Shearing Waves and the MRI Dynamo in Stratified Accretion Discs

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To Andrew Holt,
who taught me maths.
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Preface

This Thesis is equal parts numerical and analytical investigations. We must perform numerical experiments - simulations - if we wish to delve below the surface of the Sun or observe the formation of a planet in its entirety. These numerical results must then be interpreted and condensed to the point where a human can understand and predict systems qualitatively without the need for another six months of computing time - and so appears the need for analytical predictions, to verify our simulations and make clear the underlying physical mechanisms at play.

I have tried to write this Thesis to be as clear as possible in its language, with reference to the comments on lucidity made by McIntyre[73]. Further, on the theme of efficient exchange of information, I am pleased to note that this thesis was edited in ViM and Inkscape, and makes use of RAMSES, all of which have GPL-compatible licenses, as well as many smaller pieces of code that I wrote myself. For plotting I used gnuplot, which has a more restrictive free software license. I regret that I could find no usable free software to fill the roles of Mathematica and the Intel compiler for FORTRAN90.

This Thesis was prepared using the L\TeX template made available by Harish Bhandari.

This Thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration.
Shearing Waves and the MRI Dynamo in Stratified Accretion Discs

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Accretion discs efficiently transport angular momentum by a wide variety of as yet imperfectly understood mechanisms, with profound implications for the disc lifetime and planet formation. We discuss two different methods of angular momentum transport: first, generation of acoustic waves by mixing of inertial waves, and second, the generation of a self-sustaining magnetic field via the magnetorotational instability (MRI) which would be a source of dissipative turbulence. Previous local simulations of the MRI have shown that the dynamo changes character on addition of vertical stratification.

We investigate numerically 3D hydrodynamic shearing waves with a conserved Hermitian form in an isothermal disc with vertical gravity, and describe the associated symplectic structure. We continue with a numerical investigation into the linear evolution of the MRI and the undular magnetic buoyancy instability in isolated flux regions and characterise the resultant quasi-linear EMFs as a function of height above the midplane. We combine this with an analytic description of the linear modes under an assumption of a poloidal-toroidal scale separation. Finally, we use RAMSES to perform full MHD simulations in a zero net flux shearing box, followed by spatial and a novel temporal averaging to reveal the essential structure of the dynamo.

We find that inertial modes may be efficiently converted into acoustic modes for “bending waves”, despite a fundamental ambiguity in the inertial mode structure. With our linear MRI and the undular magnetic buoyancy modes we find the localisation of the instability
high in the atmosphere becomes determined by magnetic buoyancy rather than field strength for small enough azimuthal wavenumber, and that the critical Alfvén speed below which the dynamo can operate increases with increasing distance from the midplane. We calculate analytically quasi-linear EMFs which predict both a vertical propagation of toroidal field and a method for creation of radial field. From our fully nonlinear calculations we find an electromotive force in phase with the toroidal field, which is itself $3\pi/2$ out of phase with the radial (sheared) field at the midplane, and good agreement with our quasi-linear analytics.

We have identified an efficient mechanism for generating acoustic waves in a disc. In our investigation of the accretion disc dynamo, we have reproduced analytically the EMFs calculated in our simulations, given arguments based on the phase of relevant quantities, several correlation integrals and the scalings suggested by our analytic work. Our analysis contributes significantly to an explanation for the dynamo in an accretion disc.
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Chapter 1

Introduction

...And I think that confidently and concisely answers the question, “What does the red spectrum tell us about quasars?”

Arnold Rimmer, BSc., SSc.

The Universe is governed at its largest scales by gravity. Imagine a large cloud of slowly rotating gas. Under gravity the gas begins to fall into the centre of the cloud; at the same time the “ballerina effect”, the conservation of angular momentum, will force the gas to rotate faster, resisting the collapse. For the collapse to continue some angular momentum must be carried away, and so some of the gas will be spun out into a disc - an accretion disc - which then slowly deposits mass onto the newly formed central object. An accretion disc is a common structure, coming in sizes from the discs within close binary systems with an orbital period of days, through protoplanetary discs such as birthed our Solar system, up to the active galactic nuclei which are the brightest objects in the Universe.

If we wish to know how many habitable planets have formed in the Universe, if we wish to understand how stars begin, if we wish to probe active galactic nuclei - then we must understand accretion discs.

Accretion discs must transport their angular momentum outwards in order to deposit mass on the central object, and so throughout their lifetime they spread radially even as they transport mass inwards. The central question in the study
of accretion discs is how rapidly angular momentum is transported outwards and by what mechanism.

1.1 Observations

At the time of writing the Open Exoplanet Catalogue lists just over 900 confirmed extrasolar planets\footnote{The Open Exoplanet Catalogue can easily be accessed using Hanno Rein’s excellent free app for iPhone, iPad and iPod.} with over 3000 unconfirmed planetary candidates discovered by the Kepler mission. People are beginning to probe the jet structure around T Tauri-type young stellar objects using the VLA\footnote{The Open Exoplanet Catalogue can easily be accessed using Hanno Rein’s excellent free app for iPhone, iPad and iPod.}. Using ALMA it has even become possible to image the dust distribution (and hence possibly large vortices) within single protoplanetary discs\footnote{The Open Exoplanet Catalogue can easily be accessed using Hanno Rein’s excellent free app for iPhone, iPad and iPod.}. It is an exciting time to be alive.

The first observations of accretion discs were published in 1959 in the Third Cambridge Catalogue of Radio Sources (3C, \cite{24}), a survey which included the particularly bright object 3C 273. This was the first identified quasar, a type of object now understood to be an active galactic nucleus: an extremely large and hot accretion disc around a galactic central black hole. Quasars have been observed to eject matter at such speeds that - when the mass ejection is oriented towards Sol - the mass appears to be moving far faster than light. Although the theoretical results in this Thesis could well be applied to such discs around black holes we choose to give examples of lengthscales etc. in this Introduction as they relate to protoplanetary discs around T Tauri or Ae/Be (pre-main sequence) stars. A protoplanetary disc had long been theorised to be the progenitor of the Solar system on the basis of the close orbital alignment of the planets, but it is only in the last two decades that we have managed to observe distant protoplanetary discs directly. These protoplanetary discs are made up of largely hydrogen with some heavier elements: Hayashi\footnote{The Open Exoplanet Catalogue can easily be accessed using Hanno Rein’s excellent free app for iPhone, iPad and iPod.} considered the minimum mass Solar nebula required to give rise to our own planetary system and estimated that the solid fraction is around 1% by mass for protoplanetary discs. The presence of this “dust” has important observational and dynamical consequences.

Observations of protoplanetary discs are taken in three main regions of the electromagnetic spectrum:
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- **Microwave (sub-mm):** The Sub-millimetre Array (SMA) in Hawaii has a wavelength range of 0.4 - 3.0 mm, and observes cool dust (50K to 200K) between 1-10 AU radii, heated by radiation from the central star. At such low temperatures the emitted light is in the low energy sub-millimetre range, and at such large distances the gas is optically thin, allowing observations to give estimates of the total disc mass (Beckwith et al. [13] using telescopes MPIfR and IRAM, or more recently Andrews and Williams [2] using the SMA: they extrapolate to an average temperature of around 20K at 100AU).

- **Infrared:** the Spitzer Space Telescope (SST) was launched in 2003 and has a wavelength range of 3 – 160µm (infrared), observing the warm dust again heated by its proximity to the central star as well as Hα emission by hot hydrogen in the final stages of accretion; the Herschel Space observatory covered a similar range of wavelengths until it was shut down in April of this year. We refer to the five part analysis of the SST Infrared Survey by Kessler-Silacci et al. [62], which detected silicate dust of radius 1 µm (Part I) and an average temperature range in T Tauri discs of 100-200K in the approximate range of 1 - 10 AU (Part III), and observed efficient mixing of dust grains in both the radial and vertical directions (Part IV), including larger grains that are thought only to form close to the host stars at large radii. The SST has also contributed to the detection of extrasolar planets via transit photometry (whereby light from a star is periodically occluded by an orbiting planet).

- **Visible:** the Hubble Space Telescope (HST) was launched in 1990 and will remain operational until at least 2014, taking observations in the 200-2500 nm range (visible, near-infrared and near-ultraviolet). In 1994 its images of the Orion nebula confirmed the existence of protoplanetary discs (O’Dell and Wen [79]). We show such an observation in Figure 1.1.

Active galactic nuclei emit at much higher energies; for observations of these the Chandra X-ray Observatory (CXO) was launched in 1999, taking observations in the 0.08-10 keV (0.124-15.5 nm) range (X-Ray), and we show such an observation in Figure 1.2. A review of protoplanetary disc observations is available in Alexander [1].
Figure 1.1: The protostellar disc HH-30 (diameter: 450AU) as observed by Hubble (visible light): the two curved bright regions are lit by light from the central object which has reflected from dust at the disc’s surface. The dark central band is the midplane dust blocking the light from the central star. The jet strength is varying in time. Image credit: XZ Tauri: NASA, Krist, Stapelfeldt, Hester, Burrows. HH 30: NASA, Watson, Stapelfeldt, Krist and Burrows.
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Figure 1.2: The quasar PKS 1127-145 as observed by Hubble (left, visible) and Chandra (right, X-ray). The jet (the long purple feature visible only in X-Ray) is around one million light years long, and was launched from an accretion disc around the central black hole of the quasar. Image credit: NASA/CXC/A.Siemiginowska/J.Bechtold.

1.2 Accretion disc structure

In this Introduction we place our work in the context of on protoplanetary discs, but any such discussion must include the work of Shakura and Sunyaev on black hole accretion discs. We describe this work in §1.3. Discs around black holes (which include as a subcategory quasars such as that shown in Figure 1.2) are significantly hotter and brighter than protoplanetary discs, with surface temperatures ranging between $10^4 - 10^5$ K and luminosities close to the Eddington limit (i.e. inward gravitational acceleration is resisted by outward radiation pressure). A protoplanetary disc exists for between 3 and 20 Myr and has a typical radius of 300-500AU \cite{48, 87}; we shall see in §1.5.1 that the greater the distance from the central object the weaker the vertical compression by the gravitational potential, leading to the distinctive flared shape shown in Figure 1.3. At the inner edge the flow of gas must be compatible with the central star, either matching speed to its rotation or flowing along its magnetic field to the star’s poles. The eventual fate of the disc is determined by hot photoevaporative outflows as described by Hollenbach et al. \cite{54}; there exists some distance $R_g$ from the central
star at which the local gravitational potential energy from the mass of the system equals the local thermal energy in the gas. Outside of this radius the required gravitational escape energy is lower and the gas can escape the gravity well, so leaving the accretion disc. We have labelled this as a “photoevaporative flow” in Figure 1.3. Photoevaporation is negligible until all but a fraction of the mass has been accreted, 99% of the way through the disc’s lifetime, at which time it quickly opens a gap in the disc. The inner disc is then accreted as before on a ‘viscous’ timescale (see §3.3.2 for what is meant here by ‘viscous’) of around 0.1 Myr, then allowing radiation to heat the outer disc directly. The outer disc then quickly evaporates (again in around 0.1 Myr - see e.g. the review by Hollenbach et al.[55]). Dust within the disc atmosphere settles towards the midplane[20].

This short disc lifetime immediately leads to the central question of accretion disc research: how is the disc transporting so much mass inwards (and thus angular momentum outwards) so quickly? We refer to §3.3.2 of Armitage’s excellent book *Astrophysics of Planet Formation* to estimate the fluid Reynolds number of
molecular gas at 10AU as

\[ R_e = \frac{c_s H}{\nu_{\text{mol}}} \sim 10^{10}. \]

where \( c_s \) is the sound speed in the gas and \( H = H(R) \) is the disc height (later to be replaced with the closely related ‘scale height’). This gives a molecular viscous accretion timescale, the time over which angular momentum could be transported outwards by only a molecular viscosity, as \( t_{\text{mol}} \sim 10^{13} \) years. This is much, much longer than the protoplanetary disc lifetime: the observed angular momentum transport cannot be due to fluid viscous stresses.

Analysis of data from the NASA IR Telescope Facility found the innermost region of protoplanetary discs is heated to around 1000K by radiation from the central star, dropping off rapidly to several hundred Kelvin by 10AU\(^7\). Models also suggest a hot corona of around 1000K above the disc surface (e.g. Kamp and Dullemond, 2004\(^5\)). Such a high temperature will ionise some fraction of the gas: this ionisation allows for the possibility of magnetic fields and so dynamo action and MHD turbulence. The lower the ionisation, the larger the Ohmic resistivity \( \eta \) in the gas which will tend to dissipate magnetic energy and inhibit magnetic instabilities. We consult Fromang, Terquem and Balbus\(^3\) calculate the electron fraction \( x_e = n_e/n_n \) in an \( \alpha \) model of a disc (where \( \alpha \) models will be described in \( \S1.3 \)) as a function of the abundance fraction of potassium over molecular hydrogen. They calculate

\[
x^3_e = \frac{T^{1/2}}{1000} x_M x_e^2 - \frac{\zeta}{n_n} x_e + 100 T^{1/2} \frac{\zeta}{n_n} x_M = 0
\]

where \( \zeta \) is a parameter describing the penetration of X-rays into the disc. At 0.3AU and for a disc with \( \alpha = 0.01 \), \( T = 1000 \)K, and \( n_n = 3 \times 10^{15} \) we find

\[ x_e = 3.16228 \times 10^{-10} \]

which translates to an Ohmic diffusion of

\[ \eta \approx 2.34 \times 10^{13} \text{cm}^2 \text{s}^{-1}. \]
SHEARING WAVES AND THE ACCRETION DISC DYNAMO

The sound speed in molecular hydrogen will be around \(2.3\text{km s}^{-1}\) at that temperature and we assume \(H = 0.05R\), giving a magnetic Reynolds number of

\[
R_m = \frac{c_s H}{\eta} \sim 2300
\]

which is well above the critical \(R_m \approx 100\) required for the important magnetorotational instability (MRI). With the earlier estimate of the viscosity (appropriate for \(R = 10\text{AU}\), much further out) we can (under-)estimate the magnetic Prandtl number \(P_m\)

\[
P_m = \frac{\nu}{\eta} \sim 10^{-3}
\]

which is some seventeen orders of magnitude smaller than that for the interstellar medium \((P_m \approx 10^{14}[85])\) and is around that calculated for at the Earth’s core \((P_m \approx 10^{-3}\), larger than that at the base of the Solar convection zone which has \(P_m \approx 10^{-5}[81, 85]\)). In §1.8 we will discuss the effect of this magnetic Reynolds number on the MRI which we discuss in detail in this Thesis and will introduce properly in §1.8. For now, it only matters that the MRI is supposed to be a source of turbulence which transports angular momentum well.

At some radius the disc thickness increases enough to insulate its midplane from the ionising radiation from the central star. This region will have a magnetic Reynolds number lower than that above and low ionisation compared to the inner regions of the disc or the local disc surface (Gammie[39]). The precise radius at which this occurs is the subject of continuing research (e.g. Armitage et al.[8], Mohanty et al.[75]), but this low-ionisation region will be free from MRI-induced turbulence and so be relatively quiescent (hence the label in Figure 1.3 of “dead”). The accretion flow will proceed through the turbulent ionised (“active”) surface layers. The dead zone can collect mass with time and so might act as a reservoir for dust to coagulate and form planets, but would significantly complicate the problems we intend to consider. Throughout this Thesis we will assume that the gas is sufficiently ionised for the magnetic field to be well coupled to the gas (except in Chapter 2, where we consider a purely hydrodynamic problem) and neglect the influence of dust except insofar as it determines the bulk fluid
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parameters.

The observational images shown in Figures 1.1 and 1.2 both show a jet launched from the central region of the disc, perpendicular to the plane of the disc, with both jets of size comparable to or larger than the disc which launched it. Research has followed the line of Blandford and Payne [14], who showed that an open magnetic field anchored to the surface of the disc could successfully launch cold gas, provided said magnetic field is inclined sufficiently close to the horizontal plane of the disc. These open field lines become more toroidal with increasing height above the disc, making a spiral of magnetic field which can collimate the cold gas into a jet, similar to how plasma can be confined by magnetic field lines in tokamak fusion reactors (see e.g. the ITER project). This cold gas needs only have sufficient thermal energy to pass through the slow magnetocoustic point before centrifugal acceleration along the bent magnetic field lines takes over. These jets can remove both mass and angular momentum from the central regions of a disc without the need for any turbulent process and without the need for the gravitational potential to be overcome by thermal energy and so are attractive candidates for explaining the anomalous angular momentum loss from accretion discs: recent work has had its focus on either the cold launching of matter along vertical magnetic fields [95,32] or the collimation of successfully launched matter into coherent jets [70,61].

With such an inhomogeneous disc structure we must discuss the possibility of thermal convection. The measure of diffusion of momentum against the diffusion of thermal energy is due to Prandtl: let

\[ P_{\text{therm}} = \frac{\nu}{\kappa} \]

where \( \kappa \) here is the thermal diffusivity (but at all other points in the Thesis will refer to the epicyclic frequency). For a Prandtl number \( P_{\text{therm}} \) sufficiently large (i.e. strong thermal diffusion) we can make an isothermal approximation and neglect variations in temperature, although recent simulations have found that assuming a small thermal Prandtl number can increase the magnetic field strength produced by the turbulent flow by around 30% (Gressel [15], see also Bodo et al. [15]). The physical interpretation of the possible increase in magnetic
energy available to drive turbulence is further complicated by the linear work of Ryu and Goodman\cite{Ryu1988} who showed that convectively unstable modes transport angular momentum inwards, and who suggested that convection may mix angular momentum per unit mass $R^2\Omega$ rather than the angular velocity $\Omega$. We here avoid consideration of the thermal Prandtl number by arguing that for the disc location to which we have already properly restricted ourselves (the ionised region interior in radius to the dead zone) the temperature will be largely set by a balance between irradiation by the central star and optically thin cooling rather than diffusion. In this Thesis we shall consider only an isothermal gas.

The accretion disc was itself formed from a cloud of gas collapsing under its own gravity. We must consider that portions of the disc may be dense and cool enough for a second collapse to occur. Toomre\cite{Toomre1977} showed that infalling regions of radius $L_C$ and greater will be spun out (and hence stabilised) by the Coriolis force, where

$$L_C = \frac{G\Sigma}{\Omega^2}$$

whilst velocity dispersions or thermal energy could stabilise infalling regions of size $L_J$ and smaller (in the manner of the Jeans instability), where

$$L_J = \frac{\langle u^2 \rangle}{G\Sigma}.$$ 

If there is some range of wavelengths between these two the disc is expected to fragment under the gravitational instability. This is succinctly expressed as the Toomre criterion: if

$$Q = \left( \frac{L_J}{L_C} \right)^{1/2} = \frac{\langle u^2 \rangle^{1/2} \Omega}{G\Sigma} \lesssim 1$$

then the disc will quickly coagulate into isolated dense regions. The practical implementation of the disc’s changing gravitational field is difficult in both an analytic and numeric sense, in that every fluid element is in communication with every other fluid element and gravitoturbulence (where $Q \lesssim 1$ and yet the fluid cannot cool quickly enough to collapse) may result. We will take the common
approximation due to Cowling\cite{19} and neglect self-gravity of the disc in its entirety in this Thesis, even though in Chapter 2 we will consider waves relevant to planets that might have their origin in the gravitational instability. We will discuss planet formation very briefly in \S 1.4.

In truth, we have only touched upon the wealth of dynamic processes that are thought to occur in an accretion disc. In this Thesis we will examine just two of these possibilities in any detail: first the generation of fast acoustic waves in Chapter 2 and thereafter the evolution of the MRI introduced below.

### 1.3 2D disc models

The description of an accretion disc in \S 1.2 above is extremely complicated. Since the main question is one of the transport of angular momentum and mass, it is natural to write down equations which include only that as a first step. Lynden-Bell and Pringle\cite{71} considered an integral over the azimuth and the vertical extent of a thin disc to write down the evolution of surface density $\Sigma$ under a net radial flow $v_R$ to obtain

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R) = 0$$

and similarly for the evolution of the angular momentum density

$$\frac{\partial}{\partial t} (\Sigma R^2 \Omega) + \frac{1}{R} \frac{\partial}{\partial R} (R (\Sigma R^2 \Omega) v_R) = \frac{1}{2 \pi R} \frac{\partial}{\partial R} (RG) \quad (1.1)$$

where $G$ is the (again, vertically and azimuthally integrated) torque by fluid at $R - \delta R$ on fluid at $R + \delta R$. We may rearrange for $v_R$,

$$v_R = \frac{1}{R \Sigma} \frac{\partial}{\partial R} \left( \frac{1}{2 \pi R} \frac{\partial G}{\partial R} \right)$$

The torque $G$ is not necessarily viscous, but is usually written in terms of the local rate of shear $Rd\Omega/dR$

$$G = 2 \pi R^2 \Sigma \nu R \frac{d\Omega}{dR}$$
with some as yet unknown $\nu$. As might be expected from the choice of $\nu$ as notation we will shortly see that these equations are diffusive, and we may make this nature explicit by eliminating $v_R$ from the evolution equation for $\Sigma$:

$$ \frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{1}{\sigma} \frac{\partial}{\partial R} \left( R^2 \frac{d \Omega}{dR} \Sigma \nu \right) \right) = 0 $$

which is a diffusion of the surface density $\Sigma$ with some $\nu$, where $\nu$ can be some function of $\Sigma$. If $\nu \propto R^a \Sigma^b$ for some constants $a, b$ then a self-similar solution for $\Sigma$ can be found. The above thin disc equations are easily solved given some assumption about $\nu$ and can be augmented with mass or angular momentum source terms, or more complex assumptions about the heating and cooling in the disc. The diffusivity $\nu$ has been parametrised as a dimensionless “alpha” due to Shakura and Sunyaev\[90\] as

$$ \nu = \alpha H c_s $$

(1.2)

where $H$ is the disc thickness and $c_s$ is the sound speed in the disc. This is to say that the rate of diffusion is given by the size of the largest possible turbulent eddy that can exist within the disc multiplied by a typical velocity, which is some prefactor $\alpha \lesssim 1$ multiplied by the sound speed. Shakura and Sunyaev then give the prefactor $\alpha$ as

$$ \alpha = \frac{|u|^2}{v_s^2} + \frac{v_a^2}{v_s^2} $$

where $u$ is the turbulent local velocity and $v_a^2 = |B|^2/\rho$ is the square of the local Alfvén speed associated with the strength of the disc’s magnetic field. These local quantities are currently impossible to measure observationally, but observations of global disc evolution have suggested that $\alpha$ lies in the region of $0.1 \sim 0.4$ and simulations calculate $\alpha$ to be an order of magnitude smaller than this\[64\]. That this $\alpha$ relies on the ratio of the Alfvén speed to thermal sound speed was validated by the rediscovery of the MRI as a source of turbulence. We discuss the conditions for the MRI in \[1.8.1\]. In this Thesis we do not this type of $\alpha$ prescription, and from this point on $\nu$ will have its usual fluid dynamic meaning.
of the fluid kinematic viscosity.

1.4 Planetary formation and migration

In Chapter 2 we will discuss the production of acoustic waves from low-frequency inertial waves through the action of shear. Acoustic waves not only transport angular momentum efficiently but also have a role in planetary migration. We briefly recap planetary formation and migration.

The number of extrasolar planets observed by humanity grows by the year and, as is suggested by the name, the primary point of interest in a protoplanetary disc (aside from the general question of angular momentum transport which is important for all accretion discs) is the formation of planets. Currently there are two competing theories for the formation of giant planets: first, the gravitational instability as described above in §1.2, where patches of discs are not so large as to be supported by angular momentum, but not so small as to be supported by thermal energy may collapse inwards (and if they can cool sufficiently quickly form a planet). Second, “core accretion” (originally due to Safronov[89], translation): the dust in the disc may coagulate, especially if trapped in regions of high pressure[9], to form a rocky core which may then either accrete gas from the disc to become a gas giant, or, if it forms late enough, to become a rocky planet such as Earth. The mechanics of planet formation are well outside the thrust of this Thesis and we refer the reader to Papaloizou and Terquem[83] for a recent review.

One of the problems of predicting the eventual planetary system that might form from a protoplanetary disc is the fact of planetary migration. In Chapter 2 we will be discussing the generation of fast acoustic waves from slow inertial waves as they pass through a mixed Lindblad/vertical resonance, and acoustic waves generated at Lindblad and corotation resonances have previously been considered by Goldreich and Tremaine[43] in what is now known as Type I planetary migration. As is suggested by the name, there are several other types of planetary migration: Type II migration, where the planet opens a gap and migrates on the viscous timescale of the disc; Type III migration, where a density gradient across the orbit of the planet can create an exponentially increasing migration
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rate, and very recently Type IV migration which relies on the stochasticity of the background turbulence.

In Type I migration the tidal potential of the planet excites density waves at the inner and outer Lindblad resonances, locations where the epicyclic frequency equals the Doppler shifted frequency of the forcing due to the embedded planet, i.e. \( \kappa^2 - m(\Omega - \Omega_p)^2 = 0 \), where here \( m \) is an azimuthal wavenumber that serves as an index for the Lindblad resonance. The density waves at these locations then each provide a torque on the planet by fact of the enhanced gravitational attraction of the density waves close to the Lindblad resonances, and if these torques are unequal the planet will gain or lose angular momentum and so drift radially within the disc. The \( m \)th outer Lindblad resonance is in general stronger than the \( m \)th inner Lindblad resonance for a number of reasons described in Ward\[111\], essentially that the outer resonance is both stronger and closer to the perturber, and the planet therefore drifts inwards towards the central star; Papaloizou and Terquem quote the infall time for an Earth mass planet to be \( 10^5 \) years, and so there is a problem in that planets which successfully form may be destroyed before the gas disc can evaporate, although Tanaka et al.\[97, 98\] estimate \( 10^6 \) for a three dimensional disc. Forthcoming work by Baruteau and Papaloizou\[10\] (submitted) investigates the effect of these density waves might have on another planet migrating within the same system; any source of acoustic waves in a disc could affect the fate of an embedded planet.

Of course, the system that results form the mechanisms outlined above could contain several interacting planets and might evolve considerably in time, but since we are concerned only with accretion discs within their lifetimes we will go no further.

1.5 Local models of discs

To avoid considering the entire disc we will work in the “shearing sheet” approximation from Goldreich and Lynden-Bell\[12\] whereby only a small patch of a full accretion disc is considered. Recent statistical comparisons between shearing sheet and global simulations show broad agreement\[94\]. To obtain this approximation we assume an angular frequency \( \Omega \propto R^{-q} \) and expand around some fixed
radius $R_0$, where $\Omega(R_0) = \Omega_0$ for this section only. Write $x = R - R_0$, with $R_0 \gg |x|$. Then

$$
\Omega = K(R_0 + x)^{-q}
= KR_0^{-q} \left(1 + \frac{x}{R_0}\right)^{-q}
\sim KR_0^{-q} \left(1 - q\frac{x}{R_0} + O\left(\frac{x^2}{R_0^2}\right)\right)
= \Omega_0 \left(1 - \frac{q}{R_0} + O\left(\frac{x^2}{R_0^2}\right)\right)
$$

which is to say that in the frame rotating with angular frequency $\Omega_0$ there will be a shear flow $-q\Omega_0 xe_y$, where $y$ has become our local azimuthal co-ordinate and where $-\frac{d\log\Omega}{d\log R} = q = 3/2$ is the dimensionless shear parameter for a Keplerian disc. This is shown schematically in Figure 1.4. If we were to place a simple dye pattern into this flow we would get Figure 1.5. In this Thesis we will always work in the shearing sheet approximation and so consider $\Omega_0$ to be constant unless otherwise stated.

![Diagram of shearing sheet](image)

**Figure 1.4:** The shearing sheet: our radial co-ordinate $R$ has been replaced by a local co-ordinate $x$, and our azimuthal co-ordinate $\phi$ has been replaced by a local co-ordinate $y$. We have a vertical rotation vector $\Omega e_z$ and an azimuthal shear $- q \Omega xe_y$.

We would like simulations to represent a small patch of an accretion disc, and accretion discs do not have rigid walls every few scale heights in the radial direction. We obviously cannot use radially periodic boundary conditions because the
Figure 1.5: The linear background shear advecting a pattern of dye. Initially, $k_x$ is large and negative; at the point of the swing $k_x = 0$; finally, $k_x$ is large and positive.

Azimuthal shear would change discontinuously over the boundary. The solution is to use “shearing-periodic” boundary conditions, in which the box is surrounded by copies of itself moving with the shear.

Figure 1.6: Shearing periodic boundary conditions.

The pattern of dye shown in Figure 1.5 has a constant azimuthal wavenumber, $k_y$, but a radial wavenumber that changes in time as $k_x = q\Omega_0 t k_y$. At the far left $k_x$ is large and negative; the shear then proceeds to unwind the pattern until we reach the centre panel where $k_x = 0$ at $t = 0$ (in Chapter 2 we shall refer to this point the “swing”, where the pattern swings over from leading to trailing); we continue on to the far right where $k_x$ is large and positive. We shall often assume in this Thesis that all perturbation quantities are $\propto \exp(ik_y y + i(k_y q\Omega_0 t)x)$ and so gain time-dependent partial differential equations. This alteration to a Fourier ansatz is usually called a “shearing wave ansatz” and is due to Kelvin[109].
1.5.1 Local vertical gravity

We derive the vertical gravity $g_z$ for the shearing sheet. Recall the Keplerian gravity potential at a distance $r$ from some mass $M$

$$
\Phi(r) = -\frac{GM}{r}
$$

which, in cylindrical co-ordinates

$$
= -\frac{GM}{(R^2 + z^2)^{1/2}}.
$$

We then see that the vertical acceleration due to gravity is

$$
-\frac{\partial \Phi}{\partial z} = \frac{\partial}{\partial z} \left( \frac{GM}{(R^2 + z^2)^{1/2}} \right)
= -z \left( \frac{GM}{(R^2 + z^2)^{3/2}} \right)
$$

which, since $z \ll R$ in our local approximation,

$$
= -z\Omega_0^2 + O\left(\frac{z^2}{R^2}\right)
$$

Notice that the vertical gravity grows stronger with height but decreases like $R^{-3}$ with radius. If we have vertical hydrostasis

$$
c_s^2 \frac{\partial \rho}{\partial z} = -\rho \Omega_0^2 z
$$

we see that

$$
\rho(z) = \rho_0 \exp(-z^2/2H^2)
$$

where we have introduced the vertical scale height $H = c_s/\Omega_0$. The density falls off as a Gaussian from the midplane.
1.5.2 Centrifugal force balance

In the next section we will give the governing equations for the shearing sheet. The contributions from the Coriolis and centrifugal forces are somewhat subtle and we discuss them now. The vital point to remember is that at every point of an accretion disc with circular orbits we have the balance \( R\Omega^2 = \frac{d\phi}{dR} \), where we have written \( \phi_g \) for the gravitational potential due to the central star. We have as a natural consequence that \( \frac{d}{dR} (R\Omega^2) = \frac{d^2\phi}{dR^2} \) at all points, including at our fixed reference point \( R = R_0 \). If we now transform to a frame rotating with angular velocity \( \Omega_0 \) there will appear in the momentum equation a centrifugal contribution to the inertia \( \Omega \times (\Omega \times x) = -\Omega_0^2 R e_R \) which may be rewritten as a potential \( = \frac{1}{2} \Omega_0^2 R^2 \).

We now expand the centrifugal potential around the point \( R = R_0 \).

\[
\phi_c = -\frac{1}{2} (R_0 + x)^2 \Omega(R_0)^2 = -\frac{(R_0 + x)^2}{2R_0} \left. \frac{d\phi}{dR} \right|_{R_0}
\]

and the same for the gravitational potential

\[
\phi_g(R_0 + x) = \phi_g(R_0) + x \left. \frac{d\phi_g}{dR} \right|_{R_0} + \frac{1}{2} x^2 \left. \frac{d^2\phi_g}{dR^2} \right|_{R_0} + ...
\]

We write the sum of these and gather terms in powers of \( x \)

\[
\phi_c + \phi_g = \left( \phi_g(R_0) - \frac{1}{2} R_0 \left. \frac{d\phi_g}{dR} \right|_{R_0} \right) + x \left( \left. \frac{d\phi_g}{dR} \right|_{R_0} - \left. \frac{d\phi_g}{dR} \right|_{R_0} \right) + \frac{1}{2} x^2 \left( \left. \frac{d^2\phi_g}{dR^2} \right|_{R_0} - \frac{1}{R_0} \left. \frac{d\phi_g}{dR} \right|_{R_0} \right) + ...
\]

and of these three terms, the first is constant and may be discarded, the second is zero, and the third may be rewritten by using \( \frac{d}{dR} (R\Omega^2) = \frac{d^2\phi}{dR^2} \) and that \( \frac{d\log \Omega}{d\log R} = \)
In the momentum equation in the shearing sheet, this term will balance the portion of the Coriolis force which arises from the background shear, \( 2\Omega \times (-q\Omega x e_y) \).

We therefore omit these terms from the governing equations given in the next section. From this point on in the Thesis we drop the subscript on \( \Omega_0 \) and work only in the local approximation.

### 1.6 Governing Equations

From this section on we shall split the background shear off from the velocity, writing the total velocity as \( u - q\Omega x e_y \). In this Thesis we shall concern ourselves with the physical systems described by the continuity equation for the density \( \rho \),

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho (u - q\Omega x e_y) \right) = 0,
\]

and the momentum equation, where we have already noted that the centrifugal term on the left hand side will balance the radial gravitational acceleration on the right and the contribution to the Coriolis force from the background shear,

\[
\rho \left( \frac{Du}{Dt} + 2\Omega \times u - q\Omega x \frac{\partial u}{\partial y} - q\Omega u_x e_y \right)
\]

\[
= -\nabla p + \nabla \cdot \left( BB - \frac{B^2}{2} I \right) + \rho g_z e_z + \nabla \cdot \sigma,
\]

\[
= -\nabla p - \nabla \frac{B^2}{2} + B \cdot \nabla B + \rho g_z e_z + \nabla \cdot \sigma,
\]  

(1.3)

where \( u \) is the velocity, \( \Omega \) is the vertical rotation vector (which in the previous section we would have called \( \Omega_0 e_z \)), \( x \) is our position, \( B \) is the magnetic field, \( g_z \) the vertical gravitational acceleration \( (g_z = -\frac{\partial \phi}{\partial z}) \) and \( \sigma \) the stress tensor. We
also have the induction equation,

\[
\frac{\partial B}{\partial t} = q\Omega x \frac{\partial B}{\partial y} - q\Omega B_x e_y + \nabla \times \mathcal{E} + \nabla \cdot (\eta \nabla B)
\]

\[
= q\Omega x \frac{\partial B}{\partial y} - q\Omega B_x e_y + B \cdot \nabla u - u \cdot \nabla B - B \nabla \cdot u + \eta \nabla^2 B,
\]

(1.4)

(where we have introduced \( \mathcal{E} = u \times B \), the electromotive force) and an isothermal equation of state,

\[
p = c_s^2 \rho.
\]

These are the usual compressible MHD equations in a rotating frame, with some vertical gravitational acceleration \( g_z \). We usually shall not consider these equations in full; we shall linearise and take several different averages to allow ourselves to make analytic progress. In Chapter 2 we shall set the magnetic field to be zero and consider a hydrodynamic problem.

We shall also make use of the horizontally averaged induction equation. This equation governs the evolution of the large-scale magnetic field when there exists a meaningful separation between large- and small-scales. In components,

\[
\frac{\partial B_x}{\partial t} = -\frac{\partial \mathcal{E}_y}{\partial z}
\]

\[
\frac{\partial B_y}{\partial t} = -q\Omega B_x + \frac{\partial \mathcal{E}_x}{\partial z}.
\]

These equations have a conservative term apiece from the horizontally averaged EMFs together with a contribution from the background shear which we shall discuss in \[1.5\]. There are therefore two routes to a self-sustained large scale dynamo: either \( \mathcal{E}_x \) is such that it reinforce \( B_y \) directly, or \( \mathcal{E}_y \) is of such a sign that it creates a \( B_x \) which will then reinforce \( B_y \) via shear. In this Thesis we will only consider background quantities with no horizontal structure (e.g. after a horizontal average) and so do not distinguish them by an overbar or similar.
1. Introduction

1.7 Generation of acoustic waves

In the §1.4 we discussed the global effect an acoustic wave may have by transporting an embedded planet through gravitational interaction. However, acoustic waves generally appear in local MRI simulations as weak shocks (e.g. Gardiner and Stone[40]) which led Heinemann and Papaloizou[50] to consider the generation of acoustic waves by the “balanced solutions” in the shearing sheet with a nonzero pseudo potential vorticity (PPV) assumed to be generated by some MRI turbulence. They argued that for a weak magnetic field the Lorentz terms in the momentum equation

\[-\nabla \frac{B^2}{2} + B \cdot \nabla B\]

are second order and so can be neglected when linearising the momentum equation to consider single shearing waves. After some algebra the governing equations can be decoupled into three independent equations for the azimuthal velocity and two quantities \(u_{\pm} = u_x \pm c_s\). The azimuthal velocity is then governed by

\[
\frac{d^2 u_y}{dt^2} + \left(\kappa^2 + (k_x^2 + k_y^2)c_s^2\right) u_y = -\frac{i k_x c_s^2}{\rho} \omega_z
\]

where \(\omega_z\) is the pseudo potential vorticity. The balanced solution for this is easily found by neglecting the term involving the double time derivative on the left hand side. Then:

\[
u_y = -\frac{i k_x c_s^2}{\rho \left(\kappa^2 + (k_x^2 + k_y^2)c_s^2\right)} \omega_z
\]

where \(\omega_z = i k_x u_y - i k_y u_x\). These balanced solutions do not oscillate and are linear in the amplitude of the PPV (i.e. they are the ‘vortical’ modes explicitly excluded by the previous shearing sheet analysis of Narayan et al.[78] and Nakagawa and Sekiya[77]) and have zero angular momentum. During the swing two acoustic waves of equal amplitude are generated, propagating in opposite senses and so conserving net angular momentum. From the linear WKBJ analysis the amplitude of e.g. the radial velocity should continue to increase after the swing.
as $|t|^{1/2}$, but their subsequent simulations\cite{51} showed it to quickly decrease for $k_x H > 1$, very soon after the swing, despite replicating well the phase and initial amplitude of the generated acoustic waves. They found that they could match the generated wave profiles only if they continued their analysis into the weakly nonlinear regime\cite{52}, upon which point agreement became excellent.

In Chapter 2 we will perform an analysis similar to the original WKBJ treatment of Heinemann and Papaloizou for an isothermal atmosphere with vertical gravity, which we showed in \S1.5.1 will have a background density profiles $ho = \rho_0 \exp(-z^2/2H^2)$. This background has a natural basis of Hermite polynomials with index $n$ for its wave solutions, and such a decomposition has been used to investigate the possibility of over-reflection of acoustic waves incident on the corotation region by Li, Goodman and Narayan\cite{68}, and to investigate the torques on an embedded planet that could lead to migration by Tanaka et al.\cite{97,98}. The balanced solutions of the above are replaced by incompressible vortical modes, which we will remove by way of a gauge transform, and inertial modes which are predominantly vertical in the limit $|t| \to \infty$. These inertial modes only oscillate for $(k_x H)^2 q^2 < 8(2 - q)$ and - whether oscillating or not - resemble qualitatively the balanced solutions of the 2D case. In particular their energy decays as $|t|^{-1}$ as $|t| \to \infty$, and so if there is some weak low frequency source of 3D motion that can excite leading inertial waves then they can be converted into acoustic waves.

By analogy with the nonlinear investigations of Heinemann and Papaloizou we expect that the generated acoustic modes would, if evolved using the full nonlinear set of equations, quickly steepen into dissipative shocks, contributing to the nonlinear behaviour of the system. This differs sharply from the axisymmetric case: the analysis of Bate et al.\cite{11} found axisymmetric acoustic waves could propagate extremely long distances without sharpening. Again by analogy with Heinemann and Papaloizou we suppose that vertical motions from the undular magnetic buoyancy instability could provide such a source of leading inertial modes, although there remains a difficulty in that the undular magnetic buoyancy modes of largest amplitude will themselves be trailing.

In Chapter 2 we describe in more detail the physical differences between the natural initial conditions for such a shearing wave problem compared to the natural boundary conditions for the similar problem in the spatial domain considered
by Li, Goodman and Narayan.

1.8 Magnetic fields

Magnetic fields are ubiquitous and evolving on all scales, from the magnetic eruptions of Sol that batter the Earth and the geodynamo that protects us from them, through the field that threads the empty space between the spiral arms of galaxies[26], and even to the immense voids between clusters of galaxies[12]. These fields are not passive but react back on their generating flows: three Chapters of this Thesis are concerned with the regeneration of the magnetic field in accretion discs via the magnetorotational instability (derived in §1.8.1).

Observational evidence for the dynamic nature of magnetic fields comes easily from the Sun. The Solar magnetic dipole reverses every 11 years, giving a 22 year cycle. Throughout this cycle the number of Sunspots - regions of strong vertical magnetic field - on its surface waxes and wanes in a systematic way, shown in Figure 1.7. Zeeman splitting allows us to measure the polarity of these sunspots; they always appear in pairs, with the magnetic field through one pointing upwards and the magnetic field through the other pointing downwards. In each hemisphere one polarity of sunspot preferentially leads the other on the Sun’s surface. As the large scale solar cycle proceeds sunspot production increases to a maximum before ceasing completely, only to resume again with the polarities reversed. Periodically these two systems interact: magnetic reconnection above the Solar surface ejects hot ionised gas which impacts on the Earth’s magnetic field several days later - giving us the fantastic natural light show called the Aurora Borealis. In accretion discs we have the time variance in the strength of jets as shown in Figure 1.1 or observations of magnetic fields via the of Zeeman splitting of electrons passing through the magnetic field which stretches between galactic spiral arms mentioned in the previous paragraph.

1.8.1 The MRI and magnetic buoyancy

The MRI is a local magnetic instability that is now widely believed to drive accretion disc turbulence. It was described as far back as Velikhov[107] and
Figure 1.7: Observational evidence for dynamo cycles: Sunspot observations from 1874 to June 2013. Over the course of a half-cycle the location that Sunspots appear moves towards the midplane and the number of Sunspots observed increases until they cease. They reappear at the farther latitudes with opposite polarity. The Sun’s magnetic field has a clear time dependence. Image credit: Hathaway, NASA.
Chandrasekhar and brought to prominence by Balbus and Hawley. It requires solely the presence of a weak background magnetic field and rotation with a negative angular velocity gradient (such as a disc in Keplerian rotation). Energy is drawn from the background shear by stretching of magnetic field lines, and the resultant MHD state is turbulent. Despite much recent research the nonlinear behaviour of the MRI remains poorly understood - in particular its ability to sustain a large-scale magnetic field.

The essential ingredients of the MRI are differential rotation and magnetic tension. To give an example of the types of calculation we shall carry out in this Thesis we will quickly derive an extremely basic version of the toroidal MRI: consider an incompressible, differentially rotation fluid with no diffusion and a constant background toroidal field $B_y$, upon which we place a perturbation with large poloidal wavenumber ($k_x$ and $k_z \gg k_y$). After linearising the incompressible form of Equations 1.3 and 1.4 and seeking some growth rate $\gamma$ via a shearing wave ansatz we find

$$\gamma u_x - 2\Omega u_y = -ik_x p + i\omega_a b_x$$
$$\gamma u_y + (2 - q)\Omega u_x = i\omega_a b_y$$
$$\gamma u_z = -ik_z p + i\omega_a b_z$$

for momentum (where we have written the Alfvén frequency $\omega_a = k_y v_a$, the frequency at which a magnetic field line would oscillate when plucked),

$$\gamma b_x = i\omega_a u_x$$
$$\gamma b_y = i\omega_a u_y - q\Omega b_x$$
$$\gamma b_z = i\omega_a u_z$$

for induction, and

$$ik_x u_x + ik_z u_z = 0$$
for incompressibility. This leads directly to a dispersion relation

\[
(\gamma^2 + \omega_a^2)^2 + \frac{k_z^2}{k_x^2 + k_z^2} \left( \kappa^2 \gamma^2 - 2q \Omega^2 \omega_a^2 \right) = 0
\]

where \( \kappa^2 = 2(2-q)\Omega^2 \) is the square of the epicyclic frequency. This is a quadratic in \( \gamma^2 \) with purely real solutions i.e. \( \gamma \) real or imaginary, with optimal

\[
\gamma = \frac{k_z}{\sqrt{k_x^2 + k_z^2}} q \Omega \quad \text{at} \quad \omega_a = \frac{k_z}{\sqrt{k_x^2 + k_z^2}} \sqrt{(4-q)q} \Omega.
\]

This tells us several important things: the MRI grows fastest for \( k_z \gg k_x \), i.e. \(|u_z| \ll |u_x|\), predominantly horizontal perturbations, with growth rate comparable to the local angular velocity \( \Omega \), and a growth rate independent of the magnetic field strength. The fastest growth rate is found by optimising \( \omega_a \). This means that for any magnetic field strength we the optimal growth will be present, given a sufficiently large azimuthal wavenumber \( k_y \). Instability exists for any \( -\frac{d\log \Omega}{d\log R} = q < 0 \).

