Novel Geometry Gradient Coils for MRI Designed by Genetic Algorithm

Guy Barnett Williams
Downing College

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This thesis concerns the design of gradient coils for magnetic resonance imaging systems. The method of design by genetic algorithm optimisation is applied to novel gradient geometries both by use of conventional computer facilities, and, by parallelisation of the design algorithm, on a supercomputer architecture. Geometries and regions of interests which are inaccessible to analytic solution are considered, and the criteria which are difficult to include in such algorithms, such as the robustness of the design, are also included.

To exemplify this, in the first instance a two axis biplanar coil was designed and the performance of the genetic algorithm tested and evaluated. The coil was tested computationally; a working example was constructed and tested in a MRI scanner both on phantom objects and on a human knee. Consideration of the usefulness of the coil regions not optimised for linearity for image reconstruction was done. The gradient efficiencies of the final designs in the z and y directions respectively were $0.3 \text{ mTm}^{-1}\text{A}^{-1}$ and $0.4 \text{ mTm}^{-1}\text{A}^{-1}$ over a 15 cm diameter region of interest. The size of the interior of the gradient set was designed to be $40.0 \text{ cm} \times 24.4 \text{ cm} \times 40.0 \text{ cm}$, to fit within the confines of the bore of an existing scanner. The linearity in the primary direction over the region of optimisation was less than 5% for both coils.

The algorithm was extended for operation on a Hitachi SR2201 supercomputer using parallelisation. The performance in this mode was evaluated and found to be favourable in comparison with the standard computer architecture, with an increase in speed in real time of a factor of more than 40 in some configurations of the supercomputer. Various polygonal cross-section design shapes requiring the use of this improved computer performance were optimised and evaluated computationally. Such designs have previously been inaccessible to the genetic algorithm optimisation model. Tests were made between the performance of the genetic algorithm on various similar design problems, and simulated images from such gradient coils were produced. Finally an example of a transverse coaxial return path gradient coil is presented computationally. This coil had an internal diameter of 32 cm, and external diameter of 44 cm and a length of 40 cm. It achieved a strength of $0.1 \text{ mTm}^{-1}\text{A}^{-1}$ over a cylinder of diameter 20 cm and length 25 cm, with a deviation from linearity of less than 5% over this volume.
This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where explicitly indicated in the text. The number of words in this thesis (including footnotes, tables and appendices) is less than 60,000.

G. Williams

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"The time has come," the Walrus said,
"To talk of many things:
Of shoes - and ships - and sealing wax -
Of cabbages - and kings -
And why the sea is boiling hot -
And whether pigs have wings."

THE WALRUS AND THE CARPENTER,
from THROUGH THE LOOKING GLASS,
Lewis Carroll 1872.
Illustration by Sir John Tenniel.

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Abbreviations

CT        Computed Tomography
EPI       Echo Planar Imaging
GA        Genetic Algorithm
GRASS     Gradient Recalled Acquisition in the Steady State
HPCF      High Performance Computing Facility
HSLMC     Herchel Smith Laboratory for Medicinal Chemistry
MPI       Message Passing Interface
MRI       Magnetic Resonance Imaging
PET       Positron Emission Tomography
r.f.      Radio frequency
e.m.f.    Electro-motive force
r.m.s.    Root mean squared
Chapter 1

Introduction

Over the past two decades magnetic resonance imaging (MRI) has evolved into an important tool for clinical diagnostic radiology. In comparison with older, more established techniques it provides great anatomical detail and contrasts from soft tissue; it also has far fewer risks associated with it than modalities which require ionising radiation.

However, although the advantages over more conventional scanning equipment are clear, there are problems associated with the hardware necessary for MRI which has slowed its wider acceptance. The magnets themselves are very big, heavy and their interiors can be claustrophobic. The cost increases with the size of the region over which imaging is to be performed, and early designs of hardware were not optimised to make best use of the space available. Older hardware is not able to make use of more recent developments such as Echo Planar Imaging (EPI) which make greater demands of the specifications. Acoustic noise is also a problem, and increasingly so with more modern MRI imaging sequences.

All these problems are currently being addressed. Perhaps the most dramatic improvements can occur in magnet design, including open C-shaped magnets which are now available for clinical use. In this laboratory, development of novel hardware has focused on the gradient coils since these are usually the limiting factor for the implementation
of modern scan acquisitions within existing magnetic resonance systems. This thesis describes work which has been done as a member of this group within the Herchel Smith Laboratory for Medicinal Chemistry (HSLMC), in particular developing the earlier work of Dr. B. Fisher.1

Over the course of this work, advances have been made in computing power which have enabled previously unacceptably computationally expensive calculations to be performed and new geometries to be modelled. Early designs were driven largely by the accessibility of a method which could easily be implemented and this is no longer the case. Starting with a biplanar geometry, this work describes the design development of novel geometries of gradient coils, using genetic algorithm optimisation as the design method. The biplanar geometry is demonstrated and analyzed and implications for future coil design and construction is discussed.

The ultimate incarnation of this increased computer power presented within this work is that of the Hitachi SR2201 supercomputer. This thesis describes and details the migration of the algorithm to this architecture and the speed benefits of this transition; design geometries are then presented using this increase in computer performance.

1.1 Overview of thesis

Chapter 2 details some basic MRI theory and the role of the gradient coil within the MR system; in particular, the design features required of gradient coils are discussed. The complications arising from the restraints of both the requirements of the other hardware and also those from fundamental physical theory are detailed. Gradient coil designs from the literature are discussed in this context where the authors have paid particular attention to one or more of these restrictions.

Chapter 3 describes methods which have previously been used for gradient coil design and discusses their relative merits and disadvantages. The particular approach of genetic algorithm design is further detailed since it is this optimisation method that is employed throughout this thesis.

Chapter 4 applies the genetic algorithm to the problem of biplanar gradient design, using the methods previously discussed in Chapter 3. This particular geometry was of interest within the laboratory due to the restricted space within existing reduced bore hardware. Larger samples and potential research where greater flexibility was required could be explored. For example, the movement of a knee under different degrees of flexion is a possible application. There is extensive discussion of the issues arising from the design of this gradient coil, including a demonstration of the convergence properties of the genetic algorithm. The gradient coil was constructed and tested both on the bench and in the scanner, and sample images are presented.

Chapter 5 addresses the possibilities opened up for this design technique by the introduction of high performance computing. One of the major disadvantages of this approach to gradient design is the amount of time taken to produce results, and high performance computing has the potential to reduce this. The amount by which a supercomputing architecture is capable of speeding up this application is quantified, and a geometrically inaccessible problem comprising gradient units of polygonal cross-section is demonstrated computationally for transverse coils. Finally, the supercomputer is applied to a transverse gradient design problem with coaxial current return paths.

Concluding remarks and opportunities for further developments are detailed in Chapter 6, both in terms of the refinement of the current techniques and in terms of its applicability to other problems.
Chapter 2

MRI Theory

2.1 Introduction

In 1945 Bloch and Purcell working in groups at Massachusetts Institute of Technology and Stanford University independently demonstrated the property of nuclear paramagnetism; they showed that there was a small but detectable magnetic moment induced in protons within solids and liquids when placed in a magnetic field, and that by using radio frequency energy information could be gleaned about its magnitude.\(^2\)\(^3\)

Although it was to be a further 25 years before the first proton images were produced using this technique,\(^4\) and with further developments more data can be obtained from the effect, the same nuclear paramagnetic theory underpins all MR technology. It is intrinsically insensitive - it is perhaps surprising that any information can be obtained at all, even in very high magnetic fields. But by using clever manipulation, images can be obtained that rival other imaging protocols such as Computed Tomography (CT) scans or Positron Emission Tomography (PET) scans without the need for high frequency ionising radiation.

In this chapter the basic physical principles involved in this phenomenon are reviewed, and the techniques that may be employed to exploit it are detailed with particular reference to the hardware responsible for spatial localisation - the magnetic field gradient coil (or
2.2 Nuclear Paramagnetism

The nucleus of every atom has associated with it a certain intrinsic angular momentum (spin) which does not correspond to any actual motion of the nuclei. The law of conservation of momentum must however include it as a factor to be correct on the nuclear level. The reason for this strange nuclear property does not concern us here, but it can be viewed as arising from the motion of the particles which make up the nucleus on a fundamental level: the quarks which bind to form the individual nucleons.\(^6\)

The intrinsic angular momentum is characterised by a spin quantum number. For nuclei with an even number of nucleons, the spin number is zero. For odd numbered nuclei, the spin number (in its unexcited nuclear state, which is the case except under very high energy conditions\(^6\)) is half-integral.

For example the most common isotope of Hydrogen, \(^1\)H, has a spin number of \(\frac{1}{2}\), while sodium-23 (\(^{23}\)Na) has spin number \(\frac{3}{2}\). The spin magnitude is related to the associated spin quantum number by 

\[
I = \sqrt{l(l + 1)},
\]

where \(I\) is the spin number and \(l\) is the magnitude of the spin.

But \(l\) is not the only quantum number associated with spin. As well as its magnitude, the laws of quantum mechanics allow us to know the component of spin along one direction (but no others). The spin component along this direction must have a value \(m\) in the range \(-l, -l+1, ..., l-1, l\). So for a \(^1\)H nucleus, the allowed quantum numbers are \(-\frac{1}{2}\) and \(+\frac{1}{2}\). By convention, we take this component to be along the z-axis. The two numbers \(m\) and \(l\) tell us all we can know about the spin state of an individual quantum spin. This restriction gives rise to a range of possible directions for the spin vector, as shown in Fig. 2.1.

The angular momentum measured due to the spin is related by the conversion factor of Planck’s constant, \(h\): 

\[
J = hI \tag{2.1}
\]

In the absence of an applied magnetic field, all these possible directions have the same energy associated with them. However the nuclear angular momentum we have been considering gives rise to a magnetic moment which is proportional to the spin. The relationship is given by \(\mu = \gamma J\). The constant of proportionality, known as the gyromagnetic ratio (\(\gamma\)), varies between nuclei, as is shown in Table 2.2 for various cases.\(^7\)

Now, a magnetic moment \(\mu\) within a magnetic field \(B\) has an energy \(E\) associated with it, where \(E\) is given by \(E = -\mu \cdot B\). If we consider a magnetic field \(B_0\) of strength \(B_0\) applied along the z-axis, we will cause a split in the degenerate energy levels of nuclei with non-zero spin. This is known as the Zeeman effect. By creating an energy difference between quantum levels we are producing a preference for the nuclear moments to orientate closer to the direction of the applied magnetic field. This will produce a net magnetic moment as Bloch and Purcell observed, and we shall quantify this below. The nuclear energy level splitting is shown in for two different nuclei in Fig. 2.2.

For the simple case of the \(^1\)H atom (\(l=1/2\)) where there are only two levels, we can see...
from the equation \( E = -\mu \cdot B \) that the splitting is \( \gamma \hbar B_0 \), where \( \hbar \) is Planck’s constant. \( ^1H \) atoms are those which are most commonly used in imaging. The proportion of atoms in the “parallel” state as opposed to the higher energy “anti-parallel” state will be given by a Boltzmann distribution. This states that the number in each state will be proportional to \( e^{-E/kt} \), where \( E \) is the energy of the state. So the fraction of atoms in each state, \( n_1 \) and \( n_{-1} \) for parallel and anti-parallel respectively will be given by:

\[
\begin{align*}
  n_1 &= \frac{e^{\gamma \hbar B_0/2kT}}{e^{\gamma \hbar B_0/2kT} + e^{-\gamma \hbar B_0/2kT}} \\
  n_{-1} &= \frac{e^{-\gamma \hbar B_0/2kT}}{e^{\gamma \hbar B_0/2kT} + e^{-\gamma \hbar B_0/2kT}}
\end{align*}
\]

(2.2)

(2.3)

Note that for typical values for the field and temperature \( (T = 298K, B_0 = 2T) \), this is still a very small effect, with only one in every million spins aligned anti-parallel. The average magnetisation

\[
\mu = \frac{1}{2} \hbar \gamma \left( \frac{e^{\gamma \hbar B_0/2kT} - e^{-\gamma \hbar B_0/2kT}}{e^{\gamma \hbar B_0/2kT} + e^{-\gamma \hbar B_0/2kT}} \right) = \frac{1}{2} \hbar \gamma \tanh(\gamma \hbar B_0/2kT) \]

(2.4)

is also very small, even averaged over many atoms. It is the manipulation of this magnetisation which allows us to derive our magnetic resonance signal.

2.3 Semi-Classical Approach

The quantum description of the behaviour of the nuclei underpins the theory of MRI, but it is not usually the most appropriate way to represent it beyond this stage. If we can assume that there is no interaction between the spins so that they all react independently to the magnetic field, we can define a parameter for bulk magnetisation per unit volume \( M \), such that

\[
M = \sum_{\text{all spins}} \mu_i \]

(2.5)

This assumption allows us to treat the magnetisation classically, and relates our algebra to an observable quantity unlike the very small individual nuclear magnetic moments. It
is valid for all the theory which follows.

From Equation 2.4, we can see that the net magnetisation will be given by

\[
M = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \hbar \gamma \tanh(\gamma \hbar B_0 / 2kT) \end{pmatrix}
\]  

(2.6)

where \( n \) is the number of spins per unit volume of the sample. This is the macroscopic description of the magnetisation at thermal equilibrium.

The effect of a magnetic field on a magnetic dipole is to produce a torque on it given by the equation

\[
\frac{dJ}{dt} = \mu \times B
\]

(2.7)

where \( J \) is the angular momentum of the dipole. In our case the angular momentum and magnetisation are parallel and are related by \( \mu = \gamma J \), so for our net magnetisation we can write

\[
\frac{dM}{dt} = \gamma M \times B_0
\]

(2.8)

At equilibrium, the direction of \( M \) will clearly be parallel to the magnetic field, and so there will be no change in \( M \) with time. However if \( M \) is not parallel to \( B_0 \), there will be a precession of \( M \) around the axis of \( B_0 \), with an angular frequency of \( \gamma B_0 \). This is known as the Larmor frequency.

Let us consider the rotating frame about the axis of \( B_0 \) which has angular frequency \( \omega = \gamma B_0 \). We can represent the magnetisation in this rotating frame by

\[
\frac{dM}{dt} = \gamma M \times B_0 - M \times \omega = \gamma M \times (B_0 - \omega / \gamma).
\]

(2.9)

In this frame the magnetisation is stationary. However if we apply a constant \( B_1 \) field of amplitude \( B_1 \) in the plane transverse to the axis of rotation then the effect will be to rotate the magnetisation down towards the transverse plane. In the stationary frame this is the equivalent of a rotating \( B_1 \) field of frequency \( \omega \) and produces a spiralling of the magnetisation vector down towards the transverse plane. Such a \( B_1 \) field can be represented by

\[
B_1 = B_1 \cos \omega t + B_1 \sin \omega t
\]

(2.10)

This is shown for both the rotating and stationary frames in Fig. 2.3.

![Figure 2.3](image_url)

**Figure 2.3**: a) The motion of the magnetisation in the rotating frame under the influence of a \( B_1 \) field. b) The corresponding motion in the stationary frame of reference. The magnetisation is constant in the rotating frame but a radio frequency field in the stationary frame, causing a precession of the magnetisation vector.

It is easy to produce at least one component of such a \( B_1 \) field, say \( B_1 \cos \omega t \), by using an radio-frequency transmitter with polarisation in this direction. We can represent the oscillating field produced by this as a sum of a counter-rotating and co-rotating fields, i.e. \( B_1 \cos \omega t = \frac{1}{2}(e^{-j\omega t} + e^{j\omega t}) \). Now the latter term is the resonance term required above, while the first term has a negligible effect provided \( B_1 < B_0 \). Hence we can produce a perturbation in the magnetisation of our sample using an radio frequency (r.f.) pulse at the correct frequency \( \omega \). The design of the r.f. pulses is very complicated, but for the moment we need only consider two requirements, other than the correct frequency. The first is the angle that the pulse rotates the magnetisation by from the vertical, and the second is the axis about which it does so. Pulses are usually referred to by the angle by
which they rotate the magnetisation, i.e. for many sequences 90° or 180°. The axis about which this rotation acts is defined by phasing them with respect to a reference signal which is maintained at the Larmor frequency. This reference signal has the effect of defining a set of axes in the rotating frame, and is essential for interpreting the output from the system.

Once there is a component of magnetisation in the transverse plane, we are able to detect it using an r.f. receiver tuned in the same way as the r.f. transmitter above. We should view a signal oscillating at the Larmor frequency, as the constant amplitude transverse component is swept around the transverse plane. In fact it is possible to obtain both transverse components of our signal in the rotating frame from our single channel r.f. detector by comparing the phase of our output with our reference signal. This process is known as heterodyning.

So from each excitation we can acquire a two channel output, comprising our knowledge of the magnitude of the net magnetisation vector of the sample, and its phase relative to a frame of reference rotating at the Larmor frequency. This can be conveniently represented using complex notation, a format which is well suited to interpretation using Fourier analysis.

2.4 Relaxation Processes

Of course, our signal does not remain constant and will return to equilibrium over a period of time. There are three parameters associated with this process in static imaging: $T_1$, $T_2$ and $T_2'$. 

2.4.1 Longitudinal Relaxation

$T_1$ describes the characteristic time needed for the longitudinal relaxation to return to its equilibrium value. This process follows the following equation.

$$\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1}$$

(2.11)

This process occurs by the transfer of energy from the excited spins to other forms of energy in the lattice, such as thermal energy. It follows a random pattern, which is why it is best described on average by an exponential decay.

After a 90° pulse, the longitudinal magnetisation evolution would be described by

$$M_z = M_0(1 - e^{-t/T_1})$$

(2.12)

2.4.2 Transverse Relaxation

In contrast transverse relaxation is due to interactions between the spins themselves. Our signal is only present due to the fact that the precession of the spins is in phase. As they interact this phase coherence may be lost without any exchange of energy. This is also a random process and is thus also described on a macroscopic level by an exponential equation, with characteristic time $T_2$.

$$\frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2}$$

(2.13)

This interaction violates our assumption for the semi-classical model; and in some cases (e.g. magnetic quadrupole interactions) it becomes necessary to model using a more complex form than an exponential decay. However in most situations considered in MR systems this is not necessary.

Note that $T_2$ is always shorter than $T_1$ since $T_1$ also involves a loss of coherence during the exchange of energy.
There is another fundamentally different reason for transverse relaxation. This is due to $B_0$ inhomogeneity. If different spins in different parts of the sample are experiencing different magnetic fields, then the resulting different precession rates will also cause a loss of phase coherence. So we define another parameter, the effective transverse relaxation constant, as $T_1'$. If we say that the component of $T_2$ due to magnet inhomogeneity is $T_{2i}$, then the relationship between the three quantities is

$$\frac{1}{T_2'} = \frac{1}{T_2} + \frac{1}{T_{2i}}$$

(2.14)

Of course relaxation due to magnet inhomogeneity is not a random effect. The magnetisation lost can be recovered by a suitable pulse sequence, such as spin-echo (see Section 2.8.2), or minimised by shimming the magnet. This is not the case for intrinsic $T_2$ relaxation.

### 2.5 The Bloch Equations

We are now in a position to describe fully the evolution of the magnetisation. This is simplest in the rotating frame. Using Equations 2.9, 2.11 & 2.13, our three components are given by:

$$\frac{dM_x}{dt} = -\gamma M_y B_1 - (M_x - M_0)/T_1$$

(2.15)

$$\frac{dM_y}{dt} = \gamma M_x B_0 - \omega/\gamma - M_x/T_2$$

(2.16)

$$\frac{dM_z}{dt} = \gamma M_y B_1 - \gamma M_x (B_0 - \omega/\gamma) - M_y/T_2$$

(2.17)

These are known as the Bloch Equations.

### 2.6 The Magnetic Resonance Scanner

So far we have described the physical theory by which nuclear magnetic resonance signals may be detected by use of a magnetic field and probe tuned to the correct frequency. We have also described the time evolution of these signals in the undisturbed state. Now we turn our attention to the practical hardware necessary to acquire these signals.

Fig. 2.4 shows a schematic of a typical cylindrical whole body magnetic resonance scanner. The outermost surface is the magnet coils themselves, which are contained within a cryostat for the case of superconducting magnet. Within these are the shim coils. While a MR magnet is designed to be as homogeneous as possible, the introduction of a sample into the field inevitably produces inhomogeneities within this field due to susceptibility differences. These coils are used to attempt to correct for these distortions which will vary for different samples. The shim coils are coils of wire whose currents are independently controlled so as to optimise the static field. As well as such “active” shim coils, a typical magnet may also contain “passive” shims. Strips of iron may be used to direct the magnetic flux lines in order to improve the homogeneity of the magnet.

