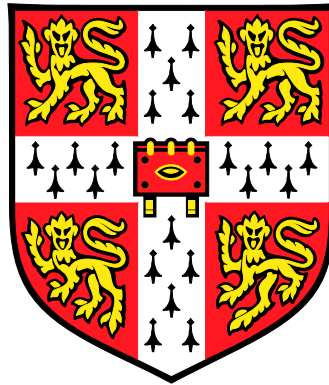# THE CONTACT PROPERTY
FOR MAGNETIC FLOWS ON SURFACES

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This dissertation is submitted for the degree of
Doctor of Philosophy

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A David e Marisa

This dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except where specifically indicated in the text.

This dissertation is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other university.

Gabriele Benedetti

October 18, 2014

The contact property for magnetic flows on surfaces

Gabriele Benedetti

SUMMARY

This work investigates the dynamics of *magnetic flows* on closed orientable Riemannian surfaces. These flows are determined by triples (M, g, σ), where M is the surface, g is the metric and σ is a 2-form on M. Such dynamical systems are described by the Hamiltonian equations of a function E on the tangent bundle TM endowed with a symplectic form ω_σ, where E is the kinetic energy. Our main goal is to prove existence results for

a) periodic orbits   and   b) Poincaré sections

for motions on a fixed energy level $\Sigma_m := \{E = m^2/2\} \subset TM$.

We tackle this problem by studying the contact geometry of the level set Σ_m. This will allow us to

a) count periodic orbits using algebraic invariants such as the Symplectic Cohomology SH of the sublevels $(\{E \leq m^2/2\}, \omega_\sigma)$;

b) find Poincaré sections starting from pseudo-holomorphic foliations, using the techniques developed by Hofer, Wysocki and Zehnder in 1998.

In Chapter 3 we give a proof of the invariance of SH under deformation in an abstract setting, suitable for the applications.

In Chapter 4 we present some new results on the energy values of contact type. First, we give explicit examples of exact magnetic systems on $\mathbb{T}^2$ which are of contact type at the strict critical value. Then, we analyse the case of non-exact systems on $M \neq \mathbb{T}^2$ and prove that, for large m and for small m with symplectic σ, Σ_m is of contact type. Finally, we compute SH in all cases where Σ_m is convex.

On the other hand, we are also interested in non-exact examples where the contact property fails. While for surfaces of genus at least two, there is always a level not of contact type for topological reasons, this is not true anymore for S^2. In Chapter 5, after developing the theory of magnetic flows on surfaces of revolution, we exhibit the first example on S^2 of an energy level not of contact type. We also give a numerical algorithm to check the contact property when the level has positive magnetic curvature.

In Chapter 7 we restrict the attention to low energy levels on S^2 with a symplectic σ and we show that these levels are of dynamically convex contact type. Hence, we prove that, in the non-degenerate case, there exists a Poincaré section of disc-type and at least an elliptic periodic orbit. In the general case, we show that there are either 2 or infinitely many periodic orbits on Σ_m and that we can divide the periodic orbits in two distinguished classes, *short* and *long*, depending on their period. Then, we look at the case of surfaces of revolution, where we give a sufficient condition for the existence of infinitely many periodic orbits. Finally, we discuss a generalisation of dynamical convexity introduced recently by Abreu and Macarini, which applies also to surfaces with genus at least two.

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Malo mortuum impendere quam vivum occidere.

Petronius, Satyricon

Then there is an intellectual or a student or two, very few of them though, here and there, with ideas in their heads that are often vague or twisted. Their "country" consists of words, or at the most of some books. But as they fight they find that those words of theirs no longer have any meaning, and they make new discoveries about men's struggles, and they just fight on without asking themselves questions, until they find new words and rediscover the old ones, changed now, with unsuspected meanings.

Italo Calvino, The Path to the Spiders' Nests

Every two or three days, at the moment of the check, he told me: "I've finished that book. Have you another one to lend me? But not in Russian: you know that I have difficulty with Russian." Not that he was a polyglot: in fact, he was practically illiterate. But he still 'read' every book, from the first line to the last, identifying the individual letters with satisfaction, pronouncing them with his lips and laboriously reconstructing the words without bothering about their meaning. That was enough for him as, on different levels, others take pleasure in solving crossword puzzles, or integrating differential equations or calculating the orbits of the asteroids.

Primo Levi, The Truce

Contents

CHAPTER 1

Introduction

1.1. Overview of the problem

Let the triple (M, g, σ) represent a magnetic system, where M is a closed manifold, g is a Riemannian metric and σ is a closed 2-form on M. We say that a magnetic system is *exact, non-exact* or *symplectic* if σ is such. Such data give rise to a Hamiltonian vector field X_E^σ on the symplectic manifold $(TM, \omega_\sigma := d\lambda - \pi^*\sigma)$, where π is the footpoint projection from TM to M and λ is the pull-back of the Liouville 1-form on the cotangent bundle via the duality isomorphism given by g. The kinetic energy $E(x, v) = \frac{1}{2} g_x(v, v)$ is the Hamiltonian function associated to X_E^σ. These systems model the motion of a charged particle on the manifold M under the influence of a magnetic force, which enters the evolution equations via the term $-\pi^*\sigma$ in the symplectic form. In the absence of a magnetic force, namely when $\sigma = 0$, the particle is free to move and we get back the geodesic flow of the Riemannian manifold (M, g). The zero section is the set of rest points for the flow and all the smooth hypersurfaces $\Sigma_m := \{E = \frac{m^2}{2}\}$, with $m > 0$, are invariant sets. Hence, we can analyse the vector fields $X_E^\sigma\big|_{\Sigma_m}$ separately for every m and compare the dynamics for different value of such parameter. In the geodesic case, no interesting phenomenon arises from this point of view. Even if the dynamics of the geodesic flow has a very rich structure and its understanding requires a deep study, the flows on Σ_m and $\Sigma_{m'}$ are conjugated up to a constant time factor and, therefore, the parameter m does not play any role in the analysis.

On the other hand, the situation for a general magnetic term σ is quite different. As the parameter m varies and passes through some distinguished values, the dynamics on Σ_m can undergo significant transformations, in the same way as the topology of the levels of a Morse function changes when we cross a critical value [**CMP04**]. How do we detect and measure these differences? Since the flows $\Phi^{X_E^\sigma}\big|_{\Sigma_m}$ can exhibit a very complicated behaviour, we have to focus our analysis on some simple property of these systems. In the present work we have chosen to look at the *periodic orbits*. In particular, we examine

a) existence and multiplicity of periodic orbits on a given free homotopy class;
b) knottedness of a single periodic orbit and linking of pairs, through the existence of global Poincaré sections and periodic points of the associated return map;
c) local properties of the flow at a periodic orbit (e.g. stability), that are related to the linearisation of the flow (elliptic/hyperbolic orbits).

Periodic orbits of magnetic flows received much attention in the last thirty years and here we seize the occasion to mention some of the relevant literature on the subject.

When M is a surface, the two classical approaches that have been pursued are Morse-Novikov theory and symplectic topology (see Taĭmanov's [**Taĭ92**] and Ginzburg's [**Gin96**] surveys for details and further references). Refinements of these old techniques and completely new strategies have been developed recently. Some authors work with (weakly) exact magnetic forms [**BT98, Pol98, Mac04, CMP04, Osu05, Con06, Pat06, FS07, Mer10, Taĭ10, AMP13, AMMP14, AB14**]. Others seek periodic solutions with low kinetic energy [**Sch06**], the majority of them assuming further that σ is symplectic [**Ker99, GK99, GK02a, Mac03, CGK04, GG04, Ker05, Lu06, GG09, Ush09**]. Schneider's approach [**Sch11, Sch12a, Sch12b**] for orientable surfaces and symplectic σ uses a suitable index theory for vector fields on a space of loops and shows in a very transparent way how the Riemannian geometry of g influences the problem. Finally, we point out [**Koh09**] where heat flow techniques are employed and [**Mer11, FMP12, FMP13**] which construct a Floer theory for particular classes of magnetic fields.

This thesis would like to give its contribution to the understanding of magnetic fields on *closed orientable surfaces* by studying the contact geometry of the level sets Σ_m. In reference to point a), b), and c) presented above, this will allow us to

a) count periodic orbits in a given free homotopy class ν using algebraic invariants such as the Symplectic Cohomology SH_ν of the sublevels $(\{E \leq \frac{m^2}{2}\}, \omega_\sigma)$;

b) find Poincaré sections starting from pseudo-holomorphic foliations [**HWZ98, HLS13**];

c) determine whether a periodic orbit is elliptic or hyperbolic by looking at its Conley-Zehnder index [**DDE95, AM**].

1.2. Contact hypersurfaces and Symplectic Cohomology

We say that a closed hypersurface Σ in a symplectic manifold (W, ω) is of *contact type* if there exists a primitive of ω on Σ which is a contact form. Hypersurfaces of contact type have been intensively studied in relation to the problem of the existence of closed orbits. Indeed, in this case the Hamiltonian dynamics on Σ is the dynamics of the Reeb flow associated to the contact form, up to a time reparametrisation. After some positive results in particular cases [**Wei78, Rab78, Rab79**], in 1978 Alan Weinstein conjectured that every closed hypersurface of contact type (under some additional homological condition now thought to be unnecessary) carries a periodic orbit [**Wei79**]. The conjecture is still open in its full generality, but for magnetic systems on orientable surfaces is a consequence of the solution to the conjecture for every closed 3-manifolds by Taubes [**Tau07**] (see also [**Hut10**] for an expository account). Such proof uses Embedded Contact Homology (another kind of algebraic invariant) to count periodic orbits. Recently, Cristofaro-Gardiner and Hutchings have refined Taubes' approach and raised the lower bound on the number of periodic orbits in dimension 3 to two [**CGH12**]. The case of irrational ellipsoids in $\mathbb{C}^2$ shows that their estimate is sharp, at least for lens spaces [**HT09**].

When Σ bounds a compact region $W^\Sigma \subset W$, we can also compare the orientation induced by the ambient manifold and the one induced by the contact form. We say that the hypersurface is of *positive* contact type if these two orientations agree and it is of *negative* contact type otherwise. The importance of such distinction relies on the fact that in the positive case, and under some additional assumption on the Chern class of ω, we can define the Symplectic cohomology SH^* of (W^Σ, ω). Its defining complex is generated by the cohomology of W^Σ and the periodic orbits on Σ. Since the cohomology of the interior is known, if we can compute SH^*, we gain information about the periodic orbits on the boundary. As is typical in Floer theory, such computation is divided into two steps:

(1) finding explicitly SH^* in simple model cases;
(2) proving that SH^* is invariant under a particular class of deformations, that bring the case of interest to one of the models.

We prove such invariance in Chapter 3 in a setting that will be useful for the applications to magnetic flows on surfaces. This result has been inspired to us by reading [**Rit10**], where a similar invariance is proven in the setting of ALE spaces (see [**Rit10**, Theorem 33 and Lemma 50]. We warmly thank Alexander Ritter for several useful discussions on this topic. Let us now give the precise statement.

THEOREM A. *Let W be an open manifold and let ω_s be a family of symplectic forms on W, with $s \in [0,1]$. Suppose $W_s \subset W$ is a family of zero-codimensional embedded compact submanifold in W, which are all diffeomorphic to a model W'. Let $(W_s, \omega_s|_{W_s}, j_s)$ be a convex deformation. Fix a free homotopy class of loops ν in $W' \simeq W_s$ such that the contact forms α_0 and α_1 are both ν-non-degenerate. If $c_1(\omega_s)$ is ν-atoroidal for every $s \in [0,1]$ and $s \mapsto \omega_s$ is projectively constant on ν-tori, then*

$$SH^*_\nu(W_0, \omega_0, j_0) \simeq SH^*_\nu(W_1, \omega_1, j_1).$$

Before commenting on this result, we clarify the terminology used. We refer to Chapter 3 for a more thorough discussion. The map j_s denotes a collar of the boundary. The normal vector field associated to j_s induces the contact form α_s on ∂W_s. By *convex* deformation, we mean a deformation for which the boundary stays of positive contact type. A contact form is ν-*non-degenerate*, if all its periodic orbits in the class ν are transversally non-degenerate. A two-form ρ on W' is said to be ν-*atoroidal* if the cohomology class of its transgression $[\tau(\rho)] \in H^1(\mathscr{L}_\nu W')$ is zero. A family of two-forms $\{\omega'_s\}$ on W' is said to be *projectively constant on ν-tori*, if the class $[\tau(\omega'_s)]$ is independent of s up to a positive factor.

Symplectic Cohomology for manifolds with boundary of positive contact type was introduced by Viterbo in [**Vit99**]. In [**Vit99**, Theorem 1.7] (see also [**Oan04**, Theorem 2.2]) he proves the invariance of SH^* for contractible loops when all the ω_s are 0-atoroidal (in other words, they are aspherical). However, we do not understand his proof. In particular, when he deals with the "Generalized maximum principle" (Lemma 1.8), the coordinate z is treated in the differentiation as if it did not depend explicitly on the variable s in contrast to what happens in the general case.

In [**BF11**], Bae and Frauenfelder prove a similar result for the Symplectic Co-homology of contractible loops on a closed manifolds and for Rabinowitz Floer Ho-mology of twisted cotangent bundles. In their setting all the ω_s are aspherical and the primitives of ω_s to the universal cover of W grow at most linearly at infinity. It would interesting to find out whether the deformation invariance that we get in Theorem A and the one Bae and Frauenfelder get in [**BF11**] fit into the long exact sequence between Symplectic Homology, Symplectic Cohomology and Rabinowitz Floer Homology [**CFO10**].

As far as the hypotheses on the symplectic forms are concerned we observe two things. First, the fact that $c_1(\omega_s)$ is ν-atoroidal has two consequences: Symplectic Cohomology is well-defined since $c_1(\omega_s)$ is also aspherical and, hence, no holomor-phic sphere can bubble off [**HS95**]; Symplectic Cohomology is $\mathbb{Z}$-graded. Second, the fact that ω_s is projectively constant on ν-tori implies that the local system of coefficients associated to $\tau(\omega_s)$, which appears as a weight in the definition of the Floer differential, is independent of s up to isomorphism.

We construct the isomorphism mentioned in Theorem A in two steps. First, in virtue of Gray's Theorem we find an auxiliary family of symplectic manifolds (W'_s, ω'_s) such that (W'_s, ω'_s) is isomorphic to (W_s, ω_s) and the support of $\omega'_s - \omega'_0$ is disjoint from the boundary. Then, we define a class of admissible paths of pairs $\{(H_s, J_s)\}$ such that H_s is uniformly small and J_s is uniformly bounded on the support of $\omega'_s - \omega'_0$. In this way the 1-periodic orbits of X_{H_s} does not depend on s. The moduli spaces of Floer cylinders corresponding to $\{(H_s, J_s)\}$ satisfy uniform bounds on the energy, since by a Palais-Smale Lemma, the time they spend on the support of $\omega'_s - \omega'_0$ is uniformly bounded. This implies that the moduli spaces are compact and, therefore, we can use them to define continuation homomorphisms between the chain complexes $SC^*_\nu(W_s, \omega'_s, j'_s, H_s, J_s)$. Such homomorphisms will be weighted using the local system of coefficients associated to $\tau(\omega_s)$, so that they will commute with the Floer differentials, yielding maps in cohomology. Two homotopies of homotopies arguments show that these maps are isomorphisms and that they commute with the direct limit.

In Chapter 4 we apply the general theory of contact hypersurfaces to magnetic systems (M, g, σ), in order to prove the existence of periodic orbits. We denote by $\mathrm{Con}^+(g, \sigma)$, respectively $\mathrm{Con}^-(g, \sigma)$, the set of all m such that Σ_m is of positive, respectively negative, contact type.

Historically, the first examples that have been studied are exact systems, since these can be equivalently described using tools from Lagrangian mechanics. In this case we can define $m_0(g, \sigma)$ and $m(g, \sigma)$ the *Mañé critical values* of the abelian and universal cover, respectively (after the reparametrisation $m \mapsto \frac{m^2}{2}$). As far as the contact property is concerned, it is known that

- for $m > m_0(g, \sigma)$, $m \in \mathrm{Con}^+(g, \sigma)$ and, up to time reparametrisation, the dynamics is given by the geodesic flow of a Finsler metric [**CIPP98**];

- for $m \leq m_0(g, \sigma)$, Σ_m is not of restricted contact type, namely the contact form cannot be extended to the interior, since there exists $M_m \subset M$ a compact manifold with non-empty boundary such that ∂M_m is the union of supports of periodic orbits whose total action is negative [**Taĭ91, CMP04**]. Each of these orbits is a *waist*: namely a local minimiser for the action functional. If $M \neq \mathbb{T}^2$, this implies that $\mathrm{Con}^+(g, \sigma) = (m_0(g, \sigma), +\infty)$, while, if $M = \mathbb{T}^2$, there are examples such that $m_0(g, \sigma) \in \mathrm{Con}^+(g, \sigma)$ [**CMP04**].

As far as the periodic orbits are concerned, it is known that

- for $m > m(g, \sigma)$ there exists at least a periodic orbit in every non-trivial free homotopy class;
- for almost every $m \leq m(g, \sigma)$ there is a contractible periodic orbit [**Con06**] and infinitely many periodic orbits in the same homotopy class of a waist [**AMP13, AMMP14**]. Each of these orbits is a mountain pass, namely is obtained by a minimax argument on a 1-dimensional family of loops.

We give a survey of the results about the contact property in Section 4.1. The only original element we add is an explicit construction of the contact structures at $m_0(g, \sigma)$ for the two-torus. Indeed, the argument in the original paper [**CMP04**] was not constructive. As in the supercritical case, the Symplectic Cohomology of the filling is isomorphic to the singular homology of the loop space. In particular, $SH_0^* \neq 0$. For restricted contact type hypersurfaces, Ritter proved in [**Rit13**, Theorem 13.3] that the non-vanishing of Symplectic Cohomology implies the non-vanishing of Rabinowitz Floer Homology (see [**CFO10**] for the relation between these two homology theories). By Theorem 1.2 in [**CF09**], the non-vanishing of RFH implies the non-displaceability of the boundary. Hence, if Ritter's theorem could be generalised to the filling of arbitrary hypersurfaces of contact type, we would have proven that the critical energy level is non-displaceable (see [**CFP10**] for a discussion of displaceability of hypersurfaces in twisted tangent bundles).

We proceed to the non-exact case in Section 4.2. The first condition we need is for the 2-form ω_σ to be exact on Σ_m. This happens if and only if $M \neq \mathbb{T}^2$. Then, we look for a contact form within the class of primitives defined in (4.8). In Corollary 4.15 and Corollary 4.16, we find that $m \in \mathrm{Con}^+(g, \sigma)$ for m large enough and we compute the Symplectic Cohomology of the filling in terms of the Symplectic Cohomology of the model cases.

PROPOSITION B. *Let (M, g, σ) be a magnetic system on a surface different from the two-torus. If m belongs to the unbounded component of $\mathrm{Con}^+(g, \sigma)$, then*

$$SH^* \left(\left\{ E \leq \frac{m^2}{2} \right\}, \omega_\sigma \right) = \begin{cases} H_{-*}(\mathscr{L}M, \mathbb{Z}) & \text{if } M \text{ has positive genus,} \\ 0 & \text{if } M = S^2. \end{cases}$$

If the magnetic form is symplectic we also show in Section 4.2.2 and Section 4.2.3 that low energy levels are

- of positive contact type, if $M = S^2$;

• of negative contact type, if M is a surface of higher genus.

In the first case, we compute SH and we again find that it vanishes. In the second case, we cannot define SH of the filling, since the Liouville vector field points inwards at the boundary. However, it would be interesting to find a compact symplectic manifold (W', ω') with boundary of positive contact type that can be glued to Σ_m in such a way that $(\{E \leq \frac{m^2}{2}\}, \omega_\sigma) \sqcup_{\Sigma_m} (W', \omega')$ is a closed symplectic manifold.

Finally, in Section 4.4, we use the computation of Symplectic Cohomology to reprove some known lower bounds on the number of periodic orbits in the exact and non-exact case. We collect them in the following proposition.

PROPOSITION C. *Let (M, g, σ) be a magnetic system and suppose that m belongs either to the unbounded component of $\mathrm{Con}^+(g, \sigma)$, or, if $M = S^2$ and σ is symplectic, to the component of $\mathrm{Con}^+(g, \sigma)$ containing 0. Then,*

- *if $M = S^2$, there exists a periodic orbit on Σ_m. If all the iterates of this periodic orbit are non-degenerate, then there exists another geometrically distinct periodic orbit;*
- *if $M = \mathbb{T}^2$, there exists a periodic orbit on Σ_m in every non-trivial free homotopy class. If such orbit is non-degenerate, then there exists another geometrically distinct periodic orbit in the same class;*
- *if M is a surface of higher genus, there exists one periodic orbits on Σ_m in every non-trivial free homotopy class.*

1.3. The contact property on surfaces of revolution

In Chapter 5 we look at the contact property on surfaces of revolution, in order to test the general results contained in Chapter 4 in a concrete case. We construct the surface $S_\gamma^2 \subset \mathbb{R}^3$ by rotating a profile curve $(\gamma, \delta) \subset \mathbb{R}^2$ parametrised by arc-length. We study the magnetic system $(S_\gamma^2, g_\gamma, \mu_\gamma)$ where g_γ is the restriction of the Euclidean metric on $\mathbb{R}^3$ to S_γ^2 and μ_γ is the area form associated to this metric. Up to a homothety, we also assume that the area of the surface is 4π. From the previous discussion, we know that there exists two positive values $m_{-,\gamma} < m_{+,\gamma}$ such that $[0, m_{-,\gamma}) \cup (m_{+,\gamma}, +\infty) \subset \mathrm{Con}^+(g_\gamma, \mu_\gamma) =: \mathrm{Con}_\gamma$. These can be taken to be the two roots of the quadratic equation $m^2 - m_\gamma m + 1 = 0$ (where $m_\gamma \geq 0$ is defined below) if such roots are real. In this case $m_\gamma \geq 2$ and the length of the gap $m_{+, gamma} - m_{-,\gamma}$ is $\sqrt{m_\gamma^2 - 4}$ and, hence, it increases with m_γ. If the roots are not real, or, in other words, $m_\gamma < 2$, then we simply have $\mathrm{Con}_\gamma = [0, +\infty)$. The number m_γ depends on the Riemannian geometry of S_γ^2. It is defined by

$$m_\gamma := \inf_{\beta \in \mathcal{P}^{(1-K)\mu_\gamma}} \|\beta\|,$$

where K is the Gaussian curvature and $\mathcal{P}^{(1-K)\mu_\gamma}$ is the set of primitives of $(1-K)\mu_\gamma$. We give an explicit formula for m_γ in terms of the function γ in Proposition 5.5. It can be used to get the following estimate on the contact property.

Proposition D. *If S_γ^2 is symmetric with respect to the equator and the curvature increases from the poles to the equator, then $m_\gamma \leq 1$. Therefore, $\mathrm{Con}_\gamma = [0, +\infty)$.*

On the other hand, for every $C > 0$, there exists a convex S_γ^2 such that $m_\gamma > C$.

The second part of the proposition above relies on the fact that m_γ can be arbitrarily big provided the curvature is sufficiently concentrated around at least one of the poles. Thus, for these surfaces, the contact property can be proved only for a small set of parameters.

Hence, we are led to ask, in general, how good is the set $[0, m_{-,\gamma}) \cup (m_{+,\gamma}, +\infty)$ in approximating Con_γ, the actual set of energies where the contact property holds. For this purpose we employ McDuff's criterion [**McD87**], which says that Σ_m is of contact type provided all the invariant measures supported on this hypersurface have positive action. Finding the actions of an invariant measure is usually a difficult task. However, for surfaces of revolution there are always some latitudes that are the supports of periodic orbits. We compute the action of such latitudes in Proposition 5.10. If the magnetic curvature $K_m : \Sigma_m \to \mathbb{R}$, defined as $K_m := m^2 K + 1$ is positive, we only have two periodic orbits which are latitudes (see Proposition 5.9). By Proposition 5.10, their action is positive. Therefore, they do not represent an obstruction to the contact property. In addition, under the same curvature assumption, we have a simple description of the dynamics of the system after reduction by the rotational symmetry. In particular, this allows us to devise a numerical strategy to compute the action of all the ergodic invariant measures as we explain in Section 5.4. The data we have collected suggest that all such actions are positive, hinting, therefore, at the following conjecture.

Conjecture E. *Let (S^2, g, σ) be a symplectic magnetic system and suppose that for some $m > 0$, the magnetic curvature is positive. Then, Σ_m is of contact type.*

The numerical computations, and possibly an affirmative answer to the conjecture, would then indicate that the system $(S_\gamma^2, g_\gamma, \mu_\gamma)$ associated to a *convex* surface would be of contact type at every energy level. This shows that the inclusion $[0, m_{-,\gamma}) \cup (m_{+,\gamma}, +\infty) \subset \mathrm{Con}_\gamma$ will be strict, in general. Establishing the conjecture would also yield another proof of Corollary 1.3 in [**Sch12a**] about the existence of two closed orbits on every energy level, when $K \geq 0$ and $f > 0$.

To complete the picture, we see in Proposition 5.11 that positive magnetic curvature is not necessary for having the contact property. Moreover, using again Proposition 5.10, we give the first known example of an energy level on a non-exact magnetic system on S^2 which is not of contact type (Proposition 5.12). This shows that the inclusion $\mathrm{Con}_\gamma \subset [0, +\infty)$ can be strict, as well.

1.4. Low energy levels of contact type on the two-sphere

In the last chapter of the thesis we focus on low energy levels of general symplectic magnetic systems on S^2. We can write $\sigma = f\mu$, where μ is the Riemannian area and $f : S^2 \to \mathbb{R}$ is a function. Without loss of generality we assume that $\int_{S^2} \sigma = 4\pi$

and that f is positive. By the previous discussion, we know that these levels are of contact type. Hence, there exists a family of primitives $m \mapsto \lambda_m$ of $\omega_\sigma|_{\Sigma_m}$ made of contact forms. In Lemma 7.4 we see that, using Gray Stability Theorem, one can find diffeomorphisms $F_m : \Sigma_1 \to \Sigma_m$ such that

$$\check{\lambda}_m := F_m^* \lambda_m = \frac{\check{\lambda}_0}{\rho_m} + o(m^2), \qquad \text{where} \ \ \rho_m := 1 - \frac{m^2}{2f}$$

The form $\check{\lambda}_0$ is an S^1-connection on $SS^2 := \Sigma_1$ with curvature σ. In Lemma 7.14 we show that the Reeb vector field of $\check{\lambda}_0/\rho_m$ can be written as the composition of the σ-Hamiltonian flow with Hamiltonian ρ_m on the base S^2 and a rotation in the fibres with angular speed ρ_m. This allows us to expand Ginzburg's action function $S_m : SS^2 \to \mathbb{R}$ for the form $\check{\lambda}_m$, defined in Section 7.2, in the parameter m around zero. This function was introduced for the first time in [**Gin87**] using local Poincaré sections. Its critical points are those periodic orbits of the Reeb flow of $\check{\lambda}_m$ which are close to a curve that winds once around a fibre of $SS^2 \to S^2$.

PROPOSITION F. *The following expansion holds*

$$S_m(x, v) = 2\pi + \frac{\pi}{f(x)} m^2 + o(m^2). \tag{1.1}$$

As a consequence, if $x \in S^2$ is a non-degenerate critical point of f, then there exists a family of loops $m \mapsto \gamma_m$, such that γ_0 winds uniformly once around $S_x S^2$ in the positive sense and the support of $F_m(\gamma_m)$ is a periodic orbit on Σ_m.

The existence of periodic orbits close to non-degenerate critical points is stated without proof in [**Gin96**].

Now we move to analyse the Conley-Zehnder indices of the Reeb flow of λ_m and present some dynamical corollaries of this analysis. As $M = S^2$, Σ_m is diffeomorphic to the lens space $L(2,1) \simeq \mathbb{RP}^3$. On lens spaces we can identify the distinguished class of *dynamically convex* contact forms. We say that a contact form is dynamically convex if the Conley-Zehnder index μ_{CZ} of every contractible periodic orbit of the Reeb flow is at least 3. Such forms were introduced by Hofer, Wysocki and Zehnder [**HWZ98**] on S^3 as a contact-invariant generalisation of convex hypersurfaces in $\mathbb{C}^2$. If $\tau \in \Omega^1(L(p,q))$ is a dynamically convex contact form, we have the following two implications on the dynamics of the associated Reeb flow R^τ.

(i) There exists a global Poincaré section of disc-type for R^τ, under the condition that, when $p > 1$ (namely $L(p,q) \neq S^3$), all periodic orbits are non-degenerate.

(ii) When $p > 1$ and all periodic orbits are non-degenerate, there exists an elliptic periodic orbit for R^τ.

Point (i) was proven in [**HWZ98**] for $p = 1$. The proof for $p > 1$ is contained in [**HLS13**]. Probably the non-degeneracy assumption can be removed by running the same lengthy approximation argument contained in [**HWZ98**]. Point (ii) was proven for convex hypersurfaces in $\mathbb{C}^2$ symmetric with respect to the origin in [**DDE95**]. A proof of the general case stated above was recently announced in [**AM**].

Dynamical convexity was proven in the context of the *standard* tangent bundle by Harris and G. Paternain [**HP08**]. They showed that, if (S^2, g) is a Riemannian two-sphere with $1/4$-pinched curvature, the geodesic flow is dynamically convex on every energy level. In this thesis we prove dynamical convexity for *twisted* tangent bundles (see also the forthcoming paper [**Ben14**]).

THEOREM G. *Let (S^2, g, σ) be a symplectic magnetic system. If m is low enough, there exists a primitive λ_m of ω_σ on Σ_m which is a dynamically convex contact form.*

We give two different arguments to show this result.

- In Section 7.3, we construct a hypersurface $\hat{\Sigma}_m \subset \mathbb{C}^2$ and a double cover $p_m : \hat{\Sigma}_m \to \Sigma_m$ such that $p_m^* \lambda_m = -\lambda_{\mathrm{st}}|_{\hat{\Sigma}_m}$. The hypersurface $\hat{\Sigma}_m$ is convex for small m, since it tends to the sphere of radius 2 when m goes to zero.
- In Section 7.4, we give a direct estimate of the Conley-Zehnder index of contractible periodic orbits of R^{λ_m}.

The latter argument follows closely the strategy of proof of [**HP08**]: the fact that the magnetic form is symplectic and the energy is low, plays the same role as the pinching condition on the curvature. Moreover, this second proof can be adapted to surfaces of higher genus to show that, if γ is a periodic solution of R^{λ_m} which is free homotopic to $|e_M|$ times a vertical fibre, then $\mu_{\mathrm{CZ}}(\gamma) \leq 2e_M + 1$ (here e_M denotes the Euler characteristic). This inequality fits into a notion of *generalised* dynamical convexity which is currently being developed by Abreu and Macarini [**AM**]. They claim that they can prove the existence of an elliptic periodic orbit in this wider setting (see Point (ii)). However, it is not known so far, if it is possible to extend results on the existence of global Poincaré sections to this case (see Point (i)).

Theorem G can be used to obtain the following information on the dynamics.

COROLLARY H. *Let (S^2, g, σ) be a symplectic magnetic system and let m be low enough. On Σ_m there are either two periodic orbits homotopic to a vertical fibre or infinitely many periodic orbits. If f has three distinct critical points $x_{\min}$, x_{Max} and x_{nondeg} such that $x_{\min}$ is an absolute minimiser, x_{Max} is an absolute maximiser and x_{nondeg} is non-degenerate, then the second alternative holds. Moreover, if Σ_m is non-degenerate*

- *there exists a Poincaré section of disc-type for the magnetic flow on Σ_m;*
- *there exists an elliptic periodic orbit γ on Σ_m and, therefore, generically, there exists a flow-invariant fundamental system of neighbourhoods for γ. Hence, the dynamical system is not ergodic with respect to the Liouville measure on Σ_m.*

We do not have any example where there are exactly two non-contractible periodic orbits on low energy levels. However, by [**Gin87**, Assertion 3] we know that these two orbits are *short*, namely they are close to curves that wind around a vertical fibre once. In the next theorem, we show that we can make rigorous a dichotomy between short and long orbits.

THEOREM I. *Let $(S^2, g, \sigma = f\mu)$ be a symplectic magnetic system. Given $\varepsilon > 0$ and a positive integer n, there exists $m_{\varepsilon,n} > 0$ such that for every $m < m_{\varepsilon,n}$ the projection $\pi(\gamma)$ of a periodic prime solution γ on Σ_m either is a simple curve on S^2 with length in $\left(\frac{2\pi-\varepsilon}{\max f} m, \frac{2\pi+\varepsilon}{\min f} m\right)$ or has at least n self-intersections and length larger than $\frac{m}{\varepsilon}$.*

This result is an adaptation to the magnetic settings of [**HS12**, Theorem 1.6] for Reeb flows of convex hypersurfaces close to $S^3 \subset \mathbb{C}^2$. That paper brings in the contact category classical results for pinched Riemannian metrics on S^2 [**Bal83**] and on spheres of any dimension [**Ban86**]. Theorem 1.6 in [**HS12**] also contain a lower bound on the linking number between short and long orbits. This statement is substituted in Theorem I by a lower bound on the number of self-intersections for long orbits. It is likely that the estimates on the linking number obtained by Hryniewicz and Salomão can be used as a black box to get our estimates for the self-intersections, but we did not pursue this strategy explicitly. Instead, our proof is based on an application of the Gauss-Bonnet formula for surfaces.

Even if we do not know if it is possible to have an energy level with exactly two periodic orbits, it has been proven by Schneider in [**Sch11**, Theorem 1.3] that there are examples with exactly two *short* orbits. In Section 7.6, we have a closer look at this problem for surfaces of revolution $(S^2_\gamma, g_\gamma, f\mu_\gamma)$, where f is a rotationally invariant function. We show the existence of a smooth family of embedded tori $C_m : [-\ell, +\ell]/_\sim \times \mathbb{T}_{2\pi} \hookrightarrow \Sigma_m$, where ℓ is the length of γ and $\sim$ is the equivalence relation which identifies $-\ell$ and $+\ell$. They are defined by the formula

$$C_m(u, \psi) := \begin{cases} (-u, -\pi/2, \psi) & \text{if } u < 0, \\ (u, \pi/2, \psi + \pi) & \text{if } u > 0, \end{cases}$$

where we have put on Σ_m the triple of coordinates (t, φ, θ). The first and third one are the latitude and longitude, respectively. The second one is the angle in the vertical fibre counted starting from the longitudinal direction. Each C_m is obtained by gluing two global Poincaré sections of cylinder-type, C_m^- and C_m^+, along their common boundary. This boundary is made by the two unique periodic orbits that project to latitudes on S^2_γ. The two *smooth* return maps defined on the interior of C_m^- and C_m^+ extend to a global *continuous* map $F_m^2 : C_m \to C_m$ (Proposition 7.35). Denote by $\Omega_f : S^2_\gamma \to \mathbb{R}$ the extension of the function $-\frac{\dot{f}}{\gamma f^3}$ to the poles and set $\Omega_f^- := \inf |\Omega_f|$. We have the following result for the return map.

PROPOSITION J. *The family of maps $F_m^2 : [-\ell, +\ell]/_\sim \times \mathbb{T}_{2\pi} \to [-\ell, +\ell]/_\sim \times \mathbb{T}_{2\pi}$ admits the expansion*

$$F_m^2(u, \psi) = \left(u, \psi + \pi \Omega_f m^2 + o(m^2)\right).$$

Hence, if Ω_f is not constant, there are infinitely many periodic orbits on every low energy level. Such condition is satisfied if, for example, $\frac{\ddot{f}}{f^3}(0) \neq -\frac{\ddot{f}}{f^3}(\ell)$. If $\Omega_f^- > 0$ (namely $\dot{f} = 0$ only at the poles and $\ddot{f} \neq 0$ there), the period T of a Reeb orbit of

R^{λ_m}, *different from a latitude, satisfies*

$$T \geq \frac{2}{\Omega_f^- m^2} + O\left(\frac{1}{m^3}\right).$$

In particular, there are only two short orbits and their projection to S_γ^2 is supported on two latitudes.

Finally, we observe that this proposition also gives potential candidates for magnetic systems with only two closed orbits. Namely, those for which Ω_f is constant.

Preliminaries

In this chapter we set the notation and recall the prerequisites needed in the subsequent discussion. The first section describes the conventions and symbols used in the paper. The second section is devoted to the basic properties of the tangent bundle of an oriented Riemannian surface. The third section introduces magnetic fields. The fourth section deals with Hamiltonian structures on three-manifolds. Magnetic flows on a positive energy level are particular instances of this general class of dynamical systems.

2.1. General notation

All objects are supposed to be smooth unless otherwise specified.

If $T > 0$ is a real number, we set $\mathbb{T}_T := \mathbb{R}/T\mathbb{Z}$.

If p, q are coprime positive integers, we write $L(p, q)$ for the associated lens space. It is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$-action given by

$$[k] \cdot (z_1, z_2) = (e^{2\pi i \frac{k}{p}} z_1, e^{2\pi i \frac{kq}{p}} z_2).$$

If γ and γ' are two knots in S^3 we denote by $\mathrm{lk}(\gamma, \gamma')$ their linking number.

If M is a manifold we write $\Omega^k(M)$ for the space of k-differential forms on M and $\Gamma(M)$ for the space of vector fields on M. The interior product between $Z \in \Gamma(M)$ and $\omega \in \Omega^k(M)$ will be written as $\imath_Z \omega$. If $\omega \in \Omega^k(M)$, we denote by $\mathcal{P}^\omega$ the set of its primitives. Namely, $\mathcal{P}^\omega = \{\tau \in \Omega^{k-1}(M) \mid \omega = d\tau\}$.

If $Z \in \Gamma(M)$, we denote by $\mathcal{L}_Z$ the associated Lie derivative and by Φ^Z the flow of Z defined on some subset of $\mathbb{R} \times M$. We write its time t flow map as Φ^Z_t. A periodic orbit for Z is a loop $\gamma : \mathbb{T}_T \to M$, such that $\dot\gamma = Z_\gamma$. When we want to make the period of the orbit explicit we use the notation (γ, T). We call $\Omega^k_Z(M)$ the space of Z-invariant k-forms: τ belongs to $\Omega^k_Z(M)$ if and only if $\mathcal{L}_Z \tau = 0$.

If ν is a free homotopy class of loops in M, we denote by $\mathscr{L}_\nu M$ the space of loops in ν.

Let $\pi : E \to M$ be an S^1-bundle over a surface M with orientation $\mathfrak{o}_M$. We denote by V the generator of the S^1-action and we endow E with the orientation $\mathfrak{O}_E := \mathfrak{o}_M \oplus -V$. An S^1-connection form on E is a $\tau \in \Omega^1(E)$ such that $\tau(V) = 1$ and $d\tau = -\pi^*\sigma$ for some $\sigma \in \Omega^2(M)$ called the curvature form.

If (M, ω) is a symplectic manifold and $H : M \to \mathbb{R}$ is a real function, the Hamiltonian vector field X_H is defined by

$$\imath_{X_H} \omega = -dH. \tag{2.1}$$

We define now some objects on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Denote by J_{st} the *standard complex structure* and by g_{st} the *Euclidean inner product*. Define the *standard Liouville form* $\lambda_{\mathrm{st}} \in \Omega^1(\mathbb{C}^n)$ as $(\lambda_{\mathrm{st}})_z(W) := \frac{1}{2}g_{\mathrm{st}}(J_{\mathrm{st}}(z), W)$, for $z \in \mathbb{C}^n$ and $W \in T_z\mathbb{C}^n \simeq \mathbb{C}^n$. Finally, the *standard symplectic form* is defined as $\omega_{\mathrm{st}} := d\lambda_{\mathrm{st}}$, or in standard real coordinates $\omega_{\mathrm{st}} = \sum_i dx^i \wedge dy^i$.

2.2. The geometry of an oriented Riemannian surface

Let M be a closed orientable surface and let $e_M \in \mathbb{Z}$ be its Euler characteristic. Let $\pi : TM \to M$ be the tangent bundle of M, $L^{\mathcal{V}}_{(x,v)} : T_xM \to T_{(x,v)}TM$ the associated vertical lift and write TM_0 for the complement of the zero section in TM. Suppose that we also have fixed an orientation $\mathfrak{o}$ on M. If $\sigma \in \Omega^2(M)$ we use the shorthand $[\sigma] := \int_M \sigma$ (where the integral is with respect to $\mathfrak{o}$).

Let g be a Riemannian metric on M. It yields an isomorphism $\flat : TM \to T^*M$ which we use to push forward the metric for tangent vectors to a dual metric g for 1-forms. Call $\sharp : T^*M \to TM$ the inverse isomorphism. We write $|\cdot|$ for the induced norms on each T_xM and T_x^*M. From the duality construction we have the identity $|l| = \sup_{|v|=1}|l(v)|$, $\forall l \in T_x^*M$. We collect this family of norms together to get a supremum norm $\|\cdot\|$ for sections:

$$\forall Z \in \Gamma(M), \quad \|Z\| := \sup_{x \in M}|Z_x|; \qquad \forall \beta \in \Omega^1(M), \quad \|\beta\| := \sup_{x \in M}|\beta_x|. \tag{2.2}$$

The Riemannian metric induces a *kinetic energy function* $E : TM \to \mathbb{R}$ defined by $E((x,v)) := \frac{1}{2}g_x(v,v)$. The level sets $\Sigma_m := \{E = \frac{1}{2}m^2\} \subset TM$ are such that

- the zero level Σ_0 is the zero section $\{(x,0)\,|\,x \in M\}$;
- for $m > 0$, $\pi_| : \Sigma_m \to M$ is an S^1-bundle.

Let ∇ be the Levi Civita connection of g and $\frac{\nabla^\gamma}{dt}$ be the associated covariant derivative along a curve γ on M. For every $(x,v) \in TM$, ∇ gives rise to a horizontal lift $L^{\mathcal{H}}_{(x,v)} : T_xM \to T_{(x,v)}TM$. It has the property that $d_{(x,v)}\pi \circ L^{\mathcal{H}}_{(x,v)} = \mathrm{Id}_{T_xM}$. Finally, denote by $\mu \in \Omega^2(M)$ the positive Riemannian area form and by K the Gaussian curvature. We combine them to obtain the curvature form $\sigma_g \in \Omega^2(M)$ associated to g. It is defined by $\sigma_g := K\mu$.

We introduce a frame of $T(TM_0)$ and a coframe of $T^*(TM_0)$ depending on $(g, \mathfrak{o})$. The first element of the frame is Y, which is defined as $Y_{(x,v)} := L^{\mathcal{V}}_{(x,v)}(v)$.

The geodesic equation

$$\frac{\nabla^\gamma}{dt}\dot{\gamma} = 0 \tag{2.3}$$

for curves γ in M gives rise to a flow on TM. The second element of the frame is the generator X of such flow. It is called the *geodesic vector field* and can be equivalently defined as $X_{(x,v)} = L^{\mathcal{H}}_{(x,v)}(v)$.

Consider the 2π-periodic flow $\Phi^V_\varphi : TM \to TM$, which rotates every fibre of π by an angle φ. We choose as the third element of our frame, V the generator of this flow. If we denote by $\jmath : TM \to TM$ the rotation of $\pi/2$ in every fibre, then $V_{(x,v)} = L^{\mathcal{V}}_{(x,v)}(\jmath_x v)$ and $V|_{\Sigma_m} \in \Gamma(\Sigma_m)$ is the generator of the S^1-action on Σ_m.

Finally, the last element of the frame is H, defined by $H_{(x,v)} := L^{\mathcal{H}}_{(x,v)}(\jmath_x v)$.

The first element of the coframe is dE, the differential of the kinetic energy.

The second element is λ, the pull-back under $\flat$ of the standard Liouville form on T^*M. It acts by $\lambda_{(x,v)}(\xi) := g_x(v, d_{(x,v)}\pi\,\xi)$ on $\xi \in T_{(x,v)}TM$. The exterior differential $\omega := d\lambda$ yields a symplectic form on TM. We denote by $\mathfrak{O}$ the orientation associated to $\omega \wedge \omega$. Every Σ_m inherits an orientation $\mathfrak{O}_{\Sigma_m}$ which is obtained from $\mathfrak{O}$ following the convention of putting the outward normal to Σ_m (namely Y) first. Observe that such orientation matches with the orientation of an S^1-bundle over an oriented surface as defined in Section 2.1.

The third element is τ, the angular component of the Levi-Civita connection. On $\xi \in T_{(x,v)}TM$, it acts by $\tau_{(x,v)}(\xi) = g_x(\jmath_x v, \frac{\hat{x}}{dt}\hat{v})$, where $(\hat{x}, \hat{v})$ is any path passing through (x,v) with tangent vector ξ.

The fourth element is given by $\eta := \jmath^*\lambda$.

We now recall the main properties of the frame and the coframe, which are a particular case of the results contained in [**GK02b**].

PROPOSITION 2.1. *The frame (X, Y, H, V) is $\mathfrak{O}$-positive with dual coframe*

$$\left(\frac{\lambda}{2E}, \frac{dE}{2E}, \frac{\eta}{2E}, \frac{\tau}{2E}\right).$$

We have the following bracket relations for the frame

$$\begin{cases} [Y, X] = X & [Y, H] = H & [Y, V] = 0 \\ [V, X] = H & [V, H] = -X & [X, H] = 2EKV. \end{cases} \tag{2.4}$$

Accordingly, we get the following structural equations for the coframe

$$\begin{cases} d\left(\dfrac{\lambda}{2E}\right) &=& \dfrac{\lambda}{2E} \wedge \dfrac{dE}{2E} - \dfrac{\eta}{2E} \wedge \dfrac{\tau}{2E}, \\[2mm] d\left(\dfrac{dE}{2E}\right) &=& 0, \\[2mm] d\left(\dfrac{\eta}{2E}\right) &=& \dfrac{\lambda}{2E} \wedge \dfrac{\tau}{2E} - \dfrac{dE}{2E} \wedge \dfrac{\tau}{2E}, \\[2mm] d\left(\dfrac{\tau}{2E}\right) &=& -2EK\dfrac{\lambda}{2E} \wedge \dfrac{\eta}{2E}. \end{cases} \tag{2.5}$$

The last equation in (2.5) can be rewritten as

$$d\left(\frac{\tau}{2E}\right) = -\pi^*\big(K\mu\big). \tag{2.6}$$

It implies that $\frac{\tau}{2E}$ is an S^1-connection form on every Σ_m with curvature σ_g.

For the following discussion it will be convenient to have also a corresponding statement for the restriction to $SM := \Sigma_1$ of the frame and coframe defined above.

COROLLARY 2.2. *The triple (X, V, H) is an $\mathfrak{O}_{SM}$-positive frame of $T(SM)$ with dual coframe (λ, τ, η). We have the following relations*

$$\begin{cases} [V, X] = H & [H, V] = X & [X, H] = KV; \\ d\lambda = \tau \wedge \eta & d\tau = K\eta \wedge \lambda = -\pi^*\sigma_g & d\eta = \lambda \wedge \tau. \end{cases} \tag{2.7}$$

DEFINITION 2.3. The volume form $\chi \in \Omega^3(SM)$ defined by

$$\chi := \lambda \wedge \tau \wedge \eta = -\tau \wedge \pi^*\mu, \tag{2.8}$$

associated to the dual coframe, is called the *Liouville volume form*.

We end up the general discussion about the geometry of M by proving some properties related to local sections of SM.

Consider an open set $U \subset M$ and denote by SU the unit sphere bundle over U. Let $Z : U \to SU$ be a section and associate to Z an angular function $\varphi_Z : SU \to \mathbb{T}_{2\pi}$. The value of φ_Z at the point (x, v) is the angle between v and Z_x. In the next lemma we compute the differential of the angular function. First, observe that, if $v \in T_xU$, we have $g_x\big((\nabla_v Z)_x, Z_x\big) = 0$. Thus, there exists $\kappa_x^Z \in T_x^*M$ such that

$$(\nabla_v Z)_x = \kappa_x^Z(v)\jmath_x Z_x, \tag{2.9}$$

LEMMA 2.4. *If Z is a section of SU, we have*

$$\begin{cases} d\varphi_Z(X)(x, v) = -\kappa_x^Z(v), \\ d\varphi_Z(V)(x, v) = 1, \\ d\varphi_Z(H)(x, v) = -\kappa_x^Z(\jmath_x v). \end{cases} \tag{2.10}$$

PROOF. We can write the angle φ_Z as an element of $S^1 \subset \mathbb{C}$:

$$(\cos\varphi_Z(x, v), \sin\varphi_Z(x, v)) = \big(g_x(v, Z_x), g_x(v, \jmath_x Z_x)\big). \tag{2.11}$$

Differentiating along X this map, we get

$$\big(d_{(x,v)}(\cos\varphi_Z)(X), d_{(x,v)}(\sin\varphi_Z)(X)\big) = \big(g_x((\nabla_v Z)_x, v), -g_x((\nabla_v Z)_x, \jmath_x v)\big).$$

On the other hand, we also have

$$\big(d_{(x,v)}(\cos\varphi_Z)(X), d_{(x,v)}(\sin\varphi_Z)(X)\big) = d\varphi_Z(X)(x, v)\cdot\big(-\sin\varphi_Z(x, v), \cos\varphi_Z(x, v)\big).$$

Thus, we can express $d\varphi_Z(X)(x, v)$ as the standard inner product in $\mathbb{R}^2$ between $\big(g_x((\nabla_v Z)_x, v), -g_x((\nabla_v Z)_x, \jmath v)\big)$ and $\big(-\sin\varphi_Z, \cos\varphi_Z\big)$.

$$\begin{aligned} d\varphi_Z(X)(x, v) &= -\sin\varphi_Z(x, v)g_x\big((\nabla_v Z)_x, v\big) + \cos\varphi_Z(x, v)g_x\big((\nabla_v Z)_x, \jmath_x v\big) \\ &= -\sin\varphi_Z(x, v)\kappa_x^Z(v)g_x(\jmath_x Z, v) - \cos\varphi_Z(x, v)\kappa_x^Z(v)g_x\big(\jmath_x Z, \jmath_x v\big) \\ &= -\kappa_x^Z(v). \end{aligned}$$

For the second statement we differentiate in t the identity $\varphi_Z \circ \Phi_t^V = \varphi_Z + t$.

