***Traité du triangle arithmétique*, 1654**

In 1654 Blaise Pascal entered into correspondence with Pierre de Fermat of Toulouse about some problems in calculating the odds in games of chance, as a result of which he wrote the *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matière*, probably in August of that year. Not published until 1665, this work, and the correspondence itself which was published in 1679, is the basis of Pascal's reputation in probability theory as the originator of the concept of expectation and its use recursively to solve the 'Problem of Points', as well as the justification for calling the arithmetical triangle 'Pascal's triangle'. These advances are considered to be the foundation of probability theory.

The *Traité du triangle arithmétique* itself is 36 pages long (setting aside *quelques autres petits traitez sur la mesme matière*) and consists of two parts. The first carries the title by which the whole is usually known, in English translation *A Treatise on the Arithmetical Triangle*, and is an account of the arithmetical triangle as a piece of pure mathematics. The second part *Uses of the Arithmetical Triangle* consists of four sections:

Use (1) *... in the theory of figurate numbers*
(2) *... in the theory of combinations*
(3) *... in dividing the stakes in games of chance*
(4) *... in finding the powers of binomial expressions*.

Pascal opens the first part by defining an unbounded rectangular array like a matrix in which 'The number in each cell is equal to that in the preceding cell in the same column plus that in the preceding cell in the same row', and he considers the special case in which the cells of the first row and column each contain 1 (Figure 1). Symbolically, he has defined $\{f_{i,j}\}$ where

$$f_{i,j} = f_{i-1,j} + f_{i,j-1}, \qquad i,j = 2, 3, 4, ...,$$
$$f_{i,1} = f_{1,j} = 1, \qquad i,j = 1, 2, 3, ...$$

1	1	1	1	1	1	1	.	.	.
1	2	3	4	5	6	.	.	.	.
1	3	6	10	15	.	.	.	.	.
1	4	10	20	.	.	.	.	.	.
1	5	15	.	.	.	.	.	.	.
1	6	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Figure 1. Pascal's arithmetical triangle

The rest of Part I is devoted to the demonstration of nineteen corollaries flowing from this definition, and concludes with a 'problem'. The corollaries include all the common relations among the *binomial coefficients* (as the entries of the triangle are now universally called), none of which was new. Pascal proves the twelfth corollary,

$$(i-1)f_{i,j} = jf_{i-1,j+1} \text{ in our notation,}$$

by explicit use of mathematical induction. The 'problem' is to find $f_{i,j}$ as a function of i and j, which Pascal does by applying the twelfth corollary recursively. Part I of the *Treatise* thus amounts to a systematic development of all the main results then known about the properties of the numbers in the arithmetical triangle.

In Part II Pascal turns to the applications of these numbers. The numbers thus defined have three different interpretations, each of great antiquity (which he does not, however, mention). The successive rows of the triangle define the *figurate numbers* which have their roots in Pythagorean arithmetic. Pascal treats these in section (1).

The second interpretation is as *binomial numbers*, the coefficients of a binomial expansion, which are arrayed in the successive diagonals, their identity with the figurate numbers having been recognized in Persia and China in the eleventh century and in Europe in the sixteenth. The above definition of $f_{i,j}$ is obvious on considering the expansion of both sides of

$$(x+y)^n = (x+y)\,(x+y)^{n-1}.$$

The fact that the coefficient of $x^r y^{n-r}$ in the expansion of $(x+y)^n$ may be expressed as

$$\frac{n(n-1)(n-2)...(n-r+1)}{1.2.3...r} = \binom{n}{r}$$

was known to the Arabs in the thirteenth century and to the Renaissance mathematician Cardano in 1570. It provides a closed form for $f_{i,j}$, with $n = i+j-2$ and $r = i-1$. Pascal treats the binomial interpretation in section (4).

The third interpretation is as a *combinatorial number*, for the number of combinations of n different things taken r at a time, ^nC_r, is equal to

$$\binom{n}{r},$$

a result known in India in the ninth century, to Hebrew writers in the fourteenth century, and to Cardano in 1550. Pascal deals with this interpretation in section (2), giving a novel demonstration of the combinatorial version of the basic addition relation

$$^{n+1}C_{r+1} = {}^nC_r + {}^nC_{r+1},$$

for, considering any particular one of the $n+1$ things, ^nC_r gives the number of combinations that include it and $^nC_{r+1}$ the number that exclude it, the two together giving the total.

In section (3) Pascal breaks new ground, and this section, taken together with his correspondence with Fermat, is the basis of his reputation as the father of probability theory. In it he amplifies and formalises the solution of the Problem of Points which he had discussed with Fermat, calling it *La règle des partis*. They had both arrived at the combinatorial solution involving the counting of all the ways in which the game could have been completed. Pascal, however, does not refer to this method explicitly in the *Traité*, preferring to prove the same result by mathematical induction based on his method of expectations. It is a brilliant display using the results recorded earlier in the *Traité*, and is prized as the birth of modern probability theory.

This introduction is based on part of a chapter in *The Cambridge Companion to Pascal* (Nicholas Hammond, ed., Cambridge University Press, forthcoming) with the permission of the publisher. A fuller description of the *Traité* may be found in *Pascal's Arithmetical Triangle* by A.W.F.Edwards (second edition, 2002, Johns Hopkins University Press).

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