Within the shim apparatus are the gradient coils. These will be discussed further in the following sections. Essentially they are coils of wire similar to the shim coils and able to produce a linear variation in the field across the sample being imaged.
Within the gradient coils lies the radio frequency probe. This is usually separated from
the gradient coils with a radio frequency shield to prevent coupling between the probe and
the gradient coils. A copper gauze will absorb high frequency signals from the probe and
prevent it from coupling with the gradient coils; however the low frequency gradient pulses
will be unaffected and allowed to pass through.

2.6.1 Radio frequency probe design in practice

The details of gradient coil design will be discussed in detail later in this chapter. However
there are also relevant issues regarding r.f. coil design which we will review here. More
detailed descriptions may be found in the literature. In order to be appropriate for a
particular system, the probe must fit geometrically within the magnet and gradient coils
with which it is to be paired. It must also be sensitive over the appropriate region being
imaged and insensitive over all other regions.

The r.f. probe works by inductive coupling of the magnetic moments within the sample
with the coils within the probe. The electromotive force (e.m.f.) induced in the probe is
proportional to the interaction between the moment and the irradiating \( B_1 \) field from the
probe when the latter is used as a transmitter. This is maximal when the field profile
of the coil is perpendicular to the main field direction. Hence we want to choose a probe
gamma which produces fields perpendicular to \( B_0 \). These probe field are conventionally
called \( B_1 \) fields. Since we ideally require constant contrast, uniformity is also required
across the sample.

Probe geometry

There are several standard designs of r.f. probes which are used within magnetic re-
sonance. The simplest is a single loop of wire (tuned using capacitors to the correct resonant
frequency) placed with its axis perpendicular to the \( B_0 \) direction. This has the charac-
teristic of a very high \( B_1 \) response close to the coil, but a rapid fall off away from the
coil.

A development of this is the solenoidal coil, which consists of many such coils arranged
such that the sample sits within the coil. This improves the homogeneity but has a worse
local signal to noise ratio relative to the surface coil.

A frequently used r.f. coil used within cylindrical gradient coils is the quadrature
birdcage coil. This gives good homogeneity and signal to noise over a large volume.
It consists of single wire “struts” which run parallel to the \( B_0 \) direction. It also has the
advantage of consisting of two separate coils rotated by 90°. By combining the signal from
both coils, an increase in signal of \( \sqrt{2} \) can be achieved for no penalty.

More complicated designs employing similar strategies to gradient coil design exist in
the literature, but most are based around the birdcage design concept. Probe analysis
and design is a growing field, driven largely by regulatory requirements regarding power
deposition which are becoming more significant for modern in vivo acquisitions.

Transmit and receive modes

Different coils can be used for the transmit and receive modes, allowing (for example)
global excitation and local acquisition. A surface coil used for signal reception in conjunc-
tion with a large global excitation coil would have the advantages of high sensitivity and
thus low signal to noise, but would retain the advantages of the uniform excitation of the
larger coil. In this way the signal would not disappear as quickly away from the receiver
coin.
2.7 Image Acquisition

Using this array of hardware, we must now consider the methods used to acquire a signal and reconstruct an image. We will first consider how to obtain spatial information from within the sample. This is the primary role of the gradient coil. It is used to “label” different parts of the image. The method by which this is done is to vary the Larmor frequency at these different points.

Let us imagine we have three orthogonal magnetic field gradient generators which are capable of generating a linear magnetic field variation in the main $B_0$-field across our sample. If we assume that the variation is symmetric about the centre of the magnet, we can write our Larmor frequency as:

$$\omega = \gamma B_0 + \gamma G \cdot r$$

(2.18)

where $\mathbf{r}$ is the vector displacement from the centre of our system and

$$\mathbf{G} = \left( \frac{dB_x}{dx}, \frac{dB_y}{dy}, \frac{dB_z}{dz} \right),$$

(2.19)

where the amplitudes of the three components are controlled independently. However the labelling is not completely straightforward to interpret. By varying the frequency we have caused spin precession at different rates. The signal we acquire using our r.f. probe is the sum of the dephased signals from different regions, and hence we are still not able to directly acquire from a given position. Let us consider two ways in which we can do so indirectly.

2.7.1 Selective Excitation

By combining gradient pulses with r.f. pulses of small bandwidth, slices of a sample can be excited rather than its entirety.

If a magnetic field gradient is applied to the sample, then the precession frequencies will vary in the direction of the gradient. A slice orthogonal to this gradient can then be selected according to the precession frequency of the spins within it. An r.f. pulse that is frequency specific can then be used to excite only the spins within this slice, leaving the others at equilibrium. All the MR signal will now arise only from within this slice.

Such an r.f. pulse is referred to as a “soft” pulse. An r.f. pulse that is not frequency specific is referred to as a “hard” pulse. Soft pulses usually require more time to produce than hard pulses.

By repeated applications of this sequence it is possible (in theory at least) to isolate, a slice, and then a line, and finally a single voxel within a sample of arbitrarily small size. However in practice this would be very slow way to produce a complete image slice, and so we use Fourier analysis to simultaneously image many voxels. The following analysis may also be formed in three dimensions, rendering slice selection unnecessary.

2.7.2 Fourier Transforms: $k$-space

The concept of encoding the amplitude of signal from different positions as linearly related frequency components with the same amplitude is exactly analogous to a Fourier Transform.

If we apply a gradient pulse of amplitude $G$ for a period of time $t$ in the direction of the $x$-axis, the effect of the gradient will be to produce a phase shift of $e^{i\gamma G t x}$ to signal arising from a position $x$. In order to consider the signal $S(t)$ that we will acquire from the r.f. probe, we must integrate over all the spins which contribute to our signal (ignoring relaxation considerations), and are left with the expression

$$S(t) = \int_{-\infty}^{\infty} \rho(x)e^{i\gamma G t x} \, dx$$

(2.20)
This is a Fourier Transform which can be inverted to give:

\[
\rho(x) = \frac{\gamma G}{2\pi} \int_{-\infty}^{\infty} S(t)e^{-\gamma G x t} dt
\]  

(2.21)

This is useful, but only relates one dimension to the time domain in which we collect our signal - it does not help us to discriminate the second dimension within the slice.

Let us consider two independent gradient systems however, which produce gradients \(G_x\) and \(G_y\). Conventionally we write

\[
k = \frac{1}{2\pi} \int_0^1 G(t')dt'
\]  

(2.22)

(2.23)

(2.24)

or in vector notation and allowing \(G\) to vary with time

\[
k = \frac{\gamma}{2\pi} \int_0^1 G(t')dt'
\]  

(2.24)

This defines a point in "k-space", and by varying our gradient vector \(G\), we can control which point in k-space our signal is arising from, and thus redefine our signal \(S(t)\) as being \(S(k)\). Why is this useful? Consider again that the phase of the signal from a point \((x, y)\) is given a phase shift of \(e^{i\tau G x}\). By integrating over our slice we produce a signal:

\[
S(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(r)e^{i2\pi k x} dr dx
\]  

(2.25)

By Fourier inversion we can derive:

\[
\rho(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k)e^{-i2\pi k x} dk dx
\]  

(2.26)

and hence derive the signal for each position from a weighted integral of the signal that arises from a plane in k-space.

A k-space map contains all the information of a positional space map, with a 2-D Fourier Transform relating the two representations.

We can think of each position in k-space as relating to a different phase gradient across the sample in real space. Fig. 2.5 shows four points in k-space, and the corresponding phases of the spins in real space.

Figure 2.5 : Representation of k-space. Different positions in k-space relate to different phase gradients across our sample as the spins precess at the Larmor frequency. The vector sum will be different in each case.

Our signal at any one time is the vector sum of the magnetisations from each spin. If they are all in phase with each other having experienced identical magnetic fields, then we are sampling at the origin of k-space.
A linear gradient has the effect of varying the Larmor frequencies for the duration of the pulse, thus slowing some spins down and speeding others up. This creates a phase gradient across the sample - and moves our position in k-space, allowing us to sample at a different position. The degree of movement in k-space will be proportional to both the strength of the pulse and its duration.

By acquiring the value of our signal in the time domain as we pass through different points in k-space, we can acquire a k-space map, which can be Fourier transformed into real space. Note that each co-ordinate position in both real and k-space has a phase and magnitude value associated with it. In the simple case of Fig. 2.5 all points within the sample in real space have equal amplitudes (spin density) and zero phase. Generally, by using suitable pulse sequences the phase can be encoded with a measurable parameter other than simple spin density.

The object of all imaging protocols is to sample k-space efficiently, whether in a single acquisition or as a result of multiple acquisitions, in order to reproduce an image from the acquired signal. Different protocols have different speeds and are sensitive to noise in different ways, but the idea of k-space to acquire images holds good for all sequences below.

### 2.7.3 Fourier Imaging

Fig. 2.6 shows a simple imaging protocol called Fourier Imaging to acquire an image of a slice through a sample. The k-space trajectories are shown in Fig. 2.7. Each excitation results in one acquisition along the $k_x$-axis. The form of Fig. 2.6 requires some explanation although no great detail will be given here. Pulse sequences are frequently presented in this form. Each horizontal represents an input or output to the system as labelled. Time runs from left to right. The sequence starts with a slice select r.f. pulse and associated $z$ gradient pulse, before the selected spins are refocused (see section 2.8.2). The multiple lines in the $y$ gradient represent different amplitude levels for the gradient pulse as the sequence is rerun multiple times for further acquisitions at different values of $k_y$.

![Figure 2.6: Fourier Imaging Pulse sequence](image)

![Figure 2.7: The path traced in k-space by the Fourier Imaging sequence](image)

The method of encoding in the $x$ direction under a constant gradient pulse is known as read encoding, and the method of using repeated acquisitions after increasing amplitude $y$ gradient pulses in the $y$ direction is called phase encoding. There is no theoretical difference between the two once the data has been acquired, although the time delay in acquisition in the phase encoding direction may cause artifacts not present in the read encoding direction.
2.7.4 Echo Planar Imaging

More modern pulse sequences, such as echo planar imaging (EPI), can trace all of k-space in a single acquisition but these put a greater strain on the hardware.

![Figure 2.8: EPI Pulse sequence](image)

**Figure 2.8:** EPI Pulse sequence

Shown in Fig. 2.8 is an EPI sequence. EPI as an imaging protocol was developed by Mansfield in 1977\(^{16}\) but it is only more recently that it has become a useful technique due to the increased efficiency of both hardware and software. It requires very high gradients in the read direction (\(k_z\)) in order to sample a significant region of k-space within the natural time limit implied by \(T_2\).

**EPI techniques underlie many modern MRI techniques, in particular functional imaging where “snap-shot” imaging is required.**

2.8 Image Contrast

We now have described theoretical methods of manipulating and measuring the magnetisation of a sample at different points in space. Now we must consider methods for obtaining measurements of some of the parameters mentioned above.

In order to derive useful parameters from an MR-active sample, a suitable pulse sequence must be used. These combine transmit and receive modes in order to derive a meaningful output. Some examples are shown below for simple contrasts of \(T_1\), \(T_2\) and \(T_2^*\). These pulse sequences are not an exhaustive list of methods for obtaining such measurements, and are meant to be illustrative.

2.8.1 Gradient Echo

If a gradient pulse is applied for a period of time and then applied for the same period of time with equal magnitude and opposite sign, then the signal will be the same as at the original application, modulated by \(e^{-\frac{2TE}{T_2}}\) where \(TE\) is the total time for this procedure.

Such a method is often placed at the end of series of pulses designed to encode a different parameter. Due to the finite run-down period for r.f. transmitters, it is not possible to start acquiring in receive mode instantly after a pulse transmission. Hence some time delay must be introduced to overcome this problem. One common way to overcome this is by this method, known as gradient echo. Two images with different \(TE\) can be used to estimate the \(T_2^*\) of the sample.

This procedure is very fast as gradient pulses are quick and easy to produce.
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\[ \text{EPI Pulse sequence} \]

Figure 2.8: EPI Pulse sequence

\[ \text{The path traced in k-space by the EPI sequence} \]

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This procedure is very fast as gradient pulses are quick and easy to produce.
2.8.2 Spin-Echo

The spin echo technique for acquisitions was developed by Hahn in 1950, and is used in many pulse sequences due to the fact that it is relatively insensitive to $B_0$ field inhomogeneity. It predates the inclusion of gradient coils in magnetic resonance systems and for systems without gradients is used extensively. It produces $T_2$ weighted images and not $T_1$ weighted images. The sequence is shown in Fig. 2.10. A 90° pulse tips the magnetisation into the transverse plane where it begins to undergo relaxation processes. Some of this is $T_2$ relaxation which is random and the signal strength cannot be recovered. However the dephasing due to magnet inhomogeneity can be reversed by applying a 180° pulse, phased along the same axis as the original 90° degree pulse. This has the effect of reversing the phases, and the continuing different rates of precession will lead to a rephasing and an "echo". The time until the echo is twice that between the 90° and the 180° pulse. The strength is proportional to $e^{-\text{TE}/T_2}$ and hence an estimate of $T_2$ can be made by using repeated measurements and comparing them.

Spin-Echo sequences are generally longer than gradient echo sequences due to the relatively long time that it takes to produce a r.f. pulse. Hence for fast imaging sequences, gradient echoes are generally more useful despite their sensitivity to inhomogeneities.

2.8.3 Inversion Recovery

Inversion Recovery is a simple way of measuring the $T_1$ relaxation constant of a sample. The relevant pulse sequence is shown in Fig. 2.11. A 180° pulse tips the magnetisation antiparallel to the field, after which it will evolve according to the equation:

$$M_z = M_0(1 - 2e^{-\text{TI}/T_1})$$  \hspace{1cm} (2.27)

But there is still no component in the transverse plane, and so we are not measuring any signal. But if we use a 90° pulse we will tip the magnetisation into the transverse plane. Afterwards the magnetisation will decay away relatively quickly since it is now subject to $T_2$ relaxation. If we repeat this procedure for several different values of TI (waiting a sufficient time for the magnetisation to return to equilibrium after each excitation), then an exponential decay in signal with $T_1$ should be seen. By comparing this exponential decay with Eqn 2.27, we can infer our $T_1$ value.
2.9 Gradient Coil Hardware in Practice

The previous section's analysis has focused on the theory of how a gradient coil can be used - in conjunction with an r.f. probe - to produce an MR image. Now we must consider how this can be achieved in practice, and the physical limitations on the hardware. The only method of varying the magnetic field within a scanner is to use an array of current carrying elements which can be designed to produce a field profile which approximates to that described above. This must be fitted in to the bore of the magnet, along with the r.f. coil described previously.

A schematic of a typical cylindrical whole body magnetic resonance scanner was shown in Fig. 2.4. Since the cost of producing and maintaining a superconducting magnet increases rapidly with size, all space within the magnet bore must be used as efficiently as possible. With this in mind, there is not much room to produce an ideal gradient from wire coils. The major issues in gradient design are summarised below.

2.9.1 Linearity

The magnetic field profile from a current element may be derived from the Biot-Savart Law, as we will see later, and hence it is not trivial to form linear profiles. There is an inherent inverse power law fall off in magnetic field strength with distance from the current element. It is impossible to produce a linear gradient over the entire space within a gradient coil due to this relationship; however regions may be optimised for linearity. This has implications for the reconstruction of images from non-optimised regions.

2.9.2 Monotonicity

While some linearity problems can be corrected for in image processing, there are other aspects which are more problematical, such as non-monotonic gradient regions. In general terms each co-ordinate within the gradient must be described uniquely by the three offsets to the precession frequency provided by the three gradients. Otherwise it is impossible to map signal from these locations uniquely into an image under any image processing protocol.

In practice this restriction implies that the gradient profiles be monotonic, and that the polarity of the gradient does not reverse, over all regions from which magnetic resonance signals can arise. Since the magnetic field produced by the coils is necessarily zero far from the coils, there will inevitably be a point at which the magnitude of the field produced starts to decrease. Also, the law of conservation of currents coupled with the geometry of some types of coils - notably transverse gradients produced by wire patterns constrained to lie on the surface of a cylinder (such as a Golay coil) or on the surface of two plates (as in the biplanar configuration described later) - forces a change in polarity at or near their edges.

With suitable analysis (see section 4.2.3), it is possible to correct distortions produced by non-uniform gradients except in two cases:

Gradient Reversal. Firstly, for gradient sets there may exist a point outside of the main imaging volume at which the direction of the gradient changes as the current return paths become closer than the useful gradient-producing wires. This results in a zero gradient at that point, and all signal becomes concentrated in the same apparent co-ordinate. With no spatial information at all, no resolution is possible. Fig. 2.12 illustrates this for this gradient set: the image from beyond this point was inverted due to the direction change.

Gradient monotonicity. Secondly, there exist points at the edges of the sets at which the gradient direction changes. This was inevitable feature for all gradient directions since the field produced must necessarily tend to zero both at the centre of the profile and at a distance from the set. When this occurs within the volume enclosed by the set it can cause
Figure 2.12: Image of the grid phantom around the point where the \( y \) gradient changes direction. The r.f. probe used was a surface coil. See Fig. 4.2c for the gradient profiles which produce this type of distortion. Note that this image is a section of a complete image and that the distortion is not occurring at the centre of an acquired image. The grid has dimensions 16.8 cm by 13.8 cm but no scale is appropriate due to the distortions.

Figure 2.13: Image of the grid phantom close to the gradient plates. The grid phantom is the same as for Fig. 2.12.

2.9.3 Rise Time

Any current loop has an intrinsic reactive component to its impedance which will limit the speed at which the current levels can be brought to their required levels in the gradient coils. While this can be corrected for by "overshooting" the applied voltage to drive the current up faster (known as pre-emphasis) the degree that this needs to be done should be minimised.

Hence the self inductance of the gradient set is another factor which needs to be minimised. This is a function of geometry and current density and so influences both the shape of gradient set to be constructed and the placement of current elements within it.

The rise time is also influenced by interactions between the gradient coils and the magnet bore. If there is any stray field produced by the gradient coils at the edge of the conducting surface of the magnet, eddy currents will be produced which will also limit the
response time of the coils. This is a good reason not to rely too heavily on pre-emphasis as these interactions are worsened by this.

Hence as well as optimising the profile within the coil it is often useful to minimise the field produced outside of it for this reason, particularly for the case of large gradient coils. This is often done by placing a secondary “shield” coil outside of the primary coil with current in the opposing direction in order to cancel the peripheral field.\textsuperscript{18-20} This degrades the internal field but is a compromise that may have to be reached. Some designs require more than one outside coil in order to perfectly cancel the extraneous field.\textsuperscript{21,22}

For gradient designs which are to be used for a permanent C-shaped magnet, the rise time will also be affected by interactions between the field produced and the highly permeable materials in the pole pieces, and this must be taken into account.\textsuperscript{23}

2.9.4 Neural Stimulation

The rate of change of magnetic flux is of importance for in vivo magnetic resonance studies due to the potential for nerve stimulation.\textsuperscript{24-26} This is closely related to the slew rate of the gradient coils - the speed at which the gradient changes direction. The peak amount of flux should be kept as low as possible so as to enable fast switching to reach this without having very large rapid flux changes. Smaller coils reduce this flux and thus allow faster sequences to be used safely.

Apart from reducing the flux simply by use of the size of the coils, some work has been done in considering where around the coil peak flux changes occur in order to include this as a factor in coil optimisation.\textsuperscript{27,28} Fortunately EPI scanners operate well below the thresholds for myocardial stimulation.\textsuperscript{29}

2.9.5 Efficiency

The amount of field produced per unit current is another important factor in determining the utility of a gradient coil. Some sequences (such as EPI) which sample very rapidly require strong gradients so as to reach all relevant parts of k-space within a short period. Higher efficiencies can simply be generated by double-winding a gradient set - effectively having a second gradient set in series. While this will double the efficiency it will quadruple the inductance and thus seriously affect the rise time. Another compromise must be made between these two factors.

2.9.6 Power Deposition

The amount of power that a gradient coil requires may be important in some circumstances. It is possible to produce gradient coils that heat up enough to melt the current elements, as well as change the electrical characteristics. In construction thermocouples are sometimes used to prevent this becoming a terminal problem - but a low power gradient coil can still be used harder and for longer. Chu\textsuperscript{30,31} has published on the properties required from materials in order to optimise heat transfer away from gradient coils. Gradient coils may be air, oil or water cooled in order to further improve the patient comfort and preserve the coil integrity when they are being pushed by intensive sequences.

