

Maximum Tension: with and without a cosmological constant

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Abstract

We discuss various examples and ramifications of the conjecture that there exists a maximum force (or tension) in general relativistic systems. We contrast this situation with that in Newtonian gravity, where no maximum force exists, and relate it to the existence of natural units defined by constants of Nature and the fact that the Planck units of force and power do not depend on Planck's constant. We discuss how these results change in higher dimensions where the Planck units of force are no longer non-quantum. We discuss the changes that might occur to the conjecture if a positive cosmological constant exists and derive a maximum force bound using the Kottler-Schwarzschild-de Sitter black hole.

1 Introduction

Some time ago it was conjectured (Gibbons 2002) that in general relativity there should be a maximum value to any physically attainable force (or tension) given by

$$F_{\max} = \frac{c^4}{4G}, \quad (1)$$

where c is the velocity of light and G is the Newtonian gravitational constant. The sharp factor of $1/4$ is supported by considering the maximum deficit angle of a cosmic string (Gibbons, 2002). This gives rise to the closely related conjecture that there is a maximum power defined by

$$P_{\max} = cF_{\max} = \frac{c^5}{4G}, \quad (2)$$

the so-called Dyson Luminosity (Dyson, 1963), or some multiple of it to account for geometrical factors $O(1)$. This would be the maximum possible luminosity in gravitational waves, or indeed other forms of radiation that an isolated system may emit (Schiller 1997, 2006, Sperhake et al 2013, Cardoso 2013). Schiller (1997, 2006) has come to the same conclusion and proposed a stronger thesis: that the existence of a maximum force implies general relativity, just as a maximum velocity characterises special relativity. This claim is much less clear since it requires in effect a proof of cosmic censorship. It is also necessary to choose quite subtle energy conditions in order to avoid the formation of sudden singularities (Barrow, 2004) where unbounded pressure forces will occur. As we will see below, it would probably be specific to $3 + 1$ dimensional spacetime.

These conjectures provoke a number of comments and a discussion of some further ramifications of bounded forces in gravitating systems.

1.1 Newtonian gravity

First, there is no corresponding principle of maximum force in Newtonian gravity. Point particle masses can get arbitrarily close to one another and so the forces between them are unbounded in principle. An important example was constructed by Xia (1992). He considered a 5-body system consisting of two counter-rotating binaries systems of equal mass points with zero net angular momentum, between which a lighter point mass oscillates back and forth along the line joining the mass centres of the two binaries. For a Cantor

set of initial conditions, the system expands to infinite size in finite time and the lighter particle undergoes an infinite number of oscillations during that period. This totally unexpected behaviour is facilitated by the arbitrarily large gravitational forces that are possible between the point particles as their separations tend to zero and the absence of any speed limit for information transmission. This unusual Newtonian behaviour has no general relativistic counterpart: two particles of mass M whose centres approach closer than $d = 4GM/c^2$ will find themselves inside a black hole horizon. This is a simple form of 'cosmic censorship' whereby horizon formation prevents the effects of an arbitrary strong infinite force being visible.

1.2 Cosmic strings

Interesting circumstantial evidence for the maximum force conjecture can be seen in the general-relativistic metrics for line sources (Marder, 1959) or cosmic strings (Vilenkin, 1981). A static source with $\rho + \sum_{i=1}^3 p_i = 0$ for its density, ρ , and principal pressures, p_i , which has a constant mass per unit length, μ , has no Newtonian gravitational source ($\nabla^2 \Phi_N = 0$ for the Newtonian potential, Φ_N). Yet, it supports a conical metric which is flat spacetime with a missing wedge angle $\Delta\theta = 8\pi G\mu c^{-2}$ which will exceed 2π and encompass the entire spacetime if $F > F_{\text{max}}$. Thus the maximum force conjecture is linked to the structure of static cosmic strings even though they exert no forces on themselves or other particles. This example supports the correctness of the 1/4 factor in eq.(1), noted in Gibbons (2002.. The dimensionless factor in eq.(2) does not seem to be so precisely determined at present due to possible dependence on geometrical factors associated with power generating configurations involving many bodies.