The third statement follows by using the identity $H = d_\jmath \jmath^{-1}(X_\jmath)$:

$$\begin{aligned} d_{(x,v)}\varphi_Z(H_{(x,v)}) &= d_{(x,v)}\varphi_Z\Big(d_{(x,\jmath_x v)}\jmath^{-1}\big(X_{(x,\jmath_x v)}\big)\Big) \\ &= d_{(x,\jmath_x v)}(\varphi_Z \circ \jmath^{-1})\big(X_{(x,\jmath_x v)}\big) \\ &= d_{(x,\jmath_x v)}\varphi_Z\big(X_{(x,\jmath_x v)}\big) \\ &= -\kappa_x^Z(\jmath_x v) \qquad\qquad\qquad\qquad\qquad\qquad \square \end{aligned}$$

We give an explicit formula for κ_x^Z in the next lemma. First, define the *geodesic curvature* of Z as the function

$$k_Z : U \longrightarrow \mathbb{R}$$
$$x \longmapsto g_x\big((\nabla_Z Z)_x, \jmath_x Z_x\big). \tag{2.12}$$

LEMMA 2.5. *At every $x \in M$, we have $\sharp \kappa_x^Z = k_Z(x) Z_x + k_{\jmath Z}(x) \jmath_x Z_x$.*

PROOF. Let us evaluate κ_x^Z on the basis $(Z_x, \jmath_x Z_x)$:

$$
\begin{aligned}
\kappa_x^Z(Z_x) &= g_x\big((\nabla_Z Z)_x, \jmath_x Z_x\big) \\
&= k_Z(x);
\end{aligned}
\quad \bigg|\quad
\begin{aligned}
\kappa_x^Z(\jmath_x Z_x) &= g_x\big((\nabla_{\jmath Z} Z)_x, \jmath_x Z_x\big) \\
&= g_x\big(\jmath_x(\nabla_{\jmath Z} Z)_x, \jmath_x(\jmath_x Z_x)\big) \\
&= g_x\big((\nabla_{\jmath Z}\jmath Z)_x, \jmath_x(\jmath_x Z_x)\big) \\
&= k_{\jmath Z}(x).
\end{aligned}
$$

$\square$

2.3. Magnetic fields

In the previous section we introduced various geometrical objects associated to (M, g). Historically, the geodesics equation (2.3) has been used as the main tool to understand the geometry of Riemannian manifolds [**Kli95, Pat99a**]. It is a classical result that the geodesic flow can be studied in the framework of symplectic geometry, since X is the $d\lambda$-Hamiltonian vector field associated to the kinetic energy function (see [**Kli95**, Chapter 3]). In other words,

$$\imath_X d\lambda = -dE. \tag{2.13}$$

From this equation we see that Σ_0 is the set of rest points for X and for every positive m, Σ_m is invariant under the geodesic flow. Hence, we can study $X\big|_{\Sigma_m}$ for a fixed value of $m > 0$. By the homogeneity of (2.3) (which is translated in the bracket relation $[Y, X] = X$), $X\big|_{SM}$ and $X\big|_{\Sigma_m}$ have the same dynamics up to reparametrisation by a constant factor. The map conjugating the two flows is given by the flow of Y:

$$d\Phi^Y_{-\log m} \cdot X\big|_{\Sigma_m} = m X\big|_{\Sigma_{SM}} \tag{2.14}$$

One of the guiding principles of this thesis is to see how the introduction of a magnetic perturbation in the geodesic equation breaks the homogeneity and gives rise to a family of systems whose dynamics changes with m.

For any $\sigma \in \Omega^2(M)$ consider the symplectic form $\omega_\sigma := d\lambda - \pi^*\sigma \in \Omega^2(TM)$. When σ is not zero, we refer to the symplectic manifold (TM, ω_σ) as a *twisted* tangent bundle to distinguish it from the *standard* tangent bundle $(TM, \omega_0 = d\lambda)$.

We will see later some remarkable differences (in terms of Symplectic Cohomology and displaceability, for example) between the geometry of the standard and twisted tangent bundle. Such phenomena show, in this setting, the contrast between symplectic and volume-preserving geometry. Indeed, we claim that

$$\omega_\sigma \wedge \omega_\sigma = \omega_0 \wedge \omega_0. \tag{2.15}$$

Hence, the twisted and the standard tangent bundle have the same volume form, although they are different in the symplectic category. To prove the claim notice that $\pi^*\sigma \wedge \pi^*\sigma = 0$ and that $d\lambda \wedge \pi^*\sigma = d(\lambda \wedge \pi^*\sigma) = 0$, since $\lambda \wedge \pi^*\sigma = 0$ by Proposition 2.1 above.

In the next proposition we prove that also the first Chern class of the underlying class of compatible almost complex structures is unaffected by the twist.

PROPOSITION 2.6. *If σ is a 2-form on M, then $c_1(TM, \omega_\sigma) = 0$.*

PROOF. Consider the vertical distribution $\mathcal{V} \to TM$. It is a Lagrangian subbundle of the symplectic bundle $(T(TM), \omega_\sigma) \to TM$. If J_σ is an almost complex structure compatible with ω_σ, then $\mathcal{V}$ is totally real with respect to J_σ. As a consequence, we have the isomorphism of complex bundles

$$(\mathcal{V}_\mathbb{C}, i) \longrightarrow (T(TM), J_\sigma)$$
$$(u + iv) \longmapsto u + J_\sigma v,$$

where $(\mathcal{V}_\mathbb{C}, i)$ is the complexification of $\mathcal{V}$. Therefore, $c_1(TM, J_\sigma) = c_1(\mathcal{V}_\mathbb{C}, i)$. On the other hand, $c_1(\mathcal{V}_\mathbb{C}, i) = 0$ since $(\mathcal{V}_\mathbb{C}, i) \simeq (\mathcal{V}_\mathbb{C}, -i)$, $c_1(\mathcal{V}_\mathbb{C}, -i) = -c_1(\mathcal{V}_\mathbb{C}, i)$ and $H^2(TM, \mathbb{Z})$ is torsion-free (see [**MS74**, Lemma 14.9]. Alternatively, notice that $\mathcal{V} \to TM$ is orientable and therefore $\det(\mathcal{V}) \to TM$ is trivial. Since taking the determinant bundle and taking the complexification do commute, we get

$$c_1(\mathcal{V}_\mathbb{C}, i) = c_1(\det(\mathcal{V}_\mathbb{C}), \det(i)) = c_1((\det\mathcal{V})_\mathbb{C}, i) = 0. \qquad \square$$

DEFINITION 2.7. We call σ a *magnetic form* and the triple (M, g, σ) a *magnetic system*. If we fix M, we write $\mathrm{Mag}(M)$ for the space of all pairs (g, σ) such that (M, g, σ) is a magnetic system. For every $(g, \sigma) \in \mathrm{Mag}(M)$, there exists a unique function $f : M \to \mathbb{R}$ called the *magnetic strength* such that $\sigma = f\mu$. We say that the magnetic system is *exact*, *non-exact*, or *symplectic*, if such is the magnetic form σ in $\Omega^2(M)$. We denote by $\mathrm{Mag}_e(M) \subset \mathrm{Mag}(M)$ the subset of exact magnetic systems.

Finally, we define the *magnetic vector field* $X_E^\sigma \in \Gamma(TM)$ as the ω_σ-Hamiltonian vector field associated to E and we refer to $\Phi^{X_E^\sigma}$ as the *magnetic flow*.

As the geodesic flow, X_E^σ comes from a second order ODE for curves in M:

$$\frac{\nabla^\gamma}{dt}\dot{\gamma} = F_\gamma(\dot{\gamma}). \tag{2.16}$$

Here $F : TM \to TM$ is the *Lorentz force*. It is a bundle map given by

$$g(F_x(v), w) := \sigma_x(v, w) \tag{2.17}$$

and can be expressed using the magnetic strength as $F_x(v) = f(x)\jmath_x v$. Using the relation between the Levi Civita connection and the horizontal lifts, one finds that

$$X_E^\sigma = X + fV. \tag{2.18}$$

The kinetic energy E is still an integral of motion for the magnetic flow: Σ_0 is the set of rest points for X_E^σ and, for $m > 0$, X_E^σ restricts to a nowhere vanishing vector field on Σ_m.

However, the bracket relations now imply

$$d\Phi^Y_{-\log m} \cdot X^\sigma_E\big|_{\Sigma_m} = (mX + fV)\big|_{SM} = mX^{\frac{\sigma}{m}}_E\big|_{SM}. \tag{2.19}$$

From this equality we infer two things. First, that, unlike the geodesic flow, the magnetic flow is not homogeneous. Second, that studying the dynamics of the magnetic system (M, g, σ) on Σ_m is the same as studying the rescaled magnetic system $(M, g, \frac{\sigma}{m})$ on SM. The advantage of the latter formulation is that it describes the dynamics of the magnetic flow by using a 1-parameter family of flows on a *fixed* closed three-manifold. We also introduce the real parameter s, which is related to m by the equation $sm = 1$. In the following discussion, we will use both parameters, choosing every time the one that most simplify the notation. As a heuristic rule, the parameter m will be more convenient for low values of E (m close to 0), while s will be more convenient for high values of E (s close to 0). When there is no risk of confusion, we adopt the shorthand $X^s := X^{s\sigma}_E\big|_{SM}$ and $\omega_s := \omega_{s\sigma}$, when we use s, and $X^m := mX^{\frac{\sigma}{m}}_E\big|_{SM} = mX + fV$ and $\omega_m := m\omega_{\frac{\sigma}{m}} = md\lambda - \pi^*\sigma$, when we use m. Notice that we have the limits

$$\lim_{s \to 0} X^s = X, \qquad \lim_{m \to 0} X^m = fV,$$
$$\lim_{s \to 0} \omega_s = d\lambda; \qquad \lim_{m \to 0} \omega_m = -\pi^*\sigma.$$

In the standard tangent bundle, the properties of the linearisation of the geodesic flow on SM are influenced by the curvature K (see also the bracket relations above). Analogously, when there is a magnetic term, for every $m > 0$, we can define the *magnetic curvature* function $K_m : SM \to \mathbb{R}$ as $K_m := m^2 K - m(df \circ \jmath) + f$, which is related to the linearisation of X^m [**Ada97**] (see also Chapter 5).

REMARK 2.8. The duality isomorphism $\flat : TM \to T^*M$ yields a symplectomorphism between the twisted tangent bundle (TM, ω_σ) and $(T^*M, \omega^*_\sigma := d\lambda^* - \pi^*\sigma)$, where λ^* is the standard Liouville form on the cotangent bundle and we denote with π also the footpoint projection $T^*M \to M$. The ω_σ-Hamiltonian system X^σ_E is sent through $\flat$ to the ω^*_σ-Hamiltonian system associated to $E^* := E \circ \flat^{-1}$. Observe that E^* is the kinetic energy function on T^*M associated to the dual metric, namely $E^*(x, l) = \frac{1}{2}g_x(l, l)$. In particular $\Sigma^*_m := \flat(\Sigma_m) = \{E^* = \frac{1}{2}m^2\}$. Unlike ω_σ, ω^*_σ depends only on σ and not on g, which enters in the picture only through the Hamiltonian E^*.

The tangent bundle and the cotangent bundle formulation are equivalent but have different advantages and disadvantages as far as the exposition of the material is concerned. In the tangent bundle we have an intuitive understanding of the flow given by the second order ODE. On the other hand, in the cotangent bundle we have seen that g and σ contributes separately to determine the magnetic flow: the former through the Hamiltonian and the latter through the symplectic form.

We have seen that looking at different energy levels of a fixed magnetic system (M, g, σ) corresponds to a suitable rescaling of the magnetic field: $X^\sigma_E\big|_{\Sigma_m} \sim X^{\frac{\sigma}{m}}_E\big|_{\Sigma_1}$.

What happens when we rescale the metric $g \mapsto m^2 g$? In the cotangent bundle formulation, we have that $E^*_{m^2 g} = \frac{E^*_g}{m^2}$. Thus, $X^\sigma_{E^*_{m^2 g}}\big|_{\Sigma^*_1} \sim X^\sigma_{E^*_g}\big|_{\Sigma^*_m}$ and, consequently,

$$X^\sigma_{E_{m^2 g}}\big|_{\Sigma_1} \sim X^\sigma_{E_g}\big|_{\Sigma_m}. \tag{2.20}$$

2.3.1. The Maupertuis' principle. In this subsection we discuss how a magnetic system (M, g, σ) is affected by the introduction of a potential $U : M \to \mathbb{R}$. In this case we speak of a *mechanical system* (M, g, σ, U). The Hamiltonian has an additional term $E(x, v) = \frac{1}{2} g_x(v, v) + U(x)$ while the symplectic form remains the same $\omega_\sigma = d\lambda - \pi^* \sigma$. In this case the topology of the energy levels depends on U. Consider the map $\pi| : \Sigma_m \to M$, where $\Sigma_m := \{E = m^2/2\}$. Its image is the sublevel $M_m := \{U \leq m^2/2\}$ and the preimage of a point $x \in M_m$ are the vectors in $T_x M$ with norm $\sqrt{m^2 - 2U(x)}$. This is either a single point, namely the zero vector of $T_x M$, if $U(x) = m^2/2$, or a circle, if $U(x) < m^2/2$. In particular, $\pi| : \Sigma_m \to M$ is an S^1-bundle if and only if $m^2/2 > \max U$. We claim that in this case the dynamics on Σ_m is equivalent, up to reparametrisation, to the dynamics of the magnetic flow associated to $(M, g_{m,U} := (m^2 - 2U)g, \; sigma)$ on SM (where the sphere bundle is taken with respect to the metric $g_{m,U}$). To prove the claim, we work in the cotangent bundle formulation. We start by observing that $\{E^* = m^2/2\} = \{\frac{1}{2}\frac{g_x(l,l)}{m^2 - 2U(x)} = \frac{1}{2}\}$. Therefore, the ω^*_σ-Hamiltonian flow of E^* and the one of $\widetilde{E}^*(x, l) := \frac{1}{2}\frac{g_x(l,l)}{m^2 - 2U(x)}$ are the same up to reparametrisation on this common energy level. Finally, we observe that $\widetilde{E}$ is the kinetic energy function for the metric $\frac{g}{m^2 - 2U}$ which is the dual of $g_{m,U}$.

From the discussion above, we see that the results contained in this thesis can also be applied to magnetic systems with potential, provided we know U and m well enough to understand the properties of $(M, g_{m,U}, \sigma)$. For example, fix $m > 0$ and take a potential U whose C^0-norm is small compared to m^2. We claim that the mechanical system (M, g, σ, U) and the magnetic system (M, g, σ) will be close to each other on Σ_m. By Equation (2.19) and (2.20), we know that the magnetic system associated to $(M, g_{m,U}, \sigma)$ is equivalent, up to reparametrisation, to the magnetic system associated to $(M, \frac{g_{m,U}}{m^2}, \frac{\sigma}{m})$. Moreover,

$$\left\| \frac{g_{m,U}}{m^2} - g \right\| \leq \frac{2\|U\|}{m^2} \|g\|. \tag{2.21}$$

Thus, if $\|U\|$ is small compared to m^2, the magnetic flow of $(M, g_{m,U}, \sigma)$ on the unit tangent bundle of $g_{m,U}$ is close, up to reparametrisation, to the magnetic flow of $(M, g, \frac{\sigma}{m})$ on the unit tangent bundle of g.

2.3.2. Physical interpretation: constrained particles and rigid bodies with symmetry. The adjective 'magnetic' for the systems we have introduced in the present section is due to the fact that they describe the following simple phenomenon in the theory of classical electromagnetism. Consider a particle q of unit mass constrained to move on a frictionless surface M in the Euclidean space $(\mathbb{R}^3, g_{\text{st}})$. The metric induces by pull-back a metric g on M and a corresponding Levi-Civita connection ∇. With this assumption the motion of q will obey to the geodesic equation (2.3). Suppose now that q has unit charge and that M is immersed

in a stationary magnetic field $\overrightarrow{B}$. In this case the particle is subject to a Lorentz force $\overrightarrow{F} = \overrightarrow{v} \times \overrightarrow{B}$. The force $\overrightarrow{F}$ is obtained from the 2-form $\sigma := \imath_{\overrightarrow{B}} \mathrm{vol}_{g_{\mathrm{st}}}$ as prescribed by Equation (2.17). Here $\mathrm{vol}_{g_{\mathrm{st}}}$ denotes the Euclidean volume form in $\mathbb{R}^3$. The motion of q will obey to the Equation (2.16).

Making the natural assumption that $\overrightarrow{B}$ is defined on the whole $\mathbb{R}^3$, we see that σ is the pull-back of a closed 2-form on $\mathbb{R}^3$ and, hence, it is *exact*. With this physical interpretation, non-exact magnetic systems on M correspond to fields $\overrightarrow{B}$ generated by a Dirac monopole located in a region outside the surface.

Even if a magnetic monopole has not been observed so far, non-exact magnetic flows on S^2 have, nonetheless, a concrete importance since they are symplectic reductions of certain physical systems with phase space $TSO(3)$, possessing an S^1-symmetry. The study of the periodic orbits of these systems was initiated in the early Eighties by Novikov in a series of papers [**NS81, Nov81a, Nov81b, Nov82**]. His interest was motivated by the fact that in these cases the Lagrangian action functional is multivalued and, hence, one needs a generalisation of the standard Morse theory (nowadays known as Novikov theory) for proving the existence of critical points.

Among the many examples considered by Novikov, here we only look at the motion of a rigid body with rotational symmetry and a fixed point. We refer the reader to [**Kha79**] for a careful explanation of the setting and for the proofs of the statement we make.

A rigid body with a fixed point is described by a positive orthonormal basis $\mathbf{e} = (e_1, e_2, e_3)$ in $\mathbb{R}^3$. Given a fixed positive orthonormal basis $\mathbf{n} = (n_1, n_2, n_3)$, to every $\mathbf{e}$ we can associate a unique element in $SO(3)$ which is the isometry of $\mathbb{R}^3$ sending $\mathbf{n}$ to $\mathbf{e}$. Hence, the configuration space of a rigid body with a fixed point can be identified with the group $SO(3)$.

The kinetic energy of the body is obtained from a left-invariant Riemannian metric g on $SO(3)$, which, in the standard basis of $\mathfrak{so}(3)$, is represented by the matrix

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{2.22}$$

for some positive numbers I_1, I_2 and I_3. Let us consider the $\mathbb{T}_{2\pi}$-action on the configuration space that rotates the rigid body around the fixed axis n_3. In $SO(3)$ this corresponds to left multiplication by the subgroup G of rotations with axis n_3. Call $Z \in \Gamma(SO(3))$ the infinitesimal generator of G.

The quotient map for this action is the S^1-bundle map

$$p : SO(3) \longrightarrow S^2 \subset \mathbb{R}^3$$

$$(e_1, e_2, e_3) \longmapsto \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} := \begin{pmatrix} g_{\mathrm{st}}(e_1, n_3) \\ g_{\mathrm{st}}(e_2, n_3) \\ g_{\mathrm{st}}(e_3, n_3) \end{pmatrix},$$

which yields the coordinates of the vector n_3 in the frame $\mathbf{n}$. Denote by $\hat{g}$ the metric induced on S^2 by p. It is obtained pushing forward the restriction of g to the orthogonal of $\ker dp$. Notice that the function $\mathbf{e} \mapsto \sqrt{g_{\mathbf{e}}(Z, Z)}$ is invariant under G, so that it descends to a function $\nu : S^2 \to (0, +\infty)$ such that $\sqrt{g_{\mathbf{e}}(Z, Z)} = \nu(p(\mathbf{e}))$.

Finally, suppose that the rigid body is immersed in a conservative force field, which is invariant under rotations around n_3. Thus, the force is described by a potential $U : SO(3) \to \mathbb{R}$ which is invariant under the $\mathbb{T}_{2\pi}$-action. Namely, there exists $\hat{U} : S^2 \to \mathbb{R}$ such that $U = \hat{U} \circ p$.

The dynamics of the body is given by the Hamiltonian vector field on $TSO(3)$ associated to the mechanical system $(SO(3), g, 0, U)$. The $\mathbb{T}_{2\pi}$-symmetry arises from a momentum map $J : TSO(3) \to \mathfrak{g} \simeq \mathbb{R}$, which is an integral of motion. For every $k \in \mathbb{R}$, we look at the dynamics on the invariant set $\{J = k\}$ projected to TS^2 via the quotient map p. It is the dynamics of the Hamiltonian vector field associated to $(S^2, \hat{g}, k\sigma_{\hat{g}}, \hat{U}_k)$, where $\hat{U}_k = \hat{U} + \frac{k^2}{2\nu^2}$. A computation shows that the Gaussian curvature of $\hat{g}$ is *always positive* and, hence, the magnetic form is *symplectic*.

Applying Maupertuis' principle, we know that, on Σ_m such that $m^2/2 > \max \hat{U}_k$, the reduced system above is equivalent to the magnetic system $(S^2, \hat{g}_{m,\hat{U}_k}, k\sigma_{\hat{g}})$. Moreover, if $\|\hat{U}_k\|$ is small compared to m^2, up to reparametrisation, the dynamics is close to that of $(S^2, \hat{g}, \frac{k}{m}\sigma_{\hat{g}})$ on the unit tangent bundle of $\hat{g}$.

2.3.3. From dynamics to geometry. We come back to the general discussion on magnetic flows associated to (M, g, σ). We saw that the dynamics of X_E^σ on TM can be studied by looking at the dynamics of $X^s := X_E^{s\sigma}\big|_{SM}$, for positive values of the parameter s. We are going to look at this new problem from a geometric point of view, by investigating what geometric structure the symplectic manifold TM induces on the three-manifold SM.

Consider the restriction $\omega_\sigma' := \omega_\sigma\big|_{SM}$. It is a closed, nowhere vanishing 2-form on the unit tangent bundle. Hence, it has an orientable 1-dimensional kernel $\ker \omega_\sigma'$ at every point. We fix for it an orientation $\mathfrak{O}_{\ker \omega_\sigma'}$ satisfying the relation

$$\mathfrak{O}_{SM} = \mathfrak{O}_{\ker \omega_\sigma'} \oplus \mathfrak{O}^{\omega_\sigma'}, \tag{2.23}$$

where $\mathfrak{O}^{\omega_\sigma'}$ is the orientation on $\frac{T(SM)}{\ker \omega_\sigma'}$ induced by ω_σ'. Since $\chi = \lambda \wedge d\lambda = \lambda \wedge \omega_\sigma'$, we readily see that

$$\imath_{X_E^\sigma\big|_{SM}} \chi = \lambda(X_E^\sigma\big|_{SM})\omega_\sigma' - \lambda \wedge \imath_{X_E^\sigma\big|_{SM}} \omega_\sigma' = \omega_\sigma'. \tag{2.24}$$

Therefore, Equation (2.23) implies that the magnetic vector field $X_E^\sigma\big|_{SM}$ is a *positive* nowhere vanishing section of $\ker \omega_\sigma'$.

From the previous discussion, we argue that the real object of interest for understanding the geometric properties of the orbits, and not their actual parametrisation, is $\ker \omega_\sigma'$ rather than $X_E^\sigma\big|_{SM}$. The 2-form ω_σ' is a particular instance of what is called a *Hamiltonian Structure* (or HS for brevity). We introduce such objects in an abstract setting in the next section.

2.4. Hamiltonian structures

We develop below the notion of HS and we identify the special subclass of HS of contact type. Proving that a HS is of contact type has striking consequences for the dynamics, since in this case we can describe the dynamical system associated to the HS by the means of a Reeb flow. This yields the following two advantages.

a) We can count periodic orbits of Reeb flows using algebraic invariants, such as Contact Homology [**Bou09**] and Embedded Contact Homology [**Tau07**], since closed orbits are generator for the corresponding chain complexes. When the contact manifold is the *positive* boundary of a symplectic manifold, like in the case of SM, one can define a further invariant called Symplectic Homology, which takes into account the interplay between the contact structure on the boundary and the symplectic structure of the filling. In the next chapter we are going to define Symplectic Homology [**Vit99, Sei08**] (or, more precisely, its cohomological version) in detail and prove an abstract invariance result which we will use in Chapter 4 to compute it in the case of magnetic systems. This will yield a lower bound on the number of periodic orbits for such systems.

b) The theory of pseudo-holomorphic foliations developed by Hofer, Wysocki and Zehnder [**HWZ98**] allows to find Poincaré sections for *dynamically convex* Reeb flows. This allows for a description of the flow in terms of an area-preserving discrete dynamical system on the two-dimensional disc. Information about periodic points of such systems are then obtained applying results by Brouwer [**Bro12**] and Franks [**Fra92**]. The abstract setting will be presented in Chapter 6 and applied to magnetic systems in Chapter 7

In the rest of this section let $(N, \mathfrak{O})$ be a three-manifold N endowed with an orientation $\mathfrak{O}$.

DEFINITION 2.9. A closed and nowhere vanishing 2-form ω on N is called a *Hamiltonian Structure*. We say that the Hamiltonian Structure is *exact*, if ω is such. The 1-dimensional distribution $\ker \omega$ is called the *characteristic distribution* associated to ω. It has an orientation $\mathfrak{O}_{\ker \omega}$ satisfying the relation $\mathfrak{O} = \mathfrak{O}_{\ker \omega} \oplus \mathfrak{O}^\omega$, where $\mathfrak{O}^\omega$ is the orientation on $\frac{TN}{\ker \omega}$ induced by ω. Alternatively, if χ is any positive volume form, $Z^\chi \in \Gamma(N)$ defined by $\imath_{Z^\chi} \chi = \omega$ is a positive section of $\ker \omega$.

DEFINITION 2.10. Let ω be a HS and let $Z \in \Gamma(N)$ be a positive section of the characteristic line bundle associated with ω. We say that a periodic orbit (γ, T) for Φ^Z is *non-degenerate* if the multiplicity of 1 in the spectrum of $d\Phi_T^Z$ is exactly one. Observe that this property does not depend on the choice of Z. If (γ, T) is transversally non-degenerate, call *transverse spectrum* the set made of the two eigenvalues of $d\Phi_T^Z$ different from 1. We say that γ is

(1) *elliptic*, if the transverse spectrum lies on the unit circle in $\mathbb{C}$;
(2) *hyperbolic*, if the transverse spectrum lies on the real line.

Finally, we say that ω is *ν-non-degenerate* if all the periodic orbits in the class ν are non-degenerate.

REMARK 2.11. If (W, ω) is a symplectic four-manifold and $N \hookrightarrow W$ is an orientable embedded hypersurface, then $\omega|_N$ is a Hamiltonian Structure on W. If we take N to be the regular level of a Hamiltonian function $H : W \to \mathbb{R}$, we can endow N with the orientation induced from $\omega \wedge \omega$ by putting the gradient of H first. Using the Hamilton equation (2.1), we see that $X_H|_N$ is a positive section of $\ker \omega|_N$. Moreover, $\omega|_N$ is ν-non-degenerate if and only if all the periodic orbits of H on N with free homotopy class ν are *transversally non-degenerate* (see next chapter).

This observation shows that the dynamics up to reparametrisation of an autonomous Hamiltonian system on a four-dimensional symplectic manifold (W, ω) at a regular level N can be read off the geometry of the oriented one-dimensional distribution $\ker \omega|_N$ associated to the Hamiltonian structure $\omega|_N$.

2.4.1. Hamiltonian structures of contact type. We now introduce Hamiltonian structures of contact type for which the characteristic distribution has a section with special properties.

DEFINITION 2.12. We say that ω is *of contact type* if it is exact and there exists a contact form $\tau \in \mathcal{P}^\omega$. Denote by R^τ the *Reeb vector field* of τ, defined by $\imath_{R^\tau} d\tau = 0$ and $\tau(R^\tau) = 1$. One among R^τ and $-R^\tau$ is a positive section of $\ker \omega$. We say that ω is of *positive* or *negative* contact type accordingly.

REMARK 2.13. If we fix a positive section Z of $\ker \omega$, being of positive (respectively negative) contact type is equivalent to finding $\tau \in \mathcal{P}^\omega$ such that $\tau(Z) : N \to \mathbb{R}$ is a positive (respectively negative) function. In this case $R^\tau = \frac{Z}{\tau(Z)}$.

EXAMPLE 2.14. If (M, g) is a Riemannian surface as above, then $\omega_0' = d\lambda|_{SM}$ is a HS of positive contact type. Indeed, X is a positive section of $\ker \omega_0'$ and $\lambda_{(x,v)}(X) = g_x(v, d_{(x,v)}\pi X) = 1$. This implies that $\lambda|_{SM}$ is a positive contact form and X is its Reeb vector field.

EXAMPLE 2.15. Suppose that $\pi : E \to M$ is an S^1-bundle over an oriented surface $(M, \mathfrak{o}_M)$. If $\sigma \in \Omega^2(M)$ is a positive symplectic form, then $\pi^*\sigma$ is a Hamiltonian structure. We claim that $\pi^*\sigma$ is exact if and only if E is non-trivial. To prove necessity, we observe that if E is trivial, the integral of $\pi^*\sigma$ over a section of π is non-zero. To prove sufficiency, we define $e \neq 0$ the Euler number of the S^1-bundle. Then, we use the classical result [**Kob56**] saying that a 2-form on M, whose integral is $2\pi e$, is a curvature form for an S^1-connection on E. Thus, it is exact. This implies that there exists $\tau \in \Omega^1(E)$ such that

$$\bullet \quad \tau(V) = 1, \qquad \bullet \quad d\tau = -\frac{2\pi e}{[\sigma]} \pi^*\sigma. \qquad (2.25)$$

In this case $-\frac{[\sigma]}{2\pi e}\tau$ is a primitive of $\pi^*\sigma$ and a contact form with Reeb vector field $-\frac{2\pi e}{[\sigma]}V$. Therefore, $\pi^*\sigma$ is a Hamiltonian structure of contact type. It is positive if and only if e is positive.

EXAMPLE 2.16. For examples of structures of contact type in the context of the circular planar restricted three-body problem, we refer to [**AFvKP12**].

The most direct way to detect the contact property is to use Remark 2.13. However, this method could be difficult to apply, especially if we want to prove that a HS is not of contact type, since we should check that every function $\tau(Z)$ vanishes at some point. This problem is overcome by the following necessary and sufficient criterion contained in McDuff [**McD87**].

PROPOSITION 2.17. *Let ω' be an exact HS and Z be a positive section of $\ker \omega'$. Then, ω' is of positive (respectively negative) contact type if and only if the action of every null-homologous Z-invariant measure is positive (respectively negative).*

We end this subsection by recalling the basic notions about invariant measures needed in the statement of the proposition above.

If Z is a vector field on a closed manifold N, a Z-*invariant measure* ζ is a Borel probability measure on N, such that

$$\forall\, h : N \to \mathbb{R}, \qquad \int_N dh(Z)\zeta = 0. \tag{2.26}$$

We denote the set of Z-invariant measure by $\mathfrak{M}(Z)$. We associate to every $\zeta \in \mathfrak{M}(Z)$ an element $\rho(\zeta)$ in $H^1(N, \mathbb{R})^* = H_1(N, \mathbb{R})$ defined as

$$\forall\, [\beta] \in H^1(N, \mathbb{R}), \qquad < \rho(\zeta), [\beta] > := \int_N \beta(Z)\zeta. \tag{2.27}$$

Suppose Z is a positive section of $\ker \omega$, with ω an exact HS on an oriented three-manifold N, $\zeta \in \mathfrak{M}(Z)$ and $\tau \in \mathcal{P}^{\omega'}$. The *action* of ζ is defined as

$$\mathcal{A}_Z^\tau : \mathfrak{M}(Z) \longrightarrow \mathbb{R}$$

$$\zeta \longmapsto \int_N \tau(Z)\zeta. \tag{2.28}$$

If ζ is *null-homologous* (namely $\rho(\zeta) = 0$), then $\mathcal{A}_Z^\tau(\zeta)$ is independent of τ and, therefore, in this case we write $\mathcal{A}_Z^\omega(\zeta) := \mathcal{A}_Z^\tau(\zeta)$.

2.4.2. The Conley-Zehnder index. We finish this section by associating an integer, called the *Conley-Zehnder index*, to a periodic orbit of a Hamiltonian structure. It encodes information about the linearisation of the system along the orbit and will play a crucial role in defining the notion of dynamical convexity for HS of contact type. We refer to [**HK99**] for proofs and further details.

We start with the definition of the *Maslov index* for a path with values in $\mathrm{Sp}(1)$, the group of 2×2-symplectic matrices. For any $T > 0$, we set

$$\mathrm{Sp}_T(1) := \{\Psi : [0, T] \to \mathrm{Sp}(1) \mid \Psi(0) = \mathrm{Id}\}.$$

We call $\Psi \in \mathrm{Sp}_T(1)$ *non-degenerate* if $\Psi(T)$ does not have 1 as eigenvalue.

Given $\Psi \in \mathrm{Sp}_T(1)$, we associate to every $u \in \mathbb{R}^2 \backslash \{0\}$ a *winding number* $\Delta\theta(\Psi, u)$ as follows. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let

$$\frac{\Psi(t)u}{|\Psi(t)u|} = e^{i\theta_u^\Psi(t)}, \tag{2.29}$$

for some function $\theta_u^\Psi : [0, T] \to \mathbb{R}$. We define $\Delta\theta(\Psi, u) := \theta_u^\Psi(T) - \theta_u^\Psi(0)$. Let

$$I(\Psi) := \left\{ \frac{1}{2\pi} \Delta\theta(\Psi, u) \ \Big| \ u \in \mathbb{R}^2 \setminus \{0\} \right\}. \tag{2.30}$$

The interval $I(\Psi)$ is closed and its length is strictly less than $1/2$. We notice that the set $e^{2\pi i I(\Psi)} \subset S^1$ is completely determined by the endpoint $\Psi(T)$. In particular, we see that Ψ is non-degenerate if and only if $\mathbb{Z} \cap \partial I(\Psi) = \emptyset$. We define the Maslov index for a non-degenerate path as

$$\mu(\Psi) := \begin{cases} 2k, & \text{if } k \in I(\Psi), \text{ for some } k \in \mathbb{Z}; \\ 2k+1, & \text{if } I(\Psi) \subset (k, k+1), \text{ for some } k \in \mathbb{Z}. \end{cases} \tag{2.31}$$

We extend the definition to the degenerate case either by taking the maximal lower semi-continuous extension μ^l or the minimal upper semi-continuous extension μ^u. This amounts to using the same recipe as in the non-degenerate case, but in the definition of μ^l, respectively of μ^u, we shift the interval $I(\Psi)$ to the left, respectively to the right, by an arbitrarily small amount. For any $k \in \mathbb{Z}$, there hold

$$\mu^l(\Psi) \geq 2k+1 \implies I(\Psi) \subset (k, +\infty) \implies \mu^*(\Psi) \geq 2k+1, \tag{2.32}$$

$$\mu^u(\Psi) \leq 2k-1 \implies I(\Psi) \subset (-\infty, k) \implies \mu^*(\Psi) \leq 2k-1, \tag{2.33}$$

where we have used μ^* to denote both μ^l and μ^u.

We move now to describe the Conley-Zehnder index for a Hamiltonian structure ω on $(N, \mathfrak{O})$. Consider the 2-dimensional distribution $\xi_\omega := \frac{TN}{\ker \omega}$ and observe that $(\xi_\omega, \overline{\omega}) \to N$ is a symplectic vector bundle, where $\overline{\omega}$ is the symplectic form on the fibres induced by ω. Let ν be a free-homotopy class of loops in N and choose γ_ν a reference loop in ν together with a symplectic trivialisation Υ_ν of ξ_ω along γ_ν. If ν is the trivial class, just choose a constant loop with the constant trivialisation. Suppose that $c_1(\xi_\omega)$ is ν-atoroidal. This means that the integral of $c_1(\xi_\omega)$ over a cylinder $i : C := [0,1] \times \mathbb{T}_1 \to N$ such that $i(0, \cdot) = \gamma_\nu(\cdot)$ depends only on $i(1, \cdot)$.

Let Z be a positive section of $\ker \omega$ and observe that $d\Phi_t^Z$ acts on $(\xi_\omega, \overline{\omega})$ as a symplectic bundle map. Let (γ, T) be a periodic orbit of Z in the class ν. Choose a cylinder $i : C \to N$ connecting γ_ν with γ and let $\Upsilon : (i^*\xi_\omega, i^*\overline{\omega}) \to (\epsilon_C^2, \omega_{\text{st}})$ be a $\overline{\omega}$-symplectic trivialisation of $i^*\xi_\omega$ on C extending Υ_ν. Here ϵ_C^2 is the trivial rank 2-vector bundle over C. We form the path of symplectic matrices $\Psi_\gamma^{C,\Upsilon} \in \text{Sp}_T(1)$

$$\Psi_\gamma^{C,\Upsilon}(t) := \Upsilon_{\gamma(t)} \circ d_{\gamma(0)}\Phi_t^Z \circ \Upsilon_{\gamma(0)}^{-1} \in \text{Sp}(1). \tag{2.34}$$

DEFINITION 2.18. The *Conley-Zehnder index* of γ is $\mu_{\text{CZ}}^*(\gamma, \Upsilon_\nu) := \mu^*(\Psi_\gamma^{C,\Upsilon})$. This number does not depend on the choice of Z and the hypothesis on the Chern class ensures that it is also independent of the pair (C, Υ). If we choose a different γ_ν with a different trivialisation Υ_ν the index gets shifted by an integer which is the same for every closed orbit γ in ν.

REMARK 2.19. If ω is of contact type, the vector bundle $(\ker \tau, \omega|_{\ker \tau})$ is symplectic and isomorphic to $(\xi_\omega, \overline{\omega})$. Hence, we can use the former bundle for the computation of the index.

Symplectic Cohomology and deformations

The material we present in this chapter build up from the work contained in [**Rit10**]. On the one hand, it grew out of several fruitful discussions with Alexander Ritter and it is likely to appear in a forthcoming paper [**BR**]. On the other hand, the statement and proof of the crucial Proposition 3.35, which enables us to prove the main result of this section (Theorem 3.30), are due exclusively to the author of the present thesis.

The focus is Symplectic Cohomology for general convex manifolds as first developed by Viterbo in [**Vit99**]. In the next chapter, we will apply the abstract results proved here to magnetic systems.

In Section 3.1 we recall the notion of convex symplectic manifold (W, ω, j) in the open and compact case and we give a notion of isomorphism in such categories. In Section 3.2, we introduce the Symplectic Cohomology of such manifolds, a set of algebraic invariants SH_ν^* counting 1-periodic orbits of Hamiltonian flows in the free homotopy class ν. We will work under the hypothesis that $c_1(\omega)$ is ν-atoroidal.

In Section 3.3 we see how to define SH_ν^* perturbing autonomous Hamiltonian functions close to the non-constant periodic orbits.

In Section 3.4, we define SH_ν^* using a subclass of autonomous Hamiltonians, whose non-constant periodic orbits are reparametrised Reeb orbits for the contact form at infinity. For such Hamiltonians we have filtrations on the complex yielding SH_ν^* given by the action (when (W, ω) a Liouville domain) and, in general, by the period of Reeb orbits. These filtrations will be used for the applications in the next chapter.

In Section 3.5 we quote the results contained in [**Rit10**], asserting that SH_ν^* does not depend on the isomorphism class of a convex manifold.

In Section 3.6, we prove the invariance of SH_ν^* for compact manifolds under convex deformations (W_s, ω_s, j_s) which are projectively constant on ν-tori.

3.1. Convex symplectic manifolds

Let us start by treating the non-compact case. First, we define the manifolds we are interested in.

DEFINITION 3.1. Let W be an open manifold (non-compact and without boundary) of dimension $2m$ and let Σ be a closed manifold of dimension $2m - 1$. We say that W is *an open manifold with cylindrical end (modelled on Σ)* if there exists a diffeomorphism $j = (r, p) : U \to (a, +\infty) \times \Sigma$. Here U is an open subset of W, such that $W \setminus j^{-1}((b, +\infty) \times \Sigma)$ is a compact subset of W with boundary $j^{-1}(\{b\} \times \Sigma)$,

for every (or, equivalently, some) $b > a$. We write $i : \Sigma \to W$ for the embedding $i := j^{-1}\big|_{\{b\} \times \Sigma}$.

We call j (or less precisely U) a *cylindrical end of* W and we denote by $\partial_r \in TU$ the vector field generated by the coordinate r.

We now fix some additional notation for open manifolds with cylindrical end.

DEFINITION 3.2. If $f : \Sigma \to (a, +\infty)$, define by $\Gamma_f : \Sigma \to (a, +\infty) \times \Sigma$ the map $\Gamma_f(x) = (f(x), x)$ and by Σ_f the image of Γ_f inside W. Denote by W^f and W_f the compact and non-compact submanifolds of W with boundary Σ_f. If we have two functions $f_0, f_1 : \Sigma \to (a, +\infty)$, with $f_0 < f_1$, let $W_{f_0}^{f_1}$ be the compact submanifold between Σ_{f_0} and Σ_{f_1}.

We proceed to consider the subclass of open manifolds for which the symplectic structure at infinity is compatible with some contact structure on the model Σ.

DEFINITION 3.3. Let (W, ω) be a symplectic manifold having a cylindrical end $j : U \to (a, +\infty) \times \Sigma$. Suppose that (W, ω) is *exact at infinity*, meaning that ω is exact on U. We say that the triple (W, ω, j) is a *convex symplectic manifold* if $\theta := \iota_{\partial_r} \omega \in \Omega^1(U)$ is a primitive for ω on U, or, equivalently, if $(j^{-1})^* \theta = e^r p^* \alpha$ for some (unique) contact form α on Σ. We denote by $R^\alpha \in \Gamma(\Sigma)$ the Reeb vector field of α and we define the auxiliary function $\rho := e^r$. We call ∂_r the *Liouville vector field* and θ the *Liouville form*.

We say that a convex symplectic manifold is a *Liouville domain*, if θ extends to W in such a way that $\omega = d\theta$ on the whole manifold.

REMARK 3.4. Since ω is a symplectic form, an α satisfying the equality above is automatically a contact form on Σ inducing the orientation obtained from $\omega^n\big|_U$ putting the vector ∂_r first. Using the language introduced in the previous chapter we can say that every Σ_f is a hypersurface of positive contact type in (W, ω).

We now define isomorphisms of convex symplectic manifolds.

DEFINITION 3.5. Let (W_i, ω_i, j_i), for $i = 0, 1$, be two convex symplectic manifolds. An *isomorphism* between (W_0, ω_0, j_0) and (W_1, ω_1, j_1) is a diffeomorphism $F : W_0 \to W_1$ with the following properties:

(1) F is a symplectomorphism, i.e. $F^* \omega_1 = \omega_0$,
(2) there exist open cylindrical ends $V_i \subset U_i$ such that $F(V_0) \subset V_1$,
(3) $j_1 \circ F \circ j_0^{-1}\big|_{V_0}(r_0, p_0) = (r_0 - f(p_0), \psi(p_0))$ with $\psi^* \alpha_1 = e^f \alpha_0$.

The third condition is equivalent to $(F\big|_{V_0})^* \theta_1 = \theta_0$. Furthermore, a simple argument shows that $\psi : \Sigma_0 \to \Sigma_1$ is indeed a diffeomorphism.

REMARK 3.6. Let (W, ω) be a symplectic manifold with cylindrical end and suppose we are given two convex ends j_0 and j_1. If $\theta_0 = \theta_1$ (or, equivalently, $\partial_{r_0} = \partial_{r_1}$) on some common cylindrical end, then $\mathrm{Id} : (W, \omega, j_0) \longrightarrow (W, \omega, j_1)$ is an isomorphism. For this reason, we will also use the notation (M, ω, θ) to designate any convex symplectic manifold (M, ω, j) such that $\theta := \iota_{\partial_r} \omega$.

For compact symplectic manifolds we can give an analogous notion of convexity.

DEFINITION 3.7. Let (W, ω) be a compact symplectic manifold with boundary Σ and let $j = (r, p) : U \to (-\varepsilon, 0] \times \Sigma$ be a collar of the boundary. We say that (W, ω, j) is a *(compact) convex symplectic manifold* if $\theta := i_{\partial_r}\omega \in \Omega^1(U)$ is a primitive for ω on U, or equivalently, if $(j^{-1})^*\theta = e^r p^*\alpha$ for some (unique) contact form α on Σ.

A standard construction called *completion* yields an open convex symplectic manifold $(\hat{W}, \hat{\omega}, \hat{j})$ starting from a compact convex symplectic manifold (W, ω, j):

$$(\hat{W}, \hat{\omega}, \hat{j}) := (W, \omega, j) \bigsqcup_j \Big((-\varepsilon, +\infty) \times \Sigma, d(e^r\alpha), \mathrm{Id} \Big), \qquad (3.1)$$

The new object is obtained by gluing along j the original convex symplectic manifold with a positive cylindrical end of the symplectisation of (Σ, α).

DEFINITION 3.8. We say that two convex compact symplectic manifolds are isomorphic if their completions are isomorphic according to Definition 3.5.

REMARK 3.9. Let (W, ω) be a symplectic manifold with convex cylindrical end $j : U \to (a, +\infty) \times \Sigma$. Let $f : \Sigma \to (a, +\infty)$ be any function and consider W^f (see Definition 3.2). Then, $(W^f, \omega|_{W^f}, j|_{W^f})$ is a compact convex manifold. We do not lose any information by passing from W to W^f. Indeed, we observe that we can get back the whole open manifold by taking the completion as explained above. In other words,

$$(\hat{W}^f, \hat{\omega}|_{W^f}, \hat{j}|_{W^f}) \simeq (W, \omega, j). \qquad (3.2)$$

We can rephrase (3.2) by saying that every open convex symplectic manifold (W, ω) is the completion of a compact convex submanifold $(W', \omega|_{W'}) \subset (W, \omega)$. This has a natural generalisation for a family of open convex symplectic manifolds (W, ω_s, j_s). Namely, there exists a corresponding family of zero-codimensional compact submanifolds $W_s \subset W$ such that ∂W_s is j_s-convex and

$$(\hat{W}_s, \hat{\omega}_s|_{W_s}, \hat{j}_s|_{W_s}) \simeq (W, \omega_s, j_s). \qquad (3.3)$$

3.2. Symplectic Cohomology of convex symplectic manifolds

Assume for the rest of this section that (W, ω, j) is an open convex symplectic manifold and ν is a free homotopy class of loops in W.

3.2.1. Preliminary conditions. We now define the Symplectic Cohomology of (W, ω, j) in the class ν under the assumption that

(S1) all the Reeb orbits of the associated contact form α in the classes $i_*^{-1}(\nu)$ are non-degenerate (in this case we say that α is *ν-non-degenerate*);

(S2) $c_1(\omega)$ is *ν-atoroidal*.

Denote by $\mathrm{Spec}(\alpha, \nu)$ the set of periods and by $T(\alpha, \nu) > 0$ the minimal period of a Reeb orbit of α in the class ν. Assumption **(S1)** guarantees that $\mathrm{Spec}(\alpha, \nu)$ is a discrete subset of $[T(\alpha, \nu), +\infty)$.

3.2.2. Admissible Hamiltonians. Symplectic Cohomology counts the number of 1-periodic orbits in the class ν for a particular kind of 1-periodic Hamiltonians. To introduce them, we need first the following two general definitions.

DEFINITION 3.10. Let (W, ω) be a symplectic manifold and consider a function $H : \mathbb{T}_1 \times W \to \mathbb{R}$. We say that a 1-periodic orbit x for X_H is *non-degenerate* if 1 does not belong to the spectrum of $d_{x(0)}\Phi_1^H$. We say that H is ν-non-degenerate, if all the 1-periodic orbits in the class ν are non-degenerate.

DEFINITION 3.11. Let (W, ω, j) be a symplectic manifold with cylindrical end. A function $H : \mathbb{T}_1 \times W \to \mathbb{R}$ is said to have *constant slope at infinity*, if there exist constants $T_H \in (0, +\infty)$ and $a_H \in \mathbb{R}$ such that $H \circ j^{-1} = T_H e^r + a_H$ on some cylindrical end $V \subset U$. The number T_H is called the *slope* of H.

We are now ready to introduce the class of Hamiltonians that we are going to use on (W, ω, j).

DEFINITION 3.12. A function $H : \mathbb{T}_1 \times W \to \mathbb{R}$ is called ν-*admissible* if it satisfies the following two properties:

(H1) it is ν-non-degenerate,

(H2) it has constant slope at infinity and T_H does not belong to $\mathrm{Spec}(\alpha, \nu)$.

We denote the set of all ν-admissible Hamiltonians by $\mathcal{H}_\nu$.

3.2.3. The action 1-form, the grading and the moduli spaces. Consider $H \in \mathcal{H}_\nu$. Its 1-periodic orbits are the zeroes of a closed 1-form $d\mathcal{A}_H^\omega$ on $\mathscr{L}_\nu W$. If $x \in \mathscr{L}_\nu W$ and $\xi \in \Gamma(x^*TW)$ is an element in $T_x(\mathscr{L}_\nu W)$, the 1-form is defined by

$$d_x\mathcal{A}_H^\omega \cdot \xi = \int_{\mathbb{T}_1} \left(\imath_{\dot{x}(t)}\omega_{x(t)} + d_{x(t)}H(t, x(t)) \right) \cdot \xi(t)\, dt. \tag{3.4}$$

The action 1-form yields a function $\mathcal{A}_H^\omega$ on the set of ν-cylinders by integration. If $u : [a, b] \to \mathscr{L}_\nu W$ is a ν-cylinder, we set

$$\mathcal{A}_H^\omega(u) := \int_a^b d_{u(s)}\mathcal{A}_H^\omega \cdot \frac{du}{ds}(s)\, ds$$

$$= -\omega(u) + \int_{\mathbb{T}_1} H(t, u(b, t))\, dt - \int_{\mathbb{T}_1} H(t, u(a, t))\, dt,$$

where $\omega(u)$ is the integral of ω over u. Observe that $\mathcal{A}_H^\omega(u)$ does not change if we homotope the cylinder u keeping the end points fixed.