It is the fast growth rate of the MRI for even weak magnetic fields which sets it up as the prime candidate for a source of turbulence in accretion discs. As a crude estimate of the vertical lengthscales on which the MRI operates in a Keplerian disc (i.e. \( q = 3/2 \)) we balance the optimal growth rate \( 3\Omega/4 \) derived above with an Ohmic diffusion rate \( k_z^2 \eta \) (assuming a magnetic field strong enough such that \( k_y \ll k_z \) for the fastest growing mode), with \( \eta \) as calculated in \$1.2 \).

Then, at 0.3AU around a central star of mass \( M_\odot \)

\[
k_z \approx \left( \frac{3}{4} \sqrt{\frac{GM_\odot}{(0.3\text{AU})^3}} \right)^{1/2} \frac{3.4 \times 10^{13} \text{ cm}^2 \text{ s}^{-1}}{2.34}
\]

\[
= 9 \times 10^{-7} \text{ cm}^{-1}
\]

\[
\Rightarrow \lambda_{\text{crit}} \approx 4.67 \times 10^{-7} \text{ AU}.
\]

We shall discuss the linear MRI further in Chapter 4 and subsequent nonlinear behaviour in Chapter 5.

We discuss also the undular magnetic buoyancy instability. Accretion discs in Keplerian rotation naturally experience a gravity in the \( e_z \) direction (as will be
described in (1.5.1). Fromang and Papaloizou showed that without this vertical gravity and without explicit diffusivities the calculated Reynolds stress scaled inversely with the numerical resolution; Davis, Stone and Pessah then found that the calculated Reynolds stress did converge without explicit diffusivities but with vertical gravity; between these two papers it is clear that the addition of vertical gravity changed the nature of the dynamo. Including this vertical gravity gives rise to the magnetic buoyancy instability due to Parker: consider a magnetised slab below an unmagnetised slab in vertical pressure equilibrium. The magnetic pressure in the lower slab will contribute to the pressure equilibrium and so decrease the thermal pressure. The decreased thermal pressure (by way of the equation of state) implies a decreased density. We may then have heavier fluid above lighter fluid. For instability the magnetic pressure gradient must be strong enough to overcome any thermal stratification yet weak enough to avoid stabilisation through magnetic tension.

The system of equations giving rise to the magnetic buoyancy dispersion relation is significantly longer and we leave its discussion to §3.6.4. Optimising over all \( k \), the Newcomb criterion for the undular instability is

\[
g_z \frac{\partial}{\partial z} \left( \log B_y \right) > \frac{c_s^2 N^2}{v_a^2}
\]

and as already stated in this Introduction we shall discuss only isothermal atmospheres with \( N^2 = 0 \) in this Thesis: any magnetic field with \( g_z (\log B_y)' > 0 \) will be unstable.

The space-time evolution of the horizontally averaged toroidal flux in the presence of both the MRI and vertical gravity (but no net magnetic flux through the simulation domain) is shown in Figure 1.8 and will be discussed in detail in Chapter 5. Toroidal flux is periodically generated at the midplane and propagates outwards. In Chapter 5 we shall try to predict the rate of vertical migration of toroidal magnetic flux in an isothermal atmosphere with vertical gravity. We comment that for non-isothermal atmospheres we would either have the undular magnetic buoyancy instability inhibited by a stable stratification or convective turbulence that would pump magnetic flux back to the midplane. It seems possible that the vertical migration of magnetic flux is fastest for an isothermal
atmosphere.

### 1.8.2 Dynamo theory

The problem of amplifying and sustaining a changing magnetic field in a fluid flow is an entire body of literature in itself known, as “dynamo theory”. Dynamo theory has applications to the Solar dynamo, the geodynamo, laboratory flows such as the von Karman Sodium experiments, and accretion discs such as we will consider in this Thesis. We have already described how angular momentum may be lost from the inner regions of a disc by means of magnetically launched jets and we explained above how the flexible nature of the MRI makes it an ideal candidate for providing disc turbulence. However, it is not obvious how these magnetic fields sustain themselves over time: the observations of the Sun and of the jet in Figure 1.1 tell us that the magnetic fields are far from static.

We gave in Equation 1.4 the induction equation that governs the time evolution of the magnetic field. Given the strong differential rotation in an accretion disc it is a simple matter to generate toroidal field from radial field via the background shear. A more difficult problem is to create radial field from an initial toroidal field; if this is possible then the magnetic field will be self-regenerating and we will have located a dynamo loop, from radial field to toroidal field to radial...
1. Introduction

field again. Classical dynamo theory has tried to close this dynamo loop via *mean field theory* described by Krause and Rädler\[65\] and Moffat\[47\]; although we do not intend to make any mean field assumptions in this Thesis we will interpret several calculated quantities in light of results from mean field theory and it is appropriate to discuss those now. The full induction equation is

$$\frac{\partial B}{\partial t} = \nabla \times \mathcal{E} + \eta \nabla^2 B,$$

which governs the evolving magnetic field on all lengthscales and timescales. Consider that we might have two such scales, well separated from one another. This is true in the Sun, where the global dipole with its 22 year cycle is well separated compared to the convective cells of radius $\sim 10^5$ km and turnover time $\sim 1$ month thought to lie under its surface - an $O(1)$ difference in lengthscale but a $250\times$ difference in timescale (with a range of much smaller scales down to surface granules with radii of around 1000km and lifetimes of around 20 minutes).

In the context of an astrophysical disc the most unstable MRI wavelength (which we derived above) will be perhaps $10\times$ larger than $4.67 \times 10^{-7}$ AU with a growth rate of $3\Omega/4$, compared to a vertical scale height of perhaps $1.5 \times 10^{-3}$ AU and a dynamo cycle which simulations have shown is of the order of 10 orbits (see e.g. Figure 1.8) - which is to say around a $3000\times$ difference in lengthscale and a $20\times$ difference in timescale. Then we may think about averaging over the small scale by some suitable spatial, temporal or ensemble average to split the induction equation into a mean and fluctuating part, with magnetic field $B = \langle B \rangle + b$ and velocity field $U = \langle U \rangle + u$, and $\langle b \rangle = \langle u \rangle = 0$. Then, without any approximation,

$$\frac{\partial}{\partial t} \langle B \rangle = \nabla \times (\langle U \rangle \times \langle B \rangle) + \nabla \times \langle u \times b \rangle + \eta \nabla^2 \langle B \rangle$$

and

$$\frac{\partial b}{\partial t} = \nabla \times (U \times b) + \nabla \times (u \times B) + \nabla \times (u \times b - \langle u \times b \rangle) + \eta \nabla^2 b.$$

Of course, the point of this procedure is generally that we are interested in the evolution of the mean field $\langle B \rangle$ and are either unwilling or unable to properly model the fluctuating fields $u$ and $b$, arising as they will from some small scale
turbulence. We place our focus on the first of these two equations: the contribution from the correlations of the mean fields $\langle U \rangle$ and $\langle B \rangle$, $\nabla \times (\langle U \rangle \times \langle B \rangle)$, will for an accretion disc give us a simple shearing term that creates toroidal field from poloidal. The contribution from the correlations of the fluctuating fields, $\nabla \times (u \times b)$, must be treated with some form of closure model. If one assumes that the magnetic field does not influence the fluid velocity field (a poor assumption) then the original induction equation is linear in $B$, the total magnetic field. The correlation of the fluctuations can then be written as an expansion in the derivatives of $\langle B \rangle$,

$$\langle u \times b \rangle_i = \left( \alpha_{ij} + \beta_{ikj} \frac{\partial}{\partial x_j} + \ldots \right) \langle B_j \rangle$$

where successive terms decay with the ratio of the small scale to the large scale.

These two terms form the basis of a good deal of mean field theory, and are referred to in the literature as the $\alpha$-effect and $\beta$-effect (the $\alpha$-effect not to be confused with the Shakura-Sunyaev $\alpha$ described in §1.3). By comparison with $\langle u \times b \rangle$ we see both $\alpha$ and $\beta$ are pseudo-tensors, changing sign under reflections, and so require some lack of mirror symmetry in the basic flow such as a nonzero helicity. We list components of interest:

- $\alpha^S = \frac{1}{2} (\alpha + \alpha^T)$: the symmetric part of the $\alpha$ tensor gives an EMF $\alpha^S \cdot \langle B \rangle$. This provides a route for poloidal field to be created from toroidal, allowing one to close the dynamo loop.

- $\alpha^A = \frac{1}{2} (\alpha - \alpha^T)$: the antisymmetric part of the $\alpha$ tensor gives an EMF $\Gamma \times \langle B \rangle$ for a suitable vector $\Gamma$. On substitution into the mean field induction equation this becomes an advection of $\langle B \rangle$ with velocity $\Gamma$, important at the base of the Solar convection zone. Drobyshevski and Yuferev\textsuperscript{[23]} give a topological argument for the downward pumping of magnetic flux by convection: Convection cells are disconnected hot broad upwellings surrounded by cool narrow downwellings. A flux tube at the top of a layer of convection will be advected to the edges of the broad upwellings and successfully carried downwards. However, the whole length of a flux tube at the bottom of a layer of convection cannot rise upwards through a single convection cell,
still being connected to the base of the convecting region by magnetic tension. The net effect will be to transport magnetic flux downwards. We will discuss such an advective term arising from an inhomogeneous background in Chapter 3.

- $\beta_{ikj} = \beta\epsilon_{ikj}$: the isotropic part of $\beta$ gives an extra “turbulent diffusion” term $\nabla \cdot (\beta \nabla \langle B \rangle)$ in the mean field induction equation.

We will make analogies between these components and the analytically calculated EMFs that are the main main result of Chapter 3. However, at no point in this Thesis do we assume an ad hoc $\alpha$- or $\beta$-effect.

1.9 Thesis outline

The structure of this Thesis is as follows: In Chapter 2 we examine the generation of acoustic waves from inertial waves in an isothermal accretion disc with vertical gravity. We work in a Lagrangian framework with a conserved symplectic form and find efficient generation of acoustic waves is possible given a weak source of inertial waves. In Chapter 3 we consider an asymptotic expansion of linear MRI/undular instability and carefully construct an analytic expression for the quasilinear EMFs given a background which varies in the poloidal plane. We then consider several limits of interest and interpret the EMFs in the context of classical dynamo theory. In Chapter 4 we examine the quadratic EMFs that arise from a single shearing wave subject to the mixed toroidal MRI/undular instability in an isothermal disc with vertical gravity. We find the localisation of the mixed instability is sensitive to the toroidal wavenumber and - for toroidal wavenumber small enough - is determined largely by magnetic buoyancy. In Chapter 5 we shall unite the predictions of Chapters 4 and 3 via nonlinear numerical simulation. Using a simple yet effective strained temporal average we reveal the phase relationship of $B_x$ and $B_y$ changes with height, and find that we can reproduce both the radial and toroidal EMFs well using our analytic prediction of Chapter 3. Our overarching aim with the three magnetic Chapters is similar to that of Mirosh et al. [74], who related double-diffusive convective instabilities to their nonlinear evolution.
The four research Chapters are preceded by this Introduction and succeeded by our Conclusions, as well as by one Appendix for each research Chapter. Each Chapter shall itself have an introduction giving relevant background and a conclusion summarising its content.
Chapter 2

Isothermal swing calculations

\[ I \text{ bounce these feelings off the Moon} \]
\[ \text{The echoes don't come back} \]

My Very Best, Elbow

2.1 Introduction

We discuss mode conversion of the ‘canonical’ non-axisymmetric acoustic and inertial modes in an isothermal accretion disc with vertical gravity, paying particular attention to the temporal structure of the inertial modes. We shall work in the shearing wave domain, using numerical integration and asymptotic analysis to investigate the conversion between inertial and acoustic modes and the effect of the vertical structure on over-reflection. We shall also introduce the conserved Hermitian form appropriate to the shearing sheet. This simple system is separable in space but exhibits a surprisingly rich mathematical structure.

Consider a 2D density wave with azimuthal wavenumber \( m \), propagating in the same direction as the flow in a Keplerian accretion disc without self-gravity. At all radii the frequency \( \omega \) will be Doppler-shifted by the background differential rotation, giving us a Doppler-shifted frequency \( \tilde{\omega} = \omega - m\Omega \), and a distinguished corotation radius where \( \tilde{\omega} = 0 \) i.e. a radius where the phase velocity in the frame
moving with the fluid vanishes. We shall be considering waves passing through corotation, as described by Mark in a discussion of over-reflection of galactic spiral density waves.

Given a local radial co-ordinate $x$, the wave obeys a local dispersion relation

$$(\omega - m\Omega)^2 = \kappa^2 + k_x^2 c_s^2 + O\left(\frac{m^2 c_s^2}{R^2}\right)$$

(e.g. Li, Goodman and Narayan) which involves $\kappa^2 = 2(2 - q)\Omega^2$, the square of the epicyclic frequency. This dispersion relation can be rearranged for the local radial wavenumber

$$\frac{c_s^2}{\Omega^2} k_x^2(R) = \left(\frac{\omega}{\Omega} - m\right)^2 - \frac{\kappa^2}{\Omega^2} + O\left(\frac{m^2 c_s^2}{R^2\Omega^2}\right).$$

It is easy to see that near corotation ($\omega/\Omega - m$) will be small and we shall have $k_x^2 < 0$, an evanescent wave. This evanescent region was described by Narayan, Goldreich and Goodman and has width $(k_y^2 c_s^2 + \kappa^2)^{1/2}/(2\Omega q k_y)$.

Upon incidence on the edge of this evanescent region most of the outgoing wave is reflected with complex amplitude $a_-$. There is an exponentially small transmission amplitude $a_+$ with

$$|a_+| = \exp(-\pi C), \quad C = \frac{1}{2q} \frac{k_y^2 H^2 + \kappa^2/\Omega^2}{|k_y H|} \quad (2.1)$$

and this process is shown schematically in Figure 2.1. Before reaching the evanescent region the wave has $\tilde{\omega} < 0$, so that the wave lags behind the background flow, and beyond the evanescent region the transmitted wave has $\tilde{\omega} > 0$ and propagates ahead of the flow. The transmitted wave can therefore be associated with a positive angular momentum density, and the initial outgoing wave and the reflected wave with negative angular momentum densities. By consideration of the conservation of angular momentum flux one may write

$$1 = |a_-|^2 - |a_+|^2 \quad (2.2)$$

N.B. in this Chapter we shall use “outgoing” to mean exclusively “propagating away from the centre of the disc” in the positive $x$ direction.
2. Isothermal swing

where the left hand side is the normalised amplitude of the initial outgoing wave. Note that this has

\[ |a_-|^2 = 1 + |a_+|^2 > 1 \]

and so we are considering over-reflection: the reflected wave has magnitude greater than the incident outgoing wave. It is easy to maximise Equation 2.1 as a function of \( k_y \); for a Keplerian disc over-reflection has a maximum when \( k_y H = 1 \) of

\[ |a_+| = \exp(-2\pi/3) \approx 0.123 \ldots \]

and Tsang and Lai\(^{105}\) generalised these analytic expressions for \( a_\pm \) to incorporate a nonzero gradient in the background potential vorticity of the disc; with a positive radial vortensity gradient over-reflection is enhanced.

This over-reflection problem assumes a fixed \( \omega \) to solve for the radial wavenumber \( k_x \) and the wave amplitudes. We can also work in the shearing wave domain, where we fix \( k_x = k_y q \Omega t \) as a function of time to solve for \( \omega \) and the wave amplitudes. The outgoing radiation condition for large \( x \) is then replaced by an initial condition at \( t = -\infty \), and the evanescent region where \( k_x^2 < 0 \) is replaced with a “swing” - the point when a wave changes from leading \( (k_x/k_y < 0) \) to trailing \( (k_x/k_y > 0) \). Nakagawa and Sekiya made use of shearing waves to rederive the above analytic transmission amplitudes by considering an integral across a Stokes line(\(^{177}\), in an Appendix), and Heinemann and Papaloizou\(^{50}\) performed a similar calculation to study the excitation of density waves by an initial vortical perturbation\(^4\).

On addition of vertical gravity the vertical structure of the wave becomes a Hermite polynomials with index \( n \), considered in the spatial domain with a fixed \( \omega \) by Li, Goodman and Narayan\(^{68}\). We shall repeat their calculation in the shearing wave domain, although we shall provide a more thorough investigation of wave mixing; differences between our inital conditions and their boundary conditions are described in Figure 2.2. After accounting for the vertical structure\(^{4}\) this is not the shearing wave counterpart of the Tsang and Lai paper, which introduced a non-trivial potential vorticity background.
their relationship for $k_x^2$ becomes

$$\frac{c_s^2}{\Omega^2} k_x \left( \frac{\omega}{\Omega} - m \right)^2 = \left[ \left( \frac{\omega}{\Omega} - m \right)^2 - \frac{k_x^2}{\Omega^2} \right] \left[ \left( \frac{\omega}{\Omega} - m \right)^2 - n \right] + O \left( \frac{m^2 c_s^2}{R^2 \Omega^2} \right),$$

where $n$ is the index of the Hermite polynomial that determines the vertical structure and is the number of vertical nodes in the radial perturbation velocity. We see that $k_x^2 < 0$ when one or the other of the brackets on the right hand side is negative, but not both; the single evanescent region around corotation has been replaced by two evanescent regions bounded by Lindblad and vertical resonances (see Figure 2.1b). In the shearing wave domain we shall not have separation into distinct spatial regions but instead a temporal separation between leading and trailing modes.

In this Chapter we shall analyse the 3D modes of an isothermal accretion disc with vertical gravity, carefully following the formalism of Friedman and Schutz\cite{30}. By ‘modes’ we do not mean modes in the sense of eigensolutions of an eigenvalue problem but rather asymptotic solutions to the system in the limit of large $|k_x|$; these asymptotic solutions shall be found by first taking this limit and then seeking wavelike behaviour. Friedman and Schutz gave a general treatment of infinitesimal Lagrangian perturbations to a compressible system with an underlying symmetry, thus providing a method for the systematic derivation of a Hermitian form $J(\cdot, \cdot)$ on the space of solutions and associated second-order conserved quantities, as well as a canonical gauge (discussed in \S 2.1.1); by considering their work in a rotating frame and with a shearing ansatz we shall project our numerically integrated solutions onto our basis of asymptotic solutions, or modes.

We shall find four modes for each fixed $(k_y, n)$ pair: two shall be highly compressible acoustic modes similar to the 2D modes discussed above, with WKBJ time dependence $\propto \exp \left( -i \int k_x' c \, dt' \right)$. These modes are dominated by radial compression and are efficient at transporting angular momentum. The Hamiltonian is not conserved in time for a shearing wave because $k_x$ changes with time. These acoustic modes will have $H \propto |t|$ for large $|t|$; were they allowed nonlinear interactions they would eventually break as discussed for axisymmetric waves in Bate et al.\cite{11}. The remaining two modes shall be nearly incompressible
2. Isothermal swing

(a) Schematic showing 2D wave tunnelling after Narayan, Goldreich and Goodman. Shown: (a) Corotation resonance, (b) inner Lindblad resonance and (c) outer Lindblad resonance. In regions I and IV we have $\tilde{\omega}^2 - \kappa^2 > 0$ and $k_x^2 > 0$, while in regions II and III we have $\tilde{\omega}^2 - \kappa^2 < 0$ and $k_x^2 < 0$. Without loss of generality we do not consider combinations of ingoing and outgoing incident waves.

(b) Schematic showing 3D wave tunnelling for $n > 0$ after Li, Narayan and Goodman. Shown: (a, b and c) as before, (d) Inner vertical resonance and (e) outer vertical resonance. In the outermost regions I$'$ and IV$'$ we have both brackets in the dispersion relation positive and $k_z^2 > 0$, in the innermost regions II$'$ and III$'$ we have both brackets negative and $k_z^2 > 0$, and in the evanescent regions V$'$ and VI$'$ we have $\tilde{\omega}^2 - \kappa^2 > 0$ but $\tilde{\omega}^2 - n\Omega^2 < 0$ and so $k_x^2 < 0$.

Figure 2.1
Figure 2.2: Chain of causation in the spatial domain. (a) An outgoing (and leading) acoustic mode arrives at the inner vertical resonance and reflects into an ingoing (and trailing) acoustic mode. (b) Part of the outgoing acoustic mode tunnels through to create a trailing inertial mode propagating from the inner Lindblad resonance towards corotation. (c) The inertial mode passes through (and is strongly attenuated by) corotation, becoming leading, and propagates to the outer Lindblad resonance (green dotted line in region (b)), reflecting into a trailing mode heading back towards corotation; this reflected wave would pass through corotation to become a leading inertial wave incident on the inner Lindblad resonance (purple dotted line in region (c)). (d) Part of the inertial mode tunnels through to the outer vertical resonance and propagates as an outgoing (and trailing) acoustic mode.

In the time domain the natural initial condition for an incident outgoing acoustic wave contains only the leading wave at (a); neither the green dotted line nor the purple dotted line is present. The natural problems in the spatial and temporal domain are distinct.
inertial modes lying predominantly in the $y-z$ plane. These modes have time dependence $\propto \exp \left( -i\theta \int \frac{1}{t'} dt' \right)$ as $|t| \to \infty$, and $H \propto |t|^{-1}$ for large $|t|$; our motivation is partly to investigate the generation of acoustic modes from ‘cost free’ inertial modes. For a pleasant discussion of acoustic and inertial modes, we refer to Section 3 of Balbus\[5\]. We mention as relevant the paper of Mamatsashvili and Rice\[77\] that considered the generation of acoustic waves in a convectively unstable isothermal atmosphere, although our method is more powerful.

2.1.1 Basic equations

To write the equations in the best form for analysis and numerical integration requires some manipulation. We consider the axisymmetric inviscid shearing-sheet momentum and mass equations with vertical gravity and an isothermal equation of state, as described in the Introduction §1.6.

\[
\rho \left( \frac{Du}{Dt} + 2\Omega \times u \right) = -\nabla (c_s^2 \rho) - \rho \nabla \Phi
\]
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0
\]

where $\Phi$ is the tidal potential that will supply our vertical gravity. We take as our basic state the usual linear shear

\[
u = -q\Omega xe_y, \text{ and } \rho = \rho_0 \exp \left( -\frac{z^2}{2H^2} \right)
\]

and perturb around this basic state to find our linear equations

\[
\dot{u}_x^n = 2u_y^n - ik_x w^n
\]
\[
\dot{u}_y^n = -(2 - q)u_x^n - ik_y w^n
\]
\[
\dot{u}_z^n = -w^n
\]
\[
\dot{w}^n = -ik_x u_x^n - ik_y u_y^n + nu_z^n
\]

where we have introduced the perturbation enthalpy $w = c_s^2 \rho' / \rho$, the shearing ansatz described in §1.5, and a Hermite ansatz for the vertical structure, with algebraic details confined to §A.1. For this Chapter only we shall also nondi-
mensionalise via $\Omega = c_s = 1$ for algebraic brevity. In the 2D version of this problem (where $n = u_z^n = 0$, Nakagawa and Sekiya\cite{17}) there is a quadratic conserved quantity representing the angular momentum density; we seek a related conserved quantity via rewriting our Eulerian equations as Lagrangian equations. We may write (before our shearing wave ansatz) that

$$u = \frac{\partial \xi}{\partial t} + (-qxe_y) \cdot \nabla \xi - \xi \cdot \nabla (-qxe_y)$$

with $\xi$ the displacement of a fluid parcel following the flow; in this Chapter we shall use $\xi$ to refer to a 3-vector (i.e. a vector with an $x$, $y$ and $z$ component) and $\xi$ (non-bold) to refer to an entire solution (for $-\infty < \Omega t < \infty$) to the perturbation equations. With our shearing and Hermite ansätz, and for each $n$,

$$u^n = \dot{\xi}^n + q\xi_x^n e_y$$

and our governing equations (dropping the superscript $n$) become

$$\ddot{\xi}_x = 2(\dot{\xi}_y + q\xi_x) - ik_x w$$

$$\ddot{\xi}_y = -2\dot{\xi}_x - ik_y w$$

$$\ddot{\xi}_z = -w$$

$$w = -ik_x \xi_x - ik_y \xi_y + n \xi_z.$$  \hspace{1cm} (2.3)

This is almost in the ideal form for analysis and integration. However, in moving from Eulerian to Lagrangian coordinates, we have gained two extra time derivatives; we now have six where previously we had four. We may regain a system with four time derivatives by considering the vorticity equation; from this point to the end of this section several pieces of algebra are mentioned but not shown; they appear in Appendix A. We may write exact expressions for the vorticity components

$$\omega_x = (2 - q)\xi_x + \mu_x,$$

$$\omega_z = (2 - q)(w - n\xi_z) + \mu_z.$$
2. Isothermal swing

with $\mu_x$ and $\mu_z$ constants of integration. If we use the definition of the perturbation vorticity to expand the left hand sides of the above equations we find that they are Equations 2.3 integrated twice in time. We eventually reach our governing equations

$$
\begin{align*}
\dot{\xi}_x &= ik_x u_z + (2-q)\xi_y - \frac{k_x}{k_y} \mu_x - \frac{\mu_z}{ik_y} \\
\dot{\xi}_y &= ik_y u_z - 2\xi_x - \mu_x \\
\dot{\xi}_z &= u_z \\
\dot{u}_z &= ik_x \xi_x + ik_y \xi_y - n \xi_z.
\end{align*}
$$

(2.4)

Referring to Friedman and Schutz we see we may remove them via a gauge transform (using incompressible “trivial modes”) taking us to the canonical gauge; outside of the canonical gauge our projection onto our basis modes would fail. We show the explicit gauge transform in §A.2.

Our initial and final governing equations are both fourth-order in time; this differs from the 2D analysis of Nakagawa and Sekiya in which the vorticity equation was used to reduce a third order system to a second order system. Since all four time derivatives will later be associated with modes that have nonzero wave action (angular momentum) $J_c$ we lack the ‘balanced solutions’, solutions with a perturbed potential vorticity but zero angular momentum, as considered by Heinemann and Papaloziou. If we set $n = 0$ we go from a fourth-order system to a third-order system and lose the gauge transform which would allow us to set $\mu_z = 0$ and remove vorticity from the perturbation equations.

2.2 Analysis

We want to examine the generation of acoustic from inertial modes and over-reflection for each fixed $(k_y, n)$. To this end we will introduce in §2.2.1 a conserved Hermitian form $J(\eta, \xi)$, which will furnish us with an idea of the orthogonality of two solutions $\eta$ and $\xi$ and a measure of the (conserved) angular momentum of the perturbation $J_c$. We will examine our equations in the limit of large $|t|$, assuming first a time dependence of $\exp(-i \int_{-\infty}^{t} \omega(t') dt')$ with $\dot{\omega}/\omega^2 \ll 1$ – i.e. a
high frequency WKBJ ansatz – to find the naive form of our acoustic disturbances (which we call ‘f’ for ‘fast’). Guided by the acoustic dispersion relation we then assume a time dependence of $|t|^\sigma$ to find the naive form of our inertial disturbances (which we call ‘s’ for ‘slow’). We write these modes with non-bold font and think of them as being solutions to the Lagrangian equations for all $t$ that have a specific asymptotic behaviour as $t \to \pm \infty$, denoted by a subscript $i$ for ‘initial’ and $e$ for ‘end’. For example, $f^+_i$ is a solution with $J_c = +1$ that has the asymptotic form of an acoustic disturbance as $t \to -\infty$, and $s^-_e$ is a solution with $J_c = -1$ that has the asymptotic form of an inertial disturbance as $t \to +\infty$.

We will then apply our Hermitian form $J(\cdot,\cdot)$ to combinations of these four modes to find an orthonormal basis for our problem consisting of $\hat{f}^\pm$ for the acoustic modes and $\hat{s}^\pm$ for the inertial modes for $t \to \pm \infty$. Since $J(\cdot,\cdot)$ is a Hermitian form rather than an inner product we do not have $J(\xi,\xi) \geq 0$; physically, this is because perturbations can carry either positive or negative angular momentum. Instead, we seek

\[
\frac{1}{2} J(\hat{f}^\pm, \hat{f}^\pm) = \pm 1, \\
J(\hat{f}^+, \hat{f}^-) = 0,
\]

and similarly for the inertial modes, together with $J(f,s) = 0$ for all acoustic and inertial modes. We will see that we have some freedom in the choice of basis from the inertial modes, which will be discussed in §2.2.4. Using $J(\cdot,\cdot)$ and our basis we may then split any solution $\xi$ to our canonical equations into four contributions such that for large positive $t$ after the swing

\[
\xi \sim a_+ \hat{f}^+_e + a_- \hat{f}^-_e + b_+ \hat{s}^+_e + b_- \hat{s}^-_e
\]

where

\[
a_+ = \left. \frac{1}{2} J(\hat{f}^+_e; \xi) \right|_{t=+\infty} \quad a_- = -\left. \frac{1}{2} J(\hat{f}^-_e; \xi) \right|_{t=+\infty} \\
b_+ = \left. \frac{1}{2} J(\hat{s}^+_e; \xi) \right|_{t=+\infty} \quad b_- = -\left. \frac{1}{2} J(\hat{s}^-_e; \xi) \right|_{t=+\infty}
\]
2. Isothermal swing

are complex amplitudes. For $q \Omega |t| \gg 1$ we will be far from the swing and this approximation should be good. We may think of a complex transition matrix $M = M(n, k_y)$ such that

$$
\begin{pmatrix}
  a_+ \\
  a_- \\
  b_+ \\
  b_-
\end{pmatrix} = M \cdot 
\begin{pmatrix}
  \alpha_+ \\
  \alpha_- \\
  \beta_+ \\
  \beta_-
\end{pmatrix}
$$

with Greek letters signifying the corresponding complex coefficients of $f_i^\pm$, $s_i^\pm$ at $t = -\infty$.

We follow Friedman and Schutz in writing $J_c(\xi) := J(\xi, \xi)/2$. Since $J(\xi, \eta)$ is conserved for all $(\xi, \eta)$ any $\xi$ that is a solution to Equations 2.4 retains its initial value of $J_c$ for all time. If we initialise our system with some $\xi_0$ such that $J_c(\xi_0) = +1$ then we have

$$1 = |a_+|^2 - |a_-|^2 + |b_+|^2 - |b_-|^2$$

which is precisely Equation 2.2 with an additional $|b_+|^2 - |b_-|^2$ from our inertial modes.

We wish to investigate the conversion between inertial and acoustic modes and the effect of the vertical structure on over-reflection. The problem appears information-rich, since for each $k_y$ and $n$ we have a choice of four waves as initial conditions and gain for each initial condition four complex amplitudes as output; we expect $M$ to be symmetric under the interchange of e.g. positive and negative modes, but this leaves us with two modes for inputs, each still with four complex amplitudes as output. Thus, we define several physical quantities with easy interpretation.

To measure possible over-reflection, we examine whether $|a_+| \geq 1$ for a negative acoustic input to test for over-reflection; concurrently we examine $|a_-|$ for the same input to examine the effect of tunnelling and passing through the corotation radius. This corresponds directly to the case considered by Li, Narayan and Goodman, aside from the differing initial conditions already described in Figure 2.2. We also measure $A = |a_+|^2 + |a_-|^2$ given a negative inertial input to examine
generation of acoustic modes from inertial modes. Finally, we measure total mode mixing via the size of the transition matrix $M$, which we call $M$. This is the square of the Frobenius norm divided by four

$$M = \frac{1}{4} ||M||^2$$

defined such that if there is no wave amplification of any kind - when $M$ is an isometry - then $M = 1$. Note that with our conservation of action equation (**)

$$|a_+|^2 - |a_-|^2 + |b_+|^2 - |b_-|^2 = 1$$

$$\Rightarrow |a_+|^2 + |a_-|^2 + |b_+|^2 + |b_-|^2 \geq 1$$

$$\Leftrightarrow \sum_{i=1}^{4} \sum_{j=1}^{4} |m_{ij}|^2 \geq 4$$

$$\Leftrightarrow M \geq 1$$

The physical interpretation of $M$ is problematic because it relies strongly on the choice of basis vectors, consistent with the ambiguity in our basis for the inertial modes discussed in §2.2.4. Happily we shall see that the other quantities defined in this section are largely insensitive to the choice of basis.

### 2.2.1 Hermitian form $J(\xi, \eta)$

We have introduced a Hermitian form $J(\cdot, \cdot)$ after Friedman and Schutz and explained its role in decomposing our late-time solutions into orthogonal basis modes. For two solutions $\xi$ and $\eta$ to our canonical equations the product $J(\xi, \eta)$ shall be constant for all time. In our geometry and with our shearing wave and Hermite ansatz,

$$J(\xi, \eta) = \frac{ik_y}{2} \left[ (\xi_x^* (\dot{\eta}_y - \eta_y) + \xi_y^* (\dot{\eta}_y + \eta_x) + n \xi_z^* \dot{\eta}_z) - (\xi \leftrightarrow \eta)^* \right]$$
2. Isothermal swing

where terms like e.g. $\xi_x^* \eta_y$ come from our rotating frame and terms like $\xi_x^* \dot{\eta}_x$ come from compression. We may use our canonical Lagrangian equations to eliminate $\dot{\eta}_x$, $\dot{\eta}_y$ and gain

$$J(\xi, \eta) = \frac{ik_y}{2} \left[ \dot{\eta}_x w^*_x - \dot{\xi}_x^* w_y + (2 - q)(\xi_x^* \eta_y - \eta_x^* \xi_y) \right]$$

where $w_\xi$ or $w_\eta$ is the enthalpy perturbation due to a displacement $\xi$ or $\eta$ respectively. This $J(\cdot, \cdot)$ is conserved by our canonical equations (see (4.3)). Our amplitude then may be written

$$J_c(\xi) = \frac{1}{2} J(\xi, \xi) = -\frac{k_y}{2} \text{Im} \left[ \dot{\xi}_x w^*_x + (2 - q) \xi_x^* \xi_y \right]$$

which has a first term from compression and a second term from (differential) rotation. We may even re-express this in Eulerian variables using $\omega_x = (2 - q) \xi_x$ and $\omega_y = (2 - q) \xi_y$:

$$J_c(\xi) = -\frac{k_y}{2} \text{Im} \left[ v_x w^* + \frac{1}{2 - q} \omega_x^* \omega_y \right].$$

2.2.2 Fast (acoustic) modes

We examine the (predominantly radial) acoustic waves. We seek solutions $\propto \exp \left( -i \int \Omega \, dt \right)$ with the usual WKBJ ansatz where $\Omega/\omega \ll 1$ and $|k_y/k_x| \ll 1$. Since we are seeking an acoustic wave we expect the dominant balance to be between the enthalpy and the direction with largest wavenumber, i.e. between $w$ and $\xi_x$. This gives

$$\ddot{\xi}_x \sim -k_x^2 \xi_x.$$
Assuming this WKBJ ansatz in the set of Equations 2.3 gives at leading order

\[-i\omega \xi_x = ik_x v_z + (2 - q) \xi_y\]
\[-i\omega \xi_y = ik_y v_z - 2 \xi_x\]
\[-i\omega \xi_z = v_z\]
\[-i\omega v_z = ik_x \xi_x + ik_y \xi_y - n \xi_z.\]

which leads directly to the dispersion relation

\[\omega^4 - (\kappa^2 + k_x^2 + k_y^2 + n)\omega^2 + n\kappa^2 - 2k_y^2 q = 0\]

with fast solution

\[\omega = \pm \frac{1}{\sqrt{2}} \left( \kappa^2 + k_x^2 + k_y^2 + n + \sqrt{(\kappa^2 + k_x^2 + k_y^2 + n)^2 - 4(n\kappa^2 - 2k_y^2 q)} \right)^{1/2}\]
\[\sim \pm |k_x| \text{ as } |t| \to \infty\]

and we take \(\omega\) to indicate the positive root for definiteness. This may then be integrated exactly (see §A.4 for details). Since \(J_c\) is conserved for a wave we easily see the associated amplitude

\[\xi_x \propto |\omega|^{-1/2}\]

which gives

\[w \propto |\omega|^{+1/2}, \text{ and } \xi_y, \xi_z \propto |\omega|^{-3/2}.\]

The naive mode structure may then be written as a vector \((\xi, \dot{\xi}_z)\)

\[f^\pm \to \frac{1}{k_x} \sqrt{\frac{2}{k_y} \frac{1}{|\omega|^{1/2}}} \begin{pmatrix} k_x \\ k_y \mp 2ik_x/\omega \\ -i \end{pmatrix} \exp \left( \mp i \int_0^t \omega(t') dt' \right)\]
as \( t \to \pm \infty \), where we have included a prefactor in anticipation of our normalisation in \([2.2.4]\) and omitted the subscript \( e \) or \( i \) as the above expression is valid for both positive and negative \( t \).

There are four roots to the dispersion relation. We derive the inertial modes in the next section, but if we were to attempt to take the negative sign for the square root above we should find

\[
\omega'_\pm = \pm \frac{1}{\sqrt{2}} \left( \kappa^2 + k_x^2 + k_y^2 + n - \sqrt{(\kappa^2 + k_x^2 + k_y^2 + n)^2 - 4(n\kappa^2 - 2k_y^2q)} \right)^{1/2}
\]

which for large \(|t|\) is \( O(1/k_x) \) and does not satisfy the WKBJ approximation. It does suggest that our remaining waves will have a time dependence like \(\exp(\int (\sigma/|t'|)dt')\) for some unknown \(\sigma\).

### 2.2.3 Slow (inertial) modes

We examine the nearly-incompressible inertial modes, assuming their displacement to be perpendicular to the wavevector i.e. \( y - z \) dominated and having time dependence

\[
\xi_x, v_z \sim |t|^{\sigma}, \text{ and } \xi_y, \xi_z \sim |t|^{\sigma+1}.
\]

Then our canonical equations at leading order in \(|t|\) give

\[
0 = ik_y q \text{sgn}(t)v_z + (2 - q)\xi_y
\]

\[
\sigma \text{sgn}(t)\xi_y = ik_y v_z - 2\xi_x
\]

\[
\sigma \text{sgn}(t)\xi_z = v_z
\]

\[
0 = ik_y q \text{sgn}(t)\xi_x + ik_y \xi_y - n\xi_z.
\]

which gives

\[
\sigma_\pm = -\frac{1}{2} \mp i\theta
\]

\[
= -\frac{1}{2} \mp i\sqrt{\frac{2(2 - q)n}{k_y^2q^2} - \frac{1}{4}}
\]
which is a constant decay of $|t|^{-1/2}$, analogous to the WKBJ amplitude of the acoustic mode, and a ‘frequency’ $\theta$, although if we have $k_y^2 q^2 > 8(2 - q)n$ then these modes do not oscillate. This $\theta$ may be real or imaginary and we choose it to border the upper-right quadrant of the complex plane (i.e. one of either $\text{Im}[\theta]$ or $\text{Re}[\theta]$ is $\geq 0$ and the other is zero). Li, Narayan and Goodman[68] defined a quantity $q = -\theta/2$ that governs the spatial oscillations of the inertial modes. Had we tried to take the slow-wave solution for $\omega$ then we would not have found these modes; they do not lie in the WKBJ regime. The structure (before any normalisation) is

$$s^\pm = \begin{pmatrix} \xi \\ \xi_z \end{pmatrix} = \frac{1}{2k_y q} \sqrt{\frac{2}{k_y}} \begin{pmatrix} -k_y (2 + q \sigma) \; \text{sgn}(t) \times |t|^\sigma \\ 2k_y q \times |t|^{\sigma+1} \\ 2(2 - q)i/(\sigma + 1) \times |t|^{\sigma+1} \\ 2(2 - q)i \; \text{sgn}(t) \times |t|^\sigma \end{pmatrix}$$

which clearly brings out the $y - z$ nature of these displacements. Once again, we have omitted the subscript $i$ or $e$, but here the omission is due to the freedom in the choice of slow mode basis to be discussed in §2.2.4. When $\theta$ is imaginary these modes do not oscillate - thus we have avoided describing them as waves.

### 2.2.4 Chosen Mode Bases

We may now largely forget our Lagrangian perturbation quantities and work with our modes $f^\pm, s^\pm$. We seek a basis $\hat{f}^\pm, \hat{s}^\pm$ for $t \to \pm \infty$, orthonormalised with respect to the Hermitian form $J(\cdot, \cdot)$, but we shall find an obvious basis for only the fast modes. The slow modes’ corresponding obvious choice - that of a wave propagating in one direction - is degenerate when $\theta = 0$ and is meaningless when $\theta$ is imaginary. We shall avoid this degeneracy by taking instead a combination of inwards- and outwards-propagating slow modes as our basis, only to find a new degeneracy at $\theta = \infty$, i.e. the line $k_y = 0$. Since we would like to compare our non-axisymmetric waves to axisymmetric waves we pose a third, physically unmotivated basis with degenerate line $\theta = -1/2$ outside of the domain (recall $\text{Re}[\theta] \geq 0$). Regardless of these ambiguities, we present all three sets of results here and note that - even close to problematic lines - the physical conclusions will
be robust.

For brevity, all calculations of $J(\cdot, \cdot)$ in this Chapter shall be placed in Appendix §A.4. Naturally we have

$$J(f, s) = 0$$

so that the acoustic and inertial modes are orthogonal and we may normalise them independently.

We normalise first the acoustic modes. Applying $J(\cdot, \cdot)$ we find

$$J(f^+, f^-) = 0$$

$$J(f^\pm, f^\mp) \sim \pm 2 \frac{k_x^2}{\omega^2} \text{ as } |t| \to \infty$$

$$\sim \pm 2$$

and so choose

$$\hat{f}^+ = f^+ \quad \text{and} \quad \hat{f}^- = f^-$$

for both $t \to -\infty$ and $t \to +\infty$. The acoustic mode with positive $J_c$ has behaviour like $\exp \left( ik_x x + i \int \omega dt' \right)$, with $\omega > 0$, both before and after the swing. We relate this to the wave reflecting from the evanescent region in the spatial domain: A wave with $J_c > 0$ (i.e. a wave with $R > R_c$) with $k_x$ and $\omega$ of opposite sign travels inwards, meets the barrier and reflects at $t = 0$, thence travelling outwards; the change in direction of phase velocity is due to the change in sign of $k_x$. Similarly, the acoustic mode with $J_c < 0$ travels outwards for negative $t$ and inwards for positive $t$. Notice that these acoustic modes are non-dispersive in the limit of large $|k_x|$ and so the group and the phase velocity are asymptotically one and the same.

We attempt to normalise the inertial modes; for $\theta$ real (i.e. the inertial waves oscillating) we have

$$J(s^\pm, s^\pm) = \pm (2 - q) \theta \text{ sgn}(t)$$

$$J(s^+, s^-) = 0$$
while for $\theta$ imaginary we have

$$J(s^+, s^+) = 0$$

$$J(s^+, s^-) = -(2 - q)\theta \text{sgn}(t)$$

and the latter of these is itself imaginary. There is an obvious naive basis in the oscillating region $\theta^2 > 0$, which is simply the slowly propagating inertial waves. Then

$$\hat{s}_i^\pm = \left(\frac{2}{(2 - q)\theta}\right)^{1/2} s^\pm$$

$$\hat{s}_c^\pm = \left(\frac{2}{(2 - q)\theta}\right)^{1/2} s^\pm$$

and we see an obvious correspondence between these waves and the acoustic waves: both are normalised by a division by their frequency and both decay as $|k_x|^{-1/2}$. The dependence on the sign of $t$ can be understood by considering $\hat{s}^+ \propto \exp\left(-i\int_1^t 1/|t'| dt'\right) = \exp\left(-i\theta \text{sgn}(t) \log |t|\right)$.

There is a natural interpretation of this basis in the physical domain: consider the phase and group velocities in the $x$ direction with a ‘frequency’ of $\theta/|t|$. Then, for e.g. $\hat{s}_c^+$, the radial phase velocity

$$c^p_x = \frac{\theta/|t|}{k_x} = \frac{\theta}{k_y q |t|^2} \text{sgn}(t)$$

and the radial group velocity

$$c^g_x = \frac{\partial}{\partial k_x} \left(\frac{\theta}{|t|}\right)$$

$$= \frac{\theta}{k_y q} \frac{\partial}{\partial t} (1/|t|)$$

$$= -\frac{\theta}{k_y q |t|^2} \text{sgn}(t) = -c^p_x$$

are equal and of opposite sign. Thinking in the physical domain, these waves would start at corotation at $t = -\infty$ and propagate (in the sense of the group velocity) away until they reach the relevant Lindblad resonance at $t = 0$; they then reflect and propagate back towards corotation.
This basis cannot apply when $\theta^2 < 0$ and $\theta$ is imaginary. From our expression for the angular momentum

$$J_c(\xi) = \frac{k_y}{2} \text{Im} \left[ \dot{\xi}_z w^*_x + (2 - q)\xi^*_x \xi_y \right]$$

we require $\xi_x$ and $\xi_y$ to be out of phase for a nonzero $J_c(s)$; the first term vanishes for nearly-incompressible inertial waves. When $\theta^2 < 0$ we would have $\xi_x$ and $\xi_y$ from the above basis in phase. We may instead combine our two naive vectors $s^+$ and $s^-$ to break this phase and so find an analogous basis

$$\hat{s}_i^\pm = \left( \frac{1}{(2 - q)\theta} \right)^{1/2} (s^+ \pm is^-)$$
$$\hat{s}_e^\pm = \left( \frac{1}{(2 - q)\theta} \right)^{1/2} (s^+ \mp is^-).$$

These bases are disjoint, but neither of them covers the line $\theta = 0$, where the modes $s^\pm$ become degenerate and this normalisation fails. We call the union of the above two bases basis one. We may write instead a basis that is deliberately non-degenerate on this line

$$\hat{s}_i^\pm = \frac{1}{\sqrt{2(2 - q)}} \left( s^+ + s^- \pm \frac{1}{\theta} (s^+ - s^-) \right)$$
$$\hat{s}_e^\pm = \frac{1}{\sqrt{2(2 - q)}} \left( s^+ + s^- \pm \frac{1}{\theta} (s^+ - s^-) \right)$$

which has time dependence like $\cos(\log |t|) \pm \sin(\log |t|)/\theta$ in the $y$-component. We call this basis as basis 2. This basis passes smoothly through the line $\theta = 0$, but are degenerate on the boundary where $k_y = 0$, $\theta = \infty$. As we approach this degeneracy the two slow modes become able to mix with each other freely (see Figure 2.5b).