1.3 Natural units

The maximum force conjecture also shows how dimensional analysis can still provide fundamental insights. If we seek to construct natural units for various physical quantities from the constants c, G and \hbar then we can form basic Planck units (Planck, 1899) of mass, length, and time in the usual way:

$$L_{pl} = \left(\frac{G\hbar}{c^3} \right)^{1/2}, \quad (3)$$

$$T_{pl} = \left(\frac{G\hbar}{c^5} \right)^{1/2}, \quad (4)$$

$$M_{pl} = \left(\frac{\hbar c}{G} \right)^{1/2}. \quad (5)$$

These examples of a fundamental length, mass and time all contain G , c and \hbar and so have a quantum significance. However, there are associated quantities, like the Planck force, $F_{pl} = c^4/G$, and power $P_{pl} = c^5/G$, that do not contain \hbar and so are entirely classical. Whenever Planck units can be found for a quantity that does not contain \hbar , this implies that it plays a fundamental role in classical gravity.

Planck's units of 1899 (Planck, 1899) were not the first set of natural units to be proposed (see Barrow and Tipler, 1986). In 1881, Johnson Stoney proposed a set of fundamental units involving c , G and e (Stoney, 1881, Barrow 1983, 2002). This was far sighted in that c did not yet have its fundamental relativistic significance and the electron, whose existence and charge e were predicted by Stoney in 1874 was not discovered by Thomson and colleagues until 1897 (Thomson, 1879). Stoney's natural units were

$$L_S = \left(\frac{Ge^2}{c^4} \right)^{1/2}, \quad (6)$$

$$T_S = \left(\frac{Ge^2}{c^6} \right)^{1/2}, \quad (7)$$

$$M_S = \left(\frac{e^2}{G} \right)^{1/2}. \quad (8)$$

The fine structure constant $e^2/\hbar c$ can be used to convert between Stoney and Planck units and their magnitudes just differ by approximately $\sqrt{137}$, (Barrow, 2002).

1.4 Higher dimensions

In N space dimensions there are generalisations of these natural units. By Gauss's theorem we have $[G] = M^{-1}L^N T^{-2}$ and $[e^2] = ML^N T^{-2}$, but $[c] = LT^{-1}$ and $[\hbar] = ML^2 T^{-1}$ as before. hence the dimensionless quantity that generalises the fine structure constant when $N = 3$ to arbitrary dimensions is (Barrow and Tipler, 1986)

$$\hbar^{2-N} e^{N-1} G^{(3-N)/2} c^{N-4}. \quad (9)$$

Only when $N = 3$ is gravity excluded. If we confine attention to combinations of G, c and \hbar then we find that in N dimensions the physical quantity that does not include a dependence on \hbar is only a force (or power) when $N = 2$. For general N , the fundamental non-quantum quantity is

$$Q \equiv mass \times (acceleration)^{N-2}. \quad (10)$$

This reduces to force in three dimensions and suggests a generalised conjecture that in N -dimensional general relativity there will be an upper bound on the magnitude of Q determined by

$$Q_{\max} = \frac{c^{2(N-1)}}{G}. \quad (11)$$

A calculation using the N -dimensional Schwarzschild metric gives the dimensionless factor. The horizon radius of the N -dimensional Schwarzschild metric is

$$r = \left[\frac{16\pi GM}{(N-1)\Omega_{N-1}c^2} \right]^{\frac{1}{N-2}}, \quad (12)$$

where

$$\Omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}.$$

We can calculate the quantity Q_N which generalises this to $N > 3$ black holes, using Emaparan and Reall (2008), to be

$$MA^{N-2} = c^{2(N-1)} \left[\frac{(N-2)8\pi G}{(N-1)\Omega_{N-1}} \right]^{N-2} \left[\frac{(N-1)\Omega_{N-1}}{16\pi G} \right]^{N-1}. \quad (13)$$

For the case of $N = 3$ we check that (when the spherical area factor reduces to $\Omega_2 = 4\pi$)

$$MA = c^4 \left[\frac{4\pi G}{\Omega_{N-1}} \right] \times \left[\frac{\Omega_{N-1}}{8\pi G} \right]^2 = \frac{c^4}{4G}. \quad (14)$$