2.9.7 Rigidity

When a current passes through a gradient coil there is a Lorentz force on all current elements within it. For currents of the order of hundreds of Amps and main field strengths of the order of several Tesla this can lead to forces of the order of hundreds of Newtons per unit length. Since these forces may be switching direction many times per second, the coils need to be very strong to retain their shape. Some studies have used numerical
analysis to quantify this movement.\textsuperscript{32,33}

For a general configuration there may also be net torques acting as forces on different parts of the gradient coil do not balance. This can either be dealt with by firmly clamping the gradient coils - which is often impractical - or by ensuring that the torques balance. This can be done by ensuring that the gradient coils are symmetric. This has no disadvantages in terms of electrical performance as long as we require the region of uniformity of the gradient coil to be in the centre of the gradient set although it can cause complications if this is not the case.\textsuperscript{34} Asymmetric gradient coils have been developed which are torque balanced however. This can be done either by constraining the wire patterns to balance the torques within the normal former,\textsuperscript{35} or by having secondary formers which act to negate the effect of the primary.\textsuperscript{36-38} In the latter case the secondary coil may also be useful as a magnetic screen.

Even in a nominally torque-balanced coil, $B_0$ inhomogeneity across the extent of the coil may cause torques to arise, so an arrangement where each part of a coil is torque balanced separately is preferable.

### 2.9.8 Acoustic Noise

The forces on the current elements described above cause acoustic noise in the scanner. It has been shown that there is a logarithmic relationship between gradient slew rate and the noise produced in decibels.\textsuperscript{39} Not only is this unpleasant for human studies but it can also reach dangerous levels for some studies such as EPI. While ear protection can be effective, for other applications such as paediatric or veterinary imaging this may not be appropriate. The Health and Safety Executive also issue limits on acceptable noise levels for exposure to workers. The noise levels vary through the magnet bore and the magnet room and arise with a frequency response primarily related to the pulse sequence used.\textsuperscript{40}

Different methods have been used to attempt to reduce the noise through gradient design. Passive methods include attempting to clamp the current elements firmly enough that they cannot vibrate, or surrounding them with acoustic-absorbptive material. Cancellation by injection of antiphase noise into the bore can produce a null zone - but this is spatially and frequency dependent and may lead to increases in amplitude if applied incorrectly.

In recent years Mansfield has pioneered active acoustic screening,\textsuperscript{41-47} where reduction in the production of noise is considered as integral to the design of a gradient coil. The primary consideration is to ensure that there is no net force and no net torque on each current loop comprising the gradient coil. This implies that the current loops must lie in a plane perpendicular to the main field direction. If this is satisfied there will be no movement of the centre of mass of the coils and no turning force for any current applied. This eliminates the need to clasp them so firmly in place. Of course the current elements still need to be held firmly within the gradient coil itself.

However sound will still arise due to the visco-elastic properties of the material in which the current elements are embedded. As current waveforms are applied to the coils their structure will compress and expand alternately and sound will be produced. Mansfield provides a mathematical description of this phenomenon.\textsuperscript{48} In general material with a higher propagation velocity for the sound waves will have a lower acoustic response at the frequencies usually applied to gradient coils.

Another method\textsuperscript{48} to change the acoustic response is to divide the current loops into three or more touching loops connected in series. While this has no effect on the magnetic properties, the acoustic response resonances will be pushed to higher frequencies. This will lower the amplitude of the sound produced at lower frequencies. This can be incorporated into any gradient design - its disadvantage lies in the use of more wire which leads to more power deposition. Such designs may also be difficult to construct robustly.

The other approach to this problem is to eliminate the sound which is produced rather
than design coils to produce less. This can be done by placing the gradient coils in a vacuum chamber. However there may still be some sound transmitted through the supporting structures, and the size of the coil will be increased.

2.9.9 Geometries of gradient coils

For reasons as described above, different sizes and geometries are applicable in different circumstances. Originally gradient coils were constructed to match the size and shape of the magnet bore so as to maximise the volume within the bore that could be imaged: cylindrical designs were used for cylindrical magnets and biplanar designs for C-shaped magnets.

Reduced bore gradient sets (such as head coils within whole body systems) however increase the efficiency and decrease the rise time for specific applications where maximum size is less important. This can enable fast imaging sequence to be employed on systems where the hardware would not otherwise permit it. Reduced bore systems for longitudinal and transverse cylindrical coils are very well established. In some circumstances gradient coils may be produced for specialist purposes which do not use the magnet's symmetry at all.

As well as simply reducing the size of a gradient set, different geometries may be useful. Since gradient efficiency is easier to achieve if the current elements are close to the region of interest, for applications where the object to be imaged is not cylindrical (for example a flexed knee or the chest) better results may be achieved by use of a biplanar coil. This is also true for circumstances where interventional procedures are required. Elliptical gradient coils have been found to have better properties than cylindrical coils for some applications.

Patient access is an important consideration in designing gradient coils. The position of the usable region relative to the edge of the coil can be critical. Chronik and Rutt have developed an ultra-short gradient coil for head imaging. The coil is not wide enough to fit over the shoulders and yet the usable region is closer to the edges of the coil than for conventional insert gradient coils. Alternatively, wide aperture gradient coils allow the ends of the coil to move outwards in order to facilitate imaging of the neck region as the diameter of the coil is not restricted by the position of the shoulders. Parabolic shapes have been employed similarly. Some designs explicitly include "holes" in the design to enable ease of access to the region of interest. Other shapes which have been used include bi-convex coils and semi-cylindrical coils.

In recent years magnet systems have been designed for specific purposes rather than for general use and such magnets may require gradient coils with unusual characteristics to be employed. Crozier has designed such a system with custom designed gradient coils for head and neck imaging. Interventional magnet coils also require special restraints on the shapes of gradient coils which can usefully and safely be used.

Some authors have developed very different types of gradient coil. Petropoulos has presented a spherical gradient coil with very good performance. In order to eliminate both sound and rise time problems Cho has presented a gradient coil with constant current which is mechanically rotated in order to provide an image. Since this has no variation in current there is no acoustic vibration, and no inductance problems with resettling currents. This is in fact a throwback to the pioneering days of MR imaging where gradients were manually rotated in order to sample k-space.

Particular applications may allow very simplified gradients to be used, as Callaghan has found with a single wire gradient. Also, as has been described, in order to provide advantages in shielding and monotonicity, it is also sometimes appropriate to use several cylindrical surfaces containing conducting paths; or in equivalent outer surfaces for other geometries. For biplanar gradients, the shielding surfaces are two planes outside of the primary pair.
these two features, adding multiple wire paths at the same position - allowing the wires to lie on top of each other - may have other advantage, for example in that the efficiency scales more favourably with the resistance of the coils. Coaxial return paths on cylindrical coils also have advantages in the size of the region of monotonicity of the coils, and their acoustic properties.

The most recent innovation in gradient design has been proposed by Petropoulos and is referred to as "phased array" gradient coils. A disadvantage of reduced bore gradient coils is the relatively small region of gradient uniformity; a major advantage is the shorter rise time. By considering several gradient coils adjacent to each other it is possible to extend the useful gradient region. However in order to retain the rise time advantages it is necessary to eliminate the mutual inductance between them. This is done by computationally calculating the amount of overlap which nulls the magnetic coupling. In this way, high performance coils can be applied to large regions of interest, such as lumbar spine imaging.

Chapter 3

Gradient Design

The earliest field gradient coil designs for magnetic resonance were effected by the expansion of the magnetic field profiles as a power series along the gradient axis: by positioning current elements appropriately higher order terms can be eliminated.

The simplest solution for an axial gradient in the cylindrical geometry found in this way is the Maxwell pair, as shown in Fig. 3.1a. By positioning the loops $\frac{\sqrt{2}}{2} a$ apart, where $a$ is the radius of the cylinder all even order terms in the gradient expansion about the mid-point on axis are eliminated, as are the third order terms. Hence its gradient profile is linear to fifth order on axis. In practice this provides a field profile which is within 5\% of an ideal linear field within a radius of $0.5a$. 

Figure 3.1: a) A Maxwell pair. b) A Golay coil.
Its transverse counterpart, the Golay coil,77 is also derived by consideration of the spherical harmonics of the system although the mathematics is more complex.78 It is shown in Fig. 3.1b, and is also corrected to fifth order.

For non-cylindrical systems analogous simple solutions exist. Anderson79 described an elemental z gradient biplanar coil by expansion of Ampere’s Law for an infinite wire. This method of design suffers from its inflexibility and coils designed using expansion techniques were generally developed before modern MR sequences requiring short rise times and a larger region of interest relative to the size of the coil were needed. Golay coils in particular are very long relative to the size of optimal gradient region.

Coils with multiple windings distributed along their length are able to surpass these early designs by enabling more flexibility in the positions in the conductors. A conventional two loop Maxwell coil (for example) would require a much larger current to achieve the same gradient strength than a z gradient coil with thirty winding loops. And while dividing these extra loops equally between the Maxwell pair positions would give a large efficiency it would not be optimal in other ways: for example, the concentration of current in those planes would increase the inductance. Also, for some sized regions of uniformity it may be more important to use the extra windings to cancel higher order terms than the third order.

Modern gradient design practice falls broadly into two categories: the first has been described as the “inverse” approach. This approach is to specify the characteristics required of a gradient coil and formulate the current distribution required to produce them. In contrast, the “forwards” approach is essentially an optimisation procedure. A current distribution is formulated, the characteristics of the resulting gradient coil are calculated, and the current distribution is then modified depending on this performance according to some optimisation technique.

We will deal with each in turn.

3.1 The “inverse” approach to gradient design

The earliest approaches to non-conventional gradient design performed a calculation to derive the optimal current density on the surface of the gradient coil. Compton divided the coil surface into 2048 elements, and calculated the effect of a unit current density within each element at each of N points within the region of uniformity. By use of Gaussian elimination and some matrix algebra it is possible to minimise the root mean squared (r.m.s.) deviation from a predefined field profile.

Later improvements on this class of solutions minimised for power and energy stored and implemented the constraints that the field be defined at certain points using Lagrange multipliers.80 These current density maps must then be approximated into discrete current paths. This is necessarily not an exact representation though for many windings it has been found to be a good one.

These methods are inverse in the sense that they start from the field required, but they operate in “coil space”. Many recent methods use Fourier analysis to transform the problem to reciprocal space. This approach was introduced by Turner.81 Target field methods represent the current density in the gradient former as a Fourier transform in the azimuthal directions. By this method it is possible to invert the calculation of the field profile and derive current densities which produce specified field profiles. The current densities are then put into discrete wires as described previously.

Further work has refined this method. By the use of Lagrange multipliers it is possible to minimise the inductance of the coil.82 However the current density function produced by this formulation is not limited in length and may contain high frequency variations. The former problem has been addressed by adding further constraints with Lagrangian multipliers which force the current density to be zero at certain points,59 and the latter by the use of a weighting function on the Fourier domain.

It has been found that for coils of low aspect ratio (i.e. the coils are short in comparison
to their diameter) that the target field method is reduced in efficiency due to the extra constraints. Other

Another approach is to minimise the total energy used by the coil rather than the inductance (subject to the uniform field conditions). This results in a slightly slower rise time but more compact coils, and the decisive advantage depends on the application to which the coils are to be put.

Shielding is also incorporated into the calculation by introducing a second conducting surface of a specified diameter and length (the screen coil or secondary coil) which fits around the primary coil. Current densities required to null the stray field can be obtained for this and its effect on the internal field allowed for.

By using other orthogonal function expansions other surfaces can be treated in the same manner, though in practice this is restricted to certain geometries. Designs have been reported for cylinders, spheres, hyperbolae, planar and biplanar shapes.

3.2 The “forwards” approach to gradient design

The other approach to gradient coil design - and to magnet field profile optimisation in general - is to specify a “cost function” comprising the factors listed previously and to attempt to minimise it by some optimisation algorithm. While for some cost functions (such as Compton) this is possible as an inverse problem, more generally it requires an iterative procedure.

By this method all the factors which could be taken into account using the inverse methods described above can be considered, but more flexibility can be introduced. Non standard geometries become more accessible and factors such as the engineering precision possible in construction can be included. In contrast to the majority of inverse methods where the current stream function is calculated rather than wire positions, an optimisation method allows the actual design which is to be constructed to be derived at the computational stage.

The disadvantage is that the problem becomes much more computationally expensive. For complex problems convergence on a global rather than local optimum solution becomes difficult to prove.

3.2.1 Optimisation Algorithms

Optimisation algorithms vary in power and applicability. The major differences between them lie in the amount of information required as inputs to the algorithm at each iteration, and the ways in which these are used to direct the search for an optimal solution.

Random Search

If we regard analytic methods as one extreme of the set of gradient design methods - where the design parameters are inferred entirely from the resultant performance - then a random search lies at the other end of the spectrum. No information is used from iteration to iteration and improvements are achieved by randomly sampling search space on each iteration and retaining the solution if it is better than the current working optimum. This may be appropriate for search spaces with few variables, but it is not very efficient. Gradient design is too complicated to make use of this method.

Direction Set Methods

These methods attempt to reduce the multi-dimensional problem to a set of linear optimisations. By recursively optimising these problems, an optimal solution can be converged on. The simplest implementation of this would be to successively optimise with respect to the design parameters. The method of optimisation along each direction can be very simple. Binary convergence has been used exclusively for some optimisations where the
design can be expressed with relative ease due to a convenient geometry and suitable choice of current element.\textsuperscript{93}

However, more intelligent use of further iterations allows successive sets of “conjugate” directions to be defined, which better characterise the shape of the cost function and allow a solution to be converged on more quickly in more complicated search spaces.\textsuperscript{94}

Powell’s method is one such algorithm and has been used in gradient coil design.\textsuperscript{95,96}

**Derivative Methods**

By considering the first derivative of the cost function a more directed search can be performed. For numerical methods the first derivatives are not usually readily available and their estimation involves recalculating the cost function for small differences in each of the parameters, which can be very time consuming.

The simplest such method is that of steepest descent where the direction travelled in search space is that of the gradient of the function - the “steepest descent”. The distances travelled in each iteration are defined to be small and the operation is repeated to attempt to close in on a local optimum. Conjugate gradient optimisations perform a similar operation, with the direction of search modified based on information from previous iterations.

Wong\textsuperscript{36,97} first used conjugate gradient descent to design cylindrical gradient coils. Since then the technique has been combined with finite element analysis.\textsuperscript{98}

**Simulated Annealing**

As the name suggests, simulated annealing uses the theory of statistical mechanics to produce a model for the sampling of search space.\textsuperscript{99} The analogy is with a crystalline material which is cooled from high temperature to absolute zero. If this is performed very slowly then atomic rearrangements will occur so as to ensure that the material is at its ground state with minimal configurational entropy. By allowing the material to pass very slowly through its freezing point, rearrangements can occur to allow this state to be found - a process known as annealing.

This is characterised by a finite probability of the system changing to a configuration which increases the entropy. This enables locally optimal configurations to be ultimately avoided.

In the case of an optimisation problem we can liken the energy of the crystals to the cost function evaluation of the problem. The effective “temperature” of our system is a measure of how likely the algorithm is to accept a solution of higher “energy”.

From statistical mechanics theory we derive the rule that a solution will always accept an adjacent solution in search space if it lowers the energy, and will accept a solution which worsens it with probability

\[ P = e^{-\Delta E/k_B T} \]

where \( k_B \) is equivalent to the Boltzmann factor, \( \Delta E \) is the change in the cost function and \( T \) is the effective “temperature”.

Our previous methods of always accepting a solution if it improved the cost function - and always rejecting possibilities which worsen it - can be regarded as being equivalent to a fast “quench” of the system, where local optima are frozen in and cannot be escaped from.

Crozier et al have used simulated annealing to design cylindrical gradient coils,\textsuperscript{100-102} some of which are of restricted length.\textsuperscript{103} Bussko et al have also used simulated annealing to design gradient coils with coaxial return paths,\textsuperscript{104} as has Peters.\textsuperscript{105}

Tomasi et al have recently used a hybrid target field and simulated annealing method to design shielded biplanar gradient coils.\textsuperscript{106} The primary plates were designed by simulated annealing and the required current distributions in the shielding plates were derived analytically. This method has had some success.
Genetic Algorithms

In the same way that simulated annealing takes its model from physics, genetic algorithms (GAs) draw inspiration from biology and the process of evolution. If we regard different potential solutions to the problem as being different "organisms" competing for supremacy within their population, with their fitness determined by how well they minimise the required cost function, we have another viable model for optimisation.

It is implemented by encoding the design parameters within a "chromosome". A population of such designs is generated and the best designs within this population are "bred" by splicing together sections of their chromosomes. The best designs are retained for a further generation while the weaker designs do not breed and perish. Designs with different but favourable characteristics are combined to see whether their features are complementary.

The details of this procedure are fully discussed in section 3.3 with particular reference to their implementation in this study. The advantages are that many regions of search space are sampled simultaneously - one disadvantage is that many cost function evaluations are required for each iteration.

Fisher et al have used a genetic algorithm approach in this laboratory.

Genetic Annealing

Methods have been developed to combine the features of both the simulated annealing and genetic algorithms. They have complementary strengths and weaknesses.

Taboo Search

The taboo search method allows more direct control over how the search space is sampled. A "blind" search in search space is considered. However, certain directions in search space are considered taboo. For example, it might be considered taboo to revisit regions of search space within a certain number of moves. "Aspiration" conditions further modify the search procedure in allowing taboo moves to be completed under certain conditions (for example that they find the best solution so far encountered).

Such a directed search has been shown to be competitive with other methods. If simulated annealing derives from physics, and genetic algorithms from biology then the taboo search method can be said to be derived from the learning processes involved in artificial intelligence.

3.2.2 Current Elements

As well as establishing a method for optimising our design problem, we must also consider the most appropriate form of the current producing element that we shall use. This can be significant in simplifying the problem. For example, in order to produce a longitudinal coil of cylindrical geometry we might decide to use current loops as our fundamental wire element. Since an analytical solution for the field produced exists on axis for this form, this may significantly simplify our optimisation. However, for more complicated gradient coils we are obliged to use less mathematically convenient formulations.

Straight wire elements

Most gradient coils (with the significant exception of Golay coils) contain relevant field producing straight wire elements within their design, and so these must be considered within the optimisation procedure. The overall B-field profiles are created by a superposition of such B-field profiles (dB) for individual wire elements. The individual elements are calculated by direct application of the Biot-Savart Law:

\[ dB = \frac{\mu_0 I dl \times r}{4\pi r^3}, \]  

(3.1)
where $B$ is the magnetic field, $I$ is the current, $r$ is the vector from the point of interest to the current element, $dl$ is the direction of the current, $\mu_0$ is the permeability of free space and $\times$ is the vector product operator. By combining many such elemental wire paths of differing dimensions and locations optimal gradients can be produced. For MRI purposes only the $B$-field component parallel to $z$ needs to be considered. Thus,

$$B_z = \frac{\mu_0 I}{4\pi r^3} (dl \times r) \cdot k. \quad \text{(3.2)}$$

The resulting $B_z$ field is zero when the wire direction ($dl$) is parallel to the $z$ direction, and maximum when it lies in the $xy$ plane.

Fig. 3.2 exemplifies the finite current element used for analyses of gradient coils in the case where the current element lies perpendicular to the main field; in many cases (such as in Chapter 4) all field profiles from the $B_z$-producing wires within the gradient coils could be modelled using this element. In itself, the current element has no physical reality since it does not include a return path for the current; nevertheless, examination of the effect of different combinations of this profile gave insight into gradient behaviour which were not available from other analysis methods. More general straight line elements have been used to model polygonal designs in the same manner.

**Current Arc elements**

Another useful current element for modelling is the arc. This is the fundamental element for Golay coil modelling, and for the design of other geometries based around this shape.

This is somewhat less amenable to calculation as the integral cannot be done analytically in the general case. The method of integration that we use here is gaussian quadrature.

The field profile produced by a pair of current arcs as found in Golay coils is shown in Fig. 3.3.

**Composite elements**

In the majority of situations, composite elements of the above components are more usefully employed. This is generally sensible if only to ensure that the current loops are physical and contain a return path. For the case of a biplanar gradient coil where the current paths are constrained to lie in rectangles (as in the following chapter), each straight current length must have a parallel twin element with the current in the opposite sense for example.

This is also applicable to the screened cylindrical case. Brey has used a concentric return path element for a cylindrical coil as the basic unit for a longer cylindrical coil. Using this element he has then used an optimisation method to construct a macroscopic coil.
3.2.3 Figures of Merit

Given the highly complex nature of the problem as described above, it is perhaps not surprising that there is no universal figure of merit to describe the performance of gradient coils. Nevertheless some authors have proposed calculations which are appropriate in different circumstances. For the inverse designs these figures are measures of how effective the design process has been as the criteria involved must be implicit in the calculation of the design. However for iterative optimisation approaches calculation of the cost function is necessary at every iteration and determines the nature of the optimal gradient coil.

The calculation of the measure of inhomogeneity is itself not consistent across different authors. Table 3.1 shows some common measures. These all have different equally valid justifications. Methods vary between calculating the fractional variation in field or the absolute, and also between optimising the magnetic field profile or the gradient profile. Many authors do not specify which measure they are using.