We may carefully construct a basis which suffers from none of these problems, and has basis 1 behaviour near $k_y \to 0$ ($\theta \to \infty$) and has basis 2 behaviour near

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\[ \theta \to 0; \text{beyond } \theta = 0 \text{ we may continue using basis 2. Consider a combination} \]

\[ \hat{s}_e^+ = \sqrt{\frac{2}{(2-q)\theta}} (c_+(\theta)s^+ + c_-(\theta)s^-) \]

and seek the mode with \( J_c = +1 \) Then

\[ J(\hat{s}_e^+, \hat{s}_e^+)^2 = \frac{2}{(2-q)\theta} J(s^+, s^+)(|c_+|^2 - |c_-|^2) \]

\[ = 2(|c_+|^2 - |c_-|^2) \]

\[ = 2. \]

This gives the simple relationship that

\[ |c_+|^2 - |c_-|^2 = 1 \]

which ensures the correct value of \( J_c \). We now seek a basis which has the desired limits as \( \theta \) varies. We consider

\[ c_- = -4\sqrt{\frac{h}{\theta}} \left( \frac{e - \theta}{1 + f\theta} \right) \]

and - on taking said limits - find that \( e = 2, f = 1/2, h = 1/256 \). We then have

\[ c_- = -\frac{1}{2} \sqrt{\frac{1}{\theta}} \left( \frac{1 - \theta/2}{1 + \theta/2} \right) \]

and so

\[ \hat{s}_e^+ = \sqrt{\frac{2}{(2-q)\theta}} \left( \sqrt{1 + \frac{11}{4\theta} \left( \frac{1 - \theta/2}{1 + \theta/2} \right)^2} s^+ - \frac{1}{2} \sqrt{\frac{1}{\theta}} \left( \frac{1 - \theta/2}{1 + \theta/2} \right) s^- \right) \]

and similarly

\[ \hat{s}_e^- = \sqrt{\frac{2}{(2-q)\theta}} \left( \frac{1}{2} \sqrt{\frac{1}{\theta}} \left( \frac{1 - \theta/2}{1 + \theta/2} \right) s^+ - \sqrt{1 + \frac{11}{4\theta} \left( \frac{1 - \theta/2}{1 + \theta/2} \right)^2} s^- \right). \]
2. Isothermal swing

As before we may find the basis for $s_i^\pm$ by swapping our expressions for $\hat{s}_i^+$ and $\hat{s}_i^-$. We call this basis 3.

2.3 Code, integration interval and timestep

We proceed to calculate $M$ and associated quantities numerically for a region of the $n - k_y$ plane. Although we took a Hermite ansatz to separate our equations we now relax our requirement that $n$ be an integer to better understand the role the vertical index plays in these swing calculations; we vary both $n$ and $k_y$ in small increments across the region of interest. Since the problem is now one dimensional, that dimension being time, we need only specify initial conditions, perform the integration, and apply $J(\cdot, \cdot)$ to the final state to calculate our complex amplitudes $a_\pm, b_\pm$.

We wish to perform our inversion when our waves are well separated in behaviour and our complex amplitudes are well converged. For simplicity we choose a fixed large integration interval such that this is true for our most extreme cases - i.e. for the small $k_y$ and large $n$ region where radial compression will be slowest to dominate the acoustic dispersion relation. Having chosen an integration interval, we shall integrate the four linked linear equations using the classical fourth-order Runge-Kutta method, which we implement in C++.

We examine the dispersion relation again

$$\omega_\pm = \pm \frac{1}{\sqrt{2}} \left( \kappa^2 + k_x^2 + k_y^2 + n + \sqrt{(\kappa^2 + k_x^2 + k_y^2 + n)^2 - 4(n\kappa^2 - 2k_y^2q)} \right)^{1/2}$$

and ask by what time will our force balance for an acoustic mode will be predominantly radial, given small $k_y$ and large $n$. We see that our most restrictive condition shall be

$$k_y^2 \gg n$$

$$\Leftrightarrow k_y^2 q^2 t^2 \gg n$$

$$\Rightarrow |t| \gg \frac{\max(\sqrt{n}/qk_y)}{\min(k_yq)}.$$
We will examine \( n \in [0.1, 15], k_y \in [0.1, 5] \) and so we want

\[
|t| \gg 20\sqrt{15}/3 \approx 26
\]

and so we pick \( t_{\text{init}} = -300, t_{\text{final}} = +300 \) unless specified otherwise.

We initialise our timestep as

\[
\delta t = \frac{1}{100} \times \frac{2\pi}{k_y q |t_{\text{init}}|} \times \min\left(1, k_y^2\right)
\]

which is a small prefactor \((1/100)\) multiplying the asymptotic period of an acoustic wave at the beginning of the integration interval. The inertial mode frequency \( \theta \sim 1/k_y \) for small \( k_y \), and so we include also a factor of \( k_y^2 \) when \( k_y < 1 \) to ensure cases with smaller \( k_y \) do not have substantially larger timesteps. The inertial modes might retain time behaviour \(|t|^{\theta}\) down to when \(|t|\) approaches \( O(1) \) we do not increase our timestep during integration. We must also check that our inertial modes are properly resolved around \( k_y q |t| = 1 \), where the mode separation ought be breaking or have broken. We could rewrite the inertial mode’s time dependence \(|t|^{\theta}\) as \( \exp(i \int \theta/|t'| \, dt') \), so the period we must resolve at \(|t| = 1\) will be

\[
\frac{2\pi |t|}{|\theta|} = \frac{2\pi}{\sqrt{2(2-q)n - \frac{1}{4}k_y^2q^2}}
\]

which becomes small only when \( k_y \) becomes large for fixed \( n \) (or vice versa); for e.g. \( q = 3/2, n = 16 \) and \( k_y \) vanishingly small we would have a period of \( \pi/2 \), still quite large. We thus do not concern ourselves with it.

We check the conservation of \( J_c(\xi) \) over the run. If the \( J_c \) calculated deviates from its initial value by more than \( 10^{-6} \) (a deviation of 0.0001% for unit initial values) at any time then the integration is halted, the timestep reduced, and the integration restarted from the very beginning of the interval. This continues until the run completes with \( J_c \) (sufficiently) constant at all times. There is also a minimum timestep implemented of \( 10^{-9} \); this was never reached in our runs.

For the purposes of checking our code and presenting specific cases we repeat our integrations in \texttt{Matlab} with several integrators; we performed chosen
runs using the integrators ode23tb, ode23s, ode113 and ode45. No significant differences were seen, although some methods required stricter tolerances to conserve $J_c$ and took longer to run. All plots in this Chapter were generated using ode45 with a RelTol value of $10^{-3}$ and an AbsTol value of $10^{-6}$ unless otherwise specified.

2.4 Results

Figure 2.3: Real part of $\xi_x$ against time in purple with the asymptotic reconstructions overlaid on top in green. The initial conditions here were an inertial mode of positive unit action, and we clearly see acoustic modes generated as $k_x$ passes through 0. The agreement quickly becomes excellent away from this point. This plot is reminiscent of the generation of acoustic modes by the balanced solutions considered by Heinemann and Papaloizou[50].

We calculate $M$ for $n \in [0.1, 1.5]$ and $k_y \in [0.1, 5]$, and we plot our results in Figures 2.4 - 2.6. The three sets of results are in good agreement, showing the insensitivity of our physical results to the basis chosen for the inertial modes even
Figure 2.4: Results for Basis 1 with \( n \) as the abscissa and \( k_y \) as the ordinate. (a) The acoustic action generated by unit inertial action, which approaches 1 for \( n = 1, k_y \to 0 \) and is much larger near the degenerate line \( \theta = 0 \), (b) the log of the norm \( M \) of \( \mathbf{M} \) concentrated around \( \theta = 0 \) which has values as high as 30, (c) the reflected amplitude \( |a_+| \) of an acoustic mode, which never goes significantly above 1 and is minimal for \( n = 1, k_y \to 0 \), and (d) the small transmitted amplitude \( |a_-| \) of an acoustic mode which falls off rapidly as \( n \) increases. The point \( n = 9, k_y = 4 \) is directly on the degenerate line and so is not considered for this basis. There is a defect, which we believe to be numerical, confined to nearby the line \( \theta = 0 \) where the inertial modes become degenerate.
2. Isothermal swing

Figure 2.5: Results for Basis 2. (a) The acoustic action generated by unit inertial action, which is maximal for \( n = 1, k_y = 0 \) and slightly exceeds 1 there, (b) the log of the norm \( M \) of \( \mathbf{M} \) concentrated around \( k_y = 0 \) with maximum values around 150, (c) the reflected amplitude \( |a_+| \) of an acoustic mode, which never goes significantly above 1 and is minimal for \( n = 1, k_y \to 0 \), and (d) the small transmitted amplitude \( |a_-| \) of an acoustic mode which falls off rapidly as \( n \) increases. Again the region where the inertial mode basis goes degenerate appears to have some defects.
Figure 2.6: Results for Basis 3. (a) The acoustic action generated by unit inertial action, maximal for $n = 1$, $k_y \to 0$ (and $A = 0.86 ...$ for $n = 1$, $k_y = 0.1$), (b) the log of the norm $M$ of $\mathbf{M}$ concentrated around $n = 0$ (note different scale to previous plots), (c) the reflected amplitude $|a_+|$ of an acoustic mode, which never goes significantly above 1 and is minimal for $n = 1$, $k_y \to 0$, and (d) the small transmitted amplitude $|a_-|$ of an acoustic mode which falls off rapidly as $n$ increases.
2. Isothermal swing

Figure 2.7: $A$ as a function of $k_y$ for the case $n = 1$ using basis 3. Clearly, $A \to 1$ as $k_y \to 0$. 
Figure 2.8: Squared amplitudes as a function of $n$ for the case $k_y = 1$ using basis 3. We show $A = |a_+|^2 + |a_-|^2$ (green line), the quantity $|a_-|^2$ (purple line) and the corresponding 2D results from [78] are shown as green and purple boxes on the ordinate. Below $n \approx 0.078$ there is net acoustic amplification and below $n \approx 0.041$ there is straight over-reflection of the incident wave.
close to degenerate lines. We also show the numerical integration and subsequent reconstruction of a typical case in Figure 2.3. Before the swing we have only an inertial mode very slowly oscillating (purple), reminiscent of the balanced solutions plotted by Heinemann and Papaloizou[50], and after the swing we see an inertial mode with a fast acoustic oscillation overlain. In the same figure we have two green lines; our expression for the inertial mode used as initial condition for \( t < 0 \) and our inversion - a sum of inertial and acoustic modes - for \( t > 0 \). The agreement is excellent for \( |t| > 10/\Omega \).

The production of acoustic action from inertial action is most successful for low \( k_y \) and low \( n \), with a real maximum located at \( n = 1, k_y \to 0 \). Our ‘mode mixing’ quantity \( M \) is shown to be strongly basis dependent; if there is a region on which the inertial modes become degenerate then \( M \) comes sharply peaked around that region. The relatively low amount of mixing for basis 3 (where \( \log M \leq 0.16 \)) may be taken as a vindication of it as an appropriate choice of basis for numerical integration.

We find that our transmission amplitude for an incoming \( |a_-| \) tends to the 2D result (equation 2.1), derived by Narayan, Goldreich and Goodman[78] and by Nakagawa and Sekiya[77], as \( n \to 0 \) continuously; this is despite the difference between the spatial and temporal initial conditions outlined in Figure 2.2. Transmission is significant for both \( n = 1 \), with a maximum amplitude of \( |a_-| \approx 0.051 \), and \( n = 2 \), with maximum transmission amplitude of \( |a_-| \approx 0.027 \), but falls off rapidly with increasing \( n \) or with increasing \( k_y \) as the width of the evanescent regions in the spatial domain broadens. However, at no point do we find over-reflection in the sense of \( |a_+| > 1 \) for integer \( n > 0 \); instead the conservation of \( J_c \) is achieved by the production of inertial modes of the necessary sign. Indeed, for all but a small region near \( n = 1, k_y = 0 \) (to be discussed on page 62) there is near-perfect reflection of the incident acoustic mode.

The combination of perfect reflection and nonzero transmission leads to an interesting possible route to instability not dissimilar to the 2D instability described by Narayan, Goldreich and Goodman[78]. In the 2D case if an acoustic mode may be trapped between the corotation region and a reflecting boundary - such as the free outer edge of the disc - then the two spatially separated modes will bounce back and forth, each amplifying the other with every iteration. This
gives rise to slowly growing exponential solutions. In the 3D case we might imagine not one boundary, but two - such as the inner edge of the disc or the gap opened by the formation of a massive planet. Then an acoustic mode trapped in the region outside corotation may feed a mode trapped inside corotation and vice versa; the interaction between the two trapped modes is thence a potential route to instability. A necessary condition for this instability would then be

\[ |a_+|^2 + |a_-|^2 > 1 \]

in the spatial domain for net amplification to occur in each interaction. This condition is satisfied only for unphysically small values of \( n \) (see Figure 2.8) where we approach the 2D case. Obviously the quantity \( A = |a_+|^2 + |a_-|^2 \) must become greater than one before \( |a_+|^2 \) does; both quantities are shown in Figure 2.8.

The region where reflection fails is also of considerable interest. For \( n = 1 \) and small \( k_y \) we see that an ingoing acoustic mode is efficiently converted into inertial modes. Within this region our inertial modes are oscillatory and we may think in terms of our intuitive first basis; on examination of the transition matrix \( M \) it can be seen that an outgoing acoustic mode converts mainly into an outgoing (in the sense of group velocity) inertial mode. In a similar fashion, an incident ingoing inertial mode is efficiently converted into an ingoing acoustic mode. This conversion becomes total for \( n = 1 \) as \( k_y \to 0 \) (see Figure 2.7). If we reconsider the dispersion relation Li, Narayan and Goodman used to locate evanescent regions for the special case with \( n = 1 \) and a Keplerian disc (where \( \kappa^2 = \Omega^2 \)) we have

\[
\frac{k^2 c_s^2 \omega^2}{\Omega^4} = \left[ \frac{\omega}{\Omega} - m \right]^2 - 1 \left[ \frac{\omega}{\Omega} - m \right]^2 - n + O(m^2 c_s^2 / R^2 \Omega^2)
\]

we see there is no evanescent region - or, more accurately, there is an evanescent region of width \( O(mH/R) = O(k_y H) \) which vanishes with vanishing \( k_y \) (see Figure 2.11). Figures 2.9 and 2.10 show that we do not see this effect for small \( k_y \) if \( n = 2 \); we can also see that the mode amplitudes for cases with weaker
shear or smaller $k_y$ take longer to converge to their asymptotic values. This explains the extremely low transmission amplitudes calculated by Li, Narayan and Goodman\cite{68} for $n = 1$. These $n = 1$ modes have $u_x \propto z$ and $u_z$ constant in $z$ such as could be excited by an embedded planet with an orbit very slightly inclined to the disc.

This total mode conversion is consistent with the previous work on the propagation of axisymmetric bending waves in a cylindrical geometry by Bate et al\cite{11}. They discuss the forced generation of waves at resonances; these waves then propagate through the disc and break via nonlinear wave steepening. They examined as a special case is the mixed vertical and Lindblad resonance (i.e. the case when $n = 1$); they find that the emitted rightward travelling wave carries an energy flux equal to that of the simultaneously emitted leftward travelling wave. Here we are not considering the generation of wave at the mixed resonance but the fate of a mode incident upon it.

Bate et al.\cite{11} followed the nonlinear propagation of isothermal modes through a disc. Wave steepening is everywhere weak for low-amplitude waves and wave breaking is correspondingly slow. With a nonzero $k_y$ the shear will quickly increase the radial wavenumber $k_x$, and for the acoustic waves we described above we have a WKBJ amplitude $u_x \sim -i\omega \xi_x \propto |t|^{1/2}$ such that the wave amplitude will quickly grow away from corotation. An amplitude growing with distance from corotation means these waves will reach the nonlinear regime (and then break) much faster than axisymmetric waves. We may imagine some weak source of nearly-axisymmetric inertial modes close to corotation which grow to some finite amplitude at the mixed resonance, again due to shear, before converting to an acoustic mode that quickly breaks. Acoustic modes that transfer angular momentum efficiently can thus be generated by any low-frequency source of inertial modes.

We compare our results in the shearing wave domain to those in the spatial domain found by Li, Goodman and Narayan\cite{68}; their relevant results are given in our current notation in Table \ref{tab:2.1}. We have good agreement with their reflection coefficients (fourth and sixth columns) and - for the first two cases - acceptable agreement with their transmission coefficients; what small disagreement there is is unsurprising given that we are calculating exponentially small coefficients.
Figure 2.9: The conversion of positive inertial action (green line) into positive acoustic action (purple line) for $n = 1$ as $k_y$ approaches zero; the diagram shows e.g. $J(\hat{s}^+, \xi)$ (green line) against time; the region around $t = 0$ shows where the waves are not well separated and our projection onto our basis invalid. The smaller the $k_y$ the slower the convergence, and the wider the region around $t = 0$ where the separation into modes breaks down. For the $k_y = 0.01$ case an estimate of interval similar to the one shown earlier gives that $|t_{\text{init}}|$ must be $\gg 60$. As $k_y \to 0$ we approach total conversion between inertial and acoustic modes.
2. Isothermal swing

Figure 2.10: The same as Figure 2.9 for $n = 2$. The incident inertial mode is now reflected rather than converted.
Figure 2.11: Schematic showing 3D wave tunnelling as in Figure 2.1, but for \( n = 1 \), \( k_y \) vanishingly small, and for Keplerian rotation \( (q = 3/2) \). The (b & d) inner vertical and Lindblad resonances now lie at the same radius, as do (c & e) the outer resonances; we never have \( k_x^2 < 0 \).

We can also make use of the corotation absorption coefficient calculated by Li, Narayan and Goodman\[68\] to obtain a second estimate for \( |a_+|^2 \). Let

\[
|a_{+2}|^2 = |M_{b+a_+}|^2 \times \exp(-2\pi \theta) \times |M_{a_--b_-}|^2 \times |a_-|^2
\]

where \( \theta \) for this set of parameters is real. The calculation for \( |a_{+2}|^2 \) is broken down as follows: The outgoing (leading) acoustic mode with amplitude \( |a_-| \) is converted into an outgoing (trailing) inertial mode via \( |M_{a_--b_-}| \), then the outgoing (trailing) inertial mode passes through (and is strongly attenuated by) corotation to become an outgoing (leading) inertial mode by way of the corotation absorption coefficient \( \exp(-2\pi \theta) \), and finally the outgoing (leading) inertial mode is converted into an outgoing (trailing) acoustic mode via \( |M_{b+a_+}| \). For the first row of Table \( 2.1 \) this gives

\[
|a_{+2}|^2 = 0.781^2 \times \exp(-6.80 \cdots) \times 0.781^2 = 4.60e-7
\]
2. Isothermal swing

Table 2.1: Table from Li, Goodman and Narayan[68] in our current notation and with corresponding shearing-wave results. The quantity $a_{+2}$ is defined in the text near to a discussion of these results on page 66.

| n   | $\Omega_\perp/\Omega$ | $q$ | $k_y$ | $|a_-|^2$ | $|a_+|^2$ | $|a_-|^2$ | $|a_+|^2$ | $|a_{+2}|^2$ |
|-----|-----------------------|-----|-------|----------|----------|----------|----------|-------------|
| 1... | 1.0                   | 1.5 | 0.3   | 3.89e-1 | 4.60e-7 | 3.89e-1 | 4.62e-7 | 4.60e-7     |
| 1.0  | 1.395                 | 0.4 | 4.94e-1| 1.60e-6 | 4.94e-1 | 1.77e-6 | 1.60e-6 |             |
| 0.8  | 1.98                  | 0.24| 9.38e-1| 7.63e-3 | 9.51e-1 | 4.55e-3 | N/A      |             |

For the second row gives

$$|a_{+2}|^2 = 0.711^2 \times \exp(-5.99 \ldots) \times 0.711^2 = 1.60e-6.$$ 

i.e. excellent agreement with the spatial results. This operation is impossible for the third row because there the inertial modes do not oscillate and so do not split intuitively into spatially separated propagating waves. We attribute the moderate disagreement between our results and theirs to the extreme nature of that case, with $\Omega_\perp = 0.8\Omega$ and $q = 1.98$, and the inherent difficulty of calculating exponentially small transmission coefficients.

2.5 Conclusions

We have investigated the mode conversion of the canonical non-axisymmetric acoustic and inertial modes in an isothermal accretion disc with vertical gravity, and paid particular attention to the to the temporal structure of the inertial modes. Using numerical integration and asymptotic analysis we have revealed the rich mathematical structure of a simple separable system. We have reproduced the results of Li, Narayan and Goodman in the shearing wave domain and performed a thorough investigation of the parameter space. We have given what we consider to be the natural method to consider over-reflection of separable waves in the shearing wave domain.

By perturbing around the inviscid momentum equation and the continuity
equation, we found a system we could analyse using shearing-waves and a Hermite basis. By integrating once in time and considering the conservation of vorticity we found our canonical Lagrangian equations after the formalism of Friedman and Schutz, as well as derived and made use of the important Hermitian form $J(\cdot, \cdot)$ for 3D shearing waves. These equations were fourth-order in time and lacked solutions akin to the ‘balanced solutions’ used by Heinemann and Papaloziou to perturb the vorticity.

In the limit of large $|k_x|$ we derived the modes, or asymptotic solutions, of the canonical Lagrangian equations and characterised them as predominantly radial acoustic modes or vertical inertial modes. Using $J(\cdot, \cdot)$, our Hermitian form, and $J_c$, the leading-order perturbation angular momentum, we sought an orthonormal basis for the modes of the problem. Finding no unambiguous choice for our inertial waves, we derived three candidate bases and explored their virtues and faults. We then used numerical integration to calculate the transition matrix $M$ as a function of $n$ and $k_y$.

Our physical results were robust and did not rely on our choice of basis for the inertial modes. We identified a potential source of instability akin to the 2D over-reflection instability discussed by Narayan, Goodman and Goldreich and found that our system only fulfilled a necessary condition for said instability for unphysically small values of $n$; we also found good agreement between our shearing-wave calculation and the spatial calculation of Li, Goodman and Narayan, after the application of their analytic corotation absorption coefficient, although our direct calculation of exponentially small transmission coefficients did not precisely reproduce their results. We found that the non-axisymmetric corrugation waves have the special property of being able to be converted between inertial and acoustic modes without loss as $k_y \to 0$. If a source - such as an inclined embedded planet - of these weak, nearly axisymmetric inertial modes exists in a disc then we have shown that they will be efficiently converted into acoustic modes that transport angular momentum and quickly break as they are sheared.
Chapter 3

Quasilinear EMF calculation with vertical gravity

[Deleted magnets, how do they work?]

Miracles, Insane Clown Posse

3.1 Motivation

We wish to make progress towards a human understanding of the accretion disc dynamo seen in turbulent MRI simulations with vertical gravity. We consider an asymptotic expansion of linear MRI/undular modes, the small parameter being the poloidal wavelength compared to the scale height $H$, and carefully construct an analytic expression for the quasilinear EMFs. We shall examine the structure of these EMFs and relate them to both classical dynamo theory and recent numerical simulations.

Numerical simulations of the MRI, to be discussed in the introduction to Chapter 5, have revealed a wealth of behaviour arising from the simple combination of rotation and shear acting on a magnetic field. As is often the case with nonlinear systems, it is difficult or impossible to make analytic progress, even though analytic models are vital to translate the results of a simulation into physical impetus
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that we may properly internalise. Ogilvie and Lesur\textsuperscript{[66]} (hereafter LO-A, for ‘analytic’) proposed a model for a dynamo based on the MRI, analytically deriving expressions for the radial and toroidal EMFs by considering the incompressible linear MRI of a non-uniform toroidal field with large poloidal wavenumber. They found an EMF that, for weak fields, could create radial field from toroidal. This radial field would then be sheared out to reinforce the original toroidal field, similar to the rejuvenating mechanism of an $\alpha - \Omega$ dynamo. We here characterise the form of their EMF as analogous to an anisotropic negative diffusion. We propose to extend their result to the isothermal case with vertical gravity, i.e. including the magnetic buoyancy instability explored in Chapter 4.

It will be useful for our later analysis if we briefly discuss previous analytic dynamo and transport models. Consider a mean magnetic field $(B_x, B_y)$ governed by

\[
\frac{\partial B_x}{\partial t} = -\frac{\partial}{\partial z} (\alpha B_y) \\
\frac{\partial B_y}{\partial t} = -q\Omega B_x
\]

i.e. the classical $\alpha - \Omega$ dynamo equations without diffusion: the $\alpha$-effect creates radial field from toroidal, and then shear (the $q\Omega$ term) turns this radial field back into toroidal. This form of closure model was discussed famously by Moffatt\textsuperscript{[47]} and Krause and Rädler\textsuperscript{[65]}, amongst others; given a large-scale flow (e.g. isotropic, rotating), the relevant form of $\alpha$ (properly $\alpha_{ij}$, a tensor) may be derived given certain assumptions about linearity. Great efforts have been made in recent years to measure $\alpha$ from simulations by use of the “test-field method” described in Rheinhardt and Brandenburg\textsuperscript{[86]}, whereby a fully nonlinear simulation is overlain with extra passive magnetic fields to derive the effective $\alpha$ caused by the turbulent flow. This method has been revised and refined (e.g. “resetting” the test field\textsuperscript{[52], 59}, applying the method of oscillating sines as in Tobias and Cattaneo\textsuperscript{[103]}) but as yet there has been no consensus as to whether an $\alpha$-effect can be quantitatively predicted given a flow (that is to say, without simply simulating the system and measuring it after the fact).

The simple system above has a translational symmetry in $z$ (with $\alpha$ presum-
3. Analytic Quasilinear EMFs

ably arising from some local helicity density). This translational symmetry would not hold were we to add vertical gravity, and the properties of the dynamo would change. Indeed, nonlinear MRI simulations with vertical gravity appear markedly different from those without, as evidenced by Fromang and Papaloizou[37] and a partial response from Davis, Stone and Pessah[22]. Fromang and Papaloizou showed that for shearing box simulations without vertical gravity and without explicit diffusivities and with zero net magnetic flux, the Reynolds stresses converged to zero with increasing numerical resolution, i.e. ideal simulations were inherently under-resolved. Davis, Stone and Pessah[22] then showed that with vertical gravity these stresses would converge to nonzero values, suggesting that the presence of the vertical scale height and density gradients had fundamentally changed the nature of the turbulence and the dynamo.

Gressel[44] applied the test-field method to a zero net flux isothermal shearing box with vertical gravity and, in the same paper, investigated a mean-field EMF of the form

$$\bar{\vec{E}} = \alpha \bar{\vec{B}} - \eta \nabla \times \bar{\vec{B}}$$

for a 1D system intending to model the horizontally averaged fields and associated EMFs of a shearing box with vertical gravity. There, $\alpha$ was allowed to vary in time with $\dot{\alpha} \sim -\frac{\partial}{\partial z} (\alpha \bar{u}_z(z))$ representing a vertical advective flux of magnetic helicity density (this evolution equation in turn based on Equation 17 of Brandenburg et al.[16]). This mean field model was able to qualitatively reproduce the “butterfly diagram” of their nonlinear shearing box simulations with vertical gravity, while the test-field method showed an $\alpha$ antisymmetric around the midplane, but also with a consistent sign reversal for $|z| < H$ i.e. near the midplane. This change in behaviour tallies with their related result that buoyancy becomes dominant only for $|z| \geq 1.5H$, with the midplane EMFs dominated by small-scale turbulence. Crucially, they found the vertical propagation of field had a pattern migration speed ‘independent of the bulk motion of the flow’.

Our analytic work in this Chapter is similar in motivation to Ferriz-Mas et al.[25], who investigated the quasi-linear EMFs resulting from magnetic buoyancy in differentially rotating stars; they aimed towards a closed dynamo cycle.
limited by the eruption of magnetic flux. Our interest in the pattern speed echoes Kitchatinov and Pipin[63], who used a spectral expansion of perturbation quantities to estimate the rise time of flux tubes through the solar convection zone, work extended by Davies and Hughes[21] examining the convection zone as a whole. Thélen[101] considered an $\alpha$-effect proportional to the derivative of the background toroidal field to mimic the mean-field action of a buoyancy instability, with $\omega$-quenching provided by the Malkus-Proctor effect (where the dynamo is quenched by the Lorentz force driving a large-scale flow[110]). We disregard the Malkus-Proctor effect since we are considering an astrophysical disc where the weak field can be assumed not to affect the strong background shear. Thélen[100] also discussed dynamo models with $(+k_x, +k_y)$ versus $(+k_x, -k_y)$ (i.e. oppositely propagating dynamo waves) and their effect on the dynamo; we shall refer to this again in §3.5.

In this Chapter we shall extend and simplify the calculation of LO-A by considering the MRI in an isothermal atmosphere with general poloidal effective gravity and varying toroidal magnetic field; we shall calculate analytically the quasi-linear radial and toroidal EMFs that arise from an arbitrary slowly-varying background. We shall find that we are able to reproduce the incompressible and zero-gravity EMFs, and find two new terms: one related to vertical migration of toroidal flux, and one classical (nonlinear in $B_y$) $\alpha$-effect - in the sense of an $\alpha - \Omega$ dynamo - both terms to be proportional to gravity (and so, for an accretion disc, weakest at the midplane and strongest for large $|z|$). We shall consider several limits of physical interest.

3.2 Governing equations and asymptotics

In fully nonlinear 3D simulations, the horizontally averaged toroidal field evolves over multiple orbits while the MRI has an optimal growth rate of $O(\Omega)$; there is a clear separation of timescales. We may therefore consider instabilities whilst treating non-perturbation quantities as a static background, and since in shearing box simulations the radial field $B_x$ is in general an order of magnitude smaller than the toroidal field $B_y$ we shall neglect $B_x$ in its entirety during this Chapter.
3. Analytic Quasilinear EMFs

At no algebraic cost we may extend our analysis to allow variation in the radial direction as well as vertical; we thus consider some general poloidal variation of a purely toroidal background field, with some general effective poloidal gravitational force. Consider a background in magnetohydrostasis, with

\[ c_s^2 \nabla_p \rho = -\frac{1}{2} |B|^2 + \rho g_p \]

where a subscript \( p \) indicates the poloidal plane and for the shearing sheet \( g_x \) will contain a contribution \(-2q \Omega^2 x\) from the Coriolis force acting on the background flow. On this background we consider some linear perturbation with all perturbation quantities \( \propto \exp(ik_y y) \). We have the momentum equations

\[
\begin{align*}
\rho (\partial_t + ik_y U_y) u_x - 2\Omega \rho u_y &= -\partial_x \Pi' + ik \cdot B b_x + \rho' g_x + \rho \nu \nabla^2 u_x \\
\rho (\partial_t + ik_y U_y) u_y + (2\Omega + \partial_z U_y) \rho u_x &= -ik_y \Pi' + ik \cdot B b_y + (b_x \partial_x + b_z \partial_z) B_y + \rho \nu \nabla^2 u_y \\
\rho (\partial_t + ik_y U_y) u_z &= -\partial_z \Pi' + ik \cdot B b_z + \rho' g_z + \rho \nu \nabla^2 u_z
\end{align*}
\]

the induction equations

\[
\begin{align*}
(\partial_t + ik_y U_y) b_x &= ik \cdot B u_x + \eta \nabla^2 b_x \\
(\partial_t + ik_y U_y) b_y + (u_x \partial_x + u_z \partial_z) B_y &= ik \cdot B u_y - B_y \Delta + b_x \partial_x U_y + \eta \nabla^2 b_y \\
(\partial_t + ik_y U_y) b_z &= ik \cdot B u_z + \eta \nabla^2 b_z
\end{align*}
\]

and the mass equation, total pressure equation, and solenoidality condition

\[
\begin{align*}
(\partial_t + ik_y U_y) \rho' &= -\rho \Delta - u_z \partial_z \rho + \nu_m \nabla^2 \rho' \\
\Pi' &= c_s^2 \rho' + B_y b_y \\
0 &= \partial_x b_x + ik_y b_y + \partial_z b_z
\end{align*}
\]

where we have written the velocity divergence \( \Delta = \partial_x u_x + ik_y u_y + \partial_z u_z \). We shall in this Chapter assume that \( U_y \) is a linear function of only \( x \), writing \( \partial_x U_y = -q \Omega \) and \( \partial_z U_y = 0 \). Note that we have included a fictitious “mass diffusion” that will
later allow a crucial simplifying assumption. For our asymptotic ordering of variables we refer to Ogilvie’s chapter in *The Solar Tachocline* [38], so as to properly select the unstable slow mode while considering a large poloidal wavenumber (in a manner similar to Gilman [41], who considered an extremely large azimuthal wavenumber); we do not wish to consider the strongly compressible acoustic waves as in Chapter 2. We write this large poloidal wavenumber as $k_p/\epsilon$, with $\epsilon \ll 1$ so that $k_p H$ is $O(1)$. With the velocity and magnetic components defined to be $O(1)$, we take the perturbation total pressure $\Pi'$ to be $O(\epsilon)$ while $\Delta = \nabla \cdot \mathbf{u}$ is $O(1)$ rather than $O(1/\epsilon)$. Our assumption of large poloidal wavenumber and a separation of spatial scales allows us to make use of a WKBJ-type ansatz, namely

$$u_x = (u_{x0}(x,z) + \epsilon u_{x1}(x,z,t) + O(\epsilon^2)) \exp \left( \int_0^t s(x,z,t') \, dt' + i\Phi(x,z)/\epsilon \right)$$

where $\nabla \Phi = k_p$, and a similar asymptotic expansion for the other components of $\mathbf{u}$ and $\mathbf{b}$. For $\Delta$ we take

$$\Delta = (\Delta_0(x,z) + \epsilon \Delta_1(x,z,t) + O(\epsilon^2)) \exp (\ldots)$$

since at $O(1/\epsilon)$ we shall take $i k_x u_{x0} + i k_z u_{z0} = 0$. For $\Pi'$ we avoid the fast mode by assuming an asymptotically small pressure perturbation

$$\Pi' = \epsilon (\Pi'_1(x,z) + \epsilon \Pi'_2(x,z,t) + O(\epsilon^2)) \exp (\ldots).$$

Altogether, this represents a nearly incompressible high wavenumber perturbation with an amplitude variation over length scales $O(H)$. Finally, since our background and poloidal wavenumber is assumed to be slowly changing then we may expand the growth rate

$$s(x,z,t) = s_0(x,z) + \epsilon s_1(x,z,t) + O(\epsilon^2)$$

which is to say that our leading-order growth rate relies only on the instantaneous properties of the background. We could extend the above approximation to include the radial field $B_z$ at first order (and in the squared Alfvén frequency, which would become $(k_y B_y + k_z B_z)^2/\rho$, but in this Thesis we make no significant
use of these terms and confine them wherever possible to Appendices.

With the above machinery, our primary goal in this Chapter will be to calculate the leading-order contributions to the radial EMF, expanded as

\[ \mathcal{E}_x = \mathcal{E}_{x0} + \epsilon \mathcal{E}_{x1} + O(\epsilon^2), \]

and the toroidal EMF,

\[ \mathcal{E}_y = \mathcal{E}_{y0} + \epsilon \mathcal{E}_{y1} + O(\epsilon^2), \]

and under these assumptions we shall also calculate the vertical EMF \( \mathcal{E}_{z0} \).

### 3.3 Leading order calculations

We shall find it convenient to use the notation \( E_t = \exp(2\text{Re}[s_0]t) \), the amplitude of the exponential squared, and \( |\tilde{u}_{z0}|^2 = |u_{z0}|^2 E_t \). Our leading-order equations are

\[
\rho \left( s_0 + ik_y U_y + k^2_p \nu \right) u_{x0} - 2\Omega \rho u_{y0} = -ik_x \Pi_0' + i \mathbf{k} \cdot \mathbf{B} b_{x0} + \rho_0' g_x \\
\rho \left( s_0 + ik_y U_y + k^2_p \nu \right) u_{y0} + (2 - q) \Omega \rho u_{x0} = +ik \cdot \mathbf{B} b_{y0} + (b_{x0} \partial_x + b_{z0} \partial_z) B_y \\
\rho \left( s_0 + ik_y U_y + k^2_p \nu \right) u_{z0} = -ik_z \Pi_0' + i \mathbf{k} \cdot \mathbf{B} b_{z0} + \rho_0' g_z
\]

for momentum,

\[
( s_0 + ik_y U_x + k^2_p \eta ) b_{x0} = i \mathbf{k} \cdot \mathbf{B} u_{x0} \\
( s_0 + ik_y U_y + k^2_p \eta ) b_{y0} + (u_{x0} \partial_x + u_{z0} \partial_z) B_y = i \mathbf{k} \cdot \mathbf{B} u_{y0} - B_y \Delta_0 - q \Omega b_{x0} \\
( s_0 + ik_y U_y + k^2_p \eta ) b_{z0} = i \mathbf{k} \cdot \mathbf{B} u_{z0}
\]

for induction, and

\[
( s_0 + ik_y U_y + k^2_p \eta_m ) \rho_0' = -\rho \Delta_0 - u_{z0} \partial_z \rho \\
0 = c_x^2 \rho_0' + B_y b_{y0} \\
0 = ik_x u_{x0} + ik_z u_{z0}
\]

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for mass (again, with a fictitious mass diffusion), total pressure and poloidal
solenoidality. These equations are algebraic, involving no derivatives of pertur-
bation quantities, and so writing e.g. \( \gamma_\nu = (s_0 + ik_\nu U_y + k^2_\nu) \) and similar may we
easily find a dispersion relation

\[
B_y \left[ \left( 1 + \frac{\omega^2_a}{\nu_\nu \gamma_\eta} + \frac{\nu^2_a}{c^2_a} \right) D_k B_y - \frac{B_y G_k}{c^2_k} - \left( \frac{2}{\nu_\nu} - q \frac{(\gamma_\eta - \gamma_\nu)}{\gamma_\nu \gamma_\eta} \right) \Omega ik_y k_z B_y \right] \left( 2\Omega ik_y k_z - \frac{G_k}{c^2_k} \right) \]

\[
-\nu_\eta \rho \left[ 1 + \frac{\omega^2_a}{\nu_\nu \gamma_\eta} + \frac{\nu_\eta \nu^2_a}{\gamma_\eta c^2_a} \right] \left( k^2 + k^2_z \frac{(\gamma_\eta - \gamma_\nu)}{\gamma_\nu} \right) + \frac{1}{\gamma_\nu} \left( \kappa^2 k^2_z + 2\Omega ik_y k_z B_y D_k B_y \right) \right] = 0
\]

and an eigenmode (given in Equation B.1). We have written \( \omega^2_a = (k \cdot B)^2 / \rho \), the
square of the Alfvén frequency; and \( \omega^2_c = \frac{\omega^2_a}{c^4_a + c^4_s} \), the square of the cusp frequency,
which appears as a unit because of its role in the toroidal MRI as the azimuthal
restoring force (Foglizzo and Tagger[29]). We have also written

\[
D_k = k_x \frac{\partial}{\partial z} - k_z \frac{\partial}{\partial x}
\]

\[
G_k = k_x g_z - k_z g_x
\]

which arise from terms like \( (u_x 0 \frac{\partial}{\partial x} + u_z 0 \frac{\partial}{\partial x}) \) and so represent derivatives and
gravitational acceleration parallel to the poloidal motion. For the toroidal MRI
without vertical gravity we would find a purely real \( \gamma \) as used in LO-A,
but here, with gravity, the growth rate is complex and the calculation significantly
more involved.

The dispersion relation in Equation 3.2 contains both the MRI and the undular
instability; if we take \( k_z \gg k_x \) to select predominantly horizontal motion (an MRI
polarisation) and neglect radial derivatives we recover the MRI dispersion relation

\[
(\gamma^2 + \omega_a^2)^2 + \kappa^2 \gamma^2 - 2q \Omega^2 \omega_a^2 + \gamma^2 (\gamma^2 + \omega_a^2 + \kappa^2) \frac{\nu^2_a}{c^2_s} = 0
\]

while if we take \( k_x \gg k_z \) to select predominantly vertical motion (a buoyant
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polarisation) we recover the dispersion relation for the undular instability

\[ \left( 1 + \frac{\omega^2}{\gamma^2} + \frac{v_a^2}{c_s^2} \right) \left( \gamma^2 + \omega^2 - \frac{g_z B_y B'_y(z)}{\rho} \right) + v_a^2 \frac{g_z^2}{c_s^2} = 0 \]  

(3.4)

with \( \gamma \) purely real. We can write a minimal condition for instability (with \( k_y \to 0 \))

\[ \frac{\partial}{\partial z} \left( \log B_y \right) < \frac{g_z}{c_s^2 + v_a^2} \]

which (after eliminating \( g_z \) via vertical magnetohydrostasis) becomes the familiar Newcomb condition for the interchange instability in an atmosphere with Brunt-Väisälä frequency \( N = 0 \)

\[ g_z \frac{\partial}{\partial z} \left( \log \frac{B_y}{\rho} \right) > \frac{c_s^2 N^2}{v_a^2} = 0. \]

Although it naively seems that our condition for instability makes the undular instability more stable with increasing height we must remember that our Alfvén speed will increase rapidly away from the midplane. We shall discuss these limits in more detail in \( \S 3.6.3 \) and \( \S 3.6.4 \).

### 3.3.1 Leading order EMFs

As in LO-A, the leading order azimuthal EMF vanishes.

\[ \frac{\mathcal{E}_{y_0}}{E_l} = \frac{1}{2} \text{Re} \left[ u_{x_0}^* b_{z_0} - u_{x_0}^* b_{z_0} \right] \]

\[ = \frac{1}{2} \text{Re} \left[ u_{x_0}^* \left( \frac{-k_z}{k_x} b_{z_0} \right) - \left( \frac{k_z}{k_x} u_{x_0}^* \right) b_{z_0} \right] \]

\[ = 0 \]
We calculate the leading order radial EMF. The algebraic process is quite lengthy, and so we keep all details in §B.0.1

\[
\frac{\mathcal{E}_{x0}}{\mathcal{E}_t} = \frac{1}{2} \Re \left[ u_{y0} b_{z0} - u_{z0}^* b_{y0} \right] \\
= \frac{1}{2} \left( \frac{1}{k_x |\gamma_\eta|^2} \Re \left[ \frac{1}{\gamma_\mu} \right] \frac{\omega_\eta^2 D_k B_y - k_y k_z}{k_x} \left( 2 - q \right) \Omega \Re \left[ \frac{i}{\gamma_\nu \gamma_\eta^*} \right] B_y \right) |u_{z0}|^2 \\
+ \frac{1}{2} \Re \left[ \left( \frac{\omega_\eta^2}{\gamma_\nu \gamma_\eta^*} - 1 \right) b_{y0} u_{z0}^* \right]
\]

and we evaluate this last line using Equation B.2. After some algebra, this gives

\[
\frac{\mathcal{E}_{x0}}{\mathcal{E}_t} = \frac{1}{2} D_k B_y \Re \left[ \frac{1}{\gamma_\eta} \right] |u_{z0}|^2 - \frac{k_y k_z}{k_x} \left( 2 - q \right) \Omega |u_{z0}|^2 \\
- \frac{1}{2k_x} B_y \left( \frac{1}{|\gamma_\eta|^2 \gamma_\eta^* + \gamma_\nu^2} \right) \left( \frac{G_k}{c_s^2} \Re \left[ \gamma_\nu \left( \gamma_\nu \gamma_\eta^* - \omega_\eta^2 \right) \left( \gamma_\eta \gamma_\nu^* + \omega_\nu^2 \right) \right] \right) \\
\left\{ \frac{G_k}{c_s^2} \Re \left[ \gamma_\nu \left( \gamma_\nu \gamma_\eta^* - \omega_\eta^2 \right) \left( \gamma_\eta \gamma_\nu^* + \omega_\nu^2 \right) \right] \right\} |u_{z0}|^2 \\
+ k_y k_z \Omega \Re \left[ \frac{i}{\gamma_\nu} \left( (2 \gamma_\eta - q(\gamma_\eta - \gamma_\nu))(\gamma_\nu \gamma_\eta^* - \omega_\eta^2) \left( \gamma_\eta \gamma_\nu^* + \omega_\nu^2 \right) \right) \right] |u_{z0}|^2
\]

the structure of which we shall analyse in §3.6. For completeness, we note that \(\mathcal{E}_{z0} = k_z \mathcal{E}_{x0}/k_x\). We would require \(\mathcal{E}_{z0}\) if, in our mean-field equations, we retained radial derivatives \(\partial_x\) - but we shall not do so in this Thesis.

