=\; \omega_{F[\![x_i]\!]_{*j}} \quad .$$

- For $f : \sigma_1 \times \cdots \times \sigma_k \to \sigma$ of $\mathscr{L}$, the j-th component of the image of $\Gamma \vdash \omega_f(t_1, \ldots, t_k) : \sigma$ under $\mathfrak{E}(F)$ is given by

$$\begin{aligned}
& \mathfrak{E}(F)\big(\omega_f(t_1, \ldots, t_k)\big)_j \\
= \;& \mathfrak{E}(F)(\omega_f)_j \{y_{1,i} := \mathfrak{E}(F)(t_1)_i\}_{1 \le i \le |\sigma_1|} \;\cdots\; \{y_{k,i} := \mathfrak{E}(F)(t_k)_i\}_{1 \le i \le |\sigma_k|} \\
= \;& \omega_{Ff_j}(\overrightarrow{x}) \{y_{1,i} := \omega_{F[\![t_1]\!]_{*i}}(\overrightarrow{y_1})\}_{1 \le i \le |\sigma_1|} \;\cdots\; \{y_{k,i} := \omega_{F[\![t_k]\!]_{*i}}(\overrightarrow{y_k})\}_{1 \le i \le |\sigma_k|} \\
= \;& \omega_{F[\![\omega_f(\overrightarrow{z})]\!]_{*j}} \{y_{1,i} := \omega_{F[\![t_1]\!]_{*i}}(\overrightarrow{y_1})\}_{1 \le i \le |\sigma_1|} \;\cdots\; \{y_{k,i} := \omega_{F[\![t_k]\!]_{*i}}(\overrightarrow{y_k})\}_{1 \le i \le |\sigma_k|} \\
\overset{(\mathscr{E}2)}{\equiv} \;& \omega_{F[\![\omega_f(t_1, \ldots, t_k)]\!]_{*j}}(\overrightarrow{x}) \quad .
\end{aligned}$$

□

Next, note that $\mathfrak{E}(F)$ maps axioms of $\mathfrak{E}(L)$ to derivable equations of $\mathfrak{E}(L')$. Given $\Gamma \vdash_{E(L)} t \equiv s : \sigma$ in $\mathfrak{E}(L)$, we have

$$\begin{aligned}
& [\![t]\!]_* \;=\; [\![s]\!]_* && \text{in } \mathscr{L} \\
\Rightarrow \quad & F[\![t]\!]_* \;=\; F[\![s]\!]_* && \text{in } \mathscr{L}' \\
\Rightarrow \quad & \omega_{F[\![t]\!]_*} \;=\; \omega_{F[\![s]\!]_*} && \text{in } \mathfrak{E}(L') \\
\Rightarrow \quad & \mathfrak{E}(F)(t)_i \;\equiv\; \mathfrak{E}(F)(s)_i && \text{in } \mathfrak{E}(L'), \quad (1 \le i \le |F\sigma|) \quad .
\end{aligned}$$

Using the definition of $\mathfrak{L}(F)$, we obtain the functor

$$\begin{array}{rcccl} \mathfrak{L}(-) & : & \mathbf{FOAT} & \to & \mathbf{FOEP} \\ & & L\colon \mathbb{L}_S \to \mathscr{L} & \mapsto & \mathfrak{L}(L) \\ & & F & \mapsto & \mathfrak{L}(F) \end{array}$$

mapping a first-order algebraic theory to its internal language, and an algebraic translation to its induced syntactic translation.

Having shown how to construct syntactic translations from algebraic translations, and vice versa, we proceed to show that these constructions are mutually inverse in a categorical sense.

Theorem 3.10 (First-Order Syntactic Categorical Type Theory Correspondence)**.** *The categories* **FOAT** *of (multi-sorted) first-order algebraic theories and algebraic translations and* **FOEP** *of (multi-sorted) first-order equational presentations and their syntactic translations are equivalent.*

Proof. The equivalence is given by the functors $\mathbb{L}(-)$ and $\mathfrak{L}(-)$ defined above, together with the natural isomorphism

$$\xi\colon \mathbf{Id}_{\mathbf{FOEP}} \to \mathfrak{L}(-) \circ \mathbb{L}(-)$$

with component at an equational presentation $\mathscr{E}$ given by the syntactic translation isomorphism $\xi_{\mathscr{E}}\colon \mathscr{E} \to \mathfrak{L}(L_{\mathscr{E}})$ witnessing the Presentation/Theory Correspondence of Theorem 3.8, and the natural isomorphism

$$\Xi\colon \mathbf{Id}_{\mathbf{FOAT}} \to \mathbb{L}(\mathfrak{L}(-))$$

with component at an algebraic theory $L\colon \mathbb{L}_S \to \mathscr{L}$ given by the algebraic translation $\Xi_L\colon \mathscr{L} \to \mathbb{L}(\mathfrak{L}(L))$, which we take to be the isomorphism witnessing the Theory/Presentation Correspondence as defined in Theorem 2.17 $((\sigma_1,\ldots,\sigma_n) \mapsto (\sigma_1,\ldots,\sigma_n), f \mapsto \langle [\omega_f(\overrightarrow{x})]_{\mathfrak{L}(L)} \rangle)$. Naturality of ξ and Ξ establishes the idea that algebraic and syntactic translations are essentially the same. Indeed, given a first-order syntactic translation $\tau\colon \mathscr{E} \to \mathscr{E}'$, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\ \tau\ } & \mathscr{E}' \\ {\scriptstyle \xi_{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle \xi_{\mathscr{E}'}} \\ \mathfrak{L}(L_{\mathscr{E}}) & \xrightarrow{\mathfrak{L}(\mathbb{L}(\tau))} & \mathfrak{L}(L_{\mathscr{E}'}) \end{array}$$

Evidently, for a sort σ of $\mathscr{E}$,

$$\mathfrak{L}(\mathbb{L}(\tau))(\sigma) \ = \ \mathbb{L}(\tau)(\sigma) \ = \ \tau(\sigma) \ = \ \xi_{\mathscr{E}'}(\tau(\sigma)) \quad .$$

Also, for an operator $\hat{\omega}\colon \sigma_1,\ldots,\sigma_k \to \sigma$ of $\mathscr{E}$, we have

$$\begin{array}{rcl}(\boldsymbol{\mathfrak{E}}(\mathbb{L}(\tau))\circ\xi_{\mathscr{E}})(\hat{\omega}) & = & \boldsymbol{\mathfrak{E}}(\mathbb{L}(\tau))\big(\omega_{f(\hat{\omega})}(x_1,\ldots,x_k)\big)\\ & = & \big\langle \omega_{\mathbb{L}(\tau)(f(\hat{\omega}))_i}(\overrightarrow{y})\big\rangle_{1\leq i\leq|\tau(\sigma)|}\\ & = & \big\langle \omega_{\tau(\hat{\omega})_i}(\overrightarrow{y})\big\rangle_{1\leq i\leq|\tau(\sigma)|} \quad .\end{array}$$

Next, for $F\colon \mathscr{L}\to\mathscr{L}'$ an algebraic translation of algebraic theories $L\colon \mathbb{L}_S\to\mathscr{L}$ and $L'\colon \mathbb{L}_{S'}\to\mathscr{L}'$, naturality of Ξ is given by the diagram

$$\begin{array}{ccc}\mathscr{L} & \xrightarrow{\;F\;} & \mathscr{L}'\\ \Big\downarrow{\scriptstyle \Xi_L} & & \Big\downarrow{\scriptstyle \Xi_{L'}}\\ \mathbb{L}(\boldsymbol{\mathfrak{E}}(L)) & \xrightarrow{\mathbb{L}(\boldsymbol{\mathfrak{E}}(F))} & \mathbb{L}(\boldsymbol{\mathfrak{E}}(L'))\end{array}$$

whose commutativity is obvious on the objects of $\mathscr{L}$. For a morphism $f:\sigma_1\times\cdots\times\sigma_n\to\sigma$, we have

$$\begin{array}{rcl}\big(\mathbb{L}(\boldsymbol{\mathfrak{E}}(F))\circ\Xi_L\big)(f) & = & \mathbb{L}(\boldsymbol{\mathfrak{E}}(F))\big\langle[\omega_f(\overrightarrow{x})]_{\boldsymbol{\mathfrak{E}}(L)}\big\rangle\\ & = & \big\langle[\boldsymbol{\mathfrak{E}}(F)\big(\omega_f(\overrightarrow{x})\big)]_{\boldsymbol{\mathfrak{E}}(L')}\big\rangle_{1\leq i\leq|\boldsymbol{\mathfrak{E}}(F)(\sigma)|}\\ & = & \big\langle[\omega_{Ff}(\overrightarrow{x_i})_i]_{\boldsymbol{\mathfrak{E}}(L')}\big\rangle_{1\leq i\leq|F\sigma|}\\ & = & (\Xi_{L'}\circ F)(f) \quad .\end{array}$$

□

Chapter 4

Second-Order Syntax and Semantics

The realm of universal algebra is traditionally restricted to first-order languages. In particular, this leaves out languages with variable-binding. Variable-binding constructs are at the core of fundamental calculi and theories in computer science and logic [Church, 1936, Church, 1940], and incorporating them into algebra has been a main foundational problem.

This chapter reviews the work of Fiore and Hur [Fiore and Hur, 2010] on a conservative extension of universal algebra from first to second order. We present in Section 4.1 the syntactic machinery surrounding second-order languages. This includes the notion of second-order equational presentation, which allows the specification of equational theories by means of schematic identities over signatures with variable-binding operators. Second-order equational logic is presented in Section 4.2 as the deductive system underlying formal reasoning about second-order structure, and its conservativity over first-order equational logic is recalled. Finally, we review the model theory of second-order equational presentations by means of second-order algebras (Section 4.3), together with its soundness and completeness.

While the main contribution of this work is the abstract categorical presentation of second-order languages via second-order algebraic theories, the details of the syntactic development of this chapter are crucial for validating the correctness of our definitions. More precisely, properly understanding the subtleties surrounding second-order syntax will enable us to define second-order algebraic theories (Chapter 5) in a way that legitimately corresponds to second-order equational presentations. At the semantic level, the model theory of second-order universal algebra as presented in this chapter will yield a definition of second-order functorial semantics (Chapter 7) proven to be its equivalent.

4.1 Second-Order Syntactic Theory

We present the syntactic theory of second-order languages, that is languages that come equipped with variable-binding constructs and parameterised metavariables. The development comprises second-order signatures on top of which second-order terms-in-context are defined. For succinctness, our exposition restricts to the mono-sorted setting. The generalisation to the multi-sorted framework can be found in the Appendix to Chapter 4 (4.A).

4.1.1 Second-order signatures

Following the development of Aczel [Aczel, 1978], a (mono-sorted) *second-order signature* $\Sigma = (\Omega, |-|)$ is specified by a set of operators Ω and an arity function $|-|\colon \Omega \to \mathbb{N}^*$. For an operator $\omega \in \Omega$, we write $\omega\colon (n_1, \ldots, n_k)$ whenever it has arity $|\omega| = (n_1, \ldots, n_k)$. The intended meaning here is that the operator ω takes k arguments binding n_i variables in the i^{th} argument.

Any language with variable binding fits this formalism, including languages with quantifiers [Aczel, 1980], a fixpoint operator [Klop et al., 1993], and the primitive recursion operator [Aczel, 1978]. The most prototypical of all second-order languages is the λ-calculus, whose second-order signature is given next.

Example 4.1. *The second-order signature Σ_λ of the mono-sorted λ-*calculus *has operators*

$$\mathtt{abs}\colon (1) \qquad \text{and} \qquad \mathtt{app}\colon (0,0) \quad ,$$

representing λ abstraction and application, respectively.

4.1.2 Second-order terms

Variables and metavariables. Unlike the first-order universe where first-order terms are built up only from variables and (first-order) operators, second-order terms have *metavariables* as additional building blocks. We use the notational convention of denoting variables similar to first-order variables by x, y, z, and metavariables by $\mathrm{M}, \mathrm{N}, \mathrm{L}$. Metavariables come with an associated natural number arity, also referred to as its *meta-arity*. A metavariable M of meta-arity m, denoted by $\mathrm{M}\colon [m]$, is to be parameterised by m terms.

Contexts. Second-order terms are considered in contexts with two *zones*, each respectively declaring metavariables and variables. Accordingly, we use the following representation for contexts

$$\mathrm{M}_1\colon [m_1], \ldots, \mathrm{M}_k\colon [m_k] \rhd x_1, \ldots, x_n \quad ,$$

where the metavariables M_i and variables x_j are assumed to be distinct.

Terms. Signatures give rise to terms. These are built up by means of operators from both variables and metavariables, and hence referred to as second-order. The judgement for *second-order terms* in context

$$\Theta \rhd \Gamma \vdash t$$

is defined similar to the second-order syntax of Aczel [Aczel, 1978] by the following rules.

(Variables) For $x \in \Gamma$,

$$\frac{}{\Theta \rhd \Gamma \vdash x}$$

(Metavariables) For $(\mathrm{M} : [m]) \in \Theta$,

$$\frac{\Theta \rhd \Gamma \vdash t_i \quad (1 \leq i \leq m)}{\Theta \rhd \Gamma \vdash \mathrm{M}[t_1, \ldots, t_m]}$$

(Operators) For $\omega : (n_1, \ldots, n_k)$,

$$\frac{\Theta \rhd \Gamma, \overrightarrow{x}_i \vdash t_i \quad (1 \leq i \leq k)}{\Theta \rhd \Gamma \vdash \omega\big((\overrightarrow{x}_1)t_1, \ldots, (\overrightarrow{x}_k)t_k\big)}$$

where $\overrightarrow{x}_i$ stands for $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$.

Terms derived according to the first two rules only via variables and metavariables are referred to as *elementary*. Hence, an empty signature with an empty set of operators generates only elementary terms.

Terms are considered up to the α-equivalence relation induced by stipulating that, for every operator $\omega : (n_1, \ldots, n_k)$, in the term $\omega\big((\overrightarrow{x}_1)t_1, \ldots, (\overrightarrow{x}_k)t_k\big)$ the variables $\overrightarrow{x}_i$ are bound in t_i.

Example 4.2. *Two sample terms for the signature Σ_λ of the mono-sorted λ-calculus of Example 4.1 follow:*

$$\mathrm{M} : [1], \mathrm{N} : [0] \rhd - \vdash \mathtt{app}\big(\mathtt{abs}\big((\mathtt{x})\mathrm{M}[\mathtt{x}]\big), \mathrm{N}[]\big),$$

$$\mathrm{M} : [1], \mathrm{N} : [0] \rhd - \vdash \mathrm{M}[\mathrm{N}[]].$$

4.1.3 Second-order substitution calculus

The second-order nature of the syntax requires a two-level substitution calculus. Each level respectively accounts for the substitution of variables and metavariables, with the latter operation depending on the former [Aczel, 1978, Klop et al., 1993, van Raamsdonk, 2003, Fiore, 2008].

Substitution. The operation of capture-avoiding simultaneous *substitution* of terms for variables maps

$$\Theta \rhd x_1, \ldots, x_n \vdash t \qquad \text{and} \qquad \Theta \rhd \Gamma \vdash t_i \quad (1 \leq i \leq n)$$

to

$$\Theta \rhd \Gamma \vdash t\{x_i := t_i\}_{i \in \|n\|}$$

according to the following inductive definition:

- $x_j\{x_i := t_i\}_{i \in \|n\|} = t_j$
- $(\mathrm{M}[\ldots, s, \ldots])\{x_i := t_i\}_{i \in \|n\|} = \mathrm{M}\big[\ldots, s\{x_i := t_i\}_{i \in \|n\|}, \ldots\big]$
- $(\omega(\ldots, (y_1, \ldots, y_k)s, \ldots))\{x_i := t_i\}_{i \in \|n\|} \;=\; \omega\big(\ldots, (y_1, \ldots, y_k)s\{x_i := t_i, y_j := z_j\}_{i \in \|n\|, j \in \|k\|}, \ldots\big)$ with $z_j \notin \mathrm{dom}(\Gamma)$ for all $j \in \|k\|$.

The (first-order) Substitution Lemma is valid in the second-order setting as well. The proof is straightforward by induction on the structure of the term t, details of which can be found in Appendix 4.B.

Lemma 4.3 (Second-Order Substitution Lemma)**.** *Given terms*

$$\Theta \rhd \Gamma \vdash s_i \quad (1 \leq i \leq n), \qquad \Theta \rhd \Gamma \vdash r_j \quad (1 \leq j \leq k), \qquad \textit{and} \qquad \Theta \rhd x_1, \ldots, x_n, y_1, \ldots, y_k \vdash t,$$

we have

$$\Theta \rhd \Gamma \vdash t\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \;=\; t\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|} \quad .$$

Metasubstitution. The operation of *metasubstitution* of abstracted terms for metavariables maps

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \Gamma \vdash t \qquad \text{and} \qquad \Theta \rhd \Gamma, \overrightarrow{x}_i \vdash t_i \quad (1 \leq i \leq k)$$

to

$$\Theta \rhd \Gamma \vdash t\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|}$$

according to the following inductive definition:

- $x\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|} = x$
- $(\mathrm{M}_l[s_1, \ldots, s_{m_l}])\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|} = t_l\Big\{x^{(i)}_j := s_j\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|}\Big\}_{j \in \|m_l\|}$
- $(\omega(\ldots, (\overrightarrow{x})s, \ldots))\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|} = \omega\big(\ldots, (\overrightarrow{x})s\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|}, \ldots\big)$

The operation of metasubstitution is well-behaved, in the sense that it is compatible with substitution (*Substitution-Metasubstitution Lemma*) and monoidal, meaning that it is associative (*Metasubstitution Lemma I*) and has a unit (*Metasubstitution Lemma II*). Syntactic proofs of all of the following are detailed in Appendix 4.B.

Lemma 4.4 (Substitution-Metasubstitution Lemma). *Given terms*

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd \Gamma \vdash t_i \quad (1 \le i \le n), \qquad \Theta \rhd \Gamma, \overrightarrow{y}_j \vdash s_j \quad (1 \le j \le k),$$

and

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd x_1,\ldots,x_n \vdash t,$$

we have

$$\begin{aligned}\Theta \rhd \Gamma \quad \vdash \quad & t\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & t\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \quad .\end{aligned}$$

Lemma 4.5 (Metasubstitution Lemma I). *Given terms*

$$\Theta \rhd \Gamma, \overrightarrow{x}_i \vdash r_i \quad (1 \le i \le k), \qquad \Theta \rhd \Gamma, \overrightarrow{y}_j \vdash s_j \quad (1 \le j \le l),$$

and

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k],\mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l] \rhd \Gamma \vdash t,$$

we have

$$\begin{aligned}\Theta \rhd \Gamma \quad \vdash \quad & t\{\mathrm{M}_i := (\overrightarrow{x}_i)r_i\}_{i\in\|k\|}\{\mathrm{N}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|l\|} \\ = \quad & t\{\mathrm{N}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|l\|}\Big\{\mathrm{M}_i := (\overrightarrow{x}_i)r_i\{\mathrm{N}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|l\|}\Big\}_{i\in\|k\|} \quad .\end{aligned}$$

Lemma 4.6 (Metasubstitution Lemma II). *Given terms*

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd \Gamma \vdash t \qquad \text{and} \qquad \mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd \Gamma, x_1^{(i)},\ldots,x_{m_i}^{(i)} \vdash \mathrm{M}_i[x_1^{(i)},\ldots,x_{m_i}^{(i)}]$$

for $1 \le i \le k$*, we have*

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd \Gamma \vdash t\{\mathrm{M}_i := (\overrightarrow{x}_i)\mathrm{M}_i[x_1^{(i)},\ldots,x_{m_i}^{(i)}]\}_{i\in\|k\|} = t \quad .$$

4.1.4 Parameterisation

Every second-order term $\Theta \rhd \Gamma \vdash t$ can be *parameterised* to yield a term $\Theta, \hat{\Gamma} \rhd - \vdash \hat{t}$, where for $\Gamma = x_1,\ldots,x_n$,

$$\hat{\Gamma} = \mathrm{X}_1\colon[0],\ldots,\mathrm{X}_n\colon[0] \qquad \text{and} \qquad \hat{t} = t\{x_i := \mathrm{X}_i[]\}_{i\in\|n\|} \quad .$$

The variable context is thus replaced under parameterisation by a metavariable context, yielding an essentially equivalent term (formally *parameterised* term) where all its variables are replaced by metavariables, which do not themselves parameterise any terms. This allows us to intuively think of metavariables of zero meta-arity as variables, and vice versa.

4.2 Second-Order Equational Logic

We add equations on top of the above constructions to yield *second-order equational presentations*, together with rules for equational derivation via *Second-Order Equational Logic*.

4.2.1 Equational Presentations

A second-order *equation* is given by a pair of second-order terms $\Theta \rhd \Gamma \vdash s$ and $\Theta \rhd \Gamma \vdash t$ in context, written as

$$\Theta \rhd \Gamma \vdash s \equiv t \quad .$$

A second-order *equational presentation* $\mathscr{E} = (\Sigma, E)$ is specified by a second-order signature Σ together with a set of equations E, the *axioms* of the presentation $\mathscr{E}$, over it. Axioms are usually denoted by

$$\Theta \rhd \Gamma \vdash_E t \equiv s$$

to distinguish them from any other equations.

Example 4.7. *The equational presentation $\mathscr{E}_\lambda = (\Sigma_\lambda, E_\lambda)$ of the mono-sorted λ-calculus extends the second-order signature Σ_λ of Example 4.1 with the following axioms.*

$$(\beta) \quad \mathrm{M}\colon [1], \mathrm{N}\colon [0] \rhd - \vdash_{E_\lambda} \mathtt{app}\big(\mathtt{abs}\big((\mathtt{x})\mathrm{M}[\mathtt{x}]\big), \mathrm{N}[]\big) \;\equiv\; \mathrm{M}\big[\mathrm{N}[]\big]$$

$$(\eta) \quad \mathrm{F}\colon [0] \rhd - \vdash_{E_\lambda} \mathtt{abs}\big((\mathtt{x})\mathtt{app}(\mathrm{F}[], \mathtt{x})\big) \;\equiv\; \mathrm{F}[]$$

It is worth emphasising that the (mono-sorted) λ-calculus is merely taken as a running example throughout this dissertation, for it is the most intuitive and widely-known such calculus. We use it as a reference as a means of familiarisation with and appreciation of second-order syntax. The expressiveness of the second-order formalism does not, however, rely exclusively on that of the λ-calculus. One can directly axiomatise, say, primitive recursion [Aczel, 1978] and predicate logic [Plotkin, 1998] as second-order equational presentations.

(Axioms)

$$\frac{\Theta \rhd \Gamma \vdash_E s \equiv t}{\Theta \rhd \Gamma \vdash s \equiv t}$$

(Equivalence)

$$\frac{\Theta \rhd \Gamma \vdash t}{\Theta \rhd \Gamma \vdash t \equiv t} \qquad \frac{\Theta \rhd \Gamma \vdash s \equiv t}{\Theta \rhd \Gamma \vdash t \equiv s} \qquad \frac{\Theta \rhd \Gamma \vdash s \equiv t \qquad \Theta \rhd \Gamma \vdash t \equiv u}{\Theta \rhd \Gamma \vdash s \equiv u}$$

(Extended metasubstitution)

$$\frac{\mathrm{M}_1 : [m_1], \dots, \mathrm{M}_k : [m_k] \rhd \Gamma \vdash s \equiv t \qquad \Theta \rhd \Delta, \overrightarrow{x}_i \vdash s_i \equiv t_i \quad (1 \leq i \leq k)}{\Theta \rhd \Gamma, \Delta \vdash s\{\mathrm{M}_i := (\overrightarrow{x}_i)s_i\}_{i \in \|k\|} \equiv t\{\mathrm{M}_i := (\overrightarrow{x}_i)t_i\}_{i \in \|k\|}}$$

Figure 4.1: Second-Order Equational Logic

4.2.2 Equational logic

The rules of *Second-Order Equational Logic* are given in Figure 4.1. Besides the rules for axioms and equivalence, the logic consists of just one additional rule stating that the operation of metasubstitution in extended metavariable context is a congruence.

The expressive power of this system can be seen through the following two sample derivable rules.

(Substitution)

$$\frac{\Theta \rhd x_1, \dots, x_n \vdash s \equiv t \qquad \Theta \rhd \Gamma \vdash s_i \equiv t_i \quad (1 \leq i \leq n)}{\Theta \rhd \Gamma \vdash s\{x_i := s_i\}_{i \in \|n\|} \equiv t\{x_i := t_i\}_{i \in \|n\|}}$$

(Extension)

$$\frac{\mathrm{M}_1 : [m_1], \dots, \mathrm{M}_k : [m_k] \rhd \Gamma \vdash s \equiv t}{\mathrm{M}_1 : [m_1 + n], \dots, \mathrm{M}_k : [m_k + n] \rhd \Gamma, x_1, \dots, x_n \vdash s^\# \equiv t^\#}$$

where $u^\# = u\{\mathrm{M}_i := (x_1, \dots, x_n)\mathrm{M}_i[y_1^{(i)}, \dots, y_{m_i}^{(i)}, x_1, \dots, x_n]\}_{i \in \|k\|}$.

4.2.3 Parameterised equations

Performing the operation of parameterisation on a set of equations E to obtain a set of *parameterised equations* $\hat{E}$, we have that all of the following are equivalent:

$$\Theta \rhd \Gamma \vdash_E s \equiv t \quad , \qquad \Theta, \hat{\Gamma} \rhd - \vdash_{\mathscr{E}} \hat{s} \equiv \hat{t}$$

$$\Theta \rhd \Gamma \vdash_{\hat{E}} s \equiv t \quad , \qquad \Theta, \hat{\Gamma} \rhd - \vdash_{\hat{E}} \hat{s} \equiv \hat{t}$$

Therefore, and without loss of generality, any set of axioms can be transformed into a parameterised set of axioms, which in essence represents the same equational presentation. One may restrict to axioms containing an empty variable context as in the *CRS*s of Klop [Klop, 1980], but there is no reason for us to do the same.

4.3 Second-Order Universal Algebra

The model theory of Fiore and Hur [Fiore and Hur, 2010] for second-order equational presentations is recalled. For our purposes, this is presented here in elementary concrete model-theoretic terms rather than in abstract monadic terms. The reader is referred to [Fiore and Hur, 2010] for the latter perspective.

4.3.1 Semantic universe

Recall that we write $\mathbb{F}$ for the free cocartesian category on an object. Explicitly, $\mathbb{F}$ has $\mathbb{N}$ as set of objects and morphisms $m \to n$ given by functions $\|m\| \to \|n\|$. The second-order model-theoretic development lies within the semantic universe $\mathbf{Set}^{\mathbb{F}}$, the presheaf category of sets in variable contexts [Fiore et al., 1999]. It is a well-known category, and the formalisation of second-order model theory relies on some of its intrinsic properties. In particular, $\mathbf{Set}^{\mathbb{F}}$ is bicomplete with limits and colimits computed pointwise [MacLane and Moerdijk, 1992]. We write $\boldsymbol{y}$ for the Yoneda embedding $\mathbb{F}^{\mathrm{op}} \hookrightarrow \mathbf{Set}^{\mathbb{F}}$.

Substitution. We recall the *substitution monoidal structure* in the semantic universe $\mathbf{Set}^{\mathbb{F}}$ as presented in [Fiore et al., 1999]. The unit is given by the *presheaf of variables* $\boldsymbol{y}1$, explicitly the embedding $\mathbb{F} \hookrightarrow \mathbf{Set}$. This object is a crucial element of the semantic universe $\mathbf{Set}^{\mathbb{F}}$, as it provides an arity for variable binding. The monoidal tensor product $X \bullet Y$ of presheaves $X, Y \in \mathbf{Set}^{\mathbb{F}}$ is given by

$$X \bullet Y = \int^{k \in \mathbb{F}} X(k) \times Y^k \quad .$$

A monoid

$$\boldsymbol{y}1 \xrightarrow{\nu} A \xleftarrow{\varsigma} A \bullet A$$

for the substitution monoidal structure equips $A \in \mathbf{Set}^{\mathbb{F}}$ with substitution structure. In particular, the map $\nu_k : \boldsymbol{y}k \to A^k$, defined as the composite

$$\boldsymbol{y}k \cong (\boldsymbol{y}1)^k \xrightarrow{\nu^k} A^k \quad ,$$

induces the embedding

$$(A^{\boldsymbol{y}n} \times A^n)(k) \to A(k+n) \times A^k(k) \times A^n(k) \to (A \bullet A)(k) \quad ,$$

which, together with the multiplication, yield a *substitution operation*

$$\varsigma_n : A^{\boldsymbol{y}n} \times A^n \to A$$

for every $n \in \mathbb{N}$. These substitution operations provide the interpretations of metavariables.

4.3.2 Second-order algebras and models

Algebras. Every second-order signature $\Sigma = (\Omega, |-|)$ induces a *signature endofunctor* $\mathscr{F}_\Sigma : \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ given by

$$\mathscr{F}_\Sigma X \;=\; \coprod_{\omega : (n_1,\ldots,n_k) \in \Omega} \; \prod_{i \in \|k\|} X^{\boldsymbol{y}n_i} \quad .$$

$\mathscr{F}_\Sigma$-algebras $\mathscr{F}_\Sigma X \to X$ provide an interpretation

$$[\![\omega]\!]_X : \prod_{i \in \|k\|} X^{\boldsymbol{y}n_i} \to X$$

for every operator $\omega : (n_1, \ldots, n_k)$ in Σ.