Du\textsuperscript{119} has published a study on the different measures of homogeneity used in gradient coil design, with regard to the amount of distortion which is seen in images produced using these coils. The conclusion was that \( \sigma_1 \) (as defined in Table 3.1) weights the deviation

![Figure 3.3](image)

\textit{Figure 3.3:} The \( \mathbf{B}_z \) field due to a current arc pair. The crossed circles represent a field direction into the page, the dotted circles represent \( \mathbf{B}_z \) components rising from the page.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Equation</th>
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<tbody>
<tr>
<td>Turner,\textsuperscript{82} Brey\textsuperscript{118}</td>
<td>( \sigma_1 = \sqrt{\frac{1}{V} \int \frac{(B(x, y, z) - B_0(x, y, z))^2}{B_0(x, y, z)} , dt} )</td>
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<tr>
<td>Du\textsuperscript{119}</td>
<td>( \sigma_2 = \sqrt{\frac{1}{V} \int (B(x, y, z) - B_0(x, y, z))^2 , dt} )</td>
</tr>
<tr>
<td>Pissanetzky,\textsuperscript{120} Fisher\textsuperscript{1}</td>
<td>( \sigma_3 = \sqrt{\frac{1}{V} \int (G(x, y, z) - G_0)^3 , dt} )</td>
</tr>
<tr>
<td>Wong,\textsuperscript{97} Shi\textsuperscript{98}</td>
<td>( \sigma_4 = \frac{1}{V} \int \frac{</td>
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<tr>
<td>Zhang,\textsuperscript{114} Andrew,\textsuperscript{121} Tomasi\textsuperscript{106}</td>
<td>( \sigma_5 = \frac{1}{V} \int \frac{(G(x, y, z) - G_0)^2}{G_0^2} , dt} )</td>
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Table 3.1: Different measures of linearity used in the literature. \( G_0 \) and \( B_0 \) represents the required ideal gradient and ideal magnetic field offset respectively; \( G \) and \( B \) represent the actual values found at any point. \( V \) represents the volume over which the integral is performed. Factors which are present to provide comparisons between different designs (such as size of the region of interest or magnitude of the gradient) are discarded.
Table 3.2: Various cost functions used in the literature. The values $\sigma_n$ are those defined in Table 3.1. $L$ represents the inductance and $R$ the resistance of the gradient coils. The values $k_n$ are constants defined for the particular application. $B_{ext}$ is the external field produced by the gradient coil, and $\eta$ is the efficiency of the gradient coil (gradient strength per unit current). Where there is no subscript on the $\sigma$, its definition was omitted by the authors.

<table>
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<tr>
<td>Crozier$^{56,101}$</td>
<td>$E = k_1 \sigma_1^2 + \frac{k_2}{G_0} + k_3 L + k_4 B_{ext}$</td>
</tr>
<tr>
<td>Crozier$^{103}$</td>
<td>$E = k_1 G_0 \sigma_3 + k_2 B_{ext}$</td>
</tr>
<tr>
<td>Adamiak$^{122}$</td>
<td>$E = \sigma_3^2 + k_1 \frac{L}{G_0} + k_2 L + k_3 R$</td>
</tr>
<tr>
<td>Buszke$^{104}$</td>
<td>$E = k_1 \sigma_1^2 + k_2 \frac{1}{G_0} + k_3 L$</td>
</tr>
<tr>
<td>Shi$^{98}$, Adamiak$^{96}$</td>
<td>$E = \sigma_2^2$</td>
</tr>
<tr>
<td>Turner$^{123}$</td>
<td>$E = -\eta^2/\sigma_1 L$</td>
</tr>
<tr>
<td>Fisher$^{108}$</td>
<td>$E = -k_1 \eta \sigma + k_2 B_{ext} + k_3 \sigma_2$</td>
</tr>
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The major differences between these $\sigma$ values lie in their applicability to local or global image distortion. Minimisation using $B$-field values will prioritise creation of a uniform field. Optimising using values of $G$ will attempt to optimise individual regions. Images produced using such a gradient coil may be locally less distorted; but there may be a translational shifts arising from inhomogeneities in adjacent regions of the coil.

There may be degeneracy in these measures in some contexts. For example, $\sigma_3$ is equivalent to $\sigma_3$ if it is used in an optimisation algorithm to design coils of predetermined efficiency, rather than calculating the average gradient $G_0$ for each instance considered.

Having specified the linearity measure, the cost function (or error function) usually contains this in some way. Table 3.2 shows different cost functions which are typically used. The final measure introduced by Turner was proposed as a comparative measure between gradient coils rather than as a cost function within an optimisation routine but it is included here for completeness.

While all such cost functions will minimise in the case of the ideal gradient coil, different measures will be more appropriate to different situations. Firstly, these differences manifest themselves in the behaviour of the cost function near the optimum. The multiplicative cost functions such as those used by Du and Wong, could also be expressed as additive logarithmic cost functions. These may be more appropriate if solutions are required that are insensitive to small changes in the design parameters.

Further differences between the cost functions are the emphasis placed on linearity and rise time. For example, for a reduced bore gradient coil it will be more critical to minimise the linearity than the inductance due to the size of the coil. For a whole body coil the reverse will be the case - the linearity will be achieved relatively easily due to the distance of the coil from the region of interest, while the rise time will be more of a factor as a result to the central region of the coils due to the smaller values of $B_0$ in this region while $\sigma_2$ minimised the distortion.
of the proximity to the cryostat and the size (and thus inductance) of the coil. Similarly, gradient coils have been produced which deliberately optimise for the strength of the coil at the expense of the linearity\textsuperscript{124} in order to make use of image processing techniques.

In recognition of the different requirements in different contexts and so as to avoid having “insert” gradient coils which are separate from the bulk of the hardware, systems have been produced which contain two independent sets of gradient coils wound around the same former: one which has a small region of high linearity and high gradient strength, and one with a large region of linearity but a poorer gradient strength,\textsuperscript{125,126} These are known as modular gradient systems. The two modes of operation can be switched between depending on the application. Gradient coils unused by a particular mode must be left open circuit to avoid coupling between them.

Du\textsuperscript{127} has also published an evaluation of various simple cost functions for a simulated annealing gradient coil optimisation procedure. By producing 3-dimensional “maps” of the designs produced (in $\eta$-$\sigma$-$L$ space) by the optimisation procedure he was able to postulate choices of cost functions for specific tolerance criteria. This demonstrates the sensitivity of cost function to specific design application.

### 3.3 Genetic Algorithms in Detail

The genetic algorithm used in this thesis is a fairly simple algorithm. It was originally implemented in this laboratory by Dr N. Dillon. More complicated genetic algorithm approaches can be found in the literature. The algorithm was originally applied to gradient coil design by Dr B. Fisher,\textsuperscript{1} who demonstrated that it is a viable alternative to other optimisation algorithms in this context.

#### 3.3.1 Overview

The basic structure of a genetic algorithm optimisation procedure is shown in Fig. 3.4. The most time consuming step within this process is the expansion of the design chromosomes into full designs and the consequent evaluation of the cost function. The “breeding” and mutating is relatively short in comparison since the number of designs is not large enough to make this a comparable factor.

![Figure 3.4: Genetic Algorithm flow chart](image)
3.3.2 Population generation

The initial population of designs is generated at random. It is important that this population is not too small so that the power of the search facility within different regions of search space is properly utilised.

The analogy with biological evolution is particularly strained over population size. A typical biological system requires very large populations to ensure that there is sufficient genetic variability, while typical population sizes for the problems described here are of the order of one hundred. It is important to realise that while there is a useful analogy between biology and this optimisation method, they are sufficiently different that conclusions cannot be drawn about the former from the latter.

As discussed in the previous section, the design parameters are encoded into chromosomes. These can be considered as binary numbers. When encoding into a fixed length chromosome it has been generally true in this study that there is some redundancy in the chromosome. For example, if wires within a design overlap it is necessary to force the exclusion of one of the two elements or to "push" them apart. This may result in one design being described by two or more potential chromosomes. This does not necessarily reduce the efficiency of the algorithm however. Having "hidden" characteristics which are not present in the current generation of designs but may re-emerge in later generations can be regarded as a useful feature.

3.3.3 Mating and Mutation

The mating algorithm is not a straightforward breeding of one half of the population. A "mating pool" is constructed which consists of all members of the population, with "fitter" designs multiply represented. Pairs are then mated at random from this pool, with half the population involved in mating. In this way, the most fit designs may have many offspring

while even the worst (which may contain some useful features) may be represented in the following generation.

The mating pair of chromosomes are split at a random point along their length and spliced together to produce two offspring. The next generation consists of the parents and offspring produced in this process. In this way the population is maintained at a constant size.

There are many different mating strategies possible. Different pairs of chromosomes could be multiply mated, or the chromosomes could be split at several points along their length. Genetic algorithm design is a research field in its own right.

Another aspect to the mating procedure is the small finite probability of a chromosome mutating. A "mutation" is defined as a bit within the chromosome changing from 1 to 0 or the reverse. This behaviour mitigates against uniformity in the population. The mutation rate is 1% in these studies.

3.3.4 Termination

The genetic algorithm terminates when the population is uniform. This is not likely to be the case for a sizeable population due to the mutation rate. In practice the plot of cost function evaluation against iteration number flattens after many iterations and the population can be seen to be largely uniform.

No formal proof of convergence exists for genetic algorithms,\textsuperscript{128} unlike simulated annealing. However empirically genetic algorithms do find good optima for problems such as those described in the following chapters, as has been demonstrated.\textsuperscript{1}
Chapter 4

Biplanar Gradient Design

The characteristics of magnetic field gradient coils are critical for obtaining good quality images by magnetic resonance imaging. Physically, more open access to the imaging volume reduces claustrophobia in human subjects and also enables more inconvenient shapes of objects to be imaged. If no other factors were relevant it would be desirable to have the coils as large as possible to give the maximum amount of space. However, electrically their performance generally depends on the maximum gradient strength that can be achieved, the rise time required to reach those levels and the linearity of the resulting gradient. Good performance in the first two of those criteria enables implementation of faster pulse sequences, such as those used in echo planar imaging; linearity is required for the acquisition of undistorted images.

The gradient strength depends on the efficiency with which the gradient coils produce a field per unit of drive current; this in turn depends on appropriate positioning of the gradient coils with respect to the region over which the gradient is required. Since field strength falls rapidly with distance from a current element, each gradient coil must be as close as possible to the region of interest; unfortunately the requirement for a linear gradient generally requires the current elements to be further away. Hence, as we have
seen, some compromise must be reached between these competing factors. The rise time is limited partly by the self-inductance of the gradient coil, and partly by eddy current interactions with other conducting elements within the magnet system. In general, the smaller the gradient system, the shorter the rise time, since interactions with the magnet casing, and the coil self inductance, are then both reduced; however since the latter is a function of geometry, small gradient sets with more complex geometries will not necessarily have a small inductance.

Gradient systems for cylindrical magnets are, conventionally, cylindrical because of the advantages conferred by the resulting symmetry. Although they may be large enough to fill the magnet bore, they may be smaller to optimise imaging of smaller regions using higher gradient strengths and shorter rise times (for example, gradient sets for head imaging). Surface gradient coils are the simplest form of gradient set and can be as simple as a single wire placed adjacent to the sample. This clearly offers excellent access to the volume being imaged. However, since they do not surround the region of interest, it is consequently difficult to achieve gradient linearity and the field falls off markedly further from the current conductor. Biplanar gradient sets consist of two planes of conductors placed on either side of the region of interest; they retain the advantages of proximity to the object of interest, but provide additional linearity because they surround the sample, giving more control over the gradient profiles produced.

This chapter describes a method for the general design of gradient sets. The work shown has previously been published. Using linear superpositions of the magnetic field produced by individual wire elements to calculate gradient profiles, a number of designs were created; these were then compared by the optimisation algorithm described below. The theoretical methods were then used to design a particular two axis biplanar gradient set based on a pair of 40 cm square plates, separated by 24.4 cm, which was large enough to accommodate a human torso or limbs with greater clearance than small bore cylindrical coils. This geometry was chosen to enable future studies to make use of these features, while still fitting within the bore of the existing magnet within the HSLMC. The provision of the third gradient axis (in the case described here the x-axis gradient) is difficult using the biplanar geometry (for systems where the main field is transverse to the axis of the coils) for reasons which are discussed in the following section. The lack of an advantage meant that this gradient set was designed with two axes and to be used in conjunction with the whole body x gradient coil within the existing scanner.

The gradient profiles were optimised for linearity over a 15 cm diameter sphere at the centre of the set (the region of interest). Although this is the nominal maximum imaging volume, the present analysis demonstrates that this can be extended by image processing. Since the width of the biplanar gradient set is the same as that of a cylindrical gradient coil of diameter 46.9 cm, the distance from the current elements to the imaging region is roughly halved, which enabled much stronger gradients to be achieved.

4.1 Theory and Computational Studies

4.1.1 Choice of Approach

The design concept described here attempted to reconcile four factors relevant to all gradient set designs, namely the strength of the gradient produced, the linearity of its profile, the rise-time, plus the robustness of the design both with respect to its tolerance to small imperfections in its construction and its mechanical performance under the influence of high, rapidly switching currents. In the particular case described below priority was given to the strength of the gradient set and its linearity.

The optimisation method involved the generation of current distributions followed by calculations of their resulting field profiles; each design was then tested computationally and was used as an input to the optimising algorithm for a further iteration if it offered an
improvement on a previously existing optimum; the specific implementation is discussed below.

### 4.1.2 Superposition of field profiles from wire elements

The optimisation algorithm created the overall B-field profiles by a superposition of the B-field profiles \(dB\) as described in section 3.2.2. The approach described here constrained the wires to lie in rectangles on the gradient plates; since the plates lie in the \(xz\)-plane, only the wires which lie parallel to the \(x\) direction are field-producing. This had the further advantages both of enhancing the field-generating efficiency and of reducing undesirable variations in field in the \(x\) direction since the \(z\) and \(y\) gradients were the gradients of interest.

Figs. 4.1 and 4.2 exemplify two simple biplanar gradient profiles based on these unit sections, both of which use lines of symmetry to produce very basic, non-optimised gradients from single current loops. Each diagram is normalised so that the length of the arrow at each point is proportional to the strength of \(B_z\)-field. These representations illustrate some general points concerning the characteristics of this configuration of the gradient coils. Thus, the field strength for a \(z\) gradient near a plate is very strong compared with that at the centre, and varies markedly in space (see Fig. 4.1c); this implies the existence of appreciable distortions of the \(z\) direction fields near the plates. Furthermore, the \(y\) gradient changes direction beyond a certain distance from the centre in the \(z\) direction (see Fig. 4.2c), which reflects a necessary current-return path within the plane; similar gradient reversals also occur for gradients from cylindrical saddle coils (see section 2.9.2). The present optimisation routines force these return paths to be located as far away as possible from the region of interest.

In principle, gradients in the \(x\) direction can be produced from a planar arrangement of wires by using four rectangular current paths per plate to create the appropriate field profile. However in practice the resultant gradient field is very small and is not significantly larger in magnitude than that produced by a typical large bore cylindrical gradient system. Fig. 3.2 illustrates the reason for this; whereas the field strength produced by a current-carrying wire varies relatively rapidly with change in \(y\) and \(z\)-coordinates, the variation with \(x\) is much less marked. This intrinsic variation is necessarily replicated for linear combinations of such elements.

### 4.1.3 Implementation of Optimisation Algorithms

The choice of algorithm used to perform the optimisation calculations is critical to both its speed and the reliability of its convergence to a satisfactory solution.\(^{97,105,108,110}\) In the present study, the routines that evaluate the utility of the designs were based on existing genetic algorithm software,\(^{108,110}\) but with the additional features described below.

The genetic algorithm first generated a set of possible design solutions at random, the initial "population"; these were then evaluated, the quality of the resulting design...
quantified, and the competing designs ranked. Desirable characteristics were pooled by “breeding”: that is by cutting the genetic code at a random point along the sequence and combining two halves of different chromosomes. Those new designs formed the next generation of solutions and the process was then repeated sequentially; the population size was kept constant throughout that sequence, and the best designs of each previous generation were retained. This approach ensured that the population did not grow to an unmanageable size after repeated iterations, and also that the best characteristics of each generation were retained in case the offspring were inferior to their parents. The genetic algorithms had a small but finite mutation rate during breeding to avoid a tendency towards uniformity in the population. This was achieved by inverting one bit within a design chromosome from zero to one (or the reverse) if it was deemed to have mutated.

Figure 4.2: a) The position of an element of a y gradient system. In this case the current elements must be symmetric in the xy plane so as to produce a uniform z profile, and the current elements in each plate must be in the same sense so as to produce a change in sign in the y direction. b) A slice through the xz plane looking down at the bottom plate. The slice is taken off centre so as to avoid the zero-field plane at the centre. c) The yz plane.

4.1.4 Specific application of the genetic algorithm to gradient design

It is now appropriate to explain the technical details of the above process. Several different methods enabled the genetic algorithm to be used to optimise the gradient coil design which can be divided into three categories: constraints which reflect the integration of the design within its “chromosome”; evaluation of how numerical factors (such as gradient efficiency and length of wire used) are combined to evaluate the “fitness” factor used within the routine for ranking a population of designs; and methods for analysing designs after they have been ranked by the genetic algorithm. These are described in turn below.

Constraints on the designs. Each code representing a discrete gradient design consisted of the co-ordinates of one of the corners, plus the lengths and widths of a certain number of rectangular wire elements, up to a given maximum. This maximum number of rectangles was intended to prevent the optimisation procedure from producing designs which have too large an inductance. The current direction was also encoded. The required symmetry was used to produce extra rectangular paths as necessary and to deem some of them illegal if they did not obey the following symmetry rules: rectangles were made to be symmetric if they crossed a line of symmetry, and deemed illegal if they crossed a line of asymmetry. This reduced the region of interest over which the Bz-field profiles needed to be calculated by a factor of 8.

The example illustrated below set the maximum number of rectangles as 20, the number of designs compared as 100 and a 1% chance of a chromosome bit inverting. The maximum number of rectangles was actually the maximum number within one quarter of the plate. More than 20 rectangles could be used despite the limit due to the symmetry requirements. For a z design there was a maximum total of 40 rectangles and for a y design the maximum total was 80, due to this symmetry. A wire diameter of 2 mm was specified so that wire tracks had to lie at least 2 mm apart. The choice of wire diameter was important for two reasons. First, the wire diameter should be large enough
to withstand the heating due to the current through it. However, the smaller the wire the easier it would be to wind into the desired design. The diameter of 2 mm was thus a compromise figure. The physical wire used however did not need to be exactly equal to the value allowed for (although it was in this case) - the thin wire approximation was used when formulating the field. This was appropriate since the distances at which the field was measured was more than an order of magnitude greater than the wire dimensions. The wire dimension of 2 mm was previously found to be a successful measure. The wire diameter did not in general need to be chosen in order to minimise the total resistance of the gradient set. It might be expected that a lower total resistance would enable a higher current to be put through the set, but in fact the gradient amplifiers were more limited by their internal resistance than by the resistance of the gradient coils. This is important since if the resistance of the set became comparable with the amplifier internal resistance then the efficiency would no longer be a reliable indicator of the maximum strength of the set.

**Explicit calculation of gradient design fitness.** The factors contributing to the utility of the gradient coils which were calculated explicitly were the efficiency, the percentage deviation from linearity of the gradient profile, and the length of wire used; these were combined to define the search spaces for the optimisation procedure. First, the algorithm attempted to generate a population of designs all of which had an efficiency that fell above a pre-determined minimum. The remaining design factors were considered after that had been achieved. The result was a genetic algorithm problem with two different search spaces: one searching for efficiency and the other solely for the remaining factors; previous studies\(^{108,110}\) have found this to be a useful division.

The fitness factor was calculated by the following formula:

\[
\text{Fitness} = \begin{cases} \text{LARGE NUMBER} & \text{Efficiency} \\ \frac{\text{Linearity Deviation}}{\text{Length of wire used}} & \text{if Efficiency} \geq \text{Minimum Efficiency} \end{cases}
\]

The linearity deviation was defined as the sum of the standard deviations in all three directions between the calculated gradient field and that which would exist if there were an ideal undistorted gradient. This deviation was expressed as a percentage to allow comparisons to be made between the distortions associated with different efficiencies. It was important that deviations in the two orthogonal directions were included since these "parasitic" gradients also distort the image. This was a modification to the algorithm from previous studies which did not include such gradients explicitly. Refering to Table 3.1, this is a modified version of \(\sigma_3\) as a measure of the linearity, referred to as \(\sigma_{3b}\).

Mathematically we can write

\[
\sigma_{3b} = \frac{1}{V} \int |G(x, y, z) - G_0|^2 \, dt
\]

where we have now calculated vectors for the calculated and desired gradients, in order to minimise the deviations that are not in the primary direction. It should be noted that for some geometries of gradient coil, consideration of the parasitic gradients is less necessary since the symmetries are such that it is difficult for such gradients to arise. This is the case for a z gradient cylindrical coil for example.