### 3.4 First-order calculation

Again, we eliminate perturbation quantities in favour of \(u_{z0}\), with algebraic details in §B.0.2. Here, we must deal with the fact that the first order equations involve terms like \(\dot{u}_{z1}\). To gain an algebraic system we Taylor expand in time so that \(u_{z1} = u_{z1}^0 + u_{z1}^1 t + O(t^2)\), and \(\dot{u}_{z1} = u_{z1}^1 + O(t)\). After much algebra, we arrive at
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those terms involving only $B_y$

\[
\mathcal{E}_{y1} = -\frac{1}{2} k_y^2 B_y \Re \left[ D_k \frac{|\tilde{u}_{z0}|^2}{\gamma_{\eta}} \right] \\
- \frac{k_y}{k_x^2} \frac{B_y c_s^2}{c_s^2 + v_a^2} \frac{1}{|\gamma_\nu \gamma_\eta + \omega_c^2|^2} \\
\times \left( \frac{G_k}{c_s^2} (|\gamma_\nu|^2 |\gamma_\eta|^2 + \omega_c^2 \Re [\gamma_\nu \gamma_\eta]) + k_y k_z \Omega \Re \left[ i \gamma_\nu^* \gamma_\eta^* \left( 2 \gamma_\eta - q(\gamma_\eta - \gamma_\nu) \right) + 2i \gamma_\eta \omega_c^2 \right] \right) \\
\times \Re \left[ \frac{1}{\gamma_{\eta}} \right] |\tilde{u}_{z0}|^2.
\]

A brief discussion of terms of order $B_x$ is given in §B.0.2.

3.5 Summary

To summarise, we have calculated the leading order EMFs to be

\[
\mathcal{E}_{x0} = \frac{1}{2} k_x D_k B_y \Re \left[ \frac{1}{\gamma_{\eta}} \right] |\tilde{u}_{z0}|^2 - \frac{k_y k_z}{k_x} \Im \left[ \gamma \right] \frac{k_x^2 (\eta - \nu)}{|\gamma_{\eta}|^2 |\gamma_\nu|^2} B_y (2 - q) \Omega |\tilde{u}_{z0}|^2 \\
- \frac{1}{2} k_x^{-1} B_y \frac{1}{|\gamma_{\eta}|^2 c_s^2 + v_a^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2} \\
\times \left\{ G_k \Re \left[ \gamma_\eta \left( \gamma_\nu \gamma_\eta^* - \omega_a^2 \right) \left( \gamma_\eta^* \gamma_\nu^* + \omega_c^2 \right) \right] \right. \\
- k_y k_z \Omega \Im \left[ \frac{1}{\gamma_\nu} \left( 2 \gamma_\eta - q(\gamma_\eta - \gamma_\nu) \right) (\gamma_\nu \gamma_\eta^* - \omega_a^2) (\gamma_\eta^* \gamma_\nu^* + \omega_c^2) \right] \left. (\gamma_\eta^* \gamma_\nu^* + \omega_c^2) \right\} |\tilde{u}_{z0}|^2
\]
and

\[
E_{y1} = \frac{-1}{2k_y^2} B_y D_k \Re \left[ \frac{\tilde{u}_{z0}}{\gamma_\eta} \right] \\
- \frac{k_y}{k_x^2} B_y \frac{c_s^2}{v_a^2} \frac{1}{|\gamma_\nu \gamma_\eta + \omega_c^2|^2} \\
\times \left( \frac{G_k}{c_s^4} (|\gamma_\nu|^2 |\gamma_\eta|^2 + \omega_c^2 \Re [\gamma_\nu \gamma_\eta]) - k_y k_z \Omega \Im \left[ \gamma_\nu \gamma_\eta^* (2\gamma_\eta - q(\gamma_\eta - \gamma_\nu)) + 2\gamma_\eta \omega_c^2 \right] \right) \\
\times \Re \left[ \frac{1}{\gamma_\eta} \right] |\tilde{u}_{z0}|^2
\]

(3.6)

These expressions have the recurring term $|\tilde{u}_{z0}|^2 / \gamma_\eta$ and the surprising feature of involving $k_y \Im[\gamma]$ which was absent from the analysis of LO-A who considered only real unstable $\gamma$. It will be shown in §3.6.3 that the dependence on the imaginary part of $\gamma$ cannot be neglected even for weak fields. After a brief analysis of the role of the magnetic Prandtl number in §3.6.5 we shall set $P_m = 1$ to simplify our expressions before continuing.

We made reference in the Introduction to this chapter to Thévenot [100], who discussed the constructive interference of dynamo waves with $(+k_x, +k_y)$ versus $(+k_x, -k_y)$ and concluded that the superposition of such waves, with unknown relative amplitudes, made calculating a net $\alpha$ impossible. We must now check that we do not suffer from a similar difficulty in our chosen context of a shearing sheet with vertical gravity. We can confidently assume that the product of $k_y k_x$ is likely to be positive at the time of interest, given that trailing waves will have had more time to grow under the influence of the mixed MRI/buoyancy instability and shearing box simulations are dominated by trailing structures with $k_y k_x > 0$; we need only worry, therefore, about the sign of the product $k_x k_z$. In such a shearing sheet we would expect no radial variation in mean-field quantities and
no radial gravity; we rewrite our expressions to take account of this, finding

$$\mathcal{E}_{x0} = \frac{1}{2} \frac{\partial}{\partial z} (B_y) \text{Re} \left[ \frac{1}{\gamma_\eta} |\tilde{u}_{z0}|^2 - \frac{k_y k_z}{k_x} \text{Im} [\gamma_\nu] B_y (2 - q) \Omega \frac{k_p^2 (\eta - \nu)}{|\gamma_\nu|^2 |\gamma_\eta|^2} |\tilde{u}_{z0}|^2 \right]$$

$$- \frac{1}{2} B_y \frac{1}{|\gamma_\eta|^2 c_s^2 + v_a^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2}$$

$$\times \left( \frac{g_z}{c_s^2} \text{Re} \left[ \gamma_\eta (\gamma_\nu \gamma_\eta^* - \omega_a^2) (\gamma_\nu^* \gamma_\eta^* + \omega_c^2) \right] \right)$$

$$+ \frac{k_y k_z}{k_x} \Omega \text{Re} \left[ \frac{i}{\gamma_\nu} ((2 - q) (\gamma_\eta) + q \gamma_\nu (\gamma_\nu^* \gamma_\eta^* - \omega_a^2) (\gamma_\eta^* \gamma_\nu^* + \omega_c^2)) \right] |\tilde{u}_{z0}|^2$$

and

$$\mathcal{E}_{y1} = -\frac{1}{2} \frac{k_y}{k_x} B_y \frac{\partial}{\partial z} \text{Re} \left[ \frac{|\tilde{u}_{z0}|^2}{\gamma_\eta} \right]$$

$$- \frac{k_y k_z}{k_x} B_y \text{Re} \left[ \frac{1}{|\gamma_\eta|^2 c_s^2 + v_a^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2} \right]$$

$$\times \left( \frac{g_z}{c_s^2} (|\gamma_\nu|^2 |\gamma_\eta|^2 + \omega_c^2 \text{Re} [\gamma_\nu \gamma_\eta]) + \frac{k_y k_z}{k_x} \Omega \text{Re} \left[ i \gamma_\nu^* \gamma_\eta^* (2 \gamma_\eta - q (\gamma_\eta - \gamma_\nu)) + 2 i \gamma_\nu \omega_c^2 \right] \right)$$

$$\times |\tilde{u}_{z0}|^2$$

but it may easily be seen (using e.g. Mathematica) that this expression contains $k_z$ only in the combination of $k_y k_z \text{Im}[\gamma]/k_x$. Neglecting radial variations, we rewrite this

$$\frac{k_y k_z \text{Im}[\gamma]}{k_x} = -\frac{k_y}{k_x} k_z^2 v_p$$

where $v_p = \text{Im}[\gamma]/k_z$ is the vertical phase speed. After numerical investigation we conclude that wave peaks travel towards the midplane (although for a weak field we can show this analytically), and $\text{Im}[\gamma]$ thenceforth appears only in even powers. Our expressions, intended for application to the shearing sheet, do not suffer from the weakness identified by Thélen.
3.6 Analysis

To help our physical intuition, we write these EMFs as classical dynamo coefficients. Neglecting radial variations, we rewrite Equation 3.5

\[ \mathcal{E}_{x0} = \eta_T \frac{\partial B_y}{\partial z} - v B_y \]

i.e. a turbulent diffusivity \( \eta_T \), present in the zero gravity case, and an advective term with effective speed \( v \) which relies upon the presence of vertical gravity (recall that \( \text{Im}[\gamma] = 0 \) for the toroidal MRI). This term involving \( v \) is associated with the antisymmetric part of \( \alpha_{ij} \) and has elsewhere been called “the \( \gamma \) effect” [23]. We avoid calling this effect \( \gamma \) because we have already used \( \gamma \) as our growth rate in this Chapter. We also rewrite Equation 3.6 as

\[ \mathcal{E}_{y1} = -\alpha_i B_y - \alpha_g B_y \]

i.e. a term involving \( \alpha_i \), present in the incompressible case and which we shall discuss in the following paragraphs, and an \( \alpha_g \) term reliant upon the presence of vertical gravity. These coefficients are not constants but depend on the strength and variation of the background magnetic field and the local wavenumber of the disturbance - any dynamo that they describe is thus nonlinear. The structure of the diffusive, advective and \( \alpha_g \) terms are as usual in the literature but it is worth spending some time discussing \( \alpha_i \). In Chapter 4 we will consider use a correlation integral \( \mathcal{J} \) to measure the creation of constructive radial field \( B_x \) by instabilities on the toroidal field \( B_y \). We defer its definition to Equation 4.11 on page 105. For now, we only consider the correlation of \( B_y \) with the term involving \( \alpha_i \equiv (k_y/k_x) \frac{\partial}{\partial z} (\text{Re} [1/2\gamma] |u_{z0}|^2) \)

\[ \mathcal{J}_{\alpha_i} = \int \text{d}z \, B_y \frac{\partial}{\partial z} (-\alpha_i B_y) \]
and by integrating by parts twice, we find that this
\[
= \frac{k_y}{k_x} \left( -B_y^2 \frac{\partial}{\partial z} \left( \text{Re} \left[ \frac{1}{2\gamma} |u_{z0}|^2 \right] \right) + \frac{\partial}{\partial z} \left( \frac{B_y^2}{2} \right) \text{Re} \left[ \frac{1}{2\gamma} |u_{z0}|^2 \right] \right) \bigg|_{\text{boundaries}}
- \frac{k_y}{k_x} \int dz \ \text{Re} \left[ \frac{1}{2\gamma} |u_{z0}|^2 \frac{\partial^2}{\partial z^2} \left( \frac{B_y^2}{2} \right) \right].
\]

We argue that if the perturbation is localised around the most unstable point then the local contribution will dominate, allowing us to neglect the boundary terms. The real part of the growth rate $\gamma$ is positive for instability, and we argued on page 80 that the combination $k_y/k_x > 0$; the effect of the $\alpha_i$ term thus depends on the sign of $(B_y^2)''$ at the most unstable location, as was found in LO-A. Since the integral will be dominated by the most unstable location this term is constructive if $(B_y^2)'' < 0$ there i.e. if the most unstable location lies near a maximum of $B_y^2$. For the pure (unstratified) MRI, this means the net effect of $\alpha_i$ depends on the strength of $B_y$: for weak fields the perturbation will be localised near a peak of $B_y^2$; $(B_y^2)''$ will be negative and so the contribution to $J$ positive and constructive, while for sufficiently strong fields the perturbation will grow fastestwhere $\omega_a$ is optimal i.e. away from the peak; $(B_y^2)''$ may be negative and so the contribution to $J$ negative and destructive. We illustrate this in Figure 3.1.

It is clear that if we are to have a net increase of magnetic energy then we must have some favourable combination of these coefficients. In all cases, the turbulent diffusivity $\eta_T$ will reduce magnetic energy and the incompressible $\alpha_i$ will act as described above. If we have $vB_y^2'(z) > 0$ then the advective term in $\mathcal{E}_x$ will be compressing magnetic field and increasing the magnetic energy, and a similar argument applies to $\alpha$ so that $\alpha B_y^2'(z) > 0$ is constructive.

In the following sections we shall discuss the signs of these terms by the two limiting polarisations, $k_z \gg k_x$ for the MRI and $k_x \gg k_x$ for the undular instability, and then make arguments about the large-scale effect on the magnetic energy. We assume that the dynamics will be dominated by the most unstable regions i.e. where $\text{Re}[\gamma]$ will be largest; we argue that - regardless of whether the undular mode in the absence of shear would be locally unstable or not - this region will be on the side of flux concentrations furthest from the midplane, since the corresponding regions closer to the midplane will be stabilised by the sign
of the magnetic pressure gradient. We deduce that in the most unstable regions
we may assume that $z B_y^2 \left( z \right) < 0$. We shall discuss first the MRI polarisation
whereby $k_z \gg k_x$, with largely horizontal motion, in §3.6.3 and second the
undular polarisation whereby $k_z \ll k_x$, with largely vertical motion, in §3.6.4.

3.6.1 No vertical gravity ($g_z = 0$): reproducing LO-A

We relate this calculation directly to the analytic result of LO-A. We set $G_k = 0$
and recall that without vertical gravity $\gamma$ will be real. We gain

$$\mathcal{E}_{x0} = \frac{1}{2k_x} D_k B_y \left| \tilde{u}_{z0} \right|^2 \frac{\gamma}{\eta}$$

and

$$\mathcal{E}_{y1} = -\frac{1}{2k_x k_x} B_y D_k \left( \frac{\left| \tilde{u}_{z0} \right|^2}{\gamma} \right)$$

a large simplification. Indeed, this can be easily related to the work of LO-A; this
expression for $\mathcal{E}_{x0}$ is precisely their Equation 68 barring our viscous correction
and extension to general poloidal variations. Regarding $\mathcal{E}_{y0}$, we find we must first
assume that we are at an extremely late time, with the perturbation localised at
a single location $z_*$. The growth rate then becomes the growth rate evaluated
at $z_*$, and the perturbation profile becomes a sharpening Gaussian around that
point,

$$u_{z0}(z) = u_{z0}^0 \exp \left( -\frac{C}{\mathcal{A}} \left( \frac{z - z_*}{\epsilon} \right)^2 \epsilon^2 t \right)$$

with $\mathcal{A}$, $\mathcal{C}$ positive constants specified by LO-A. With this ansatz, and dropping
our poloidal variations, we find

$$\mathcal{E}_{y1} = -\frac{1}{2} \frac{k_y B_y}{k_x \gamma} \partial_z \left( \exp \left( -2 \frac{C}{\mathcal{A}} \left( \frac{z - z_*}{\epsilon} \right)^2 \epsilon^2 t \right) \right) \left| \tilde{u}_{z0} \right|^2$$

$$= \frac{2\mathcal{C}}{\mathcal{A} \eta} \frac{\gamma}{\epsilon} \left( \frac{z - z_*}{\epsilon} \right)^2 \left| \tilde{u}_{z0} \right|^2$$
Figure 3.1: We illustrate the behaviour of $\alpha_i$, the incompressible contribution to $\mathcal{E}_{\nu_1}$. Consider a linear perturbation with some amplitude $|u_{z0}|^2$ (green line) on a background magnetic field $B_y$ (black line showing $B^2_y$) without any vertical gravity and so without magnetic buoyancy. We consider in (a) a weak magnetic field where the perturbation is localised about the peak of $B^2_y$, and in (b) a stronger field where the optimal location has moved to an intermediate point where $B^2_y$ is sufficiently weak. Since we have written the integrand of our correlation integral $\mathcal{J}$ as $\propto -\text{Re} \left[ \frac{|u_{z0}|^2}{\gamma} \frac{\partial^2}{\partial z^2} B^2_y(z) \right]$ we can see that the contribution of $\alpha_i$ will always be constructive in regions $I$ and $I'$, where the second derivative of $B^2_y$ is negative, and always destructive in regions $II$ and $II'$ where it is positive. This means that in (a) we have a net increase in magnetic energy as the constructive region dominates, and we have in (b) a net decrease in magnetic energy as the destructive region dominates. The net effect of $\alpha_i$ is determined by the localisation of the perturbation.
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which corresponds to their Equation 69.

### 3.6.2 Set \( P_m = 1 \)

We wish to examine the twin influences of the MRI and of the undular instability, and first simplify by assuming \( P_m = 1 \) - that is, \( \gamma_\eta = \gamma_m = \gamma_\nu \) - until §3.6.5. Our expressions are greatly simplified, with our dispersion relation becoming

\[
B_y \left[ \left( 1 + \frac{\omega_a^2}{\gamma_\nu^2} + \frac{v_a^2}{c_s^2} \right) D_k B_y - B_y \left( 2\Omega \frac{ik_y k_z}{\gamma_\nu} + \frac{G_k}{c_s^2} \right) \right] \left[ 2\Omega \frac{ik_y k_z}{\gamma_\nu} + \frac{G_k}{c_s^2} \right] - \rho \left[ 1 + \frac{\omega_a^2}{\gamma_\nu^2} \right] \left[ (k_x^2 + k_z^2)(\gamma_\nu^2 + \omega_a^2) + \left( k_x^2 \gamma_\nu^2 + 2\Omega\frac{ik_y k_z}{\gamma_\nu} B_y D_k B_y \right) \right] = 0
\]

and our EMFs becoming

\[
\mathcal{E}_{x0} = \frac{1}{2k_x} D_k B_y \text{Re} \left[ \frac{1}{\gamma_\nu} \right] |\tilde{u}_{z0}|^2
\]

\[
- \frac{1}{2} B_y \frac{c_s^2}{c_s^2 + v_a^2} |\gamma_\nu|^2 - \omega_a^2 \left( \frac{1}{k_x} \frac{G_k}{c_s^2} (|\gamma_\nu|^2 + \omega_a^2) + \frac{k_y k_z}{k_x} 4\Omega \text{Im} [\gamma] \right) \text{Re} \left[ \frac{1}{\gamma_\nu} \right] |\tilde{u}_{z0}|^2
\]

and

\[
\mathcal{E}_{y1} = \frac{1}{2k_x} B_y D_k \frac{c_s^2}{c_s^2 + v_a^2} \frac{1}{|\gamma_\nu|^2 + \omega_a^2} \left( \frac{1}{k_x} \frac{G_k}{c_s^2} (|\gamma_\nu|^4 + \omega_a^2 \text{Re} [\gamma_\nu^2]) + \frac{k_y k_z}{k_x} 2\Omega \text{Im} [\gamma] (|\gamma_\nu|^2 - \omega_a^2) \right) \text{Re} \left[ \frac{1}{\gamma_\nu} \right] |\tilde{u}_{z0}|^2
\]

and we shall use these expressions for the remainder of the Chapter except as otherwise specified.

### 3.6.3 MRI polarisation: weak field limit and \( k_z \gg k_x \)

We refer to Davis, Stone and Pessah\[22\] whose simulations showed an average midplane plasma-\( \beta \) around \( 10^2 \) to justify taking a weak-field approximation; this will further simplify our expressions and inform our physical intuition for MRI-dominated fields (naturally after taking a weak-field approximation the buoyancy...
instability weakens; later, after taking \( k_z \gg k_x \) it will disappear). We take the weak field limit carefully, keeping \( k_y v_a / c_s := \mu \to 0 \). Naturally, \( \omega_c \sim \omega_a \) in this limit, and we find

\[
\mathcal{E}_{x0} \sim \frac{1}{2 k_x} D_k B_y \text{Re} \left[ \frac{1}{\gamma_{\nu}} \right] |\tilde{u}_{z0}|^2
\]

\[
- \frac{1}{2 k_x} B_y \frac{|\gamma_{\nu}|^2 - \omega_a^2}{\gamma_{\nu}^2 + \omega_a^2} \left( \frac{G_k}{c_s^2} \left( |\gamma_{\nu}|^2 + \omega_a^2 \right) + k_y k_z 4 \Omega \text{Im}[\gamma] \right) \text{Re} \left[ \frac{1}{\gamma_{\nu}} \right] |\tilde{u}_{z0}|^2
\]

and

\[
\mathcal{E}_y \sim -\frac{k_y}{2 k_z^2} B_y \text{Re} \left[ D_k \frac{|\tilde{u}_{z0}|^2}{\gamma_{\nu}} \right]
\]

\[
- \frac{k_y}{k_z^2} B_y \frac{1}{|\gamma_{\nu}^2 + \omega_a^2|^2}
\]

\[
\times \left( \frac{G_k}{c_s^2} \left( |\gamma_{\nu}|^4 + \omega_a^2 \text{Re} \left[ \gamma_{\nu}^2 \right] \right) + 2 k_y k_z \Omega \left( |\gamma_{\nu}|^2 - \omega_a^2 \right) \text{Im}[\gamma_{\nu}] \right) \text{Re} \left[ \frac{1}{\gamma_{\nu}} \right] |\tilde{u}_{z0}|^2
\]

and now we must estimate the size of \( k_y \text{Im}[\gamma] \). We expand the MRI dispersion relation in Equation [3.3] in powers of \( v_a^2 / c_s^2 \ll 1 \), and find at leading order

\[
k_p^2 (\gamma_0^2 + \omega_a^2)^2 + k_z^2 (\kappa^2 \gamma_0^2 - 2 q \omega_a^2 \Omega^2) = 0
\]

with real solutions. We therefore expand

\[
\gamma_{\nu} = \gamma_0 + i \mu \gamma_1 + ...
\]

and find that at first order we have

\[
\gamma_1 = \frac{\gamma_0 G_k (D_k B_y (\gamma_0^2 + \omega_a^2) - 4 i \gamma_0 k_z \omega_a \Omega \sqrt{\rho})}{2 \mu c_s (k_p^2 (\gamma_0^4 - \omega_a^4) + 2 k_z^2 \omega_a^2 \Omega^2 q) \sqrt{\rho}}
\]
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and if gradients of \( B_y \) are similarly small then this

\[
\begin{align*}
&= -\frac{2}{c_s} k_z G k^2 \gamma_0^2 \omega_a \Omega \\
&= -\frac{2}{c_s} k_z \frac{G k^2 \gamma_0^2 \omega_a \Omega}{k^2 \gamma_0^2 + 2 k^2 \omega_a^2 + k^2 \kappa^2}
\end{align*}
\]

and so we see that \( k_y \text{Im}[\gamma] = \omega_a \gamma_1 / c_s \) which is non-negligible. We desire to consider the effect of the MRI here and so we take that \( k_z \gg k_x \) i.e. a horizontally polarised perturbation. Then

\[
\gamma_1 \sim -\frac{k_x}{k_z} \frac{2 g_z \omega_a \Omega}{2 \gamma_0^2 + 2 \omega_a^2 + \kappa^2}
\]

and our expressions for the EMFs become

\[
\begin{align*}
\mathcal{E}_{x0} &\sim -\frac{1}{2} \frac{\partial B_y}{\partial z} \frac{1}{\gamma_0} |\tilde{u}_{z0}|^2 \\
&\quad - \frac{1}{2} B_y \frac{g_z}{c_s^2} \frac{|\tilde{u}_{z0}|^2}{\gamma_0} \times \left\{ \frac{\gamma_0^2 - \omega_a^2}{(\gamma_0^2 + \omega_a^2)} \right\} \\
&\quad \times \left\{ \left( \frac{\gamma_0^2}{2 \gamma_0^2 + 2 \omega_a^2 + \kappa^2} \right) \right\}
\end{align*}
\]

(3.7)

and

\[
\begin{align*}
\mathcal{E}_y &\sim -\frac{1}{2} \frac{k_y}{k_x} B_y \frac{\partial}{\partial z} \left( \frac{|\tilde{u}_{z0}|^2}{\gamma_0} \right) \\
&\quad - \frac{k_y}{k_x} B_y \frac{g_z}{c_s^2} \frac{|\tilde{u}_{z0}|^2}{\gamma_0} \\
&\quad \times \left\{ \frac{1}{(\gamma_0^2 + \omega_a^2)^2} \left( \gamma_0^2 (\gamma_0^2 + \omega_a^2) - 4 \Omega^2 (\gamma_0^2 - \omega_a^2) \right) \right\}
\end{align*}
\]

(3.8)

so that for weak fields the \( v \) and \( \alpha \) terms due to gravity are appropriately \( \propto g_z / c_s^2 \).

We wish to know what sign the mean-field coefficients take, and calculate them numerically in Figure 3.2. The \( \alpha \)-effect in \( \mathcal{E}_y \) is constructive for all \( \omega_a \), while the advection term \( v \) in \( \mathcal{E}_x \) is compressive for \( \omega_a^2 > \Omega^2 / 2 \) (i.e. \( \gamma^2 - \omega_a^2 > 0 \)) and rarefactive otherwise.

Even for very weak fields and with polarisation such that the perturbation motion is predominantly horizontal we see a vertical migration of the mean-
3. Analytic Quasilinear EMFs

Figure 3.2: The weak-field growth rate $\gamma/\Omega$ (green line) for $k_z \gg k_x$ (MRI polarisation) with maximum at $\omega_a = \sqrt{15}\Omega/4$, plus the bracketed ({}) coefficients in the radial EMF given in Equation 3.7 on page 88 (blue line), and for the bracketed ({}) coefficients in the azimuthal EMF given in Equation 3.8 (purple line). A positive quantity indicates a constructive EMF. The azimuthal EMF $\alpha$-effect is constructive for all values of $\omega_a$, while the radial EMF’s migration speed $v$ is compressive for $\omega_a > \Omega/\sqrt{2}$ (i.e. $\gamma^2 - \omega^2_a > 0$) and rarefactive below that.

Field; the migration is due to the inhomogeneity of the background rather than any effect of the undular instability, which suggests that vertical migration in simulations with a stable ($N^2 > 0$) vertical stratification might have a comparable pattern speed to that seen in isothermal simulations.

3.6.4 Undular polarisation: $k_x \gg k_z$

We consider a vertically polarised perturbation ($k_x \gg k_z$) to examine only the effects of the undular instability; this effectively removes rotation and shear from the problem. Even though we disregard rotation and shear in this subsection
we still consider ourselves to be working in the shearing sheet and so continue to assume that \( k_y k_x > 0 \), even though this assumption relied on the presence of shear. With this polarisation our EMFs become

\[
\mathcal{E}_{x0} = \frac{1}{2} \frac{\partial B_y}{\partial z} \frac{|\tilde{u}_z 0|^2}{\gamma_\nu} - \frac{1}{2} B_y \frac{\gamma_s^2}{c_s^2 + v_a^2} \left( \frac{\gamma_s^2 - \omega_a^2}{\gamma_\nu + \omega_c^2} \right) \frac{g_z}{c_s^2} \frac{|\tilde{u}_z 0|^2}{\gamma_\nu}
\]

and

\[
\mathcal{E}_{y1} = -\frac{1}{2} \frac{k_y}{k_x} B_y \frac{\partial}{\partial z} \left( \frac{|\tilde{u}_z 0|^2}{\gamma_\nu} \right) - \frac{k_y}{k_x} B_y \frac{\gamma_s^2}{c_s^2 + v_a^2} \frac{g_z}{\gamma_\nu + \omega_c^2} \frac{\gamma_s^2}{c_s^2} \frac{|\tilde{u}_z 0|^2}{\gamma_\nu}
\]

and we see again that the crucial quantity is \((\gamma_\nu^2 - \omega_a^2)\). If this quantity is positive, then the migration speed \( v \propto -z \) and flux will migrate towards the midplane, while if it is negative flux will migrate away. The \( \alpha \)-effect is negative above the midplane and is thus reinforcing the toroidal field.

To examine how \( \gamma_\nu^2 - \omega_a^2 \) varies with height in this polarisation we take the \( \gamma_\nu^2 \) given by this limit of the dispersion relation and solve \( \gamma_\nu^2 - \omega_a^2 = 0 \) for \( g_z \). We find a solution

\[
g_z = c_s^2 \frac{\partial}{\partial z} (\log B_y^2) + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{B_y^2}{2} \right) - \frac{2 \rho c_s^2 B_y}{\partial_z B_y} \omega_a^2 + O(\omega_a^4)
\]

where we have expanded in terms of the Alfvén frequency which will be small for fast-growing undular perturbations. Combining this with vertical magneto-hydrostasis gives us

\[
\frac{\partial}{\partial z} (\log \rho) = \frac{\partial}{\partial z} (\log B_y^2) - \frac{2 k_y^2}{\partial_z (\log B_y)}
\]

or equivalently

\[
\frac{\partial}{\partial z} (v_a^2) = \frac{2 \omega_a^2}{\partial_z (\log B_y)} \quad (3.9)
\]

i.e. there is a height and strength at which the vertical migration of toroidal field ceases. Evolution of the field with height would then occur in a quasi-steady fashion as the field strength increasing or decreasing change the location of this “magnetic stagnation point”. This is demonstrated in Figure 3.3.
3. Analytic Quasilinear EMFs

Figure 3.3: A discussion of the magnetic stagnation point. We plot $v_a/c_s$ (blue line) for an isolated flux concentration (which we shall make use of in Chapter 4 and describe fully in §C.3), the real growth rate $\gamma/\Omega$ (light green line), the quantity $(\gamma^2 - \omega^2_a)/\Omega^2$ and $(\nu^2_a - 2\omega^2_a/(\log B_y)'$ as suggested by Equation 3.9. From top to bottom we have taken $k_y = 2\pi/5$, $k_y = \pi/5$ and $k_y = \pi/10$. Equation 3.9 is only a good approximation for small of $\omega_a$, but more interesting is the fact that the magnetic stagnation point is “stable” in the sense of the advective term $v$ moving toroidal field towards it, with $\gamma^2 - \omega^2 > 0$ above it and $\gamma^2 - \omega^2_a < 0$ below.

We see also that for larger $k_y$ the stagnation point is farther from the maximum of $B_y$. It seems that the larger the dominant $k_y$, the further the stagnation point will be from the maximum of $B_y^2$ and so the less coherent the rise of the isolated flux concentration as it migrates upwards.
3.6.5 Effects of $P_m$ on the dynamo

We reintroduce the general $P_m$ to analyse the terms specifically arising from the difference in diffusivities. We examine

$$v^P_m = \left( \frac{k_y k_z}{k_x} \frac{\text{Im}[\gamma]}{\gamma_{\eta}^2 |\gamma_{\nu}|^2} \right) \frac{k_{p}^{2}(\eta - \nu)}{|\gamma_{\eta}^2| |\gamma_{\nu}|^2} (2 - q)\Omega|\tilde{u}_{z0}|^2,$$

the second term in $\mathcal{E}_{z0}$ from Equation 3.5. We have already noted that the first bracketed term will be positive above the midplane; if $P_m > 1$, this entire term is negative and acts to compress the toroidal field.

$$v^P_m = -\frac{1}{2} \frac{1}{|\gamma_{\eta}|^2 c_s^2 + v_a^2 |\gamma_{\eta} \gamma_{\nu} + \omega_c^2|^2}$$

$$\times \left\{ \frac{k_y k_z}{k_x} \Omega \text{Re} \left[ \frac{i}{\gamma_{\nu}} q(\gamma_{\eta} - \gamma_{\nu})(\gamma_{\nu} \gamma_{\eta}^* - \omega_a^2)(\gamma_{\eta}^* \gamma_{\nu} + \omega_c^2) \right] \right\} |\tilde{u}_{z0}|^2$$

$$= -\frac{1}{2} \frac{1}{|\gamma_{\eta}|^2 c_s^2 + v_a^2 |\gamma_{\eta} \gamma_{\nu} + \omega_c^2|^2}$$

$$\times \left\{ q k_{p}^2 (\eta - \nu) \frac{k_y k_z}{k_x} \Omega \frac{1}{|\gamma_{\nu}|^2} \text{Re} \left[ i \gamma_{\nu}^* (\gamma_{\nu} \gamma_{\eta}^* - \omega_a^2)(\gamma_{\nu}^* \gamma_{\eta} + \omega_c^2) \right] \right\} |\tilde{u}_{z0}|^2$$

and it is clear that the condition that we are seeking here involves comparing the sizes of $\gamma^2$ and $\omega_a^2$. To simplify this expression we no take a weak-field limit; since we have already discussed the weak-field approximation in §3.6.3 we relegate this calculation to §B.2, and note that a sufficient condition for this term to favour dynamo action in a weak field is that $(P_m - 1)(\gamma^2 - \omega_a^2) < 0$. In the azimuthal EMF we have the following contribution to the $\alpha$-effect.

$$\alpha^P_m = \frac{k_y}{k_x} \frac{c_s^2}{c_s^2 + v_a^2 |\gamma_{\nu} \gamma_{\eta} + \omega_c^2|^2} \left( \frac{k_y k_z}{k_x} \Omega \text{Re} \left[ -i \gamma_{\nu}^* \gamma_{\eta}^* (q(\gamma_{\eta} - \gamma_{\nu})) \right] \right) \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] |\tilde{u}_{z0}|^2$$

$$= -q \Omega \frac{k_y}{k_x} \frac{c_s^2}{c_s^2 + v_a^2 |\gamma_{\nu} \gamma_{\eta} + \omega_c^2|^2} \left( \frac{k_y k_z}{k_x} \text{Im}[\gamma] \right) k_{p}^2 (\eta - \nu) \text{Re} [\gamma_{\eta} + \gamma_{\nu}] \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] |\tilde{u}_{z0}|^2$$

which is positive (i.e. destructive) if $P_m > 1$. There is therefore a conflict between the direct advective effect, which tends to increase magnetic energy when $P_m > 1$ and when $(P_m - 1)(\gamma^2 - \omega_a^2) < 0$, and the (order-of-magnitude smaller) $\alpha$-effect which tends to decrease magnetic energy when $P_m > 1$. These terms are all
linear in $\text{Im}[\gamma]$, which vanishes in the unstratified case, and so the above analysis cannot explain e.g. the $P_m$ dependence by Fromang et al.\[35\], who showed that a dynamo is easier to excite in an (unstratified) accretion disc for $P_m > 1$. Indeed, for stratified discs Oishi and Mac Low\[80\] state that for sufficiently large $R_m$ the magnetic Prandtl number $P_m$ becomes irrelevant, something not predicted by our analysis (although their conclusion is not obviously independent of the presence of a fictitious “hyperdiffusion” $\sim \eta_H \nabla^6 B$ in their induction equation).

We are led to ask whether the dominant unstable mode does satisfy $\gamma^2 < \omega^2 c$; if we consider the pure MRI with infinite $k_z$, optimal $\omega_a$ and $q = 3/2$ for example, we would have $\gamma^2 = (15/16)\omega^2 c$; we conjecture that in the low MRI dominated atmosphere this term might be constructive and become destructive farther from the midplane.

### 3.7 Limitations

Since we are examining unstable modes with $\text{Re}[\gamma] > 0$ we have not worried about dividing by powers of $\gamma$ in our EMF expressions. We have also divided by $|\gamma \gamma + \omega^2 c|^2$ and so there is another apparent singularity when $\gamma \gamma - \omega^2 c$. We show in Appendix §B.1 that if this quantity vanishes then either $u_{z0} = 0$ and the perturbation is neither MRI- nor buoyancy-type and has trivial EMFs, or that we are at some special location satisfying

$$\frac{G_k}{c_s^2} = -\frac{2}{\gamma \omega^2 \Omega i k_y k_z} - \frac{(\gamma \gamma - \omega^2 c)}{\omega^2 \Omega i k_y k_z}. $$

If the above relationship holds we may derive forms for both $E_{x0}$ and $E_{y1}$. This is sketched in an Appendix but not investigated further, since these modes are decaying by assumption and not of particular interest. The fact that our EMF expressions can become infinite will make using them to make numerical predictions difficult, as briefly discussed in §5.5.
3.8 Conclusion

We have made concrete progress towards a human understanding of the accretion disc dynamo by use of an asymptotic expansion of the mixed MRI/undular instability, thence carefully constructing an analytic expression for the quasilinear EMFs. We examined the structure of these EMFs and have related them to classical dynamo theory.

We applied a WKBJ approximation to said instability, with asymptotic parameter the large poloidal wavenumber i.e. $k_p H/\epsilon \gg 1$. We successfully derived analytic forms for both $E_x$ and $E_y$, the radial and azimuthal EMFs respectively, for a general poloidal background and with explicit diffusivities (albeit with a fictitious mass diffusivity), where the radial EMF appeared at leading order and the azimuthal EMF at first order. We have interpreted the terms in these EMFs as a classical $\alpha$-effect, a “$\gamma$-pumping” term with effective velocity $v$, a turbulent diffusivity and an $\alpha$-effect $\alpha_i$ governed by the localisation of the perturbation; we have related the turbulent diffusivity and $\alpha_i$ to the incompressible and zero-gravity work of LO-A.

We examined the magnetic Prandtl number dependence of these terms and found that the $\alpha$-effect becomes less constructive for $P_m > 1$, but that the vertical migration speed $v$ tends more to concentrate field dependent on whether we have either $P_m > 1$ or $(P_m - 1)(\gamma_c^2 - \omega_a^2) > 0$.

We examined the two extremes of polarisation by considering two simplifying limits:

First, we assumed a weak field (with fixed $\omega_a$) and a horizontal polarisation ($k_z \gg k_x$) giving us the dispersion relation of the incompressible MRI. We found that the $\alpha$-effect was uniformly constructive and that the vertical migration speed $v$ was constructive for $\omega_a^2 > \Omega^2/2$; the presence of this vertical migration speed even when the dynamics are assumed to be dominated by horizontal motions suggested to us that the vertical migration speed will be comparable even in stably stratified discs.

Second, we assumed a vertical polarisation ($k_x \gg k_z$) to examine solely the undular instability in the absence of rotation and shear. Again we found a uniformly constructive $\alpha$-effect, and found that if buoyancy is dominant the vertical migra-
tion speed has a stable “stagnation point” which will evolve with the strength of the field, the location of which is determined by the dominant azimuthal scale.

In Chapter 5, we shall proceed to apply the analytic model that we have derived to fully 3D nonlinear simulations.
SHEARING WAVES AND THE ACCRETION DISC DYNAMO
Chapter 4

Shearing waves and numerical EMFs

Confusion
Will be my epitaph

King Crimson

4.1 Introduction

We examine the quadratic EMFs that arise from a single shearing wave subject to the mixed toroidal MRI/undular instability in an isothermal disc with vertical gravity. Using numerical integration we calculate these EMFs and associated quantities whilst varying the strength and vertical location of an isolated region of toroidal magnetic field. We shall confirm that the dynamo loop proposed by Ogilvie and Lesur[66] survives even far from the midplane, and modify their argument concerning the most unstable location to account for the effects of magnetic buoyancy.

The magnetorotational instability was brought to astrophysical prominence primarily by Balbus and Hawley[6]: given a vertical field threading an accretion disc undergoing differential rotation, they described two parcels of fluid on the same vertical field line displaced outwards and inwards respectively; these fluid parcels
are then advected apart by the differential rotation. This displacement by advection bends and stretches the field line, creating magnetic tension between the two fluid parcels; the outer one is accelerated forwards and the inner one accelerated backwards. The centrifugal force then pushes these accelerated parcels even further outwards and inwards respectively, leading to the original configuration with a larger amplitude i.e. instability. The requirement for instability is that the angular velocity decreases outwards, i.e.

$$\frac{\partial \Omega^2}{\partial R} < 0$$

which is satisfied for Keplerian discs with $q = 3/2$. This criterion does not mention the strength of the magnetic field; the dispersion relation relies on the Alfvén frequency $\omega_a = k|u_a|$ and so an arbitrarily weak field may be made unstable by a correspondingly large wavenumber, assuming no diffusive effects.

For a toroidal rather than vertical initial field, Balbus and Hawley\[7\] found a transient amplification of the non-axisymmetric slow magnetoacoustic mode. If $\omega_a^2/\Omega^2$ is smaller than $\frac{\partial \log \Omega^2}{\partial \log R}$ and $|k|^2/k_z^2$ is sufficiently close to unity then there is quasi-exponential growth; this growth, if any, will occur when $|k_x/k_z|$ is small at times close to the swing - and thus the total amplification is most effective for large $k_zH$. If $\omega_a/\Omega$ is small then the growth rate is small; if it is of order unity then we see a rapid quasi-exponential growth of strength comparable with the poloidal MRI which (for the vertical case) has maximum growth rate $3\Omega/4$.

The toroidal compressible MRI is an unstable slow magnetoacoustic wave with a displacement in the direction of the shear i.e. radially. Since slow waves are not dominated by their velocity divergence ($\nabla \cdot u = O(|u|/H)$), whenever we assume in this Thesis that both $k_x$ and $k_z$ are much larger than $k_y$ we must have also that $k_z \gg k_x$ for an MRI-type polarisation.

Both the vertical and toroidal forms of the MRI draw their energy from the differential rotation acting against magnetic tension. Drawing energy from the background differential rotation must act to transport mass inwards and angular momentum outwards in a disc, and this has been confirmed by the local stresses calculated in many nonlinear simulations in recent years. For a discussion of shearing box simulations see the introduction of Chapter 5.
We extend the work of Lesur and Ogilvie\cite{67} (hereafter LO-N, for ‘numeric’, together with Lesur and Ogilvie\cite{67} which we have called LO-A for ‘analytic’) who considered a non-uniform toroidal field threading an incompressible disc without vertical gravity. Using linear (i.e. non-interacting) shearing waves they calculated quasi-linear EMFs through the course of a swing. They found that for weak fields there was the potential to both close and saturate the dynamo loop via the vertical localisation of the EMFs: if the growth rate $\gamma$ varies slowly with the background magnetic field and density then there will be a location at which the instability is strongest. If this point had $(B^2_y)''$ negative (e.g. a maximum of the field strength as would happen for weak fields) then the toroidal EMF that they calculated could create radial field of a sign that would then be sheared to reinforce the toroidal field. For stronger fields with $(\omega_{n,max})^2 > 2q\Omega^2$ the location of maximum growth rate was no longer at the maximum of the magnetic field and so the instability will grow fastest in regions where $(B^2_y)''(z)$ is positive and their calculated toroidal EMF would produce radial field of the net wrong sign, providing a saturation mechanism for the dynamo. We wish to investigate the effects of magnetic buoyancy on the most unstable location, and discuss the implications for the dynamo loop described by Ogilvie and Lesur\cite{66} (hereafter LO-A, for ‘analytic’), especially given that it is known that dynamo action exists with only buoyancy and shear (e.g. Cline, Brummell and Catteneo\cite{18}).

The presence of vertical gravity may introduce two magnetic buoyancy instabilities: the undular instability due to Parker\cite{84} and the interchange instability described by Newcomb\cite{108}. The undular instability is a 3D instability and relies on the presence of magnetic pressure to evacuate flux tubes of mass (see e.g. Thomas\cite{102} for a discussion of the undular instability in plane-parallel atmospheres). These are then buoyant and rise, bending the field lines as they do so. The interchange instability is a 2D (axisymmetric) instability and occurs when magnetic flux density decreases upwards such that two field lines, one above the other, find it energetically favourable to switch places; any bending of magnetic field lines is energetically unfavourable for the instability. Optimising over all $k$,
the criteria for instability are

\[ g_z \frac{\partial}{\partial z} (\log B_y) > \frac{c_s^2 N^2}{v_a^2} \]

for the undular instability, and

\[ g_z \frac{\partial}{\partial z} \left( \log \frac{B_y}{\rho} \right) > \frac{c_s^2 N^2}{v_a^2} \]

for the interchange instability. Here, with isothermal perturbations on an isothermal atmosphere, we have \( N^2 = 0 \). With \( k_y \neq 0 \) we are removing the interchange mode from our analysis, but we do expect to see the undular instability. The undular instability favours small \( k_y \) since it must act against magnetic tension\[29\].

The combination of these - the toroidal MRI and the undular instability of slow magnetoacoustic shearing waves - was investigated with a shearing wave ansatz by Foglizzo and Tagger\[28, 29\] using a mix of numerical and analytical methods for \( g_z \) and Alfvén speed both independent of height. The undular instability is always present for sufficiently large \( |k_x| \) unless stabilised by diffusion, and is either stabilised around the swing by the differential rotation for weak shear or combines with the toroidal MRI for strong shear. If the shear is weak, \( q \lesssim 1/2 \), then they occur at separate times, with the undular instability at large \( |k_x| \) surrounding in time a purely oscillatory region of moderate \( |k_x| \), which then surrounds in time the toroidal MRI present for \( |k_x| \) around zero. Foglizzo and Tagger also describe the polarisation of the wave - that is, the component of the Lagrangian displacement which dominates the wave: the displacement is largely vertical for large \( |k_x| \) where the undular instability dominates and largely horizontal for small \( |k_x| \) where the toroidal MRI dominates. We perform a similar calculation but with the intent of investigating the dynamo properties of these linear waves.

In this Chapter we shall extend the numerical investigation of LO-N to include the effects of gravity and magnetic buoyancy; that is, we shall study the transient amplification of linear perturbations on a toroidal magnetic field threading an isothermal accretion disc with vertical gravity. Using random initial conditions and ensemble averages we shall characterise the resultant EMFs via their
correlation with the background magnetic field with the aim of understanding the accretion disc dynamo in a disc with vertical structure. We describe our numerics in §4.4.

We shall find that the undular instability may inhibit dynamo action for large \( k_y H \) and enhance it for small \( k_y H \). We shall show that the vertical localisation of the perturbation is strongly influenced by the strength of the undular instability, either amplifying the MRI on the upper side of flux concentrations for weakly buoyant cases, or peaking the perturbation a short way above the maximum of \( B_y \). We shall relate these results to the calculated EMFs of Chapter 3.