We note that there are canonical natural isomorphisms

$$\coprod_{i \in I} (X_i \bullet Y) \;\cong\; (\coprod_{i \in I} X_i) \bullet Y$$
$$\prod_{i \in \|n\|} (X_i \bullet Y) \;\cong\; (\prod_{i \in \|n\|} X_i) \bullet Y$$

and, for all points $\eta : \boldsymbol{y}1 \to Y$, natural extension maps

$$\eta^{\# n} : X^{\boldsymbol{y}n} \bullet Y \to (X \bullet Y)^{\boldsymbol{y}n} \quad .$$

These constructions equip every signature endofunctor $\mathscr{F}_\Sigma$ with a *pointed strength*

$$\varpi_{X, \boldsymbol{y}1 \to Y} : \mathscr{F}_\Sigma(X) \bullet Y \to \mathscr{F}_\Sigma(X \bullet Y) \quad .$$

This property plays a critical role in the notion of algebra with substitution structure, which depends on this pointed strength. The extra structure on a presheaf Y in the form of a point $\varpi : \boldsymbol{y}1 \to Y$ reflects the need of fresh variables in the definition of substitution for binding operators. We refer the reader to [Fiore et al., 1999] and [Fiore, 2008] for a detailed development.

Models. A model for a second-order signature Σ is an algebra equipped with a compatible substitution structure. Formally, Σ-*models* are defined to be Σ-*monoids*, which are objects $A \in \mathbf{Set}^{\mathbb{F}}$ equipped with an $\mathscr{F}_\Sigma$-algebra structure $\alpha\colon \mathscr{F}_\Sigma A \to A$ and a monoid structure $\nu\colon \boldsymbol{y}1 \to A$ and $\varsigma\colon A \bullet A \to A$ that are compatible in the sense that the following diagram commutes.

$$\begin{array}{ccccc} \mathscr{F}_\Sigma(A) \bullet A & \xrightarrow{\varpi_{A,\nu}} & \mathscr{F}_\Sigma(A \bullet A) & \xrightarrow{\mathscr{F}_\Sigma \varsigma} & \mathscr{F}_\Sigma(A) \\ \downarrow{\scriptstyle \alpha\bullet A} & & & & \downarrow{\scriptstyle \alpha} \\ A \bullet A & & \xrightarrow{\quad\quad\varsigma\quad\quad} & & A \end{array}$$

We denote by $\mathbf{Mod}(\Sigma)$ the category of Σ-models, with morphisms given by maps that are both $\mathscr{F}_\Sigma$-algebra and monoid homomorphisms.

4.3.3 Soundness and completeness

We review the soundness and completeness of the model theory of *Second-Order Equational Logic* as presented in [Fiore and Hur, 2010].

Semantics. A model $A \in \mathbf{Mod}(\Sigma)$ for a second-order signature Σ is explicitly given by, for a metavariable context $\Theta = (\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k])$ and variable context $\Gamma = (x_1, \ldots, x_n)$, a presheaf

$$[\![\Theta \rhd \Gamma]\!]_A = \prod_{i \in \|k\|} A^{\boldsymbol{y} m_i} \times \boldsymbol{y} n$$

of $\mathbf{Set}^{\mathbb{F}}$, together with interpretation functions

$$[\![\omega]\!]_A \colon \prod_{j \in \|l\|} A^{\boldsymbol{y} n_j} \to A$$

for each operator $\omega\colon (n_1, \ldots, n_l)$ of Σ. This induces the interpretation of a second-order term $\Theta \rhd \Gamma \vdash t$ in A as a morphism

$$[\![\Theta \rhd \Gamma \vdash t]\!]_A \colon [\![\Theta \rhd \Gamma]\!]_A \to A$$

in $\mathbf{Set}^{\mathbb{F}}$, which is given by structural induction as follows:

- $[\![\Theta \rhd \Gamma \vdash x_i]\!]_A$ is the composite

$$[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\pi_2} \boldsymbol{y} n \xrightarrow{\nu_n} A^n \xrightarrow{\pi_j} A \quad .$$

- $[\![\Theta \rhd \Gamma \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}]]\!]_A$ is the composite

$$[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\langle \pi_i \pi_1, f \rangle} A^{\boldsymbol{y} m_i} \times A^{m_i} \xrightarrow{\varsigma_{m_i}} A \quad ,$$

  where $f = \left\langle [\![\Theta \rhd \Gamma \vdash t_j]\!]_A \right\rangle_{j \in \|m_i\|}$.

- For an operator $\omega\colon (n_1, \ldots, n_l)$ of Σ,

$$[\![\Theta \rhd \Gamma \vdash \omega\big((\overrightarrow{y}_1)t_1, \ldots, (\overrightarrow{y}_l)t_l\big)]\!]_A$$

  is the composite

$$[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\langle f_j \rangle_{j \in \|l\|}} \prod_{j \in \|l\|} A^{\boldsymbol{y} n_j} \xrightarrow{[\![\omega]\!]_A} A \quad ,$$

  where f_j is the exponential transpose of

$$\prod_{i \in \|k\|} A^{\boldsymbol{y} m_i} \times \boldsymbol{y} n \times \boldsymbol{y} n_j \cong \prod_{i \in \|k\|} A^{\boldsymbol{y} m_i} \times \boldsymbol{y}(n + n_j) \xrightarrow{[\![\Theta \rhd \Gamma, \overrightarrow{y}_j \vdash t_j]\!]_A} A \quad .$$

Equational models. A model $A \in \mathbf{Mod}(\Sigma)$ *satisfies* an equation $\Theta \rhd \Gamma \vdash s \equiv t$, which we write as $A \models (\Theta \rhd \Gamma \vdash s \equiv t)$, if and only if $[\![\Theta \rhd \Gamma \vdash s]\!]_A = [\![\Theta \rhd \Gamma \vdash t]\!]_A$ in $\mathbf{Set}^{\mathbb{F}}$.

For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, the category $\mathbf{Mod}(\mathscr{E})$ of $\mathscr{E}$-models is the full subcategory of $\mathbf{Mod}(\Sigma)$ consisting of the Σ-models that satisfy the axioms E.

Example 4.8. *For the signature Σ_λ of the mono-sorted λ-calculus (Example 4.1), a model*

$$\boldsymbol{y}1 \xrightarrow{\nu} A \xleftarrow{\varsigma} A \bullet A$$

$$[\![\mathtt{abs}]\!]_A\colon A^{\boldsymbol{y}1} \to A \quad , \qquad [\![\mathtt{app}]\!]_A\colon A \times A \to A$$

of $\mathbf{Mod}(\Sigma_\lambda)$ satisfies the (β) and (η) axioms of $\mathscr{E}_\lambda$ (Example 4.7) if and only if the diagrams

$$\begin{array}{ccc} A^{\boldsymbol{y}1} \times A & \xrightarrow{\varsigma_1} & A \\ {\scriptstyle [\![\mathtt{abs}]\!]_A \times id_A} \downarrow & & \| \\ A \times A & \xrightarrow{[\![\mathtt{app}]\!]_A} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{id_A} & A \\ {\scriptstyle \mathbb{I}\left([\![\mathtt{app}]\!]_A \circ (id_A \times \nu)\right)} \downarrow & & \| \\ A^{\boldsymbol{y}1} & \xrightarrow{[\![\mathtt{abs}]\!]_A} & A \end{array}$$

commute, where $\mathbb{I}(g)$ denotes the unique exponential mate of g.

Theorem 4.9 (Second-Order Soundness and Completeness)**.** *For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, the judgement $\Theta \rhd \Gamma \vdash s \equiv t$ is derivable from E if and only if $A \models (\Theta \rhd \Gamma \vdash s \equiv t)$ for all $\mathscr{E}$-models A.*

4.3.4 Conservativity

At the level of equational derivability, the extension of (first-order) universal algebra to the second-order framework, as presented in this chapter, is conservative.

Clearly, every first-order signature is a second-order signature in which all operators do not bind any variables in their arguments. Any first-order term $\Gamma \vdash t$ can therefore be represented as the second-order term $- \rhd \Gamma \vdash t$. Indeed, for a set of first-order equations, if the equation $\Gamma \vdash s \equiv t$ is derivable in first-order equational logic, then its corresponding second-order representative $- \rhd \Gamma \vdash s \equiv t$ is derivable in second-order equational logic.

The converse statement is what is known as *conservativity* of second-order equational derivability. Although this result is not directly utilised in this dissertation, we recall it for the benefit of comprehensiveness, and refer the reader to [Fiore and Hur, 2010] for the proof.

Theorem 4.10 (Conservativity)**.** *Second-Order Equational Logic (Figure 4.1) is a conservative extension of First-Order Equational Logic. More precisely, if a second-order equation between first-order terms* $- \rhd \Gamma \vdash s \equiv t$ *lying in an empty metavariable context is derivable in second-order equational logic, then* $\Gamma \vdash s \equiv t$ *is derivable in first-order equational logic.*

4.A Appendix to Chapter 4: Multi-Sorted Second-Order Syntax

We present the multi-sorted generalisation of the mono-sorted second-order syntactic theory underlying second-order equational logic.

Signatures. A multi-sorted second-order signature $\Sigma = (S, \Omega, |-|)$ is specified by a set of sorts S, a set of operators Ω, and an arity function $|-|: \Omega \to (S^* \times S)^* \times S$.

Notation. We let $|\overrightarrow{\sigma}|$ be the length of the sequence of sorts $\overrightarrow{\sigma} = \sigma_1, \ldots, \sigma_{|\overrightarrow{\sigma}|}$.

For $\omega \in \Omega$, we typically write $\omega: (\overrightarrow{\sigma_1})\tau_1, \ldots, (\overrightarrow{\sigma_n})\tau_n \to \tau$ whenever $|\omega| = \big((\overrightarrow{\sigma_1})\tau_1, \ldots, (\overrightarrow{\sigma_n})\tau_n, \tau\big)$. Similar to the mono-sorted universe, the intended meaning here is that ω is an operator of sort τ taking n arguments, each of which binds $n_i = |\overrightarrow{\sigma_i}|$ variables of sorts $\sigma_{i,1}, \ldots, \sigma_{i,n_i}$ in a term of sort τ_i.

Example 4.11.

1. Sorted λ-calculus. *The signature of the* multi-sorted λ-calculus *over a set of* base sorts B *has set of sorts* B_λ *given by*
$$\frac{\beta \in B}{\beta \in B_\lambda} \qquad \frac{\sigma, \tau \in B_\lambda}{\sigma \Rightarrow \tau \in B_\lambda} \quad .$$
Given sorts $\sigma, \tau \in B_\lambda$, *the operators of the sorted* λ*-calculus are given by* $\mathtt{abs}_{\sigma,\tau}: (\sigma)\tau \to \sigma \Rightarrow \tau$ *and* $\mathtt{app}_{\sigma,\tau}: \sigma \Rightarrow \tau, \sigma \to \tau$.

2. Predicate logic. *The signature* $\Pi = (P, \Omega_P, |-|_P)$ *of* predicate logic *consists of the set* $P = \{\mathtt{Prop}, \star\}$, *which has two sorts, and has operators in* Ω_P *equipped with the arity function* $|-|_P: (P^* \times P)^* \times P$. *One may have* simple *predicate operators* $P: \star, \ldots, \star \to \mathtt{Prop}$, *which essentially allow the formation of* atomic predicate propositions. *The signature* Π *furthermore comes equipped with the following predicate logic operators (note the binding operators of universal and existential quantification):*

$$
\begin{array}{rl}
\text{(Equality)} & = \,:\, (\star,\star) \to \mathtt{Prop} \\
\text{(Falsum)} & \bot \,:\, (\mathtt{Prop}) \to \mathtt{Prop} \\
\text{(Truth)} & \mathtt{T} \,:\, \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Negation)} & \sim \,:\, \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Conjunction)} & \wedge \,:\, \mathtt{Prop}, \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Disjunction)} & \vee \,:\, \mathtt{Prop}, \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Implication)} & \supset \,:\, \mathtt{Prop}, \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Universal Q)} & \forall \,:\, (\star)\ \mathtt{Prop} \to \mathtt{Prop} \\
\text{(Existential Q)} & \exists \,:\, (\star)\ \mathtt{Prop} \to \mathtt{Prop}
\end{array}
$$

Contexts. The typing contexts have two sorted zones, and they are represented as

$$\mathrm{M}_1 : [\overrightarrow{\sigma_1}]\tau_1, \ldots, \mathrm{M}_k : [\overrightarrow{\sigma_k}]\tau_k \rhd x_1 : \sigma'_1, \ldots, x_n : \sigma'_n \quad ,$$

where all variables and metavariables are assumed to be distinct. Metavariable typings are parameterised sorts: a metavariable of sort $[\sigma_1, \ldots, \sigma_n]\tau$, when parameterised by terms of sort $\sigma_1, \ldots, \sigma_n$, will yield a term of sort τ.

Terms. The judgement for terms in context $\Theta \rhd \Gamma \vdash t : \tau$ is defined by the rules below. As is usual in the second-order setting, terms are considered up to α-equivalence, but we shall not formalise this here.

$$\frac{}{\Theta \rhd \Gamma \vdash x : \tau} \quad \big((x : \tau) \in \Gamma\big)$$

$$\frac{\Theta \rhd \Gamma \vdash t_i : \tau_i \quad (1 \leq i \leq n)}{\Theta \rhd \Gamma \vdash \mathrm{M}[t_1, \ldots t_n] : \tau} \quad \big((\mathrm{M} : [\tau_1, \ldots, \tau_n]\tau) \in \Theta\big)$$

$$\frac{\Theta \rhd \Gamma, \overrightarrow{x_i} : \overrightarrow{\sigma_i} \vdash t_i : \tau_i \quad (1 \leq i \leq n)}{\Theta \rhd \Gamma \vdash \omega\big((\overrightarrow{x_1})t_1, \ldots, (\overrightarrow{x_n})t_n\big) : \tau} \quad \big(\omega : (\overrightarrow{\sigma_1})\tau_1, \ldots, (\overrightarrow{\sigma_n})\tau_n \to \tau\big)$$

where $\overrightarrow{x} : \overrightarrow{\sigma}$ stands for $x_1 : \sigma_1, \ldots, x_k : \sigma_k$.

Example 4.12.

1. Sorted λ-calculus. *Two sample terms for the signature of the multi-sorted λ-calculus follow:*

   $\mathrm{M} : [\sigma]\tau, \mathrm{N} : \sigma \rhd - \vdash \mathtt{app}\big(\mathtt{abs}\big((\mathtt{x})\mathrm{M}[\mathtt{x}]\big), \mathrm{N}[]\big) : \tau,$

   $\mathrm{M} : [\sigma]\tau, \mathrm{N} : \sigma \rhd - \vdash \mathrm{M}[\mathrm{N}[]] : \tau.$

2. Predicate logic. *Two sample terms for the signature Π of predicate logic are:*

   $\Theta \rhd x : \star, y : \star \ \vdash = (x, y) : \mathtt{Prop}$

   $\mathrm{M} : [\star]\mathtt{Prop} \rhd - \vdash \exists\big((\mathtt{x})\mathrm{M}[\mathtt{x}]\big) : \mathtt{Prop}$

Equational presentations. A multi-sorted second-order equational presentation $\mathscr{E} = (\Sigma, E)$ is given by a multi-sorted signature Σ together with a set E of axioms, each of which is a pair of terms in context.

Remark 4.13. The complete syntactic theory for multi-sorted second-order languages involves definitions of substitution and metasubstitution, multi-sorted second-order equational logic, and lemmas stating the well-typedness in this framework. These notions are, however, immediately generalisable from the mono-sorted setting of this chapter, and using the multi-sorted framework introduced in this appendix. For a more proper account, we refer the reader to [Fiore and Hur, 2010].

4.B Appendix to Chapter 4: Proofs of Substitution and Metasubstitution Lemmas

4.B1 Second-Order Substitution Lemma

Given terms

$$\Theta \rhd \Gamma \vdash s_i \quad (1 \le i \le n), \qquad \Theta \rhd \Gamma \vdash r_j \quad (1 \le j \le k), \qquad \text{and} \qquad \Theta \rhd x_1, \ldots, x_n, y_1, \ldots, y_k \vdash t,$$

we have

$$\Theta \rhd \Gamma \vdash t\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \;=\; t\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|} \quad .$$

Proof. We proceed by induction on the structure of the term t:

$$\begin{aligned}
\Theta \rhd \Gamma \quad \vdash &\quad x_h\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \\
= &\quad s_h\{y_j := r_j\}_{j \in \|k\|} \\
= &\quad x_h\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|}
\end{aligned}$$

$$\begin{aligned}
\Theta \rhd \Gamma \quad \vdash &\quad \mathrm{M}[\ldots, t', \ldots]\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \\
= &\quad \mathrm{M}\big[\ldots, t'\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|}, \ldots\big] \\
= &\quad \mathrm{M}\Big[\ldots, t'\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|}, \ldots\Big] \\
= &\quad \mathrm{M}[\ldots, t', \ldots]\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|}
\end{aligned}$$

$$\begin{aligned}
\Theta \rhd \Gamma \quad \vdash &\quad \omega(\ldots, (\overrightarrow{z})t', \ldots)\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|} \\
= &\quad \omega\big(\ldots, (\overrightarrow{z})t'\{x_i := s_i\}_{i \in \|n\|}\{y_j := r_j\}_{j \in \|k\|}, \ldots\big) \\
= &\quad \omega\Big(\ldots, (\overrightarrow{z})t'\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|}, \ldots\Big) \\
= &\quad \omega(\ldots, (\overrightarrow{z})t', \ldots)\Big\{x_i := s_i\{y_j := r_j\}_{j \in \|k\|}\Big\}_{i \in \|n\|}
\end{aligned}$$

□

4.B2 Substitution-Metasubstitution Lemma

Given terms

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd \Gamma \vdash t_i \quad (1 \le i \le n), \qquad \Theta \rhd \Gamma, \overrightarrow{y}_j \vdash s_j \quad (1 \le j \le k),$$

and

$$\mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k] \rhd x_1,\ldots,x_n \vdash t,$$

we have

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & t\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & t\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \quad . \end{aligned}$$

Proof. By induction on term structure:

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & x_l\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & t_l\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & x_l\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \end{aligned}$$

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & \mathrm{M}_h[t'_1,\ldots,t'_{m_h}]\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & \mathrm{M}_h\big[t'_1\{x_i := t_i\}_{i\in\|n\|},\ldots,t'_{m_h}\{x_i := t_i\}_{i\in\|n\|}\big]\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & s_h\Big\{x^{(h)}_{i'} := t'_{i'}\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i'\in\|m_h\|} \\ = \quad & s_h\Big\{x^{(h)}_{i'} := t'_{i'}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\}_{i\in\|n\|}\Big\}_{i'\in\|m_h\|} \\ = \quad & s_h\Big\{x^{(h)}_{i'} := t'_{i'}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i'\in\|m_h\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \\ = \quad & \mathrm{M}_h[t'_1,\ldots,t'_{m_h}]\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \end{aligned}$$

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & \omega(\ldots,(\overrightarrow{z})t',\ldots)\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|} \\ = \quad & \omega\Big(\ldots,(\overrightarrow{z})t'\{x_i := t_i\}_{i\in\|n\|}\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|},\ldots\Big) \\ = \quad & \omega\Big(\ldots,(\overrightarrow{z})t'\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|},\ldots\Big) \\ = \quad & \omega(\ldots,(\overrightarrow{z})t',\ldots)\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\{x_i := t_i\{\mathrm{M}_j := (\overrightarrow{y}_j)s_j\}_{j\in\|k\|}\Big\}_{i\in\|n\|} \end{aligned}$$

□

4.B3 Metasubstitution Lemma I

Given terms

$$\Theta \rhd \Gamma, \overrightarrow{x}_i \vdash r_i \quad (1 \leq i \leq k), \qquad \Theta \rhd \Gamma, \overrightarrow{y}_j \vdash s_j \quad (1 \leq j \leq l),$$

and

$$\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k], \mathrm{N}_1 \colon [n_1], \ldots, \mathrm{N}_l \colon [n_l] \rhd \Gamma \vdash t,$$

we have

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & t\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i\}_{i \in \|k\|} \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \\ = \quad & t\{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \Big\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|}\Big\}_{i \in \|k\|} \quad . \end{aligned}$$

Proof. By induction on the structure of t. The result is obvious for variable terms x. Furthermore, in the final induction step, the proof for terms involving operators follows immediately, similar to the proof of the Second-Order Substitution Lemma (Section 4.B1). We shall hence skip over this last step as well.

$$\begin{aligned} \Theta \rhd \Gamma \quad \vdash \quad & \mathrm{M}_h[t_1, \ldots, t_{m_h}]\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i\}_{i \in \|k\|} \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \\ = \quad & r_h\{x^{(h)}_{i'} := t_{i'}\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i\}_{i \in \|k\|}\}_{i' \in \|m_h\|} \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \\ = \quad & r_h\{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \Big\{x^{(h)}_{i'} := t_{i'}\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i\}_{i \in \|k\|} \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|}\Big\}_{i' \in \|m_h\|} \\ = \quad & \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \Big\{x^{(h)}_{i'} := t_{i'}\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|}\}_{i \in \|k\|}\Big\}_{i' \in \|m_h\|} \\ = \quad & \mathrm{M}_h[t_1, \ldots, t_{m_h}]\{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|} \Big\{\mathrm{M}_i := (\overrightarrow{x}_i) r_i \{\mathrm{N}_j := (\overrightarrow{y}_j) s_j\}_{j \in \|l\|}\Big\}_{i \in \|k\|} \end{aligned}$$

□

4.B4 Metasubstitution Lemma II

Given terms

$$\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd \Gamma \vdash t \quad \text{and} \quad \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd \Gamma, x^{(i)}_1, \ldots, x^{(i)}_{m_i} \vdash \mathrm{M}_i[x^{(i)}_1, \ldots, x^{(i)}_{m_i}]$$

for $1 \leq i \leq k$, we have

$$\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd \Gamma \vdash t\{\mathrm{M}_i := (\overrightarrow{x}_i) \mathrm{M}_i[x^{(i)}_1, \ldots, x^{(i)}_{m_i}]\}_{i \in k} = t \quad .$$

Proof. We again proceed by induction on term structure, skipping the first and final step:

$$
\begin{aligned}
\Theta \triangleright \Gamma \quad \vdash \quad & \mathrm{M}_h[t_1,\ldots,t_{m_h}]\{\mathrm{M}_i := (\overrightarrow{x}_i)\mathrm{M}_i[x_1^{(i)},\ldots,x_{m_i}^{(i)}]\}_{i\in k} \\
= \quad & \mathrm{M}_h[x_1^{(h)},\ldots,x_{m_h}^{(h)}]\{x_j^{(h)} := t_j\{\mathrm{M}_i := (\overrightarrow{x}_i)\mathrm{M}_i[x_1^{(i)},\ldots,x_{m_i}^{(i)}]\}_{i\in k}\}_{j\in\|m_h\|} \\
= \quad & \mathrm{M}_h[x_1^{(h)},\ldots,x_{m_h}^{(h)}]\{x_j^{(h)} := t_j\}_{j\in\|m_h\|} \\
= \quad & \mathrm{M}_h[t_1,\ldots,t_{m_h}]
\end{aligned}
$$

□

Chapter 5

SECOND-ORDER ALGEBRAIC THEORIES

We present the crux of this dissertation: a categorical-algebra viewpoint of languages with variable binding and parameterised metavariables. The core of this development is the notion of *second-order algebraic theory*, which is a presentation-independent account of second-order syntactic theory. This generalises Lawvere's fundamental work on algebraic theories [Lawvere, 2004] to the second-order setting.

We begin by recalling the notion of exponentiability (Section 5.1), which will be a fundamental property in our abstract development. The most elementary second-order algebraic theory, the *second-order theory of equality* $\mathbb{M}$, is defined explicitly in Section 5.2. Just as Lawvere theories arise from the free cartesian category on one object, second-order algebraic theories are defined on top of $\mathbb{M}$, which we show to be the free cartesian category generated by an exponentiable object. More scrutiny is devoted to this definition, as it plays a pivotal role in the definition of second-order algebraic theory (Section 5.3). At the syntactic level, the correctness of our definition is established in Section 5.4 by showing a categorical equivalence between second-order equational presentations and second-order algebraic theories (Theorem 5.8).

We restrict our treatment to the mono-sorted universe for two main reasons: to mirror Lawvere's categorical development of mono-sorted algebraic theories; and, more importantly, to remain in a simplified framework, which we hope will ease the appreciation of the subtleties of our definitions. However, just as the multi-sorted generalisation of Lawvere theories to include typing is obtained in a straightforward manner via indexing over sets of types, the generalisation of our work to the multi-sorted setting is evident. We finally point out that, having omitted the monadic view of second-order universal algebra, the important role played by the monadic perspective in our development will not be considered here.

5.1 Exponentiable objects

Categorical exponential structures are recalled. Just as the cartesian structure characterises first-order algebraic theories, *exponentiability* abstractly formalises essential second-order characteristics.

Exponential objects. For $\mathscr{C}$ a cartesian category and A, B objects of $\mathscr{C}$, an *exponential object* $A \Rightarrow B$ is a universal morphism from $- \times A \colon \mathscr{C} \to \mathscr{C}$ to B. Explicitly, $A \Rightarrow B$ comes equipped with a morphism $\mathbb{e} \colon (A \Rightarrow B) \times A \to B$ such that for any object C of $\mathscr{C}$ and $f : C \times A \to B$, there is a unique $\mathbb{l}(f) \colon C \to A \Rightarrow B$, the *exponential mate* of f, making $\mathbb{e} \circ (\mathbb{l}(f) \times A) = f$.

Exponential functors. A cartesian functor $F : \mathscr{C} \to \mathscr{D}$ is *exponential* if it preserves the exponential structure in $\mathscr{C}$. Formally, for any exponential $A \Rightarrow B$ in $\mathscr{C}$, $FA \Rightarrow FB$ is an exponential object in $\mathscr{D}$ and the exponential mate of

$$F(A \Rightarrow B) \times FA \cong F((A \Rightarrow B) \times A) \xrightarrow{F\mathbb{e}} FB$$

is an isomorphism $F(A \Rightarrow B) \to FA \Rightarrow FB$.

Exponentiable objects. Let $\mathscr{C}$ be a cartesian category. An object $C \in \mathscr{C}$ is *exponentiable* if for all objects $D \in \mathscr{C}$ the exponential $C \Rightarrow D$ exists in $\mathscr{C}$. Given an exponentiable object C, the n-ary cartesian product C^n is obviously exponentiable for all $n \in \mathbb{N}$.

5.2 The Second-Order Theory of Equality

In the notion of categorical algebraic theory, the elementary theory of equality plays a pivotal role, as it represents the most fundamental such theory. We thus proceed to identify the second-order algebraic theory of equality $\mathbb{M}$. This we do first in syntactic terms, via an explicit description of its categorical structure, and in abstract terms by establishing its universal property.

5.2.1 Definition

The syntactic viewpoint of second-order theories presented in Section 4.1 leads us to define the category $\mathbb{M}$ with set of objects given by $\mathbb{N}^*$ and morphisms $(m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$ given by tuples

$$\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_i} \vdash t_i \rangle_{i \in \|l\|}$$

of elementary terms under the empty second-order signature. The identity on $(m_1, \ldots, m_k)$ is given by

$$\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{m_i} \vdash \mathrm{M}_i[x_1, \ldots, x_{m_i}] \rangle_{i \in \|k\|} \quad ;$$

whilst the composition of

$$\langle \mathrm{L}_1 : [l_1], \ldots, \mathrm{L}_i : [l_i] \rhd x_1, \ldots, x_{m_p} \vdash s_p \rangle_{p \in \|j\|} : (l_1, \ldots, l_i) \to (m_1, \ldots, m_j)$$

and

$$\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_j : [m_j] \rhd y_1, \ldots, y_{n_q} \vdash t_q \rangle_{q \in \|k\|} : (m_1, \ldots, m_j) \to (n_1, \ldots, n_k)$$

is given via metasubstitution by

$$\langle \mathrm{L}_1 : [l_1], \ldots, \mathrm{L}_i : [l_i] \rhd y_1, \ldots, y_{n_q} \vdash t_q \{ \mathrm{M}_p := (x_1, \ldots, x_{m_p}) s_p \}_{p \in \|j\|} \rangle_{q \in \|k\|} : (l_1, \ldots, l_i) \to (n_1, \ldots, n_k) \quad .$$

The category $\mathbb{M}$ is well-defined, as the identity and associativity axioms hold because of intrinsic properties given by the Metasubstitution Lemmas (Lemmas 4.4 - 4.6), as seen in the following.