We can associate a degree $|x|$ to every 1-periodic orbit x in ν as follows. Choose a reference loop x_ν in the class ν and fix a symplectic trivialisation Υ_{x_ν} of $x_\nu^*(TW)$. Take a connecting cylinder C_x from x_ν to x and extend the trivialisation over C_x. This will induce a symplectic trivialisation Υ_x of x^*TW, whose homotopy class depends only on Υ_{x_ν} since $c_1(\omega)$ is ν-atoroidal. Writing the linearisation of the Hamiltonian flow $d\Phi_t^{X_H}$ using Υ_x yields a path of symplectic matrices along x. We call *Conley-Zehnder index* the Maslov index of this path and we denote it by $\mu_{\mathrm{CZ}}(x) \in \mathbb{Z}$. Finally, we set $|x| := \frac{\dim(W)}{2} - \mu_{\mathrm{CZ}}(x)$.

We now define a moduli space of cylinders connecting two 1-periodic orbits using a particular kind of almost complex structures compatible with convexity.

DEFINITION 3.13. Let (W, ω, j) be a convex manifold. An ω-compatible 1-periodic almost complex structure J is *convex* if for big r, J is independent of time and $(d\rho) \circ J = -\theta$.

Take a 1-periodic ω-compatible convex almost complex structure J and consider Floer's equation for cylinders $u : \mathbb{R} \times \mathbb{T}_1 \to W$

$$\partial_s u + J(\partial_t u - X_H) = 0. \tag{3.5}$$

If u is a Floer trajectory we denote its energy by

$$E(u) := \int_{\mathbb{R}} \|\partial_s u\|_J^2 \, ds = \int_{\mathbb{R} \times \mathbb{T}_1} |\partial_s u|_J^2 \, ds \, dt \tag{3.6}$$

Using Floer's equation (3.5), one gets the identity

$$E(u) = -\mathcal{A}_H^\omega(u). \tag{3.7}$$

Call $\mathcal{M}'(H, J, x_-, x_+)$ the space of Floer trajectories that converge uniformly to the 1-periodic orbits $x\pm$ for $s \to \pm\infty$. Suppose that J is *H-regular*, namely that the operator $u \mapsto \partial_s u + J(\partial_t u - X_H)$ on the space of all cylinders is regular at $\mathcal{M}'(H, J, x_-, x_+)$. This implies that all the moduli spaces are smooth manifolds and, if non-empty, $\dim \mathcal{M}'(H, J, x_-, x_+) = \mu_{\mathrm{CZ}}(x_+) - \mu_{\mathrm{CZ}}(x_-) = |x_-| - |x_+|$.

Let $\mathcal{M}(H, J, x_-, x_+)$ be the quotient of $\mathcal{M}'(H, J, x_-, x_+)$ under the $\mathbb{R}$-action obtained by shifting the variable s. If A is a homotopy class of cylinders relative ends, we denote by $\mathcal{M}^A(H, J, x_-, x_+)$ the subset of the moduli space whose elements belong to A.

3.2.4. The cochain complex and the differential. We build a complex $SC_\nu^*(W, \omega, j, H)$ as the free Λ-module generated by the 1-periodic orbits x of H. Here Λ is the Novikov ring defined as

$$\Lambda := \left\{ \sum_{i=0}^{+\infty} n_i t^{a_i} \mid n_i \in \mathbb{Z}, \ a_i \in \mathbb{R}, \ \lim_{i \to +\infty} a_i = +\infty \right\}.$$

The differential $\delta_J : SC_\nu^*(W, \omega, j, H) \to SC_\nu^{*+1}(W, \omega, j, H)$ of such complex is defined on the generators as

$$\delta_J x := \sum_{\substack{u \in \mathcal{M}(H, J, y, x) \\ |y| - |x| = 1}} \epsilon(u) \, t^{-\mathcal{A}_H^\omega(u)} y,$$

where $\epsilon(u)$ is an orientation sign defined in [**Rit13**, Appendix 2]. We extend it on the whole $SC_\nu^*(W, \omega, j, H)$ by Λ-linearity. If x and y are two periodic orbits, we denote by $< \delta_J x, y >$, the component of $\delta_J x$ along the subspace generated by y.

By a standard argument in Floer theory, in order to prove that δ_J is a well-defined map of Λ-modules and that $\delta_J \circ \delta_J = 0$, we need to prove that the elements of $\mathcal{M}^A(H, J, x_-, x_+)$ have:

a) Uniform C^0-bounds; these stem from a maximum principle.

b) Uniform C^1-bounds; these follow from Lemma 3.14 below, where we show that $c_1(\omega)$ is aspherical (see [**HS95**]).

c) Uniform energy bounds; these follow readily from (3.7).

LEMMA 3.14. *If $\sigma \in \Omega^2(W)$ is ν-atoroidal, then σ is aspherical.*

PROOF. We have an action $\sharp : \pi_2(W) \times H_2(W, \mathbb{Z}) \to H_2(W, \mathbb{Z})$ by connected sum. Moreover, if $([S^2], [M]) \in \pi_2(W) \times H_2(W, \mathbb{Z})$, then

$$[\sigma]([S^2 \sharp M]) = [\sigma]([S^2]) + [\sigma]([M]).$$

Therefore, the lemma is proven if we show that the action preserves classes represented by ν-tori. Indeed, in this case the above identity would become

$$0 = [\sigma]([S^2]) + 0.$$

Suppose M is a ν-torus parametrised by $(s, t) \mapsto \Gamma^M(s, t) = \gamma_s^M(t)$, with $\gamma_s \in \nu$ and $(s, t) \in \mathbb{T}_1 \times \mathbb{T}_1$. The connected sum between M and some S^2 can be represented by a map $\Gamma^{S^2 \sharp M} : \mathbb{T}_1 \times \mathbb{T}_1 \to W$, which concides with Γ^M outside a small square $[s_0, s_1] \times [t_0, t_1]$ and inside the square it coincides with the map

$$[s_0, s_1] \times [t_0, t_1] \xrightarrow{\sim} S^2 \setminus D^2 \longrightarrow W,$$

where D^2 is the small disc that we remove from S^2 in order to perform the connected sum. Hence, $S^2 \sharp M$ is still a torus and the curves $t \mapsto \gamma_s^{S^2 \sharp M}(t) := \Gamma^{S^2 \sharp M}(s, t)$ are still in the class ν because $\gamma_s^{S^2 \sharp M} = \gamma_s^M$ if $s \notin [s_0, s_1]$. $\square$

Thanks to the lemma, we have well-defined cohomology groups obtained from δ_J. We denote them by $SH_\nu^*(W, \omega, j, H, J)$. We put a partial order on pairs (H, J), where H and J are as above, by saying that $(H^+, J^+) \preceq (H^-, J^-)$ if and only if $T_{H^+} \leq T_{H^-}$ (we use the reverse notation for the signs, since in the definition of maps on a generator x, we take the moduli space of cylinders arriving at x).

When $(H^+, J^+) \preceq (H^-, J^-)$ we can construct continuation maps

$$\varphi_{H^-, J^-}^{H^+, J^+} : SH_\nu^*(W, \omega, j, H^+, J^+) \longrightarrow SH_\nu^*(W, \omega, j, H^-, J^-)$$

such that,

$$(H^+, J^+) \preceq (H^0, J^0) \preceq (H^-, J^-) \quad \implies \quad \varphi_{H^-, J^-}^{H^0, J^0} \circ \varphi_{H^0, J^0}^{H^+, J^+} = \varphi_{H^-, J^-}^{H^+, J^+}.$$

Such maps are defined as follows. We consider a homotopy (H^s, J^s) with s ranging in $\mathbb{R}$, which is equal to (H^-, J^-), for s very negative, and to (H^+, J^+), for s very positive. Such homotopy yields an s-dependent Floer equation

$$\partial_s u + J^s(\partial_t u - X_{H^s}) = 0. \tag{3.8}$$

As before, we denote by $\mathcal{M}^A(\{H^s\}, \{J_s\}, x_-, x_+)$ the moduli spaces of solutions in the class A, connecting a 1-periodic orbit of X_{H^-} with a 1-periodic orbit of X_{H^+}. They are smooth manifolds and, when they are non-empty, we have that $\dim \mathcal{M}^A(\{H^s\}, \{J_s\}, x_-, x_+) = \mu_{\mathrm{CZ}}(x_+) - \mu_{\mathrm{CZ}}(x_-) = |x_-| - |x_+|$. For these moduli spaces we have to show properties a), b), c) as before. Property a) stems again from

the maximum principle, which holds provided $\frac{d}{ds}T_{H^s} \leq 0$. Property *b)* is once more a consequence of Lemma 3.14. Property *c)*, stems from the modified energy-action identity

$$E(u) = -\mathcal{A}^{\omega}_{H^+}(u) + \int_{\mathbb{T}_1} (H^- - H^+)(t, x_-(t))dt + \int_{\mathbb{R} \times \mathbb{T}_1} (\partial_s H^s)(t, u(s,t))dsdt \,, \quad (3.9)$$

which replaces (3.7) for s-dependent cylinders. Notice that the rightmost term of (3.9) is bounded thanks to the uniform C^0-bounds and the fact that $\partial_s H^s \neq 0$ only on a finite interval. If we define $\eta(x_-) := \int_{\mathbb{T}_1} (H^+ - H^-)(t, x_-(t))dt$, then at the chain level

$$\varphi^{H^+, J^+}_{H^-, J^-} x := \sum_{\substack{u \in \mathcal{M}(\{H^s\}, \{J_s\}, y, x) \\ |y| - |x| = 0}} \epsilon(u)\, t^{\eta(y) - \mathcal{A}^{\omega}_{H^+}(u)} y \,. \quad (3.10)$$

An argument similar to the one at the end of Section 3.6.5, shows that such a map intertwines δ_{J^-} and δ_{J^+}, and, therefore, yields the desired map in cohomology.

The *Symplectic Cohomology (in the class ν)* is defined to be the direct limit of the direct system $\left(SH^*_\nu(W, \omega, j, H, J), \varphi^{H^+, J^+}_{H^-, J^-}\right)$:

$$SH^*_\nu(W, \omega, j) := \varinjlim_{(H,J)} SH^*_\nu(W, \omega, j, H, J).$$

We define the Symplectic Cohomology of a compact convex symplectic manifold as the Symplectic Cohomology of its completion: $SH^*_\nu(W, \omega, j) := SH^*_\nu(\hat{W}, \hat{\omega}, \hat{j})$.

3.3. Perturbing a non-degenerate autonomous Hamiltonian

We saw that the Symplectic Cohomology of (W, ω, j) is defined starting from a Hamiltonian function whose 1-periodic orbits are non-degenerate. However, if $H : W \to \mathbb{R}$ is an autonomous Hamiltonian and x is a non-constant 1-periodic orbit for the flow of X_H, x belongs to an S^1-worth of periodic orbits. Hence, x is degenerate. In this setting, we have to look to a weaker notion of non-degeneracy.

DEFINITION 3.15. Let x be a 1-periodic orbit of X_H, where H is an autonomous Hamiltonian and denote by S_x the connected component of set of 1-periodic orbits of X_H to which x belongs. We say that x is *transversally non-degenerate*

- either if x is a constant orbit and it is non-degenerate according to Definition 3.10 (namely, $d_{x(0)}\Phi^{X_H}_1$ does not have 1 in the spectrum);
- or if x is non-constant and 1 has algebraic multiplicity 2 in the spectrum of $d_{x(0)}\Phi^{X_H}_1$.

We say that H is *transversally ν-non-degenerate* if all 1-periodic orbits in the class ν are transversally non-degenerate.

REMARK 3.16. When $\dim W = 4$ the notion of transversally non-degenerate coincides with the notion given for Hamiltonian structures in Definition 2.10.

REMARK 3.17. If x is transversally non-degenerate, then $S_x = \{x(\cdot + t')\}_{t' \in \mathbb{T}_1}$. Therefore, $S_x = \{x\}$, if x is constant and $S_x \simeq \mathbb{T}_1$, if x is not constant.

If H is an autonomous Hamiltonian such that all the 1-periodic orbits of X_H in the class ν are *transversally* non-degenerate, we can construct a small perturbation $H_\varepsilon : W \times S^1 \to \mathbb{R}$ such that

- H_ε differs from H only in a small neighbourhood U_ε of the non-constant periodic orbits of X_H and $\|H - H_\varepsilon\|$ is small;
- the constant periodic orbits of X_{H_ε} are the same as the constant periodic orbits of X_H;
- for every non-constant 1-periodic orbit x of X_H there are two periodic orbits $x_{\min}$ and x_{Max} of X_{H_ε} supported in U_ε. All the non-constant periodic orbits of X_{H_ε} arise in this way.

Consider a function $H : W \to \mathbb{R}$ that satisfies the following two properties:

(**H'1**) it is transversally ν-non-degenerate,

(**H'2**) it has constant slope at infinity and T_H does not belong to $\text{Spec}(\alpha, \nu)$.

Denote the set of such functions by $\mathcal{H}'_\nu$. We readily see that if we carry out the above perturbation to $H \in \mathcal{H}'_\nu$, the resulting function H_ε belongs to $\mathcal{H}_\nu$ (see Definition 3.12) and we can use it to compute Symplectic Cohomology.

The degree of the new non-constant orbits is given by

$$\bullet \; |x_{\min}| = \frac{\dim(W)}{2} - \mu_{\text{CZ}}(x), \qquad \bullet \; |x_{\text{Max}}| = \frac{\dim(W)}{2} - \mu_{\text{CZ}}(x) - 1, \qquad (3.11)$$

where $\mu_{\text{CZ}}(x)$ is the transverse Conley-Zehnder index. It will be interesting for the applications to compute $< \delta_J x_{\text{Max}}, x_{\min} >$. Proposition 3.9(ii) in [**BO09b**] tells us that

$$< \delta_J x_{\text{Max}}, x_{\min} > = \begin{cases} 0 & \text{if } x \text{ is good}, \\ \pm 2t^{a_x} & \text{if } x \text{ is bad}, \end{cases} \qquad (3.12)$$

where a_x is a small positive number. Recall that an orbit is *bad* if it is an even iteration of a hyperbolic orbit with odd index and it is *good* otherwise.

3.4. Reeb orbits and two filtrations of the Floer Complex

In this section we restrict the admissible Hamiltonians to a subclass $\hat{\mathcal{H}}_\nu \subset \mathcal{H}_\nu$, whose non-constant periodic orbits are in strict relation with Reeb orbits of α.

DEFINITION 3.18. Fix some b belonging to the image of the function $r : U \to \mathbb{R}$ and denote $\rho_b := e^b$. Fix also a C^2-small Morse function $H_b : W^b \to \mathbb{R}$, such that close to the boundary $H_b = h_b(e^r)$ for some strictly increasing convex function h_b. Let $H : W \to \mathbb{R}$ be an element of $\mathcal{H}'_\nu$ with the following additional properties

(**$\hat{\text{H}}$1**) on W^b, $H = H_b$,

(**$\hat{\text{H}}$2**) on W_b, $H = h(e^r)$ for a function $h : [\rho_b, +\infty) \to$ which is strictly increasing and strictly convex on some interval $[\rho_b, \rho_H)$ and satisfies $h(\rho) = T_H \rho + a_H$, for some $T_H \notin \text{Spec}(\alpha, \nu)$ and $a_H \in \mathbb{R}$, on $[\rho_H, +\infty)$.

We denote by $\hat{\mathcal{H}}'_\nu$ the subset of all the Hamiltonians in $\mathcal{H}'_\nu$ satisfying these two properties and by $\hat{\mathcal{H}}_\nu \subset \mathcal{H}_\nu$ the corresponding set of perturbations inside $\mathcal{H}_\nu$.

The 1-periodic orbits of $H \in \hat{\mathcal{H}}'_\nu$ fall into two classes: constant periodic orbits inside W^b and non-constant periodic orbits inside W_b. Observe that in the region W_b, $X_H = h'R^\alpha$ where h' is the derivative of h. Hence, $x(t) = (r(t), p(t))$ is a 1-periodic orbit for X_H if and only if $r(t)$ is constant and equal to some $\log \rho_x$ and there exists a $h'(\rho_x)$-periodic Reeb orbit $\gamma : \mathbb{T}_T \to \Sigma$ such that $p(t) = \gamma(h'(\rho_x)t)$. Since h' is increasing and $h'(\rho_b) < T(\alpha, \nu)$, the non-constant periodic orbits of X_H are in bijection with all the Reeb orbits with period smaller than T_H.

Consider now the Floer Complex $(SC^*_\nu(W, \omega, j, H_\varepsilon), \delta_J)$ with $H_\varepsilon \in \hat{\mathcal{H}}_\nu$ and some J which is H_ε-admissible. In the next subsections we show that we have two ways of filtrating this complex.

The first one is a filtration by the action and it works when (W, ω) is a Liouville domain. In this case we can define the action functional $\mathcal{A}^\omega_{H_\varepsilon} : \mathscr{L}_\nu W \to \mathbb{R}$ on the space of loops in the class ν.

The second one is a filtration by the period of Reeb orbits. It works for general convex manifolds but we have to restrict the class of admissible complex structures to some $\mathcal{J}(H)$ (see Definition 3.22), so that we can apply the maximum principle (see [**BO09a**, page 654] and Lemma 3.23).

When $\nu = 0$, both filtrations imply that the singular cohomology complex $(C^{*+n}(W, \Lambda), \delta_0)$ shifted by n is a subcomplex of $(SC^*_\nu(W, \omega, j, H_\varepsilon), \delta_J)$.

3.4.1. Filtration by the action for Liouville domains. Suppose (W, ω) is a Liouville domain. This means that the Lioville form θ extends to a global primitive for ω on the whole W. If $H : \mathbb{T}_1 \times W \to \mathbb{R}$ is any Hamiltonian function, we define the action functional $\mathcal{A}^\omega_H : \mathscr{L}_\nu W \to \mathbb{R}$ by

$$\mathcal{A}^\omega_H(x) = -\int_{\mathbb{T}_1} x^*\theta + \int_{\mathbb{T}_1} H(t, x(t)) dt \,,$$

If u is a Floer cylinder connecting x_- and x_+, (3.7) yields

$$0 \le E(u) = -\mathcal{A}^\omega_H(u) = \mathcal{A}^\omega_H(x_-) - \mathcal{A}^\omega_H(x_+) \,.$$

This implies that $\mathcal{A}^\omega_H(x_-) \ge \mathcal{A}^\omega_H(x_+)$. Therefore, if H is ν-non-degenerate and J is H-admissible, then δ_J preserves the superlevels of $\mathcal{A}^\omega_H$ and we get the following corollary.

COROLLARY 3.19. *If x and y are two different 1-periodic orbits for H and $\mathcal{A}^\omega_H(y) \le \mathcal{A}^\omega_H(x)$, then*

$$< \delta_J x, y >= 0 \,. \tag{3.13}$$

We denote by $SC^{*,>a}_\nu(W, \omega, j, H)$ the subspace of $SC^*_\nu(W, \omega, j, H)$ generated by all the orbits with action bigger than some $a \in \mathbb{R}$. Corollary 3.19 shows that δ_J restricts to a linear operator on this subspace, and the inclusion induces a map in cohomology:

$$SH^{*,>a}_\nu(W, \omega, j, H, J) \longrightarrow SH^*_\nu(W, \omega, j, H, J) \,. \tag{3.14}$$

Let $(H^+, J^+) \preceq (H^-, J^-)$ and let (H^s, J^s) be a homotopy between the two pairs satisfying $\frac{d}{ds} T_{H^s} \le 0$. If u is an s-dependent Floer cylinder, relation (3.9) can be

rewritten as

$$\mathcal{A}^{\omega}_{H^+}(x_+) - \mathcal{A}^{\omega}_{H^-}(x_-) = -E(u) + \int_{\mathbb{R} \times \mathbb{T}_1} (\partial_s H^s)(t, u(s,t)) ds dt \, .$$

From this equation we see that the continuation map preserves the filtration, provided that $\partial_s H^s \leq 0$. Under this additional hypothesis, $\varphi^{H^+, J^+}_{H^-, J^-}$ induces a map

$$SH^{*, >a}_{\nu}(W, \omega, j, H^+, J^+) \longrightarrow SH^{*, >a}_{\nu}(W, \omega, j, H^-, J^-) \, . \tag{3.15}$$

Let us now specialise to Hamiltonians $H \in \hat{\mathcal{H}}'_{\nu}$. Call $a_h : [\rho_b, +\infty) \to \mathbb{R}$, the function $a_h(\rho) := h(\rho) - \rho h'(\rho)$. The absolute value of $a_h(\rho_b)$ is small and

$$a'_h(\rho) = -\rho h''(\rho) \, . \tag{3.16}$$

Hence, a_h is strictly decreasing on $[\rho_b, \rho_H]$ and $a_h \equiv a_H$ on $[\rho_H, +\infty)$.

If x is a non-constant periodic orbit in the class ν, then $x \subset W_b$ and we can write $x(t) = (\log \rho_x, \gamma_x(h'(\rho_x)t))$, for some Reeb orbit γ_x. We compute

$$\mathcal{A}^{\omega}_H(x) \; = \; - \int_0^{h'(\rho_x)} \rho_x \gamma_x^* \alpha \; + \; h(\rho_x) \; = \; a_h(\rho_x) \, . \tag{3.17}$$

Fix $a \in \mathbb{R}$ and take H^+ and H^- in $\hat{\mathcal{H}}'_0$, such that $T_{H^+} \leq T_{H^-}$ and $H^- = H^+$ on some W^c, where $a = a_{h^+}(e^c) = a_{h^-}(e^c)$. In particular, $a > \max\{a_{H^+}, a_{H^-}\}$. Take perturbations H^+_ε and H^-_ε and let (H^s_ε, J^s) be a homotopy between them such that $\partial_s H^s_\varepsilon \leq 0$ and which is constant on W^c. We readily see that

$$SC^{*, >a}_{\nu}(W, \omega, j, H^+_\varepsilon) \; = \; SC^{*, >a}_{\nu}(W, \omega, j, H^-_\varepsilon) \, .$$

Consider continuation cylinders between the generators of these two subcomplexes. The maximum principle shows that they are contained inside W^c. However, in this region $\partial_s H^s = 0$ and the transversality of the moduli spaces shows that we only have the constant cylinders. We have proved the following corollary.

COROLLARY 3.20. *Under the hypotheses of the paragraph above, the filtered continuation map $SC^{*, >a}_{\nu}(W, \omega, j, H^+_\varepsilon) \to SC^{*, >a}_{\nu}(W, \omega, j, H^-_\varepsilon)$ is the identity.*

Take now $\nu = 0$ and assume that $H \in \hat{\mathcal{H}}'_0$. We want to use the map (3.14) to single out the constant orbits of H among all contractible orbits. If $\overline{x}$ is a constant orbit, we readily compute

$$\mathcal{A}^{\omega}_H(\overline{x}) = H(\overline{x}) \geq \min_{W^b} H \, . \tag{3.18}$$

If $x(t) = (\log \rho_x, \gamma_x(h'(\rho_x)t))$ is non-constant, we know that $\mathcal{A}^{\omega}_H(x) = a_h(\rho_x)$. Moreover, (3.16) implies that $a'_h \leq -\rho_b h''$ and, therefore,

$$a_h(\rho_x) \; \leq \; a_h(\rho_b) - \rho_b(h'(\rho_x) - h'(\rho_b)) \, .$$

Since $h'(\rho_x) \geq T(\alpha, \nu) > 0$, we see that there exists $a_0 < 0$ such that

$$\mathcal{A}^{\omega}_H(x) < a_0 < \mathcal{A}^{\omega}_H(\overline{x}) \, .$$

Taking a small perturbation $H_\varepsilon \in \hat{\mathcal{H}}_0$, this inequality still holds and we find that $SC^{*, >a_0}_0(W, \omega, j, H_\varepsilon)$ is generated by the constant periodic orbits only. A further

analysis analogous to the case of closed symplectic manifolds shows that δ_J restricted to this subcomplex is the differential for the Morse Cohomology of the function $H_\varepsilon|_{W^b} = H_b$ (see [**Vit99**, Section 1.2]), where we identify constant periodic orbits of X_{H_ε} with critical points of H_ε up to a shifting of degrees by n. Thus, we have $SH_0^{*,>a_0}(W, \omega, j, H_\varepsilon, J) \simeq H^{*+n}(W, \Lambda)$. If $(H_\varepsilon^+, J^+) \preceq (H_\varepsilon^-, J^-)$ and H^+, H^- belong to $\hat{\mathcal{H}}_0'$, then $SH_0^{*,>a_0}(W, \omega, j, H_\varepsilon^+, J^+) \simeq SH_0^{*,>a_0}(W, \omega, j, H_\varepsilon^-, J^-)$, by Corollary 3.20. Since $\hat{\mathcal{H}}_0$ is cofinal in $\mathcal{H}_0$, we can take direct limits and arrive to the following corollary.

COROLLARY 3.21. *If (W, ω, j) is a Liouville domain, then, for every $* \in \mathbb{Z}$, there exists a map*

$$H^{*+n}(W, \Lambda) \longrightarrow SH_0^*(W, \omega, j). \tag{3.19}$$

If α does not have any Reeb periodic orbit γ contractible in W such that

$$\mu_{\mathrm{CZ}}(\gamma) \in \{\dim W/2 - *, \ \dim W/2 - * + 1\},$$

then such map is an isomorphism.

3.4.2. Filtration by the period. Let (W, ω) be a general open convex manifold. We restrict the class of admissible almost complex structures as follows.

DEFINITION 3.22. Fix an ω-compatible almost complex structure J_0 such that $(d\rho) \circ J_0 = -\theta$ on W_b. For every $H \in \hat{\mathcal{H}}_\nu'$ let $\mathcal{J}(H)$ be the set of the ω-compatible almost complex structures J_ε which are small compact perturbations of J_0 and which are H_ϵ-admissible for some $H_\varepsilon \in \hat{\mathcal{H}}_\nu$.

The next lemma, which is taken from [**BO09a**, page 654], gives us some information on the behaviour of Floer cylinders for (H, J_0) which are asymptotic to a reparametrisation of a Reeb orbit for $s \to -\infty$.

LEMMA 3.23. *Take $H \in \hat{\mathcal{H}}_\nu'$ and suppose that $u : (-\infty, s_*] \times \mathbb{T}_1 \to W_b$ is a nonconstant Floer half-cylinder for the pair (H, J_0). If u is asymptotic for $s \to -\infty$ to $x(t) = (r_x, \gamma_x(h'(e^{r_x})t))$, a non-constant 1-periodic orbit for X_H, then u is not contained in W^{r_x}.*

PROOF. Using the cylindrical end $j = (r, p)$, we write $u(s, t) = (r(s, t), p(s, t))$. If we employ the splitting $TW_b \simeq <\partial_r> \oplus <R^\alpha> \oplus \ker \alpha$, the Floer equation for u takes the following form:

$$\begin{cases} \partial_s r - \alpha(\partial_t p) + h'(e^r) = 0 \\ \alpha(\partial_s p) + \partial_t r = 0 \\ \pi_{\ker \alpha}(\partial_s p) + J_0 \pi_{\ker \alpha}(\partial_t p) = 0, \end{cases} \tag{3.20}$$

where $\pi_{\ker \alpha}$ is the projection on the third factor of the splitting.

Now define the real functions $\bar{r}$ and $\bar{a}$ on the interval $(-\infty, s_*]$ by integrating in the t-direction:

$$\bar{r}(s) := \int_{\mathbb{T}_1} r(s, t) dt, \qquad \bar{a}(s) := \int_{\mathbb{T}_1} p(s, \cdot)^* \alpha.$$

By Stokes's Theorem we have that

$$\overline{a}(s_1) - \overline{a}(s_0) = \int_{[s_0,s_1]\times\mathbb{T}_1} p^*d\alpha \,. \tag{3.21}$$

Since J_0 leaves $\ker\alpha$ invariant and it is ω-compatible, the third equation yields that the rightmost term in (3.21) is non-negative and, therefore, $\overline{a}$ is a non-decreasing function.

We integrate the first equation of (3.20) in t and use this monotonicity property:

$$\frac{d}{ds}\overline{r}(s) \geq -\int_{\mathbb{T}_1} h'(e^{r(s,t)})dt + \overline{a}(-\infty) = h'(e^{r_x}) - \int_{\mathbb{T}_1} h'(e^{r(s,t)})dt \,. \tag{3.22}$$

Suppose now, by contradiction that $r(s,t) \leq r_x$ for every $(s,t) \in (-\infty, s_*] \times \mathbb{T}_1$. Since $\rho \mapsto h'(\rho)$ is increasing, we find that the rightmost term in (3.22) is non-negative. Hence, $\frac{d}{ds}\overline{r}(s) \geq 0$. We have two possibilities. Either there exists a point $s_{**} < s_*$ such that $\frac{d}{ds}\overline{r}(s_{**}) > 0$, or $\frac{d}{ds}\overline{r} \equiv 0$ on the whole cylinder. In the first case, we must have $\overline{r}(s_{**}) > r_x$, which is a contradiction. In the second case, $\overline{r} \equiv r_x$ and, since $r(s,t) \leq r_x$, we see that $r(s,t) \equiv r_x$ on the whole cylinder. By the first equation in (3.20) $\overline{a}$ is also constant and by (3.21), we have that

$$\int_{(-\infty, s_*]\times\mathbb{T}_1} p^*d\alpha = 0 \,.$$

Putting this together with the fact that the function $r(s,t)$ is constant, we get that $E(u) = 0$, which contradicts the fact that u is not constant. Since both cases led to a contradiction, we see that u is not contained in W^{r_x} and the lemma is proven. $\square$

We have the following immediate corollary.

COROLLARY 3.24. *Suppose $u : \mathbb{R} \times \mathbb{T}_1 \to W$ is a non-constant Floer cylinder for the pair (H, J_0), with $H \in \hat{\mathcal{H}}'_\nu$, which connects two 1-periodic orbits for X_H. If $x_- = (r_{x_-}, \gamma_-(T_-t))$ for some T_--periodic Reeb orbit γ_-, with $T_- = h'(e^{r_{x_-}})$, then $x_+ = (r_{x_+}, \gamma_+(T_+t))$ for some T_+-periodic Reeb orbit γ_+, with $T_+ = h'(e^{r_{x_+}})$, such that*

$$T_+ > T_- \,. \tag{3.23}$$

PROOF. Consider the function $r(s,t) := r(u(s,t))$. By the lemma we just proved, $\max r > r_{x_-}$. By the maximum principle, x_+ is not contained in $W^{r_{x_-}}$. Hence, $x_+ = (r_{x_+}, \gamma_+(T_+t))$ is a non-constant periodic orbit and $r_{x_+} > r_{x_-}$. The thesis follows since $\rho \mapsto h'(\rho)$ is strictly increasing in the interval $[\rho_b, \rho_H]$. $\square$

The previous corollary gives important information on SH_ν, once we take a perturbation in order to achieve transversality.

COROLLARY 3.25. *Given $H \in \hat{\mathcal{H}}'_\nu$, there exist $H_\varepsilon \in \hat{\mathcal{H}}_\nu$ and $J_\varepsilon \in \mathcal{J}(H)$ such that the following statement is true. Let x and y be 1-periodic orbits of X_H with $S_x \neq S_y$ such that x is associated with a Reeb orbit of period T_x and y is either constant or it is associated with a Reeb orbit of period T_y, with $T_y \leq T_x$. If x_ε and y_ε are 1-periodic orbits of X_{H_ε} close to x and y, then*

$$<\delta_{J_\varepsilon} x_\varepsilon, y_\varepsilon> = 0 \,. \tag{3.24}$$

PROOF. We argue by contradiction and suppose that there exists $\varepsilon_k \to 0$ and a sequence of cylinders $u_k \in \mathcal{M}'(H_{\varepsilon_k}, J_{\varepsilon_k}, x_{\varepsilon_k}, y_{\varepsilon_k})$.

We claim that, up to taking a subsequence, there is an $m \in \mathbb{N}$ such that for every $i = 1, \ldots, m$ there exists a shift $s_k^i \in \mathbb{R}$ and for every $i = 0, \ldots, m$ there exists a 1-periodic orbit x_i for X_H with the following properties:

- $x_0 = x$ and $x_m = y$,
- for every $i = 1, \ldots, m-1$, $s_k^i < s_k^{i+1}$,
- for every $i = 1, \ldots, m$, $u_k(\cdot + s_k^i) \to u_\infty^i$, in the C_{loc}^∞-topology, for some cylinders u_∞^i satisfying the Floer Equation (3.5) for the pair (H, J_0) and with asymptotic conditions

$$u_\infty^i(-\infty) \in S_{x_{i-1}}, \qquad u_\infty^i(+\infty) \in S_{x_i}. \tag{3.25}$$

The claim follows from Proposition 4.7 in [**BO09b**] with the only difference that in our case also the complex structure is allowed to vary with k. This however does not represent a problem, since such proposition relies on Floer's compactness theorem [**Flo89**, Proposition 3(c)], in which the almost complex structure is allowed to vary as well.

Using inductively Corollary 3.24 and the fact that the positive end of u_∞^{i-1} coincides with the negative end of u_∞^i, we see that x_i is a non-constant orbit with $x_i(t) = (r_i, \gamma_i(T_i t))$ and that $T_i \leq T_{i+1}$. Since $S_x \neq S_y$, there exists a non-constant $u_\infty^{i_*}$. This implies that $T_{i_*-1} < T_{i_*}$ and, therefore, that $T_y > T_x$. This contradiction proves the corollary. $\qquad\square$

As happened for the action filtration, Corollary 3.25 shows that for any $T > 0$, δ_{J_ϵ} leaves invariant the subspace $SC_{\nu,<T}^*(W, \omega, j, H_\varepsilon)$ generated by the periodic orbits in ν associate with Reeb orbits with period smaller than T and by the constant periodic orbits (when $\nu = 0$). We call the associated cohomology group $SH_{\nu,<T}^*(W, \omega, j, H_\varepsilon, J_\varepsilon)$.

Fix $T > 0$ and take H^+ and H^- in $\hat{\mathcal{H}}_0'$, such that $T_{H^+} \leq T_{H^-}$ and $H^- = H^+$ on some W^c, where $T = (h^+)'(e^c) = (h^-)'(e^c)$. In particular, $T < \min\{T_{H^+}, T_{H^-}\}$. Take perturbations H_ε^+ and H_ε^- and let $(H_\varepsilon^s, J_\varepsilon^s)$ be a homotopy such that $\partial_s H_\varepsilon^s \leq 0$ and which is constant on W^c. We readily see that

$$SC_{\nu,<T}^*(W, \omega, j, H_\varepsilon^+) \;=\; SC_{\nu,<T}^*(W, \omega, j, H_\varepsilon^-).$$

Consider continuation cylinders between the generators of these two subcomplexes. The maximum principle shows that they are contained inside W^c. Since in this region $\partial_s H_\varepsilon^s = 0$, the transversality of the the moduli spaces of such cylinders shows that we only have constant cylinders. We have proved the following corollary.

COROLLARY 3.26. *Under the hypotheses of the paragraph above, the filtered continuation map* $SC_{\nu,<T}^*(W, \omega, j, H_\varepsilon^+) \to SH_{\nu,<T}^*(W, \omega, j, H_\varepsilon^-)$ *is the identity.*

Take now $\nu = 0$ and assume that $H \in \hat{\mathcal{H}}_0'$. If $T_0 := T(\alpha, 0)$, we find that $SC_{0,<T_0}^*(W, \omega, j, H_\varepsilon)$ is generated by the constant periodic orbits only. As before,

$SH^*_{0,<T_0}(W,\omega,j,H_\varepsilon,J_\varepsilon) \simeq H^{*+n}(W,\Lambda)$. If H^+, H^- belong to $\hat{\mathcal{H}}'_0$ and $(H^+_\varepsilon, J^+_\varepsilon) \preceq (H^-_\varepsilon, J^-_\varepsilon)$, then Corollary 3.26 implies that

$$SH^*_{0,<T_0}(W,\omega,j,H^+_\varepsilon,J^+_\varepsilon) \; = \; SH^*_{0,<T_0}(W,\omega,j,H^-_\varepsilon,J^-_\varepsilon)\,.$$

Taking direct limits we arrive at a generalisation of Corollary 3.21 for general convex manifolds.

COROLLARY 3.27. *If (W,ω,j) is a convex symplectic manifold, then, for every $* \in \mathbb{Z}$, there exists a map*

$$H^{*+n}(W,\Lambda) \longrightarrow SH^*_0(W,\omega,j)\,. \tag{3.26}$$

If the contact structure at infinity does not have any Reeb periodic orbit γ contractible in W such that

$$\mu_{\mathrm{CZ}}(\gamma) \, \in \, \{\,\dim W/2 - *,\ \dim W/2 - * + 1\,\}\,,$$

then such map is an isomorphism.

3.5. Invariance under isomorphism and rescaling

Alexander Ritter proved in [**Rit10**] that SH is an invariant of convex symplectic manifolds up to isomorphism.

THEOREM 3.28. *If $F : (W_0,\omega_0,j_0) \to (W_1,\omega_1,j_1)$ is an isomorphism of convex symplectic manifolds , then*

$$SH^*_\nu(W_0,\omega_0,j_0) \simeq SH^*_{F(\nu)}(W_1,\omega_1,j_1).$$

Moreover, SH is also invariant under multiplication of the symplectic form ω by a positive constant c.

PROPOSITION 3.29. *If (W,ω,j) is a convex symplectic manifold and c is a positive number, then*

$$SH^*_\nu(W,\omega,j) \simeq SH^*_\nu(W,c\omega,j).$$

The isomorphism is constructed from a rescaling $\rho_c : \Lambda \to \Lambda$ of the Novikov ring, defined by $\rho_c\left(\sum_i n_i t^{a_i}\right) = \sum_i n_i t^{ca_i}$. The rescaling yields the chain isomorphisms

$$SC^*_\nu(W,\omega,j,H) \; \longrightarrow \; SC^*_\nu(W,c\omega,j,cH)$$

$$\sum_k \lambda_k x_k \; \longmapsto \; \sum_k \rho_c(\lambda_k)x_k.$$

3.6. Invariance under deformations projectively constant on tori

In this section we address the problem of the invariance of SH under convex deformations (W_s,ω_s,j_s). We are going to analyse the case of compact manifolds W_s contained in a bigger open manifold W. Thanks to Remark 3.9, this is also enough for studying deformations of an open convex symplectic manifold (W,ω_s,j_s).

From the case of ALE spaces, considered in [**Rit10**] and from the discussion in the next section, we know that we cannot expect SH to be invariant under general

convex deformations. Our goal will be to prove the invariance of SH_ν under the additional assumption that the deformation is *projectively constant on ν-tori*. The relevance of this assumption is due to the fact that it ensures that the systems of local coefficients $\underline{\Lambda}_{\tau(\omega_s)}$ on $\mathscr{L}_\nu W$ are all isomorphic. Let us start by giving the relevant definitions and clarifying the statement we have just made.

The *transgression* $\tau : \Omega^2(W) \to \Omega^1(\mathscr{L}_\nu W)$ is defined by integration on paths $u : [0,1] \to \mathscr{L}_\nu W$ as

$$\tau(\sigma)(u) = \int_0^1 ds \left(\int_{\mathbb{T}_1} \sigma_u(\partial_s u, \partial_t u)\, dt \right) = \int_u \sigma = \sigma(u)$$

where we identify the path u with the underlying cylinder, which is a map from $[0,1] \times \mathbb{T}_1$ to W. The transgression yields a map $[\tau] : H^2(W, \mathbb{R}) \to H^1(\mathscr{L}_\nu W, \mathbb{R})$ between the de Rham cohomologies. We call a closed 2-form on W ν-*atoroidal* if it belongs to $\ker[\tau]$ or, equivalently, if there exists $\eta : \mathscr{L}_\nu W \to \mathbb{R}$ such that $\tau(\sigma) = d\eta$. One such primitive can be obtained by setting $\eta(x) = \sigma(u_x)$, where u_x is any cylinder connecting a fixed reference loop x_ν to x.

A family of 2-forms ω_s is called *projectively constant on ν-tori* if there exists a family of positive real numbers c_s such that $c_0 = 1$ and $c_s^{-1}[\tau(\omega_s)]$ is constant in $H^1(\mathscr{L}_\nu W)$. This is equivalent to the existence of ν-atoroidal forms σ_s, with $\sigma_0 = 0$, such that

$$(3.27) \qquad \begin{cases} \tau(\sigma_s) = d\eta_s \\ \omega_s = c_s(\omega_0 + \sigma_s). \end{cases}$$

Define $\underline{\Lambda}_{\tau(\omega_s)}$ as the trivial Λ-bundle over $\mathscr{L}_\nu W$ together with the multiplication isomorphism $t^{-\omega_s(u)} : \Lambda_x \to \Lambda_{x'}$ for every cylinder u connecting x to x'. Since ω_s is projectively constant on ν-tori, we use (3.27) to construct the multiplication map $t^{\eta_s} : \underline{\Lambda}_{\tau(\omega_0)} \to \underline{\Lambda}_{\tau(\omega_0 + \sigma_s)}$ and the rescaling map $\rho_{c_s} : \underline{\Lambda}_{\tau(c_s^{-1}\omega_s)} \to \underline{\Lambda}_{\tau(\omega_s)}$. The fact that these two maps are isomorphisms of local systems follows from the commutativity of the two diagrams below for every cylinder u connecting x to x'.

Therefore, we conclude that $\underline{\Lambda}_{\tau(\omega_s)}$ and $\underline{\Lambda}_{\tau(\omega_0)}$ are isomorphic if ω_s is projectively constant on ν-tori.

One can define a version of Symplectic Cohomology with twisted coefficients denoted by $SH_\nu^*(W, \omega, j; \underline{\Lambda}_\alpha)$ using any local system $\underline{\Lambda}_\alpha$ with structure ring Λ. The main result in [**Rit10**] says that $SH_\nu^*(W, d\lambda + \sigma, j) \simeq SH_\nu^*(W, d\lambda, j; \underline{\Lambda}_{\tau(\sigma)})$, where σ is a sufficiently small 2-form with compact support. In view of this isomorphism, it is natural to study the invariance of Symplectic Cohomology for convex perturbations (W_s, ω_s, j_s) which preserve the isomorphism class of $\underline{\Lambda}_{\tau(\omega_s)}$.

After this introductory discussion let us state the theorem we aim to prove.

THEOREM 3.30. *Let W be an open manifold and let ω_s be a family of symplectic forms on W. Suppose $W_s \subset W$ is a family of zero-codimensional embedded compact submanifold in W, which are all diffeomorphic to a model W'. Let $(W_s, \omega_s|_{W_s}, j_s)$ be a convex deformation. Fix ν a free homotopy class of loops in $W' \simeq W_s$ such that the contact forms α_0 and α_1 are both ν-non-degenerate. If $c_1(\omega_s)$ is ν-atoroidal for every $s \in [0,1]$ and $s \mapsto \omega_s$ is projectively constant on ν-tori, then*

$$SH_\nu^*(W_0, \omega_0, j_0) \simeq SH_\nu^*(W_1, \omega_1, j_1).$$

COROLLARY 3.31. *Let (W, ω_s, j_s) be a family of open convex symplectic manifolds. Fix ν a free homotopy class of loops in W such that the contact forms α_0 and α_1, associated to j_0 and j_1, are both ν-non-degenerate. If $c_1(\omega_s)$ is ν-atoroidal for every $s \in [0,1]$ and $s \mapsto \omega_s$ is projectively constant on ν-tori, then*

$$SH_\nu^*(W, \omega_0, j_0) \simeq SH_\nu^*(W, \omega_1, j_1).$$

PROOF. Use Remark 3.9 to find a family of compact manifolds $W_s \subset W$ satisfying the hypothesis of Theorem 3.30. $\qquad\square$

The theorem can also be used to prove that in the compact case, Symplectic Cohomology does not depend on the choice of a convex collar.

COROLLARY 3.32. *Let (W, ω) be a compact symplectic manifold and suppose that $c_1(\omega)$ is ν-atoroidal for some class ν. If j_0 and j_1 are two ν-non-degenerate convex collars of the boundary, then*

$$SH_\nu^*(W, \omega, j_0) \simeq SH_\nu^*(W, \omega, j_1). \tag{3.28}$$

PROOF. Let α_0 and α_1 be the two contact forms induced on the boundary. Define the family $\alpha_s := (1-s)\alpha_0 + s\alpha_1$ and observe that $d\alpha_s = i^*\omega$. Hence, $\alpha_s \wedge d\alpha_s = (1-s)\alpha_0 \wedge \omega + s\alpha_1 \wedge \omega$ is a positive contact form. Now apply Theorem 3.30 to the deformation (W, ω, j_s), where j_s is any family of convex collars inducing α_s on ∂W and connecting j_0 to j_1. $\qquad\square$

REMARK 3.33. Thanks to the corollary we just proved, if (W, ω, j) is a convex symplectic manifold with ν-non-degenerate boundary, its Symplectic Cohomology depends only on ω and not on j. Thus, we will use the notation

$$SH_\nu^*(W, \omega) := SH_\nu^*(W, \omega, j). \tag{3.29}$$

We point out that we were not able to prove a statement analogous to Corollary 3.32 for open convex symplectic manifolds, since we ignore if the space of convex cylindrical ends is path connected.

We now proceed to the proof of the theorem, by making three preliminary reductions. First, thanks to Proposition 3.29, we can assume that $c_s = 1$.

Second, it is enough to prove invariance of symplectic cohomology for all the *non-degenerate* parameters belonging to a small interval around any fixed $s_* \in [0,1]$ (a parameter s is non-degenerate if the corresponding α_s is non-degenerate, so that SH is well defined). Then, the theorem follows by observing that the compactness

of $[0,1]$ implies that there is a finite number of non-degenerate $s_i \in [0,1]$ with $0 = s_0 < s_1 < \cdots < s_k = 1$ such that

$$SH_\nu^*(W_{s_i}, \omega_{s_i}, j_{s_i}) \simeq SH_\nu^*(W_{s_{i+1}}, \omega_{s_{i+1}}, j_{s_{i+1}}). \tag{3.30}$$

Hence, the isomorphism claimed in Theorem 3.30 is obtained by taking the composition of all the intermediate isomorphisms.

Third, by the second reduction, we can make the following assumptions.

- All the W_s are contained in a fixed symplectic manifold W with cylindrical end $j = (r, p) : U \xrightarrow{\sim} \mathbb{R} \times \Sigma$ such that the boundary of W_s is contained in U and transverse to ∂_r. Thus, there exists a family of functions $f_s : \Sigma \to \mathbb{R}$ such that $W_s = W^{f_s}$. To ease the notation we also set $\Gamma_s := \Gamma_{f_s}$, $\Sigma_s := \Sigma_{f_s}$ (see Definition 3.2).
- The convex collars j_s of $\partial W_s = \Sigma_s$ extend to a family of convex neighbourhoods $j_s = (r_s, p_s) : U_s \to (-\varepsilon_s, \varepsilon_s) \times \Sigma$ of Σ_s such that $r_s(\Sigma_s) = 0$ and $\theta_s := \iota_{\partial_{r_s}} \omega_s$ is a Liouville form for ω_s. Observe that with this choice, we have $j_s^{-1}(r_s, p_s) = \Phi_{r_s}^{\partial_{r_s}}(\Gamma_s(p_s))$.

Thanks to these two assumptions, we can define the family of positive contact forms $\alpha_s = \Gamma_s^* \theta_s \in \Omega^1(\Sigma)$.

To prove the invariance when the variation of the parameter is small and $c_s = 1$, we first bring the convex manifolds in a special form through a suitable isomorphism. In the following discussion let r_0, r_1 be two real numbers such that $r_0 < r_1 < f_s(x)$, for every s in $[0,1]$ and x in Σ.

PROPOSITION 3.34. *For every parameter $s_* \in [0,1]$, there exists δ_{s_*} such that for $|s - s_*| < \delta_{s_*}$, there exists a family of symplectic forms ω_s' on W_{s_*} with the following properties:*

a) $\omega_s' = \omega_{s_*}$ on a neighbourhood of Σ_{s_*};

b) there exists a family of isomorphisms $F_s : (\hat{W}_{s_*}, \hat{\omega}_s', \hat{j}_{s_*}) \to (\hat{W}_s, \hat{\omega}_s, \hat{j}_s)$, restricting to the identity on W^{r_0}.

PROOF. Gray Stability Theorem applied to the family of contact forms α_s on Σ yields a family of functions $u_s : \Sigma \to \mathbb{R}$ and of diffeomorphisms $\psi_s : \Sigma \to \Sigma$, such that

$$\psi_s^* \alpha_s = e^{-u_s} \alpha_{s_*}, \qquad u_{s_*} = 0, \quad \psi_{s_*} = \text{Id}_\Sigma. \tag{3.31}$$

Consider the associated map between the symplectisations

$$\Psi_s : (\mathbb{R} \times \Sigma, e^{r_{s_*}} \alpha_{s_*}) \longrightarrow (\mathbb{R} \times \Sigma, e^{r_s} \alpha_s)$$
$$(r_{s_*}, p_{s_*}) \longmapsto (r_{s_*} + u_s(p_{s*}), \psi_s(p_{s*})).$$

Equation (3.31) is equivalent to the fact that $\Psi_0 = \text{Id}_{\mathbb{R} \times \Sigma}$ and Ψ_s is an exact symplectomorphism: $\Psi_s^*(e^{r_s} \alpha_s) = e^{r_{s_*}} \alpha_{s_*}$.

As Σ_s is the graph of the function f_s, the vector fields ∂_{r_s} and ∂_r are both positively transverse to Σ_s. Hence, we can define a vector field Y_s on a neighbourhood V_s of $W_{r_1}^{f_s} \subset W$ in such a way that

- it is equal to ∂_{r_s} on a neighbourhood U'_s of Σ_s compactly contained in U_s;
- it is equal to ∂_r close to Σ_{r_1};
- every flow line hits Σ_{r_1} in the past.

Such vector field is obtained by considering a convex combination

$$Y_s = a_s(r,p)\partial_{r_s} + (1 - a_s(r,p))\partial_r,$$

where $a_s : U_s \to [0,1]$ is a function with compact support, which is 1 close Σ_s. We are now ready to construct the map F_s dividing $\hat{W}_{s_*}$ and $\hat{W}_s$ in four pieces.

i) Take δ_{s_*} in such a way that, for $|s - s_*| < \delta_{s_*}$, $\Psi_s((-\frac{\varepsilon_{s_*}}{2}, \frac{\varepsilon_{s_*}}{2}) \times \Sigma) \subset U'_s$ and set

$$F_s := \Psi_{s|} : \{(r_{s_*}, p_{s_*}) \mid r_{s_*} \geq 0\} \longrightarrow \{(r_s, p_s) \mid r_s \geq u_s(p_s)\}.$$

ii) Define $A_s : \Sigma \to W$ by $A_s(p_{s_*}) := j_s^{-1} \circ \Psi_s(0, p_{s_*})$. In particular, $A_{s_*} = \Gamma_{f_{s_*}}$. Then, ∂_{r_s} is transverse to $\Sigma'_s := A_s(\Sigma)$ since it is close to $\Sigma'_{s_*} = \Sigma_{s_*}$.