4.2 Governing equations

As in Chapter 2, we consider first the vertical force balance of the basic state; we have gained a term due to the magnetic pressure of the toroidal field

\[
0 = -c_s^2 \frac{\partial \rho}{\partial z} - \frac{\partial}{\partial z} \left( \frac{B_y^2}{2} \right) + \rho g. \quad (4.1)
\]

Here, we shall pose some magnetic field \( B_y(z) \) and solve for \( \rho(z) \), keeping some constants free. Obviously if we were to set \( B_y = 0 \) or \( B_y = B_0 \) (const) we would regain our previous expression for \( \rho \),

\[
\rho(z) = \rho_0 \exp(-z^2/2H^2).
\]

and indeed we may simplify Equation 4.1 by extracting this behaviour; we write \( \rho = \tilde{\rho}(z) \exp(-z^2/2H^2) \) and \( B^2 = \tilde{B}^2(z) \exp(-z^2/2H^2) \). We then have

\[
c_s^2 \frac{\partial \tilde{\rho}}{\partial z} = \frac{z}{H^2} \frac{\tilde{B}^2}{2} - \frac{\partial}{\partial z} \left( \frac{\tilde{B}^2}{2} \right)
\]

which may be easily solved for \( \tilde{\rho} \) given a \( \tilde{B} \). For constant \( \tilde{B} \) we find the case of constant Alfvén speed, considered by Kato[60]. We wish to examine clearly the role of buoyancy in the EMFs, and therefore consider an isolated concentration of magnetic flux which may be placed with its centre at any height \( z_c \) above the
midplane. Requiring smoothness in $B(z)$ and $B'(z)$ suggests the simple choice

$$B = \begin{cases} 
\alpha \left( \frac{z}{H} - a \right)^3 \left( \frac{z}{H} - b \right)^3 & b < z/H < a, \\
0 & \text{otherwise},
\end{cases}$$

for some $a, b > 0$, i.e. an order six polynomial which goes to zero smoothly at $z = a$, $z = b$. This gives a polynomial of order 14 for $\dot{\rho}(z)$ with $b < z/H < a$, with one constant which we set by requiring $\rho(0) = 1$ (i.e. $\dot{\rho}(b) = 1$), and leaves an adjustable quantity $\alpha$ which we may use to set the maximum Alfvén speed; the rest of the gas will be unmagnetised, and there shall be a rigid wall at height $H$ above/below the edge of the flux concentration. A concrete example is given in §C.3.

We choose and fix (i.e. we do not evolve in time) a background toroidal field $B_y(z)$ in vertical magnetohydrostasis. On this background we place an infinitesimal perturbation, governed by the mass, momentum and induction equations with a shearing wave ansatz. Were we to remove shear and gravity from these equations we would expect the system to admit three pairs of waves: the fast and slow magnetoacoustic waves and the Alfvén waves. As stated, the slow mode shall be unstable to the undular instability and transiently amplified by the toroidal MRI; the other two waves remain stable. There is the possibility of wave mixing during the swing as discussed in Chapter 2 for a purely hydrodynamic background, but since the possible amplification is of order unity when compared to the tens of orders of magnitude common with the MRI and undular instability we shall disregard these other waves. We shall integrate these problems forwards in time, subject to initial values to be specified in §4.5. We have our linearised momentum
4. Numerical Quasilinear EMFs

equations,
\[
\rho \left( \frac{\partial u_x}{\partial t} - 2\Omega u_y \right) = -ik_x \rho w - ik_x B_y b_y + ik_y B_y b_x + \frac{\partial \sigma_{xj}}{\partial x_j}, \tag{4.2}
\]
\[
\rho \left( \frac{\partial u_y}{\partial t} + (2 - q)\Omega u_x \right) = -ik_y \rho w + B_x \frac{\partial B_y}{\partial z} + \frac{\partial \sigma_{yj}}{\partial x_j}, \tag{4.3}
\]
\[
\rho \frac{\partial u_z}{\partial t} = -\rho \frac{\partial w}{\partial z} + \rho w \left( g - \frac{\partial}{\partial z} (\log \rho) \right) - \frac{\partial}{\partial z} (B_y b_y) + ik_y B_y b_z + \frac{\partial \sigma_{zj}}{\partial x_j}, \tag{4.4}
\]
the induction equations, with \( R_m = c_s^2 / \eta \Omega \)
\[
\frac{\partial b_x}{\partial t} = ik_y B_y u_x + \eta \left( -k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} \right) b_x, \tag{4.6}
\]
\[
\frac{\partial b_y}{\partial t} = -q\Omega b_x - ik_x B_y u_x - \frac{\partial}{\partial z} (B_y u_z) + \eta \left( -k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} \right) b_y, \tag{4.7}
\]
\[
\frac{\partial b_z}{\partial t} = ik_y B_y u_z + \eta \left( -k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} \right) b_z, \tag{4.8}
\]
and the mass equation
\[
\frac{\partial w}{\partial t} = c_s^2 \left[ -ik_x u_x - ik_y u_y - \frac{\partial u_z}{\partial z} - u_z \frac{\partial}{\partial z} (\log \rho) \right], \tag{4.9}
\]
where both the perturbation enthalpy \( w \) and the perturbation toroidal field \( b_y \) will be calculated “off grid” to avoid spurious pressure modes. We use the compressible stress tensor, which, before linearising, is
\[
\sigma_{ij} = \rho \nu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \nabla \cdot u \delta_{ij} \right),
\]
with constant kinematic viscosity \( \nu \). On linearising we gain
\[
\sigma_{ij} = \rho \nu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \nabla \cdot u \delta_{ij} \right) - \rho \nu q \Omega \left( \delta_{ix} \delta_{jy} + \delta_{iy} \delta_{jx} \right) w
\]
where the first term comes from perturbing the velocity and the second from
\footnote{For simplicity we set the compressible bulk viscosity \( \mu_b = 0 \).}
perturbing the density. Then

\[
\frac{\partial \sigma_{xj}}{\partial x_j} = \rho \nu \left( \nabla^2 u_x + \frac{1}{3} i k_x \nabla \cdot \mathbf{u} - q i k_y w \right) + \nu \frac{\partial \rho}{\partial z} \left( \frac{\partial u_x}{\partial z} + i k_x u_x \right)
\]

\[
\frac{\partial \sigma_{yj}}{\partial x_j} = \rho \nu \left( \nabla^2 u_y + \frac{1}{3} i k_y \nabla \cdot \mathbf{u} - q i k_x w \right) + \nu \frac{\partial \rho}{\partial z} \left( \frac{\partial u_y}{\partial z} + i k_y u_y \right)
\]

\[
\frac{\partial \sigma_{zj}}{\partial x_j} = \rho \nu \left( \nabla^2 u_z + \frac{1}{3} \nabla \cdot \mathbf{u} \right) + \nu \frac{\partial \rho}{\partial z} \left( \frac{2}{3} \frac{\partial u_z}{\partial z} - \frac{2}{3} \nabla \cdot \mathbf{u} \right).
\]

For ease of comparison with LO-N we choose a fixed magnetic Prandtl number \( P_m = \nu/\eta = 4 \), along with fixed Reynolds number \( R_e = \frac{c_s^2}{\nu \Omega} \) and magnetic Reynolds number \( R_m = \frac{c_s^2}{\eta \Omega} \) which we fix in §4.5.

### 4.3 EMFs and energy equation

At the heart of every dynamo problem is the EMF present in the induction equation. In this Chapter we use quadratic EMFs to investigate the effect a linear perturbation would have on the background field, and ascertain whether we have possible dynamo action. We recall the non-linear horizontally averaged EMFs we introduced in 1.6

\[
\mathcal{E}_x = u_y b_z - u_z b_y
\]

\[
\mathcal{E}_y = u_z b_x - u_x b_z
\]

(with the horizontal average taken after multiplication) and now calculate the horizontally averaged radial EMF arising from the perturbations, denoted by a tilde. They are

\[
\tilde{\mathcal{E}}_x = \frac{1}{2} \text{Re}[u^*_y b_z - u^*_z b_y]
\]

and the horizontally averaged toroidal EMF from the same

\[
\tilde{\mathcal{E}}_y = \frac{1}{2} \text{Re}[u^*_z b_x - u^*_x b_z].
\]
4. Numerical Quasilinear EMFs

If we recall the mean-field induction equation also introduced in 1.6, assuming axisymmetry and neglecting radial gradients, we find

\[ \frac{\partial B_y}{\partial t} = -q\Omega B_x + \frac{\partial \tilde{E}_x}{\partial z} \]

and

\[ \frac{\partial B_x}{\partial t} = -\frac{\partial \tilde{E}_y}{\partial z} \]

and we may write the first of these in terms of the toroidal magnetic energy

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} B_y^2 \right) = -q\Omega B_y B_x + B_y \frac{\partial \tilde{E}_x}{\partial z} \]

This led LO-N to define the correlation integrals

\[ I = \int dz B_y \frac{\partial \tilde{E}_x}{\partial z} = -\int dz \tilde{E}_x \frac{\partial B_y}{\partial z} + \text{(boundary terms)} \tag{4.10} \]

which measures the direct effect of the perturbation on the background toroidal magnetic field, i.e. the rate at which the perturbation would increase the background magnetic energy directly, and

\[ J = \int dz B_y \frac{\partial \tilde{E}_y}{\partial z} = -\int dz \tilde{E}_y \frac{\partial B_y}{\partial z} + \text{(boundary terms)} \tag{4.11} \]

which measures the effect the perturbation would have on the background radial magnetic field which would then be sheared out to reinforce the original toroidal field. (In integrating by parts we have discarded some boundary terms, since the EMFs must vanish on the boundaries by the no-penetration condition and the horizontal field condition, described in 4.4.1.) These integrals, then, measure the correlations between the perturbation EMFs and the rise or fall in toroidal
energy in the mean field; if either is positive then there is a route to dynamo action. We go further in defining

\[ K = \int dz \text{Re}[u^*_w \left( -B_y \frac{\partial B_y}{\partial z} \right)] \]

which is the correlation of the vertical enthalpy flux with the background magnetic pressure force, with its integrand appearing in the perturbation energy density equation (Equation 4.12) given below. If \( K \) is positive then the magnetic pressure gradient is causing lighter fluid to rise and heavier fluid to sink, meaning that the perturbation is increasing the potential energy of the background state, and vice versa; since we expect the perturbation to grow in part due to the supply of buoyant potential energy from the background we expect \( K \) to be negative in general. This integral did not exist for the incompressible case without vertical gravity because the enthalpy was constant. In this chapter all vertical integrations are done with the trapezium rule.

As usual, we may combine our momentum, induction and mass equations 4.5-4.9 with \( u \), \( b \) and \( w \) respectively to give the energy density equation

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho (|u|^2 + \frac{|w|^2}{c_s^2}) + |b|^2 \right) + \frac{\partial}{\partial z} \text{Re}[u_z (\rho w^* + B_y b_y^*)] = \\
\begin{align*}
q \Omega & \text{Re} \left[ \rho u_x u_y^* - b_x^* b_y \right] + \text{Re} \left[ u_z \frac{w^*}{c_s^2} \left( \rho g - \frac{\partial \rho}{\partial z} \right) + (u_y b_z^* - u_z b_y^*) \frac{\partial B_y}{\partial z} \right] \\
& + \text{Re} \left[ \rho u_i^* \frac{\partial \sigma_{ij}}{\partial x_j} + \eta b_t^* \nabla^2 b_t \right]
\end{align*}
\]

(4.12)

and we see that the local energy density is changed by advection of total pressure (the quantity in conservative form on the first line), the mixed Maxwell/Reynolds stress due to the shear, a term due to deviation from hydrostasis, the radial emf multiplied by \( 2 \frac{\partial B_y}{\partial z} \) (in fact the negative of the integrand of the correlation integral \( I \)) and, on the final line, terms due to diffusion. We may re-write the second term of the second line using the vertical force balance to give the negative
4. Numerical Quasilinear EMFs

of the integrand of our correlation integral $K$;

$$
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho (|u|^2 + \left| \frac{w}{c_s^2} \right|^2) + |b|^2 \right) + \frac{\partial}{\partial z} \text{Re}[u_z (\rho w^* + B_y b_y^*)] =
$$

$$
q \Omega \text{Re} \left[ \rho u_x u_y^* - b_x b_y^* \right] + \text{Re} \left[ B_y u_z \frac{w}{c_s^2} + u_y b_z^* - u_z b_y^* \right] \frac{\partial B_y}{\partial z} + \text{Re} \left[ \rho u_i^* \frac{\partial \sigma_{ij}}{\partial x_j} + \eta b_i^* \nabla^2 b_i \right]
$$

We use the total energy as a check on our code, integrating the energy density equation as a function of space and time (see §C.2.4 for tests of this). We see excellent agreement except in the final decay - when the evolution is dominated by diffusion $\sim \exp(-\tilde{\nu} t^3)$ for some $\tilde{\nu}$ encapsulating all diffusive terms - where we have significant residuals due to the finite accuracy of the temporal integration. These residuals are around five orders of magnitude smaller than the peak value of $E$ and do not affect our conclusions.

4.4 Code

We have written a finite difference code that uses classical RK4 for time integration and 6th order central differences for spatial differentiation. Both the stencils for spatial differentiation and the tests used to verify the code are in Appendix C. We choose our spatial gridscale to be smaller than $R_c^{-1/2}$ to properly resolve the viscous boundary layers at the domain edges. We then calculate our timestep based on the ‘shortest lengthscale’:

$$
l = \frac{2\pi}{\left( k_x^2 + k_y^2 + \frac{4\pi^2}{\delta x^2} \right)^{1/2}} H
$$

before calculating the viscous timescale

$$
\tau_\nu = \frac{l^2}{H^2 \Omega R_c}
$$
the diffusive timescale

\[ \tau_\eta = \frac{l^2}{H^2 \Omega R_m} \]

the buoyant timescale

\[ \tau_b = \left( c_s^2 + v_{\text{amax}}^2 \right)^{1/2} \frac{1}{|z|_{\text{max}} \Omega^2} \]

and the fast magnetic-acoustic timescale

\[ \tau_a = \frac{l}{(c_s^2 + v_{\text{amax}}^2)^{1/2}}. \]

Given these, we choose a constant timestep which is one-tenth the shortest calculated timescale. We take our validation of the energy equation as evidence that we are evolving these equations correctly.

4.4.1 Boundary conditions

We will consider a domain of total height \(3H\) with an isolated flux concentration of height \(H\) in its centre, with vertical gravity and rigid walls at the domain boundary. We enforce no-penetration in both the vertical velocity and magnetic field, e.g.

\[ u_z|_{\text{boundary}} = 0 \]

\[ u_z|_{\text{boundary}+i} = - u_z|_{\text{boundary}−i} \]

where \(i\) ranges from 1 to the number of ghost cells. We have vanishing vertical gradients in the horizontal velocity and magnetic fields (i.e. no viscous stress on the boundary), and in the enthalpy; e.g.

\[ u_x|_{\text{boundary}+i} = u_x|_{\text{boundary}−i} \]
while for the enthalpy and toroidal field, off-grid quantities, we have

\[ w_{| \text{boundary}+i-1/2} = w_{| \text{boundary}+i+1/2} \]

All of these conditions are enforced every quarter-timestep.

### 4.5 Method

We would like to examine the quasilinear EMFs generated by the mixed toroidal MRI/undular instability, and to that end we shall take an ensemble average over many integrations with randomly generated initial conditions. We must clearly identify physically interesting behaviour that is due to the mixed MRI/undular instability as opposed to the pure MRI studied by LO-N and LO-A. We first inhibit the undular instability by choosing a toroidal wavenumber of \( k_y H = \pi \); after we reproduce qualitatively the \( g_z = 0 \) results of LO-N we shall reduce our toroidal wavenumber to examine the undular instability.

For ease of comparison, we choose as many of our variables to agree with LO-N as possible. Since they were considering an incompressible fluid they had no vertical scale and nondimensionalised using their box height \( L_z \); we take our analogous lengthscale to be the sonic length \( H = c_s/\Omega \), with a vertical extent of \( 3H \). We may then fix \( R_e = 4200 \) and \( R_m = 17000 \) to follow their less viscous case (their definition of \( R_e \) includes a factor of \( q \) and so is \( 3/2 \) times greater than ours). For these Reynolds numbers we use 2600 grid points in the vertical direction (or around 6 grid points per resistive lengthscale and 13 grid points per viscous lengthscale). We also consider much stronger fields than they, who examined \( v_{a \text{max}} \in [0.02, 0.3] \times q\Omega L_z \) (or, in our units and taking \( L_z \) to \( H \), \( v_{a \text{max}} \in [0.03, 0.45]c_s \)). Since we have the added complication of a finite sound speed \( c_s \), we shall consider \( v_{a \text{max}} \in [0.05, 1.3]c_s \). We shall see in Chapter \([5]\) that this is a reasonable interval to investigate.

Although the MRI will be active for all \( 0 < \omega_a^2 < 2q\Omega^2 \), LO-N showed that the sign reversal of \( J \) to be governed by the sign of \( (B_y^2)' \) at the most unstable location where growth will be concentrated. For the large \( k_z H \) case without vertical gravity the maximum growth rate will occur where \( \omega_a = \sqrt{15}/4\Omega \approx 0.97\Omega \). We
thus would expect $J$ to change sign at $v_{a\text{max}} \approx 0.31c_s$ for $k_yH = \pi$, and at $v_{a\text{max}} \approx 0.62c_s$ for $k_yH = \pi/2$ if there were no effect from magnetic buoyancy.

As described in §4.2, we are considering an isolated concentration of magnetic flux of height $H$ with its centre at distance $z_c$ from the midplane. We place rigid walls one scale height above and below the top and bottom of the flux concentration, giving a vertical extent of $3H$ in total. This mimics the situation in a vertically complete accretion disc, where each flux concentration will be surrounded by other flux concentrations; reflections from the nearby barriers will serve as waves that would be present in the disc. To understand the effects of buoyancy we shall vary the distance $z_c$ of the flux concentration from the midplane (thus varying gravity, which is $\propto -z$).

Since the undular instability acts against magnetic tension by bending field lines vertically, a large $k_yH$ inhibits the instability; Foglizzo and Tagger (FT2) gave a condition for stability (with constant gravity and Alfvén speed) as

$$k_y^2H^2 > \frac{g_z^2}{4\Omega^2 \left( c_s^2 + \frac{v_a^2}{2} \right)}$$

which is to say that the stronger the gravity, the easier it is to overcome magnetic tension and be unstable (and so larger wavenumbers become unstable), and that the stronger the field, the smaller the toroidal wavenumber must be to avoid bending field lines. In our case we have $g_z \propto -z$ and expect the critical $k_y$ for stability to increase with height, but using the above inequality as a rough prediction we expect that for $k_yH = \pi$ the undular instability is inhibited for all field strengths and heights that we consider, while for $k_yH = \pi/2$ the undular instability would be first destabilised for $z_c = 2.5H$ in the absence of rotation and shear.

We initialise our grid at $t = -10/\Omega$ with noise on the grid scale to ensure that all wavelengths and phases are present. We use the boost libraries and integers read from /Dev/urandom to act as two seeds; these seeds initialise a uniformly distributed random variable $A$ on $(0, 1)$ from which the starting amplitudes are chosen, and a uniformly distributed random variable $\Phi$ on $(0, 2\pi)$ from which starting phases are chosen. These distributions are then sampled once for each
grid point to give an incompressible initial condition at grid point \( i \) of

\[ u_x |_{i} = A_i \exp(i\Phi_i), \text{ and } u_y |_{i} = -k_x u_x / k_y \]

and all other quantities zero. This initial condition has zero correlation length at the grid scale and may be thought of as ‘blue noise’, which favours higher wavenumbers. Finally we rescale the whole perturbation such that the perturbation energy \( E = \rho(0)c_s^2 \) initially to allow us to take an ensemble average over perturbations with the same initial energy.

### 4.6 Numerical results

#### 4.6.1 Correlation Integrals

We show in Table 4.1 the change in behaviour of \( J \) as a function of Alfvén speed and height above the midplane, with \( k_y H = \pi \) (i.e. inhibiting the undular instability), corresponding to LO-N. The behaviour for \( z_c = 0.5H \) - i.e. where buoyancy is weak - is consistent with that of the case without vertical gravity. For weak fields \( J \) is positive and dynamo action is possible, and as the field strength increases \( J \) changes sign at around \( v_{a\max} \approx 0.31c_s \), the Alfvén speed predicted above using the work of LO-N. For very weak fields the form of \( J \) is dominated by fast magnetoacoustic waves which could not exist in an incompressible fluid, making the overall sign of \( J \) unclear; an example of this is plotted in Figure 4.3.

On increasing \( z_c \) the correlation integral \( J \) changes sign at a weaker \( v_{a\max} \) i.e. the effect of buoyancy here is to somewhat inhibit dynamo action.

We show in Table 4.2 the same data for \( k_y = \pi/2 \), with weaker magnetic tension and so a stronger undular instability. As expected, the critical \( v_{a\max} \) for the sign change of \( J \) doubles compared to Table 4.1 when \( z_c = 0.5 \) because we have now halved our toroidal wavenumber and Alfvén frequency. However, as the flux concentration is raised away from the midplane we see that the critical \( v_{a\max} \) is raised even further, so that the effect of buoyancy here is to reinforce dynamo action. This dependence on height is a clear indication that the undular instability playing an important role in \( J \).
Maximum Alfvén speed
\[ \frac{z_c}{H} \]
\begin{array}{ccccccc}
0.05 & 0.1 & 0.25 & 0.4 & 0.7 & 1.0 & 1.3 \\
0.5 & \sim & ✓ & ✓ & x & x & x & x
\end{array}

Table 4.1: Summary of exploration of parameter space for \( R_e = 4200, k_y H = \pi \); a green tick indicates that \( J \) was positive and radial field will reinforce the original toroidal field, a purple cross indicates the opposite, and an orange dash indicates that \( J \) was of mixed sign. The toroidal EMF is constructive for weak fields and destructive for strong fields; as the flux tube is raised the field strength at which the sign of \( J \) changes decreases. Plots for this table are shown in §C.3.

\[ \frac{z_c}{H} \]
\begin{array}{ccccccc}
0.05 & 0.1 & 0.25 & 0.4 & 0.7 & 1.0 & 1.3 \\
0.5 & \sim & ✓ & ✓ & ✓ & x & x & x
\end{array}

Table 4.2: Summary of exploration of parameter space for \( R_e = 4200, k_y H = \pi/2 \); as expected the sign of \( J \) changes at roughly twice the previous critical \( \nu_a \) when buoyancy is weak, but when buoyancy is strong dynamo action persists even for strong fields. Plots for this table are shown in §C.3.
Figure 4.1: The ensemble averages of $|J|$ against time for $k_y H = \pi/2$ and $v_{a_{\text{max}}} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\}$ (green, blue, purple, cyan, gold, pink and khaki respectively) centred at a height of $z = 0.5H$ above the midplane. The lines are solid if $J > 0$ and dotted if $J < 0$. All lines begin with $J > 0$ and eventually suddenly switch to $J < 0$; as $v_{a_{\text{max}}}$ increases the time of this switch decreases until $J$ is unambiguously negative after the swing for $v_{a_{\text{max}}} \geq 0.7c_s$. 

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Figure 4.2: $|J|$ against time as in Figure 4.1 but with the flux concentration centred on $z = 3.5H$ (note the different vertical scale). The sign of $J$ still changes, but the critical time decreases more slowly with $v_a^{\text{max}}$; even for $v_a^{\text{max}} = 1.0$ (pink line) there is a long period of exponential growth in which $J$ is positive. It is likely that these shearing waves would become nonlinear before they reach the time at which $J$ changes sign, and so possible that their constructive linear behaviour outweighs their (truncated by nonlinearity) destructive linear behaviour.
Figure 4.3: The correlation integrals $I$ (green), $J$ (blue) and $K$ (purple) against time for a single run with $k_y H = \pi$, $v_a = 0.05c_s$ and $z_c = 0.5H$ i.e. a weak field low in the atmosphere. If $I > 0$ then the line is solid, and if $I < 0$ the line is dotted, and similarly for the other integrals. $I$ is consistently negative during the peak, but $J$ and $K$ oscillate on a sonic timescale. For stronger fields this oscillation would be completely dominated by the exponential growth.
We see in Figure 4.3 that the correlation integral $I$ for a single run is unambiguously negative and destructive (and this was typical for all runs) while the correlation integrals $J$ and $K$ oscillate rapidly in time. This fast oscillation motivates the ensemble average taken across the runs; these are presented in §C.3. After this averaging $K$ was consistently negative, consistent with a buoyant release of potential energy. The changing behaviour of $J$ with time for $k_yH = \pi/2$ and with field strength is shown in Figure 4.1 where $z_c = 0.5H$, and Figure 4.2, where $z_c = 3.5H$. With this small toroidal wavenumber we see the time at which $J$ changes sign decreases with increasing Alfvén speed, but decreases slower for flux concentrations higher in the atmosphere.

LO-N found that $J$ changed sign as a function of the Alfvén frequency $\omega_a$. On the addition of vertical gravity we have found that this dependence on $\omega_a$ is itself affected by the height above the midplane. To describe the process more concretely we examine the vertical structure of the perturbation during the quasi-exponential phase of growth.

4.6.2 Vertical structure of perturbations

The “turning off” of the dynamo as described by LO-A was due to a change in location of the maximum growth rate. We will examine in this section snapshots of the vertical structure of perturbations with vertical gravity and $k_yH = \pi/2$ and make a similar deduction, with our focus on $v_{a,\max} \in \{0.25, 1.0\}c_s$ and the flux concentration centred on $z_c \in \{0.5, 3.5\}H$ i.e. a weak or strong field, high or low in the atmosphere. We choose $v_{a,\max} = 0.25c_s$ as our ‘weak’ case to avoid $J$ being dominated by the fast magneticacoustic oscillations which would obscure the interesting physics.

We show the typical structure of $\mathcal{E}_y$ resulting from a weak and low run in Figure 4.4 and the typical level of poloidal incompressibility (a key assumption in Chapter 3) in Figure 4.5. Recall that LO-A found that $\mathcal{E}_y \propto \partial |u_z|^2 / \partial z$, which is of potentially large size $\sim 2k_z |u_z|^2$ for our finite difference method. We again take an ensemble average over the hundred runs to largely remove this contribution, and in Figures 4.6 and 4.7 plot $|u_z|^2$ with $\mathcal{E}_y$ and $\text{Re}[u_z^* w]$, all renormalised such that the positive peaks of $|u_z|^2$ and $\mathcal{E}_y$ are equal to one, and $\text{Re}[u_z^* w]$ renormalised

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to a half for ease of viewing. (In general we did find that $|K|$ was much larger than $|J|$ despite $\sqrt{\rho_0 c_s^2} |\text{Re}[u^*_z w]|$ being much smaller than $|\mathcal{E}_y|$.) For dynamo action we need the integrand of $J$ to be positive - i.e. $\mathcal{E}_y \frac{\partial B_y}{\partial z} < 0$, or the blue line to be anti-correlated with the black line upon which it is superimposed. Since there are effectively eight cases to discuss we resort to bullet points for clarity:

- Consider $k_y H = \pi$, which inhibits the undular instability. The following cases are plotted in Figure 4.6
  
  - Low in the atmosphere, at $z_c = 0.5 H$, we see the case considered and explained by LO-A: for weak fields (top left quarter) the most unstable location is near the maximum of $B_y^2$ (with a small displacement from a weak undular contribution), the EMF $\tilde{\mathcal{E}}_y$ is anticorrelated with $B_y'$, and dynamo action is possible with $J > 0$. For strong fields (bottom left quarter) the most unstable location is far from the peak of $B_y^2$ (and above, due to the weak undular instability), $\mathcal{E}_y$ is not anticorrelated with $B_y'$, and dynamo action is not present with $J < 0$.
  
  - High in the atmosphere, at $z_c = 3.5 H$, we see that the most unstable location is slightly above the peak of $B_y^2$ and is only weakly determined by the field strength. We have an asymmetry in $\tilde{\mathcal{E}}_y$ that emphasises the negative contribution to $J$. (E.g. in the upper right panel we see that the negative peak of $\mathcal{E}_y$ is $1.5 \times$ larger than its positive peak.)

- Now consider $k_y H = \pi/2$, which permits the undular instability. The following cases are plotted in Figure 4.7. We note that we now have a strong negative correlation between $\text{Re}[u^*_z w]$ and $\tilde{\mathcal{E}}_y$; in those locations that $\mathcal{E}_y$ is positive (and so giving a constructive contribution to $J$ given $B_y' < 0$ in the region shown) we have $\text{Re}[u^*_z w]$ negative, that is, the flow is moving lighter fluid upwards and heavier fluid downwards. Similarly we have $\mathcal{E}_y$ negative implying $\text{Re}[u^*_z w]$ positive, and the flow acting to increase buoyant potential energy.
  
  - Low in the atmosphere (left column), at $z_c = 0.5 H$, we may again apply the argument of LO-A: for weak fields (top left quarter) the most unstable location is near the maximum of $B_y^2$, leading to a possible
dynamo with \( J > 0 \), and for strong fields (bottom left) the unstable location is such that \( \tilde{E}_y \) is positively correlated with \( B'_y \) and \( J < 0 \).

- High in the atmosphere (right column), at \( z_c = 3.5H \), the location is entirely determined by buoyancy; the most unstable location does not significantly change when the field strength is increased. Since the most unstable location for this case is similar to that for \( k_y = \pi, v_a = 0.25c_s \) and \( z_c = 3.5H \) we must explain why \( J \) is now positive. The explanation is that the asymmetry in \( \tilde{E}_y \) that gave us a negative \( J \) for the case where \( k_y H = \pi \) is not present for \( k_y H = \pi/2 \).

In the lower right quarter of Figure 4.7 we have plotted \( \tilde{E}_x \) and \( \text{Re}[u_z w^*] \) (fuchsia and cyan lines, respectively) for \( t = +40/\Omega \) when \( J < 0 \); even the slight growth in the asymmetry of \( \tilde{E}_y \) is enough to destroy our route to a dynamo.

We would like to use this asymmetry in \( \tilde{E}_y \) to explain the behaviour shown in Tables 4.1 and 4.2: the decrease in the critical \( v_{a_{\text{max}}} \), at which the net \( J \) over the run changes sign, for \( k_y H = \pi \), and the increase in the same critical \( v_{a_{\text{max}}} \) for \( k_y H = \pi/2 \).

The asymmetry in \( \tilde{E}_y \) itself may be easily understood by consideration of LO-A’s result that \( \mathcal{E}_y \sim B_y \partial_z |u_z|^2/\gamma \), with \( \gamma \) the local linear growth rate which we discussed in Chapter 3. Consider a \( |u_z|^2 \) which is locally symmetric in \( z \), with its peak slightly above the peak of \( B_y^2 \). Then LO-A’s model for the \( \partial_z |u_z|^2/\gamma \) is magnified by multiplication by the larger \( B_y \), and so unsurprising that this asymmetry occurs - indeed, it is present to a greater or lesser extent in all panels of Figures 4.6-4.7. We must explain not the asymmetry but the lack of asymmetry which allows the constructive cases to generate dynamo action.

We appeal to the \( \alpha \)-effect, derived in Chapter 3, which vanishes without vertical gravity. The (ideal) analytic form of said effect for \( k_x \gg k_z \) is

\[
\mathcal{E}_{y1}^\alpha \sim \frac{k_y B_y}{k_x} \left( \frac{v_a^2}{v_a^2 + c_s^2} \right) \frac{1}{|\gamma|^2 + \omega_c^2 |2} \left( \frac{\omega_c}{c_s^2} (|\gamma|^2 + \text{Re} \left[ \gamma^2 \omega_c^2 \right]) \text{Re} \left[ \frac{1}{\gamma} \right] |\tilde{u}_z|^2 \right)
\]

for \( k_x \gg k_z \), and we compare the sizes of this term’s coefficient for when \( k_y = \pi \) and \( k_y = \pi/2 \) - with the caveat that with a low \( k_z \) the analytic expression can
only give us qualitative guidance rather than firm numerical results. Very roughly, then, with $v_a = 0.25c_s$ and $z = 3.5H$ (all other quantities e.g. $\gamma$ measured from data) we have for $k_yH = \pi$ that

$$E_{y_1}^\alpha \approx (640 \times 10^{-6}) \rho_0^{1/2} |u_z|^2$$

while for $k_yH = \pi/2$ we have that

$$E_{y_1}^\alpha \approx (1151 \times 10^{-6}) \rho_0^{1/2} |u_z|^2$$

i.e. the $\alpha$-effect is $1.80\times$ larger for $k_yH = \pi/2$ than for $k_yH = \pi$. We may ask the same question for the $v_a^\text{max} = 1.0c_s$ case and find that we have an $\alpha$-effect that is $1.5\times$ as large for $k_yH = \pi/2$ as for $k_yH = \pi$, a difference which we attribute to the factor $1/|\gamma^2 + \omega_c^2|^2$. The contribution from the $\alpha$-effect will not have an odd symmetry around the peak of $|u_z|^2$ but will make $\tilde{E}_y$ more positive at all points. This everywhere-positive contribution will offset the negative contribution to $J$ that comes from the asymmetry in $E_y$.

Finally we discuss the changing of the sign of $J$ with time. As the instability proceeds, the $|u_z|^2$ profile will grow more sharply peaked. We estimate the contributions to $J$ from the positive and negative lobes in $\tilde{E}_y$ as

$$-(\text{Size of -ve lobe in } \tilde{E}_y \times B'_y(z = \text{peak of -ve lobe}) + \text{Size of +ve lobe in } \tilde{E}_y \times B'_y(z = \text{peak of +ve lobe})).$$

Since the most unstable location lies just on the upper edge of flux concentrations we expect the $B'_y$ multiplying the positive lobe to be larger than the $B'_y$ multiplying the negative lobe. If we follow the instability forwards in time this contribution will approach

$$-(\text{Size of -ve lobe} + \text{Size of +ve lobe}) \times B'_y(z_{\text{peak}})$$

and so even a minor asymmetry in $\tilde{E}_y$ will eventually force $J$ to change sign.

In summary, all cases at $z_c = 0.5H$ may be explained by reference to the argument of LO-A concerning the most unstable location. At $z_c = 3.5H$ the
most unstable location becomes less sensitive to field strength and the sign of $J$ is instead controlled by an asymmetry in $E_y$ which arises naturally from the undular instability; for small $k_y H$ the $\alpha$-effect can reduce this asymmetry and maintain dynamo action in face of increasing $v_a^2$. 
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Figure 4.4: Vertical structure of $E_y$ for a typical run (green line), and for the ensemble average which removes the fast oscillations (purple line). The parameters here were $k_y \pi = H$, $v_{a}^{\text{max}} = 0.25c_s$, $z_c = 0.5H$ and $t = +20/\Omega$.

4.6.3 Conclusion

We have examined the numerical quasi-linear EMFs from a single shearing wave subject to the mixed toroidal MRI/undular instability in an isothermal disc with vertical gravity. We have confirmed that the dynamo loop proposed by OL-N is essentially unchanged near the midplane and described its evolution with height.

We calculated these EMFs and associated correlation integrals, using a new, fully tested, finite difference code to vary the background magnetic field through a variety of field strengths and vertical positions, noting that the previous correlation integrals $I$ and $J$ have been joined by a further integral $K$ which measures the rate of increase of buoyant potential energy in the background; although $K$ was of consistently negative sign, we examined its integrand and showed a good negative correlation of $\text{Re}[u_z w]$ with $\tilde{E}_y$ for $k_y = \pi/2$. 
4. Numerical Quasilinear EMFs

Figure 4.5: Vertical structure of \( \text{Re}[ik_z u_z] \) and \(-\text{Re}[\partial_z u_z] \) for \( k_y H = \pi/2 \), \( v_a^\text{max} = 0.25c_s \), \( z_c = 0.5H \) and \( t = +20/\Omega \). We have poloidal incompressibility and a moderate vertical wavenumber \( k_z H \approx 8\pi \). All other cases - stronger fields or larger \( z_c \) - had smaller vertical wavenumber but similar levels of poloidal incompressibility. Since \( k_x \gg k_z \) we see that the polarisation of the perturbation is that of the undular instability.
Figure 4.6: Ensemble average of vertical structure for \( k_y H = \pi \) at \( t = +10/\Omega \), with \(|u_z|^2\) (green) and \(i_\gamma\) (blue) renormalised to peak at +1, while \(\text{Re}[u_z^* w]\) (purple) renormalised to peak at +1/2, with black line on upper panels \(B_y\) and black lines on lower panels \(B'_y\). Left column has flux centred on \(z = 0.5H\), right column has flux centred on \(z = 3.5H\); upper four panels have \(v_a^{\text{max}} = 0.25c_s\), and the lower four have \(v_a^{\text{max}} = 1.0c_s\). For strong fields the perturbation is confined such that \(\tilde{i}_\gamma\) and \(B'_y(z)\) are not anticorrelated i.e dynamo action is inhibited.
Figure 4.7: Ensemble average of vertical structure for $k_y H = \pi/2$ at $t = +20/\Omega$, with $|u_z|^2$ (green) and $E_y$ (blue) renormalised to peak at $+1$, while $\text{Re}[u^*_zw]$ (purple) renormalised to peak at $+1/2$, with black line on upper panels $B_y$ and black lines on lower panels $B'_y$. Left column has $z_e = 0.5H$, right column $z_e = 3.5H$; upper four panels have $v'^{\text{max}}_a = 0.25c_s$, the lower four $v'^{\text{max}}_a = 1.0c_s$. For high $z_e$ the most unstable location is unaffected by field strength, and $\text{Re}[u^*_zw]$ is anticorrelated with $E_y$. In bottom right panel cyan ($\tilde{E}_y$) and fuchsia ($\text{Re}[u^*_zw]$) are from $t = +40/\Omega$ when $J < 0$. 

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We have shown that the mixed MRI/undular instability proceeds on the upper side of isolated flux concentrations; a bias introduced by the undular instability. For flux concentrations low in the atmosphere we could apply the same argument regarding the most unstable location used by LO-A; the most unstable location shifts from near the peak of $B_y$ for weak fields to weaker and weaker locations as the field gets stronger and stronger, and this localisation in turn changes the sign of the correlation integral $J$ from positive to negative. We showed that perturbations on flux concentrations high in the atmosphere have an almost fixed most unstable location rather than one determined by field strength, and that dynamo action for strong fields is inhibited there not by a change in the most unstable location but by a growing asymmetry in $E_y$. For cases with a smaller toroidal wavenumber we related the survival of dynamo action far from the mid-plane to the constructive $\alpha$-effect derived in our Chapter 3, the $\alpha$-effect offsetting the effects of the asymmetry. In particular, this $\alpha$-effect meant that those runs with a small toroidal wavenumber - i.e. with a strong buoyancy instability and a low cusp frequency - showed enhanced dynamo action beyond what would be expected from the halving of the Alfvén frequency.

The low wavenumber of our perturbations high in the atmosphere suggest that our high-$k_z$ analytic predictions in Chapter 3 will be most accurate when the magnetic field is weak and low in the atmosphere.

We thus conclude that if the undular instability is able to proceed freely - i.e. if the shearing box is sufficiently wide - then the dynamo action suggested by our ensemble average of non-interacting shearing waves is enhanced. Further, we have shown that in these cases the constructive dynamo action is correlated with regions where buoyant magnetic energy is being released from the background.
Chapter 5

Shearing box simulations

I am reviewing
My simulation
And I hope it holds the answers
that I seek
With a lengthscale!
A perturbation
And its mean-field evolution I
shall treat.

With apologies to Lionel Bart
(Oliver!)

5.1 Introduction

We shall investigate the EMFs calculated from shearing box simulations with vertical gravity and zero net magnetic flux. We shall employ both horizontal and temporal averages and thence reveal the changing nature of the dynamo with z. We find excellent agreement between our analytic prediction and the mean behaviour of our turbulent simulations for $E_x$, and good agreement for $E_y$. We use our model to explain the dependence of the dynamo on box size.

The central problem of accretion disc physics is the mechanism of radial transport
of quantities such as angular momentum, often via turbulent stresses. Shearing box simulations have been used to estimate the size of turbulent stresses arising from the MRI, amongst the other things briefly described in the Introduction to this Thesis. These turbulent stresses were found by Fromang and Papaloizou [37] to converge towards zero with increasing resolution in the absence of explicit diffusivities and vertical gravity. It was then shown by Davis, Stone and Pessah [22] that the turbulent stresses of simulations with vertical gravity did converge to a finite value. Since this point the literature has had its focus on simulations including vertical gravity.

These turbulent stresses are influenced by the dynamo, which appears in ideal simulations as a “butterfly”, so named after the solar butterfly diagram of sunspot activity. It arises in the sense of horizontally averaged toroidal flux being created at the midplane and propagating symmetrically into the upper atmosphere; for an image of the dynamo without vertical gravity see Käpylä and Korpi [58], while for the best images of the dynamo with vertical gravity see Simon, Beckwith and Armitage [93] or Figure 5.1 in this Chapter. This dynamo can be strongly influenced by diffusion, even without vertical gravity: Fromang et al. [34] showed that for each fixed magnetic Prandtl number \( P_m = \nu/\eta \) there existed a minimum Reynolds number below which MRI turbulence would simply decay. With vertical gravity Simon et al. [92] showed that with a small amount of ambipolar diffusion the midplane toroidal field becomes single-signed, with the toroidal field far from the midplane of opposite sign and an oscillating region separating them (their Figure 10). From Gressel [44] we see that the ideal picture is largely unchanged on the addition of Ohmic dissipation i.e. we see the usual butterfly diagram. Wardle and Ng [112] state that for dusty protoplanetary discs with grains around 1 \( \mu \)m Ohmic dissipation should dominate the Hall term and ambipolar diffusion. In this Chapter we consider the ideal case rather than the case with Ohmic dissipation for numerical ease.

The angular momentum transport calculated depends on the domain size: the turbulent stresses and angular momentum transport more than double when moving from a radial extent of \( L_x = 8H \) to \( L_x = 32H \) in simulations by Guan and Gammie [40], with midplane correlation length \( \sim 2H \) and coronal correlation length \( \sim 10H \). Simon et al. [93] found a radial extent of \( L_x > 2\sqrt{2}H \) is required
for stresses to converge, with strong zonal flows developing for $L_x > 4\sqrt{2}H$, and
the variability of the turbulence decreasing until $L_x = 16\sqrt{2}H$. The apparent
contradiction between these results probably arises from Simon et al. having held
the horizontal aspect ratio fixed at $L_y = 2L_x$, and Guan and Gammie having held
the azimuthal extent fixed at $L_y = 20H$.

The horizontal boundary conditions for the shearing box were discussed in the
Introduction in §1.5: the toroidal direction is periodic and the radial direction
shearing periodic. The type of vertical boundary condition used strongly in-
fluences dynamo calculations without vertical gravity. For momentum, shearing
boxes may have rigid upper and lower boundaries, which might be no-penetra-
tion and no-stress, or they may have outflow boundary conditions, which allow mass,
momentum and horizontal magnetic flux to escape the box. For the magnetic
field, the upper boundary might enforce a horizontal or vertical field, and the
conserved net vertical flux penetrating the box might be zero or non-zero. In this
Chapter we shall use vertically periodic boundary conditions for all quantities,
which requires a levelling off of the gravitational potential which we describe in
§5.2.

Bodo et al.[15] refer to the results of Davis, Stone and Pessah[22] (vertically
periodic), Shi et al.[91] (outflow) and Oishi and Mac Low[80] (“periodic, perfect
conductor, or vertical field”) to argue that the momentum boundary conditions
are irrelevant to the dynamo in a shearing box with vertical gravity but without
net vertical flux, given the Gaussian decrease in density with $|z|$. They continue
on to show, using rigid vertical boundaries, that the butterfly dynamo solution can
be disrupted by temperature differences that lead to convective vertical mixing.
In the light of the Simon, Beckwith and Armitage paper mentioned above it
seems likely that their simulations are in too small a domain to be converged,
with $\sqrt{2}H \times \pi \sqrt{2}H \times 6\sqrt{2}H$, and their calculation is shown by Gressel[45] to rely
solely on the presence of rigid vertical boundaries rather than outflow boundary
conditions.

The conserved net vertical magnetic flux is an important global constraint
and is used in simulations that estimate mass and angular momentum loss due
to disc winds. Since the vertical flux is conserved in time there is a mechanism
for energy to enter the simulation at all times via the MRI. Consequently, these simulations tend to be more vigorous than ZNF simulations and have significant mass loss with outflow boundary conditions, presenting a significant numerical challenge. Suzuki, Muto and Inutsuka [96] argued that these winds are able to create a gap between the inner edge of the accretion disc and the central object for weak fields (a plasma-β of $10^4$) with maximum vertical outflow speeds around $1.5c_s$. Bai and Stone [4] increased the field strength (to a plasma-β as small as $10^2$) and found outflow velocities of up to $5c_s$. These results must be viewed in light of Fromang et al. [32] who report outflows with a fast magnetosonic transition point (i.e. the height at which the outflow passes the fast magnetoacoustic speed) which tracks the height of the computational domain, as well as net mass loss sharply decreasing with increasing horizontal extent. As time passes and more computing power becomes available these simulations will probably be supplemented by global simulations (see e.g. Flock, Henning and Klahr [27] for a non-ideal global simulation with vertical gravity and zero net vertical flux) which will avoid this dependence on horizontal extent.