The length of wire used was included since it is proportional to the resistance of the gradient set. Minimising the length of wire also aids minimisation of the inductance, since the amount of wire used will be related to the complexity of the design.

The cost function (once the minimum gradient was achieved) can be regarded, in the nomenclature of Table 3.2, as being given by:

\[
E = R \times \sigma_{3b}
\]
The LARGE NUMBER ensured that the fitness factor was large for designs which had less than the minimum efficiency and was chosen so that all such designs had a fitness factor larger than all other designs; hence whether the minimum efficiency is reached, or not, determined the order of magnitude of the fitness factor. Thus the search always retained designs above the minimum efficiency in preference to those below it. The search can thus be viewed as being in two parts: a search to find designs of the required efficiency, followed by an optimisation of the designs after that efficiency has been achieved.

The fitness factor term defined above is relatively arbitrary: there is no single absolute measure of gradient performance due to the large number of different factors involved. Since the algorithm used in the present study was intended to be applicable only to this particular design there was no need for a universal performance measure at the design stage such as those designed for comparisons between gradient sets.\textsuperscript{132} Multiplying the contributing factors together rather than summing them effectively gave an order of magnitude calculation which was appropriate given the approximations used in the calculations and the complexities of the engineering construction.

Other tests for gradient viability. As well as these inclusions explicit in the optimisation calculations, other limiting factors were also considered by appropriate adoption use of the calculation routine. For example, transient relocation of the wires within the designs by small random amounts after every tenth iteration made it possible to assess changes in the fitness factor which would arise were a wire to be misplaced due to an engineering imperfection. The higher the accuracy required, the more likely this was to have a significant effect on the fitness factor; seemingly fit designs that would have needed an unrealistic accuracy were thus discarded. This facility represents a significant advantage for the optimisation algorithm over purely analytical routines: thus it was possible to test the engineering tolerance without interrupting the design procedure.

Another modification of the basic protocol from previous studies was to stop the calculation if it appeared to have reached its best solution, and to then reseed it only with the optimum design. By increasing the search space available this avoided the trend towards population uniformity, which the random mutation rate could not always prevent.

It was found that the patterns of wires were complex and, as described later, unnecessarily so in some cases; consequently, it was not practical to construct them in the form calculated due to the cost and the degree to which the wires would have crossed. While the constraints on the chromosomes prevented overlapping wires, they did not prevent an undue degree of crossing at right angles. Hence corners became very messy. However, the designs could be simplified by placing separate wires adjacent to each other providing that this did not greatly affect the fitness; those sets were much simpler, but still retained the essence of the designs produced by the genetic algorithm. Indeed in some cases the fitness was enhanced, due to the simultaneous repositioning of wires which might otherwise have been rejected as being illegal by the checking routines used to prevent wire overlap. The above procedure was performed for \( z \) and \( y \) gradient coil designs, and Figs. 4.4 & 4.5 show respectively the final construction designs.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{construction.png}
\caption{The constructed combined \( y \) and \( z \) gradient set. The plates are 400 mm square.}
\end{figure}

The one critical performance characteristic which was not included in the optimisation process was the rise-time of the gradient coil. This was difficult to calculate, as it is a function of the interaction between the gradient coils and the magnet bore, as well as of the self-inductance of the gradient set. Fortunately, the simultaneous minimisation of
4.2 Methods

4.2.1 Construction

Fig. 4.3 illustrates a specific example of a biplanar gradient set which was built using the design methodology summarised above. The tracks used to locate the wires were milled to a depth of 4 mm in PVC sheets 9 mm thick and of size 40 cm x 40 cm. Copper wire 2 mm diameter was then wound in these grooves and glued in place using an epoxy adhesive. The y and z plates were then glued and screwed together, and the two sets of composite sheets were connected using two plastic half-cylinders whose diameter was chosen to fit just within the required separation of the plates. The plates were designed to be detachable for ease of access. The wires for each set were connected in series to the power supply.

4.2.2 Performance Testing

The gradients were tested both on the bench (for electrical performance) and for imaging performance. The power supplies used were Techron 8524 models, with a maximum current capacity of 150 A. Images were acquired for a grid phantom which comprised a regular 1 cm grid of perspex rods within a cuboid perspex box containing 0.1 mM manganese (II) chloride solution. The phantom dimensions were 16.8 cm by 13.8 cm by 3.8 cm. The MRI sequence used was a gradient echo sequence with a repeat time of 100 ms.
4.2.3 Image Interpretation

Gradient coils produce undistorted MR images only when they produce perfectly linear magnetic field profiles; otherwise there are predictable distortions in the images produced. For a gradient coil such as that described here which has a linear central field of view of 15 cm diameter, most images will suffer some distortion at their periphery; this will be the case for most reduced bore gradient coils. To calculate the spatial distortion found in images produced using a gradient of general profile a time-invariant magnetic resonance signal acquired under the influence of an $x$ gradient was considered. This was equivalent to considering the distortion caused in the read direction of a magnetic resonance image. Applying a gradient causes an offset in the local magnetic field and therefore an equivalent offset in the precession frequency. A frequency offset will manifest itself as a translational offset following Fourier transformation of the acquired signal.

For a magnetic field gradient which causes a field offset of $\Delta B_x(x)$ at position $x$ the apparent co-ordinate $x_{app}$ was written as

$$x_{app} \propto \Delta \omega_x(x) \propto \Delta B_x(x).$$

(4.3)

For a perfectly linear gradient, the apparent co-ordinate is the same as the true co-ordinate. Accordingly the constant of proportionality can be fixed for a linear gradient:

$$x_{app} = \frac{\Delta B_x(x)}{G_x},$$

(4.4)

where:

$$\Delta B_x(x) = G_x x,$$

(4.5)

The above analysis only considered the read direction for an imaging experiment. The same positional relationship also applies to the phase encode direction, and to any distortion in the slice select direction. In a sense, the three $B$-field offsets label the point and however the data is acquired the same distortions will result given the Fourier
transform relationship between position and frequency. Accordingly, for 3-dimensional data the following three equations determined where signal was distributed:

\[ x_{app} = \frac{\Delta B_x(x, y, z)}{G_x}, \]  
\[ y_{app} = \frac{\Delta B_y(x, y, z)}{G_y}, \]  
\[ z_{app} = \frac{\Delta B_z(x, y, z)}{G_z}. \]  

Note that this analysis did not require the absolute strength of the gradients that were involved in the pulse sequences that produced the images, only the relative gradient strengths within different regions in the field of view. The supposed field of view of the image was determined by \( G_x, G_y \) and \( G_z \), and in order for this analysis to be viable the gradients had to be configured appropriately; the gradients must be configured to assume that the gradient efficiencies at the centre of the image were constant throughout the imaging volume.

This analysis made it possible to correct distortions produced by non-uniform gradients except in the two cases described in section 2.9.2.

The volume of gradient monotonicity (Fig. 4.8) was calculated using field maps calculated from the wire positions provided by the design algorithm. In the example considered here, the largest cuboid that could be inscribed in this shape had dimensions 36.3 cm by 17.2 cm by 24.4 cm. This gave a useful imaging region that started 3.6 cm from the plates. However, this cuboid was not an exact representation of the volume, and signal could be retrieved from as close as 1.9 cm for a sample which only gave rise to signal for a length of 3.8 cm in the \( z \) direction. This could be achieved by a small surface coil.

Figure 4.8: The region of monotonicity of the gradient set. Image produced using SGI Volumen software. The volume between the plates is 40 cm by 40 cm by 24.4 cm. The gradient plates would be at the top and bottom of the cube i.e. the transverse axis is the \( y \)-axis.

4.3 Results

4.3.1 Biplanar configuration analysis

The biplanar gradient set developed here cannot produce a strong gradient in one of the three axes. This is still the case even for non-orthogonal gradients which can be used to produce linear images after correction; the \( x \)-component of each gradient is very small compared with the components in the other directions. Accordingly, we are constrained to use the low efficiency \( x \) gradient produced by the large bore cylindrical set built into the magnet. Due to its size this has a large rise-time as well as a low strength in comparison with the biplanar produced gradients.

This limitation had implications for how pulse sequences should be designed for use
with these coils. It is the product of the amplitude and duration of a gradient pulse which determines the degree of movement in k-space. So since the amplitude was limited, the duration had to be increased in order to have the same effect. Consequently this limitation will only affect the pulse sequence in terms of increasing its total duration. However the increased scan time will change the contrast of the sequence and so will often be inappropriate.

For a slice in a plane containing the x direction this problem has sometimes be countered by using the x gradient as the phase encode direction rather than the read direction. In this way this increase in duration of the pulse sequence will be alleviated since the actual acquisition time can then stay the same, and only the preparation stage of the pulse sequence is altered.

However the duration is still limited by the magnitude of the largest phase encode step which can limit the utility of the gradient set. One solution would be to employ an EPI sequence with the x gradient used for the small phase encode “blips”. Paradoxically the faster sequence minimises the limitation imposed by the weak gradient. This is because the phase encode gradient is imposed in small parts rather than one large pulse per acquisition. The read gradient has to be very strong to cope with the large gradient amplitudes and fast to switch direction in the required time but the phase encode steps are very small in comparison. There is one large phase encode step needed in the preparation part of the pulse sequence, but in comparison with the entire acquisition this is much less significant than for conventional sequences.

**4.3.2 Z gradient performance analysis**

The limitations of this configuration of z gradient coil were investigated by attempting to produce designs with different minimum efficiencies. At low efficiencies coils of very good linearity were achieved with few iterations while for larger efficiencies the linearity was poorer and the designs needed more iterations to produce them. Fig. 4.9 shows the results of five different runs of different minimum efficiencies. The y-axis represents the fitness factor of the best member of a population and the x-axis represents the iteration number at which this member was pre-eminent. For this purpose the length of wire used was excluded from the optimisation since the theoretical limitations on gradient strength was the criterion of interest. Hence the fitness factor on the y-axis is simply expressed as the percentage deviation from linearity. The sudden jump from zero fitness represents the point at which the minimum efficiency was first achieved and the algorithm starts to search for solutions with smaller linearity deviations. The algorithm was capable of producing a suitable design for all gradient efficiencies up to 0.4 mT m⁻¹ A⁻¹. However the linearity that was achieved became increasingly poor for more efficient sets, and the time to achieve the minimum gradient set increased. It was decided that the compromise between strength and linearity should be for a minimum efficiency of 0.3 mT m⁻¹ A⁻¹.

Note that while it might be expected that the designs should get progressively better, the buildability check could occasionally make the design worse. Thus the improvements were not relentless.

**4.3.3 Genetic algorithm consistency**

The consistency of the genetic algorithm was measured by repeating different runs to assess the extent to which they reached the same solution. This is not a proof of convergence to a global minimum - for a numerical method this is not generally possible in a finite number of iterations - but repeated experiments with different random seeds lend weight to this conclusion. If an algorithm is not repeatable it implies that the program is converging to a local minimum in search space. Fig. 4.10 exemplifies the results of such duplicate runs. It adopts the same format as Fig. 4.9, though in this case optimising for a minimum length as well. The four runs all reached the required gradient at roughly the same time
Figure 4.9: The progress of the optimisation routine for different minimum gradient specifications. and improved at roughly the same rate.

These designs themselves could be compared more closely to assess whether the emergent wire layouts were similar. Fig. 4.11 shows only the relevant $B_z$ producing wires from the design created by each run in order to simplify the representation. The wires running parallel to the z direction are not represented since they confuse the diagram and do not contribute to the $B_z$ field. All these designs have wire positions that look superficially quite different. However if the effect these wire position differences have on the gradient profiles is considered, it can be seen that they do not cause significant differences.

Some pairs of lines where opposite current directions flow very close to each other did not have a large effect on the field profile. These small rectangular wire elements would ideally be removed by the algorithm due to the minimisation criteria with respect to length. However, due to the nature of genetic algorithms and the way that they sample search space such small changes were not always considered. Similarly, the length of the wires which extended far beyond the central region of interest was not critical. As can be seen in Fig. 3.2 the variation of the field along the wire's length was not very great and so the exact positions of the wire ends was not critical.

A feature common to all the designs shown is the two compact tracks of adjacent wires, which occur at roughly the same distance from the centre in all cases. The innermost set is more closely packed than the second. Also there was a tendency for current paths within this innermost track to be made up from two wire segments instead of a continuous wire. By examining what effect making these wires continuous had on the design this feature was found to increase the linearity. There were also central current paths in these designs which lie in the opposite direction to the majority of the wires. These were also found to be increasing the linearity of the set - while also decreasing the strength slightly.

The final iteration of one computation had a slightly poorer fitness than the other three - this corresponds to the design in the bottom right of Fig. 4.11. In crude terms this solution had a poorer linearity because the algorithm had not reached the region of search space containing the solutions with central rectangular wires with current in the opposite
Figure 4.11: Design solutions for four different genetic algorithm runs. Wire sections in which the current runs from left to right are shown as solid lines, dotted lines represent wires in which the current runs from right to left. Only wires which contribute to the $B_z$ field are shown.

These small differences in design are characteristic of the way that genetic algorithms search for solutions. Genetic algorithms cover larger regions simultaneously but are less efficient at considering the benefits of small perturbations in a design. In this case the genetic algorithm gives solutions which can be modified to be easier to construct without losing the basic design structure; the tolerances are great enough for this.

4.3.4 Performance of the gradient coils

Table 4.1 shows that the constructed gradient set had a slightly different efficiency. The efficiencies were measured by measuring the current required during a magnetic resonance sequence and the discrepancies may be due to this measuring process. As expected, the inductance measurements were higher than for some other comparable gradient sets. This was due to the fact that the rise time was not included explicitly in the optimisation procedure. The linearity was good, especially since the parasitic gradients are included in this quoted figure. The linearity in the primary direction (the direction of the desired gradient) is also quoted in the table, and is less than 5% in both cases. The measured resistance would ideally be proportional to the length of wire in the two sets. However, due to the length of wire used for the leads connecting to the gradient set, the ratio is not the same. The rise time was found to be less than $50 \mu s$ for all pulse sequences used.

Comparisons with alternative gradient coils are difficult due to the different geometries and sizes employed in each case. However for comparison with conventional MR hardware performance data for the smallest Maxwell and Golay coils which give the same width as these biplanar coils (46.9 cm diameter) are shown in Table 4.2, and also the largest such coils which will fit within the biplanar set (24.4 cm). The conventional coils are multiply wound with conductor in order to provide a comparison using roughly the same length of wire.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Wire Length (m)</td>
<td>47.15</td>
<td>59.39</td>
</tr>
<tr>
<td>Measured Resistance (Ω)</td>
<td>0.61</td>
<td>0.87</td>
</tr>
<tr>
<td>Measured Inductance (mH)</td>
<td>0.442</td>
<td>0.411</td>
</tr>
<tr>
<td>Measured Grad. Strength (mT m$^{-1}$ A$^{-1}$)</td>
<td>0.34</td>
<td>0.40</td>
</tr>
<tr>
<td>Calculated Grad. Strength (mT m$^{-1}$ A$^{-1}$)</td>
<td>0.304</td>
<td>0.427</td>
</tr>
<tr>
<td>Calculated Deviation from Linearity (%)</td>
<td>8.95</td>
<td>9.27</td>
</tr>
<tr>
<td>Linearity in primary direction (%)</td>
<td>3.53</td>
<td>4.31</td>
</tr>
</tbody>
</table>

Table 4.1: Comparisons of calculated and measured gradient performance
In terms of gradient linearity, the large diameter Maxwell-Golay coils outperform the biplanar coils, as would be expected. However their efficiencies are much lower, and in the case of the Golay coil, the conductor is very long. The smaller Maxwell-Golay coils lack the lateral access which makes the biplanar coils useful, and their linearity is poorer over the design volume, particularly in the case of the Golay coils; their efficiency is greater than the biplanars however.

Fig. 4.12a is an image of the grid phantom obtained using this gradient set. The whole body x gradient was used for the slice selection, leaving this image in the yz-plane, using both designed gradients. The field of view is 25 cm. This is larger than the region of interest for which the set was optimised and so some some geometric distortion of the image is expected far from the centre of the field of view.

In order to test the potential utility of these gradient coils for human knee imaging preliminary scans have been completed using pulse sequences optimised for cartilage measurements. To make use of the opportunity to study flexed knees would require a purpose-designed r.f. coil; for comparative purposes a cylindrical quadrature probe which is normally employed within conventional hardware for knee imaging is used here. The pulse sequence used was a fat suppressed spoiled GRASS (Gradient Recalled Acquisition in the Steady State) sequence.

A slice through an acquired 3-D image is shown in Fig 4.13. This is of comparable quality to that currently achieved by conventional hardware. By use of a more open r.f. coil (for example a surface coil or Helmholtz pair) different flexion angles could be achieved.

### Table 4.2: Comparisons of biplanar and conventional Maxwell-Golay coils of equivalent lateral width

<table>
<thead>
<tr>
<th>Components</th>
<th>Efficiency (mT m⁻¹ A⁻¹)</th>
<th>Length (m)</th>
<th>Wire Length (m)</th>
<th>Linearity (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biplanar z gradient</td>
<td>0.34</td>
<td>0.40</td>
<td>47.15</td>
<td>8.95</td>
</tr>
<tr>
<td>Maxwell (16 turns)</td>
<td>0.23</td>
<td>0.41</td>
<td>47.15</td>
<td>0.92</td>
</tr>
<tr>
<td>Maxwell (30 turns)</td>
<td>1.62</td>
<td>0.21</td>
<td>46.00</td>
<td>9.10</td>
</tr>
<tr>
<td>Biplanar y gradient</td>
<td>0.40</td>
<td>0.40</td>
<td>59.39</td>
<td>9.27</td>
</tr>
<tr>
<td>Golay (7 turns)</td>
<td>0.12</td>
<td>1.20</td>
<td>56.07</td>
<td>1.78</td>
</tr>
<tr>
<td>Golay (14 turns)</td>
<td>0.86</td>
<td>0.63</td>
<td>56.70</td>
<td>27.61</td>
</tr>
</tbody>
</table>

Figure 4.12: Gradient inhomogeneity correction of a grid phantom larger than the linear gradient region of the coils. The field of view is 25 cm.

Image correction methods for non-uniform fields have previously been described with particular reference to B₀ inhomogeneity. The above analysis is used here to interpret images acquired with non-ideal gradient profiles, with the analysis formulated to make...
Figure 4.13: A transverse slice through a 3-D knee image using the biplanar gradient coil. The Field of View was 20 cm and the voxel size is 0.8 mm(x) by 0.8 mm(y) by 1.5 mm(z). The image has been shifted to be central within the displayed frame.

clear its pulse sequence independance. Using Eqs.(4.6-4.8) and an acquired image, the position is calculated where signal from co-ordinate \((x,y,z)\) would have appeared in the acquired image i.e. at \((x_{app}, y_{app}, z_{app})\), and it is then relocated to the correct position in the corrected version.

By considering every pixel in the reconstructed image, the acquired image could then be sampled to produce a corrected image. In general the apparent position did not have integer co-ordinates and linear interpolation from adjacent pixels was used to give the most accurate representation of the actual signal.

Fig. 4.14 illustrates a two dimensional simplification of the problem that shows the real spatial co-ordinates as dotted lines and the acquired (or apparent) image grid as solid lines. These are both "contour" maps of \(\Delta B_x\) and \(\Delta B_y\) overlaid on axes in real space;

![Graph showing the relationship between \(x\) and \(y\) with \(\Delta B_x\) and \(\Delta B_y\) as variables.](https://example.com/graph.png)

Figure 4.14: The real image grid with the distorted image grid overlaid on it. Note that the exact locations of the distorted pixels are not known; the field is specified at real co-ordinates and linear interpolation is used to place the distorted pixels.

the dotted lines (real co-ordinates) correspond to contour maps for undistorted gradients. The contours are spaced at integer multiples of \(G_x\) and \(G_y\); for images with odd numbers of pixels, there is a central axis of \(\Delta B_x = \Delta B_y = 0\), but for images with an even number of pixels the first contours are at \(\Delta B_x = G_x/2\) and \(\Delta B_y = G_y/2\).