For every p_{s_*} in Σ, consider the first negative time $\tau_s(p_{s_*}) < 0$ such that $\Phi^{Y_s}_{\tau_s(p_{s_*})}(A_s(p_{s_*})) \in \Sigma_{r_1}$. Let $Q_s : \Sigma \to \Sigma$ be defined as $Q_s := p \circ \Phi^{Y_s}_{\tau_s(p_{s_*})}(A_s(p_{s_*}))$. Furthermore, take a diffeomorphism $T_s^{p_{s_*}} : [-1,0] \to [\tau_s(p_{s_*}), 0]$ which is the identity near 0. We have diffeomorphisms

$$\begin{aligned} B_s : [-1,0] \times \Sigma &\longrightarrow W^{\Sigma'_s}_{r_1} \\ (t, p_{s_*}) &\longmapsto \Phi^{Y_s}_{T_s^{p_{s_*}}(t)}(A_s(p_{s_*})). \end{aligned}$$

We set $F_s := B_s \circ B_{s_*}^{-1} : W^{f_{s_*}}_{r_1} \to W^{\Sigma'_s}_{r_1}$. By definition

$$j_s \circ B_s(t, p_{s_*}) = j_s(\Phi^{Y_s}_t(A_s(p_{s_*}))) = (t + u_s(p_{s_*}), \psi_s(p_{s_*})) = \Psi_s(t, p_{s_*})$$

for t close to zero. Hence, the two definitions of F_s match along Σ_{s_*}.

iii) Close to Σ_{r_1} the map B_s takes the form $B_s(t, p_{s_*}) = (r_1 - 1 - t, Q_s(p_{s_*}))$. Therefore, in that region we have that $F_s = B_s \circ B_{s_*}^{-1}(r,p) = (r, Q_s \circ Q_{s_*}^{-1}(p))$. Now, define

$$F_s(r,p) := (r, Q_{b(r)s+(1-b(r))s_*} \circ Q_{s_*}^{-1}(p)) : W^{r_1}_{r_0} \to W^{r_1}_{r_0}$$

where $b : [r_0, r_1] \to [0,1]$ is equal to 0 near r_0 and equal to 1 near r_1.

iv) Close to Σ_{r_0} the previous definition yields $F_s(r,p) = (r, Q_{s_*} \circ Q_{s_*}^{-1}(p)) = (r,p)$. Thus, we simply let $F_s := \mathrm{Id}_{W^{r_0}} : W^{r_0} \to W^{r_0}$.

$$\square$$

By the proposition we just proved, for every non-degenerate s in $B_{s_*}(\delta_{s_*})$, $SH^*_\nu(W_s, \omega_s, j_s) \simeq SH^*_\nu(W_{s_*}, \omega'_{s_*}, j_{s_*})$ and $\omega'_{s_*} = \omega_{s_*}$. Notice that $[\omega'_s] = [\omega_s]$ so that ω'_s is still constant on ν-tori. Thus, in order to prove the invariance under a small variation of the parameter s around s_*, it is enough to show that $SH^*_\nu(W_{s_*}, \omega'_s, j_{s_*}) \simeq SH^*_\nu(W_{s_*}, \omega_{s_*}, j_{s_*})$.

The argument is contained in the next proposition.

PROPOSITION 3.35. *Let (W, ω_0, j) be a compact convex symplectic manifold and ν a free homotopy class of loops, such that α_0 is non-degenerate on ν. Suppose we have a family of $\sigma_s \in \Omega^2(W)$ for $s \in [0,1]$, supported in $W \setminus U$, such that*

a) $\sigma_0 = 0$;

b) σ_s *is ν-atoroidal;*

c) $\omega_s := \omega_0 + \sigma_s$ *is a path of symplectic forms.*

Then, the following isomorphism holds

$$SH_\nu^*(W, \omega_0, j) \simeq SH_\nu^*(W, \omega_1, j).$$

The proof of this proposition requires several steps and will occupy the rest of the present section.

3.6.1. Admissible pairs compatible with a symplectic deformation.
Consider $(\hat{W}, \hat{\omega}_s, j)$ the completion of the compact symplectic manifolds contained in the statement of the proposition. Observe that $\hat{W}$ does not depend on s, since $\operatorname{supp} \sigma_s \cap U$ is empty. In particular, $\alpha_s = \alpha_0$ and hence, SH_ν is defined for every s. Define the total support of the 1-parameter family $\{\sigma_s\}$ as

$$\operatorname{supp}\{\sigma_s\} := \overline{\bigcup_s \operatorname{supp} \sigma_s}.$$

It is compact and contained in $W \setminus U$.

We define a subclass of admissible Hamiltonians $\mathcal{H}_\nu^{\{\omega_s\}} \subset \mathcal{H}_\nu$ on $\hat{W}$, depending on the deformation $\{\omega_s\}$.

DEFINITION 3.36. Fix once and for all a function $K : W \to \mathbb{R}$ without critical points on $\operatorname{supp}\{\sigma_s\}$. Then, take a sufficiently small ε_K in such a way that, for every $s \in [0,1]$, $X_{\varepsilon_K K}^{\omega_s}$ does not have 1-periodic orbits intersecting $\operatorname{supp}\{\sigma_s\}$.

We say that a function $H \in \mathcal{H}_\nu$ belongs to $\mathcal{H}_\nu^{\{\omega_s\}}$ (in the following discussion we drop the subscript ν, since the free homotopy class is fixed), if $H = \varepsilon_K K$ on a neighbourhood of $\operatorname{supp}\{\sigma_s\}$.

Define the sets $\mathcal{P}_s$ made of pairs (H, J_s) such that

- $H \in \mathcal{H}^{\{\omega_s\}}$;
- J_s is an ω_s-compatible convex almost complex structure which is H-regular. Namely, it is regular for the solutions $u(s', t)$ of the Floer equation 3.5, which in this case reads

$$\partial_{s'} u + J_s(\partial_t u - X_H^{\omega_s}) = 0 \,.$$

We also assume that the following three properties can be achieved.

i) The projection $\mathcal{P}_s \to \mathcal{H}^{\{\omega_s\}}$ is surjective.

ii) Denote by $\mathcal{J}_s$ the image of the projection $(H, J_s) \mapsto J_s$ from $\mathcal{P}_s$ and define $\mathcal{J}^{\{\omega_s\}} := \cup_{s \in [0,1]} \mathcal{J}_s$. Then, $\mathcal{J}^{\{\omega_s\}}$ is bounded on $\operatorname{supp}\{\sigma_s\}$. This can be rephrased by saying that there exists $B \geq 1$ such that, if J_s and $J_{s'}$ belong to $\mathcal{J}^{\{\omega_s\}}$, then

$$\frac{1}{B} |\cdot|_{J_{s'}} \leq |\cdot|_{J_s} \leq B |\cdot|_{J_{s'}} \quad \text{on } \operatorname{supp}\{\sigma_s\}. \tag{3.32}$$

This property will be used below to get a uniform bound on the norm of the Hamiltonian vector field on $\operatorname{supp}\{\sigma_s\}$.

iii) Given $H \in \mathcal{H}^{\{\omega_s\}}$, denote by $\mathcal{J}_H^{\{\omega_s\}}$ the set of all J_s such that $(H, J_s) \in \mathcal{P}_s$ for some s. There exists a cylindrical end U_H such that every $J_s \in \mathcal{J}_H^{\{\omega_s\}}$ is convex on U_H, H has constant slope on U_H and J_s is uniformly bounded on U_H^c. Namely, there exists $B_H \geq 1$ such that, for every J_s and $J_{s'}$ in $\mathcal{J}_H^{\{\omega_s\}}$,

$$\frac{1}{B_H} |\cdot|_{J_{s'}} \leq |\cdot|_{J_s} \leq B_H |\cdot|_{J_{s'}} \quad \text{on } U_H^c. \tag{3.33}$$

This property implies that the distance induced by J_s on U_H^c are all Lipschitz equivalent. This will be used to get a constant ε_H, which is uniform in J_s in Lemma 3.37 below.

To ease the notation, set $V_H := U_H^c \cap \text{supp}\{\sigma_s\}^c$. It is a relatively compact neighbourhood of the periodic orbits of X_H disjoint from $\text{supp}\{\sigma_s\}$.

We readily see from the construction that each $\mathcal{P}_s$ is a cofinal set in the set of all admissible pairs for $(\hat{W}, \hat{\omega}_s, \hat{j})$. Thus, it is enough to use continuation maps between elements of $\mathcal{P}_s$ in order to define $SH_\nu^*(\hat{W}, \hat{\omega}_s, \hat{j})$.

3.6.2. A Palais-Smale lemma. Now we show that, given $H \in \mathcal{H}^{\{\omega_s\}}$, a loop x in U_H^c which is almost a critical point for $d\mathcal{A}_H^{\omega_s}$ cannot intersect the support of the perturbation. The following lemma is taken from [**Sal99**, Exercise 1.22]. The only difference here is that we require that ε_H does not depend on J_s.

LEMMA 3.37. *There exists ε_H such that, if $x \subset U_H^c$, then for every $J_s \in \mathcal{J}_H^{\{\omega_s\}}$,*

$$|d_x \mathcal{A}_H^{\omega_s}|_{J_s} < \varepsilon_H \quad \Longrightarrow \quad x \subset V_H. \tag{3.34}$$

PROOF. First, we claim that, for every $C > 0$, the set of smooth loops

$$L = \{x \subset U_H^c \mid |d_x \mathcal{A}_H^{\omega_s}|_{J_s} \leq C, \text{ for some } s \text{ and } J_s \in \mathcal{J}_H^{\{\omega_s\}}\}$$

is relatively compact for the C^0-topology. By the Arzelá-Ascoli Theorem, the claim follows by showing that the elements of L are equicontinuous with respect to the distance induced by some J_{s_*}. We compute

$$C \geq |d_x \mathcal{A}_H^{\omega_s}|_{J_s} = \left(\int_{\mathbb{T}_1} |\dot{x} - X_H(x)|_{J_s}^2 dt \right)^{\frac{1}{2}}$$

$$\geq \left(\int_{\mathbb{T}_1} |\dot{x}|_{J_s}^2 dt \right)^{\frac{1}{2}} - \|X_H\|_{J_s}.$$

By (3.33), $\|X_H\|_{J_s} \leq C'$ uniformly in J_s. Thus, the L^2-norm of $\dot{x}$ with respect to J_s is uniformly bounded and, hence, x is uniformly $1/2$-Hölder continuous for the distance induced by J_s. Since all these distances are Lipschitz equivalent thanks to (3.33) again, the equicontinuity follows.

Now, let us prove the lemma. Arguing by contradiction, we can find a sequence of loops (x_n), such that

$$\lim_{n \to +\infty} |d_{x_n} \mathcal{A}_H^{\omega_{s_n}}|_{J_{s_n}} = 0, \qquad x_n \not\subset U_H^c \cap \text{supp}\{\sigma_s\}^c = V_H. \tag{3.35}$$

By the claim, passing to a subsequence, which we denote in the same way, x_n converges in the C^0-norm to $x_\infty \not\subset V_H$. By compactness of the interval, we can

also suppose that $s_n \to s_\infty$. We claim that x_∞ is differentiable and it is a critical point of $d\mathcal{A}_H^{\omega_{s_\infty}}$. Indeed, $X_H^{\omega_{s_n}}(x_n) \to X_H^{\omega_{s_\infty}}(x_\infty)$ in the L^2-topology. By (3.35), $\dot{x}_n \to X_H^{\omega_{s_\infty}}(x_\infty)$ in the L^2-topology. Thus, x_∞ is in $W^{1,2}$ and its weak derivative is $X_H^{\omega_{s_\infty}}(x_\infty)$. Since $X_H^{\omega_{s_\infty}}(x_\infty)$ is continuous, we conclude that $\dot{x}_\infty(t)$ exists at every t and $\dot{x}_\infty(t) = X_H^{\omega_{s_\infty}}(x_\infty(t))$. Therefore, x_∞ is a periodic orbit of H, contradicting the fact that $x_\infty \not\subset V_H$. $\square$

3.6.3. Subdividing the deformation in substeps. We decompose the deformation ω_s in shorter perturbations by a suitable smooth change of parameters $s_{s_1}^{s_0} : \mathbb{R} \to [s_0, s_1]$, such that

$$s_{s_1}^{s_0}(S) = \begin{cases} s_0 & \text{if } S \leq 0 \\ s_1 & \text{if } S \geq 1. \end{cases}$$

We set $\omega_S^{s_0, s_1} := \omega_{s_{s_1}^{s_0}(S)}$. Notice that there exists $C > 0$ such that, if $J_s \in \mathcal{J}^{\{\omega_s\}}$, then

$$\left\| \dot{\omega}_S^{s_0, s_1} \right\|_{J_s} \leq C |s_1 - s_0|. \tag{3.36}$$

To ease the notation, we drop the superscript and we write ω_S for $\omega_S^{s_0, s_1}$. Moreover, we denote s_0 by $-$ and s_1 by $+$.

3.6.4. Two kind of homotopies and two kind of homotopies of homotopies. Let $H \in \mathcal{H}^{\{\omega_s\}}$ and let $(H, J_-) \in \mathcal{P}_-$ and $(H, J_+) \in \mathcal{P}_+$. A *symplectic homotopy* or *s-homotopy* $S \mapsto (\omega_S, H, J_S)$ is a path connecting (H, J_-) to (H, J_+) and constant at infinity.

Given $S \in \mathbb{R}$ and $(H^+, J_S^+) \preceq (H^-, J_S^-)$ in $\mathcal{P}_S$, a *monotone homotopy* or *m-homotopy* is a path $r \mapsto (H^r, J_S^r)$, with $r \in \mathbb{R}$, connecting (H^-, J_S^-) to (H^+, J_S^+) and constant at infinity, such that

- each H^r has constant slope at infinity;
- $\partial_r T_{H^r} \leq 0$;
- (J_S^r) is a family of ω_S-compatible convex almost complex structures satisfying bounds analogous to the ones described in condition *ii)* and *iii)* of Section 3.6.1.

Let H^- and H^+ be in $\mathcal{H}^{\{\omega_s\}}$ and let $J_-^-, J_+^-, J_-^+, J_+^+$ be almost complex structures such that $(H^+, J_-^+) \preceq (H^-, J_-^-)$ in $\mathcal{P}_-$ and $(H^+, J_+^+) \preceq (H^-, J_+^-)$ in $\mathcal{P}_+$. A *symplectic-monotone homotopy* or *sm-homotopy* is a map $(S, r) \mapsto (\omega_S, H^r, J_S^r)$ with $(S, r) \in \mathbb{R}^2$ such that

- restricting it to $r = \pm\infty$, we get two s-homotopies;
- restricting it to $S = \pm\infty$, we get two m-homotopies.

Consider a path $r \mapsto \sigma_S^r$ of 2-forms constant at infinity and such that $\sigma_S^- = 0$ and $\sigma_S^+ = \sigma^+$. Suppose that all the elements of the 2-parameters family $\omega_S^r := \omega + \sigma_S^r$ satisfy the hypotheses of Proposition 3.35. We can define a class $\mathcal{H}^{\{\omega_S^r\}}$ of admissible Hamiltonians associated to $\{\omega_S^r\}$ in the same way we defined $\mathcal{H}^{\{\omega_s\}}$ for a 1-parameter deformation. As before, this class yields a set of admissible pairs $\mathcal{P}_S^r$, which enjoys properties similar to *i)*, *ii)*, *iii)* outlined before for $\mathcal{P}^s$. Notice that when $|S|$ is

sufficiently big, $\mathcal{P}_S^r = \mathcal{P}_\pm$ does not depend on r (and on S). Given $(H, J_-) \in \mathcal{P}_-$ and $(H, J_+) \in \mathcal{P}_+$, a *symplectic-symplectic homotopy* or *ss-homotopy* is the 2-parameter family (ω_S^r, H, J_S^r) such that $J_S^r = J_\pm$ for big $|S|$ and such that restricting it to $r = \pm\infty$, we get two s-homotopies.

We remark that the Palais-Smale lemma still holds for sm- and ss-homotopies.

3.6.5. Moduli spaces for s-homotopies and chain maps.

We know that we can associate chain maps called *continuation maps* to monotone homotopies by considering suitable moduli spaces. We now carry out a similar construction to associate chain maps to symplectic homotopies (ω_S, H, J_S) as well. For this we will need first to have $|s_1 - s_0| < \delta_H$, for some $\delta_H > 0$. The general definition of the chain map will then be as the composition of intermediate maps as we did before in (3.30). More precisely, we choose δ_H in such a way that

$$\mu_H := \sup_{(S, J_s) \in \mathbb{R} \times \mathcal{J}^{\{\omega_s\}}} \left\| \dot{\omega}_S^{s_1, s_0} \right\|_{J_s} \leq C \delta_H \leq \frac{1}{\frac{\varepsilon_K}{\varepsilon_H} \|X_K\| + 1}, \tag{3.37}$$

where we have defined

$$\|X_K\| := \sup_{J_s \in \mathcal{J}^{\{\omega_s\}}} \sup_{z \in \mathrm{supp}\{\sigma_s\}} |X_K(z)|_{J_s}.$$

Observe that $\|X_K\|$ is finite thanks to condition *ii)* in Section 3.6.1.

Thus, let (ω_S, H, J_S) be a symplectic homotopy satisfying (3.37) and define the moduli spaces $\mathcal{M}(H, \{J_S\}, x_-, x_+)$ of S-dependent Floer cylinders relative to such homotopy, connecting two 1-periodic orbits x_- and x_+ of H. Choosing the family $\{J_S\}$ in a generic way, we can suppose that $\mathcal{M}(H, \{J_S\}, x_-, x_+)$ is a smooth manifold. If this space is non-empty, we have

$$\dim \mathcal{M}(H, \{J_S\}, x_-, x_+) \;=\; \mu_{\mathrm{CZ}}(x_+) - \mu_{\mathrm{CZ}}(x_-) \;=\; |x_-| - |x_+|.$$

If A is a homotopy class of cylinders relative ends, let $\mathcal{M}^A(H, \{J_S\}, x_-, x_+)$ be the subset of the moduli space whose elements belong to A.

We claim that $\mathcal{M}^A(H, \{J_S\}, x_-, x_+)$ is relatively compact in the C_{loc}^∞-topology. As before, this is achieved in a standard way once we have the three ingredients below.

a) Uniform C^0-bounds. They stem from the fact that the perturbation σ_S is compactly supported and, hence, the maximum principle still applies.

b) Uniform C^1-bounds. As before, they stem from the fact that bubbling off of holomorphic spheres generically does not occur by Lemma 3.14.

c) Uniform bounds on the energy E as defined in (3.6). These require a longer argument, which we describe below.

First, observe that Lemma 3.37 allows to bound the amount of time an S-dependent Floer cylinder u in $\mathcal{M}^A(H, \{J_S\}, x_-, x_+)$ spends on $\mathrm{supp}\{\sigma_s\}$. Define

$$S(u, H) := \{S \in \mathbb{R} \mid u(S) \cap \mathrm{supp}\{\sigma_s\} \neq \emptyset\},$$

where we have denoted by $u(S)$ the loop $t \mapsto u(S, t)$.

LEMMA 3.38. *The following estimate holds*

$$|S(u, H)| \leq \frac{E(u)}{\varepsilon_H^2},$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

PROOF. Observe that $S(u, H) \times \mathbb{T}_1 \subset \{(S, t) \mid |\partial_S u|_{J_S} \geq \varepsilon_H\}$ and then use Chebyshev's inequality. $\qquad\square$

We claim that for an S-dependent Floer cylinder u, identity (3.7) must be replaced with the more general

$$E(u) = -\mathcal{A}_H^{\omega_+}(u) + \int_{\mathbb{R}} \dot{\omega}_S(u|_{Z^S}) \, dS, \tag{3.38}$$

where $Z^S := (-\infty, S] \times \mathbb{T}_1$. Indeed,

$$E(u) = -\int_{\mathbb{R}} d_{u(S)} \mathcal{A}_H^{\omega_S} \cdot \frac{du}{dS}(S) \, dS = -\int_{\mathbb{R}} \left[\frac{d}{dS} \left(\mathcal{A}_H^{\omega_S}(u|_{Z^S}) \right) - \left(\frac{\partial \mathcal{A}_H^{\omega_S}}{\partial S} \right) (u|_{Z^S}) \right] dS$$

$$= -\mathcal{A}_H^{\omega_+}(u) + \int_{\mathbb{R}} \left(\frac{\partial \mathcal{A}_H^{\omega_S}}{\partial S} \right) (u|_{Z^S}) \, dS.$$

The claim follows by observing that $\left(\frac{\partial \mathcal{A}_H^{\omega_S}}{\partial S} \right) (u|_{Z^S}) = \dot{\omega}_S(u|_{Z^S})$.

The first summand on the right of (3.38) depends only on the homotopy class of u relative ends, so we need to bound only the second one:

$$\int_{\mathbb{R}} \dot{\omega}_S(u|_{Z^S}) \, dS = \int_0^1 \dot{\omega}_S(u|_{Z^S}) dS \leq \sup_{S \in [0,1]} \dot{\omega}_S(u|_{Z^S}),$$

where we have used the fact that $\dot{\omega}_S = 0$ for $S \notin [0, 1]$. We now estimate the last term, remembering that, for every $S \in [0, 1]$, $\dot{\omega}_S$ vanishes outside $\mathrm{supp}\{\sigma_s\}$:

$$\dot{\omega}_S(u|_{Z^S}) \leq \int_{Z^S} \left| (\dot{\omega}_S)_u(\partial_{S'} u, \partial_t u) \right| dS' dt$$

$$\leq \int_{S(u,H) \times \mathbb{T}_1} \left| (\dot{\omega}_S)_u(\partial_{S'} u, \partial_t u) \right| dS' dt$$

$$\leq \int_{S(u,H) \times \mathbb{T}_1} |\dot{\omega}_S|_{S'} |\partial_{S'} u|_{S'} |\partial_t u|_{S'} \, dS' dt$$

$$\leq \left(\sup_{J_s \in \mathcal{J}} \|\dot{\omega}_S\|_{J_s} \right) \int_{S(u,H) \times \mathbb{T}_1} |\partial_{S'} u|_{S'} |\partial_t u|_{S'} \, dS' dt.$$

Using Floer's equation, the quantity under the integral sign can be bounded by

$$|\partial_{S'} u|_{S'} |\partial_t u|_{S'} \leq |\partial_{S'} u|_{S'} \left(|\partial_{S'} u|_{S'} + |X_H(u)|_{S'} \right) \leq |\partial_{S'} u|_{S'}^2 + \varepsilon_K \|X_K\| |\partial_{S'} u|_{S'}.$$

Therefore,

$$\int_{S(u,H)\times\mathbb{T}_1} |\partial_{S'}u|_{S'}|\partial_t u|_{S'}\, dS'dt \leq \int_{S(u,H)\times\mathbb{T}_1} |\partial_{S'}u|_{S'}^2\, dS'dt + \varepsilon_K\|X_K\| \int_{S(u,H)\times\mathbb{T}_1} |\partial_{S'}u|_{S'}\, dS'dt$$

$$\leq E(u) + \varepsilon_K\|X_K\| \left(\int_{S(u,H)\times\mathbb{T}_1} |\partial_{S'}u|_{S'}^2\, dS'dt\right)^{\frac{1}{2}} |S(u,H)|^{\frac{1}{2}}$$

$$\leq E(u) + \varepsilon_K\|X_K\|E(u)^{\frac{1}{2}}\frac{E(u)^{\frac{1}{2}}}{\varepsilon_H}$$

$$= \left(1 + \frac{\varepsilon_K}{\varepsilon_H}\|X_K\|\right)E(u).$$

Putting all the estimates together, we find

$$\int_{\mathbb{R}} \dot\omega_S(u|_{Z^S})\, dS \leq \mu_H\left(1 + \frac{\varepsilon_K}{\varepsilon_H}\|X_K\|\right)E(u).$$

Plugging this inequality into (3.38), we finally get the desired bound for the energy

$$E(u) \leq \frac{-\mathcal{A}_H^{\omega_+}(u)}{1 - \mu_H\left(1 + \frac{\varepsilon_K}{\varepsilon_H}\|X_K\|\right)}. \tag{3.39}$$

After this preparation, we define η by $d\eta = \tau(\omega_+) - \tau(\omega_-)$ and claim that

$$\varphi^{(\{\omega_S\},H,\{J_S\})} : SC_\nu^*(\hat{W},\hat\omega_-,\hat j,H) \longrightarrow SC_\nu^*(\hat{W},\hat\omega_+,\hat j,H)$$

$$x \longmapsto \sum_{\substack{v\in\mathcal{M}(H,\{J_S\},x',x)\\|x'|-|x|=0}} \epsilon(v)t^{\eta(x)-\mathcal{A}_H^{\omega_+}(v)}x',$$

is well-defined and a chain map. The good definition stems from the fact that, thanks to 3.39, for every $C\in\mathbb{R}$, there are only finitely many connecting cylinders $v\in\mathcal{M}(H,\{J_S\},x',x)$ such that $\eta(x) - \mathcal{A}_H^{\omega_+}(v) \leq C$. Moreover, the properties $a)$, $b)$, $c)$ above implies that the boundary of a component of $M(H,\{J_S\},y',x)$ with $|y'| - |x| = 1$ is either empty or consists of two elements $v_{x'}^x * u_{y'}^{x'}$ and $u_y^x * v_{y'}^y$ where

$$\begin{cases} u_y^x \in \mathcal{M}(H,J_-,y,x), \\ u_{y'}^{x'} \in \mathcal{M}(H,J_+,y',x'), \\ v_{x'}^x \in \mathcal{M}(H,\{J_S\},x',x), \\ v_{y'}^y \in \mathcal{M}(H,\{J_S\},y',y). \end{cases}$$

The gluing construction in Floer theory shows also that every $v_{x'}^x * u_{y'}^{x'}$ with the properties above arises as a boundary element of a component of some $M(H,\{J_S\},y',x)$. Therefore, to see that $\varphi^{(\{\omega_S\},H,\{J_S\})}$ is a chain map, we only have to prove that

$$-\mathcal{A}_H^{\omega_+}(u_{y'}^{x'}) + \left(\eta(x) - \mathcal{A}_H^{\omega_+}(v_{x'}^x)\right) = \left(\eta(y) - \mathcal{A}_H^{\omega_+}(v_{y'}^y)\right) - \mathcal{A}_H^{\omega_-}(u_y^x). \tag{3.40}$$

First, since $\omega_+ - \omega_-$ is ν-atoroidal, we have

$$\mathcal{A}_H^{\omega_-}(u_y^x) = \mathcal{A}_H^{\omega_+}(u_y^x) + (\omega_+ - \omega_-)(u_y^x) = \mathcal{A}_H^{\omega_+}(u_y^x) + \eta(y) - \eta(x).$$

Plugging this relation into (3.40), we get

$$-\mathcal{A}_H^{\omega_+}(u_{y'}^{x'}) - \mathcal{A}_H^{\omega_+}(v_{x'}^{x}) = -\mathcal{A}_H^{\omega_+}(v_{y'}^{y}) - \mathcal{A}_H^{\omega_+}(u_y^x),$$

which is true since $v_{x'}^{x} * u_{y'}^{x'}$ and $u_y^x * v_{y'}^{y}$ are homotopic relative ends and, as we observed before, $\mathcal{A}_H^{\omega_+}$ is invariant under such homotopies. Thus, we have proved that $\varphi^{(\{\omega_S\}, H, \{J_S\})}$ is a chain map. We denote the induced map in cohomology by $\left[\varphi^{(\{\omega_S\}, H, \{J_S\})}\right]$.

3.6.6. Moduli spaces for sm-homotopies and commutative diagrams. In the previous subsection, we showed the existence of chain maps associated to s-homotopies. Hence, an sm-homotopy $(S, r) \mapsto (\omega_S, H^r, J_S^r)$ yields a diagram

$$
\begin{array}{ccc}
SC_\nu^*(W, \omega_-, j, H^+) & \xrightarrow{\varphi^{(\{\omega_S\}, H^+, \{J_S^+\})}} & SC_\nu^*(W, \omega_+, j, H^+) \\
{\scriptstyle \varphi^{(H^+, J_-^+)}_{(H^-, J_-^-)}} \downarrow & & \downarrow {\scriptstyle \varphi^{(H^+, J_+^+)}_{(H^-, J_+^-)}} \\
SC_\nu^*(W, \omega_-, j, H^-) & \xrightarrow{\varphi^{(\{\omega_S\}, H^-, \{J_S^-\})}} & SC_\nu^*(W, \omega_+, j, H^-)
\end{array}
\tag{3.41}
$$

We claim that such diagram is commutative.

The proof follows an argument similar to the one we used to define $\varphi^{(\{\omega_S\}, H, \{J_S\})}$. We subdivide the sm-homotopy in rectangles $[S_i, S_{i+1}] \times \mathbb{R}$, so that the corresponding constants $\mu_{\{H^r\}}^{S_i, S_{i+1}}$, defined in the same way as μ_H, are sufficiently small. From the commutativity of the intermediate diagrams, we infer the commutativity of the original one.

3.6.7. Moduli spaces for ss-homotopies and bijectivity of cohomology maps. Since diagram (3.41) is commutative, the system of maps $\left[\varphi^{(\{\omega_S\}, H, \{J_S\})}\right]$ yields a map $[\varphi] : SH_\nu^*(W, \omega_-, j) \to SH_\nu^*(W, \omega_+, j)$. The purpose of this subsection is to show that $[\varphi]$ is actually an isomorphism, hence proving Proposition 3.35.

From the properties of the direct limit of groups, is enough to show that each $\left[\varphi^{(\{\omega_S\}, H, \{J_S\})}\right]$ is an isomorphism. We claim that its inverse is the map which is obtained by inverting the s-homotopy. Namely, $\left[\varphi^{(\{\omega_S\}, H, \{J_S\})}\right]^{-1} = \left[\varphi^{(\{\omega_{-S}\}, H, \{J_{-S}\})}\right]$.

First, we notice that the composition of maps in cohomology intertwines with the concatenation of s-homotopies:

$$\left[\varphi^{(\{\omega_{-S}\}, H, \{J_{-S}\})}\right] \circ \left[\varphi^{(\{\omega_{-S}\}, H, \{J_{-S}\})}\right] = \left[\varphi^{(\{\omega_S\}*\{\omega_{-S}\}, H, \{J_S\}*\{J_{-S}\})}\right].$$

We omit the details of this fact. Thus, it is enough to prove that

$$\left[\varphi^{(\{\omega_S\}*\{\omega_{-S}\}, H, \{J_S\}*\{J_{-S}\})}\right] = \mathrm{Id}.\tag{3.42}$$

Consider the standard ss-homotopy that connects the concatenated s-homotopy $(\{\omega_S\}, H, \{J_S\}) * (\{\omega_{-S}\}, H, \{J_{-S}\})$ to the constant s-homotopy (ω_-, H, J_-) and notice that the chain map associated to the latter s-homotopy is the identity. Therefore, to show (3.42), we need to prove that if two s-homotopies are connected by an ss-homotopy, they induce the same map in cohomology. This is done by adjusting the argument above for s- and sm-homotopies to ss-homotopies.

The proof of Proposition 3.35 and, hence of Theorem 3.30, is completed.

CHAPTER 4

Energy levels of contact type

In this chapter we give sufficient conditions for an energy level of a magnetic system to be of contact type and we apply the results of the previous chapter to establish lower bounds on the number of periodic orbits, when the contact structure is positive.

First, we observe that because of Proposition 2.6, condition **(S2)** on the first Chern class is satisfied for every ν. Thus, if $DM := \{E \leq \frac{1}{2}\}$, it is possible to define the groups $SH_\nu^*(DM, \omega_s)$, whenever $\omega_s' := \omega_s\big|_{SM}$ is a ν-non-degenerate HS of positive contact type (see Definition 2.10).

DEFINITION 4.1. We say that a parameter s is ν-*non-degenerate* if ω_s' is transversally ν-non-degenerate.

We saw that the dynamics of the magnetic flow associated to (M, g, σ) can be studied by looking at the family of HS ω_s', where s is a real parameter. We now want to understand when such structures are of contact type so that we can apply techniques from Reeb dynamics and, in particular, use the abstract results proved in the previous chapter for the computation of Symplectic Cohomology. The first step is to determine when ω_s' is exact. The answer is given by the following proposition.

PROPOSITION 4.2. *The 2-form ω_s' is exact if and only if one of the two alternatives holds*

   (E) *either σ is exact,*
 (NE) *or σ is not exact and M is not the 2-torus.*

PROOF. We are going to prove the proposition by defining suitable classes of primitives for ω_s'. If σ is exact on M, we just consider the injection

$$\mathcal{P}^\sigma \longrightarrow \mathcal{P}^{\omega_s'}$$
$$\beta \longmapsto \lambda_{s,\beta} := \lambda - s\pi^*\beta. \tag{4.1}$$

When σ is not exact and $M \neq \mathbb{T}^2$, then $\frac{2\pi e_M}{[\sigma]}\sigma$ is a curvature form for the S^1-bundle $\pi : SM \to M$, since e_M is the Euler number of the bundle [**Kob56**]. By the discussion contained in Example 2.15 we know that $\pi^*\sigma$, and hence ω_s', is exact. More explicitly, since τ is also a connection form on SM with curvature σ_g, we have the injection

$$\mathcal{P}^{\sigma - \frac{[\sigma]}{2\pi e_M}\sigma_g} \longrightarrow \mathcal{P}^{\omega_s'}$$
$$\beta \longmapsto \lambda_{s,\beta}^g := \lambda + s\left(\frac{[\sigma]}{2\pi e_M}\tau - \pi^*\beta\right) \tag{4.2}$$

(notice, that if $M \neq \mathbb{T}^2$ and σ is also exact, this formula reduces to (4.1) above).

It only remains to show that if σ is not exact and $M = \mathbb{T}^2$, then ω_s' is not exact. This follows from the fact that $\pi : S\mathbb{T}^2 \to \mathbb{T}^2$ is a trivial bundle, as we mentioned in Example 2.15. More precisely, take $Z : \mathbb{T}^2 \to S\mathbb{T}^2$ a section of this bundle. Then, Z is a surface embedded in $S\mathbb{T}^2$ and

$$\int_{\mathbb{T}^2} Z^* \omega_s' = \int_{\mathbb{T}^2} d(Z^*\lambda) - \int_{\mathbb{T}^2} Z^*(s\pi^*\sigma) = -s \int_{\mathbb{T}^2} \sigma \neq 0. \qquad \square$$

From now on we restrict our discussion only to case **(E)** and **(NE)** contained in the statement of Proposition 4.2. We refer to them as the *exact* and *non-exact* case. Non-exact magnetic form on $\mathbb{T}^2$ will not play any role in the rest of this thesis.

DEFINITION 4.3. Let (M, g, σ) be a magnetic system of type **(E)** or **(NE)** and define

- $\mathrm{Con}^+(g,\sigma) := \left\{ s \in [0, +\infty) \mid \omega_s' \text{ is of positive contact type} \right\}$,   (4.3)

- $\mathrm{Con}^-(g,\sigma) := \left\{ s \in [0, +\infty) \mid \omega_s' \text{ is of negative contact type} \right\}$.   (4.4)

If we fix the manifold M and we vary the pairs (g, σ) defined on M, we get the space of all magnetic systems of positive, respectively negative, contact type on M:

- $\mathrm{Con}^+(M) := \bigcup_{(g,\sigma)} \left\{ (g,\sigma,s) \mid s \in \mathrm{Con}^+(g,\sigma) \right\} \subset \mathrm{Mag}(M) \times [0, +\infty)$ (4.5)

- $\mathrm{Con}^-(M) := \bigcup_{(g,\sigma)} \left\{ (g,\sigma,s) \mid s \in \mathrm{Con}^-(g,\sigma) \right\} \subset \mathrm{Mag}(M) \times [0, +\infty)$ (4.6)

We now prove the following proposition about the sets we have just introduced.

PROPOSITION 4.4. *The sets* $\mathrm{Con}^+(M)$ *and* $\mathrm{Con}^-(M)$ *are open in the space* $\mathrm{Mag}(M) \times [0, +\infty)$. *As a consequence, the sets* $\mathrm{Con}^+(g,\sigma)$ *and* $\mathrm{Con}^-(g,\sigma)$ *are open in* $[0, +\infty)$.

If $s \in \mathrm{Con}^+(g,\sigma)$ *is* ν-*nondegenerate, then* $SH_\nu^*(DM, \omega_s)$ *is well defined. The isomorphism class of* SH_ν^* *is constant on the connected components of*

a) $\mathrm{Con}^+(M) \cap \left(\mathrm{Mag}_e(M) \times [0, +\infty) \right)$;

b) $\mathrm{Con}^+(M)$, *if* M *has positive genus;*

c) $\mathrm{Con}^+(S^2) \setminus \left(\mathrm{Mag}_e(S^2) \times [0, +\infty) \cup \mathrm{Mag}(S^2) \times \{0\} \right)$.

PROOF. Since a small perturbation of a contact form is still a contact form, the first statement readily follows.

Then, observe that (DM, ω_s) is convex if and only if $s \in \mathrm{Con}^+(g,\sigma)$. Since $c_1(\omega_s) = 0$ by Proposition 2.6, its Symplectic Cohomology is well defined as soon as s is non-degenerate. In order to prove the statement about its isomorphism class, we apply Theorem 3.30. We only need to check that the cohomology class $[\tau(\omega_s)] \in H^1(\mathscr{L}_\nu TM, \mathbb{R})$ is projectively constant in each of the three cases presented above. The base M is a deformation retraction of TM and under this deformation the cohomology class $[\omega_s] \in H^2(TM, \mathbb{R})$ is sent to $[s\sigma] \in H^2(M, \mathbb{R})$. Thus, we only need to check the analogous statement for $[\tau(\sigma)] \in H^2(\mathcal{L}_\nu M, \mathbb{R})$.

In case $a)$, σ is exact on M and, therefore, $[\tau(\sigma)] = 0$.

In case $b)$, we consider any ν-torus $u : \mathbb{T}^2 \to M$. Lemma 2.2 in [**Mer10**] implies that $u^*\sigma$ is an exact form. Therefore, $[\tau(\sigma)] = 0$.

In case $c)$, we only have the class of contractible loops. There are isomorphisms $\pi_1(\mathscr{L}S^2) \overset{\sim}{\to} \pi_2(S^2) \overset{\sim}{\to} H_2(S^2, \mathbb{Z}) \overset{\sim}{\to} \mathbb{Z}$. Hence, $H_1(\mathscr{L}S^2, \mathbb{Z}) \simeq \mathbb{Z}$ and consequently $[\tau] : H^2(S^2, \mathbb{R}) \to H^1(\mathscr{L}S^2, \mathbb{R})$ is an isomorphism. Consider two triples (g, σ, s) and (g', σ', s') such that s, s' are positive and σ, σ' are not exact. Then, $[\tau(\omega_{s\sigma})]$ and $[\tau(\omega_{s'\sigma'})]$ are the same up to a positive factor if and only if the same is true for $[\sigma]$ and $[\sigma']$. This happens if and only if the two magnetic systems lie in the same connected component of $\mathrm{Mag}(S^2) \setminus \Big(\mathrm{Mag}_e(S^2) \times [0, +\infty) \cup \mathrm{Mag}(S^2) \times \{0\} \Big)$. $\qquad\square$

In the next two subsections we are going to use the injections (4.1) and (4.2) in order to find *sufficient* conditions for an energy level to be of contact type. Thanks to Remark 2.11 and Remark 2.13, we just need to study the sign of the functions

$$\lambda_{s,\beta}(X^s)_{(x,v)} = (\lambda - s\pi^*\beta)(X + sfV)_{(x,v)}$$
$$= 1 - \beta_x(v)s \qquad (4.7)$$

for case **(E)** and the functions

$$\lambda^g_{s,\beta}(X^s)_{(x,v)} = \left(\lambda + \frac{s[\sigma]}{2\pi e_M}\tau - s\pi^*\beta \right)(X + sfV)_{(x,v)}$$
$$= 1 - \beta_x(v)s + \frac{[\sigma]f(x)}{2\pi e_M}s^2 \qquad (4.8)$$

for case **(NE)**.

Before performing this task, we single out some *necessary* conditions for ω'_s to be of contact type. Indeed, the next proposition shows that the normalised Liouville measure is a null-homologous invariant measure for X^s. Combining the computation of its action (see [**Pat09**]) and McDuff's criterion, contained in Proposition 2.17), we find the mentioned necessary conditions in Corollary 4.7.

PROPOSITION 4.5. *The normalised Liouville measure $\widetilde{\chi} := \frac{\chi}{2\pi[\mu]}$ belongs to the set $\mathfrak{M}(X^s)$. Its rotation vector is the Poincaré dual of $\frac{[\omega'_s]}{2\pi[\mu]}$. Therefore $\rho(\widetilde{\chi}) = 0$ if and only if ω'_s is exact. In case **(E)**, its action is given by*

$$\mathcal{A}^{\omega'_s}(\widetilde{\chi}) = 1. \qquad (4.9)$$

*In case **(NE)**, its action is given by*

$$\mathcal{A}^{\omega'_s}(\widetilde{\chi}) = 1 + \frac{[\sigma]^2}{2\pi e_M[\mu]}s^2. \qquad (4.10)$$

PROOF. If ζ is any 1-form on SM, then by (2.24)

$$(\imath_{X^s}\zeta)\widetilde{\chi} = \zeta \wedge (\imath_{X^s}\widetilde{\chi}) = \zeta \wedge \frac{\omega'_s}{2\pi[\mu]}. \qquad (4.11)$$

Integrating this equality over SM, we see that the $\rho(\widetilde{\chi})$ is the Poincaré dual of $\frac{[\omega'_s]}{2\pi[\mu]}$.

We now proceed to compute the action of $\widetilde{\chi}$. First, define $I : SM \to SM$ the flip $I(x,v) = (x, -v)$ and observe that $I^*\chi = \chi$. We readily see that

$$\int_{SM} \beta_x(v)\chi = \int_{SM} \beta_x(-v)I^*\chi = -\int_{SM} \beta_x(v)\chi.$$

Therefore, $\int_{SM} \beta_x(v)\chi = 0$.

Consider first case **(E)** and let $\lambda_{s,\beta}$ be a primitive of ω_s'. Then,

$$\mathcal{A}_{X^s}^{\omega_s'}(\widetilde{\chi}) = \int_{SM} \lambda_{s,\beta}(X^s)\,\widetilde{\chi} = \int_{SM} (1 - \beta_x(v)s)\,\widetilde{\chi} = 1.$$

Consider now case **(NE)** and let $\lambda_{s,\beta}^g$ be a primitive of ω_s'. Then,

$$
\begin{aligned}
\mathcal{A}_{X^s}^{\omega_s'}(\widetilde{\chi}) = \int_{SM} \lambda_{s,\beta}^g(X^s)\,\widetilde{\chi} &= \int_{SM} \left(1 - \beta_x(v)s + \frac{[\sigma]f(x)}{2\pi e_M}s^2\right)\widetilde{\chi} \\
&= 1 + \frac{[\sigma]}{2\pi e_M}s^2 \int_{SM} f(x)\,\widetilde{\chi} \\
&= 1 + \frac{[\sigma]}{2\pi e_M}s^2 \frac{1}{2\pi[\mu]}2\pi \int_M f(x)\,\mu \\
&= 1 + \frac{[\sigma]^2}{2\pi e_M[\mu]}s^2. \qquad\qquad \square
\end{aligned}
$$

The corollary below relates the action of $\widetilde{\chi}$ and the contact property. For the case **(NE)** on surfaces of genus at least two, we first need the following definition [**Mof69, AK98, Pat09**].

DEFINITION 4.6. Let (M, g, σ) be a non-exact magnetic system on a surface of genus at least 2. Define its **helicity** $s_h(g, \sigma) \in (0, +\infty)$ as

$$s_h(g,\sigma) := \sqrt{\frac{2\pi|e_M|[\mu]}{[\sigma]^2}} \tag{4.12}$$

COROLLARY 4.7. *For exact magnetic systems or for non-exact magnetic systems on S^2, ω_s' cannot be of negative contact type. Namely, $\mathrm{Con}^-(g,\sigma) = \emptyset$.*

For non-exact magnetic systems on a surface of genus at least 2, we have the inclusions

- $\mathrm{Con}^+(g,\sigma) \subset (0, s_h(g,\sigma))$ $\tag{4.13}$
- $\mathrm{Con}^-(g,\sigma) \subset (s_h(g,\sigma), +\infty).$ $\tag{4.14}$

In particular, for $s = s_h(g,\sigma)$, ω_s' is not of contact type.

4.1. Contact property for case (E)

The results of this section are classical and well-known by the experts, except possibly the computation of the subcritical Symplectic Cohomology on $\mathbb{T}^2$. Anyway, we decided to include them here in order to put case **(NE)** in a wider context.

From Corollary 4.7, we know that we only need to check when $\lambda_{s,\beta}$ is a positive contact form. Identity (4.7) implies that

$$\lambda_{s,\beta}(X^s)_{(x,v)} = 1 - \beta_x(v)s \geq 1 - |\beta_x|s \geq 1 - \|\beta\|s. \tag{4.15}$$

Therefore, $\lambda_{s,\beta}(X^s)$ is positive provided $s < \|\beta\|^{-1}$. Hence, there exists a $\beta \in \mathcal{P}^\sigma$ such that $(DM, \omega_s, \lambda_{s,\beta})$ is convex provided

$$s < s_0(g, \sigma) := \left(\inf_{\beta \in \mathcal{P}^\sigma} \|\beta\| \right)^{-1}.$$

REMARK 4.8. Observe that combining [**PP97**, Theorem 1.1] with [**CIPP98**, Theorem A], one finds that $s_0(g, \sigma)$ is the *Mañé critical value of the Abelian cover* of the Lagrangian function $L_\beta(x, v) := E(x, v) - \beta_x(v)$ after the reparametrisation $s \mapsto c(s) = \frac{1}{2s^2}$.

If we call $[0, s_1(g, \sigma))$ the connected component of $\mathrm{Con}^+(g, \sigma)$ containing 0. The above computation shows that $s_0(g, \sigma) < s_1(g, \sigma)$. By Proposition 4.4, for all ν-non-degenerate $s < s_1(g, \sigma)$, $SH^*_\nu(DM, \omega_s) \simeq SH^*_\nu(DM, \omega_0, j_0)$. The latter cohomology is known by the results of Viterbo [**Vit96**], Salamon-Weber [**SW06**] and Abbondandolo-Schwarz [**AS06**] and we get the following statement.

PROPOSITION 4.9. *If $s < s_1(g, \sigma)$ is ν-non-degenerate, then (DM, ω_s) is convex and*

$$SH^*_\nu(DM, \omega_s) \simeq H_{-*}(\mathscr{L}_\nu M, \Lambda). \tag{4.16}$$

REMARK 4.10. Observe that usually the isomorphism above is stated over $\mathbb{Z}$-coefficients. However, it holds true also with Λ-coefficients since Λ is torsion free and, therefore,

$$H_*(\mathscr{L}_\nu M, \Lambda) \simeq H_*(\mathscr{L}_\nu M, \mathbb{Z}) \otimes \Lambda, \quad SH^*_\nu(DM, \omega_s) \simeq SH^*_\nu(DM, \omega_s, \mathbb{Z}) \otimes \Lambda. \tag{4.17}$$

Here we have denoted by $SH^*_\nu(DM, \omega_s, \mathbb{Z})$ the direct limit of the cohomology of the complexes $(SC^*_\nu(DM, \omega_s, H, \mathbb{Z}), \delta^{\mathbb{Z}}_J)$, where $SC^*_\nu(DM, \omega_s, H, \mathbb{Z})$ is generated over $\mathbb{Z}$ by the 1-periodic orbitsof X_H and $\delta^{\mathbb{Z}}_J$ is defined as δ_J but without using the weights $t^{-\mathcal{A}^\omega_H(u)}$. Namely,

$$\delta^{\mathbb{Z}}_J x := \sum_{\substack{u \in \mathcal{M}(H, J, y, x) \\ |y| - |x| = 1}} \epsilon(u) y,$$

The map $\delta^{\mathbb{Z}}_J$ is a well-defined differential, as (DM, ω_s) is a Liouville domain.

The value of $s_1(g, \sigma)$ can be exactly estimated when $M \neq \mathbb{T}^2$ as the following proposition due to Contreras, Macarini and G. Paternain shows (see [**CMP04**]).

PROPOSITION 4.11. *If $s \geq s_0(g, \sigma)$, there exists an invariant measure ζ_s such that*

$$\bullet \ \pi_* \rho(\zeta_s) = 0 \in H_1(M, \mathbb{R}), \qquad \bullet \ \forall \beta \in \mathcal{P}^\sigma, \ \mathcal{A}^{\lambda_{s,\beta}}_{X^s}(\zeta_s) \leq 0. \tag{4.18}$$

Therefore, if $M \neq \mathbb{T}^2$, $\mathrm{Con}^+(g, \sigma) = [0, s_0(g, \sigma))$ and $s_1(g, \sigma) = s_0(g, \sigma)$.

If $M = \mathbb{T}^2$, ω'_s is not of restricted contact type for $s \geq s_0(g, \sigma)$. However there are examples for which $s_0(g, \sigma) < s_1(g, \sigma)$.

REMARK 4.12. It is an open question to determine whether on $\mathbb{T}^2$ $\mathrm{Con}^+(g, \sigma)$ is always connected, namely to see whether $[0, s_1(g, \sigma)) = \mathrm{Con}^+(g, \sigma)$, or not.

In [**CMP04**], the existence of a magnetic system which is of contact type at $s_0(g,\sigma)$ was proven using McDuff's sufficient criterion, which we stated in Proposition 2.17. The drawback of this method is that the criterion is not constructive, since it finally relies on an application of the Hahn-Banach Theorem. In the subsection below, we outline an explicit construction of the contact form for the kind of systems considered in [**CMP04**].