We will be relating our analytic estimate from Chapter 3 to the mean behaviour of the dynamo in a shearing box with vertical gravity. We found in this analytic estimate the recurrent quantity $|u_z|^2/\gamma$ with $\gamma$ the (complex) growth rate of the mixed MRI/undular instability. This quantity could be thought of as a velocity multiplied by a mixing length $l$ similar to that considered by Balmforth and Gough [8] in the context of Solar convection; they took

$$l = \frac{\gamma}{2|u_z|}.$$

We will be investigating a model of the form $\mathcal{E}_z \propto |u_z|^2B_y$. A similar relationship with their EMFs $\propto |u|B_y$ was briefly discussed for a ZNF simulation with vertical gravity by Davis, Stone and Pessah [22]. A previous 1D mean-field model as a function of $z$ and $t$ was considered by Gressel [14] with coefficients derived from simulations using the “test-field” method rather than analytically. We also mention as important and relevant - although well outside the scope of this thesis - the work of Herault et al. [53] who demonstrate a method for identifying exact
periodic dynamo solutions in the shearing box.

In this Chapter we shall perform several ZNF simulations of a shearing box with vertical gravity, of increasing horizontal extent. We shall apply a simple but effective temporal averaging to reveal the detail of the horizontal magnetic field and the EMFs, the interplay of the MRI and the undular instability, and the changing tilt angle of the magnetic field as it rises. We shall successfully relate these twice-averaged quantities to our previous analytic work, suggesting that the toroidal EMF (which creates the radial field) relies on the largest scales available in the box. We shall briefly discuss possible future work.

5.2 Code, boundary conditions and resolution

We make use of the second-order Godunov code RAMSES first developed by Teyssier [99], together with the MHD extension to that code implemented by Fromang et al. [31] and the FARGO orbital advection scheme described by Stone et al. [4] (again, implemented for RAMSES by Fromang et al. [32]).

We choose the simplest boundary conditions available by taking a vertically periodic box, together with the usual shearing-periodic boundary conditions (described in §1.5). This has the advantage of conserving mass and net radial and vertical magnetic fluxes; since we shall initialise our simulations with no net radial flux we shall conserve net toroidal flux as well. It does require us to artificially smooth our gravitational potential to some constant value above some specified |z|. This is similar to Davis, Stone and Pessah [22]: they take as their gravitational potential

\[ \Phi(z) = \left( \left( \frac{L_z}{2z} - 1 \right)^2 + 0.01 \left( \frac{L_z}{2z} \right)^2 \right)^{1/2} - \left( \frac{L_z}{2z} \right)^2 \times \left( \frac{z^2 \Omega^2}{2} \right) \]

giving them a single point on the boundary at which \( \Phi'(z) \big|_{|z|=L_z/2} = 0 \) together with a gravitational potential which is everywhere close to Keplerian. For \( L_z = 2\sqrt{2}H \) they found this gave them a pileup of strong flux on the boundary which
was not present for $L_z = 6\sqrt{2}H$. We impose the following gravitational potential:

$$\Phi = \begin{cases} 
    z^2\Omega^2/2, & |z| \leq z_i \\
    H^2\Omega^2P(z/H), & z_i < |z| < z_o, \\
    H^2\Omega^2P(z_o/H), & |z| \geq z_o.
\end{cases}$$

where $P$ is the unique quartic polynomial satisfying $P(z_i/H) = (z_i/H)^2/2$, $P'(z_i/H) = z_i/H$, $P''(z_i/H) = 1$ and $P'(z_o/H) = P''(z_o/H) = 0$. This gives a twice-differentiable potential which smoothly joins the correct Keplerian potential to a constant, i.e. the acceleration is differentiable at all points and goes smoothly to zero at $z_o$. For our simulations here we take $z_i = 4.75H$, $z_o = 5.00H$ with the simulation boundary at $|z| = 5.25H$. We expect that this smoothing should not affect the midplane dynamics and we do indeed find results consistent with those simulations that use outflow boundary conditions. This choice of potential is similar to Fromang and Papaloizou’s investigations into dust settling using Nirvana, although ours is twice differentiable everywhere.

Our initial magnetic field is

$$\mathbf{B} = B_x e_x, \quad B_z = B_0 \sin(2\pi x/H)$$

with $B_0 = \rho^{1/2}c_s/\sqrt{500}$, i.e. an initial plasma-$\beta$ of 1000 at $z = 0$. We then seed the momentum field with random fluctuations on which the MRI will grow. We shall always choose our radial extent to be an integer multiple of $H$ to give our box zero net vertical magnetic flux.

We present three ideal simulations:

- Run 1: $3H \times 3H \times 10.5H$
- Run 2: $5H \times 5H \times 10.5H$
- Run 3: $10H \times 10H \times 10.5H$

each with 23 grid points per scale height, following Simon, Beckwith and Armitage.
5.3 Results

We plot in Figures 5.1, 5.2 and 5.3 the horizontally averaged quantities $B_y(z,t)$, $\mathcal{E}_x$ and $\mathcal{E}_y$ respectively, for Runs 1, 2, and 3. (In this Chapter we shall always consider horizontally averaged quantities unless otherwise stated.) In Figure 5.1 we have the familiar disc “butterfly diagram” wherein field propagates outward from the midplane. We see clearly the result of Simon, Beckwith and Armitage\cite{93}: the dynamo becomes more regular as the horizontal extent of the box is increased. The typical midplane Alfvén speed for all runs is around $0.1c_s$ to $0.15c_s$. We plot the horizontally and temporally averaged Alfvén speed for Run 3 in Figure 5.14, with discussion in §5.4.2.

Of the three quantities we plot below, $B_y$ has by far the cleanest structure. Field originates at the midplane with even symmetry in $z$, propagates outwards whilst increasing in strength up to its maximum at $z = 3H$, then propagates past that point decreasing in strength. Since Run 1 followed some other dynamo solution for $t \in [110, 200]$ we extend that run for a further 60 orbits to examine the symmetric solution in which we are interested. The average dynamo period is 12.8 (2.1) orbits, 10.1 (1.8) orbits and 10.7 (0.9) orbits for Runs 1, 2 and 3 respectively, with the standard deviation bracketed. (When plotted, this looks like a linear decrease in the standard deviation with $L_x$, and extrapolating this would reach zero around $L_x = 15H$ - comparable to the radial extent where Simon, Beckwith and Armitage’s “variability” smoothed out, suggesting that the temporal variability is decreasing with the spatial variability).

Our EMFs are following some similar pattern over the course of the dynamo cycle, and the wider the box the more obvious this pattern is. Surprisingly, $\mathcal{E}_x$ is obviously increasing in strength with the horizontal extent of the box. Our azimuthal EMF $\mathcal{E}_y$ drops in peak absolute value as the box widens, with its average strength rising with its structure becoming much clearer and cleaner. Above $z = 3H$ our $\mathcal{E}_y$ becomes incoherent, which we will suggest at the end of §5.4.3 to be the effect of the undular instability. Simulations with total vertical extent less than $10H$ are unlikely to capture the full dynamo behaviour. We delay discussion of the phase relationship between our variables to §5.3.1 where we discuss strained temporal averages.
Figure 5.1: Space-time diagrams of $B_y$ in units of $\rho_0 c_s$ for Runs 1, 2 and 3 (top to bottom). Dashed blue vertical lines on plot indicate the endpoints for temporal averaging described in §5.3.1; data between the solid black vertical lines are excluded from the average as the dynamo follows some other solution. The flow becomes fully nonlinear after approx. 10 orbits; for Run 3, the dynamo gains strength after approx. 60 orbits. The decrease in variability with increasing horizontal extent is clear.
Figure 5.2: Space-time diagrams of $E_x$ in units of $\rho_0 c_s^2$ for Runs 1, 2 and 3 (top to bottom). $E_x$ grows significantly stronger with increasing horizontal extent. We shall see in §5.4.3 that this is because of the increase of $|u_z|^2$ with horizontal extent.
Figure 5.3: Space-time diagrams of $\mathcal{E}_y$ in units of $\rho_0c_s^2$ for Runs 1, 2 and 3 (top to bottom). The maximum absolute value decreases with increasing horizontal extent, but the overall variability decreases. For $|z| \geq 3H$ $\mathcal{E}_y$ becomes incoherent.
5.3.1 Strained temporal averages

Both EMFs are much noisier than the toroidal magnetic field. In order to interpret any kind of systematic behaviour of the toroidal EMF at all we must first remove as much noise from the data as possible. Since the dynamo period changes from cycle to cycle we cannot e.g. simply apply a low pass filter to remove short-time oscillations. We instead average all quantities over several dynamo cycles in the expectation that the noise will be reduced and the coherent behaviour reinforced. The amplitude of the noise should decrease like $N^{-1/2}$, where $N$ is the number of dynamo cycles over which the average is taken; we should obtain a fairly smooth and (almost) periodic single dynamo cycle. In this Chapter we take $N = 12$ (where $N$ is set by our access to computational power rather than by experimental design).

The dynamo period varies each cycle by up to 25%. We therefore strain the temporal axis before averaging; we measure the length of the first cycle at the midplane and stretch or compress the remaining cycles to fit, linearly interpolating our data onto the original cycle. The points at which we chop up our dynamo are indicated by vertical blue lines on Figure 5.1 for both Runs 1 and 2 our dynamo changes nature for some intermediate time, and we exclude these periods (bounded by vertical black lines on the same Figures) from our averages. We chose these times as endpoints of cycles by considering when the midplane toroidal field changed sign, and so we expect that behaviour in the upper atmosphere might be artificially damped by this method as flux from different cycles (and so which has risen at different rates) is mapped onto one another. To see this, consider the original cycle, followed by a short cycle, followed by a long cycle. First, the short cycle is stretched to fit the original cycle. Then, during the long cycle, the flux initially not at the midplane will have been generated by the short cycle i.e. it will itself have a shorter timescale, but will then be compressed to fit onto the original cycle. Examining $B_y$ in the upper atmosphere relative to the cutting positions for Run 3 we expect this effect to be small.

These strained temporal averages are shown in Figures 5.4 and 5.5. We denote a strained temporal average by an overbar such as in $\bar{B}_y$ or $|u_z|^2$. We see that
both $\bar{B}_x$ and $\bar{B}_y$ strengthen as they rise until $|z| = 3H$, then weaken considerably. For comparison of the phases of our main variables we have plotted by eye the pattern speed of the toroidal flux. We see that $\bar{B}_x$ rises slower than $\bar{B}_y$; at the midplane we have the average field varying with $\bar{B}_y^2$ and $\bar{B}_x^2$ out of phase, but by $|z| = 3H$ we see that $\bar{B}_x$ is in anti-phase with $\bar{B}_y$. At this height we have a term in $\dot{\bar{B}}_y$ like $-q\Omega B_x$ i.e. $\sim +kB_y$, a local stretching of the field by shear. Past this point the phase relationship changes again, continuing towards a destructive phase at $|z| = 5H$ (although we note our boundary is quite close by then, and regardless both $\bar{B}_y$ and $\bar{B}_x$ have weakened considerably). This phase change is shown schematically in Figure 5.7. Above $|z| = 4.5H$ our apparent pattern speed for $\bar{B}_y$ becomes extremely large, i.e. positive toroidal flux at $|z| = 4.5H$ essentially implies positive toroidal flux for $|z| > 4.5H$, with the toroidal field growing weaker with height. This is entirely consistent with the taller boxes simulated in Simon, Beckwith and Armitage with outflow boundary conditions; our choice of a vertically periodic box does not appear to have significantly affected the dynamics.

As expected, our EMFs have the opposite symmetry in $z$ around the midplane to $B_x$ and $B_y$, with our EMFs antisymmetric in $z$ and $\bar{B}_y$ and $\bar{B}_x$ symmetric in $z$. In Figure 5.5 we see that $\bar{E}_x$ keeps pace with the rising $\bar{B}_y$ and leads it slightly in the cycle, while $\bar{E}_y$ appears to be rising at the same speed as $\bar{B}_y$. The toroidal EMF $\bar{E}_y$ has odd symmetry about the midplane but is otherwise in phase with $\bar{B}_y$, suggesting an $\alpha$-effect that is antisymmetric in $z$. Above $|z| = 3H$, where $\bar{B}_y$ begins decreasing in power, we see $\bar{E}_y$ is dominated again by noise. We plot in Figure 5.9 the changing sizes of the EMFs with the box size and with height. They both increase in strength away from the midplane, with $\bar{E}_y$ peaking around $|z| = 2.25H$ and $\bar{E}_x$ peaking around $|z| = 3.25H$. We relate these EMFs to our analytic prediction in 5.4.3.

As with the pre-strained temporal averaged data, both $\bar{E}_x$ and $\bar{E}_y$ increases in strength with increasing horizontal extent, shown in Figure 5.9 as does $|\bar{u}_z|^2$, shown in Figure 5.6. We shall explain this increase in 5.4.1 using our previous analytic work.

\footnote{We do not intend “rise” to imply advective fluid motion; this would be utterly inconsistent with the different pattern speeds of $B_x$ and $B_y$.}
For our non-magnetic quantities (not plotted), we find that $\bar{\rho}$ is consistently slightly below its initial equilibrium value for $|z| \lesssim H$, due to the higher magnetic pressure there, and consistently above its initial value outside of this. Naturally, the vertically periodic boundary condition means that the total mass in the domain cannot change. The density variations over the course of the cycle are small. We see the following cyclic behaviour in $\overline{|u_z|^2}$, shown in Figures 5.6 and 5.8. Around the (extended) midplane $|u_z|^2$ is essentially constant in time. However, $|u_z|^2$ in the upper atmosphere varies in time with $\bar{B}_y$; as a concentration of $\bar{B}_y^2$ passes $z = 3H$ the $|u_z|^2$ above this height is briefly magnified by a factor between two and three. Examining part of the kinetic energy, $\bar{\rho}|u_z|^2$ is flat around the midplane and falls off for $|z| > 2H$.

Our strained temporal averaging has revealed the essential structure of the runs with small horizontal extent to be consistent with the widest run; the dynamo becomes more coherent but does not fundamentally change with the increase in box size.

\section{Analysis}

\subsection{Correlation coefficients, $r_x > 0.948$}

We investigate the scalings of the midplane EMFs from the simulation. We shall test a model of the form

$$\bar{E}_x \sim \frac{C(t)}{\Omega} \overline{|u_z|^2} \partial_z \bar{B}_y$$

(5.1)

for $|z| < H$. This model is related to our analytic estimate from Chapter 3 if we assume that the horizontally averaged $\overline{|u_z|^2}$ can take the place of the quadratic perturbation quantity $|u_0|^2 \exp(2\text{Re}[\gamma]t)$. This model is similar to the correlations briefly considered by Davis, Stone and Pessah, who suggested that $E_x \propto |u|B_y$.

Our assumption that $\overline{|u_z|^2} \sim |u_0|^2 \exp(2\text{Re}[\gamma]t)$ physically means that the vertical velocity is being continuously driven by the MRI/undular instability and that there are no other waves of interesting magnitude in the system, an assumption supported by the work in Chapter 2. The physical interpretation of
Figure 5.4: Left, top to bottom, $\vec{B}_x$ for Runs 1, 2 and 3; right, top to bottom, $\vec{B}_y$ for the same, with two full dynamo cycles (i.e. copies of the strained temporal average) shown. On Run 3 we superimpose the pattern speed of the toroidal field, approximately $0.44c_s$ (blue line). Clearly, near the midplane $-\vec{B}_x$ leads $\vec{B}_y$ by $\pi/2$ and the magnetic vector simply rotates in the horizontal plane, but progressing away from the midplane we see that $\vec{B}_x$ has a pattern speed closer to $0.28c_s$ (black line). This phase change initially favours the dynamo and leads to the periodic maximum in the toroidal field we see at $|z| = 3H$. Above this height, the radial field dies off sharply.
Figure 5.5: Left, top to bottom, $\bar{E}_x$ for Runs 1, 2 and 3; right, top to bottom, $\bar{E}_y$ for the same, with two full dynamo cycles (i.e. copies of the strained temporal average) shown. As in Figure 5.2 (before strained temporal averaging) we see $\bar{E}_x$ gets stronger with increasing horizontal extent. Above $z = 3H$ we see $\bar{E}_y$ loses some coherence which leads to the drop in $B_x$ at that height. Below there, both $\bar{E}_x$ and $\bar{E}_y$ change with height with the same pattern speed as $\bar{B}_y$. Both $E_x$ and $E_y$ become stronger with increasing horizontal extent; this is demonstrated more clearly in Figure 5.9.
Figure 5.6: Top to bottom, $|u_z|^2$ for Runs 1, 2 and 3, with two full dynamo cycles (i.e. copies of the strained temporal average) shown. We see an increasing vertical motion with the increasing horizontal extent, particularly for $|z| \in [H, 2H]$. This is to be expected: as the horizontal extent increases longer waves can fit into the box, enhancing the undular instability. The periodic release of buoyant kinetic energy, discussed in Figure 5.8, is absent in Run 1 but clear in Run 3.
Figure 5.7: Schematic of dynamo cycle changing with height, shown in $x - y$ plane. We show in the top panel the magnetic vector for fixed $z$. We have (a) $-\vec{B}_x$ leading $\vec{B}_y$ by $\pi/2$ in the midplane (i.e. the toroidal induction equation dominated by the shear term $q\Omega \vec{B}_x$), (b) $-\vec{B}_x$ and $\vec{B}_y$ in phase at $|z| = 3H$, and (c) $\vec{B}_y$ leading $-\vec{B}_x$ for $|z| > 3H$. Note the reversed arrows in (c) when compared with (a). We show in the bottom panel the evolution of the magnetic vector following the rise of an initially toroidal flux concentration; the field strengthens and tilts until $|z| = 3H$, then straightens and weakens.
Figure 5.8: The release of magnetic buoyancy: Immediately after the peak in $\vec{B}_y^2$ (left) reaches $z = 3H$, or the peak in $100 \times \vec{B}_z^2$ (right) reaches the same, we reach the peak of $|u_z|^2/10$ (middle) at the box boundary. In the midplane $|u_z|^2$ is essentially constant in time.
5. Shearing box simulations

Figure 5.9: Top panel: $\mathcal{E}_x^2$ split into vertical bins of height $0.5H$ (e.g. $1.0H < |z| < 1.5H$ yields a data point at $|z| = 1.25H$) and averaged over all $z$ and $t$ within the bin for the strained temporally averaged data of Run 1 (green line), Run 2 (purple line) and Run 3 (blue line). $\mathcal{E}_x^2$ increases dramatically in size with increasing horizontal extent and a peak develops around $|z| = 3.25H$. Bottom panel: The same for $\mathcal{E}_y^2$. Again we see an increase in strength with increasing horizontal extent, with a peak that develops around $|z| = 2.25H$. 

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this model is simple: $E_x$ would be acting as a turbulent diffusivity created by the vertical mixing of flux. As we saw in Figure 5.6 the vertical motions are enhanced with increasing horizontal extent due to the enhanced undular instability; if we can verify this model then we have explained the increasing strength of $E_x$ with box size; this increase in turbulent diffusion would then in turn explain the decrease of variability found by Simon, Beckwith and Armitage.

Our analytics also suggest investigating the model

$$
\bar{E}_y \sim \frac{D(t)}{\Omega} \bar{B}_y \partial_z |u_z|^2
$$

(5.2)

where $D(t)$ is encapsulating e.g. the varying amplitude of the growth rate $\gamma$. We have ignored possible variation of $\gamma$ with $z$, assuming that it remains more-or-less constant for $|z| < H$. (Since $|u_z|^2$ has even symmetry around the midplane $\partial_z |u_z|^2$ can also approximate the term like $z \bar{B}_y$ from the analytic estimate from Chapter 3.)

We now ask two questions: what correlation coefficient $r_x$ do we have between $\bar{E}_x$ and $|u_z|^2 \partial_z \bar{B}_y$, i.e. to what extent does our model predict the shape of the radial EMF for $|z| < H$, and what is the mean-squared coefficient, i.e. the average size near the midplane of $C(t)$? We define the usual correlation coefficient between two variables $X$ and $Y$, given $n$ data points,

$$
\rho_{XY}(t) = \frac{\sum_{i=1}^{n} (X_i - \bar{X}_i)(Y_i - \bar{Y}_i)}{n\sigma_X \sigma_Y}
$$

where $\bar{X}_i$ is the average of $X$ for those $n$ points, and $\sigma_X$ the corresponding standard deviation. We calculate $r$ for the model of Equation 5.1 (which we denote by $r_x$) and plot it as a function of time in Figures 5.10 (pre-temporal averaging) and 5.11 (post-temporal averaging). The closer $r$ is to +1, the better the predictor of the shape of the EMFs that we have found. We then repeat the process for the model of Equation 5.2 and denote the correlation coefficient by $r_y$.

We plot $r_x$ for both the data pre-strained temporal averaging and post-strained temporal averaging in Figures 5.10 and 5.11 respectively. There is a horizontal black line at $r_x = 0.948$. When $r_x > 0.948$ we can say that at that point in time
\[ r^2_x > 0.9, \text{i.e. our model accounts for } 90\% \text{ of the variation in the data. As the size of the box is increased we see the emergence of periods of extremely good correlation even before the strained temporal averaging. Post-strained temporal averaging we have } r^2_x > 0.9 \text{ for almost all } t \text{ for Run 3; when } r_x \text{ drops sharply for short periods it is where } \tilde{E}_x \text{ has just changed sign, representing our imperfect reproduction of its phase. This excellent correlation allows us to confidently explain the increasing strength of } \tilde{E}_x \text{ with box size as due to the increasing magnitude of } |u_z|^2 \text{ with horizontal extent, itself easily explained by the enhanced undular instability.}

We plot in Figure 5.13 the least squares regression for these same models applied to Run 3 i.e.

\[ C(t) = r_x \frac{\sigma_{\text{Model}}}{\sigma_{\text{Data}}} \]

(and similarly for \( D(t) \)). These coefficients are of order unity, supporting our analytic scalings, and \( C(t) \) is very close to 1/2.

We calculate \( r \) for the model of Equation 5.2 and plot \( r_y \) (post-temporal averaging) in Figure 5.12. It is positive, but plainly our model for \( E_x \) is less good than our model for \( \tilde{E}_x \), as should be expected given the extreme variability of the original \( E_y \).

This extremely high correlation is strong support for our model for \( \tilde{E}_x \), especially with a regression coefficient which is almost constant over the cycle. Our model for \( \tilde{E}_y \) has a weaker correlation (as expected for predictions of extremely variable quantities), but is right order of magnitude.

### 5.4.2 Alfvén speed and frequency with \( z \)

We wish to make a more detailed comparison between our simulation data and our analytic estimates, with our focus on Run 3. We thus plot in Figure 5.14 the toroidal Alfvén speed \( (\tilde{B}_y^2/\rho)^{1/2} \) for the folded Run 3 data. This Alfvén speed is an increasing function of \( |z| \), with midplane values of \( v_a \approx 0.1c_s \). At \( |z| = 3H \), where \( \tilde{B}_y \) is strongest, \( v_a \approx c_s \), and above this the Alfvén speed is generally larger than the isothermal sound speed, with \( v_a \approx 7c_s \) on the vertical boundary. For Run 1 the Alfvén speed reached \( 14c_s \) on the boundary in our folded data (not
Figure 5.10: Correlation coefficient $r_x$ for $|z| < H$ for Runs 1, 2 and 3, pre-strained temporal averaging. Dotted black line is $r = 0.949$; above this line the model explains 90% of the variation in the data. The correlation coefficient moves closer to one for wider boxes. We have $r > 0.949$ for 4% of points for Run 1, for 30% of points for Run 2, and for 54% of points for Run 3 (all excluding $t < 10$ orbits). Figure 5.11 shows the same correlation coefficient calculated after the strained temporal average.
5. Shearing box simulations

Figure 5.11: Correlation coefficient $r_x$ for $|z| < H$ for Runs 1, 2 and 3, post-strained temporal averaging. Dotted black line is $r = 0.949$; above this line the model explains 90% of the variation in the data. The correlation coefficient moves closer to one for wider boxes. We have $r > 0.949$ for 12% of points for Run 1, for 63% of points for Run 2, and for 84% of points for Run 3 (all excluding $t < 10$ orbits and those periods which are following another dynamo solution, indicated in Figure 5.1 by black vertical lines). The position of the downward spikes in Runs 2 and 3 correspond to periods just after $\mathcal{E}_x$ has changed sign (indicating our model is not reproducing the phase completely accurately). Figure 5.10 shows the same correlation coefficient calculated before the strained temporal average.
Figure 5.12: Correlation coefficient $r_y$ for $|z| < H$ for Runs 1, 2 and 3, post-strained temporal averaging. Dotted black line is $r = 0.949$; above this line the model would explain 90% of the variation in the data (although we do not approach this line for this model). The correlation coefficient moves somewhat closer to one for wider boxes. The correlation coefficient for this calculated before the strained temporal average is not plotted in this Chapter.
Figure 5.13: The least-squares regression lines for the models in Equations 5.1 and 5.2 for Run 3. Both are $O(1)$, supporting the scalings suggested by our model, and $C(t)$ appears to be very close to 1/2.
We attempt to estimate the dominant Alfvén frequency in our simulations via a Fourier transform. Working with a snapshot of our complete flow - that is, before any horizontal averaging - we split our computational domain into slices of fixed $z$, then Fourier transform in $y$ (denoted by a tilde) and calculate $|\tilde{b}_z|^2(x, k_y, z)$. We then average $|\tilde{b}_z|^2$ over $x$ to gain the average distribution of power in $k_y$ for each fixed $z$ at a single time. We then average that over five such snapshots, each at roughly the same point in the dynamo cycle, to mimic our temporal average. We denote this quantity by $\langle |\tilde{b}_z|^2 \rangle_{x,t}$ to avoid confusion with our horizontally and temporally averaged quantities.

The resultant power spectra for Runs 1 and 3 are shown in Figure 5.15. For both runs we see that energy is strongly concentrated around low $k_y$; for low $z$ we see some power at higher $k_y$ from the MRI active at the midplane; as $v_a$ increases with height these higher $k_y$ become disfavoured i.e. the “average” $k_y$ is decreasing with increasing $|z|$. The influence of the box size is seen in the width of the green regions in the Figure: our wider box will have an enhanced undular instability which favours longer wavelengths. We have no peak $k_y$ in our energy spectra, and so we cannot select an obvious Alfvén frequency by this method.

### 5.4.3 Analytic comparison

We now compare our analytic result of Chapter 3 to our horizontally and strained temporally averaged data from Run 3. Our analytic estimate requires as an input an azimuthal wavenumber $k_y$, a poloidal polarisation $\lambda = k_x/k_z$ (where $\lambda \ll 1$ corresponds to the MRI and $\lambda \gg 1$ corresponds to the undular instability), and the ratio $k_y/k_x$. We saw in §5.4.2 that we have no obvious peak in power with $k_y$ for any $z$, and we are thus left with two obvious choices: We can take $k_y$ to be as small as possible given the box size, i.e. $k^\text{min}_y = 2\pi/L_y$, or we can take $k_y$ to maximise our growth rate. For the pure MRI, this would have

$$\left( k^\text{MRI}_{y a} \right)^2 = \frac{1}{\sqrt{1 + \lambda^2}} \frac{15}{16} \Omega^2$$
Figure 5.14: The toroidal Alfvén speed \( (\vec{B}_y^2/\bar{\rho})^{1/2} \) for Run 3. The black contour shows where the toroidal Alfvén speed equals the isothermal sound speed. We have a midplane \( v_a \approx 0.1c_s \) rising to boundary values of \( v_a \approx 7c_s \).
Figure 5.15: Plot of $\langle |\tilde{\theta}_z|^2 \rangle_{x,t}$ as a function of $k_y$ and $z$ for Runs 1 and 3 (Run 2 omitted for reasons of space); energy is predominantly on the largest azimuthal scales. We can see a “flat” region for $|z| < 2H$ where some larger wavenumbers are present; outside this region energy is more concentrated in the largest scales. Run 3 is consistently “larger scale” than Run 1, and has more energy in the upper atmosphere because of vertical mixing due to the enhanced undular instability. There is no nonzero peak in $k_y$. We used $t \in \{62.1, 71.6, 81.1, 90.7, 100.3\}$ orbits for our snapshots, all of which fall at approximately the same point in the dynamo cycle.
where the above expression defines $k_y^{\omega_a}$. In our calculations in this section we limit $k_y^{\omega_a}$ to lie within $[2\pi/L_y, \pi/\Delta y]$, as would be enforced by the simulation.

We must discuss the finite numerical resolution and the unknown effects of nonlinearity. First, the radial wavenumber will have an upper bound in these simulations. Our analysis assumed an EMF arising from single trailing shearing wave with $k_x \gg k_y$, but the maximum possible radial wavenumber is the Nyquist wavenumber $2\pi/2\Delta x = 23\pi/H$. If $k_y$ is as small as possible (whilst being nonzero) then this would give $\min(k_y/k_x) \approx 3\%$ for our narrowest simulation and $\min(k_y/k_x) \approx 1\%$ for our widest simulation - where the minimum value of $k_y/k_x$ lies precisely on the grid scale. To put these in to context of the previous Chapter, we could follow such a single shearing wave until $\Omega t \approx 23$ (i.e. 3.6 orbits after the swing) or $(\Omega t) \approx 75$ (i.e. 12 orbits after the swing) for the widest simulation. Such a limit on our ability to follow even the minimally shearing mode is especially concerning given that these maximum times are extremely similar to the period of the dynamo cycle, and a comparison with Figure 4.1 on page 113 clearly demonstrate that (in the absence of numerical diffusion) the linear MRI would at $\Omega t \approx 23$ probably still be undergoing transient growth, with amplitudes typically increasing around 2 orders of magnitude; if the nature of the azimuthal EMF would naturally change at a time later than this then the effect might be lost. The transient growth itself raises the second problem: these growing modes will at some point become nonlinear, and a key assumption of our analysis will fail. In short, it is not clear that with this numerical resolution we can reach the regime in which our asymptotic analysis was performed, nor that the results we obtained from the linear analysis will describe a nonlinear simulation.

We proceed to an investigation of our analytic estimate, looking at an MRI polarisation versus a buoyant polarisation, taking $k_y^{\min}$ versus a fixed $k_y^{\omega_a}$, and including terms of $O(B_x)$ versus not. For brevity, we show in this Chapter only two successful predictors of our $\bar{E}_x$ and relegate the systematic investigation of our model’s free parameters to §D.1. We show also two good predictors of $\bar{E}_y$. As a result of this investigation we neglect terms in the EMFs which involve $\bar{B}_x$ for the remainder of this Chapter.

Our analytic estimate could be used in two ways: it could be “piggybacked” on a full nonlinear simulation, with our EMFs recalculated at every $z$ and $t$ using the
horizontally and temporally averaged quantities from the full simulation; for this case our analytic EMFs are not used in the evolution of \( B_y \) but are only compared with the horizontally averaged EMFs from the simulation. Alternatively we could calculate \( \mathcal{E}_x \) and \( \mathcal{E}_y \) and use these to step \( B_x \) and \( B_y \) forward through time. This section concerns itself with the former of these, with discussion of the latter as possible future work in \( \S 5.3 \).

Our analytic estimate involves several divisions by \( |\gamma|^2 \), which can pass through zero in certain uninteresting regions. We thus soften any quantity by which we might divide: If \( |\bar{B}_y| \) is below some minimum value \( = 0.008 c_s \rho_0^{1/2} \) then we assume that any division by \( |\gamma| < 0.5 \Omega \) is in fact a division by \( 0.5 \Omega \); this smooths out our analytic estimate in uninteresting regions. We further assume that any division by \( |\gamma| < 0.01 \Omega \) is in fact a division by \( 0.01 \Omega \), merely to avoid divisions by zero which might interrupt our numerics. We perform an identical smoothing on the denominator \( |\gamma \eta \gamma_\nu + \omega_z^2|^2 \). This smoothing is purely for numeric convenience and does not affect our results.

We also smooth

\[
\frac{\partial}{\partial z} \left( \text{Re} \left[ \frac{|u_z|^2}{\gamma} \right] \right) = \frac{\partial}{\partial z} \left( \frac{|u_z|^2 \text{Re} [\gamma]}{|\gamma|^2} \right)
\]

which involves taking the derivative of the inverse of a growth rate \( \gamma \) which is calculated independently at each grid point - it is not surprising that we find we must take some smoothing. Within the derivative (and only within this derivative) we take it such that if \( |\gamma|^2 < (3/4) \Omega^2 \) we take \( |\gamma|^2 = (3/4) \Omega^2 \). We justify this more drastic alteration by saying that a smoothly varying \( |\gamma| \) could plainly be found with a more thorough optimisation over \( (k_y, \lambda, k_y/k_x) \), and that \( \gamma \) does not vary greatly in the midplane.

Our investigation finds that we can predict the radial EMF \( \bar{\mathcal{E}}_x \) extremely well by assuming \( k_y = k_y^{\omega_n} \) and \( \lambda = 0.1 \) i.e. our azimuthal wavenumber enforcing a fixed magnetic tension, and our poloidal polarisation fixed and inhibiting the undular instability in favour of the MRI. The data and analytic comparison are shown in Figure 5.16; the real part of the growth rate \( \gamma \) used to generate this plot is shown in Figure 5.19. At around \( |z| = 3H \) we see a discrepancy between the phase of the data and the phase of our analytic prediction; an estimate with
5. Shearing box simulations

\( k_y = k_y^{\omega a} \) and \( \lambda = 10 \) (buoyant polarisation) is also shown in the bottom panel. This second figure predicts the correct phase but is less smooth.

We find further that we have two good predictions for the azimuthal EMF \( \bar{E}_y \). First, we assume \( k_y = k_y^{\min} \), \( k_y/k_x = 4 \exp(-z^2/2H^2) \) and \( \lambda \) dependent on the vertical gradient of magnetic pressure in the following way: let \( \alpha = z(B_y)^{-1}\partial_z B_y/5 \) measure whether we are on the buoyant (if \( \alpha < 0 \)) or MRI (if \( \alpha > 0 \)) side of a flux concentration. We then take

\[
\lambda = \frac{10\exp(-\alpha) + \exp(\alpha)}{\exp(-\alpha) + 10\exp(\alpha)},
\]

which is an ad hoc function which gives us \( \lambda \sim 10 \) when \( B_y^2/2 \) is decreasing away from the midplane and we wish to select a buoyant mode, and \( \lambda \sim 0.1 \) for regions where buoyancy will be suppressed and we wish to select an MRI polarisation. Second, we attempt to measure \( \lambda \) from the simulation data by taking \( \lambda = \sqrt{|u_z|^2/|u_x|^2} \) at fixed snapshots as described in Figure 5.18 (recall that in Chapter 3 we had \( k_x u_x z_0 + k_z u_z z_0 = 0 \)). This choice of \( \lambda \) is motivated by the polarisation as a function of height found by Johansen and Levin[56] in their investigation of the initial development of the MRI/undular instability.

The data and analytic comparisons for \( \bar{E}_y \) are shown in Figure 5.17, with good agreement; the real part of the growth rate \( \gamma \) used to generate this plot is shown in Figure 5.19. To obtain the correct sign of EMF for \( \bar{E}_y \) it is vital that \( \omega_a^2 - |\gamma|^2 < 0 \). It is troubling that our estimates for \( \bar{E}_x \) and \( \bar{E}_y \) use the same amplitude \( |u_z|^2 \) but wildly differing wavenumbers but it seems likely that this could be reconciled (see e.g. Figure D.14).

We can see in Figure 5.19 that the undular instability becomes markedly more powerful for \( |z| > 3H \), and in fact overtakes the growth rate of the MRI at this height. We see further in Figure 5.17 that our good approximation for \( \bar{E}_y \) is largely due to the incompressible terms rather than those terms \( \propto g_z \), the vertical gravity, with the gravitational terms becoming important only outside of \( |z| < 3H \) where the dynamo does not function. It seems highly likely that, while the weak undular instability near the midplane is necessary to set the correct polarisation \( \lambda \) to reproduce the toroidal EMF \( \bar{E}_y \), the more powerful undular instability away from the midplane is destructive to the dynamo.
Figure 5.16: Top: prediction of $\bar{E}_x$ with $k_y = k_{y\nu}$, $\lambda = 0.1$, with no $B_x$, as described on page 156. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10\times$ the contribution due to gravity. Bottom: The same, with $\lambda = 10$. The upper panel reproduces the EMF reasonably well, while the lower panel gives a better match for the phase. Plainly we could obtain an even better approximation by varying the polarisation with the changing magnetic field.
5. Shearing box simulations

Figure 5.17: Prediction of $\bar{E}_y$. Top panel: $k_y = k_{y_{\min}}$, choosing $\lambda$ using the vertical derivative of the magnetic pressure as described on page 157, with no $\bar{B}_z$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity. (To see this approximation with a fixed $\lambda = 0.1$ or $\lambda = 10$ see Figures D.11 and D.12 respectively.)

Bottom panel: $k_y = k_{y_{\min}}$, with $\lambda = 0.68 + 0.028|z|/H$ as suggested by Figure 5.18. We do not quite reproduce the phase. This $\lambda$ was derived from five single snapshots rather than using the entire cycle data, and we are confident that we could properly reproduce the phase using a more nuanced prescription for $\lambda$. 

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Figure 5.18: We estimate the polarisation $\lambda$ from snapshots of the simulation data. We plot our estimator for $\lambda = \sqrt{|u_z|^2 / |u_x|^2}$, (purple line), $100 \times B_y^2$ (green line), and a simple fit to the polarisation (blue line, $\lambda = 0.68 + 0.028|z|/H$). There is only a weak variation in $\lambda$ with $z$. As in Figure 5.15, we used $t \in \{62.1, 71.6, 81.1, 90.7, 100.3\}$ orbits for our snapshots, all of which fall at approximately the same point in the dynamo cycle.
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Figure 5.19: The growth rate $\gamma$ for the four cases plotted in Figures 5.16 and 5.17. From left to right: $100 \times B_y^2$, then $\gamma$ for our $E_x$ approximations $k_y = k_y^{\omega_x}$ and $\lambda = 0.1$ (MRI polarisation), then $k_y = k_y^{\omega_x}$ and $\lambda = 10$ (buoyant polarisation), then, for our $E_y$ approximation, $k_y = k_y^{\text{min}}$ with $\lambda$ set by $\alpha$, and finally $k_y = k_y^{\text{min}}$ with $\lambda = 0.68 + 0.028|z|/H$. Buoyancy instabilities are only active for $|z| \gtrsim 3H$ and are naturally more powerful on the upper edge of flux concentrations. Even though the pure MRI with optimal $\omega_x$ has the highest growth rate the simulation appears to be dominated by small-$k_y H$ with our fitted $\lambda = 0.68 + 0.028|z|/H$. We suggest that the higher-$k_y$ contributions are damped by our (numerical) diffusivities.
5.5 Future work: 1D Model

We now know that $|u_z|^2$ is relatively constant with time and could easily it with a polynomial in $z$ or similar; we can also now reproduce the EMFs from Run 3. Using this information we could write down the following mean-field equations

$$\frac{\partial B_y}{\partial t} = -q \Omega B_x + \frac{\partial \varepsilon_x}{\partial z}$$
$$\frac{\partial B_x}{\partial t} = -\frac{\partial \varepsilon_y}{\partial z}$$

and solve them with an initial condition taken from $t = 0$ of the folded data for Run 3. An attempted implementation using RK4 in time and a simple central differencing method found the problem too lengthy for inclusion in this thesis.

For future work, we note the following:

- A staggered grid, with EMFs calculated off-grid, was necessary to avoid spurious alternating modes.
- It seems likely that $\rho$ and $\partial_z \rho$ must also change with time to properly capture the buoyancy instability. This change in time could be well approximated by assuming vertical magnetohydrostasis.
- For stability it is convenient to include the analytic diffusive term involving $\bar{B}_x$.
- Further refinement of the analytic model is needed to prevent sudden changes in $\gamma$ with $z$ which leads to our EMFs being non-differentiable. This might be solved by optimising $\gamma$ over all $(k_y, \lambda)$, or by including explicit diffusivities in our simulations.
- At each grid point and at each timestep we must solve the dispersion relation for all of its roots and select the most unstable. In this thesis we have used Laguerre’s method, which is certain to give accurate roots but is not particularly fast. If we were to optimise properly over $(k_y, \lambda)$ the computational time required would increase sharply.
5.6 Conclusion

We have investigated the EMFs calculated from ZNF shearing box simulations with vertical gravity, and employed both spatial and strained temporal averages to reveal the changing nature of the dynamo with $z$. We have found excellent agreement between our analytic prediction and the mean behaviour of our turbulent simulations for $\mathcal{E}_x$, and good agreement for $\mathcal{E}_y$. We have used our analytic prediction to explain the dependence of the dynamo on box size.

Using the Godunov code RAMSES we simulated three shearing boxes of increasing horizontal extent. We found that both the radial EMF $\mathcal{E}_x$ and the toroidal EMF $\mathcal{E}_y$ increase in amplitude with increasing horizontal extent, following the increasing amplitude of the square of the vertical velocity $|u_z|^2$ due to enhanced buoyancy.

We then employed a simple but effective strained temporal average to better analyse the horizontally averaged quantities of interest, in particular the toroidal field. This strained temporal average revealed the changing phase relationship of $\bar{B}_x$ and $\bar{B}_y$ with height, whereby the simple rotation of the magnetic vector in the midplane is replaced by $\bar{B}_x$ and $\bar{B}_y$ in perfect anti-phase at $|z| = 3H$. The strong field it creates soon dissipates and the undular instability releases kinetic energy into the upper atmosphere.

We showed that we could model the midplane radial EMF $\mathcal{E}_x$ extremely well by assuming that we could treat the average of the square of the vertical velocity as the perturbation $|u_z|^2 \exp(2\text{Re}[\gamma]t)$ of Chapter 3 with correlation coefficient $r_x$ increasing with horizontal extent. After strained temporal averaging this correlation coefficient was extremely close to unity over the entire course of the cycle, explaining over 90% of the data, with deviations due to a small phase difference between $\bar{E}_x$ and $\partial_z \bar{B}_y$. The constant of proportionality we calculated had an obvious average value very close to 0.5. Continuing with our analytic estimate we found that we could excellently reproduce the radial EMF $\bar{E}_x$ by fixing the Alfvén frequency $\omega_a$, although a more careful treatment of the poloidal polarisation $\lambda$ is needed to give a truly smooth result. We found also that we could acceptably reproduce the toroidal EMF $\bar{E}_y$ by assuming that $k_y$ was the minimum available.
in the simulation domain and measuring $\lambda$ directly from the simulation data. We related the analytic dependence of our EMFs on $|u_z|^2$, itself related to the horizontal extent and the undular instability, to the result of Simon, Beckwith and Armitage that the dynamo becomes smoother with increasing horizontal extent; as the horizontal extent increases $|u_z|^2$ increases as longer wavelengths can fit in the box, and our EMFs rely on $|u_z|^2$. Our detailed analytic models must be given with the caveat that the wavenumbers used to reproduce $\bar{\mathcal{E}}_y$ and $\bar{\mathcal{E}}_x$ differ considerably.

We examined the linear growth rate $\gamma$ of both the MRI and the undular instability to argue that the turning off of the dynamo at $|z| = 3H$ can be attributed to the quadrupling of the strength of the undular instability in that region. We showed, via our strained temporal averaging, that at this height magnetic fields dissipate and kinetic energy immediately released into the atmosphere above, a clear sign of a destructive buoyancy instability. Above this region it seems likely that there is no dynamo, although the phase relationship between $\bar{B}_y$ and $\bar{B}_x$ is still such that the $q\Omega \bar{B}_x$ term is still constructive in the $\dot{\bar{B}}_y$ equation, and recent investigations suggest that it is difficult to distinguish a dynamo from a “stoked non-dynamo” (with field being fed from below), as described by Byington, Brummel, Stone and Gough (2013, in preparation).

We have described fully the changing magnetic field with $z$ by using a simple but novel strained temporal average. We have successfully reproduced the turbulent EMFs using our analytic derivation, and related them to the dependence of the dynamo on horizontal extent of the simulation domain. We have outlined a clear path of future work which should lead to fast and simple modelling of the ZNF accretion disc dynamo with vertical gravity.
Chapter 6

Conclusion

Where do you go
Buttoned in your favourite coat
Stepping out to a different world
And you might be home late

Ocean Colour Scene

We recall and summarise the main results of this Thesis, and reprint the key Figures from each Chapter.

6.1 Shearing waves

In Chapter 2 we considered the problem of wave mixing in the shearing sheet for an isothermal atmosphere with vertical gravity, with our intent to examine the generation of acoustic oscillations by sheared inertial modes. We decomposed the vertical structure of the problem on to a basis of Hermite polynomials, and by considering the Lagrangian displacement we derived the conserved Hermitian form $J(\cdot, \cdot)$ as described by Friedman and Schutz[30]. We considered the asymptotic limit $|t| \to \infty$ and so derived two sets of modes: the inertial modes (largely incompressible, predominantly in the $y - z$ plane, and with energy $\propto |t|^{-1}$) and the acoustic modes (highly compressible, dominated by radial motion and with energy $\propto |t|$), and discussed a freedom in the choice of basis of the inertial modes. We took an initial condition of a single type of behaviour (e.g. a pure inertial mode) at $t$ large and negative, integrated it forwards in time using our Lagrangian
equations, and used our Hermitian form $J(\cdot, \cdot)$ to project it on to our basis of acoustic and inertial modes at $t$ large and positive.