Lemma 5.1. *The definition of* $\mathbb{M}$ *above yields a well-defined category.*

Proof. Because of the monoidal properties of metasubstitution given by Metasubstitution Lemma I (Lemma 4.5) and Metasubstitution Lemma II (Lemma 4.6), the associativity and identity axioms hold in $\mathbb{M}$. Indeed, given morphisms

$$\begin{aligned}
\langle \mathrm{L}_1 : [l_1], \ldots, \mathrm{L}_h : [l_h] \rhd \overrightarrow{x}_o \vdash r_o \rangle_{o \in \|i\|} &\quad : \quad (l_1, \ldots, l_h) \to (m_1, \ldots, m_i) \\
\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_i : [m_i] \rhd \overrightarrow{y}_p \vdash s_p \rangle_{p \in \|j\|} &\quad : \quad (m_1, \ldots, m_i) \to (n_1, \ldots, n_j) \\
\langle \mathrm{N}_1 : [n_1], \ldots, \mathrm{N}_j : [n_j] \rhd \overrightarrow{z}_q \vdash t_q \rangle_{q \in \|g\|} &\quad : \quad (n_1, \ldots, n_j) \to (k_1, \ldots, k_g) \quad ,
\end{aligned}$$

we have, for all $q \in \|g\|$,

$$\begin{aligned}
\mathrm{L}_1 : [l_1], \ldots, \mathrm{L}_h : [l_h] \rhd \overrightarrow{z}_q \quad \vdash & \quad t_q \{ \mathrm{N}_p := (\overrightarrow{y}_p) s_p \}_{p \in \|j\|} \, \{ \mathrm{M}_o := (\overrightarrow{x}_o) r_o \}_{o \in \|i\|} \\
= & \quad t_q \Big\{ \mathrm{N}_p := (\overrightarrow{y}_p) s_p \{ \mathrm{M}_o := (\overrightarrow{x}_o) r_o \}_{o \in \|i\|} \Big\}_{p \in \|j\|} \quad .
\end{aligned}$$

Also, for a morphism

$$\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \overrightarrow{y}_j \vdash t_j \rangle_{j \in \|l\|} : (m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$$

and identities

$$\begin{aligned}
\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \overrightarrow{x}_i \vdash \mathrm{M}_i [\overrightarrow{x}_i] \rangle_{i \in \|k\|} &\quad : \quad (m_1, \ldots, m_k) \to (m_1, \ldots, m_k) \\
\langle \mathrm{N}_1 : [n_1], \ldots, \mathrm{N}_l : [n_l] \rhd \overrightarrow{y}_i \vdash \mathrm{N}_i [\overrightarrow{y}_i] \rangle_{i \in \|l\|} &\quad : \quad (n_1, \ldots, n_l) \to (n_1, \ldots, n_l)
\end{aligned}$$

we have, for all $j \in \|k\|$,

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \overrightarrow{y}_j \vdash t_j \{ \mathrm{M}_j := (\overrightarrow{x}_i) \mathrm{M}_i [\overrightarrow{x}_i] \}_{j \in \|k\|} \; = \; t_j \quad ,$$

and for all $i \in \|l\|$,

$$\mathrm{M}_1\colon [m_1], \ldots, \mathrm{M}_k\colon [m_k] \rhd \overrightarrow{y}_i \vdash \mathrm{N}_i[\overrightarrow{y}_i]\{\mathrm{N}_j := (\overrightarrow{y}_j)t_j\}_{j\in\|l\|} \;=\; t_i\{y_p^{(j)} := y_p^{(j)}\}_{p\in\|n_j\|} \;=\; t_i \quad .$$

□

5.2.2 Cartesian structure

The category $\mathbb{M}$ comes equipped with a strict cartesian structure, with the terminal object given by the empty sequence $()$, the terminal map $(m_1, \ldots, m_k) \to ()$ being the empty tuple $\langle\rangle$, and the binary product of $(m_1, \ldots, m_k)$ and $(n_1, \ldots, n_l)$ given by their concatenation $(m_1, \ldots, m_k, n_1, \ldots, n_l)$. Any object $(m_1, \ldots, m_k)$ is thus the cartesian product of the single tuples (m_i), for $i \in \|k\|$, with projections

$$\big\langle \mathrm{M}_1\colon [m_1], \ldots, \mathrm{M}_k\colon [m_k] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash \mathrm{M}_i[x_1^{(i)}, \ldots, x_{m_i}^{(i)}]\big\rangle \colon (m_1, \ldots, m_k) \to (m_i) \quad .$$

Indeed, given morphisms

$$\big\langle \mathrm{N}_1\colon [n_1], \ldots, \mathrm{N}_l\colon [n_l] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash q_i \big\rangle \colon (n_1, \ldots, n_l) \to (m_i)$$

for $i \in \|k\|$, the mediating morphism is

$$\big\langle \mathrm{N}_1\colon [n_1], \ldots, \mathrm{N}_l\colon [n_l] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash q_i \big\rangle_{i\in\|k\|} \quad .$$

Its uniqueness is evident, as for any

$$\big\langle \mathrm{N}_1\colon [n_1], \ldots, \mathrm{N}_l\colon [n_l] \rhd x_1^{(j)}, \ldots, x_{m_j}^{(j)} \vdash t_j \big\rangle_{j\in\|k\|} \colon (n_1, \ldots, n_l) \to (m_1, \ldots, m_k)$$

whose composition with the i-th projection is $\mathrm{N}_1\colon [n_1], \ldots, \mathrm{N}_l\colon [n_l] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash q_i$, we have for each $i \in \|k\|$

$$\mathrm{N}_1\colon [n_1], \ldots, \mathrm{N}_l\colon [n_l] \rhd x_1^{(j)}, \ldots, x_{m_j}^{(j)} \vdash \mathrm{M}_i[x_1^{(i)}, \ldots, x_{m_i}^{(i)}]\{\mathrm{M}_j := (x_1^{(j)}, \ldots, x_{m_j}^{(j)})t_j\}_{j\in\|k\|} = t_i \quad ,$$

which is simply the i-th projection q_i.

5.2.3 Exponential structure

In $\mathbb{M}$, the object (0) is exponentiable. For any tuple $(m_1, \ldots, m_k)$, the exponential $(0) \Rightarrow (m_1, \ldots, m_k)$ is given by $(m_1 + 1, \ldots, m_k + 1)$, with evaluation map $\mathrm{e}_{\overrightarrow{m},1}\colon (m_1 + 1, \ldots, m_k + 1) \times (0) \to (m_1, \ldots, m_k)$ given by the k-tuple

$$\big\langle \mathrm{M}_1\colon [m_1 + 1], \ldots, \mathrm{M}_k\colon [m_k + 1], \mathrm{N}\colon [0] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash \mathrm{M}_i\big[x_1^{(i)}, \ldots, x_{m_i}^{(i)}, \mathrm{N}[]\big]\big\rangle_{i\in\|k\|} \quad .$$

For any $(n_1,\ldots,n_l)$, the exponential mate $\mathbb{I}(\langle t_i\rangle_{i\in\|k\|})$ of a map

$$\left\langle \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{M}:[0]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)}\vdash t_i\right\rangle_{i\in\|k\|}\colon(n_1,\ldots,n_l)\times(0)\to(m_1,\ldots,m_k)$$

is given by

$$\left\langle \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)},y_i\vdash t_i\{\mathrm{M}:=y_i\}\right\rangle_{i\in\|k\|} \quad .$$

Clearly, the composite

$$(n_1,\ldots,n_l)\times(0)\xrightarrow{\mathbb{I}(\langle t_i\rangle_{i\in\|k\|})\times(0)}(m_1+1,\ldots,m_k+1)\times(0)\xrightarrow{\mathrm{e}_{\vec{m},1}}(m_1,\ldots,m_k)$$

equals

$$\begin{aligned}
&\Big\langle \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{N}\colon[0]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)}\vdash\\
&\qquad\qquad \mathrm{M}_i\big[x_1^{(i)},\ldots,x_{m_i}^{(i)},\mathrm{N}[]\big]\{\mathrm{M}_j:=(x_1^{(j)},\ldots,x_{m_j}^{(j)},y_j)t_j\{\mathrm{N}:=y_j\}\}_{j\in\|k\|}\Big\rangle_{i\in\|k\|}\\
=\quad&\left\langle \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{N}\colon[0]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)}\vdash t_i\right\rangle_{i\in\|k\|} \quad .
\end{aligned}$$

Uniqueness of the exponential mate is just as clear. If

$$\left\langle \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{N}\colon[0]\rhd z_1^{(i)},\ldots,z_{m_i}^{(i)},z\vdash s_i\right\rangle_{i\in\|k\|}\colon(n_1,\ldots,n_l)\to(m_1+1,\ldots,m_k+1)$$

is such that for $i\in\|k\|$

$$\mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{N}\colon[0]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)}\vdash \mathrm{M}_i\big[x_1^{(i)},\ldots,x_{m_i}^{(i)},\mathrm{N}[]\big]\{\mathrm{M}_j:=(z_1^{(j)},\ldots,z_{m_j}^{(j)},z)s_j\}_{j\in\|k\|}=t_i \quad ,$$

then

$$\mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l],\mathrm{N}\colon[0]\rhd x_1^{(i)},\ldots,x_{m_i}^{(i)}\vdash s_i=t_i$$

for all $i\in\|k\|$.

We finally point out that more generally, for any $n\in\mathbb{N}$, the exponential $(0)^n\Rightarrow(m_1,\ldots,m_k)$ is given by the tuple $(m_1+n,\ldots,m_k+n)$.

5.2.4 Second-order features via exponentiability

The exponential structure in $\mathbb{M}$ embodies attributes intrinsic to second-order languages. First, note that for each $n\in\mathbb{N}$, the *metaweakening* operation $W_n\colon\mathbb{M}\to\mathbb{M}$ mapping $(m_1,\ldots,m_k)$ to $(m_1+n,\ldots,m_k+n)$, and a morphism $(m_1,\ldots,m_k)\to(n_1,\ldots,n_l)$ of the form

$$\left\langle \mathrm{M}_1\colon[m_1],\ldots,\mathrm{M}_k\colon[m_k]\rhd y_1^{(j)},\ldots,y_{n_j}^{(j)}\vdash t_j\right\rangle_{j\in\|l\|}$$

to

$$\Big\langle \mathrm{M'}_1 \colon [m_1' + n], \ldots, \mathrm{M'}_k \colon [m_k' + n] \rhd y_1^{(j)}, \ldots, y_{n_j}^{(j)}, z_1^{(j)}, \ldots, z_n^{(j)} \vdash \\ t_j \Big\{ \mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)}) \mathrm{M'}_i \big[x_1^{(i)}, \ldots, x_{m_i}^{(i)}, z_1^{(j)}, \ldots, z_n^{(j)} \big] \Big\}_{i \in \|k\|} \Big\rangle_{j \in \|l\|}$$

is in fact the right adjoint $(0)^n \Rightarrow (-) \colon \mathbb{M} \to \mathbb{M}$ to the functor $(-) \times (0)^n \colon \mathbb{M} \to \mathbb{M}$.

Moreover, for any $(m_1, \ldots, m_k)$, the resulting bijection

$$\mathbb{M}\big((m_1, \ldots, m_k), (0)^n \Rightarrow (0)\big) \;\cong\; \mathbb{M}\big((m_1, \ldots, m_k) \times (0)^n, (0)\big)$$

formalises the correspondence between a second-order term and its parameterisation (Section 4.1.4). Indeed, every morphism of $\mathbb{M}$ of the form

$$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k], \mathrm{N}_1 \colon [0], \ldots, \mathrm{N}_n \colon [0] \rhd - \vdash t \big\rangle \colon (m_1, \ldots, m_k) \times (0)^n \to (0)$$

is (the single tuple of) the parameterisation of its unique exponential mate

$$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash t\{\mathrm{N}_i := x_i\}_{i \in \|n\|} \big\rangle \colon (m_1, \ldots, m_k) \to (0)^n \Rightarrow (0) \quad .$$

Abstractly, every morphism $\langle s \rangle \colon (m_1, \ldots, m_k) \to (n)$ can be *parameterised* as $\oplus_n \circ \big(\langle s \rangle \times (0)^n\big)$, whose exponential mate $\mathbb{I}\big(\oplus_n \circ (\langle s \rangle \times (0)^n)\big)$ is just $\langle s \rangle$.

Finally, the exponential structure manifests itself in all second-order terms, which, when viewed as morphisms of $\mathbb{M}$, decompose via unique universal maps.

Lemma 5.2. *In the category $\mathbb{M}$, every morphism of the form*

$$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash x_i \big\rangle \colon (m_1, \ldots, m_k) \to (n)$$

decomposes as

$$(m_1, \ldots, m_k) \longrightarrow () \xrightarrow{\mathbb{I}(\pi_i^{(n)} \circ \cong)} (n) \quad ,$$

where the unlabelled morphism is the unique terminal map, and $\mathbb{I}(\pi_i^{(n)} \circ \cong)$ is the exponential mate of the i-th projection $() \times (0)^n \cong (0)^n \xrightarrow{\pi_i^{(n)}} (0)$. Moreover, every morphism

$$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}] \big\rangle \colon (m_1, \ldots, m_k) \to (n)$$

decomposes as

$$(m_1, \ldots, m_k) \xrightarrow{\langle \pi_i, t_1, \ldots, t_{m_i} \rangle} (m_i, n^{m_i}) \xrightarrow{\varsigma_{m_i, n}} (n) \quad ,$$

where n^{m_i} denotes the sequence $n, \ldots, n$ of length m_i, $\varsigma_{m_i,n}$ is the exponential mate of

$$(m_i, n^{m_i}) \times (0)^n \xrightarrow{(m_i)\times \mathbb{e}_{m_i,n}} (m_i) \times (0)^{m_i} \xrightarrow{\mathbb{e}_{m_i}} (0) \quad ,$$

and $\mathbb{e}_{m_i,n}$ is the evaluation map associated with the exponential $((0)^n \Rightarrow (0)^{m_i}) = (n)^{m_i}$.

Proof. Explicitly, $(\pi_i^{(n)} \circ \cong) \colon () \times (0)^n \to (0)$ is given by $\langle \mathrm{N}_1 \colon [0], \ldots, \mathrm{N}_n \colon [0] \rhd - \vdash \mathrm{N}_i[] \rangle$ and its unique exponential mate is $\langle - \rhd x_1, \ldots, x_n \vdash \mathrm{N}_i[]\{\mathrm{N}_j := x_j\}_{j \in \|n\|} \rangle$, which is simply $\langle - \rhd x_1, \ldots, x_n \vdash x_i \rangle$. Composing this with $\langle\rangle \colon (m_1, \ldots, m_k) \to ()$ yields $\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash x_i \rangle$. Next, the morphism $\varsigma_{m_i,n} \colon (m_i, n^{m_i}) \to (n)$ is syntactically given by

$$\langle \mathrm{M}_i \colon [m_i], \mathrm{N}_1 \colon [0], \ldots, \mathrm{N}_{m_i} \colon [0] \rhd x_1, \ldots, x_n \vdash \mathrm{M}_i[\mathrm{N}_1[x_1, \ldots, x_n], \ldots, \mathrm{N}_{m_i}[x_1, \ldots, x_n]] \rangle \quad ,$$

and thus composed with $\langle \pi_i, t_1, \ldots, t_{m_i} \rangle$

$$\begin{aligned} \Big\langle \mathrm{M}_i \colon [m_i], \mathrm{N}_1 \colon [0], \ldots, \mathrm{N}_{m_i} \colon [0] \rhd x_1, \ldots, x_n \quad \vdash \quad & \mathrm{M}_i[\mathrm{N}_1[x_1, \ldots, x_n], \ldots, \mathrm{N}_{m_i}[x_1, \ldots, x_n]] \\ & \{\mathrm{M}_i := (y_1, \ldots, y_{m_i})\mathrm{M}_i[y_1, \ldots, y_{m_i}]\} \\ & \{\mathrm{N}_j := (x_1, \ldots, x_n)t_j\}_{j \in \|m_i\|} \Big\rangle \quad , \end{aligned}$$

this equals

$$\langle \mathrm{M}_i \colon [m_i], \mathrm{N}_1 \colon [0], \ldots, \mathrm{N}_{m_i} \colon [0] \rhd x_1, \ldots, x_n \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}] \rangle \quad .$$

□

5.2.5 Universal property

The exponential structure in $\mathbb{M}$ provides a universal semantic characterisation of $\mathbb{M}$. Loosely speaking, $\mathbb{M}$ is the free strict cartesian category on an exponentiable object. We point out the analogy to the first-order theory of equality $\mathbb{L}$, which is the cartesian category freely generated by a single object.

Proposition 5.3 (Universal property of $\mathbb{M}$)**.** *The category $\mathbb{M}$, together with the exponentiable object $(0) \in \mathbb{M}$, is initial amongst cartesian categories equipped with an exponentiable object and with respect to cartesian functors that preserve the exponentiable object.*

Proof. Let $\mathscr{D}$ be a cartesian category equipped with an exponentiable object D. There is a functor $I \colon \mathbb{M} \to \mathscr{D}$ mapping the tuple $(m_1, \ldots, m_k)$ to $(D^{m_1} \Rightarrow D) \times \cdots \times (D^{m_k} \Rightarrow D)$, and defined on morphisms of $\mathbb{M}$ by structural induction as follows:

- $$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash x_i \big\rangle \colon (m_1, \ldots, m_k) \to (n) \quad \stackrel{I}{\longmapsto}$$
$$(D^{m_1} \Rightarrow D) \times \cdots \times (D^{m_k} \Rightarrow D) \xrightarrow{!^{\mathscr{D}}} 1 \xrightarrow{\mathbb{I}(\pi_i^{\mathscr{D}} \circ \cong)} (D^n \Rightarrow D)$$

- $$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}] \big\rangle \colon (m_1, \ldots, m_k) \to (n) \quad \stackrel{I}{\longmapsto}$$
$$(D^{m_1} \Rightarrow D) \times \cdots \times (D^{m_k} \Rightarrow D) \xrightarrow{\left\langle \pi_i^{\mathscr{D}}, I\langle t_1 \rangle, \ldots, I\langle t_{m_i} \rangle \right\rangle} (D^{m_i} \Rightarrow D) \times (D^n \Rightarrow D)^{m_i} \xrightarrow{\varsigma^{\mathscr{D}}_{m_i,n}} (D^n \Rightarrow D)$$

We superscript cartesian and exponential maps by $\mathscr{D}$ to distinguish them from those in $\mathbb{M}$. Note that I is cartesian by definition and moreover exponential. To see this, note that

$$I\big((0) \Rightarrow (m)\big) \;=\; I(m+1) \;=\; D^{m+1} \Rightarrow D \;\cong\; D \Rightarrow (D^m \Rightarrow D) \;=\; I(0) \Rightarrow I(m) \quad ,$$

and that the exponential mate of $I(\oplus_{1,m}) \colon (D^{m+1} \Rightarrow D) \times D \to (D^m \Rightarrow D)$ in $\mathscr{D}$ is the isomorphism

$$(D^{m+1} \Rightarrow D) \cong D \Rightarrow (D^m \Rightarrow D) \quad .$$

To see that I is indeed the unique (up to isomorphism) universal functor associated with the initiality of $\mathbb{M}$, suppose that we are given a functor $F \colon \mathbb{M} \to \mathscr{D}$ which is cartesian and exponential mapping (0) to D. Then F is isomorphic to I. This is evident on objects, as we have

$$\begin{aligned}
F(m_1, \ldots, m_k) &= F\big((m_1) \times \cdots \times (m_k)\big) \\
&\cong F(m_1) \times \cdots \times F(m_k) \\
&= F\big((0)^{m_1} \Rightarrow (0)\big) \times \cdots \times F\big((0)^{m_k} \Rightarrow (0)\big) \\
&\cong \big(F(0)^{m_1} \Rightarrow F(0)\big) \times \cdots \times \big(F(0)^{m_k} \Rightarrow F(0)\big) \\
&= (D^{m_1} \Rightarrow D) \times \cdots \times (D^{m_k} \Rightarrow D) \\
&= I(m_1, \ldots, m_k) \quad .
\end{aligned}$$

Given a morphism $\langle t \rangle \colon (m_1, \ldots, m_k) \to (n)$ of $\mathbb{M}$, the fact that $I\langle t \rangle = F\langle t \rangle$ is an immediate consequence of the cartesian and exponential property of F and I. More precisely, by induction on the structure of the term t, we have:

- The map
$$\big\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash x_i \big\rangle \colon (m_1, \ldots, m_k) \to (n)$$
decomposes as $\mathbb{I}(\pi_i^{\mathbb{M}} \circ \cong) \circ !^{\mathbb{M}}$, and since F preserves the cartesian and exponential structure, $F\big(\mathbb{I}(\pi_i^{\mathbb{M}} \circ \cong) \circ !^{\mathbb{M}}\big) = \mathbb{I}(\pi_i^{\mathscr{D}} \circ \cong) \circ !^{\mathscr{D}}$, which is exactly the image under I.

- Similarly, $\langle \mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}] \rangle \colon (m_1, \ldots, m_k) \to (n)$ decomposes via universal cartesian and exponential morphisms of $\mathbb{M}$, which are preserved by both I and F, and thus their image under them must be equal.

□

5.3 Second-Order Algebraic Theories

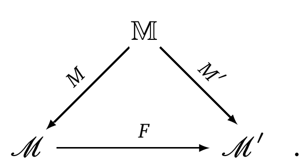
We extend Lawvere's fundamental notion of algebraic theory [Lawvere, 2004] to the second-order universe. Second-order algebraic theories are defined as second-order-structure preserving functors from the category $\mathbb{M}$ to cartesian categories.

Definition 5.4 (Second-order algebraic theories)**.** A *second-order algebraic theory* consists of a small cartesian category $\mathscr{M}$ and a strict cartesian identity-on-objects functor $M \colon \mathbb{M} \to \mathscr{M}$ that preserves the exponentiable object (0).

The most basic example of a second-order algebraic theory is the *second-order algebraic theory of equality* given by the category $\mathbb{M}$ together with the identity functor. In fact, we formally verify in Section 5.4 that this is the (second-order) algebraic theory corresponding to a second-order presentation with no operators. This is analogous to the theory of sets corresponding to $\mathbb{L}$ in the first-order setting.

Every second-order algebraic theory has an *underlying first-order algebraic theory*. To formalise this, recall that the first-order algebraic theory of equality $\mathbb{L}$ is the free strict cartesian category on an object and consider the unique cartesian functor $\mathbb{L} \to \mathbb{M}$ mapping the generating object to the generating exponentiable object (0). Then, the first-order algebraic theory underlying a given second-order algebraic theory $\mathbb{M} \to \mathscr{M}$ is given by $\mathbb{L} \to \mathscr{L}_{\mathscr{M}}$, where $\mathbb{L} \to \mathscr{L}_{\mathscr{M}} \hookrightarrow \mathscr{M}$ is the identity-on-objects, full-and-faithful factorisation of $\mathbb{L} \to \mathbb{M} \to \mathscr{M}$. In particular, the first-order algebraic theory of equality $Id_{\mathbb{L}} \colon \mathbb{L} \to \mathbb{L}$ underlies the second-order algebraic theory of equality $Id_{\mathbb{M}} \colon \mathbb{M} \to \mathbb{M}$.

Second-order algebraic translations. To complete the definition of second-order algebraic theories from a Lawvere point of view, one requires a notion of morphism between them. To this end, we define, for second-order algebraic theories $M \colon \mathbb{M} \to \mathscr{M}$ and $M' \colon \mathbb{M} \to \mathscr{M}'$, a *second-order algebraic translation* to be a cartesian functor $F \colon \mathscr{M} \to \mathscr{M}'$ such that



The category of second-order algebraic theories. We denote by **SOAT** the category of second-order algebraic theories and second-order algebraic translations, with the evident identity and composition.

5.4 Second-Order Theory/Presentation Correspondence

We illustrate how to construct second-order algebraic theories from second-order equational presentations, and vice versa, and prove that these constructions are mutually inverse. Only one direction of this correspondence is shown here, namely the passage from an algebraic theory to a presentation and back to an algebraic theory. The theory of second-order syntactic translations is required for the other direction, and this proof is thus postponed to the following chapter.

5.4.1 The theory of a presentation

Classifying categories of second-order equational presentations. For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, the *classifying category* $\mathbb{M}(\mathscr{E})$ has a set of objects $\mathbb{N}^*$ and morphisms $(m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$ given by tuples

$$\big\langle \big[\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1^{(i)}, \ldots, x_{n_i}^{(i)} \vdash t_i\big]_{\mathscr{E}} \big\rangle_{i \in \|l\|}$$

of equivalence classes of terms generated from Σ under the equivalence relation identifying two terms if and only if they are provably equal in $\mathscr{E}$ from *Second-Order Equational Logic* (Figure 4.1). Identities and composition are defined on representatives as in $\mathbb{M}$. Indeed, composition via metasubstitution respects the equivalence relation, as for

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash_{\mathscr{E}} t_1 \equiv t_2 \qquad \text{and} \qquad \mathrm{N} : [n] \vdash y_1, \ldots, y_l \vdash_{\mathscr{E}} s_1 \equiv s_2$$

the equality

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd y_1, \ldots, y_l \vdash_{\mathscr{E}} s_1\{\mathrm{N} := (x_1, \ldots, x_n)t_1\} \equiv s_2\{\mathrm{N} := (x_1, \ldots, x_n)t_2\}$$

is derivable from Second-Order Equational Logic. The categorical associativity and identity axioms making $\mathbb{M}(\mathscr{E})$ a well-defined category then follow immediately, as do the facts that $\mathbb{M}(\mathscr{E})$ comes equipped with the same cartesian structure as in $\mathbb{M}$ and that (0) is exponentiable in $\mathbb{M}(\mathscr{E})$.

Revisiting the definition of the category $\mathbb{M}$ from the viewpoint of classifying categories, observe that it classifies the most elementary second-order presentation $\mathscr{E}_0$, which has an empty set of operators and no equations. Indeed, $\mathbb{M}(\mathscr{E}_0)$ has morphisms tuples of terms (as the equivalence relation $\mathscr{E}_0$ singles out every term), and since all terms are elementary, $\mathbb{M} = \mathbb{M}(\mathscr{E}_0)$.

Classifying categories of second-order algebraic presentations are the main component when defining theories of presentations.

Lemma 5.5. *For a second-order equational presentation $\mathscr{E}$, the category $\mathbb{M}(\mathscr{E})$ together with the canonical functor $M_{\mathscr{E}}\colon \mathbb{M} \to \mathbb{M}(\mathscr{E})$ is a second-order algebraic theory.*

Proof. The functor $M_{\mathscr{E}}$ is the identity on objects and maps a tuple of terms $\langle t_1, \ldots, t_n \rangle$ to the tuple of their equivalence classes $\langle [t_1]_{\mathscr{E}}, \ldots, [t_n]_{\mathscr{E}} \rangle$. It preserves the cartesian and exponential structures of $\mathbb{M}$ as we have shown that they are, together with metasubstitution, respected by the equivalence relation $\sim_{\mathscr{E}}$. □

We refer to $M_{\mathscr{E}}\colon \mathbb{M} \to \mathbb{M}(\mathscr{E})$ as the second-order algebraic theory of $\mathscr{E}$.

Remark 5.6. Consider a second-order signature Σ and its induced second-order algebraic theory $M_{\Sigma}\colon \mathbb{M} \to \mathbb{M}(\Sigma)$. This construction is justified by considering a signature as just an equational presentation with an empty set of equations. Because of its universal property and the fact that every morphism of $\mathbb{M}$ decomposes as universal cartesian and exponential morphisms, it is clear that, since $M_{\Sigma}\colon \mathbb{M} \to \mathbb{M}(\Sigma)$ preserves the cartesian and exponential structure of $\mathbb{M}$, the algebraic theory M_{Σ} is in this case simply an inclusion functor.

5.4.2 The presentation of a theory

The *internal language* $\mathfrak{L}(M)$ of a second-order algebraic theory $M\colon \mathbb{M} \to \mathscr{M}$ is the second-order equational presentation defined as follows:

(Operators) For every $f : (m_1, \ldots, m_k) \to (n)$ in $\mathscr{M}$, we have an operator ω_f of arity $(m_1, \ldots, m_k, 0^n)$, where 0^n stands for the appearance of 0 n-times.

(Equations) Setting

$$\mathrm{t}_f = \omega_f\left((x_1^{(1)}, \ldots, x_{m_1}^{(1)})\mathrm{M}_1\left[x_1^{(1)}, \ldots, x_{m_1}^{(1)}\right], \ldots, (x_1^{(k)}, \ldots, x_{m_k}^{(k)})\mathrm{M}_k\left[x_1^{(k)}, \ldots, x_{m_k}^{(k)}\right], x_1, \ldots, x_n\right)$$

for every morphism $f : (m_1, \ldots, m_k) \to (n)$ in $\mathscr{M}$, we let $\mathfrak{L}(M)$ have equations

($\mathscr{E}$1) $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash s \equiv \mathrm{t}_{M\langle s \rangle}$
for every $\langle s \rangle : (m_1, \ldots, m_k) \to (n)$ in $\mathbb{M}$, and

($\mathscr{E}$2) $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{t}_h \equiv \mathrm{t}_g\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{n_i}^{(i)})\mathrm{t}_{f_i}\}_{i \in \|l\|}$
for every

$$\begin{aligned} h &: (m_1, \ldots, m_k) \to (n) \\ g &: (n_1, \ldots, n_l) \to (n) \\ f_i &: (m_1, \ldots, m_k) \to (n_i) \quad , \quad 1 \leq i \leq l \end{aligned}$$

such that $h = g \circ \langle f_1, \ldots, f_l \rangle$ in $\mathscr{M}$.

We write $\Sigma(M)$ and $E(M)$ for these operators and equations, respectively.

Remark 5.7. This procedure of synthesising internal languages from second-order algebraic theories yields some redundancies in the resulting set of operators. For instance, the operator $\omega_f : (m_1, \ldots, m_k, 0^n)$ induced by the morphism $f : (m_1, \ldots, m_k) \to (n)$ of $\mathscr{M}$ is essentially the same as the operator with the same arity induced by the morphism $\mathrm{e}_n \circ (f \times (0)^n) : (m_1, \ldots, m_k, 0^n) \to (0)$. By *essentially the same* we mean that the following is derivable from $(\mathscr{E}1)$ and $(\mathscr{E}2)$:

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{t}_f \equiv \mathrm{t}_{\mathrm{e}_n \circ \left(f \times (0)^n\right)} \quad .$$

5.4.3 Towards second-order syntactic categorical type theory correspondence

Having presented the transformation between second-order algebraic theories and equational presentations, we proceed to prove the first part of the mutual invertibility of these constructions.