4.1.1. Mañé Critical values of contact type on $\mathbb{T}^2$. The key observation is that when $M = \mathbb{T}^2$, $\pi^* : H^1(M,\mathbb{R}) \to H^1(SM,\mathbb{R})$ is not surjective. To pick a class which is not in the image, just consider $Z : \mathbb{T}^2 \to S\mathbb{T}^2$ a section of the S^1-bundle $\pi : S\mathbb{T}^2 \to \mathbb{T}^2$. This yields the angular coordinate $\varphi_Z : T\mathbb{T}^2_0 \to \mathbb{T}_{2\pi}$. Therefore, $d\varphi_Z \in \Omega^1(T\mathbb{T}^2_0)$ is a closed form which is not in the image of π^*. If $r \in \mathbb{R}$, we consider the family of primitives

$$\mathcal{P}^\sigma \longrightarrow \mathcal{P}^{\omega'_s}$$
$$\beta \longmapsto \lambda^{r,Z}_{s,\beta} := \lambda - s\pi^*\beta + r d\varphi_Z, \tag{4.19}$$

which reduces to the class introduced in (4.1) for $r = 0$. Namely, $\lambda^{0,Z}_{s,\beta} = \lambda_{s,\beta}$.

Recall the definition of $\kappa^Z \in \Omega^1(\mathbb{T}^2)$ from Equation (2.9). Exploiting Formula (2.10), we get on $S\mathbb{T}^2$

$$\lambda^{r,Z}_{s,\beta}(X^s)_{(x,v)} = 1 - \beta_x(v)s + r(f(x)s - \kappa^Z_x(v))$$
$$\geq 1 - |\beta_x|s + r(f(x)s - |\kappa^Z_x|). \tag{4.20}$$

Now we are going to define a distinguished class of magnetic forms σ for which the right-hand side of (4.20) is positive for $s = s_0(g,\sigma)$.

Fix $t \mapsto \gamma(t)$ an embedded contractible closed curve on $\mathbb{T}^2$ parametrised by arc length. Suppose that its period is T_γ and that its geodesic curvature k_γ satisfies

$$k_\gamma(t) - |\kappa^Z_{\gamma(t)}| \geq \varepsilon, \tag{4.21}$$

for some $\varepsilon > 0$. Fix a $B \in \Gamma(\mathbb{T}^2)$ such that

(1) γ is an integral curve for B;
(2) if $M_B := \{x \in \mathbb{T}^2 \mid |B_x| = \|B\|\}$, then $M_B = \operatorname{supp}\gamma$ and $\|B\| = 1$.

Finally, let $\beta = \flat B$ and set $\sigma := d\beta$.

We claim that $f(\gamma(t)) = k_\gamma(t)$. Indeed,

$$f|B|^2 = f\mu(B, \jmath B) = d(\flat B)(B, \jmath B) \overset{(*)}{=} B(\flat B(\jmath B)) - \jmath B(\flat B(B)) - (\flat B)([B, \jmath B])$$

$$\overset{(**)}{=} 0 - \jmath B(|B|^2) - g(B, \nabla_B \jmath B - \nabla_{\jmath B} B)$$

$$= -\jmath B(|B|^2) + g(\nabla_B B, \jmath B) - \frac{1}{2}\jmath B(|B|^2)$$

$$= |B|^3 k_B - \frac{3}{2}\jmath B(|B|^2),$$

where we used Cartan formula for the exterior derivative in $(*)$ and the symmetry of the Levi-Civita connection in $(**)$. Remember that k_B is the geodesic curvature of B defined in (2.12). The claim follows noticing that on $\operatorname{supp}\gamma$

- $|B| = \|B\| = 1$;
- $k_B = k_\gamma$ since γ is an integral line for B;
- $\jmath B(|B|^2) = 0$, because the function $|B|^2$ attains its global maximum there.

As a by-product, we obtain that $t \mapsto (\gamma(t), \dot{\gamma}(t) = B_{\gamma(t)}) \in S\mathbb{T}^2$ is a periodic orbit for X^1 because γ satisfies the magnetic equation (2.16).

We claim that $s_0(g, \sigma) = \|B\| = 1$. If we take any $\beta' \in \mathcal{P}^\sigma$,

$$\int_\gamma \beta' = \int_\gamma \beta = \int_0^{T_\gamma} \beta_{\gamma(t)}(\dot{\gamma}(t))dt = T_\gamma. \tag{4.22}$$

On the other hand,

$$\int_\gamma \beta' = \int_0^{T_\gamma} \beta'_{\gamma(t)}(\dot{\gamma}(t))dt \leq T_\gamma \|\beta'\|, \tag{4.23}$$

therefore, $\|\beta'\| \geq \|\beta\| = 1$. So, the infimum in the definition of $s_0(g, \sigma)$ is equal to 1, it is a minimum and it is attained at β. Hence, the claim is proven.

We can now estimate from below (4.20). When $s = 1$,

$$\lambda_{1,\beta}^{r,Z}(X^\sigma) \geq (1 - |\beta_x|) + r(f(x) - |\kappa_x^Z|). \tag{4.24}$$

The right-hand side is the sum of two pieces:

i) $r(f(x) - |\kappa_x^Z|)$. When $r > 0$, this quantity is bigger than $r \cdot \frac{\varepsilon}{2}$ on a neighbourhood U of supp γ;

ii) $1 - |\beta_x|$. This quantity is strictly positive outside supp γ and it vanishes on it. In particular, there exists $\delta > 0$ such that $1 - |\beta_x| \geq \delta$ on $\mathbb{T}^2 \setminus U$.

This means that the right-hand side is positive on U and on $\mathbb{T}^2 \setminus U$ as soon as

$$0 < r < \frac{\delta}{\max\{0, \sup_{x \notin U} |\kappa_x^Z| - f(x)\}}. \tag{4.25}$$

Therefore, for r in this range, we see that $(D\mathbb{T}^2, \omega_\sigma, \lambda_{1,\beta}^{r,Z})$ is convex. We conclude that $1 \in \mathrm{Con}^+(g, \sigma)$ and $s_1(g, \sigma) > s_0(g, \sigma) = 1$.

4.2. Contact property for case (NE)

To ease the notation, in this section we suppose that the magnetic form σ is rescaled by a constant factor, in such a way that $[\sigma] = 2\pi e_M$. This operation will only induce a corresponding rescaling of the parameter s and, hence, will not affect our study. We begin with two easy examples and then we move to the general discussion subdivided in the subsections below.

EXAMPLE 4.13. Denote by g_0 the metric of curvature $|K_{g_0}| = 1$ on M and consider the magnetic systems (M, g_0, μ_{g_0}). We see that $\mu_{g_0} - \frac{[\mu_{g_0}]}{2\pi e_M}\sigma_{g_0} = 0$. Hence, we can choose $\beta = 0$ and the corresponding family of primitives $\lambda_{s,0}^{g_0}$.

If $M = S^2$, then (4.8) yields $\lambda_{s,0}^{g_0}(X^s) = 1 + s^2$. Thus, $\mathrm{Con}^+(g_0, \mu_{g_0}) = [0, +\infty)$.

If M is a surface of genus at least 2, (4.8) reduces to $\lambda_{s,0}^{g_0}(X^s) = 1 - s^2$. Thus, $\mathrm{Con}^+(g_0, \mu_{g_0}) = [0, 1)$ and $\mathrm{Con}^-(g_0, \mu_{g_0}) = (1, +\infty)$. Notice that in this case, $s_h(g_0, \mu_{g_0}) = 1$ is the only value of the parameter which is not of contact type.

EXAMPLE 4.14. Consider a convex two-sphere. Namely, we endow S^2 with a metric g of positive curvature. Take the magnetic system $(S^2, g, \sigma_g = K\mu)$. As in the example above, we can choose $\beta = 0$ and the family of primitives $\lambda^g_{s,0}$. In this case, (4.8) reduces to $\lambda^g_{s,0}(X^s) = 1 + Ks^2 > 0$. Thus, $\mathrm{Con}^+(g, \sigma_g) = [0, +\infty)$. This is one of the hints to Conjecture E.

4.2.1. High energy levels. Let us start by checking when $s \in \mathrm{Con}^+(g, \sigma)$. Identity (4.8) implies that

$$\lambda^g_{s,\beta}(X^s)_{(x,v)} = 1 - \beta_x(v)s + f(x)s^2 \geq 1 - \|\beta\|s + \inf f s^2 \qquad (4.26)$$

Consider the quadratic equation $1 - \|\beta\|s + \inf f s^2 = 0$. If $\inf f \leq 0$, we have only one positive root. If $\inf f > 0$ we either have two positive roots or two non-real roots. In any case, call $s_-(g, \sigma, \beta)$ the smallest positive root and set $s_-(g, \sigma, \beta) = +\infty$ if the equation does not have real roots. Observe that $1 - \|\beta\|s + \inf f s^2 > 0$ for $s \in [0, s_-(g, \sigma, \beta))$, no matter the sign of $\inf f$. We write

$$s_-(g, \sigma) := \sup_{\beta \in \mathcal{P}^{\sigma - \sigma_g}} s_-(g, \sigma, \beta). \qquad (4.27)$$

Notice that $s_-(g, \sigma)$ is the smallest positive root (with the same convention as before if there are no real roots) of the quadratic equation $1 - m(g, \sigma)s + \inf f s^2 = 0$, where

$$m(g, \sigma) := \inf_{\beta \in \mathcal{P}^{\sigma - \sigma_g}} \|\beta\|. \qquad (4.28)$$

Thus, we conclude that there exists $\bar{s}_-(g, \sigma) \geq s_-(g, \sigma)$ such that $[0, \bar{s}_-(g, \sigma))$ is the connected component of $\mathrm{Con}^+(g, \sigma)$ containing 0.

Proposition 4.4 implies the following corollary for surfaces of high genus.

COROLLARY 4.15. *If M is a surface of genus at least 2 and $s < \bar{s}_-(g, \sigma)$ is ν-non-degenerate, then (DM, ω_s) is convex and*

$$SH^*_\nu(DM, \omega_s) \simeq SH^*_\nu(DM, \omega_0) \simeq H_{-*}(\mathscr{L}_\nu M, \Lambda). \qquad (4.29)$$

We now deal with the case of S^2. If we fix $s < s_-(g, \sigma)$, there exists $\beta \in \mathcal{P}^{\sigma - \sigma_g}$ such that $s < s_-(g, \sigma, \beta)$. Consider the Riemannian metric g_0 with $K^{g_0} = 1$ and let $g_r := rg + (1 - r)g_0$ be the linear interpolation between g and g_0. Define a corresponding family of $\beta_r \in \mathcal{P}^{\sigma - \sigma_{g_r}}$ such that $\beta_1 = \beta$. By compactness of the interval,

$$\hat{s}_-(g, \sigma, \beta) := \inf_{r \in [0,1]} s_-(g_r, \sigma, \beta_r)$$

is positive.

Take $s_* \leq s$ such that $0 < s_* < \hat{s}_-(g, \sigma, \beta)$ and consider the deformation

$$s' \in [s_*, s], \quad s' \mapsto (DS^2, \omega_{s'\sigma}, \lambda^g_{s',\beta}).$$

Since $s' \leq s$, the boundary stays convex for every s'. Moreover, the deformation is projectively constant because $s' > 0$. Then, define a second deformation

$$r \in [0, 1], \quad r \mapsto (D^{g_r}S^2, \omega^{g_r}_{s_*\sigma}, \lambda^{g_r}_{s_*,\beta_r}),$$

where we have made explicit the dependence of the disc bundle and the symplectic form on the metric using the superscript g_r. Since $s_* < \hat{s}_-(g, \sigma, \beta)$, the boundary remains convex also in this case. Moreover, the cohomology class of $\omega_{s_*\sigma}^{g_r}$ does not change.

Finally, consider a last deformation with a parameter $u \in [0, 1]$. Define the 1-parameter family of 2-forms $\sigma_u = u\sigma + (1-u)\sigma_{g_0} = \sigma_{g_0} + ud\beta_0$ and take the corresponding family

$$u \mapsto (D^{g_0}S^2, \omega_{s_*\sigma_u}^{g_0}, \lambda_{s_*,u\beta_0}^{g_0}).$$

Let $f_u^{g_0}$ be the function such that $\sigma_u = f_u^{g_0}\mu_{g_0}$. Then, $f_u^{g_0} = uf_1^{g_0} + (1-u)K^{g_0}$ and $\inf f_u^{g_0} = u\inf f_{g_0} + (1-u)K^{g_0} \geq \inf f_1^{g_0}$ (observe that $f_u^{g_0}$ and K^{g_0} have the same integral over M). Therefore, $s_* < \hat{s}_-(g, \sigma, \beta) \leq s_-(g_0, \sigma, \beta_0) \leq s_-(g_0, \sigma_u, u\beta_0)$, for every $u \in [0, 1]$. Hence, the boundary stays convex also during this last deformation and the cohomology class of $\omega_{s_*\sigma_u}^{g_0}$ does not change since $[s_*\sigma_u]$ is fixed.

Summing up, we have shown that for $s < s_-(g, \sigma)$, (g, σ, s) and (g_0, σ_{g_0}, s_*) lie in the same connected component of $\mathrm{Con}^+(S^2) \backslash \left(\mathrm{Mag}_e(S^2) \times [0, +\infty) \cup \mathrm{Mag}(S^2) \times \{0\} \right)$, for any $s_* > 0$ (remember that in Example 4.13, we proved that $s_-(g_0, \sigma_{g_0}) = +\infty$). We can now apply Proposition 4.4.

COROLLARY 4.16. *If $M = S^2$ and $0 < s < \overline{s}_-(g, \sigma)$ is non-degenerate, then (DS^2, ω_s) is convex and*

$$SH^*(D^gS^2, \omega_{s\sigma}^g) \simeq SH^*(D^{g_0}S^2, \omega_{s_*\sigma_{g_0}}^{g_0}), \tag{4.30}$$

where g_0 is the metric with constant curvature 1 and s_* is any positive number.

In the next section, we are going to compute the right-hand side of (4.30). In the remainder of this section we are going to discuss other two cases where we can use Identity (4.8) to find energy levels of contact type. The original idea is contained in [**Pat09**, Remark 2.2].

4.2.2. Low energy levels on S^2. Let us go back to Inequality (4.26) and see what happens when the polynomial $1 - s\|\beta\| + s^2 \inf f$ has a second positive root $s_+^+(g, \sigma, \beta)$. We observed that this happens if and only if $\inf f > 0$. Because of the normalisation we made at the beginning of this section, we see that this can happen only if σ is a *symplectic* form on S^2. Under these hypotheses, we have that the right-hand side of (4.26) is positive for $s \in s_+^+(g, \sigma, \beta)$. As before we denote

$$s_+^+(g, \sigma) = \inf_{\beta \in \mathcal{P}^{\sigma - \sigma_g}} s_+^+(g, \sigma, \beta), \tag{4.31}$$

which can also be defined as the biggest positive root of $1 - m(g, \sigma)s + \inf f s^2 = 0$.

We conclude that there exists $\overline{s}_+^+(g, \sigma) \leq s_+^+(g, \sigma)$ such that $(\overline{s}_+^+(g, \sigma), +\infty)$ is the unbounded connected component of $\mathrm{Con}^+(g, \sigma)$.

COROLLARY 4.17. *Let $M = S^2$ and σ be a symplectic form. If $s > \overline{s}_+^+(g, \sigma)$ is non-degenerate, then (DS^2, ω_s) is convex and*

$$SH^*(D^gS^2, \omega_{s\sigma}^g) \simeq SH^*(D^{g_0}S^2, \omega_{s_*\sigma_{g_0}}^{g_0}), \tag{4.32}$$

where g_0 is the metric with constant curvature 1 and s_* is any positive number.

Again, we refer to the next section for the computation of the right-hand side.

4.2.3. Low energy levels on a surface of high genus. We now want to investigate levels of negative contact type. Thanks to Corollary 4.7, they can arise only on a surface of genus at least 2. Let us bound $\lambda^g_{s,\beta}(X^s)$ from above using (4.8):

$$\lambda^g_{s,\beta}(X^s)_{(x,v)} = 1 - \beta_x(v)s + f(x)s^2 \leq 1 + \|\beta\|s + \sup f s^2.$$

If $\sup f \geq 0$ the right-hand side cannot be less than zero. If $\inf f < 0$, there exists a unique positive root of the associated quadratic equation $s^-_+(g,\sigma,\beta)$ such that the right-hand side is less than zero, for every $s > s^-_+(g,\sigma,\beta)$. Denote

$$s^-_+(g,\sigma) := \inf_{\beta \in \mathcal{P}^{\sigma-\sigma_g}} s^-_+(g,\sigma,\beta), \tag{4.33}$$

which can also be defined as the unique positive root of $1 + m(g,\sigma)s - \inf f s^2 = 0$. Therefore, we conclude that there exists $\overline{s}^-_+(g,\sigma) \leq s^-_+(g,\sigma)$ such that $(\overline{s}^-_+(g,\sigma), +\infty)$ is the unbounded connected component of $\mathrm{Con}^-(g,\sigma)$.

We cannot compute Symplectic Cohomology for compact concave symplectic manifolds (namely, for symplectic manifolds whose boundary is of negative contact type). A possible solution to this problem would be to consider only invariants of the boundary such as (embedded) contact homology. Another possibility would be to look for a compact convex symplectic manifold (W, ω) that could be used to cap off (DM, ω_s) from outside in order to form a closed symplectic manifold $(DM \sqcup_{SM} W, \omega_s \sqcup_{SM} \omega)$. We plan to deal with this second approach in a future research project.

REMARK 4.18. Thanks to Corollary 4.7, we have the chain of inequalities

$$0 < \overline{s}_-(g,\sigma) \leq s_h(g,\sigma) \leq \overline{s}^-_+(g,\sigma). \tag{4.34}$$

Moreover, G. Paternain proved in [**Pat09**] that

$$0 < s_c(g,\sigma) \leq s_h(g,\sigma), \tag{4.35}$$

where

$$s_c(g,\sigma) = \inf_{\widetilde{\theta} \in \mathcal{P}^{\widetilde{\sigma}}} \|\widetilde{\theta}\|_{\widetilde{g}}$$

is the critical value of the universal cover after the reparametrisation $c(s) = \frac{1}{2s^2}$. It is an open problem to study the relation between $\overline{s}_-(g,\sigma)$ and $s_c(g,\sigma)$.

4.3. Symplectic Cohomology of a round sphere

In this subsection we compute the Symplectic Cohomology of $(DS^2, \omega_{s\sigma_{g_0}})$, for $s > 0$, when g_0 is the metric of constant curvature 1.

First, we look for a primitive $\hat{\lambda}^{g_0}_{s,0}$ of $\omega_{s\sigma_{g_0}}$ on the whole TS^2_0 which extends $\lambda^{g_0}_{s,0} = \lambda + s\tau \in \Omega^1(SS^2)$. Using Identity (2.6), we readily find $\hat{\lambda}^{g_0}_{s,0} = \lambda + s\frac{\tau}{2E}$. Integrating the Liouville vector field associated to $\hat{\lambda}^{g_0}_{s,0}$ starting from SS^2 yields a

convex neighbourhood $j_{s,0}^{g_0} = (r_s, p_s) : TS_0^2 \hookrightarrow \mathbb{R} \times SS^2$ such that $SS^2 = \{r_s = 0\}$. Let us differentiate E with respect to this coordinate:

$$\frac{dE}{dr_s} = dE(\partial_{r_s}) = \omega_{s\sigma_{g_0}}(\partial_{r_s}, X_E^{s\sigma_{g_0}}) = \hat{\lambda}_{s,0}^{g_0}(X_E^{s\sigma_{g_0}}) = 2E + s^2.$$

Dividing both sides by $2E + s^2$ and integrating between 0 and r_s, we get

$$r_s = \log \sqrt{\frac{2E + s^2}{1 + s^2}}, \tag{4.36}$$

or, using the auxiliary variable, $R_s = e^{r_s}$,

$$R_s = \sqrt{\frac{2E + s^2}{1 + s^2}}.$$

From this equation we see two things. First, that R_s (or equivalently r_s) is also smooth at the zero section and, hence, it can be extended to a smooth function on the whole tangent bundle. Second, that the image of $j_{s,0}^{g_0}$ is $\left(\log\left(\frac{s}{\sqrt{1+s^2}}\right), +\infty\right) \times SS^2$. In particular, the flow of $\partial_{r_s} \in \Gamma(TS_0^2)$ is positively complete and, therefore, $j_{s,0}^{g_0}$ is actually a cylindrical end. Thus, $(TS^2, \omega_{s\sigma_{g_0}}, \hat{\lambda}_{s,0}^{g_0})$ is the symplectic completion of $(DS^2, \omega_{s\sigma_{g_0}}, \lambda_{s,0}^{g_0})$.

Let us look now at the dynamics on SS^2. We claim that all orbits are periodic and the prime orbits have all the same minimal period $T_s = \frac{2\pi}{\sqrt{1+s^2}}$. The claim can be proven either by finding explicitly all the curves with constant geodesic curvature (these are the boundaries of geodesic balls) or by using Gray Stability Theorem. The latter strategy yields an isotopy $F_{s'} : SS^2 \to SS^2$, with $s' \in [0, s]$, such that

$$F_{s'}^*(\lambda + s'\tau) = \frac{1}{\sqrt{1 + (s')^2}}\lambda.$$

Since the Reeb vector field corresponding to λ is X, whose orbits are all periodic and with minimal period 2π, the claim follows.

We are ready to compute the Symplectic Cohomology. Take as a family of increasing Hamiltonian, linear at infinity, the functions $H_s^k := k\sqrt{1 + s^2}R_s$, with $k \in \mathbb{N}$. The associated Hamiltonian flow generates an S^1-action on TS^2 with period $2\pi k$. Hence, the only 1-periodic orbits of the flow are the constant orbits, which lie on the zero section. Let us compute their Conley-Zehnder index (for the computation in the Lagrangian setting we refer to [**FMP13**, Lemma 5.4]).

The linearisation of the flow $d\Phi_t^{H_s^k} : T_{(x,0)}TS^2 \to T_{(x,0)}TS^2$ can be described as follows. First, we choose standard coordinates on TS^2, close to the point $(x, 0)$. Then, we compute the differential of the vector field $X_{H_s^k}$ at $(x, 0)$ using these coordinates. We find the matrix

$$d_{(x,0)}X_{H_s^k} = k \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & s\jmath_x \end{pmatrix}$$

Since $d_{(x,0)}\Phi_t^{H_s^k} = \exp(t \cdot d_{x_0}X_{H_s^k})$ (here exp denote the exponential of a linear endomorphism), the eigenvalues of the linearisation are 1 and e^{ikst}, both with algebraic multiplicity 2. Observe that since 1 is an eigenvalue, $(x, 0)$ is degenerate (actually,

the zero section form a Morse-Bott component of critical points since the eigenvalue in the normal directions is e^{ikst}, which is different from 1 when t is positive). For this reason, we take the lower semi-continuous extension of the index and find

$$\mu^l_{\mathrm{CZ}}(x,0) = 2\left\lfloor\frac{ks}{2\pi}\right\rfloor + 1 + (-1) = 2\left\lfloor\frac{ks}{2\pi}\right\rfloor,$$

where we used the additivity of the Conley-Zehnder index under direct products.

Consider now a time-dependent compact perturbation H^k_{s,δ_k} of H^k_s such that all the 1-periodic orbits of the new system are non-degenerate. Here $\delta_k > 0$ is a perturbative parameter that we take arbitrarily small depending on k. We claim that the direct limit for $k \to +\infty$ of the symplectic cohomology groups with Hamiltonian H^k_{s,δ_k} is zero. Indeed, let γ be a 1-periodic orbit of the Hamiltonian system associated to H^k_{s,δ_k}. Since δ_k is small, γ is close to a constant solution on the zero section. By the lower semicontinuity of the index, $|[\gamma]| = 2 - \mu_{\mathrm{CZ}}(\gamma) \leq 2 - 2\left\lfloor\frac{ks}{2\pi}\right\rfloor$. Therefore, the symplectic cohomology with Hamiltonian H^k_{s,δ_k} is zero in degree bigger than $2 - 2\left\lfloor\frac{ks}{2\pi}\right\rfloor$. Since $2 - 2\left\lfloor\frac{ks}{2\pi}\right\rfloor \to -\infty$ as $k \to +\infty$, the direct limit is zero in every degree. Thus, we have proven the following proposition.

PROPOSITION 4.19. *For every $s > 0$, there holds $SH^*(DS^2, \omega_{s\sigma_{g_0}}) = 0$.*

COROLLARY 4.20. *If (S^2, g, σ) is a non-exact magnetic system and s is a non-degenerate parameter such that $s < \hat{s}_-(g, \sigma)$ or $s > \hat{s}^+_+(g, \sigma)$, then*

$$SH^*(DS^2, \omega_s) = 0. \tag{4.37}$$

4.4. Lower bound on the number of periodic orbits

The computation of Symplectic Cohomology we performed in the previous sections can be used to prove the existence of periodic orbits on SM.

PROPOSITION 4.21. *Let (S^2, g, σ) be an exact magnetic system. If $s < s_0(g, \sigma)$, then $X^s \in \Gamma(SS^2)$ has at least one periodic orbit. If the iterates of such orbit are transversally non-degenerate, then there is another geometrically distinct periodic orbit on SS^2.*

PROOF. From [**Zil77**], we know the singular homology with $\mathbb{Z}$-coefficients of the free loop space of S^2. It is zero in negative degree and, for $k \in \mathbb{N}$, we have

$$H_k(\mathscr{L}S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k \text{ is odd,} \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is even and positive.} \end{cases} \tag{4.38}$$

Since $H_{-*}(\mathscr{L}S^2, \mathbb{Z}) \neq H^{*+2}(S^2, \mathbb{Z})$, we have $H_{-*}(\mathscr{L}S^2, \Lambda) \neq H^{*+2}(S^2, \Lambda)$ by Remark 4.10. Therefore there exists at least one prime periodic orbit γ by Corollary 3.21. Call T its period.

Suppose that γ and all its iterates are transversally non-degenerate. Construct inductively a sequence $k \mapsto H^k \in \hat{\mathcal{H}}'_0$ (see Definition 3.18) on the completion $\widehat{DS^2}$ of the disc bundle, so that

- $T_{H^k} < T_{H^{k+1}}$;

- $kT < T_{H^k}$;
- $H^k = H^{k+1}$ on $\widehat{DS^2}^{c_k}$, where $kT = (h^k)'(e^{c_k}) = (h^{k+1})'(e^{c_k})$.

Take small perturbations $H^k_\varepsilon \in \hat{\mathcal{H}}_0$ and let J^k be a H^k_ε-admissible almost complex structure. Define $SC^*(k) := SC^*_0(DS^2, \omega_s, H^k_\varepsilon)$, let δ_k be the Floer differentials $\delta_k : SC^*(k) \to SC^*(k)$, let $SH^*(k)$ be the associated cohomology groups and let $\varphi_k : SC^*(k) \to SC^*(k)$ be the continuation maps. Denote by $SH^*(\infty)$ the direct limit of $SH^*(k)$. For every $\tilde{k} = 1, \ldots, k$, we have two non-constant generators of $SC^*(k)$: $\gamma^{\tilde{k}}_{\min}$ and $\gamma^{\tilde{k}}_{\mathrm{Max}}$. They have degree $2 - \mu_{\mathrm{CZ}}(\gamma^{\tilde{k}})$ and $2 - \mu_{\mathrm{CZ}}(\gamma^{\tilde{k}}) - 1$, respectively. By Corollary 3.20, we know that φ_k is the inclusion map.

Since $SH^*(\infty)$ is non-zero in arbitrarily low degree, $\mu_{\mathrm{CZ}}(\gamma^k) > 0$ for every $k \geq 1$, by the properties of the index. We distinguish three cases.

a) $\mu_{\mathrm{CZ}}(\gamma) \geq 2$. If this happens, then for every $\tilde{k}$, $\mu_{\mathrm{CZ}}(\gamma^{\tilde{k}+1}) - \mu_{\mathrm{CZ}}(\gamma^{\tilde{k}}) > 1$. Hence, for $* \leq -2$, there can be at most one generator for $SC^*(k)$. This contradicts the fact that $SH^{-4}(\infty) \simeq \Lambda \oplus \Lambda/2\Lambda$.

b) $\mu_{\mathrm{CZ}}(\gamma) = 1$ and γ is hyperbolic. If $k > 1$, $SC^1(k)$ is generated by $\gamma_{\min}$ and $SC^0(k)$ generated by γ_{Max}, $\gamma^2_{\min}$ and p, where p is the image under the map $C^2(S^2, \Lambda) \to SC^2(k)$ of a generating cochain in $C^2(S^2, \Lambda) \simeq \Lambda$. We claim that δ_k is zero in degree 1 and 0. This would imply that the class of $\gamma_{\min}$ is non-trivial in $SH^1(k)$. Together with the fact that φ_k is the inclusion, we get a contradiction to the fact that $SH^1(\infty) = 0$.

Since there are no generators in degree 2, it is clear that δ_k is zero in degree 1. Since γ is a good orbit, $\delta_k \gamma_{\mathrm{Max}} = 0$ and since p is the image of a cochain under a chain map, we also have $\delta_k p = 0$. Finally, we compute $0 = \delta^2_k \gamma^2_{\mathrm{Max}} = \delta_k(2t^a \gamma^2_{\min})$, which implies that $\delta_k \gamma^2_{\min} = 0$ as well.

c) $\mu_{\mathrm{CZ}}(\gamma) = 1$ and γ is elliptic. Under these hypotheses, there exists a maximum $k_0 \geq 1$ such that $\mu_{\mathrm{CZ}}(\gamma^{k_0}) = 1$. If $k \geq k_0$, $SC^1(k)$ is generated by the elements $\gamma^{\tilde{k}}_{\min}$, with $\tilde{k} \in [1, k_0]$ and $SC^0(k)$ is generated by p and the elements $\gamma^{\tilde{k}}_{\mathrm{Max}}$, with $\tilde{k} \in [1, k_0]$. As the action $A^{\omega_s}_{H^k_\varepsilon}$ is decreasing with the period, we know that $\gamma^{k_0}_{\min}$ has action smaller than $\gamma^{\tilde{k}}_{\mathrm{Max}}$, for $\tilde{k} < k_0$. By Corollary 3.19, $< \delta_k \gamma^{\tilde{k}}_{\mathrm{Max}}, \gamma^{k_0}_{\min} >= 0$. Moreover, $\delta_k p = 0$ and, since γ^{k_0} is a good orbit, also $< \delta_k \gamma^{k_0}_{\mathrm{Max}}, \gamma^{k_0}_{\min} >= 0$. We conclude that $\gamma^{k_0}_{\min}$ is a cochain, which is not a coboundary in $SC^1(k)$. Together with the fact that φ_k is the inclusion, we get a contradiction to the fact that $SH^1(\infty) = 0$.

Since each case leads to a contradiction, we have proved that there exists a periodic orbit geometrically distinct from γ. $\qquad\square$

PROPOSITION 4.22. *Let $(\mathbb{T}^2, g, \sigma)$ be an exact magnetic system. If $s < s_1(g, \sigma)$, then $X^s \in \Gamma(S\mathbb{T}^2)$ has at least one periodic orbit in every non-trivial free homotopy class. If such periodic orbit is transversally non-degenerate, then there is another periodic orbit in the same class.*

PROOF. If there are no periodic orbits $SH^*_\nu(D\mathbb{T}^2, \omega_s) = 0$. If we only have one periodic orbit and such orbit is non degenerate, then $\mathrm{rk}\, SH^*_\nu(D\mathbb{T}^2, \omega_s) \in \{0, 2\}$.

In any case $\operatorname{rk} SH_\nu^*(D\mathbb{T}^2, \omega_s) \leq 2$. On the other hand, we claim that $\mathscr{L}_\nu \mathbb{T}^2$ is homotopy equivalent to $\mathbb{T}^2$. If γ_ν is a reference loop in the class ν, the two homotopy equivalences are given by

$$\mathbb{T}^2 \longrightarrow \mathscr{L}_\nu \mathbb{T}^2 \qquad\qquad \mathscr{L}_\nu \mathbb{T}^2 \longrightarrow \mathbb{T}^2$$
$$x \longmapsto \gamma_{\nu,x}(t) := \gamma_\nu(t) + x; \qquad\qquad \gamma \longmapsto \gamma(0).$$

Thanks to Proposition 4.9, this leads to the contradiction

$$2 \geq \operatorname{rk} SH_\nu^*(D\mathbb{T}^2, \omega_s) = \operatorname{rk} H_{-*}(\mathscr{L}_\nu \mathbb{T}^2, \Lambda) = \operatorname{rk} H_{-*}(\mathbb{T}^2, \Lambda) = 4. \qquad \square$$

PROPOSITION 4.23. *Let (M, g, σ) be a magnetic system on a surface of genus at least two. If σ is exact and $s < s_0(g, \sigma)$ or σ is not exact and $s < \hat{s}_-(g, \sigma)$, then $X^s \in \Gamma(SM)$ has at least one periodic orbit in every non-trivial free homotopy class.*

PROOF. If there are no orbits in the class $\nu \neq 0$, $SH_\nu^*(DM, \omega_s) = 0$. However, $\mathscr{L}_\nu M$ is homotopy equivalent to S^1, if ν is non-trivial. Thanks to Proposition 4.9 or Corollary 4.15, this leads to the contradiction

$$0 = SH_\nu^*(DM, \omega_s) = H_{-*}(\mathscr{L}_\nu M, \Lambda) = H_{-*}(S^1, \Lambda) \neq 0. \qquad \square$$

PROPOSITION 4.24. *Let (S^2, g, σ) be a non-exact magnetic system. If the parameter s belongs to $(0, \overline{s}_-(g, \sigma)) \cup (\overline{s}_+^+(g, \sigma), +\infty)$, then $X^s \in \Gamma(SS^2)$ has at least one periodic orbit. If the iterates of such periodic orbit are transversally non-degenerate, then there is another geometrically distinct periodic orbit on SS^2.*

PROOF. The proof is similar to the one of Proposition 4.21 and, therefore, we only discuss the parts where we have to argue differently from there.

Since $SH^*(DS^2, \omega_s) = 0 \neq H^{*+2}(S^2, \Lambda)$, there exists at least one prime periodic orbit γ by Corollary 3.27. Call T its period.

Suppose that γ and all its iterates are transversally non-degenerate. Construct the sequences H^k and H_ε^k as in Proposition 4.21 and let J_ε^k be a sequence in $\mathcal{J}(H^k)$ (see Definition 3.22). Define $SC^*(k)$, δ_k, $SH^*(k)$, φ_k and $SH^*(\infty)$ as before.

For every $\tilde{k} = 1, \ldots, k$, we have two non-constant generators of $SC^*(k)$: $\gamma_{\min}^{\tilde{k}}$ and $\gamma_{\text{Max}}^{\tilde{k}}$. They have degree $2 - \mu_{\text{CZ}}(\gamma^{\tilde{k}})$ and $2 - \mu_{\text{CZ}}(\gamma^{\tilde{k}}) - 1$, respectively. By Corollary 3.26, we know that φ_k is the inclusion map.

If $\mu_{\text{CZ}}(\gamma^k) \leq 0$ for every $k \geq 1$, then the degrees of the non-constant periodic orbits are all positive. By Corollary 3.27, this means that $SH^{-2}(\infty) \simeq H^0(S^2, \Lambda)$, which is a contradiction. Therefore, we assume that $\mu_{\text{CZ}}(\gamma^k) > 0$, for every $k \geq 1$. We distinguish four cases.

a) $\mu_{\text{CZ}}(\gamma) \geq 3$. We claim that there exists k_0 such that $\mu_{\text{CZ}}(\gamma^{k_0+1}) - \mu_{\text{CZ}}(\gamma^{k_0}) > 2$ and γ^{k_0} is good. This implies that $\gamma_{\text{Max}}^{k_0}$ represents a non-zero class on $SC^*(k)$, for $k \geq k_0$. Taking the direct limit, we conclude that $SH^*(\infty) \neq 0$, contradicting our assumption. Thus, we only need to prove the claim.

We use the iteration formula for the index (see [**Lon02**]):

$$\mu_{\text{CZ}}(\gamma^k) = \begin{cases} 2\lfloor k\vartheta \rfloor + 1, \ \vartheta \in (0, +\infty) \setminus \mathbb{Q}, & \text{if } \gamma \text{ is elliptic}; \\ k\mu_{\text{CZ}}(\gamma), & \text{if } \gamma \text{ is hyperbolic}. \end{cases}$$

If γ is hyperbolic, then $\mu_{\mathrm{CZ}}(\gamma^{k+1}) - \mu_{\mathrm{CZ}}(\gamma^k) = \mu_{\mathrm{CZ}}(\gamma) > 2$ and the claim is satisfied by any good iterate of γ. If γ is elliptic, then $\vartheta > 1$ and

$$\mu_{\mathrm{CZ}}(\gamma^{k+1}) - \mu_{\mathrm{CZ}}(\gamma^k) = 2 + 2\left(\lfloor (k+1)(\vartheta - 1) \rfloor - \lfloor k(\vartheta - 1) \rfloor\right).$$

Therefore, there exists k_0 such that $\lfloor (k_0 + 1)(\vartheta - 1) \rfloor - \lfloor k_0(\vartheta - 1) \rfloor \geq 1$. For such k_0 we have $\mu_{\mathrm{CZ}}(\gamma^{k_0+1}) - \mu_{\mathrm{CZ}}(\gamma^{k_0}) > 2$. Since all the iterates of an elliptic orbit are good, the claim is proven also in this case.

b) $\mu_{\mathrm{CZ}}(\gamma) = 2$. In this case $\gamma_{\min}$ has degree zero and $SC^*(k) = 0$ for $* > 0$. Therefore, $\delta_k \gamma_{\min} = 0$. Moreover, $SC^{-1}(k)$ is generated only by γ_{Max} and since γ is a good orbit, we know that $< \delta_k \gamma_{\mathrm{Max}}, \gamma_{\min} >= 0$. This implies that $\gamma_{\min}$ represents a non-zero class in $SH^0(k)$. Taking the direct limit, we conclude that $SH^0(\infty) \neq 0$, contradicting our assumption.

c) $\mu_{\mathrm{CZ}}(\gamma) = 1$ and γ is hyperbolic. This case leads to a contradiction following the same argument given in Proposition 4.21.

d) $\mu_{\mathrm{CZ}}(\gamma) = 1$ and γ is elliptic. Also this case leads to a contradiction following the same argument given in Proposition 4.21, but we have to use the filtration with the period (see Section 3.4.2), instead of the filtration with the action.

Since each case leads to a contradiction, we have proved that there exists a periodic orbit geometrically distinct from γ. $\qquad\square$

REMARK 4.25. For the case of S^2, we do not know whether the orbits we find are homotopic to a fibre or not.

CHAPTER 5

Rotationally symmetric magnetic systems on S^2

In this chapter we aim at studying the contact property when (S^2_γ, g_γ) is a surface of revolution with profile function γ and $\sigma = \mu_\gamma$ is the area form. We are going to estimate the interval $[s_-(g_\gamma, \mu_\gamma), s^+_+(g_\gamma, \mu_\gamma)]$ in terms of geometric properties of γ and try to understand, at least with numerical methods, the gaps $[s_-(g_\gamma, \mu_\gamma), \overline{s}_-(g_\gamma, \mu_\gamma)]$ and $[\overline{s}^+_+(g_\gamma, \mu_\gamma), s^+_+(g_\gamma, \mu_\gamma)]$. To ease the notation we are going to use the parameter m instead of s in this chapter. Hence, we define

$$X^m_\gamma := m X^{m\mu_\gamma}_{E_\gamma}\big|_{SS^2_\gamma}, \qquad \omega_{m,\gamma} := m\omega_{\frac{\mu_\gamma}{m}},$$

$$m_{+,\gamma} := \frac{1}{s_-(g_\gamma, \mu_\gamma)}, \qquad m_{-,\gamma} := \frac{1}{s^+_+(g_\gamma, \mu_\gamma)},$$

$$\overline{m}_{+,\gamma} := \frac{1}{\overline{s}_-(g_\gamma, \mu_\gamma)}, \qquad \overline{m}_{-,\gamma} := \frac{1}{\overline{s}^+_+(g_\gamma, \mu_\gamma)},$$

$$\mathrm{Con}_\gamma := \left\{ m \in (0, +\infty) \ \Big| \ \frac{1}{m} \in \mathrm{Con}^+(g_\gamma, \mu_\gamma) \right\} \cup \{0\}.$$

5.1. The geometry of a surface of revolution

To construct a surface of revolution, take a function $\gamma : [0, \ell_\gamma] \to \mathbb{R}$ and consider the conditions:

1) $\gamma(t) = 0$ if $t = 0$ or $t = \ell_\gamma$ and is positive otherwise,
2) $\dot{\gamma}(0) = 1$, $\dot{\gamma}(\ell_\gamma) = -1$ and $|\dot{\gamma}(t)| < 1$ for $t \in (0, \ell_\gamma)$,
3) all even derivatives of γ vanish for $t \in \{0, \ell_\gamma\}$,
4) the following equality is satisfied

$$\int_0^{\ell_\gamma} \gamma(t) \, dt = 2. \tag{5.1}$$

DEFINITION 5.1. A function γ satisfying the first three hypotheses of the list is called a *profile function*. If also the fourth one holds, we say that γ is *normalised*.

Let S^2_γ be the quotient of $[0, \ell_\gamma] \times \mathbb{T}_{2\pi}$ with respect to the equivalence relation that collapses the set $\{0\} \times \mathbb{T}_{2\pi}$ to a point and the set $\{\ell_\gamma\} \times \mathbb{T}_{2\pi}$ to another point. We call these points the *south* and the *north pole*. Outside the poles the smooth structure is given by the coordinates $(t, \theta) \in (0, \ell_\gamma) \times \mathbb{T}_{2\pi}$, which also determine a well-defined orientation on S^2_γ.

On S^2_γ we put the Riemannian metric g_γ, defined in the (t, θ) coordinates by the formula $g_\gamma = dt^2 + \gamma(t)^2 d\theta^2$. This metric extends smoothly to the poles because of conditions 2) and 3) listed before. Moreover, condition 4) yields the normalisation $\mathrm{vol}_{g_\gamma}(S^2_\gamma) = 4\pi$.

Let us denote by $(t, \theta, v_t, v_\theta)$ the associated coordinates on the tangent bundle. If $\varphi := \varphi_{\partial_t} : S\big(S_\gamma^2 \setminus \{\text{south pole, north pole}\}\big) \to \mathbb{T}_{2\pi}$ is the angular function associated to the section ∂_t, we have the relations

$$v_t = \cos\varphi, \qquad v_\theta = \frac{\sin\varphi}{\gamma(t)}. \tag{5.2}$$

As a consequence, (t, φ, θ) are coordinates on SS_γ^2, which are compatible with the orientation $\mathfrak{O}_{SS^2}$ defined in Section 2.2. Using the coframe (λ, τ, η), we can express the coordinate frame $(\widetilde{\partial}_t, \partial_\varphi, \widetilde{\partial}_\theta)$ in terms of the frame (X, V, H) and vice versa:

$$\begin{cases} \widetilde{\partial}_t &=& \cos\varphi X - \sin\varphi H \\ \partial_\varphi &=& V \\ \widetilde{\partial}_\theta &=& \gamma\sin\varphi X + \gamma\cos\varphi H + \dot\gamma V. \end{cases} \tag{5.3}$$

$$\begin{cases} X &=& \cos\varphi\widetilde{\partial}_t - \dfrac{\dot\gamma\sin\varphi}{\gamma}\partial_\varphi + \dfrac{\sin\varphi}{\gamma}\widetilde{\partial}_\theta \\ V &=& \partial_\varphi \\ H &=& -\sin\varphi\widetilde{\partial}_t - \dfrac{\dot\gamma\cos\varphi}{\gamma}\partial_\varphi + \dfrac{\cos\varphi}{\gamma}\widetilde{\partial}_\theta, \end{cases} \tag{5.4}$$

We have put a tilde on $\widetilde{\partial}_t$ and $\widetilde{\partial}_\theta$ to distinguish them from the coordinate vectors ∂_t and ∂_θ associated to the coordinates (t, θ) on S_γ^2.

As anticipated at the beginning of this chapter, we consider as the magnetic form on the surface SS_γ^2, the Riemannian area form μ_γ. This is a symplectic form which satisfies the normalisation $[\mu_\gamma] = 4\pi$ introduced in Section 4.2. In coordinates (t, θ) we have $\mu_\gamma = \gamma dt \wedge d\theta$. With this choice $f \equiv 1$ and $\mu_\gamma - \sigma_g = (1 - K)\mu_\gamma$. We recall that the Gaussian curvature for surfaces of revolution is $K = -\frac{\ddot\gamma}{\gamma}$.

If we define

$$m_\gamma := m(g_\gamma, \mu_\gamma) = \inf_{\beta \in \mathcal{P}^{(1-K)\mu_\gamma}} \|\beta\|,$$

we have the formula

$$m_{\pm,\gamma} = \frac{m_\gamma \pm \sqrt{m_\gamma^2 - 4}}{2}.$$

In particular, the length of the interval $[m_{-,\gamma}, m_{+,\gamma}]$ is equal to $\sqrt{m_\gamma^2 - 4}$, a monotone increasing function of m_γ. This observation readily gives a sufficient condition for having the contact property at every energy level.

COROLLARY 5.2. *If $m_\gamma < 2$, then $\mathrm{Con}_\gamma = [0, +\infty)$.*

In the next section we compute m_γ, showing that the infimum in its definition is actually a minimum. Namely, there exists $\beta^\gamma \in \mathcal{P}^{(1-K)\mu_\gamma}$ such that $m_\gamma = \|\beta^\gamma\|$.

5.2. Estimating the set of energy levels of contact type

Consider an arbitrary closed Riemannian manifold (M, g). Let $Z \in \Gamma(M)$ be a vector field that generates a 2π-periodic flow of isometries on M. The projection operator on the space of Z-invariant k-forms $\Pi_Z^k : \Omega^k(M) \to \Omega_Z^k(M)$ is defined as

$$\forall\tau \in \Omega^k(M), \quad \Pi_Z^k(\tau) := \frac{1}{2\pi}\int_0^{2\pi} (\Phi_{\theta'}^Z)^*\tau\, d\theta'. \tag{5.5}$$

PROPOSITION 5.3. *The operators Π_Z^k commute with exterior differentiation:*

$$d \circ \Pi_Z^k = \Pi_Z^{k+1} \circ d. \tag{5.6}$$

The projection Π_Z^1 does not increase the norm $\|\cdot\|$ defined in (2.2):

$$\forall \beta \in \Omega^1(M), \quad \|\Pi_Z^1(\beta)\| \leq \|\beta\|. \tag{5.7}$$

PROOF. For the proof of the first part we refer to [**Boo86**, Section: The De Rham groups of Lie groups]. For the latter statement we use that Φ^Z is a flow of isometries. We take some $(x,v) \in TM$ and compute

$$\begin{aligned}
\left|\Pi_Z^1(\beta)_x(v)\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \beta_{\Phi_{\theta'}^Z(x)}\left(d_x\Phi_{\theta'}^Z v\right)d\theta'\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \|\beta\|\left|d_x\Phi_{\theta'}^Z v\right|d\theta' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \|\beta\||v|\,d\theta' \\
&= \|\beta\||v|.
\end{aligned}$$

Thus, $|\Pi_Z^1(\beta)_x| \leq \|\beta\|$. The proposition follows by taking the supremum over x. $\quad\square$

Let us apply this general result to S_γ^2. Consider the coordinate vector field ∂_θ. This extends to a smooth vector field also at the poles and Φ^{∂_θ} is a 2π-periodic flow of isometries on the surface. Applying Proposition 5.3 to this case, we get the following corollary.

COROLLARY 5.4. *The 2-form $(1-K)\mu_\gamma$ is ∂_θ-invariant. Hence, $\Pi_{\partial_\theta}^1$ sends $\mathcal{P}^{(1-K)\mu_\gamma}$ into itself.*

PROOF. The first statement is true because Φ^{∂_θ} is a flow of isometries and thus μ_γ and K are invariant under the flow. To prove the second statement, we observe that, if $\beta \in \mathcal{P}^{(1-K)\mu_\gamma}$, the previous proposition yields

$$d\left(\Pi_{\partial_\theta}^1(\beta)\right) = \Pi_{\partial_\theta}^2(d\beta) = \Pi_{\partial_\theta}^2((1-K)\mu_\gamma) = (1-K)\mu_\gamma.$$

Hence, $\Pi_{\partial_\theta}^1(\beta) \in \mathcal{P}^{(1-K)\mu_\gamma}$. $\quad\square$

Suppose now that β lies in $\mathcal{P}^{(1-K)\mu_\gamma} \cap \Omega_{\partial_\theta}^1(S_\gamma^2)$. Thus, $\beta = \beta_t(t)dt + \beta_\theta(t)d\theta$. The function $\beta_\theta = \beta(\partial_\theta)$ is uniquely defined by $d\beta = (1-K)\mu_\gamma$ and the fact that it vanishes on the boundary of $[0, \ell_\gamma]$. Indeed,

$$d\beta = d\left(\beta_t dt + \beta_\theta d\theta\right) = \dot{\beta}_\theta dt \wedge d\theta = \frac{\dot{\beta}_\theta}{\gamma}\mu_\gamma \tag{5.8}$$

Recalling the formula for K, we have $\dot{\beta}_\theta = \gamma + \ddot{\gamma}$. Hence, $\beta_\theta = \Gamma + \dot{\gamma}$, where $\Gamma : [0, \ell_\gamma] \to [-1, 1]$ is the only primitive of γ such that $\Gamma(0) = -1$. Notice that

1) Γ is increasing,
2) $\Gamma(\ell_\gamma) = -1 + \int_0^{\ell_\gamma} \gamma(t)\,dt = -1 + 2 = 1$,
3) the odd derivatives of Γ vanish at $\{0, \ell_\gamma\}$.

Since β_θ and its derivatives of odd orders are zero for $t = 0$ and $t = \ell_\gamma$, the 1-form $\beta^\gamma := \beta_\theta d\theta$ is well defined also at the poles and belongs to $\mathcal{P}^{(1-K)\mu_\gamma} \cap \Omega^1_{\partial_\theta}(S^2_\gamma)$. Finally, the norm of this new primitive is less than or equal to the norm of β:

$$|\beta_{(t,\theta)}| = \sqrt{\beta_t^2 + \frac{\beta_\theta^2}{\gamma^2}} \geq \left|\frac{\beta_\theta}{\gamma}\right| = |\beta^\gamma_{(t,\theta)}|. \tag{5.9}$$

Summing up, we have proven the following proposition.