1.5 Cosmological evolution

There is also a cosmological aspect to the maximum force and power conjectures. Consider the standard Newtonian cosmological picture of an isotropic and homogeneous universe modelled by an expanding spherical ball of material with radius proportional to an expansion scale factor $a(t)$. If we assume $a(t) \propto t^n$ then the force generated is proportional to $\ddot{a} \propto t^{n-2}$. Thus F will grow as $F = F_{pl}(t/t_{pl})^{n-2}$ for $t > t_{pl}$ if $n > 2$. Likewise, the power associated with this expansion is proportional to $\dot{a}\ddot{a} \propto t^{2n-3}$ and grows with time as $P = P_{pl}(t/t_{pl})^{2n-3}$ when $t > t_{pl}$ under the weaker condition $n < 3/2$; it is constant when $n = 3/2$, (Barrow and Cotsakis, 2013)¹. Here, we see a decoupling of the force and power conditions. An example of a cosmological evolution with divergent force is given by the formation of a finite-time sudden singularity in an isotropic and homogeneous Friedmann universe where \ddot{a} (and the fluid pressure, p) diverges even though a , \dot{a} and the fluid density ρ remain finite, even though $\rho + 3p > 0$ always.

2 The effect of the cosmological constant

Recently, David Thornton (private communication) has raised the question of how the inclusion of a cosmological constant affects these conclusions about a maximum force. Recall that Einstein's theory of general relativity states that matter curves spacetime and spacetime moves matter according to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu}, \quad (15)$$

¹Recently it has been claimed (Dolgov et al, 2014) that the $n = 3/2$ expansion, which would be our example of expansion at constant power, gives a preferred fit to supernovae data in an accelerating zero-curvature universe in the recent past.

where the dimensions of the quantities involved are $[x^\mu] = L$, $[g_{\mu\nu}] = 1$, $[R_{\mu\nu}] = L^{-2}$ and the energy-momentum tensor $T_{\mu\nu}$ has the dimensions of force per unit area, $[T_{\mu\nu}] = ML^{-1}T^{-2}$ and the cosmological constant Λ has dimensions $[\Lambda] = L^{-2}$. The inverse of Einstein's constant, $\frac{c^4}{8\pi G}$, has the dimensions of force $[MLT^{-2}]$, and allows us to convert from curvature to energy density or stress.

2.1 New units and bounds

The cosmological constant Λ adds a universal repulsion $\Lambda > 0$ (or attraction if $\Lambda < 0$) to the Newtonian gravitational attraction on a mass M that is proportional to the distance \mathbf{r} (Milne and McCrea, 1934)

$$\mathbf{F}_\Lambda = M \frac{\Lambda c^2}{3} \mathbf{r}, \quad (16)$$

and introduces an additional dimensionful constant into physics. A linear combination of \mathbf{F}_Λ and the inverse-square force law of Newton is the general force law which allows spherical masses to be replaced by point masses of the same mass (Laplace, 1825, Barrow and Tipler, 1986). If $\Lambda > 0$, as indicated by observations, this may be used to define another set of fundamental 'de Sitter' units of length, time and mass by dimensional analysis:

$$L_{ds} = \sqrt{\frac{1}{\Lambda}}, \quad (17)$$

$$T_{ds} = \frac{1}{c} \sqrt{\frac{1}{\Lambda}}, \quad (18)$$

$$M_{ds} = \frac{\hbar}{c} \sqrt{\Lambda}. \quad (19)$$

We see that it is not possible to create classical quantities from de Sitter units that are independent of \hbar if they include M_{ds} and so there is no new classical counterpart of eq. (1) involving Λ , but we can investigate whether this bound (or one similar) still holds in the presence of Λ .

Discussions of the physical significance of the constants of nature often make use of a mass-radius diagram (Carr and Rees, 1979, Barrow and Tipler, 1986). All bodies, at rest, may be assigned a mass M and a radius or size R . Since inertial mass, passive gravitational mass, and active gravitational mass

are equal to a high degree of precision, the definition of mass is unambiguous. The precise definition of radius is not completely clear (radius of gyration, mean half-diameter, ...), but we will ignore that small ambiguity here. It follows that all bodies may be assigned a point in the positive quadrant of the $M - R$ plane.

Large regions of the $M - R$ plane are unoccupied by observable objects because the Heisenberg Uncertainty Principle gives the lower bound on observability of real quantum states

$$M \times R > \frac{\hbar}{c}, \quad (20)$$

and the black-hole existence condition gives the constraint ²

$$R/M > \frac{2G}{c^2}. \quad (21)$$

Most celestial objects crowd around the lines of constant (atomic or nuclear) density in the $M - R$ plane where $M \propto R^3$. The bounding rectangular hyperbola (20) and the straight line (21) intersect close to the point (M_{pl}, R_{pl}) .