We found that overreflection in the sense of a reflected acoustic mode being larger than its parent incident mode was absent for $n \geq 1$, consistent with the spatial work of Li, Narayan and Goodman\[68\]. We found that we could turn inertial action into acoustic action with approaching 100% efficiency for $n = 1$. We show again an acoustic mode generated from an inertial initial condition in Figure 6.1 and show again the complete conversion in Figure 6.2. This is a way to get energy from the flow ‘for free’; low energy inertial modes can be efficiently exchanged for high energy acoustic modes which can carry angular momentum quickly across an accretion disc. By considering the Hermitian form $J(\cdot, \cdot)$ we laid out a clear method to analyse related problems in the shearing sheet; this work could easily be extended to the magnetic case to allow us to examine the generation of fast magnetoacoustic waves by the swing, either without vertical structure or with vertical gravity with constant Alfvén speed (both separable problems).

6.2 The accretion disc dynamo

Chapters 3, 4 and 5 had their focus on the accretion disc dynamo.

In Chapter 3 we considered the MRI in an isothermal atmosphere with a general poloidal effective gravity and varying toroidal magnetic field. Assuming a small poloidal wavelength as an asymptotic parameter we calculated analytically the quasi-linear radial and toroidal EMFs that arise from a background which varies slowly compared to the wavelength of the instability. We were able to reproduce the incompressible and zero-gravity EMFs calculated by Ogilvie and Lesur\[66\] and found that the addition of vertical gravity yielded two new terms: one related to vertical migration of toroidal flux, and one $\alpha$-effect - in the sense of an $\alpha - \Omega$ dynamo - both terms to be proportional to gravity (and so, for an accretion disc, weakest at the midplane and strongest for large $|z|$).

Our main result of that Chapter was our quasilinear EMFs, which appeared
6. Conclusions

Figure 6.1: Real part of the radial Lagrangian displacement for the modes considered in Chapter 2 against time in purple, with the asymptotic reconstructions overlaid on top in green. The initial conditions here were an inertial mode of positive unit action, and we clearly see acoustic modes generated as $k_y$ passes through 0. This plot is reminiscent of the generation of acoustic modes by the balanced solutions considered by Heinemann and Papaloizou [50]. This Figure was shown in the main text as Figure 2.3.
Figure 6.2: The angular momentum contained in the acoustic oscillations at $t \gg 0$ (given an initial condition of an inertial mode with unit angular momentum) as a function of $k_y$ for Hermite index $n = 1$ (i.e. $u_x \propto z$, $u_z \propto 1$). As $k_y \to 0$ the conversion between the two becomes complete: we can efficiently exchange inertial action for acoustic action. This Figure was shown in the main text as Figure 2.7.
6. Conclusions

in the text as Equations 3.5 and 3.6:

\[
E_{x0} = \frac{1}{2k_x}D_k B_y \Re \left( \frac{1}{\gamma_\eta} \left| \tilde{u}_{z0} \right|^2 - \frac{k_y}{k_x} \Im \left[ \gamma \frac{k_x^2 (\eta - \nu)}{\gamma_\eta \gamma_\nu} B_y (2 - q) \Omega \left| \tilde{u}_{z0} \right|^2 \right] \right. \\
- \frac{1}{2k_x} B_y \frac{1}{\gamma_\eta^2 c_s^2 + v_\alpha^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2} \\
\times \left\{ \frac{G_k}{c_s^2} \Re \left[ \gamma_\eta (\gamma_\nu^* \gamma_\eta^* - \omega_a^2) (\gamma_\eta^* \gamma_\nu^* + \omega_c^2) \right] \\
- k_y k_z \Omega \Im \left[ \frac{1}{\gamma_\nu} (2\gamma_\eta - q(\gamma_\eta - \gamma_\nu))(\gamma_\nu^* \gamma_\eta^* - \omega_a^2) (\gamma_\eta^* \gamma_\nu^* + \omega_c^2) \right] \right\} |\tilde{u}_{z0}|^2
\]

and

\[
E_{y1} = -\frac{1}{2k_x^2} B_y D_k \Re \left( \frac{|\tilde{u}_{z0}|^2}{\gamma_\eta} \right) \\
- k_y \frac{k_x}{c_s^2 + v_\alpha^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2} \\
\times \left( \frac{G_k}{c_s^2} (|\gamma_\nu|^2 |\gamma_\eta|^2 + \omega_a^2 \Re [\gamma_\nu \gamma_\eta]) - k_y k_z \Omega \Im \left[ \frac{1}{\gamma_\eta} (2\gamma_\eta - q(\gamma_\eta - \gamma_\nu))(\gamma_\nu^* \gamma_\eta^* - \omega_a^2) (\gamma_\eta^* \gamma_\nu^* + \omega_c^2) \right] \right) \\
\times \Re \left[ \frac{1}{\gamma_\nu} \left| \tilde{u}_{z0} \right|^2 \right]
\]

which we then analysed in several limits. In the MRI polarisation, with predominantly horizontal motion, we found that the term in \( E_{x0} \) which resembles an advective term for \( B_y \) remains finite and does not vanish, suggesting that the vertical migration of field ought have comparable speeds in stably stratified atmospheres. In the undular polarisation, with predominantly vertical motion, we found the vertical migration of flux had a stagnation point wherever

\[
\frac{\partial}{\partial z} (v_\alpha^2) = \frac{2\omega_a^2}{\partial_z (\log B_y)}
\]

which appeared in the main text as Equation 3.9. A natural extension to this approximation would be to relax the isothermal assumption.

In Chapter 4 we wrote and tested a code to evolve the linearised MHD equations in the shearing sheet with the intent of examining the quasi-linear EMFs that result from the mixed MRI/undular instability for an isolated flux concentration. We took ensemble averages of these EMFs whilst varying \( k_y \), the background
magnetic field strength and the distance of the isolated flux concentration from the midplane. Previously, Lesur and Ogilvie\cite{Lesur_2012} argued that the MRI will be localised around the most unstable location, which for weak fields is the point of maximum $B_y^2$ and for stronger fields will be whereever $\omega_n^2 = q\Omega^2(1 - q/4)$ (and for even stronger fields the MRI would be suppressed). They continued on to define a correlation integral $J$ which was positive if the unstable location was near the maximum of $B_y^2$ and negative otherwise; if $J > 0$ then dynamo action is possible. For flux concentrations at low $|z|$ we showed that this argument held true, with the natural adjustment that for stronger fields the side of the flux concentration furthest from the midplane was more unstable because of the presence of the weak undular instability. For high $|z|$ this argument was valid only for $k_y$ large enough to inhibit the undular instability; otherwise the most unstable location was insensitive to the background magnetic field strength and the sign of $J$ was instead determined by the degree of spatial asymmetry in $E_y$. We laid this out in detail in §4.6.2.

We showed also that with the larger of two azimuthal wavenumbers $k_y$ the field strength at which $J$ changed sign decreased with increasing $|z|$, while for the smaller of two azimuthal wavenumbers $k_y$ the field strength at which $J$ changed sign increased with increasing $|z|$, and interpreted it in light of the spatial asymmetry in $E_y$. We recall this in Figure 6.3.

With our analytic estimate and our understanding of the linear MRI/undular instability we moved on to our Chapter 5, wherein we modified the Godunov code RAMSES to allow for vertically periodic boundary conditions by smoothly taking the vertical gravitational potential $\Phi$ to a constant value at some $|z|$ close to the boundary. We verified the result of Simon, Beckwith and Armitage\cite{Simon_2003} that the zero net flux dynamo becomes increasingly regular with the increasing horizontal extent of the shearing box. Using a simple yet effective strained temporal average we showed that the horizontally averaged toroidal magnetic field tilts as it propagates away from the midplane (which we show again in Figure 6.4), increasing in strength until $|z| = 3H$, after which point it decreases; this “turning off” at $|z| = 3H$ also happens to the radial magnetic field and the azimuthal EMF. We showed that the radial and azimuthal EMFs become much stronger as the horizontal extent increases, as does the mean squared vertical velocity $|u_z|^2$. The
6. Conclusions

Figure 6.3: Left: the ensemble averages of $|J|$ against time for $k_y H = \pi/2$ and increasing maximum Alfvén speed, centred at a height of $z = 0.5H$ above the midplane. Right: The same, but with the flux concentration centred on $z = 3.5H$ (note the different vertical scales). The lines are solid if $J > 0$ (dynamo action) and dotted if $J < 0$ (no dynamo action), with all lines beginning with $J > 0$ and eventually switching to $J < 0$. In the left panel for $v_{a}^{\max} \geq 0.7c_s$ (yellow line) we have $J$ unambiguously negative, while for the right panel we have a long period of exponential growth with $J > 0$ even for $v_{a}^{\max} = 1.0c_s$ (pink line). These Figures were shown in the main text as Figure 4.1 and Figure 4.2.
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evidence for these strengthening EMFs is reproduced in Figure 6.5. Led by our analytic work in Chapter 3 we investigated a midplane (|z| < 1.0H) model for the radial EMF of the form

$$\tilde{E}_x \sim \frac{C(t)}{\Omega} \frac{|u_z|^2 \partial_z \tilde{B}_y}{\Omega}$$

which appears in the text as Equation 5.1. We found a correlation coefficient of \(r^2_x > 0.9\) for over 80% of the dynamo cycle, with periods of poor correlation corresponding to sign changes in \(B_y\), as well as a least squares regression coefficient very close to 1/2. This is an excellent correlation and an order unity least squares coefficient: strong evidence that our model is the correct line of attack. We found weaker evidence in favour of our model for the azimuthal EMF

$$\tilde{E}_y \sim \frac{D(t)}{\Omega} \frac{\tilde{B}_y \partial_z |u_z|^2}{\Omega}$$

which appeared in the text as Equation 5.2. We found a positive correlation coefficient (although not one so startling as \(r^2_x > 0.9\)) and a least squares coefficient of order unity.

Comparing our full quasilinear EMFs from Chapter 3 to our data we investigated all combinations of an MRI polarisation versus an undular polarisation, a minimal \(k_y\) (given the box size) versus a \(k_y\) which fixed the Alfvén frequency, and including \(B_x\) in the analytic estimate versus not. We also attempted to measure the polarisation at several snapshots by assuming \(\lambda \sim \sqrt{|u_z|^2/|u_x|^2}\). We found that we could approximate the azimuthal EMF well by taking the \(\lambda\) measured from the simulation together with \(k_y = k_{y_{\text{min}}}\), the smallest that will fit in the simulation domain, and that we could approximate the radial EMF well by taking an MRI polarisation together with \(k_y\) that fixes \(\omega_a\); these estimators are reproduced in Figure 6.6 and fit the data remarkably well. These analytic estimates are subject to some smoothings described in §5.4.3 of Chapter 3 and could likely be improved upon with a more careful treatment of the polarisation \(\lambda\).

We have thus progressed from considering quasi-linear numerical EMFs of a single flux concentration to a good reproduction of the fully nonlinear EMFs in a shearing box simulation 10H × 10H in horizontal extent. This is far from the end of the matter: the nonlinear simulations that we have run have been ideal,
Figure 6.4: Left: \( \bar{B}_x \) for our largest simulation in Chapter 5. Right: \( \bar{B}_y \) for the same. We show two full dynamo cycles (i.e. copies of the strained temporal average) and superimpose the pattern speed of the toroidal field, approximately 0.44\( c_s \) (blue line), and the pattern speed of the radial field, closer to 0.28\( c_s \) (black line). The phase relationship of radial and toroidal field change with \( z \). This Figure was shown in the main text as Figure 5.4.

with no diffusivities, and we have repeatedly referenced the work of Fromang and Papaloizou\cite{37} which made clear that the MRI without diffusivities is not converged on the smallest scales. Our nonlinear simulations have a horizontal extent which does not reach that required by Simon, Beckwith and Armitage\cite{93} for convergence. Our analytic comparison treated the azimuthal wavenumber and polarisation crudely; it seems likely that measuring \( \lambda = \sqrt{|u_z|^2/|u_z|^2} \) at all \((z, t)\) would yield a better (or at least more consistent) estimate for the EMFs or give clues to systematic variation in \( \lambda \), and it remains to reconcile the two distinct wavenumbers used to estimate the radial and toroidal EMFs respectively. Even the simple act of running our best simulation for longer would give us a larger \( N \) in our strained temporal average, and so give us smoother data with which to work. As outlined in §5.5 we would liked to have used these calculated EMFs not just as passive verification of our nonlinear simulation but as a fully independent model of the dynamo in a shearing box with vertical gravity. Another natural question that follows immediately is whether this dynamo which we have taken strides to understand is truly representative of the dynamo of the entire accretion disc: perhaps in future we could be taking an azimuthal average of a global disc.
Figure 6.5: Top panel: $\mathcal{E}_x^2$ split into vertical bins of height $0.5H$ and averaged over all $z$ and $t$ within the bin for our run of smallest horizontal extent (green line), medium horizontal extent (purple line) and our largest horizontal extent (blue line). $\mathcal{E}_x^2$ increases dramatically in size with increasing horizontal extent and a peak develops around $|z| = 3.25H$. Bottom panel: The same for $\mathcal{E}_y^2$. Again we see an increase in strength with increasing horizontal extent, with a peak that develops around $|z| = 2.25H$. This Figure was shown in the main text as Figure 5.9.
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Figure 6.6: Left: prediction of $\tilde{E}_x$ with $k_y = k_y^{\text{wa}}$, and $\lambda = 0.1$, as described on page 156, with simulation data on the left and our prediction on the right. Right: the same, for our prediction of $\tilde{E}_y$ with $k_y = k_y^{\text{min}}$ and $\lambda = 0.68 + 0.028|z|/H$.

These Figures were shown in the main text as Figure 5.16 and Figure 5.17.

simulation and reintroducing the radial derivatives in our analytic estimates from Chapter 3. We have opened up many promising avenues of research.
Appendix A

Appendix Swing

A.1 Hermite Ansatz

We perturb around the momentum equation given in §2.1.1 and introduce the shearing wave ansatz.

\[
\begin{align*}
\dot{u}_x(z) &= 2\Omega u_y - ik_x w \\
\dot{u}_y(z) &= -(2 - q)\Omega u_x - ik_y w \\
\dot{u}_z(z) &= -\frac{\partial w}{\partial z} \\
\dot{w}(z) &= -ik_x u_x - ik_y u_y - \left( \frac{\partial u_z}{\partial z} - \frac{zu_z}{H^2} \right)
\end{align*}
\]

where we have introduced the perturbation enthalpy \( w = c_s^2 \rho'/\rho \) and the shearing wave ansatz described in §1.5. This problem is completely separable in space; we decompose our vertical structure via a Hermite ansatz whereby

\[
\begin{align*}
\dot{u}_x(z) &= \sum u^n_x \text{He}_n(z/H), & \dot{u}_y(z) &= \sum u^n_y \text{He}_n(z/H), \\
\dot{u}_z(z) &= \sum u^n_z \text{He}'_n(z/H), & \dot{w}(z) &= \sum w^n \text{He}_n(z/H)
\end{align*}
\]

and \( \text{He}_n \) are Hermite polynomials

\[
\text{He}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}
\]
(i.e. $\text{He}_0(x) = 1$, $\text{He}_1(x) = x$, $\text{He}_2(x) = x^2 - 1,...$). We nondimensionalise for this Appendix and for Chapter 2 via $c_s = \Omega = H = 1$ and gain, for each $n$, the $z$-independent equations

\begin{align*}
\dot{u}_x^n &= 2u_y^n - ik_x w^n \\
\dot{u}_y^n &= -(2 - q)u_x^n - ik_y w^n \\
\dot{u}_z^n &= -w^n \\
\dot{w}_x^n &= -ik_x u_x^n - ik_y u_y^n + nu_z^n
\end{align*}

where we have applied the identities

$$\text{He}_{n+1}(z/H) = (z/H)\text{He}_n(z/H) - \text{He}'_n(z/H)$$

and

$$\text{He}'_{n+1}(z/H) = (n+1)\text{He}_n(z/H).$$

We then continue our manipulations in \[2.1.1\] by introducing the Lagrangian displacement.

### A.2 Vorticity equation and gauge transform

We must choose the “canonical gauge” described by Friedman and Schutz. We describe what is meant by a “trivial mode”, explicitly manipulate the vorticity equation, and perform the gauge transform from non-canonical to canonical gauge. In \[2.1.1\] we had

\begin{align*}
\ddot{\xi}_x &= 2(\dot{\xi}_y + q\dot{\xi}_x) - ik_x w \\
\ddot{\xi}_y &= -2\dot{\xi}_x - ik_y w \\
\dot{\xi}_z &= -w \\
w &= -ik_x \xi_x - ik_y \xi_y + n\xi_z
\end{align*}

which has six time derivatives where our original Eulerian equations had only four. These two extra time derivatives correspond to what Friedman and Schutz called ‘trivial modes’ - exact incompressible solutions of the Lagrangian perturbation
A. Appendix: Isothermal swing calculations

equations that correspond to a vanishing Eulerian perturbation i.e. a relabelling of Lagrangian fluid elements. Since these modes must be incompressible they are easily found by inspection; in vector form of $(\xi_x, \xi_y, \xi_z, v_z)^T$, we have

$$t^z = \frac{1}{(2-q)k_y} \begin{pmatrix} 0 \\ -i \\ k_y/n \\ 0 \end{pmatrix}, \quad t^x = \frac{1}{(2-q)k_y} \begin{pmatrix} -k_y \\ k_x \\ 0 \\ 0 \end{pmatrix}$$

which have zero velocity and enthalpy perturbation. (To see this recall that $u_y = \dot{\xi}_y + q\xi_x$.)

We desire not to consider these extra modes and eliminate them via the vorticity equation followed by an explicit gauge transform. Explicitly, our perturbation vorticity is

$$\omega_x = (\dot{\xi}_y + q\xi_x - ik_y\dot{\xi}_z)$$

and

$$\omega_z = ik_x(\dot{\xi}_y + q\xi_x) - ik_y\dot{\xi}_x$$

and $\omega_y$ fully determined by $\nabla \cdot \omega = 0$. The vorticity equation in the shearing sheet, before linearising, is

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u - \omega \nabla \cdot u.$$

After linearising (but before the shearing wave and Hermite ansätze) this becomes

$$\frac{\partial \omega}{\partial t} - ik_y q x \omega = (2-q) \left( \frac{\partial u}{\partial z} - (\nabla \cdot u)e_z \right) - q \omega_x e_y.$$ 

We examine the $x$ and $z$ components

$$\frac{\partial \omega_x}{\partial t} - ik_y q x \omega_x = (2-q) \frac{\partial u_x}{\partial z},$$

$$\frac{\partial \omega_z}{\partial t} - ik_y q x \omega_z = (2-q) \left(-ik_x u_x - ik_y u_y \right).$$
For shearing waves the second term on the left hand side cancels with the time derivative of the shearing ansatz \( \propto \exp(ik_y q t x) \); for a Hermite ansatz the \( z \)-derivatives disappear since \( \omega_x \propto \text{He}_n^l \) and \( \omega_z \propto \text{He}_n \); for a Lagrangian system these may all be integrated once in time. This gives

\[
\omega_x = (2 - q) \xi_x + \mu_x
\]

and

\[
\omega_z = (2 - q) (w - n \xi_z) + \mu_z
\]

with \( \mu_x, \mu_z \) both constants of integration (with implicit superscripts of \( n \)). The above relations between \( \omega \) and \( \xi \) are now easily confirmed by substitution into our Lagrangian equations. Referring again to Friedmann and Schutz we see that an absence of trivial modes corresponds to these constants of integration vanishing, motivating us at the end of this section to remove them via a gauge transform. We expand the above two equations

\[
ik_y \dot{\xi}_z - (\dot{\xi}_y + q \xi_z) = (2 - q) \xi_x + \mu_x
\]

\[
\iff \dot{\xi}_y = ik_y \dot{\xi}_x - 2 \xi_x - \mu_x
\]

and

\[
ik_x (\dot{\xi}_y + q \xi_x) - i k_y \dot{\xi}_x = -(2 - q) (ik_x \xi_x + ik_y \xi_y) + \mu_z
\]

\[
\iff ik_y \dot{\xi}_x = i k_x \dot{\xi}_y + (2 - q) ik_y \xi_y + 2ik_x \xi_x - \mu_z
\]

\[
\iff \dot{\xi}_x = ik_x \dot{\xi}_z + (2 - q) \xi_y - \frac{k_x}{k_y} \mu_x - \frac{\mu_z}{ik_y}.
\]

With these two first-order equations we may replace the \( \ddot{\xi}_x \) and \( \ddot{\xi}_y \) equations:

\[
\dot{\xi}_x = ik_x v_z + (2 - q) \xi_y - \frac{k_x}{k_y} \mu_x - \frac{\mu_z}{ik_y},
\]

\[
\dot{\xi}_y = ik_y v_z - 2 \xi_x - \mu_x,
\]

\[
\ddot{\xi}_z = u_z
\]

\[
\dot{u}_z = ik_x \xi_x + ik_y \xi_y - n \xi_z.
\]
A. Appendix: Isothermal swing calculations

We now have four time derivatives and two constants of integration $\mu_x$ and $\mu_z$, and so immediately apply a gauge transformation to remove the constants; consider the substitution

\[
\begin{pmatrix}
\xi_x \\
\xi_y \\
\xi_z \\
u_z
\end{pmatrix} = \begin{pmatrix}
\eta_x \\
\eta_y \\
\eta_z \\
v_z
\end{pmatrix} + \mu_xt^x + \mu_zt^z
\]

giving a modified $\dot{\xi}_x$ equation,

\[
\dot{\eta}_x = ik_xu_z + (2 - q)\eta_y + (2 - q)(t_y^z\mu_x + t_y^z\mu_z) - \frac{k_x}{k_y}\mu_x - \frac{\mu_z}{ik_y}
\]

\[
= ik_xu_z + (2 - q)\eta_y,
\]

a modified $\dot{\xi}_y$ equation,

\[
\dot{\eta}_y + \mu_xt^x = ik_yu_z - 2\eta_x - 2\mu_xt^x - \mu_x
\]

\[
\Leftrightarrow \dot{\eta}_y = ik_yu_z - 2\eta_x + \frac{(2 - q)k_y}{(2 - q)k_y}\mu_x - \mu_x
\]

\[
= ik_yu_z - 2\eta_x,
\]

and leaving the final two equations unchanged; to summarise, we have

\[
\dot{\eta}_x = ik_xv_z + (2 - q)\eta_y
\]

\[
\dot{\eta}_y = ik_yv_z - 2\eta_x
\]

\[
\dot{\eta}_z = v_z
\]

\[
\dot{v}_z = ik_x\eta_x + ik_y\eta_y - n\eta_z.
\]

which are simply our governing equations with $\mu_x = \mu_z = 0$. Thus we see that we may set these constants to zero and consider only the modes of the ‘canonical gauge’ of Friedman and Schutz and refer to these equations as our ‘canonical Lagrangian equations’. We shall now forget $\eta$ and analyse $\xi$ assuming this gauge in the next section.
A.3 Conservation of $J(\cdot, \cdot)$

We mention for completeness a possible Lagrangian density for this system:

$$\mathcal{L}_n(\xi, \dot{\xi}, t) = \frac{1}{2} \left( |\dot{\xi}_x|^2 + |\dot{\xi}_y|^2 + n|\dot{\xi}_z|^2 + 4\Re \left[ \dot{\xi}_x \dot{\xi}_y^* \right] + 2q|\xi_x|^2 \right) - \frac{1}{2} |ik_x \xi_x + ik_y \xi_y - n\xi_z|^2.$$ 

To check that our Hermitian form is conserved we take the time derivative.

$$\frac{\partial}{\partial t} J(\xi, \eta) \prop \frac{d}{dt} \left[ \dot{\xi}_x \dot{\eta}_x + \dot{\xi}_y \dot{\eta}_y + \dot{\xi}_z \dot{\eta}_z + n(\dot{\xi}_x \dot{\eta}_z - \dot{\xi}_z \dot{\eta}_x) \right] + 2 \frac{d}{dt} \left[ \dot{\xi}_y \eta_x - \dot{\xi}_x \eta_y \right]$$

the symmetric terms cancel,

$$= \left[ \dot{\xi}_x \dot{\eta}_x + \dot{\xi}_y \dot{\eta}_y + \dot{\xi}_z \dot{\eta}_z + 2(\dot{\xi}_x \dot{\eta}_x - \xi_x w_x) \right] - (\eta \leftrightarrow \xi^*)$$

the $q\xi_x^* \eta_x$ term is symmetric and cancels, as do the Coriolis terms after some inspection.

$$= \left[ \dot{\xi}_x (-ik_x \eta_x) + \dot{\xi}_y (-ik_y \eta_y) + n \dot{\xi}_z (-w_z) \right] - (\eta \leftrightarrow \xi^*, i \leftrightarrow -i),$$

as required.
A. Appendix: Isothermal swing calculations

A.4 Phase calculation \( \int \omega(t') \, dt' \)

In calculating the extreme cases of e.g. high \( n \), low \( k_y \) and low \( n \), high \( k_y \) of the isothermal swing problem we are moving around the \( \omega \)-plane in ways which are difficult to quantify after taking a limit of large \( |t| \), varying the integration time required to correctly deduce the asymptotic phase by some laborious-to-calculate amount. We therefore make no approximation and integrate the phase exactly.

We begin from

\[
\omega = \frac{1}{\sqrt{2}} \left( \kappa^2 + k^2_x + k^2_y + n + \sqrt{(\kappa^2 + k^2_x + k^2_y + n)^2 - 4(n\kappa^2 - 2k^2_yq)} \right)^{1/2}
\]

and note that

\[
\int_0^t \omega(t') \, dt' = \frac{1}{k_y q} \int_0^{k_x} \omega(k_x') \, dk_x'.
\]

We introduce the constants \( A = \kappa^2 + k^2_y + n \) and \( B = (n\kappa^2 - 2k^2_yq) \) and make the substitution \( k_x' = \sqrt{A} \sinh \phi' \):

\[
\int_0^t \omega(t') \, dt' = \frac{\sqrt{A/2}}{k_y q} \int_0^\phi \cosh(\phi') \sqrt{A \cosh^2(\phi') + \sqrt{A^2 \cosh^4(\phi') - 4B}} \, d\phi'.
\]

At this point we turn to Mathematica. After considerable simplification by hand we arrive at

\[
\int_0^t \omega(t') \, dt' = \frac{1}{4k_y q} \left( \frac{2\sqrt{2}k_x}{\sqrt{\tau^2 + \sqrt{\tau^4 - 4B}}} \left[ \tau^2 - B \frac{(\tau^4 - 2B) + \tau^2 \sqrt{\tau^4 - 4B}}{(\tau^4 - B)\sqrt{\tau^4 - 4B} + \tau^2 (\tau^4 - 3B)} \right] \right)
\]

\[
+ 2\sqrt{B} \left( \log \left[ A - \frac{4B}{\tau^2 + \sqrt{\tau^4 - 4B}} - \frac{2\sqrt{2Bk_x}}{\sqrt{\tau^2 + \sqrt{\tau^4 - 4B}}} \right] - \log[P] \right)
\]

\[
+ A \left( \log \left[ k_x + \sqrt{\tau^4 - 4B} + \sqrt{2\tau^2 + \sqrt{\tau^4 - 4B}} \right] - \log[P] \right)
\]
which is real, with the notational simplifications that

\[
A = k_y^2 + \kappa^2 + n \\
B = \kappa^2 n - 2qk_y^2 \\
P = \sqrt{A^2 - 4B} \\
\tau^2 = A + k_x^2
\]

This integral might appear to be complex; the second line involves \( \sqrt{B} \) where \( B \) may be negative. To see that it is real, consider large \( |t| \) and \( B < 0 \); we gain in the second bracket

\[
\log(A - i\sqrt{2|B|}) - \log(A^2 + 4|B|)/2
\]

which is purely imaginary and cancels the \( i \) from the \( \sqrt{B} \) in front.
A. Appendix: Isothermal swing calculations

A.3 Normalisation of waves

We show that fast and slow waves are generally orthogonal.

\[ J(f^+, s^+) = \frac{ik_y}{2} \left[ \dot{s}_z w_f - \dot{f}_z^* w_s + (2 - q)(f_x^* s_y - s_x f_y^*) \right] \]

\[ = \frac{ik_y}{2} \sqrt{\frac{2}{k_y \omega}} \left[ ik_x s_z + \left( \frac{\omega}{k_x} \right) w_s + (2 - q) \left( s_y - s_x \left( \frac{\omega k_y + 2ik_x}{\omega k_x} \right) \right) \right] \]

\[ \times \exp \left( -i \int_t^{t'} \omega(t') dt' \right) \]

\[ = \frac{ik_y}{2} \sqrt{\frac{2}{k_y \omega}} \left[ ik_x s_z + (2 - q)s_y + O(|t|^{\sigma - 1}) \right] \]

\[ \times \exp \left( -i \int_t^{t'} \omega(t') dt' \right) \]

\[ = \frac{ik_y}{2} \sqrt{\frac{2}{k_y \omega}} \left[ s_x + O(|t|^{\sigma - 1}) \right] \]

\[ \times \exp \left( -i \int_t^{t'} \omega(t') dt' \right) \]

\[ = o(1) \]

Calculation of naive fast waves

\[ J(f^+, f^+) = -k_y \text{Im} \left[ \dot{\xi}_x w^*_x + (2 - q)\xi^*_x \xi_y \right] \]

\[ = -k_y \text{Im} \left[ \dot{\xi}_x w^*_x + O(t^{-3/2}) \right] \]

\[ \sim -k_y \text{Im} \left[ \dot{\xi}_x (-ik_x \xi_x)^* \right] \]

\[ = -2 \text{Im} \left[ (\mp k_x/\omega^{3/2}) \left( ik_x/\omega^{1/2} \right) \right] \]

\[ \sim \pm \frac{k^2}{\omega^2} \sim \pm 2 \]
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and

\[
J(f^+, f^-) = \frac{ik_y}{2} \left[ \hat{f}^i w^*_i - f^+_i w_- + O \left( t^{-3/2} \right) \right]
\]
\[
\sim \frac{ik_y}{2} \left[ \hat{f}^- (-ik_x \xi^+_x)^* - f^+_x (-ik_x \xi^-_x) \right]
\]
\[
= -\frac{k_x k_y}{2} \left[ \hat{f}^i \xi^+_x + f^+_x \xi^-_x \right]
\]
\[
= -\frac{k_x k_y}{2} \left[ \mp \omega \frac{\omega}{k_x} \mp \omega \frac{\omega}{k_x} \right] = 0
\]
as required.

Calculation of naive slow waves: for the oscillating case

\[
J(s^+, s^-) = -k_y \text{Im} \left[ \hat{s}^i w^*_i + (2 - q) \xi^+_x \xi^+_y \right]
\]
\[
= -k_y \text{Im} \left[ O \left( t^{\sigma^+_x + \sigma^-_y - 1} \right) + (2 - q) \xi^+_x \xi^+_y \right]
\]
\[
\sim -(2 - q) k_y \text{Im} \left[ \xi^+_x \xi^+_y \right]
\]
\[
\sim -2(2 - q) \text{Im} \left[ \left( \frac{2 + q \sigma^+_x}{2q} \right)^* \text{sgn}(t) \right]
\]
\[
\sim (2 - q) \text{Im} \left[ \sigma^+_x \right] \text{sgn}(t)
\]
\[
\sim \pm (2 - q) \theta \text{sgn}(t)
\]

and

\[
J(s^+, s^-) = \frac{ik_y}{2} \left[ \hat{s}^i w^*_i - \hat{s}^+_i w^- + (2 - q) (s^+_i s^-_y - s^-_i s^+_y) \right]
\]
\[
= \frac{ik_y}{2} \left[ O \left( t^{\sigma^+_y + \sigma^-_x - 1} \right) + (2 - q) (s^+_x s^-_y - s^-_x s^+_y) \right]
\]
\[
= (2 - q) i \left[ \left( \frac{2 + q \sigma^+_y}{2q} \right) - \left( \frac{2 + q \sigma^-_y}{2q} \right) \right] t^{\sigma^+_y + \sigma^-_x - 1} \text{sgn}(t)
\]
\[
= (2 - q) i \left[ \sigma^-_x - \sigma^+_y \right] t^{\sigma^+_y + \sigma^-_x + 1} \text{sgn}(t)
\]
A. Appendix: Isothermal swing calculations

but for \( \theta^2 > 0 \) we have \( \sigma_+^* = \sigma_- \) and so this vanishes. When \( \theta^2 < 0 \) we have \( \text{Im}[\sigma] = 0 \) and so

\[
J(s^+, s^+) = 0
\]

\[
J(s^+, s^-) = (2 - q) \frac{i}{2} [\sigma_+ - \sigma_+] |t|^\sigma_+ \sigma_- sgn(t)
\]

\[
= (2 - q) \frac{i}{2} [2i\theta] sgn(t)
\]

\[
= -(2 - q)\theta sgn(t)
\]

which leads easily to the non-oscillating expression for basis 1. We demonstrate now that basis 2 is orthonormal: consider the oscillating case (\( \theta \) real); then

\[
J(s^\pm, s^\pm) = \frac{1}{2(2 - q)} \left[ J\left(s^+ \left(1 \pm \frac{\tau}{\theta}\right), s^+ \left(1 \pm \frac{\tau}{\theta}\right)\right) + J\left(s^- \left(1 \mp \frac{\tau}{\theta}\right), s^- \left(1 \mp \frac{\tau}{\theta}\right)\right)\right]
\]

\[
= \frac{1}{2(2 - q)} \left[ \left|1 \pm \frac{\tau}{\theta}\right|^2 - \left|1 \mp \frac{\tau}{\theta}\right|^2 \right] J(s^+, s^+)
\]

\[
= \theta \tau \left[ 1 \pm \frac{\tau}{\theta} \right] J(s^+, s^+)
\]

and \( \theta \) is real, so this

\[
= \frac{\theta \tau}{2} \left[ \pm 2 \frac{\tau}{\theta} \pm 2 \frac{\tau}{\theta} \right]
\]

\[
= \pm 2.
\]

Consider also when \( \theta \) is imaginary. Then we should have

\[
J(s^\pm, s^\pm) = \frac{1}{2(2 - q)} \left[ J\left(s^+ \left(1 \pm \frac{\tau}{\theta}\right), s^- \left(1 \mp \frac{\tau}{\theta}\right)\right) + J\left(s^- \left(1 \mp \frac{\tau}{\theta}\right), s^+ \left(1 \pm \frac{\tau}{\theta}\right)\right)\right]
\]

and by the properties of a Hermitian form

\[
= \frac{1}{2(2 - q)} \left[ \left(1 \pm \frac{\tau}{\theta}\right)^* \left(1 \mp \frac{\tau}{\theta}\right) J(s^+, s^-) + \left(1 \mp \frac{\tau}{\theta}\right)^* \left(1 \pm \frac{\tau}{\theta}\right) J(s^-, s^+)\right]
\]

\[
= \frac{1}{2(2 - q)} \left[ \left(1 \pm \frac{\tau}{\theta}\right)^* \left(1 \mp \frac{\tau}{\theta}\right) - \left(1 \mp \frac{\tau}{\theta}\right)^* \left(1 \pm \frac{\tau}{\theta}\right)\right] J(s^+, s^-).
\]
Since $\theta$ here is imaginary we have that this

$$= \frac{1}{2(2-q)} \left[ \left(1 \mp \frac{\tau}{\theta}\right)^2 - \left(1 \pm \frac{\tau}{\theta}\right)^2 \right] J(s^+, s^-)$$

$$= -\frac{\theta \tau}{2} \left[ \left(1 \mp \frac{\tau}{\theta}\right)^2 - \left(1 \pm \frac{\tau}{\theta}\right)^2 \right]$$

$$= -\frac{\theta \tau}{2} \left[ \mp 2 \frac{\tau}{\theta} \mp 2 \frac{\tau}{\theta} \right]$$

$$= \pm 2$$

as required.
Appendix B

Appendix Analytic EMFs

Quasilinear EMFs

B.0.1 Zeroth order

Since the algebraic method used to derive the Quasilinear EMFs is rather involved, we give the full derivation here and sketch the important steps in the main text. We take our governing equations (first given in §3.2): the momentum equations,

\[
\begin{align*}
\rho \left( \partial_t + i k_y U_y \right) u_x - 2 \Omega \rho u_y &= -\partial_x \Pi' + i k \cdot B b_x + \rho' g_x + \rho \nu \nabla^2 u_x \\
\rho \left( \partial_t + i k_y U_y \right) u_y + (2 \Omega + \partial_z U_y) \rho u_x &= -i k_y \Pi' + i k \cdot B b_y + (b_x \partial_x + b_z \partial_z) B_y \\
&\quad + \rho \nu \nabla^2 u_y \\
\rho \left( \partial_t + i k_y U_y \right) u_z &= -\partial_z \Pi' + i k \cdot B b_z + \rho' g_z + \rho \nu \nabla^2 u_z
\end{align*}
\]

the induction equations

\[
\begin{align*}
(\partial_t + i k_y U_y) b_x &= i k \cdot B u_x \\
(\partial_t + i k_y U_y) b_y + (u_x \partial_x + u_z \partial_z) B_y &= i k \cdot B u_y - B_y \Delta + b_x \partial_x U_y + \eta \nabla^2 b_y \\
(\partial_t + i k_y U_y) b_z &= i k \cdot B u_z + \eta \nabla^2 b_z
\end{align*}
\]
and the mass equation, total pressure equation, the solenoidality condition, and an expression for the velocity divergence

\[
(\partial_t + ik_y U_y) \rho' = -\rho \Delta - u_z \partial_z \rho + \nu_m \nabla^2 \rho'
\]

\[
\Pi' = c_s^2 \rho' + B_y b_y
\]

\[
0 = \partial_x b_x + ik_y b_y + \partial_z b_z
\]

\[
\Delta = \frac{\partial u_x}{\partial x} + ik_y u_y + \frac{\partial u_z}{\partial z}
\]

Recall our ansatz

\[
u = [u_0(y, z) + \epsilon u_1(y, z, t) + O(\epsilon^2)] \quad \exp \left[ i \epsilon^{-1} \Phi(x, z) + \int_0^t s(y, z, t') dt' \right]
\]

\[
b = [b_0(y, z) + \epsilon b_1(y, z, t) + O(\epsilon^2)] \quad \exp \left[ i \epsilon^{-1} \Phi(x, z) + \int_0^t s(y, z, t') dt' \right]
\]

\[
\rho' = [\rho_0(y, z) + \epsilon \rho_1(y, z, t) + O(\epsilon^2)] \quad \exp \left[ i \epsilon^{-1} \Phi(x, z) + \int_0^t s(y, z, t') dt' \right]
\]

\[
\Pi' = \epsilon [\Pi_0(y, z) + \epsilon \Pi_1(y, z, t) + O(\epsilon^2)] \quad \exp \left[ i \epsilon^{-1} \Phi(x, z) + \int_0^t s(y, z, t') dt' \right]
\]

\[
\equiv c_s^2 \rho' + B_y b_y
\]

\[
\Delta = [\Delta_0(y, z) + \epsilon \Delta_1(y, z, t) + O(\epsilon^2)] \quad \exp \left[ i \epsilon^{-1} \Phi(x, z) + \int_0^t s(y, z, t') dt' \right].
\]

We take also a wavevector

\[
k_x/\epsilon = \partial_x \Phi/\epsilon, \quad k_z/\epsilon = \partial_z \Phi/\epsilon
\]

and get at \(O(1/\epsilon)\) in the expansion that

\[
ik_x u_{x0} + ik_z u_{z0} = 0.
\]

which leads to our zeroth-order equations

\[
\rho \left( s_0 + ik_y U_y + k^2 \nu \right) u_{x0} - 2\Omega \rho u_{y0} = -ik_x \Pi'_0 + i k \cdot B b_{x0} + \rho_0 g_x
\]

\[
\rho \left( s_0 + ik_y U_y + k^2 \nu \right) u_{y0} + (2 - q) \Omega \rho u_{x0} = +ik \cdot B b_{y0} + (b_{x0} \partial_x + b_{z0} \partial_z) B_y
\]

\[
\rho \left( s_0 + ik_y U_y + k^2 \nu \right) u_{z0} = -ik_z \Pi'_0 + i k \cdot B b_{x0} + \rho_0 g_z
\]
\[ (s_0 + ik_y U_x + k^2 \eta) b_x \]  
\[ = i k \cdot B u_{x0} \]  
\[ (s_0 + ik_y U_y + k^2 \eta) b_y + (u_{x0} \partial_x + u_{z0} \partial_z) B_y \]  
\[ = i k \cdot B u_{y0} - B_y \Delta_0 - q \Omega b_{z0} \]  
\[ (s_0 + ik_y U_y + k^2 \eta) b_{z0} \]  
\[ = i k \cdot B u_{z0} \]

and

\[ (s_0 + ik_y U_y + k^2 \eta_m) \rho'_0 = -\rho \Delta_0 - u_{z0} \partial_z \rho, \]
\[ 0 = c_s^2 \rho'_0 + B_y b_{y0}, \]
\[ 0 = ik_x u_{x0} + ik_z u_{z0}. \]

For brevity we now write e.g. \( \gamma_\nu = (s_0 + ik_y U_y + k^2 \nu) \). We must later make a crucial simplifying assumption that \( \eta_m = \eta \), i.e. there is some “mass diffusion” equal to the Ohmic resistivity. If we think of this as a matrix problem

\[ \mathbf{A} \mathbf{e}_0 = 0 \]

with \( \mathbf{A} \) the coefficients in the above equations, we find easily our eigenvector (not normalised)

\[
\begin{pmatrix}
    u_{x0} \\
    u_{y0} \\
    u_{z0} \\
    b_{x0} \\
    b_{y0} \\
    b_{z0} \\
    \Pi' \\
    \rho_0 \\
    \Delta_0
\end{pmatrix} =
\begin{pmatrix}
    -k_z A/k_x \\
    \frac{1}{\gamma_\nu} \left( (2 - q) \Omega \frac{k_x}{k_x} + \frac{ik_y B_y D_k B_k}{\rho \gamma_\eta} \right) A - \frac{ik_y B_y C}{\rho \gamma_\eta} \\
    \frac{1}{k_x} \left( \frac{\rho}{\gamma_\nu} \left( \gamma_\nu \gamma_\eta + \omega_a^2 \right) A - \frac{B_y g_x C}{c_s^2} \right) \\
    \frac{ik_y B_y A}{\gamma_\eta} \\
    \frac{i}{k_x} \left( \frac{\rho}{\gamma_\nu} \left( \gamma_\nu \gamma_\eta + \omega_a^2 \right) A - \frac{B_y g_x C}{c_s^2} \right) \\
    \frac{C B_y / c_s^2}{-\gamma_\nu C - \frac{B_y D_k B_k}{\rho k_x A}}
\end{pmatrix}
\]

with

\[
A = k_x \gamma_\eta \left[ 1 + \frac{\omega_a^2}{\gamma_\nu \gamma_\eta} + \frac{\gamma_m v_x^2}{\gamma_\eta c_s^2} \right],
\]

\[
C = \left[ \left( 1 + \frac{\omega_a^2}{\gamma_\nu \gamma_\eta} + \frac{v_x^2}{c_s^2} \right) D_k B_y - \frac{B_y G_k}{c_s^2} - \left( \frac{2 \Omega}{\gamma_\nu} - q \left( \frac{\gamma_\eta - \gamma_\nu}{\gamma_\nu} \right) \right) i k_x B_y \right]
\]
In seeking our dispersion relation we note our final elimination, which will become useful later. Our final two equations are

\[
\left[ \left( 1 + \frac{\omega^2}{\gamma \nu \gamma_\eta} + \frac{v^2_a}{c_s^2} \right) D_k B_y - \frac{B_y G_k}{c_s^2} - \left( \frac{2}{\gamma \nu} - \frac{q (\gamma_\eta - \gamma_\nu)}{\gamma_\nu \gamma_\eta} \right) \Omega i k_y k_z B_y \right] u_{z0} = 0
\]

\[
+k_x \gamma_\eta \left[ 1 + \frac{\omega^2}{\gamma \nu \gamma_\eta} + \frac{\gamma_\eta v^2_a}{\gamma_\eta c_s^2} \right] b_y = 0
\]

and

\[
\rho \left[ (k_x^2 + k_z^2) \frac{(\gamma_\eta \gamma_m + \omega^2_s)}{\gamma_\eta} + \frac{1}{\gamma \nu} \left( \kappa^2 k_z^2 + 2\Omega \frac{i k_y k_x B_y}{\rho \gamma_\eta} D_k B_y \right) \right] u_{z0} = 0
\]

\[
+k_x B_y \left[ 1 + \frac{\omega^2}{\gamma \nu \gamma_\eta} + \frac{\gamma_\eta v^2_a}{\gamma_\eta c_s^2} \right] \left[ (k_x^2 + k_z^2) \frac{(\gamma_\eta \gamma_m + \omega^2_s)}{\gamma_\eta} + \frac{1}{\gamma \nu} \left( \kappa^2 k_z^2 + 2\Omega \frac{i k_y k_x B_y}{\rho \gamma_\eta} D_k B_y \right) \right] b_{z0} = 0
\]

with consequent dispersion relation

\[
k_x \gamma_\eta b_{y0} = - D_k B_y \left\{ \frac{\gamma \nu \gamma_m G_k}{c_s^2} + (2\Omega \gamma_\eta - q(\gamma_\eta - \gamma_\nu)) i k_y k_z B_y \right\} u_{z0}
\]

\[
= - D_k B_y \left[ G_k c_s^2 + v^2_a \left\{ \frac{\gamma \nu \gamma_m G_k}{c_s^2} + (2\Omega \gamma_\eta - q(\gamma_\eta - \gamma_\nu)) \Omega i k_y k_z \right\} \left( \gamma_\nu ^* \gamma_\nu ^* + \omega^2_s \right) \right] u_{z0}
\]

We calculate now the radial EMF $\mathcal{E}_x$. We begin from

\[
\frac{\mathcal{E}_x}{E_t} = \frac{1}{2} \Re \left[ u_{y0} b_{z0}^* - u_{z0}^* b_{y0} \right]
\]

where $E_t = \exp(2\Re[s_0]t)$. We start by eliminating $b_{z0}$ and $u_{y0}$ in favour of $u_{z0}$.
B. Appendix: Analytic EMFs

and $b_{y0}$, gaining

$$\frac{\mathcal{E}_{x0}}{E_t} = \frac{1}{2} \left( \frac{1}{k_x |\gamma_\eta|^2} \text{Re} \left[ \frac{1}{\gamma_\nu} \omega_a^2 D_k B_y - \frac{k_y k_z}{k_x} (2 - q) \Omega \text{Re} \left[ \frac{i}{\gamma_\nu \gamma_\eta^*} B_y \right] |u_{z0}|^2 \right] \right)$$

$$+ \frac{1}{2} \text{Re} \left[ \left( \frac{\omega_a^2}{\gamma_\nu \gamma_\eta^*} - 1 \right) b_{y0} u_{x0}^* \right]$$

and we evaluate this last line using Equation B.2. After some algebra, this gives

$$\frac{\mathcal{E}_{x0}}{E_t} = \frac{1}{2} k_x B_y \text{Re} \left[ \frac{1}{\gamma_\eta} |u_{z0}|^2 - \frac{k_y k_z}{k_x} B_y (2 - q) \Omega \text{Re} \left[ \frac{i}{\gamma_\nu \gamma_\eta^*} \right] |u_{z0}|^2 \right]$$

$$- \frac{1}{2} k_x B_y |\gamma_\eta|^2 c_s^2 + v_a^2 |\gamma_\eta \gamma_\nu + \omega_c^2|^2$$

$$\times \left\{ \frac{G_k}{c_s^2} \text{Re} \left[ \gamma_\eta \left( \gamma_\nu \gamma_\eta^* - \omega_a^2 \right) \left( \gamma_\eta^* \gamma_\nu^* + \omega_c^2 \right) \right] \right\} |u_{z0}|^2$$

which gives us the result shown in the text.