PROPOSITION 5.5. *There exists a unique $\beta^\gamma \in \mathcal{P}^{(1-K)\mu_\gamma}$ that can be written in the form $\beta^\gamma = \beta_\theta^\gamma(t)d\theta$. It satisfies $m_\gamma = \|\beta^\gamma\|$. Moreover $\beta_\theta^\gamma = \Gamma + \dot\gamma$ and*

$$\|\beta^\gamma\| = \sup_{t \in [0,\ell_\gamma]} \left|\frac{\Gamma(t) + \dot\gamma(t)}{\gamma(t)}\right|. \tag{5.10}$$

Thus, we see that we can compute m_γ directly from the function γ. As an application, we now produce a simple case where m_γ can be bounded from above.

PROPOSITION 5.6. *Suppose $\gamma : [0, \ell_\gamma] \to \mathbb{R}$ is a normalised profile function such that $\gamma(t) = \gamma(\ell_\gamma - t)$. If K is increasing in the variable t, for $t \in [0, \ell_\gamma/2]$, then $m_\gamma \leq 1$. Therefore, $\mathrm{Con}_\gamma = [0, +\infty)$.*

PROOF. We observe that the functions Γ, $\dot\gamma$ and, hence, β_θ^γ are odd with respect to $t = \ell_\gamma/2$. This means, for example, that $\beta_\theta^\gamma(t) = -\beta_\theta^\gamma(\ell_\gamma - t)$. Therefore, in order to compute m_γ, we can restrict the attention to the interval $[0, \ell_\gamma/2]$. We know that $\beta_\theta^\gamma(0) = \beta_\theta^\gamma(\ell_\gamma/2) = 0$ and we claim that, if K is increasing, β_θ^γ is positive in the interior. Indeed, observe that $\{t \in [0, \ell_\gamma/2] \,|\, \dot\beta_\theta^\gamma = 0\}$ is an interval and that $\dot\beta_\theta^\gamma$ is positive before this interval and it is negative after. Therefore, either β_θ^γ is constantly zero or it does not have a local minimum in the interior. Thus, it cannot assume negative values.

Let us estimate $\beta_\theta^\gamma/\gamma$ at an interior absolute maximiser t_0. The condition $\frac{d}{dt}\big|_{t=t_0} \frac{\beta_\theta^\gamma}{\gamma} = 0$ is equivalent to

$$\frac{\big(\Gamma(t_0) + \dot\gamma(t_0)\big)\dot\gamma(t_0)}{\gamma^2(t_0)} = 1 - K(t_0). \tag{5.11}$$

Since $\Gamma(t_0) + \dot\gamma(t_0) = \beta_\theta^\gamma(t_0) \geq 0$ and $\dot\gamma(t_0) > 0$, we see that $1 - K(t_0) \geq 0$. Moreover, using that $\Gamma(t_0) < 0$, we get

$$\left(\frac{\dot\gamma(t_0)}{\gamma(t_0)}\right)^2 > 1 - K(t_0). \tag{5.12}$$

Finally, exploiting Equation (5.11) again, we find

$$m_\gamma = \frac{\beta_\theta^\gamma(t_0)}{\gamma(t_0)} = \big(1 - K(t_0)\big)\frac{\gamma(t_0)}{\dot\gamma(t_0)} \leq \sqrt{1 - K(t_0)} \leq 1. \tag{5.13}$$

The fact that $\mathrm{Con}_\gamma = [0, +\infty)$ now follows from Corollary 5.2.   □

To complement the previous proposition we show that if, on the contrary, we assume that the curvature of S^2_γ is sufficiently concentrated at one of the poles, m_γ can be arbitrarily large. We are going to prove this behaviour in the class of

convex surfaces since in this case we also have a strategy, explained in Section 5.4, to compute numerically the action of invariant measures. Using McDuff's criterion, this will enable us to estimate the gaps $[m_{-,\gamma}, \overline{m}_{-,\gamma}]$ and $[\overline{m}_{+,\gamma}, m_{+,\gamma}]$.

Before we need a preliminary lemma. Recall that S^2_γ is convex, i.e. $K \geq 0$, if and only if $\ddot\gamma \leq 0$.

LEMMA 5.7. *For every $0 < \delta < \frac{\pi}{2}$ and for every $\varepsilon > 0$, there exists a normalised profile function $\gamma_{\delta,\varepsilon}$ such that $S^2_{\gamma_{\delta,\varepsilon}}$ is convex and*

$$\dot\gamma_{\delta,\varepsilon}(\delta) < \varepsilon. \tag{5.14}$$

PROOF. Given δ and ε, we find $a \in (\frac{2\delta}{\pi}, 1)$ such that $0 < \cos\left(\frac{\delta}{a}\right) < \varepsilon$. This is achieved by taking $\frac{\delta}{a}$ very close to $\pi/2$ from below. Consider the profile function $\gamma_a : [-\frac{\pi a}{2}, \frac{\pi a}{2}] \to [0, a]$ of a round sphere of radius a, where the domain is taken to be symmetric to zero to ease the following notation. It is defined by $\gamma_a(t) := a \cos(\frac{t}{a})$. Then, $\dot\gamma_a(-\frac{\pi a}{2} + \delta) = \cos\left(\frac{\delta}{a}\right) < \varepsilon$ and so, up to shifting the domain again, Inequality (5.14) is satisfied. However, γ_a is not normalised since

$$\int_{-\frac{\pi a}{2}}^{\frac{\pi a}{2}} \gamma_a(t)dt = 2a^2 < 2.$$

In order to get the normalisation in such a way that Inequality (5.14) is not spoiled, we are going to stretch the sphere in the interval $\left(-(\frac{\pi a}{2} - \delta), \frac{\pi a}{2} - \delta\right)$.

We claim that, for every $C > 0$ there exists a diffeomorphism $F_C : \mathbb{R} \to \mathbb{R}$ with the property that

- it is odd: $\forall t \in \mathbb{R}$, $F_C(t) = -F_C(-t)$;
- for $t \geq \frac{\pi a}{2} - \delta$, $F_C(t) = t + C$ and for $t \leq -(\frac{\pi a}{2} - \delta)$, $F_C(t) = t - C$;
- $\dot F_C \geq 1$;
- for $t \leq 0$, $\ddot F_C(t) \geq 0$ and for $t \geq 0$, $\ddot F_C(t) \leq 0$.

Such a map can be constructed as a time C flow map Φ^ψ_C, where $\psi : \mathbb{R} \to \mathbb{R}$ is an odd increasing function such that, for $t \geq \frac{\pi a}{2} - \delta$, $\psi(t) = 1$.

Consider the function $\gamma^C_a : [-C - \frac{\pi a}{2}, C + \frac{\pi a}{2}] \to \mathbb{R}$, where $\gamma^C_a(s) := \gamma_a(F^{-1}_C(s))$. One readily check that γ^C_a (up to a shift of the domain) is a profile function satisfying the convexity conditions and for which (5.14) holds. To finish the proof it is enough to find a positive real number C_2 such that $\int_\mathbb{R} \gamma^{C_2}_a = 2$. Since we know that $\gamma^0_a = \gamma_a$ and $\int_\mathbb{R} \gamma_a < 2$, it suffices to show that

$$\lim_{C \to +\infty} \int_\mathbb{R} \gamma^C_a(s)ds = +\infty.$$

Observe that $b := \gamma^C_a(-C - \frac{\pi a}{2} + \delta) = \gamma_a(-\frac{\pi a}{2} + \delta) = a\sin(\frac{\delta}{a}) > 0$. Then, for $s \in [-C - \frac{\pi a}{2} + \delta, C + \frac{\pi a}{2} - \delta]$, $\gamma^C_a(s) \geq b$ and we have the lower bound

$$\int_\mathbb{R} \gamma^C_a(s)ds \geq \int_{-C-\frac{\pi a}{2}+\delta}^{C+\frac{\pi a}{2}-\delta} \gamma^C_a(s)ds \geq \int_{-C-\frac{\pi a}{2}+\delta}^{C+\frac{\pi a}{2}-\delta} b\,ds = 2(C + \frac{\pi a}{2} - \delta)b.$$

The last quantity tends to infinity as C tends to infinity.   □

PROPOSITION 5.8. *For every $C > 0$, there exists a convex surface with total area 4π such that $m_\gamma > C$.*

PROOF. Fix an $\varepsilon_0 < 1$. Take any $\delta < \sqrt{1 - \varepsilon_0}$ and consider the normalised profile function $\gamma_{\delta,\varepsilon_0}$ given by the lemma. We know that

$$\gamma_{\delta,\varepsilon_0}(\delta) = \int_0^\delta \dot{\gamma}_{\delta,\varepsilon_0}(t)dt \leq \int_0^\delta 1\,dt = \delta.$$

In the same way we find $\Gamma_{\delta,\varepsilon_0}(\delta) \leq -1 + \delta^2$. From this last inequality we get

$$\Gamma_{\delta,\varepsilon_0}(\delta) + \dot{\gamma}_{\delta,\varepsilon_0}(\delta) < -1 + \delta^2 + \varepsilon_0 < 0.$$

This yields the following lower bound for $m_{\gamma_{\delta,\varepsilon_0}}$:

$$m_{\gamma_{\delta,\varepsilon_0}} \geq \left| \frac{\Gamma_{\delta,\varepsilon_0}(\delta) + \dot{\gamma}_{\delta,\varepsilon_0}(\delta)}{\gamma_{\delta,\varepsilon_0}(\delta)} \right| \geq \frac{1 - \varepsilon_0}{\delta} - \delta.$$

The proposition is proven taking δ small enough. $\square$

To sum up, we saw that the rotational symmetry gives us a good understanding of the set $[0, m_{-,\gamma}) \cup (m_{+,\gamma}, +\infty)$. Understanding the set $[0, \overline{m}_{-,\gamma}) \cup (\overline{m}_{-,\gamma}, +\infty)$, or even better Con_γ, is more subtle. In Section 5.4 we perform this task only numerically and when the magnetic curvature $K_m = m^2 K + 1$ is positive.

As a first step, in the next section we will briefly study the symplectic reduction associated to the symmetry and the associated reduced dynamics (for the general theory of symplectic reduction we refer to [**AM78**]). Proposition 5.10 and the numerical computation outlined in Section 5.4 suggest that, if $K_m > 0$, the contact property holds. In particular, if $K \geq 0$, every energy level should be of contact type. To complete the picture, we show in Proposition 5.11 that the assumption on the magnetic curvature is not necessary and in Proposition 5.12 that there are cases where the magnetic curvature is not positive and that are not of contact type.

5.3. The symplectic reduction

Observe that the flow Φ^{∂_θ} on S_γ^2 lifts to a flow $d\Phi^{\partial_\theta}$ on SS_γ^2. Since Φ^{∂_θ} is a flow of isometries, $d\Phi_{\theta'}^{\partial_\theta}$ in coordinates is simply translation in the variable θ: $d\Phi_{\theta'}^{\partial_\theta}(t, \varphi, \theta) = (t, \varphi, \theta + \theta')$. Hence, $d\Phi^{\partial_\theta}$ is generated by $\widetilde{\partial}_\theta$. As the flow $\Phi^{\widetilde{\partial}_\theta} = d\Phi^{\partial_\theta}$ is 2π-periodic and acts freely on SS_γ^2, we can take its quotient $\widehat{SS_\gamma^2}$ with respect to this $\mathbb{T}_{2\pi}$-action. Furthermore, the quotient map $\widehat{\pi} : SS_\gamma^2 \to \widehat{SS_\gamma^2}$ is a submersion. The variables t and φ descend to coordinates defined on $\widehat{SS_\gamma^2}$ minus two points, which are the fibres of the unit tangent bundle over the south and north pole. In these coordinates we simply have $\widehat{\pi}(t, \varphi, \theta) = (t, \varphi)$. In particular, $\widehat{SS_\gamma^2}$ is diffeomorphic to a 2-sphere.

Any $\tau \in \Omega_{\widetilde{\partial}_\theta}^k(SS_\gamma^2)$ such that $\imath_{\widetilde{\partial}_\theta}\tau = 0$ passes to the quotient and yields a well-defined form on $\widehat{SS_\gamma^2}$. The 2-form $\imath_{\widetilde{\partial}_\theta}\nu_\gamma$ falls into this class and, hence, there exists $\Theta_\gamma \in \Omega^2(\widehat{SS_\gamma^2})$ such that $\imath_{\widetilde{\partial}_\theta}\nu_\gamma = \widehat{\pi}^*\Theta_\gamma$. Moreover, the form Θ_γ is symplectic on $\widehat{SS_\gamma^2}$. On the other hand, X_γ^m is also $\widetilde{\partial}_\theta$-invariant thanks to Equation (5.4). So there

exists $\widehat{X}_\gamma^m \in \Gamma(\widehat{SS_\gamma^2})$ such that $d\widehat{\pi}(X_\gamma^m) = \widehat{X}_\gamma^m$. We claim that this new vector field is Θ_γ-Hamiltonian. We start by noticing that if β^γ is as defined in Proposition 5.5 and $\lambda_m^\gamma := m\lambda - \pi^*\beta^\gamma + \tau$, then $\lambda_m^\gamma \in \Omega^1_{widetilde\partial_\theta}(SS_\gamma^2)$. Then, Cartan identity implies

$$\imath_{X_\gamma^m}(\imath_{\widetilde{\partial}_\theta}\nu_\gamma) = -\imath_{\widetilde{\partial}_\theta}(\imath_{X_\gamma^m}\nu_\gamma) = -\imath_{\widetilde{\partial}_\theta}\omega_{m,\gamma} = -\imath_{\widetilde{\partial}_\theta}(d\lambda_m^\gamma) = -\mathcal{L}_{\widetilde{\partial}_\theta}\lambda_m^\gamma + d(\imath_{\widetilde{\partial}_\theta}\lambda_m^\gamma)$$
$$= d(\lambda_m^\gamma(\widetilde{\partial}_\theta)). \qquad (5.15)$$

Define $I_{m,\gamma} := \lambda_m^\gamma(\widetilde{\partial}_\theta)$. Since $I_{m,\gamma}$ is $\widetilde{\partial}_\theta$-invariant, there exists $\widehat{I}_{m,\gamma} : \widehat{SS_\gamma^2} \to \mathbb{R}$ such that $I_{m,\gamma} = \widehat{\pi}^*\widehat{I}_{m,\gamma}$. Thus, reducing equality (5.15) to $\widehat{SS_\gamma^2}$, we find that $\widehat{X}_\gamma^m$ is the Θ_γ-Hamiltonian vector field generated by $-\widehat{I}_{m,\gamma}$. Using Equation (5.3), we find the coordinate expression

$$\widehat{I}_{m,\gamma}(t,\varphi) = m\gamma(t)\sin\varphi - \Gamma(t). \qquad (5.16)$$

As a by-product, we also observe that $I_{m,\gamma}$ is an integral of motion for X_γ^m.

Let us consider now the two auxiliary functions $\widehat{I}_{m,\gamma}^\pm : [0,\ell_\gamma] \to \mathbb{R}$ defined by $\widehat{I}_{m,\gamma}^\pm(t) := \widehat{I}_{m,\gamma}(t,\pm\pi/2) = \pm m\gamma(t) - \Gamma(t)$. We know that

$$\widehat{I}_{m,\gamma}^-(t) \le \widehat{I}_{m,\gamma}(t,\varphi) \le \widehat{I}_{m,\gamma}^+(t), \qquad (5.17)$$

with equalities if and only if $\varphi = \pm\pi/2$. On the one hand, we have $\widehat{I}_{m,\gamma}^+ \ge -1$ and $\widehat{I}_{m,\gamma}^+(t) = -1$ if and only if $t = \ell_\gamma$. On the other hand, $\widehat{I}_{m,\gamma}^+$ attains its maximum in the interior. Indeed,

$$\frac{d}{dt}\widehat{I}_{m,\gamma}^+ = m\dot{\gamma} - \gamma. \qquad (5.18)$$

and so $\frac{d}{dt}\widehat{I}_{m,\gamma}^+(0) = m > 0$. Since $\widehat{I}_{m,\gamma}^+(0) = 1$, the maximum is also strictly bigger than 1. A similar argument tells us that the maximum of $\widehat{I}_{m,\gamma}^-$ is 1 and it is attained at 0 and the minimum of $\widehat{I}_{m,\gamma}^-$ is strictly less than -1 and it is attained in $(0,\ell_\gamma)$. As a consequence, $\max\widehat{I}_{m,\gamma} = \max\widehat{I}_{m,\gamma}^+ > 1$ and $\min\widehat{I}_{m,\gamma} = \min\widehat{I}_{m,\gamma}^- < -1$.

In the next proposition we deal with the critical points $\widehat{\mathrm{Crit}}_{m,\gamma}$ of $\widehat{I}_{m,\gamma}$ and study their relations with the critical points $\widehat{\mathrm{Crit}}_{m,\gamma}^\pm$ of $\widehat{I}_{m,\gamma}^\pm$. We are going to show that, if $K_m > 0$, the only elements in $\widehat{\mathrm{Crit}}_{m,\gamma}$ are the (unique) maximiser and the (unique) minimiser. In this case the dynamics of $\widehat{X}_\gamma^m$ is very simple: besides the two rest points, all the other orbits are periodic and wind once in the complement of these two points.

PROPOSITION 5.9. *There holds*

$$\widehat{\mathrm{Crit}}_{m,\gamma} = \widehat{\mathrm{Crit}}_{m,\gamma}^- \times \left\{-\frac{\pi}{2}\right\} \bigcup \widehat{\mathrm{Crit}}_{m,\gamma}^+ \times \left\{+\frac{\pi}{2}\right\}.$$

Moreover, $t_0 \in \widehat{\mathrm{Crit}}_{m,\gamma}^\pm$ if and only if $\pm m\dot{\gamma}(t_0) = \gamma(t_0)$ and $\{(t_0,\pm\pi/2,\theta)\}$ is the support of a closed orbit for X_γ^m. All the periodic orbits, whose projection to S_γ^2 is a latitude, arise in this way. Every regular level of $\widehat{I}_{m,\gamma}$ is the support of a closed orbit of $\widehat{X}_\gamma^m$ and its preimage in SS_γ^2 is an X_γ^m-invariant torus.

Finally, if $K_m > 0$, $\widehat{\mathrm{Crit}}_{m,\gamma}^\pm$ contains only the absolute maximiser (respectively minimiser) of $\widehat{I}_{m,\gamma}^\pm$. We denote this unique element by $t_{m,\gamma}^\pm$.
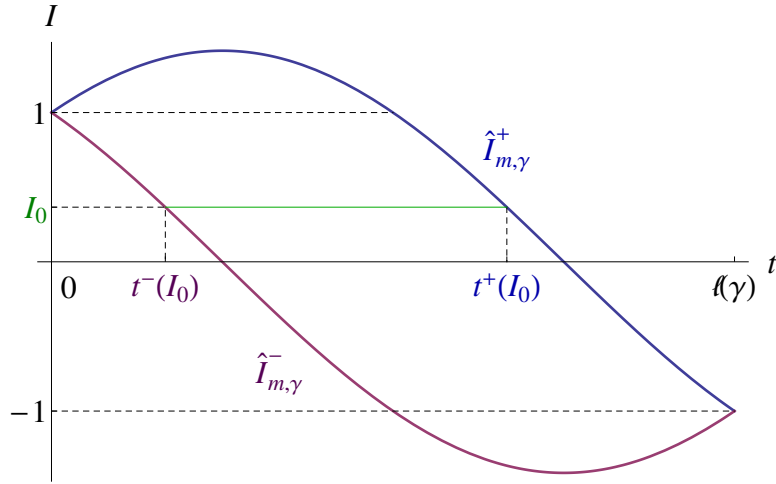
FIGURE 1. Graphs of the functions $\widehat{I}_{m,\gamma}^-$ and $\widehat{I}_{m,\gamma}^+$

PROOF. The first statement follows from the fact that $\partial_\varphi \widehat{I}_{m,\gamma} = 0$ if and only if $\varphi = \pm\pi/2$. Recalling (5.18) we see that $\pm m\dot\gamma(t) = \gamma(t)$ is exactly the equation for the critical points of $\widehat{I}_{m,\gamma}^\pm$. The statement about the relation between closed orbits of X_γ^m and latitudes follows from the fact that at a critical point of $\widehat{I}_{m,\gamma}$, $\widehat{X}_\gamma^m = 0$. Hence, on its preimage X_γ^m is parallel to $\widetilde{\partial}_\theta$. By the implicit function theorem the regular level sets of $\widehat{I}_{m,\gamma}$ and $I_{m,\gamma}$ are closed submanifolds of codimension 1. In the latter case they are tori since X_γ^m is tangent to them and nowhere vanishing.

We prove now uniqueness under the hypothesis on the curvature. We carry out the computations for $\widehat{I}_{m,\gamma}^+$ only. To prove that the absolute maximiser is the only critical point, we show that if t_0 is critical, the function is concave at t_0. Indeed,

$$\frac{d^2}{dt^2}\widehat{I}_{m,\gamma}^+(t_0) = m\ddot\gamma(t_0) - \dot\gamma(t_0) = m\gamma(t_0)\left(\frac{\ddot\gamma(t_0)}{\gamma(t_0)} - \frac{\dot\gamma(t_0)}{m\gamma(t_0)}\right)$$
$$= -m\gamma(t_0)\left(K(t_0) + \frac{1}{m^2}\right) < 0. \qquad \square$$

The picture above shows qualitatively $\widehat{I}_{m,\gamma}^-$ and $\widehat{I}_{m,\gamma}^+$ when K_m is positive.

In order to decide whether $\omega_{m,\gamma}'$ is of contact type or not, the first thing to do is to compute the action of latitudes.

PROPOSITION 5.10. *Take* $t_0 \in (0, \ell_\gamma)$ *such that* $\dot\gamma(t_0) \neq 0$ *and let* $m_{t_0} := \left|\frac{\gamma(t_0)}{\dot\gamma(t_0)}\right|$. *The lift of the latitude curve* $\{t = t_0\}$ *parametrised by arc length and oriented by* $(\text{sign}\,\dot\gamma(t_0))\partial_\theta$ *is the support of a periodic orbit for* $X_\gamma^{m_{t_0}}$. *We call* ζ_{t_0} *the associated invariant probability measure. Its action is given by*

$$\mathcal{A}_{X_\gamma^{m_{t_0}}}^{\omega_{m_{t_0},\gamma}'}(\zeta_{t_0}) = \frac{\gamma(t_0)^2 - \dot\gamma(t_0)\Gamma(t_0)}{\dot\gamma(t_0)^2} \tag{5.19}$$

and $I_{m_{t_0},\gamma}\big|_{\text{supp}\,\zeta_{t_0}} = \dot\gamma(t_0)\mathcal{A}(\zeta_{t_0})$. *As a result, if* $K_{m_{t_0}} > 0$, *then* $\mathcal{A}(\zeta_{t_0}) > 0$.

PROOF. The curve $u \mapsto \left(t_0, \operatorname{sign} \dot\gamma(t_0)\frac{\pi}{2}, \frac{m \operatorname{sign} \dot\gamma(t_0)}{\gamma(t_0)} u\right)$ is a periodic orbit by the previous proposition. On the support of this orbit we have

$$m_{t_0} v_\theta = \left|\frac{\gamma(t_0)}{\dot\gamma(t_0)}\right| \frac{\operatorname{sign} \dot\gamma(t_0)}{\gamma(t_0)} = \frac{1}{\dot\gamma(t_0)} \tag{5.20}$$

and, as a consequence,

$$\lambda^\gamma_{m_{t_0}}(X^{m_{t_0}}_\gamma) = \frac{\gamma(t_0)^2}{\dot\gamma(t_0)^2} - \beta^\gamma_\theta(t_0)\frac{1}{\dot\gamma(t_0)} + 1 = \frac{\gamma(t_0)^2 - \dot\gamma(t_0)\Gamma(t_0)}{\dot\gamma(t_0)^2}. \tag{5.21}$$

Since this is a constant, we get Identity (5.19) for the action.

The second identity is proved using the definition of $I_{m_{t_0},\gamma}$:

$$I_{m_{t_0},\gamma}\big|_{\operatorname{supp}\zeta_{t_0}} = \left|\frac{\gamma(t_0)}{\dot\gamma(t_0)}\right| \gamma(t_0) \operatorname{sign} \dot\gamma(t_0) - \Gamma(t_0) = \dot\gamma(t_0)\frac{\gamma(t_0)^2 - \dot\gamma(t_0)\Gamma(t_0)}{\dot\gamma(t_0)^2}.$$

Under the curvature assumption, $I_{m_{t_0},\gamma}$ is maximised or minimised at $\operatorname{supp}\zeta_{t_0}$ according to the sign of $\dot\gamma(t_0)$. In both cases $I_{m_{t_0},\gamma}\big|_{\operatorname{supp}\zeta_{t_0}}$ and $\dot\gamma(t_0)$ have the same sign. Hence, also the third statement is proved. $\square$

This proposition shows that, when $K_m > 0$, the action of the periodic orbits that project to latitudes is not an obstruction for $\omega'_{m,\gamma}$ to be of contact type. Thus, as we also discuss in the next subsection, one could conjecture that under this hypothesis $\omega'_{m,\gamma}$ is of contact type. On the other hand, we claim that having $K_m > 0$ is not a necessary condition for the contact property to hold. For this purpose it is enough to exhibit a non-convex surface for which $m_\gamma < 2$. This can be achieved as a consequence of the fact that the curvature depends on the second derivative of γ, whereas m_γ depends only on the first derivative.

We can start from $S^2_{\gamma_0}$, the round sphere of radius 1, and find a non-convex surface of revolution S^2_γ, which is C^1-close to the sphere and coincides with it around the poles. This implies that $m_\gamma = \sup_{t\in[0,\ell_\gamma]} \left|\frac{\Gamma(t) + \dot\gamma(t)}{\gamma(t)}\right|$ can be taken smaller than 2 since it is close as we like to $m_{\gamma_0} = 0$. Hence, every energy level of $(S^2_\gamma, g_\gamma, \mu_\gamma)$ is of contact type and the following proposition is proved.

PROPOSITION 5.11. *The condition $K_m > 0$ is not necessary for Σ_m to be of contact type.*

On the other hand, we now show that is not true, in general, that the contact property holds on every energy level.

PROPOSITION 5.12. *There exists a symplectic magnetic system (S^2, g, σ) that has an energy level not of contact type.*

PROOF. We will achieve this goal by finding m and γ such that X^m_γ has a closed orbit projecting to a latitude with negative action. Then, the proof is complete applying Proposition 2.17.

Fix some $\varepsilon \in (0,1)$. We claim that, for every $\delta \in (0,\pi/2)$, there exists a normalised profile function $\gamma_{\delta,\varepsilon}$ such that

$$\dot\gamma_{\delta,\varepsilon}(\delta) < -\varepsilon. \tag{5.22}$$

Such profile function can be obtained as in Lemma 5.7. Take an $a \in \left(\frac{\delta}{\pi}, \frac{\delta}{\pi/2} \right)$ (this is equivalent to $\delta \in \left(\frac{\pi a}{2}, \pi a \right)$) such that $-\sin \left(\frac{\delta}{a} - \frac{\pi}{2} \right) < -\varepsilon$. Consider the profile function $\gamma_a : [-\pi a/2, \pi a/2] \to \mathbb{R}$ of a round sphere of radius a. Thanks to the last inequality, $\dot{\gamma}_a(\delta - \frac{\pi a}{2}) < -\varepsilon$ and, therefore, γ_a satisfies (5.22) (up to a shift of the domain). Now we stretch an interval compactly supported in $(\delta - \frac{\pi a}{2}, \pi a/2)$ by a family of diffeomorphisms F_C as we did in Lemma 5.7 (though here the condition on the second derivative of F_C is not necessary). In this way we obtain a family of profile functions γ_a^C satisfying (5.22). Since the area diverges with C, we find $C_2 > 0$ such that $\gamma_a^{C_2}$ is normalised. This finishes the proof of the claim.

Since $\left| \dot{\gamma}_{\delta,\varepsilon}(\delta) \right| \leq 1$, we have that $\gamma_{\delta,\varepsilon}(\delta) \leq \delta$ and $\Gamma_{\delta,\varepsilon}(\delta) \leq -1 + \delta^2$. The latitude at height $t = \delta$ of such surface is a closed orbit for $X_{\gamma_{\delta,\varepsilon}}^{m_{\delta,\varepsilon}}$, where $m_{\delta,\varepsilon} := \frac{\gamma_{\delta,\varepsilon}(\delta)}{|\dot{\gamma}_{\delta,\varepsilon}(\delta)|}$. Using formula (5.19) we see that the action of the corresponding invariant measure $\zeta_{\delta,\varepsilon}$ is negative for δ small enough:

$$\mathcal{A}(\zeta_{\delta,\varepsilon}) \leq \frac{\delta^2 - (-\varepsilon)(-1 + \delta^2)}{\varepsilon^2} = -\frac{1}{\varepsilon} + o(\delta) < 0. \qquad \square$$

5.4. Action of ergodic measures

When $K_m > 0$, we also have a way to compute numerically the action of ergodic invariant measures. We consider only ergodic measures since they are the extremal points of the set of probability invariant measures by Choquet's Theorem and, therefore, it is enough to check the positivity of the action of these measures, in order to apply Proposition 2.17. Every ergodic measure ζ is concentrated on a unique level set $\{I_{m,\gamma} = I(\zeta)\}$, for some $I(\zeta) \in \mathbb{R}$. Moreover, if $I(\zeta) = I(\zeta')$ there exists a rotation $\Phi_{\theta'}^{\widetilde{\partial}_\theta}$ such that $\left(\Phi_{\theta'}^{\widetilde{\partial}_\theta} \right)_* \zeta = \zeta'$. Since the action is $\widetilde{\partial}_\theta$-invariant, we deduce that it is a function of $I(\zeta)$ only and we can define $\mathcal{A} : [\min I_{m,\gamma}, \max I_{m,\gamma}] \to \mathbb{R}$. We already have an expression for the action at the minimum and maximum of $I_{m,\gamma}$. We now give a formula for the action when $I \in (\min I_{m,\gamma}, \max I_{m,\gamma})$.

Every integral line $z : \mathbb{R} \to SS^2$ of X_γ^m, such that $I_{m,\gamma}(z) = I$ oscillates between the latitudes at height $t^-(I)$ and $t^+(I)$. Their numerical values can be easily read off from the graphs of $\widehat{I}_{m,\gamma}^\pm$ drawn in Figure 1. If we take z with $t(z(0)) = t^-(I)$, there exists a smallest $u(I) > 0$ such that $t\big(z(u(I))\big) = t^+(I)$. By Birkhoff's ergodic theorem

$$\mathcal{A}(I) = \frac{1}{u(I)} \int_0^{u(I)} \left(m^2 - m \frac{\beta_\theta(t) \sin \varphi}{\gamma(t)} (z(u)) + 1 \right) du. \qquad (5.23)$$

Using this identity, we computed with *Mathematica* the action when S_γ^2 is an ellipsoid. We found that is positive on every energy level in the interval $[m_{-,\gamma}, m_{+,\gamma}]$. By Proposition 2.17, this shows numerically that $\mathrm{Con}_\gamma = [0, +\infty)$, hence corroborating the conjecture that Σ_m is of contact type if $K_m > 0$. On the other hand, we know that, when the ellipsoid is very thin, its curvature is concentrated on its poles and hence, by Proposition 5.8, the set $[m_{-,\gamma}, m_{+,\gamma}]$ is not empty. Therefore, these data would also show numerically that the inclusion $[0, m_{-,\gamma}) \cup (m_{+,\gamma}, +\infty) \subset \mathrm{Con}_\gamma$ can be strict.

Dynamically convex Hamiltonian structures

In this chapter we introduce dynamically convex Hamiltonian structures, for which we have sharper results on the set of periodic orbits. They were introduced by Hofer, Wysocki and Zehnder in [**HWZ98**] as a way of generalising the contact structures arising on the boundary of convex domains in $\mathbb{C}^2$ (see Example 6.4 below) with a notion which makes sense in the contact category. Later on, this notion has been extended by Hryniewicz, Licata and Salomão [**HLS13**] to lens spaces and, at the moment this thesis is being written, Abreu and Macarini [**AM**] are developing a general theory of dynamical convexity, which will eventually embrace also non-exact magnetic systems on surfaces of higher genus (see Section 7.4.1).

The abstract results we present below will play a crucial role in Chapter 7, when we will analyse symplectic magnetic systems on S^2.

DEFINITION 6.1. Let ω be a HS of contact type on $L(p,q)$. We say that ω is *dynamically convex* if, for every contractible periodic orbit γ, $\mu^l_{\mathrm{CZ}}(\gamma) \geq 3$. By abuse of terminology, we will call a contact primitive of ω dynamically convex, as well.

REMARK 6.2. Thanks to (2.32) the condition on the index is equivalent to $I(\Psi^{C,\Upsilon}_\gamma) \subset (1, +\infty)$ (see (2.34)).

REMARK 6.3. If $\pi : S^3 \to L(p,q)$ is the quotient map, then ω is dynamically convex if and only if $\pi^*\omega$ is dynamically convex.

EXAMPLE 6.4. If $\Sigma \hookrightarrow \mathbb{C}^2$ is a closed hypersurface bounding a convex domain, then $\lambda_{\mathrm{st}}\big|_\Sigma$ is dynamically convex.

EXAMPLE 6.5. We remarked in Example 2.14 that, if (M, g) is a Riemannian surface, $\lambda\big|_{SM}$ is a contact form. Let $M = S^2$ and notice that $SM \simeq L(2,1)$. Harris and G. Paternain showed in [**HP08**] that, if $\frac{1}{4} \leq K < 1$, then $\lambda\big|_{SS^2}$ is dynamically convex.

For further examples of dynamical convexity, in the context of the circular planar restricted three-body problem and of the rotating Kepler problem, we refer the reader to [**AFF$^+$12**] and [**AFFvK13**], respectively.

As we show in the next two sections, dynamical convexity has two main consequences for the associated Reeb flow: the existence of a symplectic Poincaré section and the existence of an elliptic periodic orbit.

6.1. Poincaré sections

The main result on the dynamics of dynamically convex Hamiltonian Structures reads as follows.

THEOREM 6.6 ([**HWZ98**],[**HLS13**]). *Let ω be a HS on $L(p,q)$ and suppose, furthermore, that, if $p > 1$, ω is non-degenerate. If ω is dynamically convex, then there exists a Poincaré section of disc-type $D^2 \to L(p,q)$ (which is a p-sheeted cover on ∂D^2) for the characteristic distribution of ω.*

REMARK 6.7. The case of S^3, namely $p = 1$, is due to Hofer, Wysocki and Zehnder, while the case $p > 1$ is due to Hryniewicz, Licata and Salomão. It is likely that the non-degeneracy condition for $p > 1$ can be dropped by repeating the lengthy argument contained in [**HWZ98**, Section 6–8]. However, for the applications to periodic orbits of a degenerate system we can simply apply Theorem 6.6 after we have lifted the problem from $L(p,q)$ to S^3. The details of such argument will be explained below.

For the convenience of the reader we now recall the notion of Poincaré section.

DEFINITION 6.8. Let N be a closed 3-manifold. A *Poincaré section* for $Z \in \Gamma(N)$ is a compact surface $i : S \to N$ such that

- it is an embedding on the interior $\mathring{S} := S \setminus \partial S$;
- $i(\partial S)$ is the disjoint union of a finite collection of embedded loops $\{\gamma_k\}$ and $i|_{i^{-1}(\gamma_k)} : i^{-1}(\gamma_k) \to \gamma_k$ is a finite cover;
- the vector field Z is transverse to $i(\mathring{S})$ and every γ_k is the support of a periodic orbit for Z;
- every flow line of Z hits the surface in forward and backward time.

If $z \in \mathring{S}$, let $t(z)$ be the smallest positive number such that $F_{\mathring{S}}(z) := \Phi^Z_{t(z)}(z) \in \mathring{S}$. The map $F_{\mathring{S}} : \mathring{S} \to \mathring{S}$ defines a diffeomorphism called the *Poincaré first return map*.

Finally, if Z is a positive section for $\ker \omega$, where ω is a HS on N, a map i satisfying the requirements above is called a Poincaré section for ω.

If Z and N are as in Definition 6.8, the discrete dynamical system $(\mathring{S}, F_{\mathring{S}})$ carries important information about the qualitative dynamics on N, since periodic points of $F_{\mathring{S}}$ correspond to the periodic orbits of Z different from the γ_k's.

When Z is a positive section of $\ker \omega$, with $\omega \in \Omega^2(N)$ a Hamiltonian Structure, $(\mathring{S}, i^*\omega)$ is a symplectic manifold with finite area and $F_{\mathring{S}}$ is a symplectomorphism.

If we suppose, in addition, that S is a disc, then $N \simeq L(p,q)$ (see [**HLS13**]) and Proposition 5.4 in [**HWZ98**] implies that $F_{\mathring{S}}$ is C^0-conjugated to a homeomorphism of the disc preserving the standard Lebesgue measure. The work of Brouwer ([**Bro12**]) and Franks ([**Fra92**] and [**Fra96**]) on area-preserving homeomorphisms of the disc imply that either $F_{\mathring{S}}$ has only a single periodic point, which is a fixed point for $F_{\mathring{S}}$, or there are infinitely many periodic points.

COROLLARY 6.9. *Suppose $\omega \in \Omega^2(L(p,q))$ is a HS having a Poincaré section of disc-type. Then, $\ker \omega$ has 2 periodic orbits γ_1 and γ_2 whose lifts to S^3, $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$, form a Hopf link (namely they are unknotted and $|\mathrm{lk}(\widehat{\gamma}, \widehat{\gamma}')| = 1$). Either these are the only periodic orbits of $\ker \omega$ or there are infinitely many of them. The second alternative holds if there exists a periodic orbit whose lift to S^3 is knotted or if there are two periodic orbits γ and γ' such that $|\mathrm{lk}(\widehat{\gamma}, \widehat{\gamma}')| \neq 1$.*

EXAMPLE 6.10. Let us verify the above general results in a concrete example. For any pair (p, q) of positive real numbers, consider the ellipsoid

$$\Sigma_{p,q} := \left\{ (z_1, z_2) \in \mathbb{C}^2 \ \Big| \ \frac{|z_1|^2}{2p} + \frac{|z_2|^2}{2q} = 1 \right\} \subset \mathbb{C}^2.$$

Since $\Sigma_{p,q}$ is convex, we know by Example 6.4 that it is also dynamically convex and, hence, by Theorem 6.6, its Reeb flow has a Poincaré section of disc-type. We can construct explicitly one such disc as follows. First, using the Hamilton equation for the Hamiltonian $(z_1, z_2) \mapsto \frac{|z_1|^2}{p} + \frac{|z_2|^2}{q}$, we get $R^\lambda = \frac{\partial_{\theta_1}}{p} + \frac{\partial_{\theta_2}}{q}$, where θ_i is the angular coordinate in the z_i-plane. Thus, $\Phi_t^{R^\lambda}(z_1, z_2) = (e^{\frac{it}{p}} z_1, e^{\frac{it}{q}} z_2)$ and we see that, for any $\theta \in \mathbb{T}_{2\pi}$, $S_\theta := \{(z_1, z_2) \in \Sigma_{p,q} \,|\, \theta_1 = \theta\}$ is a Poincaré section of disc-type parametrised by z_2. The return map is $F_{\mathring{S}_\theta}(z_2) = e^{2\pi i \frac{p}{q}} z_2$, namely a rotation by the angle $2\pi \frac{p}{q}$. Such rotation has the origin as the unique fixed point if and only if $\frac{p}{q}$ is irrational. When $\frac{p}{q}$ is rational, every orbit is periodic.

If ω is dynamically convex and, when $p > 1$, it is also assumed to be non-degenerate, Theorem 6.6 directly implies that the hypotheses of Corollary 6.9 are satisfied and, thus, we have a dichotomy between two and infinitely many periodic orbits for $\ker \omega$. We now aim to prove that the dichotomy still holds for a dynamically convex HS, without any non-degeneracy assumption.

We restrict to $p = 2$, since $SS^2 \simeq L(2, 1)$ and we will need only this case in the applications.

Regard $L(2, 1)$ as the quotient of S^3 by the antipodal map $A : S^3 \to S^3$. The quotient map $\pi : S^3 \to L(2, 1)$ is a double cover with A as the only non-trivial deck transformation. There is a bijection $Z \mapsto \widehat{Z}$ between $\Gamma(L(2, 1))$ and $\Gamma_A(S^3) \subset \Gamma(S^3)$ the subset of A-invariant vector fields. The antipodal map permutes the flow lines of $\widehat{Z}$. Moreover, a lift of a trajectory for Z is a trajectory for $\widehat{Z}$ and the projection of a trajectory for $\widehat{Z}$ is a trajectory for Z. In the next lemma we restrict this correspondence to prime contractible periodic orbits of Z. We are grateful to Marco Golla for communicating to us the elegant proof about the parity of the linking number presented below.

LEMMA 6.11. *There is a bijection between contractible prime orbits z of Z and pairs of antipodal prime orbits $\{\widehat{z}, A(\widehat{z})\}$ of $\widehat{Z}$ such that $\widehat{z}$ and $A(\widehat{z})$ are disjoint. Furthermore, the linking number $lk(\widehat{z}, A(\widehat{z}))$ between them is even.*

PROOF. Associate to a contractible periodic orbit z its two distinct lifts $\widehat{z}_1$ and $\widehat{z}_2 = A(\widehat{z}_1)$. Since z is contractible both lifts are closed. They are also prime since a lift of an embedded path is still embedded. Suppose that the two lifts intersect. This implies that there exist points t_1 and t_2 such that $\widehat{z}_1(t_1) = \widehat{z}_2(t_2)$. Applying π to this equality, we find $z(t_1) = z(t_2)$ and so $t_1 = t_2$ modulo the period of z. Hence, $\widehat{z}_1 = \widehat{z}_2$ contradicting the fact that the two lifts are distinct.

For the inverse correspondence, associate to two antipodal disjoint prime periodic orbits $\{\widehat{z}, A(\widehat{z})\}$ their common projection $\pi(\widehat{z})$. The projected curve is contractible since its lifts are closed. Furthermore, it is also prime since, if $\pi(\widehat{z})(t_1) = \pi(\widehat{z})(t_2)$,

either $\widehat{z}(t_1) = A(\widehat{z})(t_2)$ and $\widehat{z} \cap A(\widehat{z}) \neq \emptyset$ or $\widehat{z}(t_1) = \widehat{z}(t_2)$ and $t_1 = t_2$ modulo the period of $\widehat{z}$.

We now compute the linking number between the two knots. Consider S^3 as the boundary of B^4 the unit ball inside $\mathbb{R}^4$ and denote still by A the antipodal map on B^4, which extends the antipodal map on S^3. Take an embedded surface $S_1 \subset B^4$ such that $\partial S_1 = \widehat{z}_1$ and transverse to the boundary of B^4. By a small perturbation we can also assume that $0 \in B^4$ does not belong to the surface. The antipodal surface $S_2 := A(S_1)$ has the curve $\widehat{z}_2$ as boundary and $lk(\widehat{z}_1, \widehat{z}_2)$ is equal to the intersection number between S_1 and S_2. By perturbing again S_1 we can suppose that all the intersections are transverse. Indeed, if we change S_1 close to a point z of intersection, this will affect $S_2 = A(S_1)$ only near the antipodal point $A(z) = -z$, which is different from z since the origin does not belong to S_1. Now that transversality is achieved, we claim that the number of intersections is even. This stems from the fact that, if $z \in S_1 \cap S_2$, then $A(z) \in A(S_1) \cap A(S_2) = S_2 \cap S_1$ and z and $A(z)$ are different since $z \neq 0$. As a consequence, the algebraic intersection number between the two surfaces is even as well and the lemma follows. $\square$

REMARK 6.12. In the proof of the lemma, the sign of the intersection between S_1 and S_2 at z is the same as the sign at $A(z)$, since A preserves the orientation. Thus, we cannot conclude that the total intersection number is zero. Indeed, for any $k \in \mathbb{Z}$ one can find a pair of antipodal knots, whose linking number is $2k$.

PROPOSITION 6.13. If ω is a dynamically convex HS on $L(2,1)$, then $\ker \omega$ has either two or infinitely many periodic orbits. In the first case, the two orbits are non-contractible.

PROOF. By Corollary 6.9 there exist two prime closed orbits $\widehat{z}_1$ and $\widehat{z}_2$ of $\ker \pi^* \omega$ forming a Hopf link and if there is any other periodic orbit geometrically distinct from these two, $\ker \pi^* \omega$ has infinitely many periodic orbits.

We claim that $z_1 := \pi(\widehat{z}_1)$ and $z_2 := \pi(\widehat{z}_2)$ are geometrically distinct closed orbits for $\ker \omega$ on $L(2,1)$. If, by contradiction, z_1 coincides with z_2, by Lemma 6.11, $\widehat{z}_1$ and $\widehat{z}_2$ are antipodal and their linking number is even. This is a contradiction since $|lk(\widehat{z}_1, \widehat{z}_2)| = 1$. Therefore, we conclude that z_1 and z_2 are distinct. On the other hand, if $\ker \pi^* \omega$ has infinitely many periodic orbits the same is true for $\ker \omega$. Hence, also $\ker \omega$ has either 2 or infinitely many distinct periodic orbits.

If $\ker \omega$ has a prime contractible periodic orbit w, its lifts $\widehat{w}_1$ and $\widehat{w}_2$ are disjoint, antipodal and prime periodic orbits for $\ker \pi^* \omega$ by Lemma 6.11. Since $lk(\widehat{w}_1, \widehat{w}_2)$ is even, $\{\widehat{w}_1, \widehat{w}_2\} \neq \{\widehat{z}_1, \widehat{z}_2\}$ and, therefore, there are at least three distinct periodic orbits for $\ker \pi^* \omega$. So there are infinitely many periodic orbits for $\ker \pi^* \omega$ and, hence, also for $\ker \omega$. $\square$

6.2. Elliptic periodic orbits

Generally speaking, once we have established the existence of a periodic orbit (γ, T) for a flow Φ^Z, we can use it to understand the behaviour of the dynamical

system in a neighbourhood of γ by looking at the spectral properties of $t \mapsto d_{\gamma(0)}\Phi_t^Z$, the linearisation of the flow at the periodic orbit.

We have seen in Definition 2.10 that when γ is a non-degenerate periodic orbit of a Hamiltonian Structure ω, we have only two possibilities: either the transverse spectrum lies on the real line (γ hyperbolic) or it lies on the unit circle of the complex plane (γ elliptic).

As a sample of the applications one might get in the first case we mention the work of Hofer, Wysocki and Zehnder in [**HWZ03**]. The authors proved in Theorem 1.9 that a *generic* Reeb flow R^τ on S^3 without Poincaré sections of disc-type, has a hyperbolic orbit whose stable and unstable manifolds intersect in a homoclinic orbit. Following [**Mos65**, Chapter III], this yields the existence of a *Bernoulli shift* embedded in the flow of R^τ through the construction of local Poincaré sections. Since a Bernoulli shift has infinitely many periodic points, Φ^{R^τ} has infinitely many periodic orbits as well.

In contrast to the chaotic behaviour caused by the presence of the Bernoulli-shift, when γ is elliptic the flow is expected to be stable and quasi-periodic close to the periodic orbit. As explained in [**Mos65**, Section 2.4.d], if the transverse spectrum does not contain any root of unity, the KAM theorem implies the existence of a fundamental system of open neighbourhoods of the orbit, whose boundaries are invariant under the flow. In particular, each of these neighbourhoods is R^τ-invariant. Hence, γ is a stable periodic orbit and, as a result, the Reeb flow is non-ergodic.

Existence of elliptic periodic orbits for the Reeb flow on the boundary of convex domains in $\mathbb{C}^2$ has been proved in particular cases and it is an open problem raised by Ekeland [**Eke90**, page 198] to determine whether an elliptic orbit is to be found on every system of this kind. Its existence has been showed if

- the curvature satisfies a suitable pinching condition [**Eke86**];
- the hypersurface is symmetric with respect to the origin [**DDE95**].

The latter case is the most interesting to us since, in view of the double cover $S^3 \to SS^2$, lifted contact forms will automatically be invariant with respect to the antipodal map.

Recently, Abreu and Macarini have announced an extension of the results contained in [**DDE95**] to dynamically convex symmetric systems.

THEOREM 6.14 ([**AM**]). *If ω is a dynamically convex Hamiltonian Structure on $L(p, q)$, with $p > 1$, then it has an elliptic periodic orbit.*

In the same work, the authors will also define a notion of dynamical convexity for contact forms not necessarily on lens spaces. This has potential applications to symplectic magnetic fields on surfaces of genus at least two on low energy levels. We will comment on such generalisation in Section 7.4.1.

Low energy levels of symplectic magnetic flows on S^2

In this chapter we study in more detail *low* energy levels of magnetic flows (S^2, g, σ) when σ is a symplectic form normalised in such a way that $[\sigma] = 4\pi$. Throughout this chapter we are going to use the parameter m and the associated notation introduced in Section 2.3 just before Remark 2.8.

REMARK 7.1. Consider more generally a mechanical system (S^2, g, σ, U) with σ symplectic. We saw in Section 2.3.1 that when m is small and $\|U\|$ is small compared to m^2, the dynamics on Σ_m is the dynamics of a corresponding magnetic system on low energy level. Thus, we can apply to this case the results presented below.

Moreover, we saw in Section 2.3.2 a concrete example of mechanical system $(S^2, \hat{g}, \frac{k}{m}\sigma_{\hat{g}}, \hat{U}_k)$ obtained by reduction from $(SO(3), g, 0, U)$. The associated magnetic system is $(S^2, \hat{g}_{m,\hat{U}_k}, \frac{k}{m}\sigma_{\hat{g}})$, with $\|\hat{g}_{m,\hat{U}_k} - \hat{g}\| \leq \frac{2\|\hat{U}_k\|}{m^2}$ and

$$\frac{2\|\hat{U}_k\|}{m^2} = \frac{2\|U + \frac{k^2}{2\nu^2}\|}{m^2} \leq \frac{2\|U\|}{m^2} + \frac{k^2}{m^2}\left\|\frac{1}{\nu^2}\right\|.$$

Thus, we are allowed to approximate the reduced dynamics of $(SO(3), g, 0, U)$ with momentum k and energy $\frac{m^2}{2}$, with a symplectic magnetic system on a low energy level provided $\|U\|$ and k^2 are small with respect to m^2.