Theorem 5.8 (Theory/presentation correspondence)**.** *Every second-order algebraic theory $M : \mathbb{M} \to \mathscr{M}$ is isomorphic to the second-order algebraic theory $M_{\mathfrak{L}(M)} : \mathbb{M} \to \mathbb{M}(\mathfrak{L}(M))$ of its associated second-order equational presentation.*

Proof. We prove the correspondence via an explicit description of the isomorphism and its inverse. Define the identity-on-objects functor

$$\mu_M : \mathscr{M} \to \mathbb{M}(\mathfrak{L}(M))$$

by mapping $f : (m_1, \ldots, m_k) \to (n)$ of $\mathscr{M}$ to

$$\left\langle \left[\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{t}_f \right]_{\mathfrak{L}(M)} \right\rangle : (m_1, \ldots, m_k) \to (n) \quad .$$

Functoriality of μ_M is implied by the equational theory of $\mathfrak{L}(M)$. More precisely, the identity $id^{\mathscr{M}}_{(m_1,\ldots,m_k)}$ on $(m_1, \ldots, m_k)$ in $\mathscr{M}$ is mapped to the k-tuple of equivalence classes of

$$\begin{aligned} \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x^{(i)}_1, \ldots, x^{(i)}_{m_i} \quad \vdash \quad & \mathrm{t}_{\pi_i^{(\mathscr{M})}} \\ = \quad & \mathrm{t}_{M(\pi_i^{(\mathbb{M})})} \\ = \quad & \mathrm{t}_{M\langle \mathrm{M}_i[x^{(i)}_1, \ldots, x^{(i)}_{m_i}]\rangle} \\ \overset{\mathscr{E}1}{\equiv} \quad & \mathrm{M}_i[x^{(i)}_1, \ldots, x^{(i)}_{m_i}] \quad , \end{aligned}$$

for $1 \le i \le k$ and $\pi_i^{(-)} : (m_1, \ldots, m_k) \to (m_i)$ the canonical projection in $-$, which makes the above tuple indeed the identity in $\mathbb{M}(\mathfrak{L}(M))$. Similarly, preservation of composition is a consequence of $(\mathscr{E}2)$ of $\mathfrak{L}(M)$. Consider, without loss of generality, the morphisms $\langle f_1, \ldots, f_l \rangle : (m_1, \ldots, m_k) \to (n_1, \ldots, n_l)$

and $g\colon (n_1,\dots,n_l) \to (n)$ of $\mathscr{M}$. Then $\mu_M(g)\circ\mu_M(\langle f_1,\dots,f_l\rangle)$ is given by the equivalence class of

$$\begin{array}{rcl} \mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n & \vdash & \mathrm{t}_g\{\mathrm{N}_i := (\overrightarrow{y}_i)\mathrm{t}_{f_i}\}_{i\in\|l\|} \\ & \overset{\mathscr{E}2}{\equiv} & \mathrm{t}_{g\circ\langle f_1,\dots,n_l\rangle} \quad , \end{array}$$

making $\mu_M(g)\circ\mu_M(\langle f_1,\dots,f_l\rangle) = \mu_M(g\circ\langle f_1,\dots,n_l\rangle)$.

This definition is strong enough to yield an algebraic translation from $M\colon \mathbb{M}\to\mathscr{M}$ to the classifying algebraic theory $M_{\mathfrak{L}(M)}\colon \mathbb{M}\to\mathbb{M}(\mathfrak{L}(M))$, since for any $\langle t\rangle\colon (m_1,\dots,m_k)\to(n)$ in $\mathbb{M}$, the morphism $M\langle t\rangle\colon (m_1,\dots,m_k)\to(n)$ in $\mathscr{M}$ is mapped under μ_M to the equivalence class of

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash \mathrm{t}_{M\langle t\rangle} \quad ,$$

which by ($\mathscr{E}1$) is provably equal to t, whose equivalence class is the image of t under $M_{\mathfrak{L}(M)}$.

In the other direction, define the identity-on-objects mapping

$$\bar{\mu}_M : \mathbb{M}(\mathfrak{L}(M)) \to \mathscr{M}$$

by induction on the structure of representatives of equivalence classes $[-]_{\mathfrak{L}(M)}$ as follows:

- $\big[\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash x_i\big]_{\mathfrak{L}(M)}$ is mapped to

$$(m_1,\dots,m_k) \xrightarrow{!^{(\mathscr{M})}} () \xrightarrow{\mathbb{I}(\pi_i^{(\mathscr{M})}\circ\cong)} (n) \quad .$$

- $\big[\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash \mathrm{M}_i[t_1,\dots,t_{m_i}]\big]_{\mathfrak{L}(M)}$ is mapped to

$$(m_1,\dots,m_k) \xrightarrow{\left\langle \pi_i^{(\mathscr{M})},\bar{\mu}_M([t_1]_{\mathfrak{L}(M)}),\dots,\bar{\mu}_M([t_{m_i}]_{\mathfrak{L}(M)})\right\rangle} (m_i, n^{m_i}) \xrightarrow{\varsigma^{(\mathscr{M})}_{m_i,n}} (n) \quad .$$

- For $f : (n_1,\dots,n_l)\to(j)$ in $\mathscr{M}$,

$$\big[\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash \omega_f\big((\overrightarrow{y}_1)t_1,\dots,(\overrightarrow{y}_l)t_l,s_1,\dots,s_j\big)\big]_{\mathfrak{L}(M)}$$

  is mapped under $\bar{\mu}_M$ to the composite

$$\begin{array}{l} (m_1,\dots,m_k) \xrightarrow{\left\langle \bar{\mu}_M[t_1]_{\mathfrak{L}(M)},\dots,\bar{\mu}_M[t_l]_{\mathfrak{L}(M)},\bar{\mu}_M[s_1]_{\mathfrak{L}(M)},\dots,\bar{\mu}_M[s_j]_{\mathfrak{L}(M)}\right\rangle} (n+n_1,\dots,n+n_l,n^j) \\ \qquad\qquad\qquad\qquad \downarrow{\scriptstyle (0)^n\Rightarrow\left(\mathrm{e}_j\circ(f\times(0)^j)\right)} \\ \qquad\qquad\qquad\qquad (n) \end{array}$$

Note that equivalence classes of elementary terms s are simply mapped to $M\langle s\rangle$ under $\bar{\mu}_M$.

We show that the mapping $\bar{\mu}_M$ is: (i) well-defined, (ii) functorial, and (iii) an algebraic translation $\mathbb{M}(\mathfrak{L}(M)) \to \mathscr{M}$.

(i) To verify that $\bar{\mu}_M$ is well-defined, we show that equal terms (that is representatives of equivalence classes $[-]_{\mathfrak{L}(M)}$) according to axioms ($\mathscr{E}1$) and ($\mathscr{E}2$) of $\mathfrak{L}(M)$ are mapped under $\bar{\mu}_M$ to equal morphisms of $\mathscr{M}$. Consider axiom ($\mathscr{E}1$), and let $\langle s\rangle : (m_1,\ldots,m_k) \to (n)$ be a morphism of $\mathbb{M}$. Then the image of $\big[\mathrm{t}_{M\langle s\rangle}\big]_{\mathfrak{L}(M)}$ under $\bar{\mu}_M$ is the composite

$$(m_1,\ldots,m_k) \xrightarrow{\mathbb{I}(id_{(m_1,\ldots,m_k,0^n)})} (0)^n \Rightarrow (m_1,\ldots,m_k,0^n) \xrightarrow{(0)^n\Rightarrow\left(\mathrm{e}_n\circ(M\langle s\rangle\times(0)^n)\right)} (n) \quad ,$$

which is simply $M\langle s\rangle$, and is in turn the image of $\langle s\rangle$ under $\bar{\mu}_M$ as s is an elementary term. For the axiom ($\mathscr{E}2$), let $g : (n_1,\ldots,n_l) \to (n)$, $h : (m_1,\ldots,m_k) \to (n)$, and $f_i : (m_1,\ldots,m_k) \to (n_i)$ (for $1 \le i \le l$) be morphisms of $\mathscr{M}$ such that $g \circ \langle f_1,\ldots,f_l\rangle = h$. Then

$$\begin{aligned}
& \bar{\mu}_M\Big(\big[\mathrm{t}_g\{\mathrm{M}_i := (\overrightarrow{x_i})\mathrm{t}_{f_i}\}_{i\in\|l\|}\big]_{\mathfrak{L}(M)}\Big) \\
= \; & \big((0)^n \Rightarrow \big(\mathrm{e}_n \circ \big(g \times (0)^n\big)\big)\big) \circ \big((0)^n \Rightarrow \big(\mathrm{e}_n \circ \big(\langle f_1,\ldots,f_l\rangle \times (0)^n\big)\big)\big) \circ \mathbb{I}\big(id_{(m_1,\ldots,m_k,0^n)}\big) \\
= \; & \big((0)^n \Rightarrow \big(\mathrm{e}_n \circ \big((g \circ \langle f_1,\ldots,f_l\rangle) \times (0)^n\big)\big)\big) \circ \mathbb{I}\big(id_{(m_1,\ldots,m_k,0^n)}\big) \\
= \; & \big((0)^n \Rightarrow \big(\mathrm{e}_n \circ \big(h \times (0)^n\big)\big)\big) \circ \mathbb{I}\big(id_{(m_1,\ldots,m_k,0^n)}\big) \\
= \; & \bar{\mu}_M\Big(\big[\mathrm{t}_h\big]_{\mathfrak{L}(M)}\Big) \quad .
\end{aligned}$$

(ii) For the identity condition of functoriality, note that the identity in $\mathbb{M}(\mathfrak{L}(M))$ is given by the equivalence class of an elementary term, and by definition, a morphism $f = \langle [t]_{\mathfrak{L}(M)}\rangle$ of $\mathbb{M}(\mathfrak{L}(M))$, for t an elementary term, is simply mapped to $M(\langle t\rangle)$ under $\bar{\mu}_M$. Therefore, for any $(m_1,\ldots,m_k)$ in $\mathbb{M}(\mathfrak{L}(M))$, and since M is a functor, we have that

$$\bar{\mu}_M\big(id^{\mathbb{M}(\mathfrak{L}(M))}_{(m_1,\ldots,m_k)}\big) = M\big(id^{\mathbb{M}}_{(m_1,\ldots,m_k)}\big) = id^{\mathscr{M}}_{(m_1,\ldots,m_k)} \quad ,$$

where the superscript in $id^{\mathscr{C}}$ identifies the category $\mathscr{C}$ the identity is being taken in. Next, for compositionality, note that, by its definition, $\bar{\mu}_M$ commutes with metasubstitution. More precisely, from the equational theory of $\mathfrak{L}(M)$, any morphism of $\mathbb{M}(\mathfrak{L}(M))$ can be written as $[\mathrm{t}_h]_{\mathfrak{L}(M)}$, for $h = g \circ f$ a morphism of $\mathscr{M}$. By definition, this is mapped under $\bar{\mu}_M$ to

$$\bar{\mu}_M[\mathrm{t}_g]_{\mathfrak{L}(M)} \circ \bar{\mu}_M[\mathrm{t}_f]_{\mathfrak{L}(M)} \quad .$$

(Recall point (i) above for more details.)

(iii) The functor $\bar{\mu}_M$ is an algebraic translation. This is an immediate consequence of the fact that it maps a morphism $\langle [s]_{\mathfrak{E}(M)} \rangle$, for s elementary, to $M\langle s \rangle$, therefore making

$$\bar{\mu}_M\big(M_{\mathfrak{E}(M)}(\langle s \rangle)\big) = M\langle s \rangle \quad .$$

We finally proceed to show that the algebraic translations μ_M and $\bar{\mu}_M$ are mutually inverse. Trivially, this is the case on their restrictions on objects. It remains to verify the same on morphisms.

Indeed, the image of a morphism $f : (m_1, \ldots, m_k) \to (n)$ of $\mathscr{M}$ under $\bar{\mu}_M \circ \mu_M$ is given by

$$(m_1, \ldots, m_k) \xrightarrow{\mathbb{I}(id_{(m_1,\ldots,m_k,0^n)})} (0)^n \Rightarrow (m_1, \ldots, m_k, 0^n) \xrightarrow{(0)^n \Rightarrow (\mathrm{e}_n \circ (f \times (0)^n))} (n)$$

which is equal to $\mathbb{I}\big(\mathrm{e}_n \circ (f \times (0)^n)\big)$, which is simply f.

In the other direction, we show, by induction on the structure of the term t, that for a morphism $\langle [t]_{\mathfrak{E}(M)} \rangle : (m_1, \ldots, m_k) \to (n)$ of $\mathbb{M}(\mathfrak{E}(M))$,

$$(\mu_M \circ \bar{\mu}_M)\langle [t]_{\mathfrak{E}(M)} \rangle = \langle [t]_{\mathfrak{E}(M)} \rangle \quad .$$

- For $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash x_i$, $(\mu_M \circ \bar{\mu}_M)\langle [x_i]_{\mathfrak{E}(M)} \rangle$ is given by the single tuple of the equivalence class of the term

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash \mathrm{t}_{M\langle x_i \rangle} \quad ,$$

  which by axiom $(\mathscr{E}1)$ of $\mathfrak{E}(M)$ is equal to x_i.

- The image of $\big\langle \big[\mathrm{M}_i[t_1, \ldots, t_{m_i}]\big]_{\mathfrak{E}(M)} \big\rangle : (m_1, \ldots, m_k) \to (n)$ under $\mu_M \circ \bar{\mu}_M$ is given, by induction on $t_1, \ldots, t_{m_i}$, by the single tuple containing the equivalence class of the term

$$\begin{aligned} \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \quad & \vdash \quad \mathrm{t}_{M\left\langle \mathrm{M}_i\left[\mathrm{N}_1[\overrightarrow{x}], \ldots, \mathrm{N}_{m_i}[\overrightarrow{x}]\right]\right\rangle} \{\mathrm{M}_i := (\overrightarrow{y}_i)\mathrm{t}_{M\langle \mathrm{M}_i[\overrightarrow{y}_i] \rangle}\} \\ & \qquad\qquad \{\mathrm{N}_j := (\overrightarrow{x})t_j\}_{j \in \|m_i\|} \\ & \overset{\mathscr{E}1}{\equiv} \quad \mathrm{M}_i\big[\mathrm{N}_1[\overrightarrow{x}], \ldots, \mathrm{N}_{m_i}[\overrightarrow{x}]\big] \{\mathrm{M}_i := (\overrightarrow{y}_i)\mathrm{M}_i[\overrightarrow{y}_i]\} \\ & \qquad\qquad \{\mathrm{N}_j := (\overrightarrow{x})t_j\}_{j \in \|m_i\|} \\ & = \quad \mathrm{M}_i[t_1, \ldots, t_{m_i}] \quad . \end{aligned}$$

- For $f : (n_1, \ldots, n_l) \to (j)$ in $\mathscr{M}$, the image of

$$\big\langle \big[\omega_f\big((\overrightarrow{y_1})t_1, \ldots, (\overrightarrow{y_l})t_l, s_1, \ldots, s_j\big)\big]_{\mathfrak{E}(M)} \big\rangle : (m_1, \ldots, m_k) \to (n)$$

under $\mu_M \circ \bar{\mu}_M$ is the single tuple containing the equivalence class of the term

$$\begin{aligned}
\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \quad \vdash \quad & \mathrm{t}_{(0)^n \Rightarrow \left(\oplus_j \circ (f \times (0)^j)\right)} \{\mathrm{N}_p := (\overrightarrow{y}_p) t_p\}_{p \in \|l\|} \\
& \qquad \{\mathrm{N}'_q := (\overrightarrow{x}) s_q\}_{q \in \|j\|} \\
\equiv \quad & \mathrm{t}_f \{z_i := \mathrm{N}'_i[x_1, \ldots, x_n]\}_{i \in \|j\|} \\
& \{\mathrm{N}_p := (\overrightarrow{y}_p) t_p\}_{p \in \|l\|} \{\mathrm{N}'_q := (\overrightarrow{x}) s_q\}_{q \in \|j\|} \\
= \quad & \omega_f \left((\overrightarrow{y_1}) \mathrm{N}_1[\overrightarrow{y_1}], \ldots, (\overrightarrow{y_l}) \mathrm{N}_l[\overrightarrow{y_l}], z_1, \ldots, z_j \right) \\
& \{z_i := \mathrm{N}'_i[x_1, \ldots, x_n]\}_{i \in \|j\|} \\
& \{\mathrm{N}_p := (\overrightarrow{y}_p) t_p\}_{p \in \|l\|} \{\mathrm{N}'_q := (\overrightarrow{x}) s_q\}_{q \in \|j\|} \\
= \quad & \omega_f \left((\overrightarrow{y_1}) t_1, \ldots, (\overrightarrow{y_l}) t_l, s_1, \ldots, s_j \right) \quad .
\end{aligned}$$

□

Finally, we jump ahead and point out that we have in fact defined natural isomorphisms

$$\mu_{(-)} : Id_{\mathbf{SOAT}} \to \mathbb{M}(\mathfrak{L}(-)) \qquad \text{and} \qquad \bar{\mu}_{(-)} : \mathbb{M}(\mathfrak{L}(-)) \to Id_{\mathbf{SOAT}}$$

with components at a second-order algebraic theory $M : \mathbb{M} \to \mathscr{M}$ given respectively by the algebraic translations μ_M and $\bar{\mu}_M$ defined in the proof above. We postpone the proof of this naturality to the next chapter, where functoriality of $\mathbb{M}(-)$ and $\mathfrak{L}(-)$ will be established by defining syntactic translations of internal languages as the image of algebraic translations.

Chapter 6

Second-Order Syntactic Translations

Algebraic theories come with an associated notion of algebraic translation, their morphisms. In the second-order universe, the syntactic morphism counterpart has yet to be formalised. Abstractly, comparison of equational presentations could be provided via the algebraic translation between the corresponding classifying algebraic theories. However, as is often the case for the computer scientist, an explicit transformation at the syntactic level may be preferable.

In this chapter, we distill a notion of second-order syntactic translation between second-order equational presentations that corresponds to the canonical notion of morphism between second-order algebraic theories. These syntactic translations provide a mathematical formalisation of notions such as encodings and transforms. The correctness of our definition is once again established by showing a categorical equivalence between algebraic and syntactic translations. This completes the *Second-Order Syntactic Categorical Type Theory Correspondence*, by which second-order algebraic theories and their algebraic translations correspond to second-order equational presentations and their syntactic translations.

We start by defining syntactic translations of second-order signatures (Section 6.1) and second-order equational presentations (Section 6.2). The explicit way of going from an algebraic to a syntactic translation, and back, is demonstrated in Section 6.3. We conclude by showing that these transformations between algebraic and syntactic translations provide an equivalence (Section 6.4).

This work involves a high attention to detail to develop the underlying syntactic machinery. We stress that the correctness of the notions introduced here can only be established and understood via a magnified look at the rigorous, yet subtle, technicalities.

6.1 Second-Order Signature Translations

We introduce the canonical notion of morphism between second-order signatures via second-order syntactic translations.

6.1.1 Signature translations

A *syntactic translation* $\tau\colon \Sigma \to \Sigma'$ between second-order signatures is given by a mapping from the operators of Σ to the terms of Σ' as follows:

$$\omega\colon (m_1,\ldots,m_k) \quad \mapsto \quad \mathrm{M}_1\colon [m_1],\ldots,\mathrm{M}_k\colon [m_k] \rhd - \vdash \tau_\omega$$

Note that the term associated to an operator has an empty variable context and that the metavariable context is determined by the arity of the operator.

6.1.2 Extended translation on terms

A second-order syntactic translation $\tau\colon \Sigma \to \Sigma'$ extends to a mapping from the terms of Σ to the terms of Σ'

$$\begin{array}{rcl} T_\Sigma & \to & T_{\Sigma'} \\ \Theta \rhd \Gamma \vdash t & \mapsto & \Theta \rhd \Gamma \vdash \tau(t) \end{array}$$

according to the following definition by induction on term structure:

- $\tau(x) = x$
- $\tau\big(\mathrm{M}[t_1,\ldots,t_m]\big) = \mathrm{M}\big[\tau(t_1),\ldots,\tau(t_m)\big]$
- $\tau\big(\omega\big((x_1^{(1)},\ldots,x_{n_1}^{(1)})t_1,\ldots(x_1^{(k)},\ldots,x_{n_k}^{(k)})t_k\big)\big) = \tau_\omega\big\{\mathrm{M}_i := (x_1^{(i)},\ldots,x_{n_i}^{(i)})\tau(t_i)\big\}_{i\in\|k\|}$

We refer to this mapping as the *translation extension* or the induced *translation of terms*.

Substituting for variables in a term followed by syntactic translation of the resulting term amounts to the same as term translation followed by substitution, and similarly for metasubstitution. This subtlety is crucial when defining morphisms of signatures as syntactic translations.

Lemma 6.1 (Compositionality). *The extension of a syntactic translation between second-order signatures commutes with substitution and metasubstitution.*

Proof. See Appendix 6.A1 for a detailed syntactic proof. ☐

To familiarise the reader with these syntactic definitions, we provide examples of second-order signature translations.

Example 6.2.

(1) The simplest way to translate a second-order signature is to map it to itself. Every operator can be mapped to the 'simplest' term induced by that operator. More formally, for Σ a second-order signature, the mapping

$$\omega\colon n_1,\ldots,n_l \quad \mapsto \quad \mathrm{N}_1\colon[n_1],\ldots,\mathrm{N}_l\colon[n_l] \rhd - \vdash \omega\big((\overrightarrow{y}_1)\mathrm{N}_1[\overrightarrow{y}_1],\ldots,(\overrightarrow{y}_n)\mathrm{N}_l[\overrightarrow{y}_l]\big)$$

defines a second-order syntactic translation. We will later show that this defines the identity *syntactic translation.*

(2) It is well-known that the basic mono-sorted λ-calculus may be used to model simple arithmetic structures and operations. For instance, Church numerals *are a way of formalising natural numbers via the λ-calculus. The Church numeral n is roughly a function which takes a function f as argument and returns the n-th composition of f. The encoding of basic operations on natural numbers, such as addition and multiplication, via Church's λ-calculus can be formalised as a syntactic translation as follows:*

$$\begin{aligned} \mathtt{add}\colon(0,0) \quad &\mapsto \quad \mathrm{M}\colon[0],\mathrm{N}\colon[0] \rhd - \vdash \lambda f\,x.\mathrm{M}f(\mathrm{N}f\,x) \\ \mathtt{mult}\colon(0,0) \quad &\mapsto \quad \mathrm{M}\colon[0],\mathrm{N}\colon[0] \rhd - \vdash \lambda f.\mathrm{M}(\mathrm{N}f) \end{aligned}$$

(3) For a more concrete example, consider the Continuation Passing Style (CPS) transform [Plotkin, 1998]. A formalisation of the CPS transform for the λ-calculus can be given via a syntactic translation. We provide it in informal notation for ease of readability.

$$\begin{aligned} \mathtt{app}\colon(0,0) \quad &\mapsto \quad \mathrm{M}:[0],\mathrm{N}\colon[0] \rhd - \vdash \lambda k.\mathrm{M}[]\big(\lambda m.m(\lambda l.\mathrm{N}[]l)k\big) \\ \mathtt{abs}\colon(1) \quad &\mapsto \quad \mathrm{F}\colon[1] \rhd - \vdash \lambda k.k\big(\lambda x.(\lambda l.\mathrm{F}[x]l)\big) \end{aligned}$$

6.2 Second-Order Equational Translations

6.2.1 Equational translations

A *syntactic translation* $\tau\colon\mathscr{E}\to\mathscr{E}'$ between second-order equational presentations $\mathscr{E}=(\Sigma,E)$ and $\mathscr{E}'=(\Sigma',E')$ is a signature translation which preserves the equational theory of $\mathscr{E}$ in the sense that axioms are mapped to theorems. Formally, it is a syntactic translation $\tau\colon\Sigma\to\Sigma'$ such that, for every axiom $\Theta\rhd\Gamma\vdash_{\mathscr{E}} s\equiv t$ in E, the judgement $\Theta\rhd\Gamma\vdash_{\mathscr{E}'}\tau(s)\equiv\tau(t)$ is derivable from E'.

The condition that only axioms are required to be mapped to theorems is strong enough to ensure that all theorems of $\mathscr{E}$ are also mapped to theorems of $\mathscr{E}'$, as shown by the next Lemma.

Lemma 6.3. *The extension of a syntactic translation between second-order equational presentations preserves second-order equational derivability.*

Proof. One needs to only check the extended metasubstitution derivation rule of Second-Order Equational Logic (Figure 4.1). Indeed, having

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \Gamma \vdash_{\mathscr{E}'} \tau(s) \equiv \tau(t) \quad \text{and} \quad \Theta \rhd \Gamma', x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash_{\mathscr{E}'} \tau(s_i) \equiv \tau(t_i) \quad (1 \leq i \leq k)$$

implies

$$\Theta \rhd \Gamma, \Gamma' \vdash_{\mathscr{E}'} \tau(s)\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau(s_i)\}_{i \in \|k\|} \equiv \tau(t)\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau(t_i)\}_{i \in \|k\|}$$

by extended metasubstitution, which, by the Compositionality Lemma (Lemma 6.1), further gives

$$\Theta \rhd \Gamma, \Gamma' \vdash_{\mathscr{E}'} \tau\big(s\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau(s_i)\}_{i \in \|k\|}\big) \equiv \tau\big(t\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau(s_i)\}_{i \in \|k\|}\big) \quad .$$

□

6.2.2 The category of second-order equational presentations

Syntactic translation composition. The *composite* of equational translations $\tau : \mathscr{E}_1 \to \mathscr{E}_2$ and $\tau' : \mathscr{E}_2 \to \mathscr{E}_3$ is the translation $(\tau' \circ \tau) : \mathscr{E}_1 \to \mathscr{E}_3$ defined by mapping an operator ω of $\mathscr{E}_1$ to the term $\tau'(\tau_\omega)$ of $\mathscr{E}_3$. Its extension on a term t is simply $\tau'(\tau(t))$, which can be verified by structural induction.

- $(\tau' \circ \tau)(x) \;=\; x \;=\; \tau'\big(\tau(x)\big)$

- $$\begin{aligned}(\tau' \circ \tau)\big(\mathrm{M}[t_1, \ldots, t_m]\big) &= \mathrm{M}\big[(\tau' \circ \tau)(t_1), \ldots, (\tau' \circ \tau)(t_m)\big] \\ &= \mathrm{M}\big[\tau'\big(\tau(t_1)\big), \ldots, \tau'\big(\tau(t_m)\big)\big] \\ &= \tau'\big(\tau\big(\mathrm{M}[t_1, \ldots, t_m]\big)\big)\end{aligned}$$

- $$\begin{aligned}(\tau' \circ \tau)\big(\omega\big(\ldots, (x_1^{(i)}, \ldots, x_{m_i}^{(i)})t_i, \ldots\big)\big) &= (\tau' \circ \tau)_\omega\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})(\tau' \circ \tau)(t_i)\}_{i \in \|k\|} \\ &= \tau'(\tau_\omega)\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau'\big(\tau(t_i)\big)\}_{i \in \|k\|} \\ &= \tau'\big(\tau_\omega\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau(t_i)\}_{i \in \|k\|}\big) \\ &= \tau'\big(\tau\big(\omega\big(\ldots, (x_1^{(i)}, \ldots, x_{m_i}^{(i)})t_i, \ldots\big)\big)\big)\end{aligned}$$

Because τ and τ' preserve equational derivability, the equation $\Theta \rhd \Gamma \vdash \tau'\big(\tau(s)\big) \equiv \tau'\big(\tau(t)\big)$ is a theorem of $\mathscr{E}_3$ whenever $\Theta \rhd \Gamma \vdash s \equiv t$ is an axiom of $\mathscr{E}_1$, and thus, the composite $(\tau' \circ \tau)$ is an equational translation.

Furthermore, composition of equational translations is an associative operation:

$$\big((\tau'' \circ \tau') \circ \tau\big)(\omega) \;=\; (\tau'' \circ \tau')(\tau_\omega) \;=\; \tau''\big(\tau'(\tau_\omega)\big) \;=\; \tau''\big((\tau' \circ \tau)(\omega)\big) \;=\; \big(\tau'' \circ (\tau' \circ \tau)\big)(\omega) \quad ,$$

where of course all composites above are assumed to be well-defined.

The syntactic identity translation. For a second-order equational presentations $\mathscr{E}$, the *syntactic identity translation* $\tau^{\mathscr{E}} : \mathscr{E} \to \mathscr{E}$ is defined by mapping an operator $\omega : (m_1, \ldots, m_k)$ to the term

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd - \vdash \omega\big((x_1^{(1)}, \ldots, x_{m_1}^{(1)})\mathrm{M}_1[x_1^{(1)}, \ldots, x_{m_1}^{(1)}], \ldots, (x_1^{(k)}, \ldots, x_{m_k}^{(k)})\mathrm{M}_1[x_1^{(k)}, \ldots, x_{m_k}^{(k)}]\big) \quad .$$

The extension of $\tau^{\mathscr{E}}$ on terms is just the identity mapping, which is easily verified by structural induction:

- $\tau^{\mathscr{E}}(x) \;=\; x$
- $\tau^{\mathscr{E}}(\mathrm{M}[t_1, \ldots, t_m]) \;=\; \mathrm{M}[\tau(t_1), \ldots, \tau(t_m)] \;=\; \mathrm{M}[t_1, \ldots, t_m]$
- $\tau^{\mathscr{E}}\big(\omega\big(\ldots, (\overrightarrow{x_i})t_i, \ldots\big)\big) \;=\; \tau^{\mathscr{E}}_\omega\{\mathrm{M}_i := (\overrightarrow{x_i})\tau^{\mathscr{E}}(t_i)\}_{i \in \|k\|}$
  $= \omega\big(\ldots, (\overrightarrow{x_i})\mathrm{M}_i[\overrightarrow{x_i}], \ldots\big)\{\mathrm{M}_i := (\overrightarrow{x_i})t_i\}_{i \in \|k\|}$
  $= \omega\big(\ldots, (\overrightarrow{x_i})t_i, \ldots\big)$

This immediately implies that an axiom $\Theta \rhd \Gamma \vdash_{\mathscr{E}} s \equiv t$ is mapped to itself under $\tau^{\mathscr{E}}$, making it an equational translation.