Since the signal at the real co-ordinates had to be calculated and the signal at the distorted co-ordinates was known, linear interpolation was used from the four points which surround each real image point (or eight points in the 3-dimensional case). However this did not provide a correct estimate for the signal at that point. If the distorted local gradient was higher than anticipated then the contours were closer together and the same magnetic resonance signal from each voxel in real space was spread over more distorted image voxels, thus reducing its amplitude in each one. An amplitude scaling factor was
accordingly applied to scale with the contour density at each real point. This was provided by the Jacobean transform, normalised to be unity when the gradients were undistorted. For 3-dimensions:

\[
J(x,y,z) = \frac{1}{G_x G_y G_z} \begin{vmatrix}
\frac{\partial G_x}{\partial x} & \frac{\partial G_y}{\partial y} & \frac{\partial G_z}{\partial z} \\
\frac{\partial G_y}{\partial y} & \frac{\partial G_y}{\partial y} & \frac{\partial G_y}{\partial y} \\
\frac{\partial G_z}{\partial z} & \frac{\partial G_z}{\partial z} & \frac{\partial G_z}{\partial z}
\end{vmatrix}
\]  

(4.9)

This approach did entail a disadvantage in that it only sampled the acquired image in order to reconstruct the image, rather than using the acquired image in its entirety. If signal was retrieved from the apparent co-ordinates immediately surrounding the real co-ordinate but the real image voxel contained many distorted image voxels then all the information stored in those other pixels was lost. If the voxel considered produced a uniform signal, then the Jacobean provided an exact correction. Otherwise the undersampling could potentially lead to an image voxel which did not contain all the information actually present in the volume element. This problem was dealt with by increasing the resolution of the real image grid into which the signal was extrapolated. In contrast, in regions where the resolution of the distorted image was lower than that of the real grid, as occurred when the resolution was increased, then signal values were interpolated. In this way regions of the reconstructed image may misleadingly appear to contain more information than they actually do.

Thus images that have been restored using this algorithm accordingly required a degree of interpretation since regions of them may contain estimated signal values and other regions may have been undersampled. Fortunately these regions were known from the value of the Jacobean at that pixel. For a Jacobean greater than one there was a chance of undersampling; for a Jacobean less than one the linear interpolation was occurring over more than one pixel boundary.

Similarly, the signal to noise of these images would be expected to vary, decreasing in areas of high gradient strength. Although the signal level was increased by the multiplication of the Jacobean, the noise was increased in the same proportion. So the signal to noise ratio was inversely proportional to the square root of the Jacobean at each pixel.

Flow data were not considered by this algorithm which only correctly reregistered signal which was stationary throughout the acquisition routine. If flow had existed through the region then the signal would have been exposed to varying gradients which flow measurement algorithms assume to be constant with position. Flow data should thus be disregarded unless it is measured in a region of gradient linearity the size of which is greater than the product of the maximum flow rate and the length of the pulse sequence (this would ensure that all signal measured will have experienced the same gradient efficiency).

Another type of artifact introduced by this procedure occurred beyond the point where the gradient folded back on itself, as previously described. Under the reconstruction algorithm described above, signal was copied to these forbidden regions from the centre of the image where the expected B-field corresponds to the B-field found at the edge. Since no signal could come from these regions at all, these artifacts were eliminated by blanking all pixels beyond the region of monotonicity in the reconstructed image.

By using this distortion correction procedure the image in Fig. 4.12a was processed further to give Fig. 4.12b, giving virtually straight edges. The intensity variation was due to r.f. probe inhomogeneity rather than the gradient set.

However for images with a sample closer to the plates, the correction software did not adequately correct distorted images. This was because misplacements of the wires were more important as the distance from them became comparable to the size of the distortion. This caused the calculated field maps to be significantly inaccurate, and thus the correction software to break down.
4.4 Discussion

This chapter concerns designs of biplanar gradient sets for magnetic resonance imaging specifically to suit the geometry of the specimen being imaged, contrasting with the standard approach of fitting the subject of the image to a standard geometry of gradient set.

The particular geometry that was considered is the biplanar configuration. Conventional magnetic resonance gradient sets are cylindrical due to the symmetry advantages conferred. However by using two plates, as in this study, it has been possible to place the field producing elements closer to the imaging region for objects which do not have cylindrical symmetry. This enabled stronger gradients than would have been possible from a cylindrical coil large enough to accommodate an equivalent object. Biplanar gradients also allow better access to the imaging volume than cylindrical gradients, allowing the potential for imaging the human torso, or for some degree of interventional studies on the human knee.

Such gradient coils for magnetic resonance imaging can be categorised in terms of their magnetic field generating efficiency, their rise-time, how large the region over which a linear field is produced, and the degree of this linearity. In this analysis the efficiency and linearity have been focused on, and in particularly the implications of accepting greater tolerances for the linearity of the gradient coil have been considered.

The gradient calculation was performed by using superpositions of the Biot-Savart Law from all the current elements within a design. The field arising from unit current elements was considered and the net effect of the sum of such elements calculated to produce a magnetic field map for a theoretical gradient set. This not only produced a reliable method for design purposes but also allowed insights to be gained about the nature of biplanar gradient coils, highlighting the expected regions of linearity and those where a correction algorithm were useful for interpreting data.

Different combinations of these current elements were compared for their suitability for producing magnetic field gradients by means of a genetic algorithm optimisation procedure. Some factors such as gradient efficiency and the linearity were explicitly included while others such as rise-time were implicit in the restrictions on the number of wire loops and length of wire that could be used. In addition, the choice of an optimisation procedure allowed non-invertible factors such as the ease of construction to be included in the procedure.

The intrinsic insensitivity of genetic algorithms to small changes in design optima was also well suited to this problem since designs were not needed to be of high precision given the engineering limitations. Tests were performed to check the consistency of the genetic algorithm method, the results of which are illustrated in Figs. 4.10 and 4.11. These demonstrated that crudely similar wire layouts were produced for the same input parameters; and that these produced very similar electrical performance, as would be expected from a robust algorithm.

The particular gradient set constructed was large enough to contain a human torso. It exemplified the design procedures detailed above and performed close to its predicted limitations. This paper has also described a general method for image correction for use in regions of poor gradient linearity. Due to the relatively large volume within a gradient coil that has a monotonic gradient profile in comparison with the volume that has a linear gradient profile, this is a powerful and useful algorithm as exemplified in Fig. 4.12. Here the straight edges of the phantom are virtually restored. Further from the centre corrections are less successful due to uncorrectable features as illustrated in Figs. 2.12 and 2.13, or inaccuracies in the construction causing inaccuracies in the fieldmaps used. Closer to the plates the latter factor becomes more significant.

The use of this correction routine makes necessary a greater degree of interpretation than is used for usual magnetic resonance images. As previously described the variable
gradient strengths in different parts of the image create regions of different signal to noise, and also regions of different resolution. The latter effect causes problems when transferring the data to a corrected image both if the resolution is too high or too low, as described earlier. Thus when using the correction algorithm it is important to consider the resolution of the grid onto which the uncorrected image is to be copied. Also, flow data are not reliably corrected for. Although - subject to the conditions above - the algorithm does correctly restore the phase information within an image, any effect on this due to the motion of spins through the gradient set will not be restored.

The correction algorithm was successful to such an extent that we could consider the size of the monotonic region between such uncorrectable features to be the more important factor than linearity in limiting the performance of gradient coils. This has implications for the appropriate choice of geometry for the coils. For example, the current return path in the y gradient coil (see Fig. 4.2c) intrinsically limits the potential volume for imaging, and given the correction methods that we have employed, is the limiting factor rather than the linearity of the gradient set. The size of the monotonic region could be increased by making the rectangular current elements lie in the xy-plane, winding them around thicker plates and imaging in the space between them. This would prevent the gradient reversal illustrated in Fig. 2.12. However it would reduce the gradient strength and, since there would be less control over the field, the linearity would be poorer.

The study of such putative gradient coils where geometry is of more concern than the case of design is the subject of the next chapter.

Chapter 5

Gradient System Design of Complex Shape

After the findings from the previous chapter regarding the usefulness of the genetic algorithm, we can consider a more complex approach to gradient coil design and test the limits of our algorithm's effectiveness. The genetic algorithm approach has shown that for the biplanar geometry it can produce good results. If we progressively relax our constraints on the geometry, the next logical step is to see how "free" we can allow our wire paths to be. If we are able to abandon the symmetries of the magnet and its geometrical conveniences, is there a benefit in magnetic field performance for some regions of interest?

Such a philosophy has been used before, but most usually in order to produce coils with a uniform region away from the central region, or to impose another fixed but alien geometry on the problem, as is the case in the previous chapter. This approach attempts to reverse the design problem and allow the geometry to vary as much as possible to find a solution. Such a methodology is difficult for analytic strategies since such problems are less readily invertible. However the optimization's search space becomes more complex with more degrees of freedom for less constrained designs.
Hence in order to do this more powerful computing facilities are required than have previously been applied in this thesis; or to gradient design in general. In this chapter we first show how a supercomputer can be used to speed up the design process, and then consider possible geometries which may be interesting for magnetic resonance imaging. The first designs considered are relatively accessible configurations, and a generalisation is then attempted. Finally, we demonstrate some designs which have been produced using the supercomputing facilities in this University, namely the massively parallel SR2201.

The two classes of gradient coil that we consider are referred to as "block" gradient coils and "arc segment" gradient coils.

5.1 Gradient Design using High Performance Computing

In order to experiment with the expansive gradient designs, high performance computer resources are required. These algorithms are very intensive and even on the fastest available single processors the time taken is prohibitive.

Fortunately genetic algorithms are well suited to parallelisation. This is the means by which a large task is distributed across various processors (known as nodes). Algorithms which are well suited to this are those whose most computationally expensive part occurs within a loop, iterations of which are independent of each other. This is the case for genetic algorithms, where each member of the population has to be evaluated separately.

Access has been available to a Hitachi SR2201 supercomputer. This is part of the Cambridge High Performance Computing Facility (HPCF). It comprises 256 nodes and is capable of peak transfer rates of 300 MBytes/s between those nodes. Each node has 256 Mb of main memory and a peak computational performance of 300 MFLOPS. The total supercomputer is usually divided into partitions. The production partitions generally used in this thesis contained either 32 or 64 nodes.

A common measure of performance of parallelised code is the linear speedup. This is the ratio of the time taken for a task on a single processor to that across multiple processors. Perfect parallelisation would imply a speedup equal to the number of nodes. In practice there can be superlinear speedup due to such issues as memory accesses being made more efficient.

Preliminary testing was conducted using modified source code from Chapter 4 both on the supercomputer and on a cluster of Linux machines. Parallelisation was performed using Message Passing Interface (MPI) protocols. The Linux cluster contained four 333 MHz Intel machines each with 256 Mb of main memory.

Results are shown in Table 5.1. As can be seen the speedup is approximately equal to the number of nodes for the small cluster of Linux machines but poorer for the larger supercomputer. There are various explanations for this. Firstly, the parallelisation is not complete. Each member is evaluated in parallel but collation of the results is currently done on a single processor. More critically, the optimum solution is recalculated if it differs after each iteration. This is to reduce memory usage during each iteration. There is no way of knowing which member of the population will be the current optimum until all have

<table>
<thead>
<tr>
<th></th>
<th>No. of iterations</th>
<th>Real Time (hrs)</th>
<th>CPU time (hrs)</th>
<th>Real Time /iteration</th>
<th>CPU time /iteration</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single linux node</td>
<td>74</td>
<td>70.8</td>
<td>70.7</td>
<td>0.96</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>Linux (4 nodes)</td>
<td>272</td>
<td>65.8</td>
<td>254.3</td>
<td>0.24</td>
<td>0.93</td>
<td>3.97</td>
</tr>
<tr>
<td>Single SR2201 node</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1.0</td>
<td>1.0</td>
<td>8</td>
</tr>
<tr>
<td>SR2201 (64 nodes)</td>
<td>305</td>
<td>8</td>
<td>512</td>
<td>0.026</td>
<td>1.68</td>
<td>41.67</td>
</tr>
</tbody>
</table>

Table 5.1: Comparisons of real time and CPU time for a parallel code application for a linux cluster and supercomputer partition.

\[\text{http://www.hpcf.cam.ac.uk}\]
been evaluated. Rather than keep all field profiles in memory, the optimum is regenerated and displayed. Memory considerations are less of a consideration with a parallel machine, however if the number of chromosomes is greater than the number of nodes the problem of overwriting still occurs. An alternative strategy would be to parallelise the algorithm at a lower level (for example to parallelise the loop which calculates the magnetic field) however this was rejected as unnecessary.

Another factor is that shorter runs, such as occur with an 8 hour CPU limit as exists on the SR2201, tend to be less representative of the progress of the algorithm over a longer period. Unfortunately the production partitions were only available in 8 hour units, with processes forcibly ended once the time was finished.

Nonetheless, even with only a 41-fold increase in power, parallelisation speeds up the linear algorithm greatly.

5.2 "Block" gradients

As described in section 4.2.3, the limiting factor for the size of usable volume of a transverse gradient coil such as a Golay coil or the biplanar coil previously described, is determined (in the transverse case) by the necessity for the current return paths to lie within the same plane (or on the same cylinder) as the primary wires. This necessarily inverts the gradient at the edges of the coil and inverts the image, as in Fig. 2.12. For imaging required to be near the edge of the gradient coil (for example neck imaging in a head gradient system) this can render studies impossible. Gradient coils which provide a large useful volume relative to their size are thus important.\textsuperscript{121}

However using very similar formulism to the previous chapter, we can explore the possibilities of winding around blocks, keeping the wire return paths perpendicular to the main field at all times. An elemental “block” gradient set is shown in Fig. 5.1. Some studies have been performed on developing similar cylindrical coils.\textsuperscript{121} They enable greater control over the length of the coil since the return paths do not lie within the same plane - they would otherwise force an increase in the length of the gradient coil to keep the return paths away from the region of linearity which they would degrade. The acoustic response of such coils are also lower due to local torque balancing in the coil. However these coils are inevitably thicker than conventional coils and thus use more room in the magnet bore. While the optimised region can be kept further from the return paths at the edges of the coil, it is also closer to the return paths in the centre - this will reduce the efficiency of the coil.

![An elemental block z gradient](image)

But this geometry does retain the open access enjoyed by the biplanar set. And since the axis of the coils is now parallel to the main field we would also expect less interaction with the conducting surfaces in the magnet. Major flux lines lie within the blocks and parallel to the magnet bore rather than perpendicular to the conducting surfaces of the magnet assembly. Conversely, we have less control over the linearity than before and so would expect a more distorted image.

The C-code which was used within the previous chapter is easily adapted to this problem, and the wire positions for construction were determined by use of the same genetic algorithm optimisation package. The specific implementation here follows closely that described in Chapter 4 except in the encoding of wire positions. This is simplified since there is only one position to encode rather than having to specify a complete wire
path within a plane. A facility to encode the thickness of the blocks was included, but not used in the test case described below for convenience of construction. The buildability constraints were imposed by randomly shifting the position of the windings around the blocks in the z direction, similarly to the method described in Chapter 4.

5.2.1 A small scale block gradient coil

In this study, in order to test the modelling software, a longitudinal gradient coil was constructed which was just large enough to contain a grid phantom, used for testing purposes. The dimensions of the cuboidal grid phantom were 16.8 cm x 13.0 cm x 3.0 cm and the size of the region between the gradient blocks was 20.0 cm x 15.0 cm x 6.0 cm. The thickness of each block was 4.3 cm. After construction this was placed inside an existing reduced bore gradient set (diameter 35 cm), which was used to provide the x and y gradients. Signals were collected by a large volume quadrature resonator itself larger than the test gradient coil.

While the advantages discussed above apply largely to transverse gradient coils, a longitudinal coil was produced as the test object. This is due to dimensions available within the 35 cm gradient set with the grid phantom inside. The purpose here is to test the modelling software, and there is a greater extent within the r.f. probe in the axial direction, allowing modelling to be performed on more than the 3.0 cm width of the phantom. The edges of this are very close to the test gradient coil which would cause further acquisition problems due to proximity with the metal of the coils.

The "block" gradient coil former was made of wood, with the wire paths milled out. The copper wires (the same specifications as for the biplanar coils in the previous chapter) were glued in place with Araldite.

Results

The genetic algorithm was run for 400 iterations. This took 10 hours of computing time on a 120 MHz Pentium processor running the Linux operating system. This test system was adequately modelled on a single processor, with no need for the supercomputer.

Fig. 5.2a is an image in the yz plane, and a simulated image using calculated field maps is shown in Fig. 5.2b. The acquired image was produced using a gradient echo sequence. The grid phantom was the same as that used in Chapter 4 and had dimensions 16.8 cm by 13.8 cm by 3.8 cm, with perspex rods placed 1 cm apart. Both images show a hyper-intense region at the centre of the image, which is a region where the gradient is low and a larger volume contributes to each voxel.

Figure 5.2 : Images produced using the physical gradient coil, and by a simulation of the expected gradient profile. The z direction of the magnet is the vertical axis of this image, and the y direction is the horizontal axis.

The simulated image was produced using an idealised high resolution image of the grid phantom, with the "pegs" spaced exactly 1 cm apart. By using the calculated field
maps for the gradient coil, the magnetic field offsets at the corners of each pixel were found. Since, as described in section 4.2.3, the apparent x co-ordinate is proportional to the B-field offset produced by the gradient coil, the position of each distorted "real" pixel can be found on the simulated image. If we assume a perfect y gradient, the distorted pixel will be in the shape of a trapezium. It is simple to calculate the area of this, and distribute the intensity from the real image onto the simulated image grid proportionately to the area of the real pixel which falls at that position.

It can be seen from these images that the simulation has similar distortions to the real image. This produces confidence that the programs are operating correctly. The quality of the acquired image is poor however, reflecting the fact that the r.f. probe was placed outside of the gradient set, and inevitably close to it. The effect of the current carrying conductors of the longitudinal gradient coil within it has disrupted the signal acquisition. Shimming was also ineffective in improving the image quality.

The region of monotonicity of the z gradient, calculated from the theoretical gradient profiles, fills 76% of the total volume between the plates.

5.2.2 Possible developments for longitudinal coils

By widening the gap between the blocks we can design a coil which could be used for neck imaging for example. This would take advantage of the extended monotonic region between the plates. However, as discussed above, the advantages are primarily acoustic. The gradient strength will be lower, but due to the loops being wound parallel to the magnet bore axis, the rise time is likely to be reduced as eddy currents will not be induced in the magnet cryostat to the same extent.

5.2.3 Possible developments for transverse coils

While this approach works for longitudinal gradients, transverse gradients are much harder to control since the wire positioning parameters do not offer much flexibility. While the region of monotonicity can be quite good, the linearity and strength are not. However let us consider an array of these blocks, arranged around the edge of a cylinder as in Fig. 5.3. This would reduce the access, but may enable greater control of the field profile within it. The symmetries are marked on this figure for the case of 16 blocks. Those which lie on the line of anti-symmetry must necessarily have zero current (those blocks labelled as number 4), however the other blocks are free in their shape, current levels and positioning as long as they conform to the two symmetry lines.

The symmetry lines refer to the symmetries in the current levels, and to the symmetries in the field produced. However the current directions do not follow this rule, and the shapes of the blocks do not conform to this formulation either. For the block shapes, both lines are symmetry lines. For the current directions, the symmetry line is such that current sense is preserved under reflection about the symmetry line and inverted under reflection about the anti-symmetry line.

As described in the previous section, it was found while experimenting with the elemental gradient coil that it was very quiet. This is due to the lack of airspace within the gradient coil, as this "speaker" effect can generate a large proportion of the noise in an MRI experiment. The reasons for this are discussed in section 2.9.8.

It is possible to design a more complicated gradient coil using an array of these straight wire elements in order to allow more control over the field profile and thus a more useful central field. It would retain the advantages of a concentric return path in forcing the transverse gradient reversal point further towards the edge of the coil. There are various configurations that we can consider.
Fixed current block

The simplest configuration is to have identically wound blocks equally spaced around the circumference of the region of interest. This would work well for longitudinal gradient coils, since it retains some cylindrical symmetry. However for a transverse coil where there must be a symmetry line across a diameter of the cylinder, equal blocks will not produce the required variation.

We could make them do so by changing the current levels within them, making the current level different for each block. In the simplest case this would be to counter-wind the blocks on one half of the cylinder, and keep the blocks on the centre line with zero current. Alternatively, it may be that a sinusoidal variation in current in the blocks around the circumference of the cylinder is most appropriate, so an arrangement with current levels such as this could also be considered.

Varied current blocks

Of course the current ratio for each block can be optimised by use of the genetic algorithm. As we shall see later, the variation of the current levels is by no means necessarily a simple sinusoidal relationship. Inclusion in the chromosome allows free variation in the current level, without greatly increasing the computational task by including many more parameters. As previously, symmetry is utilised so that only a quarter of the blocks needed to be encoded.

Varied block shapes

The next logical step is to allow the shape of the blocks and the winding positions on them to vary, symmetry permitting. It may be that blocks closer to the line of antisymmetry require fewer windings than those further away. Also, if screening is being included in the

Having varying current levels introduces new problems into the gradient design, since it is no longer simple to connect the gradient coils in series. Achieving different current levels by connecting each block to a separate amplifier, or across a voltage divider, have problems due to the different response times of amplifiers and the reactive component of the impedance. These problems could be reduced by acting to minimise the mutual inductance between the elements. An alternative is to multiply wind some blocks which will allow exact fractional currents to be used. This restricts the range of current levels but avoids potential problems with unbalanced gradients. However in this case the wire size must be very small in comparison with the distance to the region of linearity, otherwise the accuracy of the positioning of the wires will become an important factor.