7.1. Contact forms on low energy levels

Given $\beta \in \mathcal{P}^{\sigma - \sigma_g}$, fix for the rest of the chapter a primitive $\lambda^g_{m,\beta} = m\lambda - \pi^*\beta + \tau$ of ω'_m. We know that there exists $m_\beta > 0$ such that, for $m \in [0, m_\beta)$, the function

$$h_{m,\beta}(x, v) := \lambda^g_{m,\beta}(X^m)_{(x,v)} = m^2 - \beta_x(v)m + f(x) \tag{7.1}$$

is positive and, as a consequence $\lambda^g_{m,\beta}$ is a positive contact form. Denote by $R^{m,\beta} := R^{\lambda^g_{m,\beta}}$, the Reeb vector field, by $\Phi^{m,\beta}$ its flow and by $\xi^{m,\beta} := \ker \lambda^g_{m,\beta}$ the contact distribution. We readily compute

$$R^{m,\beta} = \frac{X^m}{h_{m,\beta}} = \frac{m}{h_{m,\beta}}X + \frac{f}{h_{m,\beta}}V. \tag{7.2}$$

We proceed to relate the time parameter of a flow line for $\Phi^{m,\beta}$ and the length of its projection on S^2.

LEMMA 7.2. *If $\gamma : [0, T] \to SS^2$ is a flow line of $\Phi^{m,\beta}$, then*

$$\frac{m}{\max h_{m,\beta}}T \leq \ell(\pi(\gamma)) \leq \frac{m}{\min h_{m,\beta}}T, \tag{7.3}$$

where ℓ denotes the length of a curve in S^2 with respect to the metric g.

PROOF. We compute the norm of the tangent vector of $\pi(\gamma)$:

$$\left|\frac{d}{dt}\pi(\gamma)\right| = \left|d_\gamma\pi\left(\frac{d}{dt}\gamma\right)\right| = \left|\frac{m}{h_{m,\beta}}\gamma\right| = \frac{m}{h_{m,\beta}}. \tag{7.4}$$

Plugging this identity into the formula for the length of $\pi(\gamma)$, we get

$$\ell(\pi(\gamma)) = \int_0^T \left|\frac{d}{dt}\pi(\gamma)\right| dt = \int_0^T \frac{m}{h_{m,\beta}} dt \tag{7.5}$$

and the lemma follows bounding the integrand from below and from above. $\qquad\square$

We define a global ω'_m-symplectic trivialisation of $\xi^{m,\beta}$ in the next lemma.

LEMMA 7.3. *The contact structure $\xi^{m,\beta}$ admits a global ω'_m-symplectic frame:*

$$\begin{pmatrix} \check{H}^{m,\beta} := \dfrac{H + \beta_x(\jmath_x v)V}{\sqrt{h^{m,\beta}}} \\[2ex] \check{X}^{m,\beta} := \dfrac{X + (\beta_x(v) - m)V}{\sqrt{h^{m,\beta}}} \end{pmatrix}.$$

Call $\Upsilon^{m,\beta} : \xi^{m,\beta} \to (\epsilon^2_{SS^2}, \omega_{st})$ the symplectic trivialisation associated to this frame. It is given by $\Upsilon^{m,\beta}(Z) = \sqrt{h^{m,\beta}}(\eta(Z), \lambda(Z)) \in \mathbb{R}^2$.

PROOF. To find a basis for $\xi^{m,\beta}$, we set $\widetilde{H}^{m,\beta} := H + a_H V$ and $\widetilde{X}^{m,\beta} := X + a_X V$, for some $a_H, a_X \in \mathbb{R}$. Imposing $\lambda^g_{m,\beta}(\widetilde{H}^{m,\beta}) = 0$, we get

$$0 = m\lambda(H + a_H V) - \pi^*\beta(H + a_H V) + \tau(H + a_H V) = 0 - \beta_x(\jmath_x v) + a_H.$$

Hence, we have $a_H = \beta_x(\jmath_x v)$. In the same way we find $a_X = \beta_x(v) - m$. In order to turn this basis into a symplectic one, we compute

$$\begin{aligned} \omega'_m(\widetilde{H}^{m,\beta}, \widetilde{X}^{m,\beta}) &= \omega'_m(H + a_H V, X + a_X V) \\ &= \omega'_m(H, X) + a_X \omega'_m(H, V) + a_H \omega'_m(V, X) \\ &= -(-f) + a_X \cdot (-m) + a_H \cdot 0 \\ &= h_{m,\beta}. \end{aligned}$$

Thus $(\check{H}^{m,\beta}, \check{X}^{m,\beta})$, as defined in the statement of this lemma, is a symplectic basis. We now find the coordinates of $Z = a^1 \check{H}^{m,\beta} + a^2 \check{X}^{m,\beta}$ with respect to this basis:

$$\eta(Z) = \eta(a^1 \check{H}^{m,\beta} + a^2 X^{m,\beta}) = a^1 \eta(\check{H}^{m,\beta}) + a^2 \eta(X^{m,\beta}) = \frac{a^1}{\sqrt{h_{m,\beta}}} + a^2 \cdot 0.$$

In the same way, $\lambda(Z) = \frac{a^2}{\sqrt{h_{m,\beta}}}$, so that $(a^1, a^2) = \sqrt{h_{m,\beta}}(\eta(Z), \lambda(Z))$. $\qquad\square$

When $m = 0$, the objects defined above reduce to

$$\lambda^g_{0,\beta} = -\pi^*\beta + \tau \qquad\qquad R^{0,\beta} = V,$$

$$\check{H}^{0,\beta} = \frac{1}{\sqrt{f}}\left(H + \beta_x(\jmath_x v)V\right) \qquad\qquad X^{0,\beta} = \frac{1}{\sqrt{f}}\left(X + \beta_x(v)V\right).$$

In particular, $\lambda_{0,\beta}^g$ is an S^1-connection form on SS^2 with curvature σ. Therefore, we can think of $m \mapsto \lambda_{m,\beta}^g$ as a deformation of an S^1-connection with positive curvature through contact forms. This observation leads to the following result.

LEMMA 7.4. *There exists a diffeomorphism $F_{m,\beta} : SS^2 \to SS^2$ and a real function $q_{m,\beta} : SS^2 \to \mathbb{R}$ such that*

$$F_{m,\beta}^* \lambda_{m,\beta}^g = e^{q_{m,\beta}} \lambda_{0,\beta}^g. \tag{7.6}$$

The family of diffeomorphisms is generated by the vector field

$$Z^{m,\beta} := \frac{\check{H}^{m,\beta}}{\sqrt{h_{m,\beta}}} = \frac{1}{h_{m,\beta}}\Big(H + \beta_x(\jmath_x v)V\Big) \in \ker \lambda_{m,\beta}^g. \tag{7.7}$$

The map

$$[0, m_\beta) \longrightarrow C^\infty(SS^2, \mathbb{R})$$

$$m \longmapsto q_{m,\beta}$$

is smooth and admits the Taylor expansion at $m = 0$

$$q_{m,\beta} = \frac{1}{2f}m^2 + o(m^2). \tag{7.8}$$

PROOF. We apply Gray Stability Theorem (see for example [**Gei08**, Theorem 2.2.2]) to the family $m \mapsto \lambda_{m,\beta}^g$ and get the equation

$$\imath_{Z^{m,\beta}} \omega'_m = -\frac{d}{dm}\lambda_{m,\beta}^g = -\lambda, \qquad \text{with } Z^{m,\beta} \in \ker \lambda_{m,\beta}^g, \tag{7.9}$$

for $Z^{m,\beta}$ and the equation

$$\begin{cases} \dfrac{\partial}{\partial m}q_{m,\beta} = \left(\dfrac{d}{dm}\lambda_{m,\beta}^g\right)(R^{m,\beta})_{F_{m,\beta}} = \dfrac{m}{h_{m,\beta} \circ F_{m,\beta}} \\ q_{0,\beta} = 0 \end{cases} \tag{7.10}$$

for the function $q_{m,\beta}$. There exists a unique pair $(Z^{m,\beta}, q_{m,\beta})$ satisfying such relations. By Lemma 7.3, we know that $Z^{m,\beta} = a_H \check{H}^{m,\beta} + a_X \check{X}^{m,\beta}$. Using the fact that $(\check{H}^{m,\beta}, \check{X}^{m,\beta})$ is an ω'_m-symplectic basis, we get

$$a_H = \imath_{Z^{m,\beta}}\omega'_m(X^{m,\beta}) = -\lambda(X^{m,\beta}) = -\frac{1}{\sqrt{h_{m,\beta}}}$$

$$a_X = -\imath_{Z^{m,\beta}}\omega'_m(\check{H}^{m,\beta}) = \lambda(\check{H}^{m,\beta}) = 0.$$

Since the function $(m, z) \mapsto \frac{d}{dm}q_{m,\beta}(z)$ is smooth on $[0, m_\beta) \times SS^2$, the same is true for $(m, z) \mapsto q_{m,\beta}(z)$. Therefore, the map $m \mapsto q_{m,\beta}$ is smooth in the C^∞-topology and we can expand it at $m = 0$. From (7.10) we see that $q_{0,\beta} = 0$, $\frac{\partial}{\partial m}q_{0,\beta} = 0$ and $\frac{\partial^2}{\partial m^2}q_{0,\beta} = \frac{1}{h_{0,\beta} \circ F_{0,\beta}} = \frac{1}{f}$. Thus, (7.8) follows. $\square$

After this preliminary discussion we are ready to prove the results on periodic orbits we outlined in the introduction. In the next section we start with the expansion in the parameter m of the action function $S_m : SS^2 \to \mathbb{R}$ defined by Ginzburg in [**Gin87**]. We point out that this results holds true also for magnetic systems on surfaces of higher genus.

PROPOSITION 7.5. *There exists a smooth family of smooth functions $m \mapsto S_m$, where $S_m : SS^2 \to \mathbb{R}$, such that*

- *the critical points of S_m are the support of those periodic orbits of X^m which are close to a vertical fibre;*
- *the following expansion holds*

$$S_m = 2\pi + \frac{\pi}{f}m^2 + o(m^2). \tag{7.11}$$

COROLLARY 7.6. *If $x \in S^2$ is a non-degenerate critical point of $f : S^2 \to \mathbb{R}$, then there exists a smooth family of curves $m \mapsto \gamma_m$, such that*

- *γ_0 winds uniformly once around $S_x S^2$ in the positive sense;*
- *the support of γ_m is a periodic orbit for X^m.*

REMARK 7.7. Corollary 7.6 was already known by the experts, even if there is no explicit proof in the literature. Ginzburg mentions it in its survey paper [**Gin96**] in the paragraph after Remark 3.5 on page 136. Moreover, Castilho, using Arnold's approach of the guiding centre approximation [**Arn86**], constructs a local normal form for a magnetic system on low energy levels which is similar to the one we found in Lemma 7.4 (see [**Cas01**, Theorem 3.1]). It could be used to prove a local version of Proposition 7.5 and, hence, the corollary.

In Section 7.3 and Section 7.4, we are going to give two independent proofs of the following theorem.

THEOREM 7.8. *The 1-form $\lambda_{m,\beta}^g \in \Omega^1(SS^2)$ is dynamically convex for small m.*

Combining this theorem with the abstract results contained in Chapter 6 we get the result below.

COROLLARY 7.9. *Let (S^2, g, σ) be a symplectic magnetic system. If m is small enough, then the magnetic flow on Σ_m has either two or infinitely many periodic orbits. In the first case, the two orbits are non-contractible. Therefore, the second alternative holds if there exists a prime contractible periodic orbit.*

If we suppose in addition that Σ_m is a non-degenerate level, then

a) there is an elliptic periodic orbit (hence the system is not ergodic);

b) there is a Poincaré section of disc-type.

REMARK 7.10. The existence of two periodic orbits on low energy levels of symplectic magnetic systems is not new. In [**Gin87**, Assertion 3], Ginzburg proves the existence of 2 periodic orbits on S^2 and of 3 periodic orbits on surfaces of genus at least 1. Such orbits are close to the fibres of $\Sigma_m \to M$. Furthermore, when the energy level is non-degenerate, he improves these lower bounds by finding at least $4 - e_M$ periodic orbits, 2 of which are elliptic (see the characterisation of the return map as the Hessian of the action at the end of page 104 in [**Gin87**]).

REMARK 7.11. Example 4.13 exhibits energy levels where all orbits are periodic and no prime periodic orbit is contractible. Theorem 1.3 in [**Sch11**] (see also Section

7.6 below) shows that there exists an energy level with only two periodic orbits close to the fibres. It is still an open problem to find an energy level on a non-exact magnetic system on S^2 with exactly two periodic orbits. We point out that in the class of exact magnetic systems such example has been found. Indeed, Katok showed in [**Kat73**] that there are Randers metrics on S^2 with only two closed geodesics (see also [**Zil83**]) and G. Paternain in [**Pat99b**, Section 2] showed that every geodesic flow of a Randers metric arises, up to time reparametrisation, on a supercritical energy level of some exact magnetic field on S^2.

Combining Corollary 7.6 with Corollary 7.9, we can formulate a sufficient condition for the existence of infinitely many periodic orbits.

COROLLARY 7.12. *Let* $(S^2, g, \sigma = f\mu)$ *be a symplectic magnetic system. If the function* $f : S^2 \to \mathbb{R}$ *has three distinct critical points* $x_{\min}$, x_{Max} *and* x_{nondeg} *such that* $x_{\min}$ *is an absolute minimiser,* x_{Max} *is an absolute maximiser and* x_{nondeg} *is non-degenerate, then there exist infinitely many periodic orbits of the magnetic flow on every sufficiently small energy level.*

In Section 7.5 we prove Theorem I, establishing a dichotomy between short and long periodic orbits. We rephrase it here for the convenience of the reader.

THEOREM 7.13. *Suppose* $(g, \sigma) \in \text{Mag}(S^2)$ *and* σ *is symplectic. Given* $\varepsilon > 0$ *and a positive integer* n*, there exists* $m_{\varepsilon,n} > 0$ *such that for every* $m < m_{\varepsilon,n}$ *the projection* $\pi(\gamma)$ *of a periodic prime solution* $\gamma : \mathbb{T}_T$ *of* X^m *either is a simple curve on* S^2 *with length in* $(\frac{2\pi - \varepsilon}{\max f} m, \frac{2\pi + \varepsilon}{\min f} m)$ *or has at least* n *self-intersections and length larger than* $\frac{m}{\varepsilon}$*.*

This result was inspired to us by [**HS12**], where a completely analogous statement for Reeb flows on convex hypersurface in $\mathbb{C}^2$ close to the round S^3 is proved. In that paper the authors show that short orbits are unknotted, the linking number between two short orbits is 1, while the linking number between shorts and long orbits goes to infinity as the hypersurface approaches the round S^3. In view of Corollary 7.18, it is likely that Theorem 7.13 can also be proven as a corollary of Theorem 1.6 in [**HS12**] by making explicit the relation between linking numbers on S^3 and self-intersections of the projected curve on S^2.

Finally, the aim of Section 7.6 is to prove Proposition J regarding rotationally symmetric magnetic flows. In such proposition we formulate a sufficient condition for having infinitely many periodic orbits and a sufficient condition for having exactly two short periodic orbits, on every low energy level.

7.2. The expansion of the Ginzburg action function

We recall the definition of S_m. Set $\check{\lambda}_0 := \lambda_{0,\beta}^g$, $\check{\lambda}_m := e^{q_{m,\beta}}\check{\lambda}_0$. Let $\check{R}^m = R^{\check{\lambda}_m}$ be the associated Reeb vector field. For each $(x, v) \in SS^2$ we construct a local Poincaré section as follows. If y is a curve on S^2, denote by $P_t^y : S_{y(0)}S^2 \to S_{y(t)}S^2$ the parallel transport with respect to the S^1-connection $\check{\lambda}_0$. Let $B_\delta^2 \subset S^2$ be a small

geodesic ball centered at x. For every $x' \in B_\delta^2$, let $y_{x'}^x : [0,1] \to S^2$ be the geodesic connecting x to x' and define

$$i_{(x,v)} : B_\delta^2 \longrightarrow SS^2$$
$$x' \longmapsto \left(x', P_1^{y_{x'}^x}(v) \right).$$

The map $i_{(x,v)}$ is a local section of the S^1-bundle SS^2 such that $i_{(x,v)}(x) = (x,v)$. Therefore, it is transverse to $\check{R}^m$ for every m sufficiently small and there exists a first positive return time $t_m(x,v)$ such that $\Phi_{t_m(x,v)}^{\check{R}^m} = i_{(x,v)}(x'_m(x,v))$, for some $x'_m(x,v) \in B_\delta^2$. Define the path $\check{\gamma}_m(x,v) : [0,1] \to SS^2$ by

$$\check{\gamma}_m(x,v)(r) := \left(y_{x'_m(x,v)}^x(r), P_r^{y_{x'_m(x,v)}^x}(v) \right)$$

and define $\overline{\check{\gamma}_m}(x,v)$ to be the inverse path. Notice that $\frac{d}{dr}\check{\gamma}_m(x,v) \in \ker \check{\lambda}_0 = \ker \check{\lambda}_m$. The loop $\gamma_m(x,v) : [0,2] \to SS^2$ is obtained by concatenation

$$\gamma_m(x,v)(r) = \begin{cases} \Phi_{t_m(x,v)r}^{R^m}(x,v) & \text{for } r \in [0,1], \\ \overline{\check{\gamma}_m}(x,v)(r-1) & \text{for } r \in [1,2]. \end{cases} \qquad (7.12)$$

Denote by $\check{C}^\infty(\mathbb{T}_2, SS^2)$ the space of piecewise smooth loops of period 2 in SS^2 and notice that the family of maps $m \mapsto \left(\gamma_m : SS^2 \to \check{C}^\infty(\mathbb{T}_2, SS^2) \right)$ is smooth. Finally, define S_m as the composition of the action functional associated with $\check{\lambda}_m$ and the above embedding of SS^2 inside the space of loops:

$$S_m := \mathcal{A}^{\check{\lambda}_m} \circ \gamma_m : SS^2 \longrightarrow \mathbb{R}$$
$$(x,v) \longmapsto \int_{\mathbb{T}_2} \gamma_m(x,v)^* \check{\lambda}_m = t_m(x,v) + \int_{\mathbb{T}_1} \overline{\check{\gamma}_m}(x,v)^* \check{\lambda}_m$$
$$= t_m(x,v).$$

It is proven in [**Gin96**, Lemma 3, page 104] that, if (x,v) is a critical point of S_m, then $\Phi_{t_m(x,v)}^{\check{R}^m} = (x,v)$ and, therefore, (x,v) is in the support of a periodic orbit.

Notice that since the family $m \mapsto S_m$ is smooth, it admits an approximating Taylor expansion truncated at any order. To find such expansion, we observe that $\check{\lambda}_m = \frac{\check{\lambda}_0}{\pi^* \rho_m} + o(m^2)$ by Lemma 7.4, where $\rho_m : S^2 \to \mathbb{R}$ is given by $\rho_m(x) := 1 - \frac{m^2}{2f(x)}$. Thus, we prove Proposition 7.5 in two steps:

i) we find an expansion for $S'_m : SS^2 \to \mathbb{R}$, the action function for the 1-form $\check{\lambda}'_m := \frac{\check{\lambda}_0}{\pi^* \rho_m}$ and we see that is equal to the desired expansion of S_m;

ii) we show that $S_m - S'_m = o(m^2)$.

For the first step we need the lemma below.

LEMMA 7.14. *Let $\pi : E \to M$ be an S^1-bundle over a closed orientable surface. Let τ be an S^1-connection on E with positive curvature form σ. Fix $\rho : M \to (0, +\infty)$ a positive function and define the contact form $\tau_\rho := \frac{\tau}{\pi^* \rho} \in \Omega^1(E)$. Then, the Reeb vector field of τ_ρ splits as*

$$R_{(x,v)}^{\tau_\rho} = -L_{(x,v)}^{\mathcal{H}}\left((X_\rho)_x \right) + \rho(x) V_{(x,v)}, \qquad (7.13)$$

where $L^{\mathcal{H}}$ is the horizontal lift with respect to $\ker \tau$ and $X_\rho \in \Gamma(M)$ is the σ-Hamiltonian vector field associated to ρ (namely, $\imath_{X_\rho}\sigma = -d\rho$).

PROOF. Since σ is positive, τ is a contact form. Hence, also τ_ρ, which is obtained multiplying τ by a positive function, is a contact form as well. Without loss of generality we write $R^{\tau_\rho}_{(x,v)} = L^{\mathcal{H}}_{(x,v)}(Z_{(x,v)}) + a(x,v)V_{(x,v)}$, where $Z(x,v) \in T_x M$ and $a(x,v) \in \mathbb{R}$ are to be determined. Imposing that $1 = \tau_\rho(R^{\tau_\rho})$, we obtain that $a(x,v) = \rho(x)$. Imposing that $0 = \imath_{R^{\tau_\rho}}d\tau_\rho$, we get

$$0 = \imath_{L^{\mathcal{H}}(Z)+\pi^*\rho V}\left(\frac{-\pi^*\sigma}{\pi^*\rho} - \pi^*\left(\frac{d\rho}{\rho^2}\right) \wedge \tau\right)$$

$$= \pi^*\left(\frac{-\imath_Z\sigma + d\rho}{\rho}\right) - \pi^*\left(\frac{d\rho(Z)}{\rho^2}\right)\tau.$$

This implies that $-\imath_Z\sigma + d\rho = 0$ and $\frac{d\rho(Z)}{\rho^2} = 0$. The first condition implies that $-Z$ is the σ-Hamiltonian vector field with Hamiltonian ρ. Since the Hamiltonian function is a constant of motion, the second condition is also satisfied and the lemma is proven. □

Let us proceed to the proof of the proposition following steps *i)* and *ii)*.

PROOF OF PROPOSITION 7.5. Let us find the expansion for $m \to S'_m$. Since we already know that this expansion will be uniform, we can fix the point $(x,v) \in SS^2$ and expand the function $m \mapsto S'_m(x,v)$. We denote with a prime all the objects associated with $\check{\lambda}'_m$. Denote by $(\widetilde{x}(t), \widetilde{v}(t))$ the flow line of R'^m going through (x,v) at time 0. Consider $P^{\widetilde{x}}_{t'_m(x,v)}(v) \in S_{x'_m(x,v)}S^2$, the parallel transport of v along $\widetilde{x}$. The angle between $P^{\widetilde{x}}_{t'_m(x,v)}(v)$ and $\check{\gamma}'_m(x,v)(1)$ is equal to the integral of the curvature σ on a disc bounding the concatenated loop $\widetilde{x} * \pi\left(\overline{\check{\gamma}'_m}(x,v)\right)$. Since this loop is contained inside a ball of radius $d(x, x'_m(x,v))$, we can bound the area of the disc by some constant times $d(x, x'_m(x,v))^2$. However, $\frac{d}{dt}\widetilde{x} = X_{\rho_m} = -m^2 X_{2/f}$ implies that $d(x, x'_m(x,v)) \leq t_m(x,v) \cdot m^2 \left\|X_{2/f}\right\|$. Hence, the area of the disc goes to zero faster than m^3. As a consequence, the angle between $P^{\widetilde{x}}_{t'_m(x,v)}(v)$ and $\check{\gamma}'_m(x,v)(1)$ is of order $o(m^3)$. Consider the continuous loop $\mathring{\gamma}'_m(x,v)$ obtained by the concatenation of three paths:

- the path $(\widetilde{x}(t), P^{\widetilde{x}}_t(v))$;
- a path contained in the fibre over $x'_m(x,v)$ and connecting $P^{\widetilde{x}}_{t'_m(x,v)}(v)$ with $\check{\gamma}'_m(x,v)(1)$ following the shortest angle;
- the path $\overline{\check{\gamma}'_m}(x,v)$.

Since $\pi(\mathring{\gamma}'_m(x,v)) = \pi(\gamma'_m(x,v))$, we can consider the increment in the angle between $\mathring{\gamma}'_m(x,v)$ and $\gamma'_m(x,v)$ along this loop. As $\mathring{\gamma}'_m(x,v)(0) = \gamma'_m(x,v)(0)$, this angle will be a multiple of 2π. For $m = 0$ we readily see that the angle is 2π, hence it remains 2π by continuity. However, the derivative of the increment of this angle is

- $t'_m(x,v)\rho_m(x)$, over $\widetilde{x}$;
- of order $o(m^3)$, over the path contained in the fibre of $x'_m(x,v)$;
- zero, over $\pi(\overline{\check{\gamma}'_m}(x,v))$.

Putting things together, we find $2\pi = t'_m(x,v)\rho_m(x) + o(m^3)$, which implies Step i):

$$S'_m(x,v) = t'_m(x,v) = 2\pi\left(1 - \frac{m^2}{2f(x)}\right) + o(m^3).$$

For Step ii) we observe that there exists a smooth path $m \mapsto \check{\lambda}''_m \in \Omega^1(SS^2)$ such that $\check{\lambda}_m = \check{\lambda}'_m + m^3\check{\lambda}''_m$. As a consequence, $R^{\check{\lambda}_m} = R^{\check{\lambda}'_m} + m^3 R''_m$, for some smooth family $m \mapsto R''_m \in \Gamma(SS^2)$. This implies that $t_m(x,v) - t'_m(x,v) = o(m^2)$ and, hence, $S_m = t_m = t'_m + o(m^2) = S'_m + o(m^2)$. This establish Step ii) and the entire proposition. $\qquad\square$

We now have all the tools to prove the corollary.

PROOF OF COROLLARY 7.6. Applying the proposition we just proved, we find that $S_m = 2\pi + m^2 \check{S}_m$, where $\check{S}_0 = \frac{\pi}{f}$. Therefore, the critical points of S_m are the same as the critical points of $\check{S}_m$. Notice that the critical points of $\check{S}_0$ are the vertical fibres $S_x S^2$, where $x \in S^2$ is a critical point of f. Let x be a non-degenerate critical point of f and consider a tubular neighbourhood $\mathbb{T}_{2\pi} \times B_\delta$ of $S_x S^2$ with coordinates (ψ, z). For every $\psi \in \mathbb{T}_{2\pi}$ consider the restriction of $\check{S}_m$ to $\{\psi\} \times B_\delta$ and call it $\check{S}_m^\psi$. Since x is a non-degenerate critical point, the functions $\check{S}_0^\psi$ have $0 \in B_\delta$ as non-degenerate critical point. Therefore, by the inverse function theorem, for every sufficiently small m there is a path $z_m : \mathbb{T}_{2\pi} \to B_\delta$ such that $z_m(\psi)$ is the unique critical point of S_m^ψ. We claim that $\gamma_m(\psi) := (\psi, z_m(\psi)) \in \mathbb{T}_{2\pi} \times B_\delta$ is a critical point for $\check{S}_m$. Indeed, the function $S_m \circ \gamma_m : \mathbb{T}_{2\pi} \to \mathbb{R}$ has at least one critical point ψ_*, so that $d_{\gamma_m(\psi_*)} S_m\big(\frac{d}{d\psi}\gamma_m(\psi_*)\big) = 0$. Since $\frac{d}{d\psi}\gamma_m(\psi_*)$ is transverse to $\{0\} \times T_{z_m(\psi_*)} B_\delta \subset T_{\gamma_m(\psi_*)}\mathbb{T}_{2\pi} \times B_\delta$ and

$$d_{\gamma_m(\psi_*)} S_m\big|_{\{0\} \times T_{z_m(\psi_*)} B_\delta} = d_{z_m(\psi_*)} S_m^{\psi_*} = 0,$$

we have that $\gamma_m(\psi_*)$ is a critical point for S_m. However, the critical points for S_m comes in S^1-families since they correspond to periodic orbits of $\check{R}^m$. This implies the claim and finishes the proof of the corollary. $\qquad\square$

REMARK 7.15. The approximation of the magnetic flow with the Reeb flow of $R^{\check{\lambda}'_m}$ is called the *guiding centre approximation*. A precise formulation can be found in [**Arn97**]. It was used by Castilho in [**Cas01**] to prove a theorem on region of stabilities for the magnetic flow via Moser's invariant curve theorem [**Mos77**]. Recently, Raymond and Vũ Ngọc [**RVN13**] employed this approach to study the semiclassical limit of a magnetic flow with low energy close to a non-degenerate minimum of f.

7.3. Contactomorphism with a convex hypersurface

The main goal of this section is to construct a convex hypersurface in $\mathbb{C}^2$ which is a contact double cover of $(SS^2, \lambda^g_{m,\beta})$. Example 6.4 will then imply that $\lambda^g_{m,\beta}$ is dynamically convex and give a first proof of Theorem 7.8.

PROPOSITION 7.16. *If $m \in [0, m_\beta)$, there exists a double cover $p_{m,\beta} : S^3 \to SS^2$ and an embedding $v_{m,\beta} : S^3 \to \mathbb{C}^2$ bounding a region starshaped around the origin*

and such that $p_{m,\beta}^* \lambda_{m,\beta}^g = -v_{m,\beta}^* \lambda_{\mathrm{st}}$. *Furthermore, as m goes to zero, $v_{m,\beta}$ tends in the C^2-topology to the embedding of S^3 as the Euclidean sphere of radius 2. In particular, $v_{m,\beta}$ is a convex embedding for m sufficiently small.*

We construct the double cover $p_{m,\beta}$ in three steps. Let (S^2, g_0, μ_0) be the magnetic system on the round sphere of radius 1 given by the area form. We denote by SS_0^2 the unit sphere bundle, by $\jmath_0$ the rotation by $\pi/2$ and by τ_0 the vertical form associated with the metric g_0. We have already seen in Lemma 7.4 that there exists a contactomorphism $F_{m,\beta}$ between $e^{q_{m,\beta}} \lambda_{0,\beta}^g$ and $\lambda_{m,\beta}^g$. Our next task is to relate $\lambda_{0,\beta}^g$ with τ_0. For this purpose, we need the following proposition due to Weinstein [**Wei75**]. For a proof we refer to [**Gui76**, Appendix B].

PROPOSITION 7.17. *Suppose $E_i \to S^2$, with $i = 0, 1$, are two S^1-bundles endowed with S^1-connection forms $\tau_i \in \Omega^1(E_i)$. Call $\sigma_i \in \Omega^2(S^2)$ their curvature forms and suppose they are both symplectic and such that*

$$\left| \int_{S^2} \sigma_0 \right| = \left| \int_{S^2} \sigma_1 \right|, \tag{7.14}$$

Then, there is an S^1-equivariant diffeomorphism $B : E_0 \to E_1$ such that $B^* \tau_1 = \tau_0$.

Thanks to the normalisation $[\sigma] = 4\pi$ and the fact that $\lambda_{0,\beta}^g$ is an S^1-connection form on SS^2, we get the following corollary.

COROLLARY 7.18. *There is an S^1-equivariant diffeomorphism $B_\beta : SS_0^2 \to SS^2$ such that $B_\beta^* \lambda_{0,\beta}^g = \tau_0$.*

What we have found so far tells us that we only need to study the pull-back of τ_0 to S^3. This will be our next task. The ideas we use are taken from [**CO04, HP08**].

We identify $\mathbb{C}^2$ with the space of quaternions by setting $\mathbf{1} := (1, 0)$, $\mathbf{i} := (i, 0)$, $\mathbf{j} := (0, 1)$ and $\mathbf{k} := (0, i)$. With this choice left multiplication by $\mathbf{i}$ corresponds to the action of J_{st}. Let $v : S^3 \to \mathbb{C}^2$ be the inclusion of the unit Euclidean sphere. Identify the Euclidean space $\mathbb{R}^3$ with the vector space spanned by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ endowed with the restricted inner product. We think the round sphere (S^2, g_0) as embedded in this version of the Euclidean space. Thus, the unit sphere bundle SS_0^2 is embedded in $\mathbb{R}^3 \times \mathbb{R}^3$ as the pair of vectors (u_1, u_2) such that $u_1, u_2 \in S^2$ and $g_{\mathrm{st}}(u_1, u_2) = 0$.

If $z = (u_1, u_2) \in SS_0^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and $Z = (v_1, v_2) \in T_z SS_0^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$, then

$$(\tau_0)_z(Z) = g_0 \Big(v_2 - g_{\mathrm{st}}(v_2, u_1) u_1, \jmath_{0 u_1}(u_2) \Big) = g_{\mathrm{st}}(v_2, \jmath_{0 u_1}(u_2)) \tag{7.15}$$

as a consequence of the relation between the Levi Civita connections on S^2 and $\mathbb{R}^3$.

For any $U \in S^3$, we define a map $C_U : \mathbb{R}^3 \to \mathbb{R}^3$ using quaternionic multiplication and inverse by $C_U(U') = U^{-1} U' U$. The quaternionic commutation relations and the compatibility between the metric and the multiplication tell us that C_U restricts to an isometry of S^2. Hence, dC_U yields a diffeomorphism of the unit sphere bundle onto itself given by $(u_1, u_2) \mapsto d_{u_1} C_U(u_2) = (C_U(u_1), C_U(u_2))$. Moreover, since C_U is an isometry, $(dC_U)^* \tau_0 = \tau_0$.

We are now ready to define the covering map $p_0 : S^3 \to SS_0^2$. It is given by $p_0(U) := d_{\mathbf{i}} C_U(\mathbf{j})$. Let us compute the pull-back of τ_0 by p_0.

PROPOSITION 7.19. *The covering map p_0 relates τ_0 and λ_{st} in the following way:*

$$p_0^* \tau_0 = -4\upsilon^* \lambda_{\mathrm{st}}. \tag{7.16}$$

PROOF. First of all we prove that both sides of (7.16) are invariant under right multiplication. For every $U \in S^3$, we define $R_U : S^3 \to S^3$ as $R_U(U') := U'U$. Thus, the identity $p_0 \circ R_U = dC_U \circ p_0$ holds. Let us show that $p_0^* \tau_0$ is right invariant:

$$R_U^* \big(p_0^* \tau_0\big) = (p_0 \circ R_U)^* \tau_0 = (dC_U \circ p_0)^* \tau_0 = p_0^* \big((dC_U)^*(\tau_0)\big) = p_0^* \tau_0.$$

On the other hand, $\upsilon^* \lambda_{\mathrm{st}}$ is also right invariant:

$$\begin{aligned}
\big(R_U^*(\upsilon^* \lambda_{\mathrm{st}})\big)_{U'}(W) = \big(\upsilon^* \lambda_{\mathrm{st}}\big)_{R_U(U')}(dR_U W) &= \frac{1}{2} g_{\mathrm{st}}\big((\mathbf{i}U')U, WU\big) \\
&= \frac{1}{2} g_{\mathrm{st}}\big(\mathbf{i}U', W\big) \\
&= \big(\upsilon^* \lambda_{\mathrm{st}}\big)_{U'}(W),
\end{aligned}$$

where we used that $R_U : \mathbb{C}^2 \to \mathbb{C}^2$ is an isometry.

Therefore, it is enough to check equality (7.16) only at the point $\mathbf{1}$. A generic element W of $T_{\mathbf{1}} S^3$ can be written as $s\mathbf{i} + w\mathbf{j} = s\mathbf{i} + \mathbf{j}\overline{w}$, where $w := w_1 \mathbf{1} + w_2 \mathbf{i}$ and $\overline{w} := w_1 \mathbf{1} - w_2 \mathbf{i}$ with s, w_1 and w_2 real numbers. On the one hand,

$$(\upsilon^* \lambda_{\mathrm{st}})_{\mathbf{1}}(W) = \frac{1}{2} g_{\mathrm{st}}\big(\mathbf{i1}, W\big) = \frac{1}{2} g_{\mathrm{st}}\big(\mathbf{i}, s\mathbf{i} + w\mathbf{j}\big) = \frac{s}{2}. \tag{7.17}$$

On the other hand, we have that $d_{\mathbf{1}} p_0(W) = (\mathbf{i}W - W\mathbf{i}, \mathbf{j}W - W\mathbf{j})$. From the definition of τ_0 we see that we are only interested in the second component:

$$\mathbf{j}W - W\mathbf{j} = \mathbf{j}(s\mathbf{i} + \mathbf{j}\overline{w}) - (s\mathbf{i} + w\mathbf{j})\mathbf{j} = -2s\mathbf{k} + (w - \overline{w}) = -2s\mathbf{k} + 2w_2 \mathbf{i}.$$

Now we apply formula (7.15) with $(u_1, u_2) = (\mathbf{i}, \mathbf{j})$ and $v_2 = -2s\mathbf{k} + 2w_2 \mathbf{i}$. In this case $\jmath_{0_{u_1}}$ is left multiplication by $\mathbf{i}$, so that $\jmath_{0_{u_1}}(u_2) = \mathbf{ij} = \mathbf{k}$ and we find that

$$g_{\mathrm{st}}\big(-2s\mathbf{k} + 2w_2 \mathbf{i}, \mathbf{k}\big) = -2s. \tag{7.18}$$

Comparing (7.17) with (7.18) we finally get $(p_0^* \tau_0)_{\mathbf{1}} = -4(\upsilon^* \lambda_{\mathrm{st}})_{\mathbf{1}}$. $\square$

Putting things together, we arrive at the following intermediate step.

PROPOSITION 7.20. *There exists a covering map $p_{m,\beta} : S^3 \to SS^2$ and a real function $\widehat{q}_{m,\beta} : S^3 \to \mathbb{R}$ such that*

$$p_{m,\beta}^* \lambda_{m,\beta}^g = -4 e^{\widehat{q}_{m,\beta}} \upsilon^* \lambda_{\mathrm{st}}.$$

Moreover the function $\widehat{q}_{m,\beta}$ tends to 0 in the C^∞-topology as m goes to zero.

PROOF. Lemma 7.4 gives us $F_{m,\beta} : SS^2 \to SS^2$ and $q_{m,\beta} : SS^2 \to \mathbb{R}$. 7.18 gives us $B_\beta : SS_0^2 \to SS^2$. If we set $p_{m,\beta} := F_{m,\beta} \circ B_\beta \circ p_0 : S^3 \to SS^2$, then

$$\begin{aligned}
p_{m,\beta}^* \lambda_{m,\beta}^g = p_0^* \big(B_\beta^*(F_{m,\beta}^* \lambda_{m,\beta}^g)\big) &= p_0^* \big(B_\beta^*(e^{q_{m,\beta}} \lambda_{0,\beta}^g)\big) \\
&= p_0^* (e^{q_{m,\beta} \circ B_\beta} \tau_0) \\
&= -4 e^{q_{m,\beta} \circ B_\beta \circ p_0} \upsilon^* \lambda_{\mathrm{st}}.
\end{aligned}$$

Defining $\widehat{q}_{m,\beta} := q_{m,\beta} \circ B_\beta \circ p_0$ we only need to show that $\widehat{q}_{m,\beta}$ goes to 0 in the C^∞ topology. This is true since, by Lemma 7.4, the same holds for $q_{m,\beta}$. □

PROOF OF PROPOSITION 7.16. The final step in the proof is to notice that contact forms of the type $\rho v^* \lambda_{\mathrm{st}} \in \Omega^1(S^3)$ with $\rho : S^3 \to (0, +\infty)$ arise from embeddings of S^3 in $\mathbb{C}^2$ as the boundary of a star-shaped domain. To see this, define $v_{\sqrt{\rho}} : S^3 \hookrightarrow \mathbb{C}^2$ as $v_{\sqrt{\rho}}(z) := \sqrt{\rho(z)} v(z)$. A computation shows that $v_{\sqrt{\rho}}^* \lambda_{\mathrm{st}} = \rho v^* \lambda_{\mathrm{st}}$.

Using this observation we see that $\widehat{p}_{m,\beta}^* \lambda_{m,\beta}^g = -v_{m,\beta}^* \lambda_{\mathrm{st}}$ with $v_{m,\beta} := v_{\sqrt{\rho_{m,\beta}}}$ and $\rho_{m,\beta} := 4e^{\widehat{q}_{m,\beta}}$. For small m, $\rho_{m,\beta}$ is C^2-close to the constant 4 and, therefore, the embedding $v_{m,\beta}$ is C^2-close to the sphere of radius 2. This shows that $v_{m,\beta}(S^3)$ for small m and the proof is complete. □

REMARK 7.21. In general the convexity of the embedding v_ρ can be verified defining the function $Q_\rho : \mathbb{C}^2 \to [0, +\infty)$, where $Q_\rho(z) := |z|^2 / \rho(\frac{z}{|z|})$. Then, $v_{\sqrt{\rho}}(S^3) = \{Q_\rho = 1\}$ and $v_{\sqrt{\rho}}(S^3)$ is convex if and only if the Hessian of Q_ρ is positive.

7.4. A direct estimate of the index

In this subsection we present an alternative proof of Theorem 7.8 showing that $\lambda_{m,\beta}^g$ is dynamically convex via a direct estimate of the index, as prescribed by Definition 6.1. The advantage of this method is that it generalises to systems on surfaces of genus at least two as we discuss in Subsection 7.4.1. We can think of this proof as the magnetic analogue of what Harris and Paternain did for the geodesic flow, where the pinching condition on the curvature plays the same role as assuming m small and σ symplectic. After writing this proof, we discovered a similar argument in [**HS12**, Section 3.2].

To compute the index, we consider for each $z = (x, v) \in SS^2$ the path

$$\Psi_z^{m,\beta}(t) := \Upsilon_{\Phi_t^{m,\beta}(z)}^{m,\beta} \circ d_z \Phi_t^{m,\beta} \circ \left(\Upsilon_z^{m,\beta} \right)^{-1} \in \mathrm{Sp}(1).$$

We define the auxiliary path $B_z^{m,\beta}(t) := \dot{\Psi}_z^{m,\beta}(\Psi_z^{m,\beta})^{-1} \in \mathfrak{gl}(2, \mathbb{R})$. The bracket relations (2.7) for (X, V, H) allow us to give the following estimate for this path.

LEMMA 7.22. *We can write* $B_z^{m,\beta} = J_{\mathrm{st}} + \rho_z^{m,\beta}$, *where* $\rho_z^{m,\beta} : \mathbb{R} \to \mathfrak{gl}(2, \mathbb{R})$ *is a path of matrices, whose supremum norm is of order* $O(m)$ *uniformly in z as m goes to zero. In other words, there exist $\overline{m} > 0$ and $C > 0$ not depending on z, but only on $\sup f$, $\inf f$, $\|df\|$ and $\|\beta\|$, such that*

$$\forall m \leq \overline{m}, \quad \|\rho_z^{m,\beta}\| \leq Cm. \tag{7.19}$$

PROOF. For any point $z \in SS^2$ and for any vector $\overrightarrow{a_0} = (a_0^1, a_0^2) \in \mathbb{R}^2$ we have a path $\overrightarrow{a} = (a^1, a^2) : \mathbb{R} \to \mathbb{R}^2$ defined by the relation

$$\overrightarrow{a} = \Psi_z^{m,\beta} \overrightarrow{a}_0. \tag{7.20}$$

Using the definition of $\Psi_z^{m,\beta}$, we see that $\overrightarrow{a}$ satisfies

$$Z_z^{\overrightarrow{a_0}}(t) := d_z \Phi_t^{m,\beta} \left(a_0^1 \check{H}_z^{m,\beta} + a_0^2 \check{X}_z^{m,\beta} \right) = a^1(t) \check{H}_{\Phi_t^{m,\beta}(z)}^{m,\beta} + a^2(t) X_{\Phi_t^{m,\beta}(z)}^{m,\beta}.$$

If we differentiate with respect to t the identity

$$a_0^1 \check{H}_z^{m,\beta} + a_0^2 \check{X}_z^{m,\beta} = d_{\Phi_t^{m,\beta}(z)} \Phi_{-t}^{m,\beta} Z_z^{\vec{a_0}}(t), \tag{7.21}$$

we get the following differential equation for $\vec{a}$:

$$\begin{aligned}
0 = {}& \dot{a}^1(t) \check{H}_{\Phi_t^{m,\beta}(z)}^{m,\beta} + \dot{a}^2(t) \check{X}_{\Phi_t^{m,\beta}(z)}^{m,\beta} \\
& + a^1(t) \left[R^{m,\beta}, \check{H}^{m,\beta} \right]_{\Phi_t^{m,\beta}(z)} + a^2(t) \left[R^{m,\beta}, \check{X}^{m,\beta} \right]_{\Phi_t^{m,\beta}(z)}.
\end{aligned} \tag{7.22}$$

To estimate the first Lie bracket, we observe that $m \mapsto [R^{m,\beta}, \check{H}^{m,\beta}]$ is a $\Gamma(SS^2)$-valued map which is continuous in the C^0-topology since the maps $m \mapsto R^{m,\beta}$ and $m \mapsto \check{H}^{m,\beta}$ are continuous in the C^1-topology. As a result, we have that $\left[R^{m,\beta}, \check{H}^{m,\beta} \right] = \left[R^{0,\beta}, \check{H}^{0,\beta} \right] + \rho$, where ρ is an $O(m)$ in the C^0-topology, as m goes to zero. Furthermore,

$$\begin{aligned}
\left[R^{0,\beta}, \check{H}^{0,\beta} \right] = \left[V, \frac{1}{\sqrt{f}} (H + (\beta \circ \jmath) V) \right] &= \frac{1}{\sqrt{f}} \Big([V, H] + V(\beta \circ \jmath) V \Big) \\
&= \frac{1}{\sqrt{f}} \Big(-X - \beta V \Big) \\
&= -X^{0,\beta} \\
&= -X^{m,\beta} + \rho'.
\end{aligned}$$

Putting things together, $\left[R^{m,\beta}, \check{H}^{m,\beta} \right] = -X^{m,\beta} + \rho_1$, where ρ_1 is an $O(m)$ in the C^0-topology. In a similar way we find that $\left[R^{m,\beta}, X^{m,\beta} \right] = \check{H}^{m,\beta} + \rho_2$. Substituting these expressions for the Lie brackets inside (7.22), we find

$$\dot{a}^1 \check{H}^{m,\beta} + \dot{a}^2 \check{X}^{m,\beta} = a^1 \check{X}^{m,\beta} - a^2 \check{H}^{m,\beta} - a^1 \rho_1 - a^2 \rho_2.$$

Applying the trivialisation $\Upsilon^{m,\beta}$, we get

$$\dot{\vec{a}} = (J_{\mathrm{st}} + \rho_z^{m,\beta}) \vec{a}, \tag{7.23}$$

where $\rho_z^{m,\beta} \in \mathfrak{gl}(2, \mathbb{R})$ is of order $O(m)$.

On the other hand, differentiating Equation (7.20), we have

$$\dot{\vec{a}} = \dot{\Psi}_z^{m,\beta} \vec{a}_0 = \dot{\Psi}_z^{m,\beta} \left(\Psi_z^{m,\beta} \right)^{-1} \vec{a} = B_z^{m,\beta} \vec{a}. \tag{7.24}$$

Thus, comparing (7.23) and (7.24), we finally arrive at $B_z^{m,\beta} = J_{\mathrm{st}} + \rho_z^{m,\beta}$. $\qquad \square$

The previous lemma together with the following proposition reduces dynamical convexity to a condition on the period of Reeb orbits. First, we need the following notation. For each $Z \in \Gamma(N)$ we call $T_0(Z)$ the minimal period of a closed contractible orbit of Φ^Z. We set $T_0(Z) = 0$, if Φ^Z has a rest point. We remark that the map $Z \mapsto T_0(Z)$ is lower semicontinuous with respect to the C^0-topology.

PROPOSITION 7.23. *Let C be the constant contained in (7.19). If the inequality*

$$\frac{2\pi}{T_0(R^{m,\beta})} < 1 - Cm, \tag{7.25}$$

is satisfied, then $\lambda_{m,\beta}^g$ is dynamically convex.

PROOF. Let γ be a contractible periodic orbit for $R^{m,\beta}$ with period T and such that $\gamma(0) = z$. Consider $\Psi_z^{m,\beta}\big|_{[0,T]} \in \mathrm{Sp}_T(1)$ defined as before and fix a $u \in \mathbb{R}^2 \setminus \{0\}$. If $\theta_u^{\Psi_z^{m,\beta}}$ is defined by Equation (2.29), we can bound its first derivative by means of Lemma 7.22, as follows:

$$
\begin{aligned}
\dot{\theta}_u^{\Psi_z^{m,\beta}} &= \frac{g_{\mathrm{st}}\big(\dot{\Psi}_z^{m,\beta} u, J_{\mathrm{st}} \Psi_z^{m,\beta} u\big)}{|\Psi_z^{m,\beta} u|^2} = \frac{g_{\mathrm{st}}\big(B_z^{m,\beta} \Psi_z^{m,\beta} u, J_{\mathrm{st}} \Psi_z^{m,\beta} u\big)}{|\Psi_z^{m,\beta} u|^2} \\
&= \frac{g_{\mathrm{st}}\big((J_{\mathrm{st}} + \rho_z^{m,\beta}) \Psi_z^{m,\beta} u, J_{\mathrm{st}} \Psi_z^{m,\beta} u\big)}{|\Psi_z^{m,\beta} u|^2} \\
&= \frac{|\Psi_z^{m,\beta} u|^2 + g_{\mathrm{st}}\big(\rho_z^{m,\beta} \Psi_z^{m,\beta} u, J_{\mathrm{st}} \Psi_z^{m,\beta} u\big)}{|\Psi_z^{m,\beta} u|^2} \\
&\geq 1 - \|\rho_z^{m,\beta}\|.
\end{aligned}
$$

Hence, we can estimate the normalised increment in the interval $[0, T]$ by

$$
\Delta(\Psi_z^{m,\beta}\big|_{[0,T]}, u) = \frac{1}{2\pi} \int_0^T \dot{\theta}_u^{\Psi_z^{m,\beta}}(t) dt \geq (1 - Cm)\frac{T}{2\pi}.
$$

Therefore, by Remark (6.2), $\mu_{\mathrm{CZ}}^l(\gamma) \geq 3$ provided $(1 - Cm)\frac{T}{2\pi} > 1$. Asking this condition for every contractible periodic orbit is the same as asking Inequality (7.25) to hold. The proposition is thus proved. $\qquad\square$

We are now ready to reprove Theorem 7.8.

SECOND PROOF OF 7.8. Thanks to Proposition 7.23, it is enough to show that Inequality (7.25) holds for small m. First, we compute the periods of contractible orbits for $R^{0,\beta} = V$. A loop going around the vertical fibre k times with unit angular speed has period $2\pi k$ and is contractible if and only if k is even. Hence, $T_0(R^{0,\beta}) = 4\pi$.