Note that $\tau^{\mathscr{E}}$ is indeed the identity in the space of equational translations and their composition, since for any $\tau : \mathscr{E}_1 \to \mathscr{E}_2$ and $\omega : (m_1, \ldots, m_k)$ an operator of $\mathscr{E}_1$, we have

$$\tau^{\mathscr{E}_2}\big(\tau(\omega_1)\big) \quad = \quad \tau(\omega_1) \quad ,$$

and

$$\begin{aligned}
\tau\big(\tau^{\mathscr{E}_1}(\omega)\big) \quad &= \quad \tau\big(\omega\big(\ldots, (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\mathrm{M}_i[x_1^{(i)}, \ldots, x_{m_i}^{(i)}], \ldots\big)\big) \\
&= \quad \tau_\omega\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\tau\big(\mathrm{M}_i[x_1^{(i)}, \ldots, x_{m_i}^{(i)}]\big)\}_{i \in \|k\|} \\
&= \quad \tau_\omega\{\mathrm{M}_i := (x_1^{(i)}, \ldots, x_{m_i}^{(i)})\mathrm{M}_i[x_1^{(i)}, \ldots, x_{m_i}^{(i)}]\}_{i \in \|k\|} \\
&= \quad \tau_\omega \quad .
\end{aligned}$$

The category of second-order equational presentations. We denote by **SOEP** the category of second-order equational presentations and second-order syntactic translations. The previous discussion surrounding composition and identity ascertains that this is a well-defined category.

6.3 Syntactic and Algebraic Translations

The notion of syntactic translation between second-order equational presentations introduced above is justified by establishing its equivalence with that of algebraic translation between the associated second-order algebraic theories. With this end in mind, we illustrate how to construct syntactic translations from algebraic translations, and vice versa.

6.3.1 Induced algebraic translations

A syntactic translation $\tau : \mathscr{E} \to \mathscr{E}'$ of second-order equational presentations $\mathscr{E} = (\Sigma, E)$ and $\mathscr{E}' = (\Sigma', E')$ induces the algebraic translation

$$\mathbb{M}(\tau) \colon \mathbb{M}(\mathscr{E}) \to \mathbb{M}(\mathscr{E}')$$

mapping $\langle [t_1]_{\mathscr{E}}, \dots, [t_l]_{\mathscr{E}} \rangle$ to $\langle [\tau(t_1)]_{\mathscr{E}'}, \dots, [\tau(t_l)]_{\mathscr{E}'} \rangle$. Note that the induced algebraic translation $\mathbb{M}(\tau)$ is essentially specified by the extension of the syntactic translation τ on terms. This definition respects equivalence since the extension of τ preserves equational derivability, and thus $\Theta \rhd \Gamma \vdash_{\mathscr{E}} s \equiv t$ implies $\Theta \rhd \Gamma \vdash_{\mathscr{E}'} \tau(s) \equiv \tau(t)$. From the Compositionality Lemma (Lemma 6.1), we know that extensions of syntactic translations commute with substitution and metasubstitution, which easily yields functoriality of $\mathbb{M}(\tau)$. Finally, we point out that, since translation extensions act as the identity on elementary terms, the functor $\mathbb{M}(\tau)$ commutes with the theories $M_{\mathscr{E}} \colon \mathbb{M} \to \mathbb{M}(\mathscr{E})$ and $M_{\mathscr{E}'} \colon \mathbb{M} \to \mathbb{M}(\mathscr{E}')$, making it indeed an algebraic translation.

This development gives a functor

$$\begin{array}{rcl} \mathbb{M}(-) \quad : \quad \mathbf{SOEP} & \to & \mathbf{SOAT} \\ \mathscr{E} & \mapsto & M_{\mathscr{E}} \colon \mathbb{M} \to \mathbb{M}(\mathscr{E}) \\ \tau \colon \mathscr{E} \to \mathscr{E}' & \mapsto & \mathbb{M}(\tau) \colon \mathbb{M}(\mathscr{E}) \to \mathbb{M}(\mathscr{E}') \end{array}$$

mapping an equational presentation to its classifying theory, and a syntactic translation to its induced algebraic translation. Since the extension of the syntactic identity translation $\tau^{\mathscr{E}} \colon \mathscr{E} \to \mathscr{E}$ is the identity on terms, it is mapped under $\mathbb{M}(-)$ to the identity algebraic translation $\mathbb{M}(\tau^{\mathscr{E}})$ mapping $\langle \dots, [t]_{\mathscr{E}}, \dots \rangle$ to itself. Also, given syntactic translations $\tau \colon \mathscr{E}_1 \to \mathscr{E}_2$ and $\tau' \colon \mathscr{E}_2 \to \mathscr{E}_3$, we have

$$\mathbb{M}(\tau' \circ \tau)\big([t]_{\mathscr{E}_1}\big) = \big[(\tau' \circ \tau)(t)\big]_{\mathscr{E}_3} = \big[\tau'\big(\tau(t)\big)\big]_{\mathscr{E}_3} = \mathbb{M}(\tau')\big(\big[\tau(t)\big]_{\mathscr{E}_2}\big) = \big(\mathbb{M}(\tau') \circ \mathbb{M}(\tau)\big)\big([t]_{\mathscr{E}_1}\big) \quad ,$$

which establishes functoriality of $\mathbb{M}(-)$.

6.3.2 Induced syntactic translations

An algebraic translation $F : \mathscr{M} \to \mathscr{M}'$ between second-order algebraic theories $M : \mathbb{M} \to \mathscr{M}$ and $M' : \mathbb{M} \to \mathscr{M}'$ induces the syntactic translation

$$\mathfrak{L}(F) : \mathfrak{L}(M) \to \mathfrak{L}(M') \quad ,$$

which, for a morphism $f : (m_1, \dots, m_k) \to (n)$ of $\mathscr{M}$, maps the operator ω_f of $\mathfrak{L}(M)$ to the term

$$\mathrm{M}_1 : [m_1], \dots, \mathrm{M}_k : [m_k], \mathrm{N}_1 : [0], \dots, \mathrm{N}_n : [0] \rhd - \vdash \mathrm{t}_{Ff} \{x_i := \mathrm{N}_i[]\}_{i \in \|n\|} \quad ,$$

where we recall that

$$\mathrm{t}_{Ff} = \omega_{Ff}\big((x_1^{(1)}, \dots, x_{m_1}^{(1)})\mathrm{M}_1[x_1^{(1)}, \dots, x_{m_1}^{(1)}], \dots, (x_1^{(k)}, \dots, x_{m_k}^{(k)})\mathrm{M}_k[x_1^{(k)}, \dots, x_{m_k}^{(k)}], x_1, \dots, x_n\big) \quad .$$

We verify that $\mathfrak{L}(F)$ is indeed an equational translation by looking at the induced translations on the terms of the left- and right-hand side of the axioms of $\mathfrak{L}(M)$. Recall from Section 5.4.2 that these axioms are given by ($\mathscr{E}1$) and ($\mathscr{E}2$). Consider ($\mathscr{E}1$), which states that for $\langle s \rangle : (m_1, \dots, m_k) \to (n)$ of $\mathbb{M}$, we have the equation $\mathrm{M}_1 : [m_1], \dots, \mathrm{M}_k : [m_k] \rhd x_1, \dots, x_n \vdash s \equiv \mathrm{t}_{M\langle s \rangle}$ in $\mathfrak{L}(M)$. Since s is elementary, its image under the translation $\mathfrak{L}(F)$ is also given by $M\langle s \rangle$. On the other hand, note that $\mathfrak{L}(f)(\mathrm{t}_{M\langle s \rangle}) = \mathrm{t}_{(F \circ M)\langle s \rangle} = \mathrm{t}_{M'\langle s \rangle}$. From the axiom ($\mathscr{E}1$) of $\mathfrak{L}(M')$, we have that $s \equiv \mathrm{t}_{M\langle s \rangle}$, and therefore

$$\mathrm{M}_1 : [m_1], \dots, \mathrm{M}_k : [m_k] \rhd x_1, \dots, x_n \vdash \mathfrak{L}(F)(s) \equiv \mathfrak{L}(F)(\mathrm{t}_{M\langle s \rangle})$$

in $\mathfrak{L}(M')$. Similarly, for the axiom ($\mathscr{E}2$) of $\mathfrak{L}(M)$, and in the notation of Section 5.4.2, we have that $\mathfrak{L}(F)(\mathrm{t}_h) = \mathrm{t}_{Fh}$, and on the other hand:

$$\begin{aligned} & \mathfrak{L}(F)\Big(\mathrm{t}_g\{\mathrm{M}_i := (\overrightarrow{x_i})\mathrm{t}_{f_i}\}_{i \in \|l\|}\Big) \\ = \ & \mathfrak{L}(F)(\mathrm{t}_g)\{\mathrm{M}_i := (\overrightarrow{x_i})\mathfrak{L}(F)(\mathrm{t}_{f_i})\}_{i \in \|l\|} \\ = \ & \mathrm{t}_{Fg}\{\mathrm{M}_i := (\overrightarrow{x_i})\mathrm{t}_{Ff_i}\}_{i \in \|l\|} \quad . \end{aligned}$$

Hence, the image of axiom ($\mathscr{E}2$) of $\mathfrak{L}(M)$ under the translation $\mathfrak{L}(F)$ is just axiom ($\mathscr{E}2$) of $\mathfrak{L}(M')$. This makes $\mathfrak{L}(F)$ indeed an equational translation.

We have essentially defined the functor

$$\begin{array}{rcl} \mathfrak{L}(-) \ : \ \mathbf{SOAT} & \to & \mathbf{SOEP} \\ M : \mathbb{M} \to \mathscr{M} & \mapsto & \mathfrak{L}(M) \\ F : \mathscr{M} \to \mathscr{M}' & \mapsto & \mathfrak{L}(F) : \mathfrak{L}(M) \to \mathfrak{L}(M') \end{array}$$

mapping a second-order algebraic theory to its internal language, and an algebraic translation to its induced syntactic translation.

Some more simple syntactic manipulation is needed to show that $\mathfrak{L}(-)$ is functorial. Given a morphism $f : (m_1, \ldots, m_k) \to (n)$ in $\mathscr{M}$, the translation $\mathfrak{L}(id_{\mathscr{M}})$ maps the operator ω_f of $\mathfrak{L}(M)$ to the term

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k], \mathrm{N}_1 : [], \ldots, \mathrm{N}_n : [] \rhd - \vdash \mathrm{t}_f\{x_i := \mathrm{N}_i[]\}_{i \in \|n\|} \quad ,$$

which is the image of ω_f under the syntactic identity translation $\tau^{\mathfrak{L}(M)}$. Moreover, given algebraic translations $F : \mathscr{M}_1 \to \mathscr{M}_2$ and $G : \mathscr{M}_2 \to \mathscr{M}_3$ and a morphism $g : (n_1, \ldots, n_j) \to (l)$, the image of ω_g of $\mathfrak{L}(M_1)$ under the composite translation $\mathfrak{L}(G) \circ \mathfrak{L}(F)$ is given by the term

$$\begin{aligned} \mathrm{N}_1 : [n_1], \ldots, \mathrm{N}_j : [n_j], \mathrm{L}_1 : [], \ldots, \mathrm{L}_l : [] \rhd - \quad \vdash \quad & \mathfrak{L}(G)\big(\mathrm{t}_{Fg}\{x_i := \mathrm{L}_i[]\}_{i \in \|l\|}\big) \\ = \quad & \mathfrak{L}(G)(\omega_{Fg}) \\ = \quad & \mathrm{t}_{(G \circ F)(g)}\{x_i := \mathrm{L}_i[]\}_{i \in \|l\|} \\ = \quad & \mathfrak{L}(G \circ F)(\omega_g) \quad . \end{aligned}$$

6.4 Second-Order Syntactic categorical Type Theory Correspondence

6.4.1 Second-order presentation/theory correspondence

Second-order syntactic translations embody the mathematical machinery that enables us to compare second-order equational presentations at the syntactic level without having to revert to their categorical counterparts. In particular, the question of when two presentations are *essentially the same* can now be answered via the notion of syntactic translation.

Analogous to the first-order setting (Definition 3.7), a second-order syntactic translation $\tau : \mathscr{E} \to \mathscr{E}'$ is said to be an *isomorphism*, if it has an inverse $\bar{\tau}$ yielding the syntactic identity translation on $\mathscr{E}$ (respectively $\mathscr{E}'$) when composed to the left (respectively right) with τ.

This is used to show the second direction of the invertibility of constructing theories from presentations, and vice versa. More precisely, we prove that every second-order equational presentation is isomorphic to the second-order equational presentation of its associated algebraic theory.

Keeping this objective in mind, define, for a given second-order equational presentation $\mathscr{E}$ with classifying algebraic theory $M_{\mathscr{E}} : \mathbb{M} \to \mathbb{M}(\mathscr{E})$, the *natural translation*

$$\nu_{\mathscr{E}} : \mathscr{E} \to \mathfrak{L}(M_{\mathscr{E}})$$

by mapping an operator $\omega\colon (m_1,\dots,m_k)$ of $\mathscr{E}$ to the term

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd - \vdash \mathrm{t}_{\left\langle [\tau^{\mathscr{E}}_{\omega}]_{\mathscr{E}} \right\rangle} \quad ,$$

where we remind the reader that $\tau^{\mathscr{E}}(\omega)$ is the image of ω under the identity translation $\tau^{\mathscr{E}}$, and hence $\left\langle [\tau^{\mathscr{E}}(\omega)]_{\mathscr{E}} \right\rangle\colon (m_1,\dots,m_k) \to (0)$ is a morphism of $\mathbb{M}(\mathscr{E})$.

The fact that the natural translation $\nu_{\mathscr{E}}$ is an equational translation relies on the following special property of its extension on terms.

Lemma 6.4. *For any second-order equational presentation $\mathscr{E}$, the extension of the natural translation $\nu_{\mathscr{E}}\colon \mathscr{E} \to \mathfrak{L}(M_{\mathscr{E}})$ on a term*

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash s$$

of $\mathscr{E}$ is given by the term

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash \mathrm{t}_{\langle [s]_{\mathscr{E}} \rangle}$$

of $\mathfrak{L}(M_{\mathscr{E}})$.

Proof. A detailed syntactic proof can be found in Appendix 6.A2. □

Given an axiom $\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash t \equiv t'$ of $\mathscr{E}$ then, the operators $\omega_{\langle [t]_{\mathscr{E}} \rangle}$ and $\omega_{\langle [t']_{\mathscr{E}} \rangle}$ are obviously equal, which makes the terms $\mathrm{t}_{\langle [t]_{\mathscr{E}} \rangle}$ and $\mathrm{t}_{\langle [t']_{\mathscr{E}} \rangle}$ of $\mathfrak{L}(M_{\mathscr{E}})$ syntactically equal. This implies the equational derivability of

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash_{\mathfrak{L}(M_{\mathscr{E}})} \mathrm{t}_{\langle [t]_{\mathscr{E}} \rangle} \equiv \mathrm{t}_{\langle [t']_{\mathscr{E}} \rangle} \quad ,$$

which, together with Lemma 6.4, yields

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k] \rhd x_1,\dots,x_n \vdash_{\mathfrak{L}(M_{\mathscr{E}})} \nu_{\mathscr{E}}(t) \equiv \nu_{\mathscr{E}}(t') \quad ,$$

making $\nu_{\mathscr{E}}$ indeed an equational translation.

In the other direction, define the *opposite natural translation*

$$\bar{\nu}_{\mathscr{E}}\colon \mathfrak{L}(M_{\mathscr{E}}) \to \mathscr{E}$$

by mapping, for a morphism $\langle [t]_{\mathscr{E}} \rangle\colon (m_1,\dots,m_k) \to (n)$ of $\mathbb{M}(\mathscr{E})$, the operator $\omega_{\langle [t]_{\mathscr{E}} \rangle}\colon (m_1,\dots,m_k,0^n)$ to

$$\mathrm{M}_1\colon [m_1],\dots,\mathrm{M}_k\colon [m_k],\mathrm{N}_1\colon [0],\dots,\mathrm{N}_n\colon [0] \rhd - \vdash t\{x_i := \mathrm{N}_i[\,]\}_{i\in\|n\|} \quad .$$

We point out that this mapping is well-defined in the sense that it respects the equivalence with respect to $\mathscr{E}$, as from Second-Order Equational Logic we know that the operation of substitution in extended metavariable context is a congruence.

To verify that, according to this definition, $\bar{\nu}_{\mathscr{E}}$ is really an equational translation, one needs to show that the two axioms ($\mathscr{E}1$) and ($\mathscr{E}2$) of $\mathfrak{E}(M_{\mathscr{E}})$ are mapped under $\bar{\nu}_{\mathscr{E}}$ to theorems of $\mathscr{E}$. A similar argument to the verification of the preservation of equations of an induced syntactic translation (Section 6.3.2) can be used, and so we skip over the details here.

Theorem 6.5 (Second-order presentation/theory correspondence)**.** *Every second-order equational presentation $\mathscr{E}$ is isomorphic to the second-order equational presentation $\mathfrak{E}(M_{\mathscr{E}})$ of its associated algebraic theory $M_{\mathscr{E}} : \mathbb{M} \to \mathbb{M}(\mathscr{E})$.*

Proof. As anticipated, the isomorphism is witnessed by the natural translation $\nu_{\mathscr{E}} : \mathscr{E} \to \mathfrak{E}(M_{\mathscr{E}})$ with its inverse given by the opposite natural translation $\bar{\nu}_{\mathscr{E}} : \mathfrak{E}(M_{\mathscr{E}})$. Indeed, an operator $\omega : (m_1, \ldots, m_k)$ of $\mathscr{E}$ is mapped under the composite $\bar{\nu}_{\mathscr{E}} \circ \nu_{\mathscr{E}}$ to

$$\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd - \vdash \bar{\nu}_{\mathscr{E}}\big(\omega_{\langle[\tau^{\mathscr{E}}(\omega)]_{\mathscr{E}}\rangle}\big) \;=\; \tau^{\mathscr{E}}(\omega) \quad .$$

In the other direction, for a morphism $\langle [s]_{\mathscr{E}} \rangle : (m_1, \ldots, m_k) \to (n)$ of $\mathbb{M}(\mathscr{E})$, the operator $\omega_{\langle [s]_{\mathscr{E}} \rangle}$ is mapped under $\nu_{\mathscr{E}} \circ \bar{\nu}_{\mathscr{E}}$ to

$$\begin{array}{rcl} \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k], \mathrm{N}_1 : [0], \ldots, \mathrm{N}_n : [0] \rhd - & \vdash & \nu_{\mathscr{E}}\big(s\{x_i := \mathrm{N}_i[\,]\}_{i \in \|n\|}\big) \\ & = & \nu_{\mathscr{E}}(s)\{x_i := \nu_{\mathscr{E}}(\mathrm{N}_i[\,])\}_{i \in \|n\|} \\ & = & \nu_{\mathscr{E}}(s)\{x_i := \mathrm{N}_i[\,]\}_{i \in \|n\|} \\ & = & \mathrm{t}_{\langle [s]_{\mathscr{E}} \rangle}\{x_i := \mathrm{N}_i[\,]\}_{i \in \|n\|} \\ & = & \tau^{\mathfrak{E}(M_{\mathscr{E}})}(\omega_{\langle [s]_{\mathscr{E}} \rangle}) \quad . \end{array}$$

$\square$

6.4.2 Syntactic/algebraic translation correspondence

The constructions of induced algebraic and syntactic translations are shown to be mutually inverse, thereby establishing them as the correct notions of morphisms of, respectively, algebraic theories and equational presentations. This equivalence is one of the main results of this dissertation, namely the *Second-Order Syntactic Categorical Type Theory Correspondence*.

Theorem 6.6 (Second-Order Syntactic Categorical Type Theory Correspondence)**.** *The categories* **SOAT** *and* **SOEP** *are equivalent.*

Proof. The equivalence is given by the functors

$$\boldsymbol{\mathfrak{L}}(-)\colon \mathbf{SOAT} \to \mathbf{SOEP} \qquad \text{and} \qquad \mathbb{M}(-)\colon \mathbf{SOEP} \to \mathbf{SOAT}$$

together with the natural transformation $\mu\colon \mathrm{Id}_{\mathbf{SOAT}} \to \mathbb{M}(\boldsymbol{\mathfrak{L}}(-))$ with component at a second-order algebraic theory $M\colon \mathbb{M} \to \mathscr{M}$ given by the isomorphism

$$\mu_M\colon \mathscr{M} \to \mathbb{M}(\boldsymbol{\mathfrak{L}}(M))$$

defining the Theory/Presentation Correspondence of Theorem 5.8, and also the natural transformation $\nu\colon \mathrm{Id}_{\mathbf{SOEP}} \to \boldsymbol{\mathfrak{L}}(\mathbb{M}(-))$ with component at a second-order equational presentation $\mathscr{E} = (\Sigma, E)$ given by the isomorphism

$$\nu_{\mathscr{E}}\colon \mathscr{E} \to \boldsymbol{\mathfrak{L}}(M_{\mathscr{E}})$$

defining the Presentation/Theory Correspondence of Theorem 6.5. From the very definitions of the functors $\mathbb{M}(-)$ and $\boldsymbol{\mathfrak{L}}(-)$ and the isomorphisms $\mu_{(-)}$ and $\nu_{(-)}$, the diagrams

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\;F\;} & \mathscr{M}' \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \mu_{M'}} \\ \mathbb{M}(\boldsymbol{\mathfrak{L}}(M)) & \xrightarrow{\mathbb{M}(\boldsymbol{\mathfrak{L}}(F))} & \mathbb{M}(\boldsymbol{\mathfrak{L}}(M')) \end{array} \qquad \begin{array}{ccc} \mathscr{E} & \xrightarrow{\;\tau\;} & \mathscr{E}' \\ {\scriptstyle \nu_{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle \nu_{\mathscr{E}'}} \\ \boldsymbol{\mathfrak{L}}(M_{\mathscr{E}}) & \xrightarrow{\boldsymbol{\mathfrak{L}}(\mathbb{M}(\tau))} & \boldsymbol{\mathfrak{L}}(M_{\mathscr{E}'}) \end{array}$$

commute for any second-order algebraic translation F between algebraic theories $M\colon \mathbb{M} \to \mathscr{M}$ and $M'\colon \mathbb{M} \to \mathscr{M}'$, and any second-order syntactic translation $\tau\colon \mathscr{E} \to \mathscr{E}'$ of equational presentations $\mathscr{E}$ and $\mathscr{E}'$, thereby establishing naturality of μ and ν.

Consider the diagram above on the left; its commutativity is trivial on the objects of $\mathscr{M}$. Given a morphism $f : (m_1, \ldots, m_k) \to (n)$ of $\mathscr{M}$, its image under $\mu_{M'} \circ F$ is the morphism

$$\big\langle \big[\mathrm{t}_{Ff} \big]_{\boldsymbol{\mathfrak{L}}(M)} \big\rangle : (m_1, \ldots, m_k) \to (n) \quad .$$

Going the other way, the image of f under $\mathbb{M}(\boldsymbol{\mathfrak{L}}(F)) \circ \mu_M$ is given by

$$\begin{aligned} & \mathbb{M}(\boldsymbol{\mathfrak{L}}(F))\big\langle \big[\mathrm{t}_f \big]_{\boldsymbol{\mathfrak{L}}(M)} \big\rangle \\ = \;& \big\langle \big[\boldsymbol{\mathfrak{L}}(F)\big(\mathrm{t}_f\big) \big]_{\boldsymbol{\mathfrak{L}}(M)} \big\rangle \\ = \;& \big\langle \big[\boldsymbol{\mathfrak{L}}(F)(\omega_f)\{\mathrm{N}_i := x_i\}_{i \in \|n\|} \big]_{\boldsymbol{\mathfrak{L}}(M)} \big\rangle \\ = \;& \big\langle \big[\mathrm{t}_{Ff} \big]_{\boldsymbol{\mathfrak{L}}(M)} \big\rangle \quad . \end{aligned}$$

To verify the commutativity of the diagram to the right, note that the image of an operator $\omega\colon n_1, \ldots, n_l$

of $\mathscr{E}$ under the composite $\nu_{\mathscr{E}'} \circ \tau$ is the term

$$\mathrm{N}_1 : [n_1], \ldots, \mathrm{N}_l : [n_l] \rhd - \vdash \mathrm{t}_{\langle [\tau(\omega)]_{\mathscr{E}'} \rangle} \quad .$$

On the other hand, the image of ω under $\mathfrak{L}(\mathbb{M}(\tau)) \circ \nu_{\mathscr{E}}$ is given by

$$\begin{aligned} & \mathfrak{L}(\mathbb{M}(\tau))\big(\mathrm{t}_{\langle [\mathrm{t}_\omega]_{\mathscr{E}} \rangle}\big) \\ = \quad & \mathrm{t}_{\mathbb{M}(\tau)\langle [\mathrm{t}_\omega]_{\mathscr{E}} \rangle} \\ = \quad & \mathrm{t}_{\langle [\tau(\mathrm{t}_\omega)]_{\mathscr{E}'} \rangle} \\ = \quad & \mathrm{t}_{\langle [\tau(\omega)]_{\mathscr{E}'} \rangle} \quad . \end{aligned}$$

□

6.A Appendix to Chapter 6

6.A1 Proof of Compositionality Lemma

For compositionality with substitution, we show by structural induction on terms t that

$$\tau\big(t\{x_i := t_i\}_{i\in\|n\|}\big) \quad = \quad \tau(t)\big\{x_i := \tau(t_i)\big\}_{i\in\|n\|} \quad .$$

$$\begin{array}{ll}
- & \tau\big(x_j\{x_i := t_i\}_{i\in\|n\|}\big) \\
= & \tau(t_j) \\
= & x_j\{x_i := \tau(t_i)\}_{i\in\|n\|} \\
= & \tau(x_j)\{x_i := \tau(t_i)\}_{i\in\|n\|}, \quad \text{for } j \in \|n\|.
\end{array}$$

$$\begin{array}{ll}
- & \tau\big(\mathrm{M}[s_1,\ldots,s_m]\big\{x_i := t_i\big\}_{i\in\|n\|}\big) \\
= & \tau\big(\mathrm{M}\big[s_1\big\{x_i := t_i\big\}_{i\in\|n\|},\ldots,s_m\big\{x_i := t_i\big\}_{i\in\|n\|}\big]\big) \\
= & \mathrm{M}\big[\tau\big(s_1\big\{x_i := t_i\big\}_{i\in\|n\|}\big),\ldots,\tau\big(s_m\big\{x_i := t_i\big\}_{i\in\|n\|}\big)\big] \\
= & \mathrm{M}\big[\tau(s_1)\big\{x_i := t_i\big\}_{i\in\|n\|},\ldots,\tau(s_m)\big\{x_i := t_i\big\}_{i\in\|n\|}\big] \\
= & \mathrm{M}\big[\tau(s_1),\ldots,\tau(s_m)\big]\big\{x_i := t_i\big\}_{i\in\|n\|} \\
= & \tau\big(\mathrm{M}[s_1,\ldots,s_m]\big)\big\{x_i := t_i\big\}_{i\in\|n\|}
\end{array}$$

$$\begin{array}{ll}
- & \tau\big(\omega\big(\ldots,(y_1,\ldots,y_k)s,\ldots\big)\big\{x_i := t_i\big\}_{i\in\|n\|}\big) \\
= & \tau\big(\omega\big(\ldots,(z_1,\ldots,z_k)s\big\{x_i := t_i\big\}_{i\in\|n\|}\big\{y_j := z_j\big\}_{j\in\|k\|},\ldots\big)\big) \\
= & \tau_\omega\Big\{\mathrm{M} := (z_1,\ldots,z_k)\tau\big(s\big\{x_i := t_i\big\}_{i\in\|n\|}\big\{y_j := z_j\big\}_{j\in\|k\|}\big)\Big\} \\
= & \tau_\omega\Big\{\mathrm{M} := (z_1,\ldots,z_k)\tau(s)\big\{x_i := t_i\big\}_{i\in\|n\|}\big\{y_j := z_j\big\}_{j\in\|k\|}\Big\} \\
\stackrel{\alpha}{=} & \tau_\omega\Big\{\mathrm{M} := (y_1,\ldots,y_k)\tau(s)\big\{x_i := t_i\big\}_{i\in\|n\|}\Big\} \\
= & \tau\big(\omega\big(\ldots,(y_1,\ldots,y_k)s,\ldots\big)\big)\big\{x_i := t_i\big\}_{i\in\|n\|}
\end{array}$$

Similarly, for compositionality with metasubstitution, we show by induction on the structure of terms t that

$$\tau\big(t\big\{\mathrm{M}_i := (x_1^{(i)},\ldots,x_{k_i}^{(i)})t_i\big\}_{i\in\|n\|}\big) \quad = \quad \tau(t)\big\{\mathrm{M}_i := (x_1^{(i)},\ldots,x_{k_i}^{(i)})\tau(t_i)\big\}_{i\in\|n\|} \quad .$$

$$\begin{array}{ll}
- & \tau\big(x\big\{\mathrm{M}_i := (x_1^{(i)},\ldots,x_{k_i}^{(i)})t_i\big\}_{i\in\|n\|}\big) \\
= & \tau(x)
\end{array}$$

$$
\begin{aligned}
&= x\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\\
&= \tau(x)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|}
\end{aligned}
$$

- $\tau\big(\mathrm{M}_j[s_1,\dots,s_{m_j}]\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|}\big)$

$$
\begin{aligned}
&= \tau\Big(t_j\Big\{x_l^{(j)} := s_l\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|}\Big\}_{l\in\|m_j\|}\Big)\\
&= \tau(t_j)\Big\{x_j^{(j)} := \tau\big(s_l\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|}\big)\Big\}_{l\in\|m_j\|}\\
&= \tau(t_j)\Big\{x_j^{(j)} := \tau(s_l)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\Big\}_{l\in\|m_j\|}\\
&= \mathrm{M}_j\big[\tau(s_1),\dots,\tau(s_{m_j})\big]\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\\
&= \tau\big(\mathrm{M}_j[s_1,\dots,s_{m_j}]\big)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}
\end{aligned}
$$

- $\tau\big(\omega(\dots,(y_1,\dots,y_m)s,\dots)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|}\big)$

$$
\begin{aligned}
&= \tau\big(\omega(\dots,(y_1,\dots,y_m)s\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})t_i\}_{i\in\|n\|},\dots)\big)\\
&= \tau_\omega\Big\{\mathrm{N} := (y_1,\dots,y_m)\tau\big(s\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\big)\Big\}\\
&= \tau_\omega\Big\{\mathrm{N} := (y_1,\dots,y_m)\tau(s)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\Big\}\\
&= \tau_\omega\{\mathrm{N} := (y_1,\dots,y_m)\tau(s)\}\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}\\
&= \tau\big(\omega(\dots,(y_1,\dots,y_m)s,\dots)\big)\{\mathrm{M}_i := (x_1^{(i)},\dots,x_{k_i}^{(i)})\tau(t_i)\}_{i\in\|n\|}
\end{aligned}
$$

6.A2 Proof of Lemma 6.4

We proceed by induction on the structure of the term s.