One may strengthen these conditions by making more restrictive assumptions about the body. For example, Buchdahl (1959) obtained the bound

$$R > \frac{9}{4} \frac{GM}{c^2} \quad (22)$$

for isotropic fluid spheres in the case of vanishing cosmological constant. Inclusion of the cosmological constant modifies this to, (Mak et al, 2000)

$$R > \frac{2GM}{c^2} \frac{1}{\left(1 - \Lambda R^2 - \frac{1}{9} \left(1 - \frac{c^2 \Lambda R^3}{GM}\right)^2\right)}. \quad (23)$$

Cosmic repulsion from a positive cosmological constant will blow apart a body (Hayward et al, 1994, Maeda et al, 1998, Cissoko et al, 1998, Nakao et al 1991, 1993), unless its representative point lies below the horizontal line

$$R = \sqrt{\frac{3}{\Lambda}}. \quad (24)$$

²For simplicity we restrict attention to spherically symmetric metrics. For non-spherically symmetric metrics one might replace R by Thorne's Hoop radius, (Gibbons, 2009, Cvetič et al, 2011).

The representative points of all ordinary bodies must lie below the line (24). and are thus confined inside a triangular region bounded by the curves (20), (24) and (24). The intersection of (20) and (24) gives a lower bound for the mass of any body:

$$M > \frac{\hbar}{c} \sqrt{\frac{\Lambda}{3}}, \quad (25)$$

while the intersection of (21) and (24) is gives an upper bound for the mass of any body:

$$M < \frac{c^2}{G} \sqrt{\frac{3}{\Lambda}}. \quad (26)$$

2.2 Kottler-Schwarzschild-de Sitter black holes

In order to obtain a maximum force estimate in the presence of a cosmological constant, consider the Kottler-Schwarzschild-de Sitter solution of Einstein's equations. This is the spherically symmetric vacuum solution with non-zero cosmological constant:

$$ds^2 = -c^2 \Delta(r) dt^2 + \frac{dr^2}{\Delta(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (27)$$

with

$$\Delta = 1 - \frac{2GM}{c^2 r} - \frac{1}{3} \Lambda r^2. \quad (28)$$

In order to have a static region, $\Delta(r)$ must have two positive roots, The smaller, at $r = r_B$, gives the radius of a black hole event horizon and the larger, at $r = r_C$, of a cosmological event horizon. The condition for two roots is (Gibbons and Hawking, 1977)

$$3M\sqrt{\Lambda} < \frac{c^2}{G}. \quad (29)$$

The limiting case was originally found by Nariai (1951) and occurs when

$$r_C = r_B = \frac{1}{\sqrt{\Lambda}}, \quad (30)$$

and the solution is the metric product $dS_2 \times S^2$

Thus, we see that

$$\frac{2GM}{c^2} \leq r_B \leq \frac{1}{\sqrt{\Lambda}}, \quad \frac{1}{\sqrt{\Lambda}} \leq r_C \leq \sqrt{\frac{3}{\Lambda}}. \quad (31)$$

In terms of a force we have

$$F_\Lambda = \frac{Mc^2}{3}\Lambda r, \quad (32)$$

and so

$$F_\Lambda(r_B) < \frac{1}{3}Mc^2\sqrt{\Lambda} < \frac{c^4}{9G}. \quad (33)$$

and can be compared with the conjectured maximum force in the absence of a Λ term given in eq. (1).

3 Conclusions

We have extended the evidence for the existence of a maximum force, in general relativity in various ways. We showed how the existence of such a fundamental bound is linked to the existence of natural units of force that exclude Planck's constant. Extensions to arbitrary space dimensions reveal a new quantity that is a candidate for a universal upper bound in general relativity. We also discussed why a maximum force cannot exist in Newtonian gravity and how counterexamples describing gravitating with arbitrarily large forces are avoided by the presence of event horizons. We discussed the development of strong forces in cosmology and at sudden finite-time singularities. Finally, we extended the discussion of the maximum force conjecture to include a cosmological constant. This leads to an additional system of natural units and new bounds on the maximum and minimum masses of bodies in the universe. Using the Kottler-Schwarzschild-de Sitter black hole solution in general relativity we derived a new maximum force bound in the presence of a positive cosmological constant.

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