Using the lower semicontinuity of the minimal period, we find that

$$
\limsup_{m \to 0} \left(\frac{2\pi}{T_0(R^{m,\beta})} + Cm \right) \leq \frac{2\pi}{4\pi} + 0 = \frac{1}{2} < 1
$$

and, therefore, the inequality is still true for m small enough. $\qquad\square$

7.4.1. Generalised dynamical convexity. Abreu and Macarini [**AM**] have recently defined a generalisation of dynamical convexity to arbitrary contact manifolds. We give a brief sketch of it in this subsection since it applies to low energy values of symplectic magnetic systems on surfaces of genus at least two.

Let (N, ξ) be a closed contact manifold and let ν be a free homotopy class of loops in N. Suppose that the contact homology $HC^\nu(N, \xi)$ of (N, ξ) in the class ν is well defined and that $c_1(\xi)$ is ν-atoroidal, so that the homology is also $\mathbb{Z}$-graded.

DEFINITION 7.24. Define k_- and k_+ in $\mathbb{Z} \cup \{-\infty, +\infty\}$ by

- $k_- := \inf_{k \in \mathbb{Z}} \left\{ HC_k^\nu(N, \xi) \neq 0 \right\}$, $\bullet$ $k_+ := \sup_{k \in \mathbb{Z}} \left\{ HC_k^\nu(N, \xi) \neq 0 \right\}$.   (7.26)

A ν-nondegenerate contact form α supporting ξ is called *positively ν-dynamically convex* if $k_- \in \mathbb{Z}$ and $\mu_{\mathrm{CZ}}(\gamma) \geq k_-$ for every Reeb orbit of α in the class ν. Similarly, α is called *negatively ν-dynamically convex* if $k_+ \in \mathbb{Z}$ and $\mu_{\mathrm{CZ}}(\gamma) \leq k_+$ for every Reeb orbit of α in the class ν.

Notice that a contact form α supporting (S^3, ξ_{st}), that is dynamically convex according to Definition 6.1, is indeed positively dynamically convex according to Definition 7.24, with $k_- = 3$. The new notion of convexity allows Abreu and Macarini to prove the existence of an elliptic orbit for Reeb flows on Boothby-Wang manifolds.

DEFINITION 7.25. A contact manifold (N, ξ) is called *Boothby-Wang* if it admits a supporting contact form β whose Reeb flow gives a free S^1-action.

THEOREM 7.26 ([**AM**]). *Let $(N, \xi = \ker \beta)$ be a Boothby-Wang contact manifold and let ν be the free homotopy class of the simple closed orbits of R^β. Assume that one of the following hypotheses holds:*

- *M/S^1 admits a Morse function whose critical points have all even index;*
- *all the iterates of ν are non-contractible.*

If $G < S^1$ is a non-trivial finite subgroup, then the Reeb flow of every G-invariant positively (respectively, negatively) ν-dynamically convex contact form α supporting ξ up to isotopy has an elliptic closed orbit γ representing ν such that $\mu_{\mathrm{CZ}}(\gamma) = k_-$ (respectively, $\mu_{\mathrm{CZ}}(\gamma) = k_+$).

Let M be a surface with genus at least two and let σ be a symplectic magnetic form such that $[\sigma] = 2\pi e_M$. In Subsection 4.2.3 we proved that when m is low the primitives of ω'_m of the type $\lambda^g_{m,\beta}$ are contact forms. Consider the cover $p : N \to SM$ which restricts to the standard $|e_M|$-sheeted cover above every fibre S_xM. Let ν_M be the class on N corresponding to the lift of a curve in SM winding $|e_M|$ times around S_xM. Then, $(N, \ker \lambda^g_{0,\beta})$ and the class ν_M satisfy the hypotheses of the theorem (a curve going around S_xM once is a free element in $\pi_1(SM)$ as can be seen from the long exact sequence of homotopy groups for the fibration $SM \to M$). Moreover, Abreu and Macarini computed $HC^\nu_*(N, \ker p^*\lambda^g_{m,\beta})$ and proved that $k_+ = 2e_M + 1$.

Therefore, in order to prove that $p^*\lambda^g_{m,\beta}$ is negatively ν-dynamically convex, and be able to apply the theorem, it is enough to show that, for every periodic orbit (γ, T) of $R^{\lambda^g_{m,\beta}}$ homotopic to a curve winding $|e_M|$ times around the fibres of $SM \to M$, the following inequality holds:

$$\mu_{\mathrm{CZ}}(\gamma) \leq 2e_M + 1. \tag{7.27}$$

Now we briefly explain how to adapt the argument used on S^2 to get Inequality (7.27). First of all, $R^{m,\beta}$ converges to $-V$ (and not to V), for m tending to 0. This difference of sign leads to the estimate $\|B_z^{m,\beta} - (-J_{\mathrm{st}})\| = Cm$ (compare with Lemma 7.22). We can use this inequality to obtain the upper bound

$$\Delta(\Psi_z^{m,\beta}\big|_{[0,T]}, u) \leq (-1 + Cm)\frac{T}{2\pi}, \qquad \forall u \in \xi_{m,\beta}\big|_{\gamma(0)}.$$

On the other hand, by the lower semicontinuity of the minimal period in the class ν_M, we have $T > 2\pi(|e_M| - \varepsilon)$, for an arbitrary ε and $m < m_\varepsilon$. Putting together these two inequalities we get

$$I\left(\Psi_z^{m,\beta}\big|_{[0,T]}\right) \subset (-\infty, e_M + Cm(|e_M| - \varepsilon) + \varepsilon) \subset (-\infty, e_M + 1)$$

for $m < m_\varepsilon$ small enough. This inclusion implies (7.27), thanks to (2.32).

7.4.2. A geometric estimate of the minimal period. We end this section by giving a geometric proof of Inequality (7.25) for small values of m giving an *ad hoc* proof of the lower semicontinuity of the minimal period in this case. The construction we present here will turn useful again in Section 7.5.

We consider a finite collection of closed discs $\mathbf{D} := \{D_i \mid D_i \subset S^2\}$ such that the open discs $\dot{\mathbf{D}} := \{\dot{D}_i\}$ cover S^2. We also fix a collection of vector fields of unit norm $\mathbf{Z} = \{Z_i \mid Z_i \in \Gamma(D_i), |Z_i| = 1\}$. Let δ be the Lebesgue number of the cover $\mathbf{D}$ with respect to the Riemannian distance. Finally, let $\varphi_i : SD_i \to \mathbb{T}_{2\pi}$ be the angular function associated to Z_i. We set

$$C_{(\mathbf{D},\mathbf{Z})}^\beta(m) := \sup_i \left(\sup_{D_i} \frac{|d\varphi_i(X)|}{h_{m,\beta}} \right)$$

and we observe that $m \mapsto C_{(\mathbf{D},\mathbf{Z})}^\beta(m)$ is locally bounded around $m = 0$. We now write down the condition of lower semicontinuity for the minimal period explicitly. Let $\varepsilon > 0$ be arbitrary. We claim that there exists $m_\varepsilon > 0$ such that, for $m < m_\varepsilon$, the period T of a contractible periodic orbit γ of $R^{m,\beta}$ is bigger than $4\pi - \varepsilon$. The claim will follow from the next two lemmas.

LEMMA 7.27. *Suppose that (γ, T) is a periodic orbit for $R^{m,\beta}$ and that $\pi(\gamma)$ is not contained in any D_i. If T_* is any positive real number and $m < \frac{2\delta \inf h_{m,\beta}}{T_*}$, then $T > T_*$.*

PROOF. By assumption $\pi(\gamma)$ is not contained in any ball of radius δ. Thus,

$$2\delta \leq \ell(\pi(\gamma)) \leq \frac{m}{\inf h_{m,\beta}} T < \frac{2\delta}{T_*} T,$$

where the second inequality is obtained using Lemma 7.27. Multiplying both sides by $\frac{T_*}{2\delta}$ yields the desired conclusion. □

LEMMA 7.28. *Suppose that (γ, T) is a periodic orbit for $R^{m,\beta}$ and that $\pi(\gamma)$ is contained in some D_{i_0}. Let $\widetilde{\varphi}_{i_0} : [0, T] \to \mathbb{R}$ be a lift of $\varphi_{i_0} \circ \gamma\big|_{[0,T]} : [0, T] \to \mathbb{T}_{2\pi}$ and let $2\pi N := \widetilde{\varphi}_{i_0}(T) - \widetilde{\varphi}_{i_0}(0)$, with $N \in \mathbb{Z}$. For every $\varepsilon' > 0$, there exists $m'_{\varepsilon'} > 0$ small enough and independent of (γ, T) such that $N \geq 1$ and*

$$T \geq 2\pi N(1 - \varepsilon'). \tag{7.28}$$

PROOF. Using the fundamental theorem of calculus we get

$$\widetilde{\varphi}_{i_0}(T) - \widetilde{\varphi}_{i_0}(0) = \int_0^T \frac{d\varphi_{i_0}}{dt} dt = \int_0^T d\varphi_{i_0}(R^{m,\beta}) dt.$$

Since $d\varphi_{i_0}(R^{m,\beta}) = md\varphi_{i_0}\left(\frac{X}{h_{m,\beta}}\right) + \frac{f}{h_{m,\beta}}$ and $\frac{f}{h_{0,\beta}} = 1$, we have, for m small enough,

$$d\varphi_{i_0}(R^{m,\beta}) \geq \inf_{SS^2} \frac{f}{h_{m,\beta}} - mC^\beta_{(\mathbf{D},\mathbf{Z})}(m) > 0. \tag{7.29}$$

This implies that $N \geq 1$ and, moreover, that

$$2\pi N \leq \int_0^T d\varphi_{i_0}(R^{m,\beta})dt \leq \left(\sup_{SS^2} \frac{f}{h_{m,\beta}} + mC^\beta_{(\mathbf{D},\mathbf{Z})}(m)\right)T.$$

Exploiting $\frac{f}{h_{0,\beta}} = 1$ again, we see that there exists $m'_{\varepsilon'}$ such that, if $m < m'_{\varepsilon'}$,

$$2\pi N(1 - \varepsilon') < \frac{2\pi N}{\sup_{SS^2} \frac{f}{h_{m,\beta}} + mC^\beta_{(\mathbf{D},\mathbf{Z})}(m)} \leq T. \qquad \square$$

We now prove the claim taking $m_\varepsilon := \min\left\{\frac{2\delta \inf h_{m,\beta}}{4\pi - \varepsilon}, m'_{\varepsilon'}\right\}$, where $m'_{\varepsilon'}$ is obtained from Lemma 7.28 with a value of ε' given by the equation $4\pi(1 - \varepsilon') = 4\pi - \varepsilon$. If γ is not contained in a D_i, the claim follows from Lemma 7.27 with $T_* = 4\pi - \varepsilon$. If γ is contained in some D_{i_0}, then N is even since γ is contractible. Therefore, $T > 4\pi(1 - \varepsilon') = 4\pi - \varepsilon$ and the claim is proved also in this case.

7.5. A dichotomy between short and long orbits

In this section we prove Theorem 7.13 (and, hence, Theorem I). The statement will readily follow from the general Corollary 7.30 after we show that

- periodic orbits, whose projection on the base S^2 has a fixed number of self-intersections, have bounded length (Corollary 7.33);
- the projection on S^2 of periodic orbits with period close to 2π is a simple curve (Lemma 7.34).

The foundational result for our proof is Proposition 1 of [**Ban86**].

PROPOSITION 7.29 (Bangert). *Let $\overline{\Phi}: \mathbb{R} \times M \to M$ be a C^1-flow and $\overline{p} \in M$ a periodic point of prime period $\overline{T} > 0$. Then, for every $\varepsilon > 0$ there exist a neighbourhood $\mathfrak{U}$ of $\overline{\Phi}$ in the weak C^1-topology on $C^1(M \times \mathbb{R}, M)$ and a neighbourhood U of $\overline{p}$ in M such that the following is true: If a flow $\Phi \in \mathfrak{U}$ has a periodic point $p \in U$ with prime period T then $T > \varepsilon^{-1}$ or there exists a positive integer k such that $|kT - \overline{T}| < \varepsilon$ and the linear map $d_{\overline{p}}\Phi_{\overline{T}}: T_{\overline{p}}M \to T_{\overline{p}}M$ has an eigenvalue which generates the group of k-th roots of unity.*

We can apply this proposition to $\overline{T}$-periodic flows. A flow is $\overline{T}$-periodic if and only if through every point there is a prime periodic orbit whose period is $\overline{T}$.

COROLLARY 7.30. *Let $\overline{\Phi}$ be a $\overline{T}$-periodic flow on a compact manifold M. Then, for every $\varepsilon > 0$ there exists a C^1-neighbourhood $\mathfrak{U}$ of $\overline{\Phi}$, such that if γ is a periodic orbit of $\Phi \in \mathfrak{U}$ with prime period T, either $T > \varepsilon^{-1}$ or $|T - \overline{T}| < \varepsilon$.*

Let us go back to symplectic magnetic systems on low energy levels on S^2. We observed before that for $m = 0$ the Reeb vector field is equal to the vector field V. Hence its flow is 2π-periodic and we can apply Bangert result for m small.

COROLLARY 7.31. *For every $\varepsilon > 0$, there exists $m_\varepsilon > 0$, such that, if $m < m_\varepsilon$, every prime periodic orbit γ of $R^{m,\beta}$ on SS^2 either has period bigger than ε^{-1} or in the interval $(2\pi - \varepsilon, 2\pi + \varepsilon)$.*

In the next lemma we prove that knowing the number of self intersections of $\pi(\gamma)$ gives a uniform bound on $\ell(\pi(\gamma))$. Thanks to Lemma 7.2, this will imply a bound on the period, as well.

LEMMA 7.32. *Let $x : I \to S^2$ be a closed curve which is a smooth immersion except possibly at n_0 points and has no more than n_1 self-intersections. Suppose moreover that, when defined, the geodesic curvature of x is bounded away from zero. Namely, $\min k_x > 0$. Then,*

$$\ell(x) \leq \frac{\pi(4n_1 + 2 + n_0) + (n_1 + 1)|\min K| \operatorname{vol}_g(S^2)}{\min k_x} \tag{7.30}$$

PROOF. We argue by induction on n_1. Suppose $n_1 = 0$, then x bounds a disc D and we can apply the Gauss-Bonnet formula:

$$\int_x k_x(t)dt = 2\pi - \int_D K dA - \sum_{i=1}^{n_0} \vartheta_i.$$

Since $\vartheta_i \geq -\pi$, the right-hand side is smaller than $2\pi + |\min K| \operatorname{vol}_g(S^2) + n_0\pi$. The left-hand side is bigger than $\ell(x) \min k_x$ and dividing both sides by $\min k_x$ we get (7.30) in this case.

Take now a curve x with $n_1 \geq 1$ self-intersections. Then, there exists a smooth loop x_1 inside x. Up to a change of basepoint, x is the concatenation of x_1 and x_2, where x_2 has at most $n_1 - 1$ self-intersections and $n_0 + 1$ corners. By induction we can apply inequality (7.30) to the two pieces:

$$\ell(x_1) \leq \frac{3\pi + |\min K| \operatorname{vol}_g(S^2)}{\min k_{x_1}},$$

$$\ell(x_2) \leq \frac{\pi(4n_1 - 2 + n_0 + 1) + n_1|\min K| \operatorname{vol}_g(S^2)}{\min k_{x_2}}.$$

Since $\min k_x = \min\{\min k_{x_1}, \min k_{x_2}\}$ we can substitute it in the denominators above without affecting the inequality. Adding up the resulting relations we get

$$\ell(x) = \ell(x_1) + \ell(x_2) \leq \frac{\pi(3 + 4n_1 - 2 + n_0 + 1) + (n_1 + 1)|\min K| \operatorname{vol}_g(S^2)}{\min k_x},$$

which is the desired result. $\square$

We get immediately the following corollary.

COROLLARY 7.33. *Let γ be a periodic orbit of $R^{m,\beta}$ such that $\pi(\gamma)$ has at most n self-intersections. Then,*

$$\frac{\ell(\pi(\gamma))}{m} \leq \frac{4n + 2 + (n + 1)|\min K| \operatorname{vol}_g(S^2)}{\min f}$$

and, therefore,

$$T \leq \max h_{m,\beta} \frac{4n + 2 + (n + 1)|\min K| \operatorname{vol}_g(S^2)}{\min f} =: C_n. \tag{7.31}$$

PROOF. To get the first inequality, we apply Lemma 7.32 to $\pi(\gamma)$, which has at most n self-intersections and no corners, and observe that $k_{\pi(\gamma)} = \frac{f(\pi(\gamma))}{m}$. The second inequality follows from Lemma 7.2.   □

If we fix $n \in \mathbb{N}$, Corollaries 7.33 and 7.31 show that, for m small enough, a solution with n self-intersections can exist if and only if its period is close to 2π. The next step will be to prove that this happens only if $\pi(\gamma)$ is a simple curve.

LEMMA 7.34. *For every $\varepsilon' > 0$, there exists $\overline{m}_{\varepsilon'} > 0$ such that, if $m \leq \overline{m}_{\varepsilon'}$ and γ is a prime periodic orbit whose projection $\pi(\gamma)$ has at least 1 self-intersection, the period of γ is bigger than $4\pi - \varepsilon'$.*

PROOF. We use the same strategy of Section 7.4.2 and we consider a collection of discs and unit vector field $(\mathbf{D}, \mathbf{Z})$. The lemma is proven once we show that, if $\pi(\gamma) \subset D_i$, for some D_i, then $N \geq 2$. We achieve this goal by making a particular choice of $(\mathbf{D}, \mathbf{Z})$. Namely, we take D_i contained in some orthogonal chart (x_i^1, x_i^2) and we take $Z_i := \partial_{x_i^1}/|\partial_{x_i^1}|$.

Suppose without loss of generality that the self-intersection is happening at $t = 0$. Thus, we know that there exists $t_* \in (0, T)$ such that $\pi(\gamma(0)) = \pi(\gamma(t_*))$. By precomposing the coordinate chart with an orthogonal linear map we can assume that the tangent vector at zero is parallel to $\partial_{x_i^1}$, namely that $\varphi_i(0) = 0$. We consider the function $x_i^2 \circ \gamma : \mathbb{T}_T \to \mathbb{R}$ and compute its derivative. Since the parametrisation is orthogonal we know that

$$\frac{d}{dt}\pi(\gamma) = \left|\frac{d}{dt}\pi(\gamma)\right| \left(\cos\varphi_i \frac{\partial_{x_i^1}}{|\partial_{x_i^1}|} + \sin\varphi_i \frac{\partial_{x_i^2}}{|\partial_{x_i^2}|}\right).$$

However, in coordinates we always have

$$\frac{d}{dt}\pi(\gamma) = \frac{d}{dt}(x_i^1 \circ \gamma)\partial_{x_i^1} + \frac{d}{dt}(x_i^2 \circ \gamma)\partial_{x_i^2}.$$

Comparing the two expressions we find

$$\frac{d}{dt}(x_i^2 \circ \gamma) = \frac{\left|\frac{d}{dt}\pi(\gamma)\right|\sin\varphi_i}{|\partial_{x_i^2}|}. \tag{7.32}$$

A point t_0 is critical for $x_i^2 \circ \gamma$ if and only if $\varphi_i(t_0) = 0$ or $\varphi_i(t_0) = \pi$. Computing the second derivative at a critical point we get

$$\frac{d^2}{dt^2}\Big|_{t=t_0}(x_i^2 \circ \gamma) = \frac{\left|\frac{d}{dt}\pi(\gamma)\right|}{|\partial_{x_i^2}|}\cos\varphi_i\dot{\varphi}_i\Big|_{t=t_0}. \tag{7.33}$$

Since $\dot{\varphi}_i > 0$ by (7.29), we know that t_0 is a strict local minimum if $\varphi_i(t_0) = 0$ and a strict local maximum if $\varphi_i(t_0) = \pi$. From the previous discussion we know that $t = 0$ is a strict minimum and that $x_i^2 \circ \gamma(0) = x_i^2 \circ \gamma(t_*)$. Hence, there must exist two strict maxima $t_1 \in (0, t_*)$ and $t_2 \in (t_*, T)$. At these points $\varphi_i = \pi$ and, thus, $N \geq 2$.   □

We are now ready to prove Theorem 7.13.

PROOF OF THEOREM 7.13. Given ε and n, take any $\varepsilon'_{\varepsilon,n}$ such that the following inequalities hold for small m:

- $\varepsilon'_{\varepsilon,n} < \min\{1, \varepsilon\}$,   • $C_{n-1} < \dfrac{1}{\varepsilon'_{\varepsilon,n}}$,   • $\varepsilon'_{\varepsilon,n} \max h_{m,\beta} < \varepsilon$,

where C_{n-1} is the constant defined in (7.31). Define $\widetilde{m}_{\varepsilon,n} = \min\{m_{\varepsilon'_{\varepsilon,n}}, \overline{m}_{\varepsilon'_{\varepsilon,n}}\}$, where $m_{\varepsilon'_{\varepsilon,n}}$ and $\overline{m}_{\varepsilon'_{\varepsilon,n}}$ are given by Corollary 7.31 and Lemma 7.34, respectively.

If $m < \widetilde{m}_{\varepsilon,n}$ and γ is a periodic orbit with period smaller than $\frac{1}{\varepsilon'_{\varepsilon,n}}$, using Corollary 7.31, we find that $T \in (2\pi - \varepsilon'_{\varepsilon,n}, 2\pi + \varepsilon'_{\varepsilon,n})$. Thus $T < 4\pi - \varepsilon'_{\varepsilon,n}$ and, by Lemma 7.34, $\pi(\gamma)$ is a simple curve in S^2. We use Lemma 7.2 to estimate the length of $\pi(\gamma)$ in this case:

$$\frac{2\pi - \varepsilon'_{\varepsilon,n}}{\max h_{m,\beta}} m \leq \ell(\pi(\gamma)) \leq \frac{2\pi + \varepsilon'_{\varepsilon,n}}{\min h_{m,\beta}} m. \tag{7.34}$$

Shrinking the interval $[0, \widetilde{m}_{\varepsilon,n}]$ if necessary, we find some $m_{\varepsilon,n} \leq \widetilde{m}_{\varepsilon,n}$, such that, if $m \in [0, m_{\varepsilon,n}]$, inequalities (7.34) imply

$$\frac{2\pi - \varepsilon}{\max f} m \leq \ell(\pi(\gamma)) \leq \frac{2\pi + \varepsilon}{\min f} m,$$

as required. If the period T is bigger than $\frac{1}{\varepsilon'_{\varepsilon,n}} > \frac{1}{\varepsilon}$, then $\pi(\gamma)$ has at least n self-intersections by the condition $C_{n-1} < \frac{1}{\varepsilon'_{\varepsilon,n}}$ and Corollary 7.33. In this case we also have

$$\ell(\pi(\gamma)) \geq \frac{m}{\varepsilon'_{\varepsilon,n} \max h_{m,\beta}} \geq \frac{m}{\varepsilon}.$$

This finishes the proof of the theorem. $\qquad\square$

7.6. A twist theorem for surfaces of revolution

As observed in Remark 7.11, we do not have any example of a contact-type energy level for a non-exact magnetic system on S^2 with exactly two periodic orbits. This is due to the fact that the only case where we can compute all the trajectories of the magnetic flow is Example 4.13, where all the orbits are shown to be periodic.

Observe, indeed, that knowing that the flow of $X^m \in \Gamma(SS^2)$ is a perturbation of a flow with exactly two periodic orbits, whose iterations are assumed to be non-degenerate, is not enough to deduce that the magnetic flow has only two periodic orbits. The only thing that we can deduce is that X^m has exactly two *short* periodic orbits [**Sch11**]. However, *long* periodic orbits may appear in the perturbed system.

On the other hand, if the perturbation satisfies a suitable *twist condition*, we can prove the existence of infinitely many long periodic orbits for the new flow.

In this section we are going to use such perturbative approach on surfaces of revolution. Unlike Chapter 5, we consider more general systems of the kind $(S^2_\gamma, g_\gamma, f\mu_\gamma)$, where $f : S^2_\gamma \to (0, +\infty)$ is any positive function invariant under the rotations around the axis of symmetry. This means that f depends on the t-variable only and we can write $f : [0, \ell_\gamma] \to (0, +\infty)$.

We know that for m small, $\lambda^g_{m,\beta\gamma}$ is a dynamically convex contact form and, hence, at least in the non degenerate case, Φ^{m,β^γ} has a Poincaré section of disc-type

$i_{m,\beta^\gamma} : D \to SS^2_\gamma$. Moreover, since Φ^{0,β^γ} is 2π-periodic, we know that the return map $F_{m,\beta^\gamma} : \mathring{D} \to \mathring{D}$ for $m = 0$ is the identity. Then, if we could find explicitly a *smooth* isotopy $m \mapsto i_{m,\beta^\gamma}$ of Poincaré sections, we could try to expand the maps F_{m,β^γ} in the parameter m around zero to get some information on the dynamics. However, it is in general a difficult task to construct by hand a Poincaré section and the rotational symmetry does not seem to help finding a disc in this case. On the other hand, we claim that such symmetry allows to find a Poincaré section of *annulus* type, as we will show in the next subsection.

Historically, annuli were the first kind of Poincaré sections to be studied. The discovery of such section in the restricted 3-body problem [**Poi87**] led Poincaré to formulate his Last Geometric Theorem [**Poi12**]. It asserts the existence of infinitely many periodic points of the return map F, if such map satisfies the so-called *twist condition*: namely, F extends continuously to the boundary of the annulus and it rotates the inner and outer circle by different angles. One year later Birkhoff proved Poincaré's Theorem [**Bir13**] and in subsequent research found a section of annulus type for the geodesic flow of a convex two-sphere [**Bir66**, Chapter VI.10].

In the last subsection we are going to give a condition on γ and f that ensures that the Poincaré maps for small values of m are twist. By contrast, the complementary condition will single out a class of magnetic systems whose long periodic orbits have period or order $O(m^{-2})$ or higher. It would be interesting to compare this estimate with the abstract divergence rate coming from Bangert's proof of Proposition 7.29. A direction of future research would be to look for systems with exactly 2 periodic orbits within this class.

From now on, we consider a fixed profile function γ and the rotationally invariant primitive $\beta^\gamma = \beta^\gamma_\theta d\theta$, so that we can safely suppress the symbols γ and β from the subscripts and superscripts.

7.6.1. Definition of the Poincaré map. Take the loop $c : [-\ell, \ell]/_\sim \to SS^2$, where $\sim$ is the relation that identifies the boundary of the interval. It is defined in the coordinates (t, φ, θ), as

$$c(u) := \begin{cases} (-u, -\pi/2, 0), & \text{if } u < 0, \\ (u, \pi/2, \pi), & \text{if } u > 0. \end{cases}$$

We extend it smoothly for $u \in \{-\ell, 0, \ell\}$ and we observe that $t(c(u)) = |u|$. Using the $\mathbb{T}_{2\pi}$-action given by the rotational symmetry we can move transversally c in SS^2 and form an embedded 2-torus $C : [-\ell, \ell]/_\sim \times \mathbb{T}_{2\pi} \to SS^2$:

$$C(u, \psi) := \begin{cases} (-u, -\pi/2, \psi), & \text{if } u < 0, \\ (u, \pi/2, \psi + \pi), & \text{if } u > 0. \end{cases}$$

In particular, notice that $\psi = \theta$, for $u < 0$ and $\psi = \theta + \pi$, for $u > 0$. Since m is sufficiently small, Proposition 5.9 implies that $X^m_\gamma := mX + fV$ has only two periodic orbits ζ^-_m and ζ^+_m supported on latitudes t^-_m and t^+_m.

The sign tells us whether the projection of $\zeta_m^\pm$ rotates in the same direction as ∂_θ or not. The numbers $t_m^\pm$ satisfy the equation

$$\pm m\dot\gamma(t) = f(t)\gamma(t).$$

With our choices we have $t_m^+ < t_m^-$. These two orbits lie inside the image of the torus C. If we define $u_m^- := -t_m^-$ and $u_m^+ := t_m^+$, then $\psi \mapsto C(u_m^\pm, \psi)$ is a reparametrisation of $\zeta_m^\pm$. Hence, the 2-torus is divided into the union of two closed cylinders C_m^- and C_m^+ with the two orbits as common boundary. Each of the cylinders is a Poincaré section for X^m and we would like to compute its first return map $F_m^\pm : \mathring{C}_m^\pm \to \mathring{C}_m^\pm$. If we look at the first return map $F_m : [-\ell, \ell]/_\sim \setminus \{u_m^-, u_m^+\} \times \mathbb{T}_{2\pi} \to [-\ell, \ell]/_\sim \times \mathbb{T}_{2\pi}$, we see that this map swaps the cylinders and $F_m^\pm = F_m^2|_{C_m^\pm}$. We have $F_0(u, \psi) = (-u, \psi + \pi)$ and, as we expected, $F_0^\pm(u, \psi) = (u, \psi)$. We know that F_m is a *smooth* family of *smooth* maps and we claim that it extends on the whole torus to a *smooth* family of *continuous* maps.

PROPOSITION 7.35. *The family $m \mapsto F_m$ admits an extension to a smooth family of continuous maps $m \mapsto \widehat{F}_m : [-\ell, \ell]/_\sim \times \mathbb{T}_{2\pi} \to [-\ell, \ell]/_\sim \times \mathbb{T}_{2\pi}$.*

For the proof of the proposition, we need first to compute the projection of the differential of the Reeb vector field R^m at $\zeta_m^\pm$ on the contact distribution ξ_m.

LEMMA 7.36. *At a point $(t_m^\pm, \pm\frac{\pi}{2}, m\frac{\theta}{\gamma(t_m^\pm)})$ we have the identity*

$$\begin{pmatrix} [\check{H}^m, R^m] \\ [\check{X}^m, R^m] \end{pmatrix} = \begin{pmatrix} 0 & \frac{f}{h_m} - m\frac{H(h_m)}{h_m^2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \check{H}^m \\ \check{X}^m \end{pmatrix}. \tag{7.35}$$

PROOF. Thanks to Lemma 7.3, the matrix above is given by

$$\sqrt{h_m} \begin{pmatrix} \eta([\check{H}^m, R^m]) & \lambda([\check{H}^m, R^m]) \\ \eta([\check{X}^m, R^m]) & \lambda([\check{X}^m, R^m]) \end{pmatrix}. \tag{7.36}$$

Thus, we have to compute the Lie brackets only up to multiples of V. Below we use the symbol $\doteq$ between two vectors that are equal up to a multiple of V. Observe that on the support of $\zeta_m^\pm$ we have the identities

- $\beta = \pm\frac{\beta_\theta}{\gamma}$, • $\beta \circ \jmath = 0$, • $V(\beta) = \beta \circ \jmath = 0$, • $H = -\pm\widetilde{\partial}_t$, • $X = \pm\frac{\widetilde{\partial}_\theta}{\gamma}$.

We compute

$$\begin{aligned}
[\check{H}^m, R^m] &= \left[\check{H}^m, \frac{m}{h_m}X + \frac{f}{h_m}V\right] \\
&\doteq m\check{H}^m\left(\frac{1}{h_m}\right)X + \frac{m}{h_m}[\check{H}^m, X] + \frac{f}{h_m}[\check{H}^m, V] \\
&\doteq m\frac{1}{\sqrt{h_m}}H\left(\frac{1}{h_m}\right)X + \frac{m}{h_m\sqrt{h_m}}[H, X] + \frac{f}{h_m\sqrt{h_m}}[H, V] \\
&\doteq m\frac{1}{\sqrt{h_m}}H\left(\frac{1}{h_m}\right)X + \frac{f}{h_m\sqrt{h_m}}X.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
[\check{X}^m, R^m] &= \left[\check{X}^m, \frac{m}{h_m}X + \frac{f}{h_m}V\right] \\
&\doteq m\check{X}^m\left(\frac{1}{h_m}\right)X + \frac{m}{h_m}[\check{X}^m, X] + \frac{f}{h_m}[\check{X}^m, V] \\
&\doteq 0 + \frac{m}{h_m\sqrt{h_m}}(\beta - m)[V, X] + \frac{f}{h_m\sqrt{h_m}}[X, V] \\
&= -\frac{m^2 - m\beta + f}{h_m\sqrt{h_m}}H \\
&= -\frac{1}{\sqrt{h_m}}H.
\end{aligned}
$$

Applying the trivialisation Υ^m, we get the desired formula.   □

We now prove a lemma which yields a local version of the proposition, under a further assumption on the differential of the vector field.

LEMMA 7.37. *Consider coordinates $(z = (x, y), \varphi)$ on $\mathbb{R}^2 \times \mathbb{T}_{2\pi}$ and let (r, θ) be polar coordinates on the first factor. Let $B_\delta := \{r < \delta\} \subset \mathbb{R}^2$ be the open disc of radius δ and $\mathbb{T}_\delta^3 := B_\delta \times \mathbb{T}_{2\pi}$ be the solid torus with section B_δ. Suppose that $m \mapsto Z_m \in \Gamma(\mathbb{T}_\delta^3)$ is a smooth family of smooth vector fields and decompose $Z_m = Z_m^{\mathbb{R}^2} + d\varphi(Z_m)\partial_\varphi$. Assume that*

a) $\forall (m, \varphi)$, $(Z_m^{\mathbb{R}^2})_{(0,\varphi)} = 0$ and the endomorphism $d_{(0,\varphi)}^{\mathbb{R}^2}Z_m^{\mathbb{R}^2}$ is antisymmetric;

b) the smooth family of functions $a_m : \mathbb{T}_{2\pi} \to \mathbb{R}$ defined by $d_{(0,\varphi)}^{\mathbb{R}^2}Z_m^{\mathbb{R}^2} = a_m J_{\mathrm{st}}$ is uniformly bounded from below by a positive constant.

Under these hypotheses, there exists some $\delta' < \delta$ and a smooth family of first return maps on the set $\{(0, y, \varphi) \mid 0 < |y| < \delta'\}$. Such family extends to a smooth family of continuous maps $P_m : \{(0, y, \varphi) \mid |y| < \delta'\} \to \{(0, y, \varphi) \mid |y| < \delta'\}$.

PROOF. We claim that the family of functions $m \mapsto d\theta(Z_m^{\mathbb{R}^2})$ extends to a smooth family of continuous functions $m \mapsto \widehat{a}_m$ on the whole $\mathbb{T}_\delta^3$. Indeed, observe that $d_{(z,\varphi)}\theta(\cdot) = \omega_{\mathrm{st}}^{\mathbb{R}^2}(z, \cdot)/r^2$. As a consequence,

$$
\begin{aligned}
d_{(z,\varphi)}\theta(Z_m^{\mathbb{R}^2}) = \frac{\omega_{\mathrm{st}}^{\mathbb{R}^2}(z, Z_m^{\mathbb{R}^2})}{r^2} &= \frac{\omega_{\mathrm{st}}^{\mathbb{R}^2}\left(z, d_{(0,\varphi)}^{\mathbb{R}^2}Z_m^{\mathbb{R}^2}z + o(r)\right)}{r^2} \\
&= \frac{g_{\mathrm{st}}\left(z, -J_{\mathrm{st}}d_{(0,\varphi)}^{\mathbb{R}^2}Z_m^{\mathbb{R}^2}z\right)}{r^2} + \frac{\omega_{\mathrm{st}}^{\mathbb{R}^2}\left(z, o(r)\right)}{r^2} \\
&= a_m + o(1).
\end{aligned}
$$

Since the term $o(1)$ is uniform in m and φ, the claim follows.

From the fact that a_m is bounded away from zero, we deduce that on some small $\mathbb{T}_{\delta'}^3$, $d_{(z,\varphi)}\theta(Z_m) = d_{(z,\varphi)}\theta(Z_m^{\mathbb{R}^2})$ is also bounded away from zero and, hence, there is a well defined first return time $t_m(y, \varphi)$ for the flow of Z_m on $\{(0, y, \varphi)) \mid 0 < |y| < \delta'\}$. It is uniquely determined by the equation

$$
\pi = \int_0^{t_m(y,\varphi)} d_{\Phi_t^{Z_m}(0,y,\varphi)}\theta(Z_m^{\mathbb{R}^2})\,dt = \int_0^{t_m(y,\varphi)} \widehat{a}_m(\Phi_t^{Z_m}(0, y, \varphi))\,dt.
$$

Hence, $t_m(y, \varphi)$ is defined also for $z = 0$ and $m \mapsto t_m$ is a smooth family of continuous functions because the same is true for $m \mapsto \widehat{a}_m$. The required extension of the Poincaré map is given by $P_m(y, \varphi) = \Phi^{Z_m}_{t_m(y,\varphi)}(0, y, \varphi)$. $\qquad\square$

We are now ready to prove the proposition.

PROOF OF PROPOSITION 7.35. We need only to consider the system close to the latitudes $\zeta^{\pm}_m$. Since m is small, ζ^-_m and ζ^+_m will be near the north and south pole, respectively. We only analyse the case of the south pole, since for the north pole we can use a similar argument. Consider the vector field $\mathring{X}^m := b_m \check{X}^m$, where $b_m : SS^2 \to (0, +\infty)$ is a rotationally invariant function to be determined later. Notice that such vector field is transverse to C at the south pole, namely when $u = 0$. Therefore, the map $(x_m, u_m, \psi) \mapsto \Phi^{\mathring{X}^m}_{x_m}(C(u_m + u^+_m, \psi))$ yields a well-defined smooth family of diffeomorphisms between a neighbourhood of the fibre over the south pole and $B_\delta \times \mathbb{T}_{2\pi} \subset \mathbb{R}^2 \times \mathbb{T}_{2\pi}$. We use these maps to pull back the Reeb vector field to $B_\delta \times \mathbb{T}_{2\pi}$. We still denote by R^m the pull-back.

We show that R^m satisfies the hypotheses of Lemma 7.37. On $x_m = 0$, we have

- $\partial_{x_m} = \mathring{X}^m = b_m \check{X}^m$ $\qquad\qquad$ • $\partial_{u_m} = \widetilde{\partial}_t = -H = -\sqrt{h_m}\check{H}^m$,

so that $(\partial_{x_m}, \partial_{u_m})$ is a basis of $\xi_m|_{\{x_m=0\}}$. Denote by

$$A := \begin{pmatrix} b_m & 0 \\ 0 & \sqrt{h_m} \end{pmatrix}$$

the change of basis matrix. We also have dual relations on ξ^*_m:

- $dx_m = \dfrac{\sqrt{h_m}}{b_m}\lambda$ $\qquad\qquad$ • $du_m = -\dfrac{\sqrt{h_m}}{\sqrt{h_m}}\eta$.

Notice that, since b_m and $\sqrt{h_m}$ are rotationally invariant, at $\{x_m = 0\}$, we have $[\partial_{x_m}, R^m] = b_m[\check{X}^m, R^m]$ and $[\partial_{u_m}, R^m] = -\sqrt{h_m}[\check{H}^m, R^m]$. Then, we compute the matrix of $d_{(0,\psi)}(R^m)^{\mathbb{R}^2}$ in the (x_m, u_m) coordinates

$$\begin{aligned}
d_{(0,\psi)}(R^m)^{\mathbb{R}^2} &= \begin{pmatrix} dx_m\big([\partial_{x_m}, R^m]\big) & dx_m\big([\partial_{u_m}, R^m]\big) \\ du_m\big([\partial_{x_m}, R^m]\big) & du_m\big([\partial_{u_m}, R^m]\big) \end{pmatrix} \\
&= A^{-1} \begin{pmatrix} \sqrt{h_m}\lambda\big([\check{X}^m, R^m]\big) & \sqrt{h_m}\lambda\big([\check{H}^m, R^m]\big) \\ \sqrt{h_m}\eta\big([\check{X}^m, R^m]\big) & \sqrt{h_m}\eta\big([\check{H}^m, R^m]\big) \end{pmatrix} A \\
&= A^{-1} \begin{pmatrix} 0 & \frac{f}{h_m} - m\frac{H(h_m)}{h^2_m} \\ -1 & 0 \end{pmatrix} A \\
&= \begin{pmatrix} 0 & -\frac{\sqrt{h_m}}{b_m}\left(\frac{f}{h_m} - m\frac{H(h_m)}{h^2_m}\right) \\ \frac{b_m}{\sqrt{h_m}} & 0 \end{pmatrix}
\end{aligned}$$

Therefore, we need $\dfrac{b_m}{\sqrt{h_m}} = \dfrac{\sqrt{h_m}}{b_m}\left(\dfrac{f}{h_m} - m\dfrac{H(h_m)}{h^2_m}\right)$. Namely,

$$b_m = +\sqrt{f - m\frac{H(h_m)}{h_m}}.$$

In this case we have

$$d_{(0,\psi)}(R^m)^{\mathbb{R}^2} = \sqrt{\frac{f}{h_m} - m\frac{H(h_m)}{h_m^2}} J_{\mathrm{st}}$$

which gives at once that both hypothesis *a)* and *b)* are satisfied. Thanks to Lemma 7.37, the Reeb flow admits a smooth family of continuous first return maps on $\{(0, u_m, \psi) \mid |u_m| < \delta'\} \subset C$. Since this set is an open neighbourhood of ζ_m^+ in C, the proposition follows. $\qquad\square$

REMARK 7.38. We do not know if the extended return maps can be taken to be smooth.

The rotational symmetry of the system implies that F_m can be written in the form $F_m(u, \psi) = (u_m(u), \psi + \psi_m(u))$. When u and $u_m(u)$ are not in $\{-\ell, 0, \ell\}$, then we can also define the longitude θ of the starting point and the longitude $\theta + \theta_m(u)$ of the ending point. If u and $u_m(u)$ have the same sign, then $\theta_m(u) = \psi_m(u)$, while if u and $u_m(u)$ have opposite sign $\theta_m(u) = \psi_m(u) + \pi$.

The system has also another type of symmetries given by the reflections along planes in $\mathbb{R}^3$ containing the axis of rotation. In coordinates these maps can be written as $(t, \theta) \mapsto (t, 2\theta_0 - \theta)$, for some $\theta_0 \in \mathbb{T}_{2\pi}$. Trajectories of the flow are sent to trajectories of the flow travelled in the opposite direction. Therefore, if u and $u_m(u)$ are not in $\{-\ell, 0, \ell\}$, then $u_m(u_m(u)) = u$ and $\theta_m(u_m(u)) = \theta_m(u)$. This implies that $\psi_m(u_m(u)) = \psi_m(u)$ for *every* $u \in [-\ell, +\ell]/_\sim$. Thus, $F_m^2(u, \psi) = (u, \psi + 2\psi_m(u))$.

When $|u|$ is different from 0 and ℓ, and m is suitably small (m will be smaller and smaller as $|u|$ is close to 0 or ℓ), u and $u_m(u)$ have different signs and we can write an integral formula for $\psi_m(u) = \theta_m(u) + \pi$. Without loss of generality, we fix some $u < 0$ for the forthcoming computation. Let $z_m^u : [0, R] \to SS^2$ be the only solution of the magnetic flow, passing through $C(u, 0)$ at time 0 and through $C(u_m(u), \psi_m(u))$ at time R. Then, $\varphi \circ z_m^u : [0, R] \to [-\pi/2, \pi/2]$ is a monotone increasing function, such that $\varphi(z_m^u(0)) = -\pi/2$ and $\varphi(z_m^u(R)) = \pi/2$. Indeed, thanks to Equation (5.4), its derivative satisfies the equation

$$\frac{d}{dr}\varphi = f - \frac{\dot{\gamma}}{\gamma} m \sin\varphi$$

(where we dropped the composition with z_m^u in the notation). Hence, we can reparametrise z_m^u using the variable φ and get the following expression for $\theta_m(u)$:

$$\theta_m(u) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{d\varphi}\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{dr}\frac{dr}{d\varphi}\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{m\sin\varphi}{\gamma f - \dot{\gamma} m \sin\varphi}\, d\varphi, \qquad (7.37)$$

where we substituted $\frac{d\theta}{dr} = \frac{m\sin\varphi}{\gamma}$, using Equation (5.4) again.

In the next section we expand $\theta_m(u)$ with respect to the variable m around $m = 0$. We aim at computing the first non-zero term.

7.6.2. The Taylor expansion of the return map. For small m we have that $t(z_m^u) = |u| + O_u(m)$ (the size of $O_u(m)$ depending on u), hence γf is big compared to $\dot{\gamma} m \sin\varphi$. Thus, we expand the denominator in the integrand in (7.37), up to a

term of order m^2, using the formula for the geometric series:

$$
\frac{m \sin \varphi}{\gamma f - \dot{\gamma} m \sin \varphi} = \frac{m \sin \varphi}{\gamma f} \left(1 + \frac{\dot{\gamma} m \sin \varphi}{\gamma f} \right) + o_u(m^2)
$$

$$
= m \frac{1}{\gamma f} \sin \varphi + m^2 \frac{\dot{\gamma}}{(\gamma f)^2} \sin^2 \varphi + o_u(m^2). \tag{7.38}
$$

Discarding the $o_u(m^2)$ remainder, we plug (7.38) into (7.37) and compute the two resulting integrals separately.

We use integration by parts for the first one

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} m \frac{1}{\gamma f} \sin \varphi \, d\varphi = m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{d}{dt} \frac{1}{\gamma f} \right) \frac{dt}{d\varphi} \cos \varphi \, d\varphi
$$

$$
= m \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{d}{dt} \frac{1}{\gamma f} \right) \frac{m \cos \varphi}{f} \cos \varphi \, d\varphi + o_u(m^2)
$$

$$
= m^2 \left(\frac{d}{dt} \frac{1}{\gamma f} \right) \frac{1}{f} (|u|) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \varphi \, d\varphi + o_u(m^2)
$$

$$
= \frac{m^2 \pi}{2} \left(\frac{d}{dt} \frac{1}{\gamma f} \right) \frac{1}{f} (|u|) + o_u(m^2).
$$

For the second integral we readily find

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} m^2 \frac{\dot{\gamma}}{(\gamma f)^2} \sin^2 \varphi \, d\varphi = m^2 \frac{\dot{\gamma}}{(\gamma f)^2} (|u|) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \varphi \, d\varphi + o_u(m^2)
$$

$$
= \frac{m^2 \pi}{2} \frac{\dot{\gamma}}{(\gamma f)^2} (|u|) + o_u(m^2).
$$

Putting things together we get

$$
\theta_m(u) = \frac{m^2 \pi}{2} \left(\left(\frac{d}{dt} \frac{1}{\gamma f} \right) \frac{1}{f} + \frac{\dot{\gamma}}{(\gamma f)^2} \right) (|u|) + o_u(m^2)
$$

$$
= \frac{m^2 \pi}{2} \left(-\left(\frac{\dot{\gamma}}{\gamma^2 f} + \frac{\dot{f}}{\gamma f^2} \right) \frac{1}{f} + \frac{\dot{\gamma}}{(\gamma f)^2} \right) (|u|) + o_u(m^2)
$$

$$
= \frac{m^2 \pi}{2} \left(-\frac{\dot{f}}{\gamma f^3} \right) (|u|) + o_u(m^2).
$$

By Proposition 7.35, we know that $\psi_m(u) = \theta_m(u) + \pi$ is a smooth family of functions. Hence, the expansion above translates in an expansion for ψ_m that holds on the whole torus and such that the remainder is of order $o(m^2)$ uniformly in u. Indeed, observe that $\dot{f}(0) = \dot{f}(\ell) = 0$ and, therefore, the function $-\dot{f}/(\gamma f^3)$ extends smoothly at the poles, taking the value $-\ddot{f}/f^3$ at the south pole and $\ddot{f}/f^3$ at the north pole. Call this extension $\Omega_f : S^2 \to \mathbb{R}$ and set $\Omega_f^- := \inf |\Omega_f|$. We have arrived at the final result for this section.

PROPOSITION 7.39. *The family* $F_m^2 : [-\ell, +\ell]/_\sim \times \mathbb{T}_{2\pi} \to [-\ell, +\ell]/_\sim \times \mathbb{T}_{2\pi}$ *admits the expansion*

$$
F_m^2(u, \psi) = \left(u, \psi + \pi \Omega_f m^2 + o(m^2) \right). \tag{7.39}
$$

COROLLARY 7.40. *If* $-\frac{\dot{f}}{\gamma f^3}$ *is not constant, the magnetic flow has infinitely many periodic orbits on every low energy level. Such condition is satisfied if, for example,* $\frac{\ddot{f}}{f^3}(0) \neq -\frac{\ddot{f}}{f^3}(\ell)$ *and in this case both* F_m^+ *and* F_m^- *are twist maps for small m.*

If $\Omega_f^- > 0$ *(namely* $\dot{f} = 0$ *only at the poles and* $\ddot{f} \neq 0$ *there), the period* T *of an orbit different from the latitudes* $\zeta_m^\pm$ *satisfies*

$$T \geq \frac{2}{\Omega_f^- m^2} + O\left(\frac{1}{m^3}\right). \tag{7.40}$$

In particular, ζ_m^+ *and* ζ_m^- *are the only two short orbits.*

From the expansion (7.39), it follows that a necessary condition for having ψ_m constant for some small m is to require that

$$-\frac{\dot{f}}{\gamma f^3} = 2k,$$

for some $k \in \mathbb{R}$. We are interested in this condition, because the only way of having exactly two closed periodic orbits for X^m is to have $\psi_m = a_m \in \mathbb{R}/\mathbb{Q}$. The equation above can be rewritten as

$$\frac{d}{dt}\frac{1}{f^2} = k\gamma$$

and integrating in the variable t we find the solution

$$f = \frac{1}{\sqrt{k\Gamma + h}}, \tag{7.41}$$

where $\Gamma : [0, \ell] \to \mathbb{R}$ is a primitive of γ such that $\Gamma(0) = 0$, $\Gamma(\ell) = \frac{\text{vol}}{2\pi}$ and $h \in \mathbb{R}$ is any constant such that

$$\begin{cases} h > -\frac{k\,\text{vol}}{2\pi}, & \text{if } k < 0, \\ h > 0, & \text{if } k \geq 0, \end{cases}$$

so that the quantity under square root is strictly positive.

A direction of future research would be to study magnetic systems with f given by (7.41) for suitable choices of γ, k and h, to see if the flow of any of them can be written down explicitly. In this way one could check if $\psi_m = a_m \in \mathbb{R}/\mathbb{Q}$, for some value of m.

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