- $\nu_{\mathscr{E}}(x_i)$

$$
\begin{aligned}
&= x_i\\
&\overset{(\mathscr{E}1)}{=} \mathrm{t}_{M_{\mathscr{E}}\langle x_i\rangle}\\
&= \mathrm{t}_{\langle [x_i]_{\mathscr{E}}\rangle}
\end{aligned}
$$

- $\nu_{\mathscr{E}}\big(\mathrm{M}_i[t_1,\dots,t_{m_i}]\big)$

$$
\begin{aligned}
&= \nu_{\mathscr{E}}\big(\mathrm{M}_i\big[\mathrm{N}_1[x_1,\dots,x_n],\dots,\mathrm{N}_{m_i}[x_1,\dots,x_n]\big]\{\mathrm{M}_i := (y_1,\dots,y_{m_i})\mathrm{M}_i[y_1,\dots,y_{m_i}]\}\\
&\qquad \{\mathrm{N}_j := (x_1,\dots,x_n)t_j\}_{j\in\|m_i\|}\big)\\
&= \nu_{\mathscr{E}}\big(\mathrm{M}_i\big[\mathrm{N}_1[x_1,\dots,x_n],\dots,\mathrm{N}_{m_i}[x_1,\dots,x_n]\big]\big)\{\mathrm{M}_i := (y_1,\dots,y_{m_i})\nu_{\mathscr{E}}\big(\mathrm{M}_i[y_1,\dots,y_{m_i}]\big)\}\\
&\qquad \{\mathrm{N}_j := (x_1,\dots,x_n)\nu_{\mathscr{E}}(t_j)\}_{j\in\|m_i\|}\\
&= \mathrm{M}_i\big[\mathrm{N}_1[x_1,\dots,x_n],\dots,\mathrm{N}_{m_i}[x_1,\dots,x_n]\big]\{\mathrm{M}_i := (y_1,\dots,y_{m_i})\mathrm{M}_i[y_1,\dots,y_{m_i}]\}
\end{aligned}
$$

$$\{\mathrm{N}_j := (x_1, \dots, x_n)\nu_{\mathscr{E}}(t_j)\}_{j \in \|m_i\|}$$

$$\overset{(\mathscr{E}1)}{=} \mathsf{t}_{\langle [t^*]_{\mathscr{E}} \rangle}\{\mathrm{N}_j := (x_1, \dots, x_n)\mathsf{t}_{\langle [t_j]_{\mathscr{E}} \rangle}\}_{j \in \|m_i\|}$$

$$\text{(for } t^* = \mathrm{M}_i[\mathrm{N}_1[x_1, \dots, x_n], \dots, \mathrm{N}_{m_i}[x_1, \dots, x_n]])$$

$$\overset{(\mathscr{E}2)}{=} \mathsf{t}_{\langle [\mathrm{M}_i[t_1, \dots, t_{m_i}]]_{\mathscr{E}} \rangle}$$

- $$\nu_{\mathscr{E}}(\omega((\overrightarrow{y}_1)t_1, \dots, (\overrightarrow{y_l})t_l))$$

$$= \mathsf{t}_{\langle [\tau^{\mathscr{E}}_{\omega}]_{\mathscr{E}} \rangle}\{\mathrm{N}_i := (\overrightarrow{y_i})\nu_{\mathscr{E}}(t_i)\}_{i \in \|l\|}$$

$$= \mathsf{t}_{\langle [\tau^{\mathscr{E}}_{\omega}]_{\mathscr{E}} \rangle}\{\mathrm{N}_i := (\overrightarrow{y_i})\mathsf{t}_{\langle [t_i]_{\mathscr{E}} \rangle}\}_{i \in \|l\|}$$

$$\overset{(\mathscr{E}2)}{=} \mathsf{t}_{\langle [\tau^{\mathscr{E}}_{\omega}]_{\mathscr{E}} \rangle \circ \langle [t_1]_{\mathscr{E}}, \dots, [t_l]_{\mathscr{E}} \rangle}$$

$$= \mathsf{t}_{\langle [\omega((\overrightarrow{y}_1)t_1, \dots, (\overrightarrow{y_l})t_l)]_{\mathscr{E}} \rangle}$$

Chapter 7

Second-Order Functorial Semantics

The main objective of universal algebra is the formalisation of algebraic structures and their models. Lawvere's seminal thesis gives a categorical presentation of the notion of algebraic model. We show that his functorial semantics for algebraic theories admits generalisation to the second-order universe, in which a *second-order (set-theoretic) functorial model* of a second-order algebraic theory is given in terms of a suitable functor from the algebraic theory to **Set**.

This constitutes the essence of this chapter. Having shown the syntactic correctness of the definition of second-order algebraic theory, we establish its semantic correctness, by which Second-Order Functorial Semantics is shown to correspond to the set-theoretic model-theory of second-order universal algebra.

We start by recalling the theory of clones from classical universal algebra (Section 7.1). Clone structures abstractly describe second-order algebraic structures and will be shown to provide semantics to second-order equational presentations equivalent to that of second-order set-theoretic models. In Section 7.2, we show that exponentiable objects induce clones, and use this to introduce a notion of classifying clone for classifying second-order algebraic theories. The formal definition of second-order functorial models is given in Section 7.3. In line with one of the main themes throughout this dissertation, we explicitly describe the transition from classifying clones of equational presentations (and thereby set-theoretic algebras) to abstract functorial models of their classifying algebraic theories, and vice versa (Section 7.4). We then show that these constructions are mutually inverse, thereby establishing the *Second-Order Semantic Categorical Type Theory Correspondence*.

We conclude by using the theory of functorial semantics to provide a different point of view on the theory of syntactic translations (Section 7.5). As algebraic translations are essentially functorial models, we show that syntactic translations can be thought of as syntactic models of equational presentations. We refer to this development as *Translational Semantics*.

7.1 Clone Structures

We recall and develop some aspects of the theory of *clones* from universal algebra [Cohn, 1965]. Clones provide a presentation of algebras that abstracts away from the details of their corresponding syntactic equational presentations. In modern first-order universal algebra, one understands by a *clone* on a set S the set of all *elementary operations* on S, which includes projections $S^n \to S$ for any $n \in \mathbb{N}$ and is closed under multiple finitary function composition. A formal categorical definition of clones suitable to our second-order setting follows next.

7.1.1 Categorical clones

Clones. A *clone* in a cartesian category $\mathscr{C}$ is an $\mathbb{N}$-indexed collection $\{C_n\}_{n\in\mathbb{N}}$ of objects of $\mathscr{C}$ equipped with *variable maps* $\iota_i^{(n)} : 1 \to C_n$, $(i \in \|n\|)$, for each $n \in \mathbb{N}$, and *substitution maps* $\varsigma_{m,n} : C_m \times (C_n)^m \to C_n$ for each $m, n \in \mathbb{N}$, such that the following commute:

$$\begin{array}{ccc} C_n \times 1 & \xrightarrow{\;id_{C_n}\times\langle\iota_1^{(n)},\dots,\iota_n^{(n)}\rangle\;} & C_n \times (C_n)^n \\ & {\scriptstyle \pi_1}\searrow \qquad \swarrow{\scriptstyle \varsigma_{n,n}} & \\ & C_n & \end{array}$$

$$\begin{array}{ccc} 1 \times (C_n)^m & \xrightarrow{\quad\pi_2\quad} & (C_n)^m \\ {\scriptstyle \iota_i^{(m)}\times id_{(C_n)^m}}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ C_m \times (C_n)^m & \xrightarrow{\quad\varsigma_{m,n}\quad} & C_n \end{array}$$

$$\begin{array}{ccc} C_l \times (C_m)^l \times (C_n)^m & \xrightarrow{\;\varsigma_{l,m}\times id_{(C_n)^m}\;} & C_m \times (C_n)^m \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varsigma_{m,n}} \\ C_l \times (C_n)^l & \xrightarrow{\quad\varsigma_{l,n}\quad} & C_n \end{array}$$

where φ is the morphism $id_{C_l} \times \langle \varsigma_{m,n} \circ (\pi_i \times id_{(C_n)^m}) \rangle_{i\in\|l\|}$.

As is well-known, every clone $\{C_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}$ canonically extends to a functor $\mathbb{F} \to \mathscr{C}$ defined by mapping n to C_n. Moreover, given another cartesian category $\mathscr{D}$, any cartesian functor $F : \mathscr{C} \to \mathscr{D}$ preserves the clone structure in $\mathscr{C}$, in the sense that every clone $\{C_n\}_{n\in\mathbb{N}}$ of $\mathscr{C}$ induces the clone $\{F(C_n)\}_{n\in\mathbb{N}}$ with structure maps given by $F(\iota_i^{(n)})$ and $F(\varsigma_{m,n} \circ \cong)$ (for $m, n \in \mathbb{N}$ and $i \in \|n\|$), where $\cong$ is the canonical isomorphism $F(C_m) \times \big(F(C_n)\big)^m \to F\big(C_m \times (C_n)^m\big)$.

Categories of clones. Given a cartesian category $\mathscr{C}$, the category **Clone**($\mathscr{C}$) is defined to have objects clones $\{C_n\}_{n\in\mathbb{N}}$ of $\mathscr{C}$. A *clone homomorphism* $\{C_n\}_{n\in\mathbb{N}} \to \{D_n\}_{n\in\mathbb{N}}$ is an $\mathbb{N}$-indexed family of morphisms $\{h_n : C_n \to D_n\}_{n\in\mathbb{N}}$ of $\mathscr{C}$ such that for all $m, n \in \mathbb{N}$ the following commute:

$$\begin{array}{ccc} 1 & \xrightarrow{\iota_i^{(C)}} & C_n \\ & \underset{\iota_i^{(D)}}{\searrow} & \downarrow{\scriptstyle h_n} \\ & & D_n \end{array} \qquad \begin{array}{ccc} C_m \times (C_n)^m & \xrightarrow{\varsigma^{(C)}_{m,n}} & C_n \\ {\scriptstyle h_m \times (h_n)^m}\downarrow & & \downarrow{\scriptstyle h_n} \\ D_m \times (D_n)^m & \xrightarrow{\varsigma^{(D)}_{m,n}} & D_n \end{array}$$

7.1.2 Clones for equational presentations

Signature clones. A clone for a second-order signature Σ in a cartesian category $\mathscr{C}$ is given by a clone $\{S_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}$, together with, for each $n \in \mathbb{N}$, *natural operator maps*

$$\tilde{\omega}_n : S_{n+n_1} \times \cdots \times S_{n+n_l} \to S_n$$

for every operator $\omega : n_1, \ldots, n_l$ of Σ, such that, for all $n, m \in \mathbb{N}$, the diagram

$$\begin{array}{ccccc} & & \prod_{i\in\|l\|} S_{n+n_i} \times (S_{m+n_i})^{n+n_i} & & \\ & {\scriptstyle \langle \mathrm{id}\times\upsilon_{n_i}\rangle_{i\in\|l\|}}\nearrow & & \searrow{\scriptstyle \prod_{i\in\|l\|}\varsigma_{n+n_i,m+n_i}} & \\ \prod_{i\in\|l\|} S_{n+n_i} \times (S_m)^n & & & & \prod_{i\in\|l\|} S_{m+n_i} \\ {\scriptstyle \tilde{\omega}_n\times\upsilon_0}\downarrow & & & & \downarrow{\scriptstyle \tilde{\omega}_m} \\ S_n \times (S_m)^n & & \xrightarrow{\qquad\varsigma_{n,m}\qquad} & & S_m \end{array}$$

commutes, where for each $k \in \mathbb{N}$, the morphism υ_k is given by

$$(S_m)^n \;\cong\; (S_m)^n \times 1 \xrightarrow{(S_j)^n \times \langle \iota^{(m+k)}_{m+i}\rangle_{i\in\|k\|}} (S_{m+k})^n \times (S_{m+k})^k \;\cong\; (S_{m+k})^{n+k} \quad ,$$

and j is the inclusion $\|m\| \hookrightarrow \|m+k\|$. Note that at 0, υ_0 is just the identity on $(S_m)^n$.

We write Σ-**Clone**($\mathscr{C}$) for the category of Σ-clones in $\mathscr{C}$, with morphisms given by clone homomorphisms which commute with the natural operator maps $\tilde{\omega}_n$ for every operator ω of Σ and $n \in \mathbb{N}$.

Remark 7.1. The naturality condition on the operator maps above refers to the canonical action for any $f : m \to n$ in $\mathbb{F}$ given by the composite

$$C_m \;\cong\; C_m \times 1 \xrightarrow{C_m \times \langle \iota^{(n)}_{f1}, \ldots, \iota^{(n)}_{fm}\rangle} C_m \times (C_n)^m \xrightarrow{\varsigma_{m,n}} C_n$$

that is available in any clone.

We say that a Σ-clone $\{S_n\}_{n\in\mathbb{N}}$ in a cartesian category $\mathscr{C}$ is *preserved* under a functor $F\colon \mathscr{C} \to \mathscr{D}$ if $\{F(S_n)\}_{n\in\mathbb{N}}$ is a Σ-clone in the cartesian category $\mathscr{D}$ with structure maps given by the image under F of the structure maps associated to the clone $\{S_n\}_{n\in\mathbb{N}}$. It is evident that clones are necessarily preserved under cartesian functors.

Lemma 7.2. *Cartesian functors preserve clones for second-order signatures.*

Term interpretations. A Σ-clone $\{S_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}$ induces an interpretation of terms in $\mathscr{C}$. For the metavariable context $\Theta = (\mathrm{M}_1\colon [m_1], \ldots, \mathrm{M}_k\colon [m_k])$ and variable context $\Gamma = (x_1, \ldots, x_n)$, the interpretation of a term $\Theta \rhd \Gamma \vdash t$ under the clone $\{S_n\}_{n\in\mathbb{N}}$ is a morphism

$$[\![\Theta \rhd \Gamma \vdash t]\!]_S \colon \prod_{i\in\|k\|} S_{m_i} \to S_n$$

given by induction on the structure of the term t as follows:

- $[\![\Theta \rhd \Gamma \vdash x_i]\!]_S$ is the composite

$$\textstyle\prod_{i\in\|k\|} S_{m_i} \xrightarrow{\ !\ } 1 \xrightarrow{\iota_i^{(n)}} S_n \quad .$$

- $[\![\Theta \rhd \Gamma \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}]]\!]_S$ is the composite

$$\textstyle\prod_{i\in\|k\|} S_{m_i} \xrightarrow{\langle \pi_i, [\![\Theta\rhd\Gamma\vdash t_1]\!]_S, \ldots, [\![\Theta\rhd\Gamma\vdash t_{m_i}]\!]_S \rangle} S_{m_i} \times (S_n)^{m_i} \xrightarrow{\varsigma_{m_i,n}} S_n \quad .$$

- For an operator $\omega\colon n_1, \ldots, n_l$, $[\![\Theta \rhd \gamma \vdash \omega((\overrightarrow{y_1})t_1, \ldots, (\overrightarrow{y_l})t_l)]\!]_S$ is the composite

$$\textstyle\prod_{i\in\|k\|} S_{m_i} \xrightarrow{\langle [\![\Theta\rhd\Gamma_{n_i}\vdash t_i]\!]_S \rangle_{i\in\|l\|}} \prod_{i\in\|l\|} S_{n+n_i} \xrightarrow{\tilde{\omega}} S_n \quad ,$$

  where for $i \in \|l\|$, Γ_{n_i} is the context $\Gamma, y_1^{(i)}, \ldots, y_{l_i}^{(i)}$.

Given a functor $F\colon \mathscr{C} \to \mathscr{D}$, we say that the term interpretation $[\![\Theta \rhd \Gamma \vdash t]\!]_S$ under the Σ-clone $\{S\}_{n\in\mathbb{N}}$ in $\mathscr{C}$ is *preserved* under F if $F[\![\Theta \rhd \Gamma \vdash t]\!]_S = [\![\Theta \rhd \Gamma \vdash t]\!]_{FS}$ in $\mathscr{D}$. It is again straightforward to observe that term interpretations are preserved under cartesian functors.

Lemma 7.3. *A cartesian functor $F\colon \mathscr{C} \to \mathscr{D}$ preserves interpretations for terms of a second-order signature induced by a Σ-clone in a cartesian category $\mathscr{C}$.*

Presentation clones. For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, an *$\mathscr{E}$-clone* in a cartesian category $\mathscr{C}$ is a Σ-clone $\{S_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that for all axioms $\Theta \rhd \Gamma \vdash_E s \equiv t$ of $\mathscr{E}$, the morphisms $[\![\Theta \rhd \Gamma \vdash s]\!]_S$ and $[\![\Theta \rhd \Gamma \vdash t]\!]_S$ are equal in $\mathscr{C}$. In this case, we say that the clone $\{S_n\}_{n\in\mathbb{N}}$ satisfies the axioms of $\mathscr{E}$.

We write $\mathscr{E}$-**Clone**$(\mathscr{C})$ for the full subcategory of Σ-**Clone**$(\mathscr{C})$ consisting of the Σ-clones in $\mathscr{C}$ which satisfy the axioms of the presentation $\mathscr{E} = (\Sigma, E)$.

7.1.3 Clone semantics

Clones for second-order signatures provide an axiomatisation for variable binding, parameterised metavariables and simultaneous substitution. We recall here that they are in fact an abstract, yet equivalent, formalisation of (set-theoretic) second-order model theory as presented in Chapter 4.

Proposition 7.4. *For Σ a mono-sorted second-order signature, the category* **Mod**(Σ) *of set-theoretic algebraic models for Σ is equivalent to the category* Σ-**Clone**(**Set**) *of Σ-clones in* **Set**.

Proof. A detailed development of this equivalence appears in [Fiore et al., 1999]. □

One needs an additional argument to show that the same holds when adding equations, that is that clones and algebras for second-order equational presentations are equivalent. To this end, let $\mathscr{E} = (\Sigma, E)$ be a second-order equational presentation and $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash_{\mathscr{E}} s \equiv t$ an equation of $\mathscr{E}$. Recall from Chapter 4 that a set-theoretic algebra A of **Mod**$(\mathscr{E})$ satisfies all equations of $\mathscr{E}$, and therefore the respective term interpretations $[\![s]\!]_A$ and $[\![t]\!]_A$ are equal morphisms

$$\prod_{i\in\|k\|} A^{\boldsymbol{y}m_i} \times \boldsymbol{y}n \to A$$

in $\mathbf{Set}^{\mathbb{F}}$. Consequently, their corresponding exponential transposes $\mathbb{1}[\![s]\!]_A$ and $\mathbb{1}[\![t]\!]_s$ are equal morphisms

$$\prod_{i\in\|k\|} A^{\boldsymbol{y}m_i} \to A^{\boldsymbol{y}n} \quad .$$

Now, under the equivalence of Proposition 7.4, the Σ-algebra A corresponds to the Σ-clone $\hat{A} = \{A(n)\}_{n\in\mathbb{N}}$ in **Set**, which induces the term interpretations $[\![s]\!]_{\hat{A}}$ and $[\![t]\!]_{\hat{A}}$ given by the component at (0) of $\mathbb{1}[\![s]\!]_A$ and $\mathbb{1}[\![t]\!]_A$, respectively. Therefore,

$$[\![s]\!]_{\hat{A}} = [\![t]\!]_{\hat{A}} \colon \prod_{i\in\|k\|} A(m_i) \to A(n)$$

in **Set**. We have thus shown that an equation of $\mathscr{E} = (\Sigma, E)$ satisfied by a Σ-algebra A is also satisfied by the induced Σ-clone $\hat{A}$.

The other direction is given by soundness and completeness. Suppose the judgement

$$\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k] \rhd x_1, \ldots, x_n \vdash_{\mathscr{E}} s \equiv t$$

is satisfied by a Σ-clone, then we know from soundness and completeness of Second-Order Equational Logic (Theorem 4.9) that it is necessarily satisfied by all (Σ, E)-algebras.

A second-order term equation is hence satisfied by a signature algebra if and only if it is satisfied by the corresponding signature clone in **Set**. This, together with Proposition 7.4, yields an alternative, yet equivalent, semantics of second-order equational presentations via abstract clone structures.

Proposition 7.5. *For $\mathscr{E} = (\Sigma, E)$ a second-order equational presentation, the categories* $\mathbf{Mod}(\mathscr{E})$ *of second-order $\mathscr{E}$-algebras and $\mathscr{E}$-*$\mathbf{Clone}(\mathbf{Set})$ *of set-theoretic $\mathscr{E}$-clones are equivalent.*

7.2 Classifying Clones

Before formalising second-order functorial model theory, we show that every second-order algebraic theory, and in particular those that classify second-order equational presentations, come equipped with a canonical clone structure induced by their universal exponentiable object. This will enable us to link functorial models directly to (set-theoretic) algebraic models via these so-called classifying clone structures.

7.2.1 The clone of elementary operations

Let $\mathscr{C}$ be a cartesian category. An exponentiable object C of $\mathscr{C}$ canonically induces the clone

$$\begin{aligned} \langle C \rangle &= \{C^n \Rightarrow C\}_{n \in \mathbb{N}} \\ \langle C \rangle_n &= C^n \Rightarrow C \end{aligned}$$

with variable maps $\iota_i^{(n)} \colon 1 \to \langle C \rangle_n$ given by the unique exponential mates of the cartesian projections

$$1 \times C^n \cong C^n \xrightarrow{\pi_i^{(n)}} C \quad .$$

The substitution map $\varsigma_{m,n} \colon \langle C \rangle_m \times \langle C \rangle_n^m \to \langle C \rangle_n$ is given by the exponential mate of

$$(C^m \Rightarrow C) \times (C^n \Rightarrow C^m) \times C^n \xrightarrow{(C^m \Rightarrow C) \times ev_{n,m}} (C^m \Rightarrow C) \times C^m \xrightarrow{ev_m} C \quad ,$$

where $ev_{n,m} \colon (C^n \Rightarrow C^m) \times C^n \to C^m$ is the evaluation map associated with the exponential $C^n \Rightarrow C^m = (C^n \Rightarrow C)^m$.

We refer to $\langle C \rangle$ as the *clone of elementary operations* on the object C of $\mathscr{C}$. Thus, as it is the case with every clone, the family $\langle C \rangle$ canonically extends to a functor $\mathbb{F} \to \mathscr{C}$ mapping n to $\langle C \rangle_n$ and $f : n \to m$ to $C^f \Rightarrow C \colon \langle C \rangle_n \to \langle C \rangle_m$.

7.2.2 Classifying clones

A clone for a second-order signature Σ is a clone of elementary operations equipped with appropriate extra structure for the operators of Σ.

Classifying signature clones. Let Σ be a second-order signature and $\mathbb{M}(\Sigma)$ its classifying category. The *classifying clone* of a second-order signature Σ is given by the clone of operations $\langle 0 \rangle = \{(n)\}_{n \in \mathbb{N}}$ on the universal exponentiable object (0) of $\mathbb{M}(\Sigma)$, together with the family

$$\{\tilde{f_\omega}\}_{\omega\colon (n_1,\ldots,n_l) \in \Sigma} \quad ,$$

where for an operator $\omega\colon (n_1, \ldots, n_l)$, f_ω is given by the morphism

$$\langle \omega(\ldots, (x_1, \ldots, x_{n_i})\mathrm{N}_i[x_1, \ldots, x_{n_i}], \ldots) \rangle \colon (n_1, \ldots, n_l) \to (0)$$

of $\mathbb{M}(\Sigma)$ and the instance at $j \in \mathbb{N}$ of the family

$$\tilde{f_\omega} = \{(\tilde{f_\omega})_j\}_{j \in \mathbb{N}}$$

is given by

$$(j + n_1, \ldots, j + n_l) \cong (0)^j \Rightarrow (n_1, \ldots, n_l) \xrightarrow{(0)^j \Rightarrow f_\omega} (0)^j \Rightarrow (0) \cong (j) \quad .$$

It is evident to see that our definition of a classifying clone satisfies the properties of clone structures.

Lemma 7.6. *The canonical classifying clone of a second-order signature Σ in its classifying category $\mathbb{M}(\Sigma)$ is a Σ-clone.*

Classifying term interpretation. The classifying clone $\langle 0 \rangle$ induces a canonical interpretation of terms in $\mathbb{M}(\Sigma)$. For $\Theta = (\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k])$ and $\Gamma = (x_1, \ldots, x_n)$, a term $\Theta \rhd \Gamma \vdash t$ has interpretation $[\![t]\!]_{\langle 0 \rangle}$ under the classifying clone simply given by the morphism

$$\langle t \rangle \colon (m_1, \ldots, m_k) \to (n)$$

in $\mathbb{M}(\Sigma)$. We verify this by induction on the structure of t:

- $[\![\Theta \rhd \Gamma \vdash x_i]\!]_{\langle 0 \rangle}$ is given by

$$(m_1, \ldots, m_k) \xrightarrow{!} 1 \xrightarrow{\iota_i^{(n)}} (n) \quad ,$$

which by definition is equal to

$$(m_1, \ldots, m_k) \xrightarrow{!} 1 \xrightarrow{\mathbb{I}(\pi_i^{(n)} \circ \cong)} (n) \quad ,$$

and this, in return, is equal to $\langle x_i \rangle$ by Lemma 5.2.

- Similarly, $[\![\Theta \rhd \Gamma \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}]]\!]_{\langle 0 \rangle}$ is the composite

$$(m_1, \ldots, m_k) \xrightarrow{\left\langle \pi_i, [\![t_1]\!]_{\langle 0 \rangle}, \ldots, [\![t_{m_i}]\!]_{\langle 0 \rangle} \right\rangle} (m_i, n^{m_i}) \xrightarrow{\varsigma_{m_i,n}} (n) \quad ,$$

which by induction on the t_j's ($j \in \|m_i\|$) and by Lemma 5.2 is equal to $\langle \mathrm{M}_i[t_1, \ldots, t_{m_i}] \rangle$.

- For $\omega \colon n_1, \ldots, n_l$, $[\![\Theta \rhd \Gamma \vdash \omega\big((\overrightarrow{y_1})t_1, \ldots, (\overrightarrow{y}_l)t_l\big)]\!]_{\langle 0 \rangle}$ is the composite

$$(m_1, \ldots, m_k) \xrightarrow{\left\langle [\![t_1]\!]_{\langle 0 \rangle}, \ldots, [\![t_l]\!]_{\langle 0 \rangle} \right\rangle} (n + n_1, \ldots, n + n_l) \xrightarrow{\tilde{\omega}} (n) \quad .$$

By definition of classifying clones, $\tilde{\omega} = (0)^n \Rightarrow \mathrm{t}_\omega$, and by induction, the above composite simply amounts to $\big\langle \omega\big((\overrightarrow{y_1})t_1, \ldots, (\overrightarrow{y_l})f_l\big) \big\rangle$.

Classifying presentation clones. For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, we define its classifying clone in its classifying category $\mathbb{M}(\mathscr{E})$ in a similar fashion, namely by the clone of operations $\langle 0 \rangle$ together with the family $\{(\tilde{f}_\omega)_n\}_{n \in \mathbb{N}}$, where for $\omega \colon n_1, \ldots, n_l$, the morphism f_ω is taken to be the tuple of the equivalence of the same term as in the definition of classifying signature clones, more precisely

$$\big\langle \big[\omega\big(\ldots, (x_1, \ldots, x_{n_i})\mathrm{N}_i[x_1, \ldots, x_{n_i}], \ldots\big)\big]_{\mathscr{E}} \big\rangle \colon (n_1, \ldots, n_l) \to (0) \quad .$$

A similar inductive argument shows that the interpretation for a term $\Theta \rhd \Gamma \vdash t$ induced by the classifying clone $\langle 0 \rangle$ in $\mathbb{M}(\mathscr{E})$ is the morphism $\langle [\Theta \rhd \Gamma \vdash t]_{\mathscr{E}} \rangle$.