While this gives a design with few parameters - and hence for the genetic algorithm a short chromosome - it does not necessarily reflect the optimal approach to the problem. It may be better to allow the current levels to be optimised during the design.
algorithm, it may be that a rectangular shape of block is not optimal for minimising the field levels at the cryostat.

This formulism allows much more flexibility in the design of the gradient coils, but also greatly increases the complexity of the design and thus the chromosome. Except for freeing the wire loops from the restriction of being wound around blocks and enabling them to run outside of the plane perpendicular to the field, no more degrees of freedom can be added. The exact implementation in this case is not quite so liberal. The cross-section of the blocks can vary in shape as an irregular octagon. Effectively, the corners of each block can be removed with any angle of cut, and the radial thickness is also allowed to vary. This complexity is necessarily reflected in the size of the chromosome. With identical blocks, the size of the chromosome is 172 bits for the cases considered later. For this formulism the chromosome size is 828 bits.

5.2.4 An analytic method for balancing the current levels

Another modification which can be included to the algorithm can ensure that one aspect definitely produces an optimal solution. This is the ratios of the current levels in the different current blocks. While these can be included as part of the optimisation there is also a method of calculating the optimal levels for a given configuration of wires analytically. This shortens the chromosome (improving the speed of convergence of the algorithm), and can also be used as a check on the efficiency of the convergence of the genetic algorithm for this problem. The mathematics of this procedure is described in Appendix A.

This same calculation can also be used to investigate what current levels are optimal to create a transverse gradient.

Let us postulate a current $rmg$ which does not obey Kirchoff's Laws; the current amplitude is allowed to vary around the ring. What distribution of current is required to produce an optimal transverse gradient in the plane of the ring? This can give some insight into the likely current levels required if the current levels are to be fixed without optimisation.

This is clearly an unphysical model and not useful, other than for demonstrative purposes.

![Figure 5.4: The variation in current level around a ring](image)

The results are somewhat surprising. Fig. 5.4 shows the variation in current levels postulated for ring of 256 current elements. Only the first 64 are shown as symmetry determines the level of the other quarters. As can be seen, the current level varies markedly in adjacent elements. The reason for this is that the algorithm is using the negative regions to cancel non-linearities in the gradient profile. This is a feature of many analytic algorithms for such situations where further constraints to reduce high frequency variations are not imposed. This suggests that the fixed current transverse coils where each block has the same design will be difficult to produce.
5.2.5 Implementation

Using the increase in computing power described in section 5.1 we can now implement some of the geometries described by these larger chromosomes. The cost function used was very similar to that used in Chapter 4 with the following exceptions. First, the gradient homogeneity was implicitly calculated differently within the analytic current calculations. Due to the form of the equations, the current levels are calculated to minimise \( \sigma_l \) rather than \( \sigma_{3b} \). Both measures are minimised for a perfectly linear region, and the genetic algorithm still uses \( \sigma_{3b} \) within its cost function so the effects of this will be minimal.

Second, an optional screening facility is added. This aims to minimise the stray field at a given radius outside of the coil. This was done by using the product of the r.m.s. field at a certain radius away from the centre of the gradient coil, and extending a length longer than the coil. The screening radius was chosen to be equivalent to the distance to the cryostat. For designs which were optimised using the current calculation, the screening is performed by setting the “target” field at the external points to be zero. This will minimise the r.m.s. deviation at those points.

The cost function was the product of the r.m.s. deviation at the screening distance and \( \sigma_{3b} \). In the absence of screening, the cost function was just \( \sigma_{3b} \).

The dimensions of the test gradient coil were chosen so as to be accommodated in a conventional whole body magnet. The blocks were made large so as to give a comparison with existing gradient coil designs of a similar size. If these designs were constructed they would not have many of the advantages over other coils which this formulism can enable, but if these tests are able to produce results similar to existing algorithms then the method can be established. The parameters used to describe the coil are shown in Table 5.2.

Six classes of designs were initially programmed for the supercomputer.

- Identical rectangular blocks with varying current only
- Identical rectangular blocks with varying current only with shielding on.

<table>
<thead>
<tr>
<th>Number of blocks in the gradient coil</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum No. of windings on each block</td>
<td>32 (16 \times 2 for symmetry)</td>
</tr>
<tr>
<td>Separation between opposite blocks</td>
<td>32 cm</td>
</tr>
<tr>
<td>Max transverse dimensions of blocks</td>
<td>16 cm by 8 cm</td>
</tr>
<tr>
<td>Longitudinal length of blocks</td>
<td>40 cm</td>
</tr>
<tr>
<td>Cylindrical Field of View:</td>
<td></td>
</tr>
<tr>
<td>FOV diameter</td>
<td>20 cm</td>
</tr>
<tr>
<td>FOV length</td>
<td>25 cm</td>
</tr>
<tr>
<td>Screening cylinder:</td>
<td></td>
</tr>
<tr>
<td>Diameter</td>
<td>1 m</td>
</tr>
<tr>
<td>Length</td>
<td>50 cm</td>
</tr>
</tbody>
</table>

Table 5.2: The design parameters for block gradient evaluation. The size was chosen so that a putative coil could be constructed and lie within a conventional whole body scanner.
• Blocks of varying shape and current level.

• Blocks of varying shape and current level with shielding on.

• Blocks of varying shape with calculated current levels.

• Blocks of varying shape with calculated current levels, and shielding on.

Following the results shown in Fig. 5.4, fixed current blocks were not considered. The genetic algorithm was run with a population of 128, and a 1% mutation rate. The population was chosen to be 128 since it was a multiple of 32 - the number of nodes being used.

The results are summarised in Tables 5.3 and 5.4. The optimum deviation is \( \sigma_{3b} \) expressed as a percentage of the gradient achieved. The primary deviation quoted is the more usual \( \sigma_3\), which is quoted over both the large volume chosen for optimisation and also over a smaller sphere of diameter 16 cm, which is half the diameter of the interior of the gradient coil.

These results produce linear gradients, particularly over the central region, though possibly the algorithm is only finding a local minimum in search space. This is addressed in section 5.2.6.

For the latter two design categories the genetic algorithm is finding a similar solution for the full optimisation method to that which it is finding for the calculated current levels, suggesting that the optimisation performs well in comparison with the analytic method.

The cost functions for the rectangular blocks were initially much better than those for more flexible shapes. This was concerning since the rectangular cross-section is a subset of the more general geometry, and the optimal solution for the more flexible configuration should be better since there is more flexibility in the design. However the reason lay in the method by which the stability of the solution was calculated. In the case where the blocks contained independent wire positions and shapes the random fluctuations were applied to each block independently, which is a more realistic measure of the robustness of the optimal solution. By seeding the more general design of variable shape with these “better” designs the optimum solution degraded over each buildability iteration. Similarly extending the buildability algorithm for the fixed shape blocks degraded the value of the cost function achieved.

This is a good indicator of how important the constructional accuracy can be for gradient coils, and the buildability calculation is an important feature for the robustness of these designs.

The screening was particularly ineffective for the variable current designs as the results in each case are very similar irrespective of whether screening was used or not. This is due to the limited amount of control possible with the return paths for this geometry. The other design shapes are not affected by the rigid shapes imposed in this formulation.

The runs were repeated on the supercomputer until they stopped improving. Since time on the machine is allocated in runs of 8 hours (which is equivalent to 256 hours of CPU time for 32 nodes), all figures in the CPU time column are multiples of 256 hours and the results were analyzed at the end of each run to see whether it was worth using another run.

<table>
<thead>
<tr>
<th></th>
<th>Optimum Deviation (%)</th>
<th>Primary Deviation (%)</th>
<th>( B_{\text{ext}} ) (( \mu \text{T} ))</th>
<th>( B_{\text{ext}} ) (( \mu \text{T} ))</th>
<th>Iterations (hrs)</th>
<th>CPU time (hrs)</th>
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</thead>
<tbody>
<tr>
<td>Variable current</td>
<td>9.3</td>
<td>3.0</td>
<td>1.3</td>
<td>2.02</td>
<td>4.3</td>
<td>597</td>
</tr>
<tr>
<td>Variable shape (1)</td>
<td>20.6</td>
<td>8.0</td>
<td>0.75</td>
<td>1.75</td>
<td>3.5</td>
<td>493</td>
</tr>
<tr>
<td>Variable shape (2)</td>
<td>20.9</td>
<td>6.8</td>
<td>0.96</td>
<td>1.69</td>
<td>3.4</td>
<td>476</td>
</tr>
<tr>
<td>Calculated current</td>
<td>19.9</td>
<td>7.5</td>
<td>0.81</td>
<td>1.67</td>
<td>3.4</td>
<td>350</td>
</tr>
</tbody>
</table>

Table 5.3: The optimum designs produced for an unscreened transverse gradient coil for the different configurations.
### Table 5.4: The optimum designs produced for a screened transverse gradient coil for the different configurations

<table>
<thead>
<tr>
<th>Variable</th>
<th>Deviation (%)</th>
<th>Primary Deviation (%)</th>
<th>$B_{ext}$ max ($\mu$T A$^{-1}$)</th>
<th>$B_{ext}$ ($\mu$T A$^{-1}$)</th>
<th>Iterations</th>
<th>CPU time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>10.8</td>
<td>3.4</td>
<td>2.01</td>
<td>6.4</td>
<td>592</td>
<td>256</td>
</tr>
<tr>
<td>Shape (1)</td>
<td>38.0</td>
<td>13.1</td>
<td>1.07</td>
<td>3.0</td>
<td>701</td>
<td>1536</td>
</tr>
<tr>
<td>Shape (2)</td>
<td>33.4</td>
<td>10.9</td>
<td>1.15</td>
<td>3.2</td>
<td>701</td>
<td>1536</td>
</tr>
<tr>
<td>Calculated Current</td>
<td>34.2</td>
<td>11.8</td>
<td>1.00</td>
<td>2.7</td>
<td>485</td>
<td>$\approx 1100$</td>
</tr>
</tbody>
</table>

#### 5.2.6 Tests for Genetic Algorithm Consistency

In order to test further whether the genetic algorithm had produced repeatable results, further runs were completed for the most intensive test, that of the varying block shape and current level. A comparison of the two pairs of runs is shown in Fig. 5.5.

As can be seen, the two curves converge on the same value of deviation (which corresponds to the optimum value as described in Tables 5.3 and 5.4). The sudden jump from zero deviation represents the first iteration that has achieved the minimum efficiency required.

While it is not possible to prove that these routines are converging on the optimal solution, they are certainly converging on each other.

#### 5.2.7 An unscreened transverse gradient coil example

A gradient coil which contains narrower blocks was designed with the same algorithm. This coil has the advantages described earlier of being a "birdcage" arrangement, and need not be constructed as a continuous cylinder. Aside from the difference in width of the blocks,

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1The supercomputer froze during one of the runs, and hence this time is interpolated.

---

The shape of the blocks produced is shown in Fig. 5.6. The block shapes do not deviate greatly from their greatest extent. In other words, the return wires are pushed as far as possible away from the field of view. The r.m.s. field at the screening distance was 0.86 $\mu$T A$^{-1}$.

One very interesting feature of this shape is that the block shapes have tended to form
a straight edge parallel to the line of anti-symmetry of the gradient coil. While all virtually all other edges lie as far as possible away from each other, this feature is striking. The 83% volume of monotonicity within the gradient coils compares very well with existing transverse gradient coils where the return coil paths do not lie concentrically. For example, Golay coils have 38% monotonicity, and the biplanar gradient coil previously described has a figure of 56% (this is the usable volume common to the z gradient as well).

Fig. 5.7 shows a simulated image created using this transverse gradient coil design. The image is “positioned” to lie coronally in the centre of the gradient coil. It is in the same form as the grid phantom used earlier but sized 40 cm by 30 cm. The horizontal direction in the figure represents the axial direction within the gradient coil, while the vertical is the x direction and the direction of the gradient coil. The z gradient is assumed to be perfectly linear. The dark “pegs” are 1 cm apart to aid viewing of the distortions.

Since this phantom almost completely fills the interior of the coil, we do expect some distortion. Notably at the axial extremes of the figure there are points of inversion of the gradient, as previously described. Close to the coil windings the field varies rapidly and signal is also lost. However there is a large region which is linear for imaging. There is visible distortion towards the edges of the 20 cm diameter of the optimisation region of the coil however.

5.2.8 Screened transverse gradient coils

A screened gradient can be designed using the same process as above. However on the evidence shown previously about the efficiency of the screening method, such a design is unlikely to be much more effectively screened. The linearity would be marginally poorer while the external field would be lower.

The screening in this case is not directly comparable to that achieved in analytic designs. There is no specification that the field must be zero at the target points. The inclusion in the cost function makes this factor “trade off” against linearity. Modification of the cost function will better the screening while adversely affecting other characteristics.
For example, if the screening in the cost function were increased in importance, it might be found that the block shapes were pushed inwards more. In this case, the blocks tend to maximise their size.

As shown by the variable current designs, the screening appears to be largely dependent on geometry. More sophisticated cost functions are needed to properly optimise the value in this formulation. Screening can also only be properly performed by use of a further concentric surface outside of this gradient coil in which the wire positions are not determined by the primary windings.

5.3 Arc Segment Gradient Coil Design

Having considered the implications of "block" gradient coils for transverse gradient coil design, we will now return to considering arc sections. These are a more natural geometry for such a cylindrical coil and their curvature may provide advantages over the straight line blocks previously described. We will consider arcs whose curvature centres around the central axis of the magnet bore. Similar to the previous section, we will first consider curved blocks of fixed internal and external radii whose width and position can vary around this axis, subject to the symmetry conditions of the magnet.

Frese and Stetter have shown that for a transverse gradient coil an optimal return path for producing a linear field is represented by a pair of 120° arcs, connected by radial current joins. Other authors have shown that 120° arcs are optimal for cancelling the third order harmonics in the magnetic field produced by the current elements. Consequently, as discussed in section 3.2.2, this has been used as a basic unit for modelling this class of transverse gradient coil. The problem addressed here is whether advantage can be gained by departing from this unit - it may be that a more complex shape will produce an acceptably linear field over a larger volume, or with better screening, as Frese and Stetter themselves postulated.

The constraints on the design are that it must consist of concentric arcs centred around the axis of the magnet. Inner and outer radii are allowed to vary, as is the angular width of the arc (although outer and inner arcs must subtend the same angle). The symmetry considerations impose that the arcs must be symmetrical about the line of symmetry of the transverse coil, and must not cross its line of antisymmetry. The arcs must also all be coplanar. In this section we first validate the field generating code using comparisons against existing code and analytic solutions to some special cases.

Having established this, some theoretical examples of such gradient coils are presented.

5.3.1 Validation

The code used in this section is new and has not been validated as the implementations previously described have been so a more thorough check must be made on the code, a summary of which is included here.

Straight line modelling

Previously in this thesis, similar algorithms have been used for straight line elements. Here we must use arc elements, and their method of calculation is different and uses gaussian quadrature. We can test the current modelling software by comparison with the straight wire modelling by breaking down the current arcs into many straight line sections which approximate to the arc. While this takes an unacceptable amount of time for modelling, it can be used to test the implementation of the algorithms.

The simulated Golay coil was chosen to be of internal diameter 32.0 cm, the same as the block coil arrangements (as in Table 5.2) and arc coils being considered, and was modelled using the straight line simulation and the current arc calculations. The current arcs were integrated using gaussian quadrature of order 30. In the former case the curve was broken
Figure 5.8: The variation with longitudinal distance of the magnetic field generated by a current loop. The numerically calculated solution is indistinguishable from the analytically generated value on this plot, and for the purposes of gradient design for which these elements were used.

down into 664 straight lines (a complete current ring would be 1000). In comparison with the Golay coils, the deviation was 79.19\% as against 79.54\% for the prescribed field of view, and the strength was 0.0330 mTm\(^{-1}\)A\(^{-1}\) as opposed to 0.0326 mTm\(^{-1}\)A\(^{-1}\) for the Golay coil. This represents good correlation between the two methods of calculating the field. Since the straight wire method is well-established within this thesis, agreement implies that the arc calculations are yielding the correct results.

Comparison with analytic solutions

On the axis of a complete current loop, the field value can be computed analytically and is given by

\[ B_z = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}}. \] (5.1)

A validation using this equation is shown in Fig. 5.8. The close correlation between the two lines indicates that the two algorithms are producing very similar results. We might expect closer to the wires that the gaussian quadrature might break down - as at such points the magnetic field is not smoothly varying which is one of the conditions for convergence of the algorithm. However if points are not sampled too close to the wire then this will not be a problem. It is a comparable problem to the infinitely thin wire approximation that we use for the calculation of the fields.

### Table 5.5: The results of modelling segments of varying angular width for different target gradient strengths.

<table>
<thead>
<tr>
<th>Angle subtended ((^{\circ}))</th>
<th>Strength (mTm(^{-1})A(^{-1}))</th>
<th>Deviation (%)</th>
<th>Primary Deviation (%)</th>
<th>Iterations</th>
<th>CPU time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>177</td>
<td>0.1</td>
<td>98.30</td>
<td>29.85</td>
<td>226</td>
<td>512</td>
</tr>
<tr>
<td>149</td>
<td>0.05</td>
<td>23.81</td>
<td>7.47</td>
<td>529</td>
<td>512</td>
</tr>
</tbody>
</table>

5.3.2 Multiple Arc modelling

Using this protocol, a gradient coil was designed with blocks of variable width, but fixed inner and outer radii. It has been suggested that 120\(^{\circ}\) arcs, as described by Frese-Stetter as the first instance of their invention, produce the optimal field variation by cancelling the relevant harmonics over the central sphere within a gradient coil. However there are various reasons why this may not be the case.

Firstly, the Golay expansions cancel the harmonics and these are expanded spherically. They may not be the most appropriate over the specific field of view chosen. Secondly, as seen in the previous section, the physical robustness of the design may critically affect the results.

The modelling was performed on the SR2201 supercomputer used previously, with similar parameters to that used for the block designs. In particular the internal and external radii of the coils were held to be 32 cm and 44 cm respectively. The length was
40 cm. The external radius is slightly smaller than that for the previous block designs. Previous authors\(^47,121\) have noted that the ratio of internal to external radii should not be less than 1.4 for coils with 120\(^\circ\) arcs, and this design is marginally below this limit. This is due to the physical constraints within the magnet at HSLMC.

The details of the encoding into the genetic algorithm chromosome are as follows. Each current arc has start and finish angles specified by two independent values specified within the chromosome to be between +90\(^\circ\) and -90\(^\circ\). In this configuration, the arc segments were not allowed to overlap so there are various rules for interpreting cases where the chromosome specifies this. If a segment lies entirely within another, then it is deemed to disappear. If it overlaps, then the region of overlap is deemed not to contain any part of either segment and the two segments are shortened to allow for this.

Due to symmetry considerations, blocks which lie across the symmetry axis (which is at angle zero) are made to be symmetric. Other segments which do not cross this line are repeated in accordance with the symmetry. To allow segments to lie dormant, if both angles in the chromosome are negative, then segments are similarly discarded. As described previously, these “hidden” characteristics can aid the progress of the design.

In addition, every wire is encoded in a similar manner to the block gradient coil, with reference only to the position along the z-axis. The results are shown in Table 5.5. All results converged on a single element of differing angle rather than the multiple arc solution, and increasing the desired strength caused the angle subtended by the element to be increased. Since an optimal arc spacing would be 120\(^\circ\) to minimise the deviation, the implication is that the required gradient strengths are greater than can be produced without sacrificing the linearity. The linearities are also poor.

This is disappointing since it does not provide any unusual insight into creating large linear regions away from the central volume. In particular, the facility to create a ring of

<table>
<thead>
<tr>
<th>Strength (mTm(^{-1})A(^{-1}))</th>
<th>Deviation (%)</th>
<th>Primary Deviation (%)</th>
<th>Iterations</th>
<th>CPU time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>13.96</td>
<td>4.88</td>
<td>193</td>
<td>1024</td>
</tr>
<tr>
<td>0.05</td>
<td>13.45</td>
<td>4.80</td>
<td>176</td>
<td>768</td>
</tr>
</tbody>
</table>

Table 5.6: Gradient design summaries for different target gradient strengths for the angular segment designs where the segments are allowed to overlap and have varying radial width.

many segments was not found to be a useful solution, only to use one large arc segment. It is consistent with analytic analyses of this problem though.