**B.0.2 Next order**

To find the leading order $\mathcal{E}_y$, we proceed to next order. We would begin from

$$\frac{\mathcal{E}_{y1}}{E_t} = \frac{1}{2} \text{Re} \left[ u_{z1}^* b_{x0} + u_{x0}^* b_{z1} - u_{x1}^* b_{z0} - u_{x0}^* b_{x1} \right]$$

but a quick perusal of the first order equations given in B.0.1 shows that this will give a non-algebraic system of equations i.e. we will have to deal with terms such as $u_{x1}$. To regain an algebraic system of equations, we expand all variables in
increasing powers of $t$. We have

$$A \cdot e_1 = \begin{pmatrix}
-u_{x1} - s_1 u_{x0} - \Pi_{0,x} - \Pi_0 s_{0,x} t + \frac{2i}{R_e} k_p \cdot \nabla u_{x0} \\
-u_{y1} - s_1 u_{y0} - ik_y \Pi_0 + \frac{2i}{R_e} k_y \cdot \nabla u_{y0} \\
-u_{z1} - s_1 u_{z0} - \Pi_{0,z} - \Pi_0 s_{0,z} t + \frac{2i}{R_e} k_p \cdot \nabla u_{z0} \\
-b_{x1} - s_1 b_{x0} + \frac{2i}{R_m} k_p \cdot \nabla b_{x0} \\
-b_{y1} - s_1 b_{y0} + \frac{2i}{R_m} k_p \cdot \nabla b_{y0} \\
-b_{z1} - s_1 b_{z0} + \frac{2i}{R_m} k_p \cdot \nabla b_{z0} \\
-\rho'_1 - s_1 \rho'_0 + \frac{2i}{R_m} k_y \cdot \nabla \rho'_1 \\
\Delta_0 - ik_y u_{y0} - u_{x0,x} - u_{z0,z} - (u_{x0} \partial_x + u_{z0} \partial_z) s_0 t \\
\Pi_0
\end{pmatrix}.$$

We shall proceed for both orders at once, substituting in the right hand sides only at the final step. We find for our matrix equations that

$$A \cdot e_{10} = \begin{pmatrix}
-u_{x11} - \Pi_{0,x} + \frac{2i}{R_e} k_p \cdot \nabla u_{x0} \\
-u_{y11} - ik_y \Pi_0 + \frac{2i}{R_e} k_y \cdot \nabla u_{y0} \\
-u_{z11} - \Pi_{0,z} + \frac{2i}{R_e} k_p \cdot \nabla u_{z0} \\
-b_{x11} + \frac{2i}{R_m} k_p \cdot \nabla b_{x0} \\
-b_{y11} + \frac{2i}{R_m} k_p \cdot \nabla b_{y0} \\
-b_{z11} + \frac{2i}{R_m} k_p \cdot \nabla b_{z0} \\
-\rho'_1 - s_1 \rho'_0 + \frac{2i}{R_m} k_y \cdot \nabla \rho'_1 \\
\Delta_0 - ik_y u_{y0} - u_{x0,x} - u_{z0,z} \\
\Pi_0
\end{pmatrix}$$

and

$$A \cdot e_{11} = \begin{pmatrix}
-s_{11} u_{x0} - \Pi_0 s_{0,x} \\
-s_{11} u_{y0} \\
-s_{11} u_{z0} - \Pi_0 s_{0,z} \\
-s_{11} b_{x0} \\
-s_{11} b_{y0} \\
-s_{11} b_{z0} \\
-s_{11} \rho'_0 \\
-(u_{x0} \partial_x + u_{z0} \partial_z) s_0 \\
\Pi_0
\end{pmatrix}$$

where we have assumed that the growth rate $s = s_0 + \epsilon s_{11} t + ...$.
We now eliminate for each order of $t$. We begin from

$$
\frac{\mathcal{E}^j}{E_t} = \frac{1}{2} \text{Re} \left[ u^{j*}_{z1} b_{x0} + u^{*}_{z0} b^{j}_{z1} - u^{j*}_{x1} b_{z0} - u^{*}_{x0} b^{j}_{x1} \right]
$$

where $j \in 0, 1$ denotes the relevant power of $t$. In what follows we write $I^j_x$ or $I^j_z$ for the RHS of the $x-$ and $z-$ induction equation at order $j$, and $L^j$ the RHS of the $\Delta$ equation. We eliminate $b^{j}_{z1}$ and $b^{j}_{x1}$, then $u^{j}_{x1}$, and then finally exchange all zeroth order quantities for $u_{z0}$ as above. We are left with

$$
\frac{\mathcal{E}^j}{E_t} = \frac{1}{k_x} \left( k_y B_y \text{Re} \left[ \frac{1}{\gamma \eta} \right] \text{Re} \left[ L^j u^{*}_{z0} \right] + \frac{1}{2} \text{Re} \left[ \frac{k_x I^j_x + k_z I^j_z}{\gamma \eta} u^{*}_{z0} \right] \right)
$$

which may now be evaluated. At $O(\epsilon, 1)$ we have

$$
I^0_x = -b_{x11} + \frac{2i}{R_m} k_p \cdot \nabla b_{x0}
$$

$$
I^0_z = -b_{z11} + \frac{2i}{R_m} k_p \cdot \nabla b_{z0}
$$

$$
L^0 = \Delta_0 - ik_y u_{z0} - u_{x0,x} - u_{z0,z}
$$

and at $O(\epsilon, t)$ we have

$$
I^1_x = -s_{11} b_{x0}
$$

$$
I^2_x = -s_{11} b_{z0}
$$

$$
L^1 = -(u_{x0} \partial_x + u_{z0} \partial_z) s_0
$$

$$
= -u_{z0} D_k s_0 / k_x
$$

so that

$$
\frac{\mathcal{E}^0}{E_t} = \frac{1}{k_x} \left( k_y B_y \text{Re} \left[ \frac{1}{\gamma \eta} \right] \text{Re} \left[ \left( \Delta_0 - ik_y u_{z0} - \frac{D_k u_{z0}}{k_x} \right) u^{*}_{z0} \right] - \frac{1}{2} \text{Re} \left[ \frac{k_x b_{x11} + k_z b_{z11}}{\gamma \eta} u^{*}_{z0} \right] \right)
$$

which may be manipulated into

$$
\frac{\mathcal{E}^0}{E_t} = \frac{1}{k_x} \left( k_y B_y \text{Re} \left[ \frac{1}{\gamma \eta} \right] \text{Re} \left[ \left( \Delta_0 - ik_y u_{z0} - \frac{D_k u_{z0}}{k_x} \right) u^{*}_{z0} \right] - \frac{1}{2} k_y B_y \text{Re} \left[ \frac{L^1}{\gamma \eta} u^{*}_{z0} \right] \right)
$$

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i.e.

\[
\frac{\mathcal{E}_{y1}^0}{E_t} = \frac{1}{k_x} k_y B_y \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] \text{Re} \left[ (\Delta_0 - ik_y u_{y0}) u_{z0}^* \right] - \frac{k_y B_y}{2k_x^2} \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] D_k |u_{z0}|^2
\]

\[
+ \frac{1}{2k_x^2} k_y B_y \text{Re} \left[ \frac{D_k s_0}{\gamma_{\eta}^2} \right] |u_{z0}|^2
\]

and recalling the original zeroth order equation for \( b_{y0} \) we may eliminate \( \Delta_0 - ik_y u_{y0} \) in favour of first \( b_{y0} \) and then \( u_{z0} \). After some algebra we arrive at

\[
\frac{\mathcal{E}_{y1}^0}{E_t} = -\frac{ky}{k_x^3} \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] - B_y c_s^2 \frac{1}{|\gamma_{\nu}|^2 + c_a^2 |\gamma_{\nu} \gamma_{\eta} + \omega_{\nu}^2|^2}
\]

\[
\times \left( G_k \left( |\gamma_{\nu}|^2 |\gamma_{\eta}|^2 + \omega_{\nu}^2 \text{Re} [\gamma_{\nu} \gamma_{\eta}] \right) + k_y k_z \Omega \text{Re} \left[ \frac{i \gamma_{\nu}^* \gamma_{\eta}^*}{\gamma_{\eta}} \right] (2 \gamma_{\eta} - q(\gamma_{\eta} - \gamma_{\nu})) + 2i \gamma_{\eta} \omega_{\nu}^2 \right) |u_{z0}|^2
\]

\[
+ \frac{k_y}{k_x^2} \left( q \Omega B_y k_z k_y \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] \text{Re} \left[ \frac{i}{\gamma_{\eta}} \right] |u_{z0}|^2
\]

\[- \frac{B_y}{2} \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] D_k |u_{z0}|^2 + \frac{1}{2} B_y \text{Re} \left[ \frac{D_k s_0}{\gamma_{\eta}^2} \right] |u_{z0}|^2
\]

which we will simplify further shortly. We move on to the calculation at \( O(\epsilon, t) \):

\[
\frac{\mathcal{E}_{y1}^1}{E_t} = \frac{1}{k_x} \left( k_y B_y \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] \text{Re} \left[ L^1 u_{z0}^* \right] + \frac{1}{2} \text{Re} \left[ \frac{k_z I_z^1 + k_z I_z^1}{\gamma_{\eta}} u_{z0}^* \right] \right)
\]

\[
= \frac{1}{k_x} \left( k_y B_y \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] \text{Re} \left[ L^1 u_{z0}^* \right] \right)
\]

\[
= -\frac{k_y}{k_x^2} B_y \text{Re} \left[ \frac{1}{\gamma_{\eta}} \right] \text{Re} \left[ D_k s_0 \right] |u_{z0}|^2
\]

and we now wish to combine these two. It is useful to note that ahead of time
that
\[
\frac{1}{2E_t} \text{Re} \left[ D_k \left( \frac{|u_{z0}|^2}{\gamma_0} E_t \right) \right] \\
= \frac{1}{2} \text{Re} \left( \frac{1}{\gamma_0} D_k |u_{z0}|^2 \right) - \frac{1}{2} |u_{z0}|^2 \text{Re} \left( \frac{1}{\gamma_0^2} D_k \gamma_0 \right) + |u_{z0}|^2 \text{Re} \left( \frac{D_k s_0}{\gamma_0} \right) t \\
= \frac{1}{2} \text{Re} \left( \frac{1}{\gamma_0} D_k |u_{z0}|^2 \right) - \frac{1}{2} |u_{z0}|^2 \text{Re} \left( \frac{1}{\gamma_0^2} (D_k s_0 + D_k i k_y q \Omega x) \right) + |u_{z0}|^2 \text{Re} \left( \frac{D_k s_0}{\gamma_0} \right) t \\
= \frac{1}{2} \text{Re} \left( \frac{1}{\gamma_0} D_k |u_{z0}|^2 \right) - \frac{1}{2} |u_{z0}|^2 \text{Re} \left( \frac{1}{\gamma_0^2} D_k s_0 \right) - \frac{1}{2} |u_{z0}|^2 \text{Re} \left( \frac{1}{\gamma_0^2} D_k i k_y q \Omega \right) \\
+ |u_{z0}|^2 \text{Re} \left( \frac{D_k s_0}{\gamma_0} \right) t.
\]

Now when we combine these two
\[
\frac{\mathcal{E}_{y1}}{E_t} = \frac{k_y}{k_x} \text{Re} \left( \frac{1}{\gamma_0} \left( \frac{1}{k_x} D_k B_y |u_{z0}|^2 - \frac{1}{k_x} B_y \frac{c_s^2}{c_a^2} |\gamma_0 \gamma_0 + \omega_a^2|^2 \right) \times \left( \frac{G_k}{c_s^2} (\gamma_0 \gamma_0 |\gamma_0|^2 + \omega_a^2 \text{Re} [\gamma_0 \gamma_0]) + k_y k_z \Omega \text{Re} \left[ i \gamma_0 \gamma_0 (2 \gamma_0 - q (\gamma_0 - \gamma_0)) + 2 i \gamma_0 \omega_a^2 \right] \right) |u_{z0}|^2 \right) \\
+ \frac{k_y}{k_x} \left( q \Omega B_y k_z k_y \text{Re} \left( \frac{1}{\gamma_0} \right) \text{Re} \left( \frac{i}{\gamma_0} \right) |u_{z0}|^2 - \text{Re} \left( \frac{1}{\gamma_0} \right) |u_{z0}|^2 D_k B_y \right) \\
- \frac{B_y}{2} \text{Re} \left( \frac{1}{\gamma_0} \right) D_k |u_{z0}|^2 + \frac{1}{2} B_y \text{Re} \left( \frac{D_k s_0}{\gamma_0^2} \right) |u_{z0}|^2 \right) \\
- \frac{k_y}{k_x} B_y \text{Re} \left( \frac{1}{\gamma_0} \right) \text{Re} [D_k s_0] |u_{z0}|^2 t.
\]
we see that

\[
\mathcal{E}_y = -\frac{1}{2} k_y B_y \Re \left[ D_k \frac{|u_{z0}|^2 E_t}{\gamma} \right]
\]

\[
- \frac{k_y}{k_x^2} B_y \Re \left[ \frac{1}{\gamma} \frac{c_s^2}{c_s^2 + v_a^2} \frac{1}{|\gamma_x \gamma + \omega_x|^2} \right]
\times \left( \frac{G_k}{c_s^2} (|\gamma|^2 |\gamma|^2 + \omega_c^2 \Re [\gamma_x \gamma]) + k_y k_z \Omega \Re \left[ i \gamma^* \gamma^* \left( 2 \gamma - q(\gamma - \gamma) \right) + 2 i \gamma^* \omega_c^2 \right] \right) |u_{z0}|^2 E_t
\]

which is the result shown in the text. It bears repeating that in Chapter 3 we will use

\[
\frac{\partial \gamma}{\partial x} = 0
\]

while

\[
\frac{\partial s_0}{\partial x} = i k_y q \Omega
\]

but that in our exponent for \(|u_{z0}|^2\) we consider only \(\Re[s_0]\). This is easily understood when we consider that \(\gamma\) is calculated from the coefficients of the perturbation quantities; if the background quantities do not vary radially then neither can \(\gamma\), and in Chapter 5 (specifically §5.4.3) we will take “background” to mean “horizontally averaged”.

We have neglected a term like \(-B_x \Delta_0\) in the \(e_{10}\) equation involving \(b_{x1}\). Had we included this term we would have a contribution to \(\mathcal{E}_{y1}\) using terms involving only \(B_x\). Neglecting molecular diffusions \(\eta\) and \(\nu\),

\[
\mathcal{E}_{y1}^{B_x} = -\frac{1}{2} k_x B_x \Re \left[ \frac{1}{\gamma} \frac{|u_{z0}|^2}{\gamma} \right]
\]

\[
+ \frac{1}{2 k_x} \frac{v_a^2 B_x}{c_s^2} \left( \frac{c_s^2}{c_s^2 + v_a^2} \right) \frac{1}{|\gamma_x |\gamma + \omega_x|^2} \left[ G_k \Re \left[ \left( \gamma^* \gamma^* - \omega_x^2 \right) \left( \gamma^* \gamma^* + \omega_x^2 \right) \right] \right)
\]

\[
+ k_y k_z \Omega \Re \left[ i \gamma^* \gamma^* \left( 2 \gamma - q(\gamma - \gamma) \right) \left( \gamma^* \gamma^* + \omega_x^2 \right) \right] |u_{z0}|^2
\]

\[
+ \Re \left[ \frac{|u_{z0}|^2}{2 \gamma} \right] \frac{g_z}{c_s^2} B_x
\]

which is a turbulent diffusion as in \(\mathcal{E}_{x0}\) and two advective terms.
B. Appendix: Analytic EMFs

B.1 Singularity when $\gamma_\eta \gamma_\nu = -\omega_c^2$

If $\gamma_\eta \gamma_\nu = -\omega_c^2$ then Equation \[B.2] becomes (with $\gamma_m = \gamma_\eta$)

$$\left[ -\frac{B_yG_k}{c_s^2} + \left( \frac{2}{\gamma_\nu} - q\frac{(\gamma_\eta - \gamma_\nu)}{\gamma_\nu \gamma_\eta} \right) \Omega ik_yk_zB_y \right] u_{z0} = 0$$

and so if we wish to have a non-trivial solution with $u_{z0} = 0$ we must assume that we are at some special location satisfying

$$\frac{G_k}{c_s^2} = -\frac{2}{\gamma_\nu} \Omega ik_yk_z - q\frac{(\gamma_\eta - \gamma_\nu)}{\omega_c^2} \Omega ik_yk_z$$

which may be substituted into the other possible choice for \[B.2\]

$$\rho \left[ \left( k_x^2 + k_z^2 \right) \frac{(\gamma_\nu \gamma_\eta + \omega_a^2)}{\gamma_\eta} + \frac{1}{\gamma_\nu} \left( k_x^2 k_z^2 + 2\Omega \frac{ik_yk_zB_y}{\rho \gamma_\eta D_k B_y} \right) \right] u_{z0} + k_x B_y \left[ 2\Omega \frac{ik_yk_z}{\gamma_\nu} + \frac{G_k}{c_s^2} \right] b_{y0} = 0$$

which becomes

$$\left[ k_p^2 \frac{\omega_a^2}{\gamma_\eta} + \frac{k_x^2 \kappa^2}{\gamma_\nu} - 2\Omega \frac{ik_yk_zB_y}{\rho \omega_c^2} D_k B_y \right] u_{z0} + ik_x k_y k_z q \Omega v_a^2 \frac{(\gamma_\eta - \gamma_\nu)}{B_y \omega_c^2} b_{y0} = 0.$$  

For simplicity, we shall assume that $k_p^4(\eta - \nu)^2 > 4\omega_c^2$ i.e. purely decaying solutions, which ensures that $\gamma_\eta$ and $\gamma_\nu$ are real. In our original calculation of $E_{z0}$ we desired to evaluate

$$\frac{1}{2} \text{Re} \left[ \left( \frac{\omega_a^2}{\gamma_\eta \gamma_\nu} - 1 \right) b_{y0} u_{z0}^* \right] = \frac{1}{2} \text{Re} \left[ \left( \frac{\omega_a^2}{\omega_c^2} - 1 \right) \times \frac{IB_y \gamma_\nu^2}{k_x k_y k_z q \Omega v_a^2 (\gamma_\eta - \gamma_\nu)} \times \left( k_p^2 \frac{\omega_a^2}{\gamma_\eta} \frac{v_a^2}{c_s^2 + v_a^2} + \frac{k_x^2 \kappa^2}{\gamma_\nu} - \frac{2\Omega}{\rho \omega_c^2} B_y D_k B_y \right) \right] |u_{z0}|^2$$

and with purely decaying solutions this now has an elegant phase relationship

$$= \frac{1}{q} \left( \frac{\omega_a^2}{\omega_c^2} + 1 \right) \frac{1}{k_x (\gamma_\eta - \gamma_\nu)} D_k B_y |u_{z0}|^2$$
so that purely decaying modes at this special location have a radial EMF

\[
\frac{\mathcal{E}_{x0}}{E_t} = \frac{1}{k_x} D_k B_y \left( \frac{\omega_a^2}{2 \gamma_{\nu}^3} - \frac{1}{q} \left( 2 + \frac{v_{a}^2}{c_s^2} \right) \frac{1}{(\gamma_{\eta} - \gamma_{\nu})} \right) |u_{z0}|^2
\]

and some similar expression may be calculated for \( \mathcal{E}_{y1} \). Since we have no particular reason to be interested in this special location we do not pursue the matter further.

### B.2 Effect of \( P_m \) on the dynamo: weak field limit

We examine the second term in \( \mathcal{E}_{x0} \) that

\[
v_2^{P_m} = \frac{1}{2} \frac{1}{|\gamma_{\eta}|^2 \frac{c_a^2}{c_s^2} + v_{a}^2 |\gamma_{\eta} \gamma_{\nu} + \omega_{\nu}^2|^2} \times \left\{ q k_{p}^2 (\eta - \nu) \frac{k_y k_z}{k_x} \Omega \frac{1}{|\gamma_{\nu}|^2} \Re \left[ i \gamma_{\nu}^2 (\gamma_{\nu}^* \gamma_{\eta} - \omega_a^2) \left( \gamma_{\eta}^* \gamma_{\nu} + \omega_a^2 \right) \right] \right\} |\bar{u}_{z0}|^2
\]

and it is clear that the condition that we are seeking here involves comparing the sizes of \( \gamma \) and \( \omega_a^2 \). In an attempt to simplify this expression we take a weak-field limit; as was explained in \([3.6.3]\) the combination \( k_y \Im [\gamma] \) does not vanish. We gain

\[
v_2^{P_m} = -\frac{1}{2} \frac{1}{|\gamma_{\eta}|^2 \frac{c_a^2}{c_s^2} + v_{a}^2 |\gamma_{\eta} \gamma_{\nu} + \omega_{\nu}^2|^2} \times \left\{ q k_{p}^2 (\eta - \nu) k_y k_z \Omega \frac{1}{|\gamma_{\nu}|^2} \Im [\gamma] \left( \left( \gamma_{\nu}^2 - \omega_a^2 \right) \left( \gamma_{\nu} \gamma_{\eta} + \omega_a^2 \right) + 2 \gamma_{\eta} \gamma_{\nu} \left( \gamma_{\nu}^2 - \omega_a^2 \right) \right) \right\} |\bar{u}_{z0}|^2
\]

and we may make the substitution \( \gamma_{\eta} = \gamma_{\nu} + k^2 (\eta - \nu) \) to rewrite this

\[
v_2^{P_m} = -\frac{1}{2} \frac{1}{|\gamma_{\eta}|^2 \frac{c_a^2}{c_s^2} + v_{a}^2 |\gamma_{\eta} \gamma_{\nu} + \omega_{\nu}^2|^2} \times \left\{ q k_{p}^2 (\eta - \nu) k_y k_z \Omega \frac{1}{|\gamma_{\nu}|^2} \Im [\gamma] \right. \\
\left. \times \left( \left( \gamma_{\nu}^2 - \omega_a^2 + k^2 (\eta - \nu) \right) \left( \gamma_{\nu} \gamma_{\eta} + \omega_a^2 \right) + 2 \gamma_{\eta} \gamma_{\nu} \left( \gamma_{\nu}^2 - \omega_a^2 \right) \right) \right\} |\bar{u}_{z0}|^2
\]

\[
v_2^{P_m} = -\frac{q \Omega}{2} \frac{1}{|\gamma_{\eta}|^2 \frac{c_a^2}{c_s^2} + v_{a}^2 |\gamma_{\eta} \gamma_{\nu} + \omega_{\nu}^2|^2} \frac{1}{k_x} k_y k_z \Im [\gamma] \\
\times \left\{ k_{p}^2 (\eta - \nu) (\gamma_{\nu}^2 - \omega_a^2) \left( 3 \gamma_{\nu} \gamma_{\eta} + \omega_a^2 \right) + k_{p}^2 (\eta - \nu)^2 (\gamma_{\nu} \gamma_{\eta} + \omega_a^2) \right\} |\bar{u}_{z0}|^2
\]
and it is now clear that a sufficient condition for this term to favour dynamo action is that $(P_m - 1)(\gamma_v^2 - \omega_a^2) < 0$. 

B. Appendix: Analytic EMFs
Appendix C

Appendix Numerical EMFS

C.1 Stencils

We use centred finite-differencing in this code, and place our enthalpy perturbation off-grid to avoid spurious pressure modes. We therefore need several stencils. For differencing on-grid quantities ‘on the grid’ we take e.g.

\[ \frac{\partial u}{\partial z} \bigg|_{i} = \frac{1}{\delta z} \left( \frac{3}{4} (u_{i+1} - u_{i-1}) + \frac{3}{20} (u_{i+2} - u_{i-2}) + \frac{1}{60} (u_{i+3} - u_{i-3}) \right) + O(\delta z^7) \]

while for differencing off-grid quantities ‘on the grid’ we take e.g.

\[ \frac{\partial \rho}{\partial z} \bigg|_{i} = \frac{1}{\delta z} \left( \frac{75}{64} \left( \rho_{i+1/2} - \rho_{i-1/2} \right) + \frac{25}{384} (u_{i+3/2} - u_{i-3/2}) + \frac{3}{640} (u_{i+5/2} - u_{i-5/2}) \right) + O(\delta z^7). \]

We have also the second derivative, which only ever needs an “on grid” stencil,

\[ \frac{\partial^2 u}{\partial z^2} \bigg|_{i} = \frac{1}{\delta z^2} \left( -\frac{49}{18} u_{i} + \frac{3}{2} (u_{i+1} + u_{i-1}) - \frac{3}{20} (u_{i+2} + u_{i-2}) + \frac{1}{90} (u_{i+3} + u_{i-3}) \right) + O(\delta z^8) \]

and finally we must define a suitable average for when we require an off-grid quantity at an on-grid location or vice versa:

\[ \rho_{i} = \frac{75}{128} (\rho_{i+1/2} + \rho_{i-1/2}) - \frac{25}{256} (\rho_{i+3/2} + \rho_{i-3/2}) + \frac{3}{256} (\rho_{i+5/2} + \rho_{i-5/2}) + O(\delta z^6). \]

This stencil has errors of \(O(\delta z^6)\), apparently larger than those stencils above. However, if we attempt to widen this stencil any further then the coefficients are not monotonically decreasing with order of \(\delta z\); we hence stop here to avoid
potential numerical instability.

C.2 Code tests

For each term in the perturbation equations we attempt to find some analytic expression which involves the term being tested and other, previously tested terms; this systematic introduction of new terms made code testing relatively fast and error-free. Although we have tested each term in the perturbation equations this way, we present only three such tests for brevity together with a demonstration that we are correctly integrating the energy equation; all other tests had excellent agreement with analytic predictions.

C.2.1 Viscous decay

We suppress all terms except the simplest form of viscous decay in the momentum equation, and set \( q = 0 \). This gives us

\[
\begin{align*}
\frac{\partial u_z}{\partial t} &= -\frac{\partial w}{\partial z} + \nu \frac{\partial^2 u_z}{\partial z^2} \\
\frac{\partial w}{\partial t} &= -\frac{\partial u_z}{\partial z}
\end{align*}
\]

(for the other two components \( u_x \) and \( u_y \) we see excellent agreement; we are testing here the spatial discretisation) which leads to a dispersion relation

\[
\omega^2 + i k_z^2 \nu \omega - k_z^2 c_s^2 = 0
\]

and with the parameters \( k_z = 6\pi, \text{ Re } = 100 \) we find the energy evolution shown in Figure C.1, the energy correctly decays following the dispersion relation until, after 30 orders of magnitude, longwave noise on the gridscale is revealed. This is an unavoidable consequence of spatial discretisation. When we initialise our code with random initial conditions this noise is perfectly acceptable; when we initialise our code with a high vertical wavenumber we shall examine the structure of the mode to ensure that it is indeed of the vertical scale desired.
Figure C.1: Viscous decay proceeds until - after ten orders of magnitude - it levels off.

C.2.2 Separable Toroidal MRI

We suppress gravity and all vertical gradients of background quantities, giving the system

\[
\frac{\partial u_x}{\partial t} - 2\Omega u_y = -ik_x w - ik_x \frac{B_y}{\rho} b_y + ik_x B_y \frac{b_x}{\rho} + \frac{1}{\rho} \frac{\partial \sigma_{xj}}{\partial x_j},
\]

\[
\frac{\partial u_y}{\partial t} + (2 - q)\Omega u_x = -ik_y w + \frac{1}{\rho} \frac{\partial \sigma_{yj}}{\partial x_j},
\]

\[
\frac{\partial u_z}{\partial t} = -\frac{\partial w}{\partial z} - \frac{1}{\rho} B_y \frac{\partial b_y}{\partial z} + ik_y B_y b_z + \frac{1}{\rho} \frac{\partial \sigma_{zj}}{\partial x_j},
\]

\[
\frac{\partial b_x}{\partial t} = ik_y B_y u_x + \eta \left(-k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2}\right) b_x,
\]

\[
\frac{\partial b_y}{\partial t} = -q\Omega b_x - ik_x B_y u_x - B_y \frac{\partial u_z}{\partial z} + \eta \left(-k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2}\right) b_y,
\]

\[
\frac{\partial b_z}{\partial t} = ik_y B_y u_z + \eta \left(-k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2}\right) b_z,
\]

\[
\frac{\partial w}{\partial t} = -ik_x u_x - ik_y u_y - \frac{\partial u_z}{\partial z}.
\]

We then integrate this system using \textit{Mathematica} with an explicit Fourier ansatz in the vertical direction, all perturbation quantities \(\propto \exp(ik_z z)\), and integrate...
it in our code using periodic vertical boundary conditions. With periodicity we may reduce the size of the domain to two scale heights and thus take a higher resolution (and impose a higher Reynolds number). The agreement is excellent (see Figure C.2), although since the toroidal MRI favours high vertical wavenumber our chosen \( k_z \) is outcompeted by a faster growing (and much larger) \( k_z \) that grows from the grid noise at late times; changing the vertical resolution changes the randomly chosen \( k_z \). Again, this is not a problem for cases with random initial conditions.

![Figure C.2: Re[\( u_z \)] as a function of time for the vertically homogenous toroidal MRI. There is excellent agreement until a high-wavenumber perturbation grows from the gridscale.](image)

### C.2.3 Magnetic Hermite Mode

As in Chapter 1, we may gain a completely separable system if we choose to have an Alfvén speed constant with height; this corresponds to choosing a constant \( \tilde{B}_y \), and was examined by Kato [60] in the global context of QPO frequencies. Here, we are testing solely the vertical derivatives of the background quantities, and so set \( k_x = k_y = 0 \), and find \( u_x = u_y = b_x = b_z = 0 \). We have the background...
state

\[ 0 = - \left( c_s^2 + \frac{v_a^2}{2} \right) \frac{\partial \rho}{\partial z} - \rho z \]

which gives

\[ \rho = \exp \left( - \frac{z^2}{2H_m^2} \right) \]

and

\[ B = v_a \exp \left( - \frac{z^2}{4H_m^2} \right), \quad \text{where} \quad H_m^2 = \frac{(c_s^2 + v_a^2/2)}{\Omega^2} \]

and the perturbation equations

\[ \frac{\partial u_z}{\partial t} = - \frac{\partial w}{\partial z} + w \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{B^2}{2} \right) - \frac{1}{\rho} \frac{\partial}{\partial z} (B b_y) \]

\[ \frac{\partial b_y}{\partial t} = - \frac{\partial}{\partial z} (B u_z) \]

\[ \frac{\partial w}{\partial t} = - \frac{\partial u_z}{\partial z} - u_z \frac{\partial}{\partial z} (\log \rho) \]

and with harmonic time behaviour this becomes

\[ \frac{\partial^2 u_z}{\partial z^2} - \frac{z}{H_m^2} \frac{\partial u_z}{\partial z} + \frac{(\omega^2 - \Omega^2)}{(c_s^2 + v_a^2)} u_z = 0 \]

i.e. \( u_z \propto \text{He}_n(z/H_m) \) with \( H_m^2 = (c_s^2 + v_a^2/2)/\Omega^2 \) the ‘magnetic scale height’. This gives the dispersion relation

\[ \omega^2 = \Omega^2 + n \frac{(c_s^2 + v_a^2)}{H_m^2} \]

and - as shown in Figure [C.3] we have excellent agreement between our numerics and analytics.

### C.2.4 Validation of Energy equation

We compare the evolution of the energy with the integration of the energy equation 4.12 using random initial conditions and with \( k_y H = \pi/2 \), a magnetic bump between \( z = 0 \) and \( z = 1.0H \) and \( \text{Re} = 4200 \). The agreement is excellent (see Figure [C.4]) until the evolution is dominated by diffusion driven by shear; we attribute this to residuals accumulated in the temporal integration of the energy equation.
Figure C.3: \(\text{Re}[u_z]\) against time for the magnetic Hermite mode with \(k_x = k_y = 0\), \(n = 4\), \(v_a = 0.5c_s\) i.e. \(\omega = (7/3)\Omega\). There is excellent agreement.
Figure C.4: Comparison of the energy $E$ (purple) and the integral of the energy equation (green) for linear scale above and logarithmic scale below - there is excellent agreement until the dynamics are dominated by radial diffusion (see text). Also shown in lower panel is a line mimicking the general decay of a shearing wave (blue).
C.3 Isolated magnetic bump

We give the form of the isolated magnetic bump in full here. We must solve the magnetohydrostatic vertical balance.

$$c_s^2 \frac{\partial \rho}{\partial z} = -\frac{\partial}{\partial z} \left( \frac{B_y^2}{2} \right) - \rho z \Omega^2$$

It is easier to remove the exponential decay of $\rho$ by making it explicit; take $\rho = \rho_0 \tilde{\rho}(z/H) \exp(-z^2/2H^2)$ and $B_y^2 = \tilde{B}_y^2 \exp(-z^2/2H^2)$. Then

$$\rho_0 c_s^2 \frac{\partial \tilde{\rho}}{\partial z} = \frac{z}{H^2} \frac{\tilde{B}_y^2}{2} - \frac{\partial}{\partial z} \left( \frac{\tilde{B}_y^2}{2} \right)$$

which is extremely easy to solve for $\rho(z)$ given a wide range of $\tilde{B}_y$ e.g. any polynomial function. The special case when $\tilde{B}_y$ is constant and nonzero gives the constant Alfvén speed case. We keep it simple and focus on

$$\tilde{B}_y = \rho_0^{1/2} \alpha \left( \frac{z}{H} - a \right)^3 \left( \frac{z}{H} - b \right)^3$$
with $b > a$. On substitution into the magnetohydrostatic vertical balance equation above, followed by an integration in $z$, this gives

$$\tilde{\rho} \left( \frac{z}{H} \right) = 1 - \frac{\alpha^2}{24024} \left( \left( \frac{z}{H} \right) - a \right)^6 \times \left( (a^8 - 12a^7b + 65a^6b^2 - 208a^5b^3 - 572a^3b^5 + 429(a^4 + 28^2 + a^2b^2)b^4) + 6(a^7 - 12a^6b + 65a^5b^2 - 208a^4b^3 + 429a^3b^4 - 572(21 + a^2)b^5 + 429ab^6) \left( \frac{z}{H} \right) \right. $$

$$+ 21(a^6 - 12a^5b + 65a^4b^2 - 208a^3b^3 + 429(20 + a^2)b^4 - 572ab^5 - 143b^6) \left( \frac{z}{H} \right)^2 $$

$$+ 56(a^5 - 12a^4b + 65a^3b^2 - 26(165 + 8a^2)b^3 + 429ab^4 + 286b^5) \left( \frac{z}{H} \right)^3 $$

$$+ 126(a^4 - 12a^3b + 65(22 + a^2)b^2 - 208ab^3 - 286b^4) \left( \frac{z}{H} \right)^4 $$

$$+ 84(3a^3 - 858b - 36a^2b + 195ab^2 + 520b^3) \left( \frac{z}{H} \right)^5 $$

$$+ 462(26 + a^2 - 12ab - 65b^2) \left( \frac{z}{H} \right)^6 $$

$$+ 792(a + 14b) \left( \frac{z}{H} \right)^7 $$

$$- 1716 \left( \frac{z}{H} \right)^8 \right).$$

As an example, if we place a concentration of flux just off the midplane, between $z = 0$ and $z = 1.0H$, we gain

$$\tilde{\rho} = 1 - \alpha^2 \left( \frac{z}{H} \right)^6 \left( \frac{1}{2} - 3 \left( \frac{z}{H} \right)^2 + \frac{59}{8} \left( \frac{z}{H} \right)^2 - \frac{28}{3} \left( \frac{z}{H} \right)^3 \right) $$

$$+ 6 \left( \frac{z}{H} \right)^4 - \frac{13}{11} \left( \frac{z}{H} \right)^5 - \frac{3}{4} \left( \frac{z}{H} \right)^6 + \frac{6}{13} \left( \frac{z}{H} \right)^7 - \frac{1}{14} \left( \frac{z}{H} \right)^8 \right)$$

which is unpleasant but not complicated. The initial constant has come from requiring that $\tilde{\rho}(0) = 1$ so that the density in the region with field is continuous with the region without field. We may then write

$$\rho = \begin{cases} 
\exp(-z^2/2H^2) & z < 0, \\
\tilde{\rho}(z/H) \exp(-z^2/2H^2) & 0 \leq z \leq 1.0H, \\
\tilde{\rho}(1) \exp(-(z^2 - H^2)/2H^2) & z > 1.0H. 
\end{cases}$$
and

\[ B_y = \begin{cases} 
\alpha^2(z - a)^2(z - b)^2 & b < z < a, \\
0 & \text{otherwise.}
\end{cases} \]

The value of \( \alpha \) is then determined through binary search within the code to enforce the desired maximum Alfvén speed; for relevant values of \( \alpha \) it is near to \( B/\sqrt{\rho} \) evaluated at \( z = (a + b)/2 \). We can choose any \( v_{\alpha}^{\text{max}} \) simply by adjusting \( \alpha \).

**Ensemble results**

For completeness we include all eight sets of plots (four for \( k_y H = \pi \), four for \( k_y H = \pi/2 \)) of \( J \) given a height and a maximum Alfvén speed. We take an ensemble average across the runs and superimpose it on the data as a solid green line. Since there is considerable variation from run to run, we present also the square roots of the two semivariances plotted as an envelope around the ensemble average as a solid purple line. (The positive semivariance of a sample is the variance of all points above the mean of the sample, and the negative semivariance is the variance of all points below; were we to plot the square root of the variance rather than the two semivariances then it would obscure the net sign of \( J \) in some of the runs.) For extra clarity we apply a Bezier smoothing to the ensemble averages to remove any remnant of the physically uninteresting fast magnetoacoustic oscillations.
C. Appendix: Numerical EMFs

Figure C.5: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi$, Re = 4200, with bump centered on $z = 0.5H$. Reading across from the top we have $v_{a}^{\text{max}} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\} c_s$. The sign changes from positive to negative between $v_{a}^{\text{max}} = 0.25c_s$ and $0.4c_s$. 
Figure C.6: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi$, Re = 4200, with bump centered on $z = 1.5H$. Reading across from the top we have $\nu_a^{\text{max}} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\} c_s$. 
Figure C.7: $J$ for $t \in [-10, 140]/\Omega$ for $k_yH = \pi$, Re = 4200, with bump centered on $z = 2.5H$. Reading across from the top we have $v_a^{\text{max}} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\}c_s$. 
Figure C.8: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi$, $Re = 4200$, with bump centered on $z = 3.5H$. Reading across from the top we have $v_{a,x}^{max} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\}c_s$. 
Figure C.9: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi/2$, $Re = 4200$, with bump centered on $z = 0.5H$. Reading across from the top we have $v_{\max} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\} c_s$. The sign change in $J$ now occurs between 0.4 and 0.7 since the azimuthal wavenumber has halved.
Figure C.10: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi/2$, $Re = 4200$, with bump centered on $z = 1.5H$. Reading across from the top we have $v_{\alpha}^{\max} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\}c_s$. 

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Figure C.11: $J$ for $t \in [-10, 140]/\Omega$ for $k_yH = \pi/2$, Re = 4200, with bump centered on $z = 2.5H$. Reading across from the top we have $\nu_a^{\max} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\}c_s$. 
Figure C.12: $J$ for $t \in [-10, 140]/\Omega$ for $k_y H = \pi/2$, $Re = 4200$, with bump centered on $z = 3.5H$. Reading across from the top we have $v_{a \max} \in \{0.05, 0.1, 0.25, 0.4, 0.7, 1.0, 1.3\} c_s$. The sign change in $J$ now occurs at $v_{a \max} = 1.0 c_s$. 
Appendix D

Appendix Shearing Box Simulations

D.1 Various analytic predictions

In §5.4.3 we discussed applying our analytic prediction to our simulation data, and mentioned that the model has several free parameters: the poloidal polarisation $\lambda = k_x/k_z$, the azimuthal wavenumber $k_y$, and the small parameter $k_y/k_x$. We showed that we could reproduce $\bar{E}_x$ extremely well for $\lambda > 1$ and $k_y$ such that $\omega_a = \sqrt{15}\Omega/4$ was fixed, and that we could reproduce $\bar{E}_y$ well by varying $\lambda$ with $z(\bar{B}_y)^{-1}\partial_z\bar{B}_y$, a function of the magnetic pressure. We did not include $\bar{B}_x$ in our calculations.

Here we present a graphical exploration of the above switches: $k_y \in \{k_{y}^{\omega}, k_{y}^{\min}\}$, $\lambda \in \{0.1, 10\}$, and whether or not we include $\bar{B}_x$ in our calculations. Since we have presented and discussed the best approximation we could find, we present this exploration without explanation. Our graphs are ordered as follows: all predictions for $\bar{E}_x$, first excluding, then including $\bar{B}_x$, followed by all predictions for $\bar{E}_y$, first excluding, then including $\bar{B}_x$. 

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Figure D.1: Prediction of $\tilde{E}_x$ with $k_y = k_\omega y$, $\lambda = 0.1$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity. This graph also appears in Figure 5.16.

Figure D.2: Prediction of $\tilde{E}_x$ with $k_y = k_\omega y$, $\lambda = 10$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity. This graph also appears in Figure 5.16.
D. Appendix: Shearing box simulations

Figure D.3: Prediction of $\bar{E}_x$ with $k_y = k_y^{\text{min}}$, $\lambda = 0.1$, with no $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity.

Figure D.4: Prediction of $\bar{E}_x$ with $k_y = k_y^{\text{min}}$, $\lambda = 10$, with no $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity.
Figure D.5: Prediction of $\bar{E}_x$ with $k_y = k_y^{\omega^2}$, $\lambda = 0.1$, including contributions from $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity.

Figure D.6: Prediction of $\bar{E}_x$ with $k_y = k_y^{\omega^2}$, $\lambda = 10$, including contributions from $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10 \times$ the contribution due to gravity.
Figure D.7: Prediction of $\bar{E}_x$ with $k_y = k_y^{\text{min}}$, $\lambda = 0.1$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10\times$ the contribution due to gravity.

Figure D.8: Prediction of $\bar{E}_x$ with $k_y = k_y^{\text{min}}$, $\lambda = 10$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and $10\times$ the contribution due to gravity.
Figure D.9: Prediction of $\tilde{E}_y$ with $k_y = k_{y}^{\omega_a}$, $\lambda = 0.1$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.

Figure D.10: Prediction of $\tilde{E}_y$ with $k_y = k_{y}^{\omega_a}$, $\lambda = 10$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.
D. Appendix: Shearing box simulations

Figure D.11: Prediction of $\bar{\mathcal{E}}_y$ with $k_y = k_y^{\min}$, $\lambda = 0.1$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.

Figure D.12: Prediction of $\bar{\mathcal{E}}_y$ with $k_y = k_y^{\min}$, $\lambda = 10$, with no $\tilde{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.
Figure D.13: Prediction of $\bar{E}_y$ with $k_y = k_y^{\omega a}$, $\lambda = 0.1$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.

Figure D.14: Prediction of $\bar{E}_y$ with $k_y = k_y^{\omega a}$, $\lambda = 10$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.
D. Appendix: Shearing box simulations

Figure D.15: Prediction of $\bar{E}_y$ with $k_y = k_y^{\text{min}}$, $\lambda = 0.1$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.

Figure D.16: Prediction of $\bar{E}_y$ with $k_y = k_y^{\text{min}}$, $\lambda = 10$, including contributions from $\bar{B}_x$. From left to right: the simulation data, the analytical estimate, the incompressible contribution, and the contribution due to gravity.
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