A derivable judgement $\Theta \rhd \Gamma \vdash_{\mathscr{E}} s \equiv t$ of $\mathscr{E}$ is therefore satisfied by the classifying clone of $\mathscr{E}$ in $\mathbb{M}(\mathscr{E})$, since $\langle [\Theta \rhd \Gamma \vdash s]_{\mathscr{E}} \rangle$ and $\langle [\Theta \rhd \Gamma \vdash t]_{\mathscr{E}} \rangle$ are equal morphisms in $\mathbb{M}(\mathscr{E})$, and therefore $[\![s]\!]_{\langle 0 \rangle} = [\![t]\!]_{\langle 0 \rangle}$. Classifying clones therefore provide *sound semantics* for second-order equational presentations in their classifying categories.

7.3 Second-Order Functorial Semantics

We extend Lawvere's functorial semantics for algebraic theories [Lawvere, 2004] from first to second order.

Definition 7.7 (Second-Order Functorial Model)**.** A second-order *functorial model* of a second-order algebraic theory $M\colon \mathbb{M} \to \mathscr{M}$ is given by a cartesian functor $\mathscr{M} \to \mathscr{C}$, for $\mathscr{C}$ a cartesian category. We write $\mathbb{Mod}(M, \mathscr{C})$ for the category of functorial models of M in $\mathscr{C}$, with morphisms (necessarily monoidal) natural transformations between them. A second-order *set-theoretic functorial model* of a second-order algebraic theory $M\colon \mathbb{M} \to \mathscr{M}$ is simply a cartesian functor from $\mathscr{M}$ to **Set**. We write $\mathbb{Mod}(M)$ for the category of set-theoretic functorial models of M in **Set**.

Note that, just as in Lawvere's first-order definition, we mereley ask for preservation of the cartesian structure rather than strict preservation. Consequently, functorial models of the same second-order algebraic theory may differ only by the choice of the cartesian product in **Set**. However, as we pointed out earlier, since the cartesian structure in **Set** is not strictly associative (whereas it is strictly associative in any first- and second-order algebraic theory), asking for preservation in the definition of a functorial model avoids the creation of unnatural categories of models.

7.4 Second-Order Semantic Categorical Type Theory Correspondence

We show that classifying clones, and thus second-order algebras, correspond to second-order functorial models.

Proposition 7.8. *Let $\mathscr{E} = (\Sigma, E)$ be a second-order equational presentation and $M_{\mathscr{E}}\colon \mathbb{M} \to \mathbb{M}(\mathscr{E})$ its classifying algebraic theory, and let $\mathscr{C}$ be a cartesian category. The category of $\mathscr{E}$-clones $\mathscr{E}$-***Clone***$(\mathscr{C})$ and the category of second-order functorial models $\mathbb{Mod}(M_{\mathscr{E}}, \mathscr{C})$ are equivalent.*

Proof. We provide an explicit description of the equivalence functors. Define

$$\Upsilon \ : \mathbb{Mod}(M_{\mathscr{E}}, \mathscr{C}) \longrightarrow \mathscr{E}\text{-}\mathbf{Clone}(\mathscr{C})$$

by mapping a cartesian functor $F\colon \mathbb{M}(\mathscr{E}) \to \mathscr{C}$ to the clone

$$\hat{F} := \{F(n)\}_{n\in\mathbb{N}}$$

whose structure maps are given by the image under F of the structure maps of the canonical classifying clone $\langle n\rangle$ of $\mathbb{M}(\mathscr{E})$. This makes $\hat{F}$ indeed a clone for the signature Σ, as, by Lemma 7.2, cartesian functors preserve clone structures. $\hat{F}$ is moreover a clone for the equational presentation $\mathscr{E}$, as it satisfies all equations in $\mathscr{C}$: given an equation $\Theta \rhd \Gamma \vdash s \equiv t$ of $\mathscr{E}$, we have $F\langle[s]_{\mathscr{E}}\rangle = F\langle[t]_{\mathscr{E}}\rangle$ (since

$\langle[s]_{\mathscr{E}}\rangle = \langle[t]_{\mathscr{E}}\rangle$), and therefore we get, by Lemma 7.3, that $[\![s]\!]_{\hat{F}} = F[\![s]\!]_{\langle 0\rangle} = F[\![t]\!]_{\langle 0\rangle} = [\![t]\!]_{\hat{F}}$.

On morphisms of $\mathbb{M}\mathrm{od}(M_{\mathscr{E}}, \mathscr{C})$, Υ is defined by mapping a monoidal natural transformation $\alpha \colon F \to G$ to $\{\alpha_n\}_{n\in\mathbb{N}} \colon \{F(n)\}_{n\in\mathbb{N}} \to \{G(n)\}_{n\in\mathbb{N}}$. This is indeed a homomorphism of $\mathscr{E}$-clones because α is natural and the clone structure maps of $\hat{F}$ and $\hat{G}$ are the images of those of $\langle n\rangle$ under F and G. Furthermore, note that Υ is functorial: the identity natural transformation $id^{(F)} \colon F \to F$ is mapped under Υ to $\{id^{(F)}_n\}_{n\in\mathbb{N}}$, where each $id^{(F)}_n \colon F(n) \to F(n)$ is simply the identity morphism in $\mathscr{C}$. Similarly, for natural transformations $\alpha \colon F \to G$ and $\beta \colon G \to H$, the image of the composite $\beta \circ \alpha$ under Υ is $\{(\beta\circ\alpha)_n\}_{n\in\mathbb{N}} = \{\beta_n \circ \alpha_n\}_{n\in\mathbb{N}}$.

In the other direction, define

$$\bar{\Upsilon} \; : \mathscr{E}\text{-}\mathbf{Clone}(\mathscr{C}) \longrightarrow \mathbb{M}\mathrm{od}(M_{\mathscr{E}}, \mathscr{C})$$

by mapping an $\mathscr{E}$-clone $\{C_n\}_{n\in N}$ to the functor $F^{(C)} \colon \mathbb{M}(\mathscr{E}) \to \mathscr{C}$, which maps $(m_1, \ldots, m_k)$ to $C_{m_1} \times \cdots \times C_{m_k}$. For $\Theta = (\mathrm{M}_1 \colon [m_1], \ldots, \mathrm{M}_k \colon [m_k])$ and $\Gamma = (x_1, \ldots, x_n)$, the image of the morphism

$$\langle[\Theta \rhd \Gamma \vdash t]_{\mathscr{E}}\rangle \colon (m_1, \ldots, m_k) \to (n)$$

under $F^{(C)}$ is defined to be the interpretation $[\![t]\!]_C$ of the term t under the clone C. This definition respects the equivalence relation of $\mathscr{E}$ as given an equation $\Theta \rhd \Gamma \vdash_{\mathscr{E}} s \equiv t$, we know that $[\![s]\!]_{\langle n\rangle} = [\![t]\!]_{\langle n\rangle}$ since $\langle n\rangle$ is an $\mathscr{E}$-clone, and therefore $F^{(C)}\langle[s]_{\mathscr{E}}\rangle = F^{(C)}\langle[t]_{\mathscr{E}}\rangle$ in $\mathscr{C}$. Moreover, note that $F^{(C)}$ is cartesian by definition.

On morphisms of $\mathscr{E}$-**Clone**$(\mathscr{C})$, $\bar{\Upsilon}$ is defined by mapping a clone homomorphism

$$\{h_n\}_{n\in\mathbb{N}} \colon \{C_n\}_{n\in\mathbb{N}} \to \{D_n\}_{n\in\mathbb{N}}$$

to $\bar{h} \colon F^{(C)} \to F^{(D)}$, with component at $(m_1, \ldots, m_k)$ given by $\bar{h}_{(m_1,\ldots,m_k)} = h_{m_1} \times \ldots, \times h_{m_k}$. Because clone homomorphisms commute with the clone structure maps, we are ensured that $\bar{h}$ is a natural transformation. This can be seen more explicitly by induction on the term structure:

- For $\langle[x_i]_{\mathscr{E}}\rangle \colon (m_1, \ldots, m_k) \to (n)$, the diagram

$$\begin{array}{ccccc}
C_{m_1} \times \cdots \times C_{m_k} & \xrightarrow{\;!\;} & 1 & \xrightarrow{\iota_i^{(C)}} & C_n \\
{\scriptstyle h_{m_1}\times\cdots\times h_{m_k}}\big\downarrow & & {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle h_n} \\
D_{m_1} \times \cdots \times D_{m_k} & \xrightarrow{\;1\;} & 1 & \xrightarrow{\iota_i^{(D)}} & D_n
\end{array}$$

  by uniqueness of the terminal map ! and because h is a homomorphism of clones and hence commutes with the clone structure maps $\iota_i^{(-)}$.

- Similarly, for $\langle [\mathrm{M}_i[t_1,\ldots,t_{m_i}]]_{\mathscr{E}}\rangle : (m_1,\ldots,m_k) \to (n)$, the following diagram commutes

$$\begin{array}{ccccc}
C_{m_1}\times\cdots\times C_{m_k} & \xrightarrow{\left\langle \pi_i^{(C)}, F^{(C)}\langle[t_1]_{\mathscr{E}}\rangle,\ldots,F^{(C)}\langle[t_{m_i}]_{\mathscr{E}}\rangle\right\rangle} & C_{m_i}\times(C_n)^{m_i} & \xrightarrow{\varsigma^{(C)}_{m_i,n}} & C_n \\
{\scriptstyle h_{m_1}\times\cdots\times h_{m_k}}\downarrow & & {\scriptstyle h_{m_i}\times(h_n)^{m_i}}\downarrow & & \downarrow{\scriptstyle h_n} \\
D_{m_1}\times\cdots\times D_{m_k} & \xrightarrow[\left\langle \pi_i^{(D)}, F^{(D)}\langle[t_1]_{\mathscr{E}}\rangle,\ldots,F^{(D)}\langle[t_{m_i}]_{\mathscr{E}}\rangle\right\rangle]{} & D_{m_i}\times(D_n)^{m_i} & \xrightarrow{\varsigma^{(D)}_{m_i,n}} & D_n
\end{array}$$

by induction on $F^{(-)}\langle[t_j]_{\mathscr{E}}\rangle$ for all $j \in \|m_i\|$, by universality of the cartesian map $\pi_i^{(D)}$, and because h_n commutes with the clone structure maps ς.

- For $\omega : n_1,\ldots,n_l$ and $\langle[\omega\big((\overrightarrow{y}_1)t_1,\ldots,(\overrightarrow{y}_l)t_l\big)]_{\mathscr{E}}\rangle$, the following diagram commutes for the same reasons as above:

$$\begin{array}{ccccc}
C_{m_1}\times\cdots\times C_{m_k} & \xrightarrow{\left\langle F^{(C)}\langle[t_1]_{\mathscr{E}}\rangle,\ldots,F^{(C)}\langle[t_l]_{\mathscr{E}}\rangle\right\rangle} & C_{n+n_1}\times\cdots\times C_{n+n_l} & \xrightarrow{\tilde{\omega}^{(C)}} & C_n \\
{\scriptstyle h_{m_1}\times\cdots\times h_{m_k}}\downarrow & & {\scriptstyle h_{n+n_1}\times\cdots\times h_{n+n_l}}\downarrow & & \downarrow{\scriptstyle h_n} \\
D_{m_1}\times\cdots\times D_{m_k} & \xrightarrow[\left\langle F^{(D)}\langle[t_1]_{\mathscr{E}}\rangle,\ldots,F^{(D)}\langle[t_l]_{\mathscr{E}}\rangle\right\rangle]{} & D_{n+n_1}\times\cdots\times D_{n+n_l} & \xrightarrow{\tilde{\omega}^{(D)}} & D_n
\end{array}$$

That $\bar{\Upsilon}$ is functorial follows from the fact that natural transformations in $\mathbb{M}\mathrm{od}(M_{\mathscr{E}},\mathscr{C})$ are monoidal. More precisely, an identity homomorphism of clones $\{id_n\}_{n\in\mathbb{N}}$ is mapped under $\bar{\Upsilon}$ to the identity natural transformation with component at $(m_1,\ldots,m_k)$ given by $id_{m_1}\times\cdots\times id_{m_k}$, which is equal to $id_{(m_1,\ldots,m_k)}$. Similarly, a composite of clone homomorphisms $\{(g\circ h)_n\}_{n\in\mathbb{N}}$ is mapped to $\overline{(g\circ h)}$ with component at $(m_1,\ldots,m_k)$ given by

$$(g\circ h)_{m_1}\times\cdots\times(g\circ h)_{m_k} \;=\; (g\circ h)_{(m_1,\ldots,m_k)} \;=\; g_{(m_1,\ldots,m_k)}\circ h_{(m_1,\ldots,m_k)} \quad .$$

Now, we proceed to show that the functors Υ and $\bar{\Upsilon}$ are equivalences. A functorial model $F : \mathbb{M}(\mathscr{E}) \to \mathscr{C}$ is mapped under $\bar{\Upsilon}\circ\Upsilon$ to $F^{(\hat{F})} : \mathbb{M}(\mathscr{E}) \to \mathscr{C}$, which maps an object $(m_1,\ldots,m_k)$ to $F(m_1)\times\cdots\times F(m_k) \cong F(m_1,\ldots,m_k)$ and a morphism $\langle[\Theta \rhd \Gamma \vdash t]_{\mathscr{E}}\rangle$ to $[\![t]\!]_{\hat{F}} = F[\![t]\!]_{\langle 0\rangle} = F\langle[\Theta \rhd \Gamma \vdash t]_{\mathscr{E}}\rangle$. A natural transformation $\alpha : F \to G$ is mapped under $\bar{\Upsilon}\circ\Upsilon$ to $\hat{\alpha} : F^{(\hat{F})} \to F^{(\hat{G})}$ and, because it is monoidal, has component at $(m_1,\ldots,m_k)$ given by $\hat{\alpha}_{(m_1,\ldots,m_k)} = \alpha_{m_1}\times\cdots\times\alpha_{m_k} = \alpha_{(m_1,\ldots,m_k)}$. In the other direction, an $\mathscr{E}$-clone $\{C_n\}_{n\in\mathbb{N}}$ is mapped under $\Upsilon\circ\bar{\Upsilon}$ to the clone $\hat{F}^{(C)} = \{F^{(C)}(n)\}_{n\in\mathbb{N}} = \{C_n\}_{n\in\mathbb{N}}$, and an $\mathscr{E}$-clone homomorphism $\{h_n\}_{n\in\mathbb{N}} : \{C_n\}_{n\in\mathbb{N}} \to \{D_n\}_{n\in\mathbb{N}}$ to $\{\bar{h}_{(n)}\}_{n\in\mathbb{N}} = \{h_n\}_{n\in\mathbb{N}}$.

□

If we take the cartesian category $\mathscr{C}$ to be **Set**, we then immediately get from Proposition 7.8 together with Proposition 7.5 the correspondence between set-theoretic functorial models, models for equational presentations, and set-theoretic clone structures.

Theorem 7.9 (Second-Order Semantic Categorical Type Theory Correspondence)**.** *For every second-order equational presentation $\mathscr{E}$, the category* $\mathbf{Mod}(\mathscr{E})$ *of $\mathscr{E}$-models and the category of second-order functorial models* $\mathbb{M}\mathrm{od}(M_{\mathscr{E}})$ *are equivalent.*

From the Second-Order Syntactic Categorical Type Theory Correspondence, we also immediately get the following equivalent formulation of the above semantic correspondence.

Corollary 7.10. *For every second-order algebraic theory $M : \mathbb{M} \to \mathscr{M}$, the category of second-order functorial models* $\mathbb{M}\mathrm{od}(M)$ *and the category of algebraic models* $\mathbf{Mod}(\mathfrak{L}(M))$ *are equivalent.*

7.5 Translational Semantics

Second-order functorial semantics enables us to take a model of an algebraic theory in any cartesian category $\mathscr{C}$. We illustrate that this way of abstractly defining algebras for theories has a syntactic counterpart via syntactic translations, which we refer to as second-order *translational semantics*.

To this end, consider two second-order equational presentations $\mathscr{E}$ and $\mathscr{E}'$, their corresponding classifying algebraic theories $M_{\mathscr{E}} : \mathbb{M} \to \mathbb{M}(\mathscr{E})$ and $M_{\mathscr{E}'} : \mathbb{M} \to \mathbb{M}(\mathscr{E}')$, and let $\tau : \mathscr{E} \to \mathscr{E}'$ be a second-order syntactic translation. Note that its induced algebraic translation $\mathbb{M}(\tau) : \mathbb{M}(\mathscr{E}) \to \mathbb{M}(\mathscr{E}')$, which commutes with the theories $M_{\mathscr{E}}$ and $M_{\mathscr{E}'}$, is by definition a second-order functorial model of the theory $M_{\mathscr{E}}$ in the cartesian category $\mathbb{M}(\mathscr{E}')$. The canonical notion of a morphism of (second-order) algebraic theories is thereby intuitively providing a model of one algebraic theory into another.

From the categorical equivalence of the Syntactic Categorical Type Theory Correspondence (Theorem 6.6), second-order syntactic translations can be thought of as *syntactic* notions of models of one equational presentation into another. Therefore, by explicitly defining the translation $\tau : \mathscr{E} \to \mathscr{E}'$, we implicitly provide a model of the presentation $\mathscr{E}$ in $\mathscr{E}'$.

We have in this work reviewed first- and second-order set-theoretic semantics for equational presentations, as well as categorical semantics, and finally introduced second-order functorial semantics. Through the development of syntactic translations, we have thus introduced a less abstract, more concrete way of giving semantics to equational presentations. We refer to this as (second-order) *Translational Semantics*.

Chapter 8

Concluding Remarks

We have incorporated second-order languages into universal algebra by developing a programme from the viewpoint of Lawvere's algebraic theories.

The pinnacle of our work is the notion of *second-order algebraic theory*, which we defined on top of a base category, the second-order theory of equality $\mathbb{M}$, representing the elementary operators and equations present in every second-order language. We showed that $\mathbb{M}$ can be described abstractly via the universal property of being the free cartesian category on an exponentiable object.

At the syntactic level, we established the correctness of our definition by showing a categorical equivalence between second-order equational presentations and second-order algebraic theories. This equivalence, referred to as the Second-Order Syntactic Categorical Type Theory Correspondence, involved distilling a notion of syntactic translation between second-order equational presentations that corresponds to the canonical notion of morphism between second-order algebraic theories. Syntactic translations provide a mathematical formalisation of notions such as encodings and transforms for second-order languages.

On top of this syntactic correspondence, we furthermore established the Second-Order Semantic Categorical Type Theory Correspondence. This involved generalising Lawvere's notion of functorial model of algebraic theories to the second-order setting. By this semantic correspondence, second-order functorial semantics were shown to correspond to the model theory of second-order universal algebra.

We now show that the core of the theory surrounding first-order algebraic theories extends to the second-order universe. Instances of this development are the existence of algebraic functors (Section 8.1) and monad morphisms (Section 8.2) in the second-order universe. Moreover, we define a notion of syntactic translation homomorphism that allows us to establish a 2-categorical type theory correspondence (Section 8.3). To keep the illustrative nature of these concluding remarks, our treatment of the various examples will remain at a rather superficial level, with many of the proofs omitted.

We conclude this chapter by briefly outlining two directions for future research (Section 8.4). The first proposal is the extension of categorical universal algebra to include dependently-sorted syntax. The second proposal is the formalisation of the theory of syntactic translations in terms of a framework that allows generic characterisation of relationships amongst algebraic languages.

8.1 Second-Order Algebraic Functors

8.1.1 First-order algebraic categories and their morphisms

The concept of an algebraic functor arising from morphisms of Lawvere theories has been developed by Lawvere [Lawvere, 2004] and revisited many times since then [Borceux, 1994, Adamek et al., 2009]. It is the canonical notion of morphism between algebraic categories.

Definition 8.1 (Algebraic Category)**.** A category is called *algebraic* if it is equivalent to the category of functorial models $\mathbf{FMod}(L)$ for some algebraic theory $L\colon \mathbb{L} \to \mathscr{L}$.

The simplest example of an algebraic category is the category **Set** of sets. Its associated algebraic theory is simply $\mathbb{L}$, together with the identity functor $\mathrm{Id}_{\mathbb{L}}\colon \mathbb{L} \to \mathbb{L}$. Every functorial model $A_{\mathrm{Id}_{\mathbb{L}}}\colon \mathbb{L} \to \mathbf{Set}$ is determined up to isomorphism by the set $A_{\mathrm{Id}_{\mathbb{L}}}(1)$, since any $n \in \mathbb{L}$ is the n-th cartesian product of the *generator* 1. Therefore, we have an equivalence $\mathbf{FMod}(\mathrm{Id}_{\mathbb{L}}) \to \mathbf{Set}\colon A_{\mathrm{Id}_{\mathbb{L}}} \mapsto A_{\mathrm{Id}_{\mathbb{L}}}(1)$.

The categories of algebras presented in this dissertation are algebraic. Consider for instance a mono-sorted first-order equational presentation $\mathscr{E} = (\Sigma, E)$ and its classifying algebraic theory $L_{\mathscr{E}}\colon \mathbb{L} \to \mathbb{L}(\mathscr{E})$. The category $\mathscr{E}$-**Alg** of algebras for the equational presentation $\mathscr{E}$ is algebraic, since by the First-Order Semantic Categorical Type Theory Correspondence (Theorem 2.18) it is equivalent to the category $\mathbf{FMod}(L_{\mathscr{E}})$ of functorial models for its classifying algebraic theory.

Remark 8.2 (Representable functorial models). We recall that the Yoneda transformation yields canonical functorial models induced by objects of a Lawvere theory. This construction is used in deriving left adjoints for first-order algebraic functors. For $L\colon \mathbb{L} \to \mathscr{L}$ an algebraic theory, an object $l \in \mathscr{L}$ induces the algebra $Y_L(l) = \mathscr{L}(l, -)\colon \mathscr{L} \to \mathbf{Set}$, which, together with the Yoneda transformation, defines a full and faithful functor $Y_L\colon \mathscr{L}^{\mathrm{op}} \to \mathbf{FMod}(L)$.

The theory surrounding algebraic categories is very rich, particularly since they can be given a very elegant characterisation via universal properties. More specifically, algebraic categories are free completions of small cocartesian categories under sifted colimits, see [Adamek et al., 2009]. They can also be characterised as those cocomplete categories which have a strong generator consisting of perfectly presentable objects, see [Adamek et al., 2009].

We are interested in the natural concept of morphism between such categories. This is given via preservation of the cartesian structure - the categorical characterisation of canonical algebraic structure.

Definition 8.3 (Algebraic Functor)**.** Let $F\colon L \to L'$ be an algebraic translation of (mono-sorted first-order) Lawvere theories $L\colon \mathbb{L} \to \mathscr{L}$ and $L'\colon \mathbb{L} \to \mathscr{L}'$. The functor

$$\mathbf{FMod}(F)\colon \mathbf{FMod}(L') \to \mathbf{FMod}(L)\colon G \mapsto G \circ F$$

is called an *algebraic functor*.

We obtain the following commutative diagram, where the unlabelled arrows are the canonical (monadic) forgetful functors:

$$\begin{array}{ccc} \mathbf{FMod}(L') & \xrightarrow{\mathbf{FMod}(F)} & \mathbf{FMod}(L) \\ & \searrow \quad \swarrow & \\ & \mathbf{Set} & \end{array}$$

Within the development surrounding algebraic categories, algebraic functors have been defined to be those functors which preserve limits, filtered colimits and epimorphisms. It is indeed the case that a functor of algebraic categories is algebraic (in this latter sense) if and only if it is induced by a morphism of algebraic theories, making the two definitions equivalent. For a proof of the following fundamental result, see for example [Borceux, 1994] or [Adamek et al., 2009].

Theorem 8.4. *A functor between algebraic categories $F\colon \mathscr{A}_2 \to \mathscr{A}_1$ preserves limits, filtered colimits and epimorphisms if and only if there exists an algebraic translation $G\colon L_1 \to L_2$ of algebraic theories $L_1\colon \mathbb{L} \to \mathscr{L}_1$ and $L_2\colon \mathbb{L} \to \mathscr{L}_2$ and equivalences $E_1\colon \mathbf{FMod}(L_1) \to \mathscr{A}_1$ and $E_2\colon \mathbf{FMod}(L_2) \to \mathscr{A}_2$ such that $F \circ E_2 = E_1 \circ \mathbf{FMod}(G)$ up to natural isomorphism.*

It is therefore legitimate to use the notation $\mathbf{FMod}(G)$ for algebraic functors, with G being the algebraic translation inducing it. Moreover, it can be shown that algebraic functors have left adjoints. This is an immediate consequence of the Adjoint Lifting Theorem.

Theorem 8.5 (Adjoint Lifting Theorem)**.** *Let $F \circ U = V \circ G$ be a commutative diagram of functors, where U and V are monadic, and G is the functor $\mathscr{C} \to \mathscr{D}$. If the category $\mathscr{C}$ has coequalisers, then G has a left adjoint as soon as F has a left adjoint.*

Proposition 8.6. *Let $F\colon L_1 \to L_2$ be an algebraic translation of algebraic theories $L_1\colon \mathbb{L} \to \mathscr{L}_1$ and $L_2\colon \mathbb{L} \to \mathscr{L}_2$. Then its induced algebraic functor $\mathbf{FMod}(F)\colon \mathbf{FMod}(L_2) \to \mathbf{FMod}(L_1)$ has a left adjoint $\widetilde{F}\colon \mathbf{FMod}(L_1) \to \mathbf{FMod}(L_2)$.*

This left adjoint $\widetilde{F}$ is in fact the essentially unique functor which preserves sifted colimits and makes the following diagram commute up to natural isomorphism.

$$\begin{array}{ccc} \mathscr{L}_1^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & \mathscr{L}_2^{\mathrm{op}} \\ {\scriptstyle Y_{L_1}}\downarrow & & \downarrow{\scriptstyle Y_{L_2}} \\ \mathbf{FMod}(L_1) & \xrightarrow{\widetilde{F}} & \mathbf{FMod}(L_2) \end{array}$$

The algebraic importance of these left adjoints is pointed out by Lawvere in his thesis [Lawvere, 2004]. As an example, the adjoint to the algebraic functor induced by an algebraic translation from the theory of monoids to the theory of rings essentially assigns to a given monoid M the monoid ring $Z[M]$ with integer coefficients. As Lawvere also points out, the fact that these adjoints form the commutative diagram above implies, for instance, that a free ring can be constructed either as the monoid ring of a free monoid, or as the tensor ring of a free abelian group. These are well-known facts from universal algebra, but given a more abstract formulation via algebraic functors and their adjoints.

We finally recall that the resulting *algebraic adjunction* is monadic, which is an immediate consequence of the following observation. For a detailed proof of this, we refer the reader to [MacLane, 1998] and [Borceux, 1994].

Proposition 8.7. *Let $U = V \circ G$ be a commutative diagram, where G is a functor $\mathscr{C} \to \mathscr{D}$. Suppose both U and V are monadic. If the category $\mathscr{C}$ has coequalisers, then the functor G is monadic as well.*

8.1.2 Second-order algebraic functors

Just as in the first-order case, every algebraic translation $F: M \to M'$ between second-order algebraic theories $M: \mathbb{M} \to \mathscr{M}$ and $M': \mathbb{M} \to \mathscr{M}'$ contravariantly induces a *second-order algebraic functor* $\mathbb{M}\mathrm{od}(F): \mathbb{M}\mathrm{od}(M') \to \mathbb{M}\mathrm{od}(M)$; $S \mapsto S \circ F$ between the corresponding categories of second-order functorial models. We also obtain the fundamental left adjoint to second-order algebraic functors.

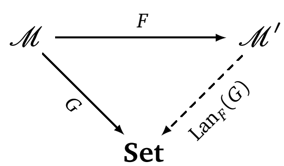
Theorem 8.8. *The algebraic functor $\mathbb{M}\mathrm{od}(F): \mathbb{M}\mathrm{od}(M') \to \mathbb{M}\mathrm{od}(M)$ induced by a second-order algebraic translation $F: M \to M'$ has a left adjoint, and the resulting adjunction is monadic.*

Proof sketch. It has been shown by Fiore and Hur [Fiore and Hur, 2008a] that in the situation

$$\begin{array}{ccc} \mathbb{M}\mathrm{od}(M') & \xrightarrow{\mathbb{M}\mathrm{od}(F)} & \mathbb{M}\mathrm{od}(M) \\ & \searrow \quad \swarrow & \\ & \mathbf{Set}^{\mathbb{F}} & \end{array}$$

the forgetful functors, given by the unlabelled arrows above, have left adjoints, and that the adjunction is monadic. Furthermore, the functorial model categories $\mathbb{M}\mathrm{od}(M)$ and $\mathbb{M}\mathrm{od}(M')$ have all coequalisers

[Fiore and Hur, 2008a]. The left adjoint to $\mathbb{M}\mathrm{od}(F)$ is given by $\mathbf{Lan}_F(-)\colon \mathbb{M}\mathrm{od}(M) \to \mathbb{M}\mathrm{od}(M')$, which maps a functorial model $G\colon \mathscr{M} \to \mathbf{Set}$ to the left Kan extension $\mathbf{Lan}_F(G)$ of G along $F\colon \mathscr{M} \to \mathscr{M}'$, that is

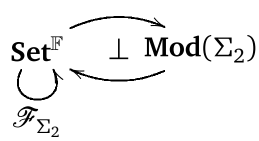


Finally, from Proposition 8.7, we get that the adjunction $\mathbf{Lan}_F(-) \dashv \mathbb{M}\mathrm{od}(F)$ is monadic. □

8.1.3 Syntactically induced second-order algebraic functors

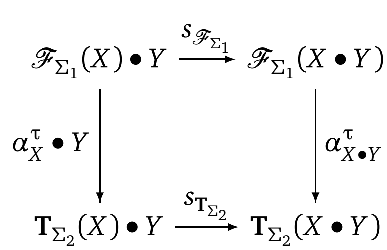
Syntactic translations of second-order equational presentations similarly yield a notion of algebraic functor which is naturally isomorphic to the one introduced above. We begin by observing that second-order syntactic signature translations behave essentially as natural transformations between the corresponding signature endofunctors and their induced monads.