The standard Golay coil (which has no outer arcs) has a strength of 0.03 mTm\(^{-1}\)A\(^{-1}\). This is weaker than the results shown here, but much more wire is used in this design. The deviation over the standard region of interest considered here is 79\% for the Golay coil, in comparison with 24\% for the 0.05 mTm\(^{-1}\)A\(^{-1}\) case described here. This coil is also much shorter than the Golay coil.

However we can generalise this formulation to allow these arcs to overlap, and also to have variable radial thickness.\(^74\) This allows the design to vary further from the analytic solution, and these extra degrees of freedom may be exploited by the design.

5.3.3 Overlapping arc sections

The encoding of these blocks in the chromosome is simpler than the non-overlapping case and has only the addition of the internal and external radii as additional parameters. The conditions regarding overlapping blocks can be discarded.

The results of this configuration are summarised in Table 5.6. The design for 0.1 mTm\(^{-1}\)A\(^{-1}\) is also represented in Fig. 5.9.

The most striking aspect of this design is its similarity to the more conventional designs. The majority of the arcs achieve their maximum radial thickness and are similar in angular
width, although there is some "tapering" towards the edges. However there are four smaller blocks which are not familiar to the other design shape. In terms of performance however, the difference is marked. The single arc design struggled to achieve a strength of $0.1 \text{ mTm}^{-1}\text{A}^{-1}$, while this configuration achieved this with a similar deviation to that for the weaker strength design. In part this is due to the multiple winding possible with the three blocks but in practice few of the the current loops around the blocks have an opposite sign to that of the main field producing direction. This implies that the loops are required for achieving the field strength rather than refining the profile - and thus that the wires positioning is a more important factor in making this a superior design.

A simulated image is shown in Fig. 5.10. The parameters of this phantom are the same as that shown in Fig. 5.7. As can be seen there is less distortion at positions closer to the coil, which is reflected in the better deviation calculation for this configuration. However, as is inevitable, there is still a point of inversion at the edges of the coil.

The r.m.s. field at the screening distance was $0.96 \mu\text{T}A^{-1}$, although this was not optimised for.

5.4 Discussion

This chapter has investigated the possibilities of further genetic algorithm exploration of designs for magnetic resonance imaging gradient coils. The classes of geometry presented are novel (although similar formulations and subsets of them have previously been presented), as is the method of approach to them. As discussed previously, genetic algorithms cover search space in a relatively unconventional manner and do not converge to local minima. Conversely they are not optimal for refining a solution which is close to an optimal position already. The repeat studies performed within this chapter illustrate this well, and empirically it was found that similar cost function values were derived for identical starting conditions.

The application of high performance computing to this problem has allowed more de-
The consistency of the convergence of the genetic algorithm approach has also been demonstrated for these large chromosome problems. There are many further refinements that may be addressed with this technique. Frese and Stetter varied the azimuthal position of the inner and outer arcs, and also their angular width along the length of the gradient coil, for example. The principles and methodology within this chapter has been protected for future application.139

Comparisons with other instances of novel gradient coils are notoriously difficult due to the differing measures used by different authors, however for comparable sizes of gradient coils the strength and linearity are similar.43,47,140 However, Bowtell has recently published a gradient coil design which is a subset of this current geometry74 and we can compare this geometry with that described here. In that paper an analytic solution is presented where current paths for different arcs always lie within the same plane. The dimensions of the design produced are shown in Table 5.7. The axial positions of the current elements are not explicitly quoted in the paper but the genetic algorithm can be tested to reproduce a good solution for these. We can consider two formulations: first, the design algorithm as used previously is run for the same dimensions of the coil and the same number of coil elements as in the published work. Second, the code is modified to fix the arc angles to be at their optima as calculated analytically - and quoted in the paper - and current elements on each arc are constrained to lie in the same plane. Effectively this is testing whether there is an advantage for this cost function in allowing the arc angles to be generated rather than fixed to their analytic optima.

As expected, both designs are good. For the fixed arc angles, the optimal deviation was 5.49% (with a linearity of 2.31% in the primary direction). For the arc angles optimised by the genetic algorithm the optimal deviation was 4.64% (with a primary linearity of 1.49%). This is as we would hope, since there are more degrees of freedom in the latter case. However neither formulation optimises for inductance which is used in the Bowtell analysis. The simulated phantom image was also produced for the Bowtell formulation and is shown in Fig. 5.12. This demonstrates the good linearity in the central region. The grid phantom is smaller in this case relative to the size of the coil and the optimised region is also smaller as a fraction of the total volume than for the arc segment coils. Slightly
Number of arc segments: 20
5 per quarter
5 per half if segments cross symmetry axis
Max. No. windings per arc segment: 12
6 x 2 for symmetry
Internal diameter: 38.8 cm
External diameter: 67.4 cm
Length: 38.0 cm

Cylindrical Field of View:
FOV diameter: 18 cm
FOV length: 18 cm

Table 5.7: Bowtell\textsuperscript{144} coaxial gradient coil dimensions

poorer peripheral linearity is expected due to this smaller central region and this is also visible in comparing the two images.

The improvement on complicating the design is minimal in this case. However, this is a comparatively small field of view in comparison with that which was considered earlier, and that the genetic algorithm improves on a partially analytic solution is a positive result.

The classes of gradient coils described here have characteristically large volumes of monotonicity compared with their size. This makes them most useful for applications where the sample (or subject) is large relative to the hardware and selective excitation is not easily achieved. They are also very short gradient coils, having a length of only 40 cm relative to an external field of 44 cm (internal 32 cm). In many new designs of magnet systems length is an important consideration.

The buildability considerations have also demonstrated their great importance, as the optimal designs were far different from the fixed current blocks, simply due to the more simplistic consideration that was used in those supercomputer runs. The buildability factor was not used in the variation of the shape of the blocks - a further study should include this as a priority. However the variation in the positioning in the z direction may be sufficient to allow for uncertainty in the exact transverse position of the wires, and this should be tested. Unfortunately, since the angular width of the blocks is greater than 90°, a different transverse design would be needed for the x and y gradients. A second design could however be generated with this program with the modification of including the wire positions of the other coil as constraints. The equivalent problem on a single cylinder is addressed by Hughes et al, for example.\textsuperscript{141} The “block” designs would be more difficult to produce a second design using a similar method. An alternative method would be to simultaneously design both transverse gradient using blocks which were common to both designs.

A major factor not considered in this analysis is the inductance of the coils. Unfortunately, while possible, its calculation is not trivial for these complex shapes. However Bowtell\textsuperscript{144} has presented a method for calculating the inductance of the arc segments using...
reciprocal space. The important factor is the rise time of the coils, and while the inductance is significant for this it is not the sole determining factor. The field characteristics at the cryostat are also significant due to induced currents in the bore, and these are considered.

Also, the length of the wire used has not been minimised. By including a multiplicative length factor into the cost function, a coil was redesigned with the same dimensions. The linearity was worsened by 5.74% (summed over the three directions) while the number of current elements reduced from 90 to 66. This implies that linearity is slightly improved at the expense of a lot more wire.

In conclusion, the computational examples produced within this chapter are an improvement on conventional designs and while they are not radically different the methodology employed is valid and applicable to more ambitious applications. However, the fundamental constraints imposed by basic electromagnetics are very serious ones and improvements are much more easily made for niche applications (such as increasing the monotonicity or decreasing the length) than for other considerations such as gradient strength which is more easily optimised.

Chapter 6

Conclusions

The topic of magnetic field modelling for MRI is an active one, in particular since the cost of magnetic resonance scanners has fallen over the past decade and the market is now able to support scanners which are optimised for specific applications. The development of taxing fast pulse sequences has required improvements in hardware, such as magnets and gradient coils. With the greater acceptance of MRI scanners, patient ease and comfort is also an ever more important consideration. All these factors require novel gradient coils to be constructed. Gradient design is a diverse area of research, with many competing methods and approaches. The work presented in this thesis represents a continuation of the program of research into the genetic algorithm optimisation approach. It broadens its applicability to more general geometries than have previously been studied, or given the increase in computer facilities available, even been amenable to this optimisation method.

The general requirements of gradient coils have been comprehensively described and the competing methods of approach to optimise the coils discussed. The restraints imposed by the underlying physics theory have been thoroughly considered, and ways to avoid the problems detailed.

The major advantages of this general method of using genetic algorithms include that
the accuracy of construction needed can be included simply into the algorithm. This was demonstrated clearly in Chapter 5 by the fact that an apparently superior design deteriorated when more realistic criteria were applied for the engineering precision. Another feature of the genetic algorithm method of sampling search space is that fine adjustments tend to be ignored or found very slowly. This is because unlike most optimisation methods, genetic algorithms do not focus on adjacent regions of search space. Hence “nearby” solutions are often not considered. However this is not necessarily a disadvantage for situations where engineering precision means that highly specific designs will not necessarily be replicated in manufacture.

In Chapter 4 a biplanar design was considered as an example of what this technique can achieve. The genetic algorithm was successfully tested for reliability of convergence, and a final design was constructed and tested both on phantoms and a human knee. By consideration of the inevitable non-linearities in the gradient coils, regions larger than the volume optimised to be linear can be used to acquire images within certain restrictions. While there are problems associated with this (such as flow), and it is certainly preferable to image from a linear region, commercial MRI systems do employ such reconstruction techniques routinely (for example, GradWarp on GE Signa scanners). As such, gradient coils can be designed to make use of such an algorithm, and the choice of geometry so as to optimise the volume within the coils that can be used for imaging is important. By consideration of the effect of non-linearities of the coils, correction algorithms were applied to images produced using this set from the non-optimised regions with some very good results. Analysis of the regions from which it was impossible to recover useful data has allowed a consideration of how to image beyond the nominal limits of the coils.

In order to make these more difficult problems accessible, a supercomputer was employed. This required parallelisation of the genetic algorithm, which has not previously been applied to gradient design. An analysis of the effectiveness of this new method was performed and found to be sufficient to enable a speed-up in real time that would allow the employment of the routine to complex design problems. After testing of the code and the use of several model systems, two classes of gradient coil were designed which had more degrees of freedom than previous designs. The algorithm was found to be consistent when tested by repeat studies. These were found computationally to be comparable with the existing optimal designs in the literature, establishing this as a viable technique. In both case, transverse geometries only were considered. This is due to the nature of the problem - longitudinal gradient coils are usually axially symmetric, and unless a region of interest away from the centre of the magnet bore is required there is no reason to make them otherwise. However this simplification of the problem makes them less interesting from a computational point of view. The broken symmetry in the transverse case, coupled with the attempt to create a coil of approximately cylindrical dimensions which produces a linear field, makes this a more complex problem.

Both by computer simulation and construction and testing of gradient coils, this thesis has demonstrated the power of the genetic algorithm method, beyond that previously established, and its potential much wider use for more complex design problems in conjunction with supercomputers.

6.1 Future Work

As mentioned above, modern magnetic resonance scanners are being designed with criteria such as patient access and comfort as an ever increasing priority.\textsuperscript{142-144} The development described through this thesis of a design approach to producing gradient coils which depart from the restrictions imposed by conventional hardware and design approached is thus timely.

There are many aspects that have not been factored into the designs. The inductance,
as discussed in the previous chapter, is an area which should be considered for a future study.

The inductance could be calculated as follows. In place of the magnetic field being calculated using the Biot-Savart law (see equation 3.1), the vector potential can instead be evaluated using:

\[ dA = \frac{\mu J \, dr}{4\pi r} \]

where \( J \, dr = J \, dl \) where the thin wire approximation can be used. \( r \) is the distance between the volume element being integrated and the position at which \( A \) is being calculated.

This equation is derived from the differential equation

\[ \nabla^2 A = -\mu J. \]

Now we can use the following equation for the inductance\(^{145}\)

\[ L = \frac{1}{\mu} \int A \cdot J \, dr, \]

and integrate around the path of the wires. Thus the relevant positions at which \( A \) must be calculated are those within the wire itself, since these are the co-ordinates where \( J \) is non-zero. This means that the thin-wire approximation cannot be used.

This is computationally expensive, and other techniques which use reciprocal space have easier access to the value.\(^{82}\) The inductance is highly dependent on the close positioning of the wires and it may require too much sampling within the wire for the vector field in order to produce a useful value. Calculation of contributions to \( A \) will require an explicit consideration of the current density and relative permeability of the wire. Equation 6.1 breaks down for \( r \) approaching zero as the thin wire approximation is no longer usable. Equation 6.2 must be solved directly for contributions arising from within the wire itself for small \( r \).

The nature of the cost function and its applicability to genetic algorithms is also not something that has been analyzed in detail in this study.

The genesis of the work of section 5.2 was the ease of construction of winding around solid cuboidal blocks. If the gradient coils discussed in this chapter were constructed, further issues concerning the engineering of the solution may come to light. Little work to date has been done in considering these problems from an engineering perspective and such an approach could reduce costs.

The gradient coils described are not the limit of this method's potential. It has been demonstrated that powerful computers are capable of implementing even very expensive design algorithms. Indeed, a project to design and build an integrated short axis magnet and gradient coil using these techniques is currently underway within this University.\(^{146}\)

Such a system may require image processing in order to be useful, using algorithms similar to that described in section 4.3.5, although there will be the initial complication that correction for \( B_0 \) inhomogeneity is sequence dependant.
Appendix A

An analytic method for balancing the current levels

This algorithm is used within Chapter 5 for calculating the optimum current ratios of blocks to provide the most homogeneous field.

The current level calculation can be cast as a matrix minimisation. If the rows of matrix $A$ contain represent column vectors containing the $B_z$ field perturbations caused by each independent current level, and column vector $X$ contains the current levels, then the magnetic field produced can be written as below:

$$AX = B$$ (A.1)

If we define our required gradient to be $m$ and the column vector containing the distance in the gradient direction from the centre of the imaging volume to be $Q$, then we can also define the required B-field, $B_0$ to be:

$$B_0 = mQ$$ (A.2)
Now our requirements for the optimal current levels are that it minimise the rms deviation from $B_0$, which is to minimise the expression

$$f = |AX - mQ|^2 \quad (A.3)$$

In addition to this the average gradient is required to be $m$. In order to achieve this we need to consider the variation of equation $A.3$ with respect to $m$. Recasting it out of matrix notation the expression becomes

$$f = \sum_i (B_i - mQ_i)^2. \quad (A.4)$$

Differentiating with respect to $m$ we derive

$$\sum_i 2Q_i (B_i - mQ_i) = 0$$

or in matrix form:

$$m\sum_i Q_i Q_i = \sum_i Q_i B_i$$

Since we know $B = AX$ this can be rewritten as

$$mQ^T Q - Q^T AX = 0.$$ 

or, as we will see, more usefully as

$$g = mQ^T Q - X^T A^T Q = 0. \quad (A.5)$$

This equation represents a method of calculating the gradient $m$ for a given current configuration $X$. Conversely for a fixed $m$, as in our case, this represents a constraint on the possible configurations of $X$. Our problem has become one of minimising equation $A.3$ subject to equation $A.5$. Lagrangian multipliers can be applied to this problem. If we expand our expression we obtain

$$f - \lambda g = X^T A^T AX - 2mX^T A^T Q + m^2 Q^T Q - \lambda \left( mQ^T Q - Q^T AX \right).$$

Differentiating with respect to the elements of $X$ results in the following matrix equation:

$$A^T AX - 2m A^T Q + \lambda A^T Q = 0. \quad (A.6)$$

This can be rearranged to give

$$X = (m \mp \lambda) \left( A^T A \right)^{-1} A^T Q. \quad (A.7)$$

By premultiplying this by $Q^T A$ and substituting using equation $A.5$ we obtain

$$- (m \mp \lambda) \left[ (Q^T A) \left( A^T A \right)^{-1} (A^T Q) \right] = mQ^T Q. \quad (A.8)$$

Combining these two equations to eliminate $(m \mp \lambda)$ solves for $X$. At this stage we will introduce new variables in order to simplify the algebra. We define

$$Y = A^T Q,$$

$$\alpha = Q^T Q,$$

$$Z = \left( A^T A \right)^{-1} Y.$$ 

This reduces the solution to be

$$X_0 = \frac{m\alpha}{Y^T Z} Z \quad (A.9)$$
and simplifies Equation A.5 to

$$Y^T X = ma.$$  \hspace{1cm} (A.10)

Now this is the optimum solution but to be consistent with our previous method of representing the currents within the blocks represented in Chapter 5 we need the current levels constrained to be less than unity; to enable greater fractional currents would render the solution incomparable with the existing method. To consider how we can find the optimal solution within this further constraint we need to look at what the problem represents graphically.

Equation A.3 can be re-written

$$f = (X - e)^T A^T A (X - e) + c$$

where

$$e = m (A^T A)^{-1} A^T Q = mZ$$

and

$$c = -e^T (A^T A) e - m^2 Q^T Q.$$

Since $A^T A$ is a real symmetric matrix, and is also positive definite, it follows that the contours of $f$ will lie on a hyper-ellipsoid in “$X$-space” whose centre is at $e$ and whose principal axes will be the eigenvectors of $A^T A$. We know it must be positive definite due to the nature of the problem: it is not possible that increasing the current levels indefinitely along certain eigenvectors will continually decrease the deviation as would be the case otherwise. The global minimum of $f$ will be at $X = e$ and will have value $c$.

By contrast the condition imposed by equation A.10 represents a hyperplane. It is within this plane that our minimum $X_0$ is located. Our new condition represents a “hypercube” centred around the origin. This is all represented in 2-dimensions on Fig A.1.

Our first problem is to determine whether the optimum current level lies within the hypercube. In this case the gradient efficiency criteria specified are not stringent enough.
Now if $\beta$ is less than one the point is within the hypercube, and if greater then it lies outside of it. If it is greater then the problem cannot be solved for that gradient strength and the co-ordinates of the vertex should be taken as optimal.

In the case that we have a point within the hypercube and a point outside of it we can easily find a point on its surface. We now use a perturbations from this start point in order to find the minimal value on the surface. The hyperellipsoidal contours of $f$ are useful to us in that within such a field there is only one minimum - and also along any direction of travel there will only be one such minimum. So if we consider a perturbation which does not leave either the hyperplane or the surface of the hypercube and is as close as possible to the direction of steepest descent $(-\nabla f)$ we will further optimise our current values.

Mathematically this can be represented as follows. If we let

$$G_0 = -\nabla f = 2mATQ - 2ATAX$$  \hspace{1cm} (A.11)

then the allowed direction of travel can be represented by

$$G = G_0 - \sum_i \lambda_i N_i - \mu Y$$  \hspace{1cm} (A.12)

where $N_i$ are the normals to the planes of the hypercube through which $G_0$ would take our solution if unrestrained, $Y$ is the normal to the hyperplane and $\mu$ and $\lambda_i$ are unknown.

To find the variables we use the conditions

$$N_i^T G = 0$$

$$Y^T G = 0.$$

Substitution results in the following solutions for $\mu$ and $\lambda_i$:

$$\lambda_i = N_i^T G_0 - \mu N_i^T Y$$  \hspace{1cm} (A.13)

$$\mu \left[ Y^T Y - \sum_i (N_i^T Y)^2 \right] = Y^T G_0 - \sum_i (N_i^T Y) (N_i^T G_0)$$  \hspace{1cm} (A.14)

Equations A.12-A.14 allow our direction of travel to be found, and our solution point moved along this direction until it either reaches the minimum value or hits the edge of the hypercube. The minimum point can be found by differentiating

$$f = (X + \gamma G)^T A^T A (X + \gamma G) - 2m(X + \gamma G)^T + m^2Q^T Q$$

with respect to $\gamma$. This results in an expression for $\gamma$:

$$\gamma = \frac{mG^T A^T Q - G^T A^T AX}{G^T A^T AG}.$$  \hspace{1cm} (A.15)

Since $-\nabla f$ changes over the hypercube this should repeated until it is not possible to travel in the direction of $-\nabla f$ at that point.

While this procedure may seem computationally very expensive, once the large matrices $A$ and $Q$ have been reduced to $A^T A$ and $A^T Q$, the order of all matrices is only that of the number of independent current levels in the gradient coil, and so the calculation is relatively fast. Similarly since $f$ only contains one minimum, the number of iterations required to find the minimum on the surface of the hypercube is limited.
Appendix B

Design Details for the Gradient Coils

This appendix details the exact designs of the gradient coils described in this thesis. For the biplanar coils, the design was milled into 400 mm square plates. The co-ordinates given are from the centre of the plate, which is at (0,0). "Clockwise" and "Anti-clockwise" refer to the sense in which the wires were wound, i.e. all but one of the rectangles is in the same sense.

All distances are in millimetres unless otherwise stated.

The block gradient coils describe the 4 distinct designs of blocks illustrated in Fig. 5.6. Block #0 is furthest from the line of anti-symmetry. Only half of the wire paths are shown, as they are repeated through symmetry in the z-axis.

The arc segment designs are similarly described, although some of the windings are counter-wound, indicated by a minus sign in brackets after the position. Fig. 5.9 represents this information graphically.
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