Syntactic translations as natural transformations. For second-order signatures Σ_1 and Σ_2, let $\mathscr{F}_{\Sigma_1}$ be the signature endofunctor induced by Σ_1 (Section 4.3.2), and $\mathbf{T}_{\Sigma_2}$ the (underlying functor of the) induced monad corresponding to Σ_2. More precisely, in the situation



$\mathbf{T}_{\Sigma_2}$ is the monad induced by the above adjunction, so that $\mathbf{T}_{\Sigma_2}$-**Alg** $\cong$ $\mathbf{Mod}(\Sigma_2)$. Furthermore, recall from Chapter 4 that objects of $\mathbf{Mod}(\Sigma_2)$ are algebras for the signature endofunctor $\mathscr{F}_{\Sigma_2}$ equipped with compatible monoid structure.

A translation $\tau\colon \Sigma_1 \to \Sigma_2$ induces a natural transformation $\alpha^\tau\colon \mathscr{F}_{\Sigma_1} \to \mathbf{T}_{\Sigma_2}$, which is *strong* in the sense that



commutes for the canonical pointed strengths $s_{\mathscr{F}_{\Sigma_1}}$ and $s_{\mathbf{T}_{\Sigma_2}}$.

Natural transformations induced in this way by syntactic translations contravariantly induce algebraic functors between categories of set-theoretic algebras, as described next.

Algebraic functors between categories of signature models. For $\tau : \Sigma_1 \to \Sigma_2$ a second-order translation with induced natural transformation $\alpha^\tau : \mathscr{F}_{\Sigma_1} \to \mathbf{T}_{\Sigma_2}$, let $A \in \mathbf{Mod}(\Sigma_2)$ be a Σ_2-model, with monoid structure $\nu_A : \boldsymbol{y}1 \to A$ and $\varsigma_A : A \bullet A \to A$, and $\mathscr{F}_{\Sigma_2}$-algebra structure map given by $\varphi_A : \mathscr{F}_{\Sigma_2} A \to A$. Denote by $\delta_A : \mathbf{T}_{\Sigma_2} A \to A$ the corresponding $\mathbf{T}_{\Sigma_2}$-algebra structure map induced by the categorical equivalence $\mathbf{Mod}(\Sigma_2) \cong \mathbf{T}_{\Sigma_2}\text{-}\mathbf{Alg}$.

Composing this $\mathbf{T}_{\Sigma_2}$-algebra structure map δ with natural transformations $\mathscr{F}_{\Sigma_1} \to \mathbf{T}_{\Sigma_2}$ essentially defines the mapping of algebraic functors. More precisely, a second-order signature translation $\tau : \Sigma_1 \to \Sigma_2$ yields the algebraic functor

$$\mathbf{Mod}(\tau) : \mathbf{Mod}(\Sigma_2) \to \mathbf{Mod}(\Sigma_1)$$

by mapping $A \in \mathbf{Set}^{\mathbb{F}}$ with structure maps $\nu_A : \boldsymbol{y}1 \to A$, $\varsigma_A : A \bullet A \to A$, and $\varphi_A : \mathscr{F}_{\Sigma_2} A \to A$ to the algebra with same underlying presheaf A and same monoid maps ν_A and ς_A, but with $\mathscr{F}_{\Sigma_1}$-algebra structure map given by the composite

$$\mathscr{F}_{\Sigma_1} A \xrightarrow{\alpha^\tau_A} \mathbf{T}_{\Sigma_2} A \xrightarrow{\delta_A} A \quad .$$

This morphism is compatible with the monoid structure given by ν_A and ς_A because of the strength of the natural transformation α^τ discussed above.

Observe that the substitution structure remains ‘constant’ under the algebraic functor $\mathbf{Mod}(\tau)$, just as it is under syntactic translations. The compatibility of the monoid structure with the structure map of the signature endofunctor can be viewed as an abstract description of the compositionality of syntactic translations with substitution and metasubstitution (Lemma 6.1). The algebraic functor $\mathbf{Mod}(\tau)$ clearly commutes with the canonical forgetful functors into $\mathbf{Set}^{\mathbb{F}}$. Using a similar argument as in Section 8.1.1, one can immediately derive a left adjoint to $\mathbf{Mod}(\tau)$, with the resulting adjunction being monadic.

Algebraic functors between categories of presentation algebras. We use the notion of algebraic equational systems developed by Fiore and Hur in [Fiore and Hur, 2007, Fiore and Hur, 2008a] to derive algebraic functors induced by syntactic translations of second-order equational presentations.

Definition 8.9 (Equational System)**.** An *equational system* $\mathbb{S}$ is given by a pair of functors $L, R : F\text{-}\mathbf{Alg} \to D\text{-}\mathbf{Alg}$ between categories of algebras for endofunctors over some base category $\mathscr{C}$. In the framework of equational presentations, the *functorial signature* F is a generalisation of the concept of endofunctor induced by an algebraic signature; the so-called *functorial terms* L, R generalise the notion of equation; and the endofunctor D corresponds to the arity of the equation. The category $\mathbb{S}$-**Alg** of algebras for the equational system $\mathbb{S}$ is given by the equaliser $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow F\text{-}\mathbf{Alg}$ of L, R. More explicitly, an $\mathbb{S}$-algebra is simply an F-algebra $(A, a\colon FA \to A)$ such that $L(A,a)$ and $R(A,a)$ are equal D-algebras on A.

Example 8.10 (Second-Order Equational Systems)**.** *Let $\mathscr{E} = (\Sigma, E)$ be a (mono-sorted) second-order equational presentation and $\boldsymbol{y}1$ be the presheaf of variables defined in Section 4.3. The second-order equational system $\mathbb{S}_{\mathscr{E}}$ associated with $\mathscr{E}$ is given by the signature endofunctor $\mathscr{F}_\Sigma$ of its underlying signature Σ, together with the functor $\Gamma_{\mathscr{E}}\colon \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ defined by $\Gamma_{\mathscr{E}}(A) := \coprod_{(\Theta \rhd \Gamma \vdash t \equiv s) \in E} (A \bullet A) + \boldsymbol{y}1$, and the pair of functors $L_{\mathscr{E}}, R_{\mathscr{E}}\colon \mathscr{F}_\Sigma\text{-}\mathbf{Alg} \rightrightarrows \Gamma_{\mathscr{E}}\text{-}\mathbf{Alg}$, where*

$$
\begin{aligned}
L_{\mathscr{E}}(A, [\![-]\!]_A) &:= \big(A, \big[\, [\![t]\!]_A \big]_{(t \equiv s) \in E}\big) \\
R_{\mathscr{E}}(A, [\![-]\!]_A) &:= \big(A, \big[\, [\![s]\!]_A \big]_{(t \equiv s) \in E}\big)
\end{aligned}
$$

*The category $\mathbb{S}_{\mathscr{E}}$-**Alg** of algebras for the second-order equational system $\mathbb{S}_{\mathscr{E}}$ is then the equaliser $\mathbb{S}_{\mathscr{E}}\text{-}\mathbf{Alg} \hookrightarrow \mathscr{F}_\Sigma\text{-}\mathbf{Alg}$ of $L_{\mathscr{E}}, R_{\mathscr{E}}\colon \mathscr{F}_\Sigma\text{-}\mathbf{Alg} \rightrightarrows \Gamma_{\mathscr{E}}\text{-}\mathbf{Alg}$.*

For a second-order signature Σ, the equational systems formalism allows one to write

$$
\mathbf{Mod}(\Sigma) \xhookrightarrow{\text{eq}} \mathscr{F}'_\Sigma\text{-}\mathbf{Alg} \rightrightarrows \Gamma_\Sigma\text{-}\mathbf{Alg} \quad ,
$$

where $\mathscr{F}'_\Sigma(X) = \mathscr{F}_\Sigma(X) + V + X \bullet X$, and the parallel pair encodes the equations of Σ-monoids. For a second-order equational presentation $\mathscr{E} = (\Sigma, E)$, we further have

$$
\begin{array}{ccccc}
 & & \mathbf{Mod}(\mathscr{E}) & & \\
 & & \big\downarrow {\scriptstyle \text{eq}} & & \\
\Gamma_\Sigma\text{-}\mathbf{Alg} & \leftleftarrows & \mathscr{F}'_\Sigma\text{-}\mathbf{Alg} & \rightrightarrows & \Gamma_E\text{-}\mathbf{Alg} \quad ,
\end{array}
$$

where the left parallel pair encodes the Σ-monoids (or substitution structure) as above, and the parallel pair to the right encodes the equations in E. We therefore get the equivalent equaliser diagram

$$
\mathbf{Mod}(\mathscr{E}) \xhookrightarrow{\text{eq}} \mathscr{F}'_\Sigma\text{-}\mathbf{Alg} \rightrightarrows (\Gamma_\Sigma + \Gamma_E)\text{-}\mathbf{Alg} \quad ,
$$

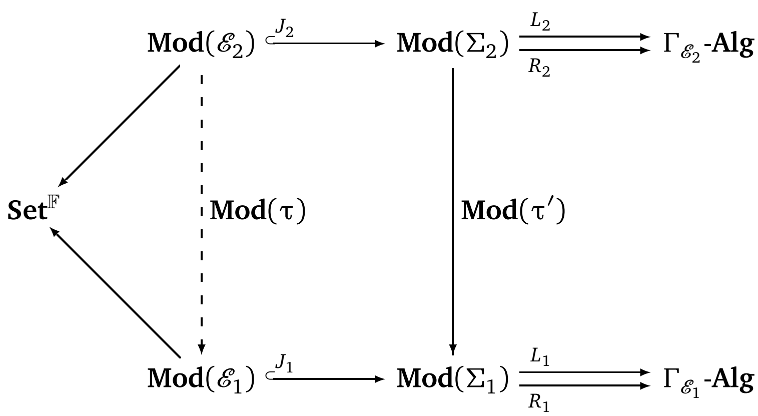
so that in fact one has

$$
\mathbf{Mod}(\mathscr{E}) \xhookrightarrow{\text{eq}} \mathbf{Mod}(\Sigma) \rightrightarrows \Gamma_{\mathscr{E}}\text{-}\mathbf{Alg} \quad .
$$

The previous discussion shows that the elegance of this abstract formalism of equational systems lies (partly) in the fact that the category $\mathbb{S}_{\mathscr{E}}$-**Alg** of algebras for $\mathbb{S}_{\mathscr{E}}$ is in fact isomorphic to the category **Mod**($\mathscr{E}$) of models for the equational presentation $\mathscr{E}$. We recall some relevant fundamental results.

Proposition 8.11. *The category* $\mathbb{S}_{\mathscr{E}}$-**Alg** *is a cocomplete, full reflective subcategory of* $\mathscr{F}_{\Sigma}$-**Alg**. *Moreover, the forgetful functor* $\mathbb{S}_{\mathscr{E}}$-**Alg** $\to$ **Set**$^{\mathbb{F}}$ *has a left adjoint, and the resulting adjunction is monadic.*

Now, we use this framework to derive algebraic functors between categories of models for second-order equational presentations, or equivalently, for equational systems. To this end, let $\mathscr{E}_1 = (\Sigma_1, E_1)$ and $\mathscr{E}_2 = (\Sigma_2, E_2)$ be second-order equational presentations, and $\tau\colon \mathscr{E}_1 \to \mathscr{E}_2$ a syntactic translation. Consider the following diagram:



Here, $\tau'\colon \Sigma_1 \to \Sigma_2$ is the restriction of τ to the underlying signatures of $\mathscr{E}_1$ and $\mathscr{E}_2$, and **Mod**(τ') is the induced algebraic functor **Mod**(Σ_2) $\to$ **Mod**(Σ_1), as derived above. Without going into the details here, it can be shown that **Mod**($\mathscr{E}_2$) together with the composite functor **Mod**(τ')$\circ J_2$ equalise the pair L_1, R_1. This is intuitively because axioms of $\mathscr{E}_1$ are mapped via the syntactic translation τ to theorems of $\mathscr{E}_2$. Hence, one gets the unique functor **Mod**(τ) making the above diagram commute. Furthermore, by the Adjoint Lifting Theorem (Theorem 8.4) and the monadicity result of Proposition 8.7, this functor will have a left adjoint, and the resulting adjunction is monadic.

We refer to **Mod**(τ): **Mod**($\mathscr{E}_2$) $\to$ **Mod**($\mathscr{E}_1$) as the second-order *syntactic algebraic functor* induced by the syntactic translation $\tau\colon \mathscr{E}_1 \to \mathscr{E}_2$. Using the Second-Order Semantic Categorical Type Theory Correspondence (Theorem 7.9), this functor can be shown to be naturally isomorphic to the composite

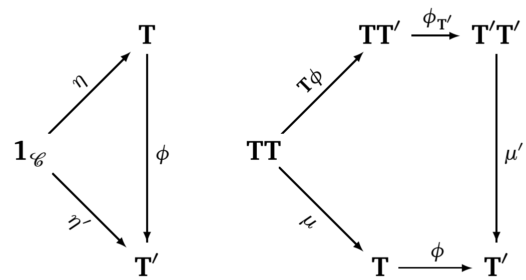
$$\mathbf{Mod}(\mathscr{E}_2) \;\cong\; \mathbb{M}\mathrm{od}(M_{\mathscr{E}_2}) \xrightarrow{\mathbb{M}\mathrm{od}(\mathbb{M}(\tau))} \mathbb{M}\mathrm{od}(M_{\mathscr{E}_1}) \;\cong\; \mathbf{Mod}(\mathscr{E}_1) \quad ,$$

where for $i = 1, 2$, $M_{\mathscr{E}_i}\colon \mathbb{M} \to \mathbb{M}(\mathscr{E}_i)$ is the algebraic theory classifiying $\mathscr{E}_i$, $\mathbb{M}(\tau)$ is the algebraic translation induced by τ, and $\mathbb{M}\mathrm{od}(\mathbb{M}(\tau))$ is its induced second-order algebraic functor.

8.2 Second-Order Monad Morphisms

We use the dual of the canonical definition of morphism between monads as in [Street, 1972] to recall the relation between monads induced by algebraic translations of algebraic theories.

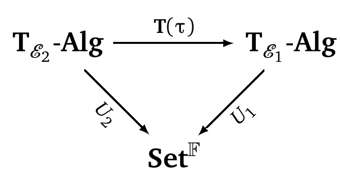
Definition 8.12 (Monad Morphism). Let $(\mathbf{T}, \eta, \mu)$ and $(\mathbf{T}', \eta', \mu')$ be monads on a category $\mathscr{C}$. A *monad morphism* $\phi\colon \mathbf{T} \to \mathbf{T}'$ is a natural transformation making the following diagrams commute.



Given two monad morphisms $\phi_1, \phi_2\colon \mathbf{T} \rightrightarrows \mathbf{T}$, a *homomorphism* of monad morphisms is a natural transformation $\sigma\colon \mathbf{1}_{\mathscr{C}} \to \mathbf{1}_{\mathscr{C}}$ such that $\phi_2 \circ \mathbf{T}\sigma = \sigma_{\mathbf{T}'} \circ \phi_1$.

Algebraic functors induce monad morphisms, but even stronger, these two fundamental notions of morphisms correspond bijectively to one another [Borceux, 1994]. We quickly illustrate these constructions and results in the second-order algebraic universe.

Recall that the category $\mathbf{Mod}(\mathscr{E})$ of models for a second-order equational presentation $\mathscr{E} = (\Sigma, E)$ is isomorphic to the category $\mathbf{T}_{\mathscr{E}}$-$\mathbf{Alg}$ of algebras for the monad $\mathbf{T}_{\mathscr{E}}$ induced by the adjunction $\mathbf{Mod}(\mathscr{E}) \leftrightarrows \mathbf{Set}^{\mathbb{F}}$. Using the same framework and notation of Section 8.1.3, consider the diagram



where $\mathbf{T}(\tau)$ is the algebraic functor obtained via composition of $\mathbf{Mod}(\tau)$ with the evident categorical equivalences, and is therefore naturally isomorphic to $\mathbf{Mod}(\tau)$ and $\mathbb{M}\mathrm{od}(\mathbb{M}(\tau))$. Let F_1 and F_2 be the left adjoints to the forgetful functors U_1 and U_2, respectively. Moreover, for $i = 1, 2$, let the canonical natural transformations of the adjunction $F_i \dashv U_i$ be given by $\alpha_i\colon \mathbf{1}_{\mathbf{Set}^{\mathbb{F}}} \to U_i \circ F_i$ and $\beta_i\colon F_i \circ U_i \to \mathbf{1}_{\mathbf{T}_{\mathscr{E}_i}\text{-}\mathbf{Alg}}$.

We define the monad morphism $\tau_{\mathbf{T}}\colon \mathbf{T}_{\mathscr{E}_1} \to \mathbf{T}_{\mathscr{E}_2}$ induced by the algebraic functor $\mathbf{T}(\tau)$ to be the composite

$$\mathbf{T}_{\mathscr{E}_1} \xrightarrow{\mathbf{T}_{\mathscr{E}_1}\alpha_2} \mathbf{T}_{\mathscr{E}_1}\mathbf{T}_{\mathscr{E}_2} = U_1F_1U_2F_2 = U_1F_1U_1\mathbf{T}(\tau)F_2 \xrightarrow{U_1\beta_1\mathbf{T}(\tau)F_2} U_1\mathbf{T}(\tau)F_2 = U_2F_2 = \mathbf{T}_{\mathscr{E}_2} \quad .$$

Indeed, it is straightforward to verify that the natural transformation $\tau_{\mathbf{T}}$ is a monad morpshism according to Definition 8.12. Moreover, the algebraic functor $\mathbf{T}(\tau)$ maps a $\mathbf{T}_{\mathscr{E}_2}$-algebra (A, φ) to the $\mathbf{T}_{\mathscr{E}_1}$-algebra $(A, \varphi \circ (\tau_{\mathbf{T}})_A)$. These constructions in fact define the bijective correspondence between (second-order) algebraic functors and monad morphisms. For a proof of a similar version of the following result, we refer the reader to [Borceux, 1994].

Proposition 8.13. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be second-order equational presentations. Monad morphisms $\mathbf{T}_{\mathscr{E}_1} \to \mathbf{T}_{\mathscr{E}_2}$ and algebraic functors $\mathbf{Mod}(\mathscr{E}_2) \to \mathbf{Mod}(\mathscr{E}_1)$ are in bijective correspondence.*

Corollary 8.14. *Given second-order algebraic theories $M_1 : \mathbb{M} \to \mathscr{M}_1$ and $M_2 : \mathbb{M} \to \mathscr{M}_2$, we have that monad morphisms $\mathbf{T}_{\mathfrak{E}(M_1)} \to \mathbf{T}_{\mathfrak{E}(M_2)}$ and algebraic functors $\mathbb{M}\mathrm{od}(M_2) \to \mathbb{M}\mathrm{od}(M_1)$ are in bijective correspondence.*

8.3 2-Categorical Type Theory Correspondence

By considering natural transformations between algebraic translations, one can form the 2-category of second-order algebraic theories. This can be mirrored syntactically by formalising a concept of *translation homomorphism*. We recall some basics of 2-category theory first.

8.3.1 Preliminaries on 2-categories

Recall that a 2-category is a category equipped with a notion of mapping between its morphisms. Each hom-set itself carries the structure of a category. Abstractly, a 2-category is a category enriched over **Cat**, the category of small categories, with the monoidal structure given by products of categories. We quickly review the more explicit definition of the basic elements of 2-category theory [Borceux, 1994].

2-categories. A 2-category C consists of a class ob(C) of objects or *0-cells*, together with, for each pair of 0-cells $A, B \in$ ob(C), a small category $\mathsf{C}(A, B)$ whose objects, denoted by $f : A \to B$, are called *1-cells*, and whose morphisms, denoted by $\alpha : f \Rightarrow g$, are called *2-cells*. Composition of 2-cells is referred to as *vertical* composition and denoted by $\bullet$. From the axioms of 2-category theory (see e.g. [Borceux, 1994]), it follows that 0-cells and 1-cells constitute a category, referred to as the underlying category of the 2-category.

Given small categories $\mathscr{A}$ and $\mathscr{B}$, one may take 1-cells to be functors $\mathscr{A} \to \mathscr{B}$ and 2-cells to be natural transformations. The most prototypical example of a 2-category is Cat, the 2-category of all small categories, functors and natural transformations.

Example 8.15.

(1) The 2-category SOAT *has 0-cells given by second-order algebraic theories, 1-cells given by their algebraic translations, and 2-cells given by natural transformations. Composition of 1-cells and 2-cells are the usual composition of functors and natural transformations, respectively.*

(2) We define the 2-category SOALG *to have second-order algebraic categories as 0-cells, second-order algebraic functors as 1-cells, and again natural transformations as 2-cells.*

We aim to construct the 2-category SOEP of second-order equational presentations. Translation homomorphisms (defined in the next section) will be taken to be the 2-cells.

2-functors. Given two 2-categories C and D, a 2-functor $F\colon \mathsf{C} \to \mathsf{D}$ assigns to every 0-cell A of C a 0-cell $FA \in \mathrm{ob}(\mathsf{D})$, and to every pair of objects A, B of C a functor $F_{A,B}\colon \mathsf{C}(A,B) \to \mathsf{D}(FA, FB)$ satisfying the canonical requirements of compatibility with composition and identity.

Biequivalence. We start by defining *internal equivalence* of 0-cells. Two objects A and B of a 2-category C are internally equivalent in C if there is a pair of 1-cells $f : A \to B$ and $g : B \to A$ such that $g \circ f \cong \mathrm{id}_A$ in the category $\mathsf{C}(A,A)$, and $f \circ g \cong \mathrm{id}_B$ in $\mathsf{C}(B,B)$. Two 2-categories C and D are *biequivalent* if there is a 2-functor $F\colon \mathsf{C} \to \mathsf{D}$ which is

1. locally an equivalence, that is for all $A, B \in \mathrm{ob}(\mathsf{C})$, the functor $F_{A,B}\colon \mathsf{C}(A,B) \to \mathsf{D}(FA, FB)$ is an equivalence;
2. surjective up to internal equivalence, that is for all $D \in \mathrm{ob}(\mathsf{D})$, there exists an object $C \in \mathrm{ob}(\mathsf{C})$ such that FC is internally equivalent to D in D.

8.3.2 Translation homomorphisms

Suppose we are given two syntactic translations $\tau_1, \tau_2\colon \mathscr{E}_1 \rightrightarrows \mathscr{E}_2$ of second-order equational presentations $\mathscr{E}_1 = (\Sigma_1, E_1)$ and $\mathscr{E}_2 = (\Sigma_2, E_2)$. A syntactic *translation homomorphism* $h\colon \tau_1 \to \tau_2$ is given by an $\mathbb{N}^*$-indexed collection of Σ_2 term tuples

$$\left\{ \left\langle \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1^{(i)}, \ldots, x_{m_i}^{(i)} \vdash h^{(i)}_{(m_1,\ldots,m_k)} \right\rangle_{i \in \|k\|} \right\}_{(m_1,\ldots,m_k) \in \mathbb{N}^*} ,$$

such that, for all terms $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash t$ of Σ_1, the diagram

$$\begin{array}{ccc}
(m_1, \ldots, m_k) & \xrightarrow{\left\langle [h^{(i)}_{(m_1,\ldots,m_k)}]_{\mathscr{E}_2} \right\rangle_{i \in \|k\|}} & (m_1, \ldots, m_k) \\
{\scriptstyle \langle [\tau_1(t)]_{\mathscr{E}_2} \rangle} \big\downarrow & & \big\downarrow {\scriptstyle \langle [\tau_2(t)]_{\mathscr{E}_2} \rangle} \\
(n) & \xrightarrow{\langle [h_{(n)}]_{\mathscr{E}_2} \rangle} & (n)
\end{array}$$

commutes in the classifying category $\mathbb{M}(\mathscr{E}_2)$ of the presentation $\mathscr{E}_2$. Note that this commutativity condition can be expressed syntactically via substitution and metasubstitution, but we shall not go into these explicit details here.

Remark 8.16. The choice of the terminology *homomorphism* for morphisms of syntactic translations is no coincidence. We observed in Section 7.5 that syntactic translations can be thought of as syntactic models of equational presentations in equational presentations. Therefore, an appropriate notion of morphism should commute with the syntactic model structures; more precisely, with the terms defining the mappings of translations.

A syntactic translation homomorphism is the appropriate formalisation of the notion of morphism between second-order syntactic translations. Indeed, for equational presentations $\mathscr{E}_1$ and $\mathscr{E}_2$, we obtain the category $\mathbf{Trans}(\mathscr{E}_1, \mathscr{E}_2)$ with objects second-order syntactic translations $\mathscr{E}_1 \to \mathscr{E}_2$, and morphisms given by translation homomorphisms.

Proposition 8.17. *The category* $\mathbf{Trans}(\mathscr{E}_1, \mathscr{E}_2)$ *is equivalent to the category* $\mathbf{AlgTrans}(M_{\mathscr{E}_1}, M_{\mathscr{E}_2})$ *whose objects are algebraic translations between the classifying algebraic theories* $M_{\mathscr{E}_1} : \mathbb{M} \to \mathbb{M}(\mathscr{E}_1)$ *and* $M_{\mathscr{E}_2} : \mathbb{M} \to \mathbb{M}(\mathscr{E}_2)$*, and whose morphisms are natural transformations.*

Moreover, we obtain a 2-categorical structure over second-order equational presentations (0-cells), syntactic translations (1-cells), and their homomorphisms (2-cells). We denote the resulting 2-category by SOEP. Using the previous Proposition, together with the Second-Order Syntactic Categorical Type Theory Correspondence (Theorem 6.6), we obtain the following fundamental result.

Theorem 8.18 (2-Categorical Type Theory Correspondence)**.** *The 2-categories* SOAT *and* SOEP *are biequivalent.*

8.4 Future Directions

We discuss two directions for future research. The first of these proposals is to extend the categorical algebra framework further beyond the second-order universe to include type dependency. The second is to develop a unified mathematical framework for theories of translations.

8.4.1 Dependently-sorted algebraic theories

We advocate the following general methodology for investigating categorical algebraic frameworks for syntactic equational presentations $\mathscr{T}$:

1. Construct the base category representing the elementary theory of equality corresponding to $\mathscr{T}$. Morphisms of that category are equivalence classes of terms built over the signature of $\mathscr{T}$ excluding its operators.

2. Classify the base category via a universal structure.

3. Define the algebraic theory corresponding to $\mathscr{T}$ as a suitable structure-preserving functor from the base category into a suitably structured category.

4. Define a functorial model to be a structure preserving functor from the algebraic theory to **Set**.

In the universe of dependently-sorted syntax, a mathematical formulation of a system of dependent sorts has already been developed [Jacobs, 1999, Fiore, 2008]. We recall this framework in the first-order setting.

First-order sort dependency. In dependently-sorted syntax, a variable $x : \sigma$ may occur in another sort $\sigma'(x)$: `sort` [Cartmell, 1986]. Formally, one can specify a first-order dependently-sorted signature to be given by:

- a countable sequence of judgements $(\Gamma_i \vdash S_i)_{i\geq 1}$ such that every $(\Gamma_{n+1} \vdash S_{n+1})$ is derivable from $(\Gamma_1 \vdash S_1, \ldots, \Gamma_n \vdash S_n)$; together with

- a countable sequence of operator judgements $(\Delta_i \vdash F_i)_{i\geq 1}$ such that every $(\Delta_{n+1} \vdash F_{n+1})$ is derivable from $(\Gamma_i \vdash S_i)_{i\geq 1}$ and $(\Delta_1 \vdash F_1, \ldots, \Delta_n \vdash F_n)$.

Abstract syntax and model theory for dependently-sorted algebra has been developed by Cartmell in [Cartmell, 1986] and [Fiore, 2008] (see also [Pitts, 2000] and [Taylor, 1999]). It would be interesting to investigate the combination of these approaches in the view of the aforementioned methodology to unify them in the context of Lawvere's framework for categorical algebra.

8.4.2 Towards a unified theory of translations

We believe that the notions of algebraic and syntactic translations between algebraic theories and equational presentations, respectively, will gain importance in the ever more pressing problem of organising and relating theories of computations. By a unified theory of translations we mean a formal mathematical framework which characterises translations and develops their properties and relationships. For instance, one my want to define an even more general notion of translation which allows interpreting *different* algebraic systems in one another. Other developments in this framework include the following:

- Give concrete descriptions of so-called *universal translations*, which include notions of initial and terminal translations and (co)limit constructions on translations.

- In particular, and because of the essentially (co)cartesian structure of algebraic systems, develop a concrete notion of product and coproduct of translations. Moreover, one may investigate the structure of tensor products of syntactic translations.

- One may seek general criteria for achieving certain canonical relationships amongst algebraic systems, such as inclusions, equivalences, and conservative extensions.

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