$\mathcal{F}$-Saturation Games

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Abstract

We study *$\mathcal{F}$-saturation games*, first introduced by Füredi, Reimer and Seress [4] in 1991, and named as such by West [5]. The main question is to determine the length of the game whilst avoiding various classes of graph, playing on a large complete graph. We show lower bounds on the length of path-avoiding games, and more precise results for short paths. We show sharp results for the tree avoiding game and the star avoiding game.

1. Introduction

For $\mathcal{F}$ a family of graphs, we say a graph G is *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$ as a subgraph. We say $G \subset H$ is an *$\mathcal{F}$-saturated* subgraph of H if G is a maximal $\mathcal{F}$-free subgraph of H. For a discussion of saturated graphs see for example Bollobás [2]. Take a graph H, $|H| = n$, and let $\mathcal{F}$ be a family of graphs. Following the definition of the triangle free game of Füredi, Reimer and Seress [4], and building on the notation of West [5], we define the *$\mathcal{F}$-saturation game* as follows.

We have two players, *Prolonger* and *Shortener*, who we take to be male and female respectively. We define a graph process $\mathcal{G}_i$. We initially set $\mathcal{G}_0 = E_n$, the empty graph on n vertices. The process ends at time t^* if G_{t^*} is an $\mathcal{F}$-saturated subgraph of H. Otherwise, at time $2t$, Prolonger chooses an edge $uv \in H \backslash \mathcal{G}_{2t}$ and $\mathcal{G}_{2t} \cup uv$ is $\mathcal{F}$-free, and $\mathcal{G}_{2t+1} = G_{2t} \cup uv$. Similarly, at time $2t+1$ Shortener chooses an edge from $H \backslash \mathcal{G}_{2t+1}$ to add, such that the graph process remains $\mathcal{F}$-free. Prolonger's goal is to maximise t^*, whilst Shortener wishes to minimise t^*. Our results will not depend on which of the two players moves first, and so we refer to this game as $\mathcal{G}(H; \mathcal{F})$. We say the value of t^* under optimal play by both Prolonger and Shortener

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is the *score* or *game saturation number* of $\mathcal{G}(H;\mathcal{F})$, denoted by $\text{sat}_{\mathcal{G}}(H;\mathcal{F})$. When only one graph is excluded, we write $\mathcal{G}(H;F) := \mathcal{G}(H;\{F\})$.

Füredi, Reimer and Seress [4] concentrate on the game $\mathcal{G}(K_n, K_3)$. They exhibit a strategy for Prolonger which demonstrates that $\text{sat}_{\mathcal{G}}(K_n, K_3) \geq (\frac{1}{2} + o(1))n\log_2 n$. They attribute to Erdős a lost proof that Shortener has a strategy showing $\text{sat}_{\mathcal{G}}(K_n, K_3) \leq \frac{n^2}{5}$. Bíró, Horn and Wildstrom [1] show that $\text{sat}_{\mathcal{G}}(K_n, K_3) \leq \frac{26n^2}{121} + o(n^2)$.

Motivated by these results, we study the case where $\mathcal{F}$ is the path on k vertices P_k, $\mathcal{F}$ is the class of all trees on k vertices or $\mathcal{F}$ is the star $K_{1,k}$.

2. Our results

As is standard, for any $k \in \mathbb{N}$ we denote a path on k vertices by P_k. To illustrate the difficulties encountered by Prolonger, we first study a variant where on his turn, he is permitted to decline to pick any edge, and set $\mathcal{G}_{2t+1} = \mathcal{G}_{2t}$. Since Shortener is still required to add edges, this graph process will still become $\mathcal{F}$-saturated and thus have a score as defined for the $\mathcal{F}$-saturation game. We will refer to this game as $\mathcal{G}_{-P}$. Since we have given Prolonger additional options, it is clear that any strategy he might use in $\mathcal{G}$ is valid in $\mathcal{G}_{-P}$, and so we have that $\text{sat}_{\mathcal{G}_{-P}}(H;\mathcal{F}) \geq \text{sat}_{\mathcal{G}}(H;\mathcal{F})$

Theorem 1. *For all $n \geq k$, we have $\frac{1}{4}n(k-2) \leq \text{sat}_{\mathcal{G}_{-P}}(K_n; P_k) \leq \frac{1}{2}n(k-1)$.*

Returning to $\mathcal{G}(K_n, P_k)$, we have results only for small values of k. Whilst these results are quite precise, they are predicated on a complete categorisation of the connected P_k-saturated graphs. Obtaining results of this precision for larger k thus seems challenging.

Theorem 2. *For all $n > 0$, we have $\frac{4}{5}n - \frac{14}{5} \leq \text{sat}_{\mathcal{G}}(K_n, P_4) \leq \frac{4}{5}n + 1$.*

Theorem 3. *For all $n > 0$, we have $n - 1 \leq \text{sat}_{\mathcal{G}}(K_n, P_5) \leq n + 2$.*

For larger classes of graphs, we have substantially precise bounds for all k. We define $\mathcal{T}_k$ to be the family of all trees on k vertices.

Theorem 4. *For all $n, k \in \mathbb{N}$, we write $n = a(k-1) + b$ for $a \in \mathbb{N}$ and $0 \leq b < k-1$. Then:*

$$\textit{If } b \neq 1: \qquad \text{sat}_{\mathcal{G}}(K_n, \mathcal{T}_k) = a\binom{k-1}{2} + \binom{b}{2}$$
$$\textit{If } b = 1: \qquad a\binom{k-1}{2} - (k-3) \leq \text{sat}_{\mathcal{G}}(K_n, \mathcal{T}_k) \leq a\binom{k-1}{2}$$

In fact, the primary constraint of $\mathcal{T}_k$ saturation is to exclude a $K_{1,k-1}$. If only this graph is excluded, we have a precise bound:

Theorem 5. *For $n \geq 3k^2 - 3k - 4$, we have the following bounds:*

$$\frac{1}{2}kn \geq \mathrm{sat}_{\mathcal{G}}(K_n, K_{1,k+1}) \geq \frac{1}{2}\left(kn - 2(k-1)\right).$$

3. Avoiding P_k in $\mathcal{G}_{-P}$

Proof of Theorem 1. The upper bound is the saturation result of Erdős and Gallai [3]. To obtain the lower bound, we exhibit a strategy for Prolonger that guarantees the required length of game. We say a graph is *everywhere traceable* if for every vertex v in the graph there is a Hamiltonian path starting at v. Hence if a graph is Hamiltonian it is everywhere traceable. We will show that the following strategy for Prolonger guarantees that the score will be large enough:

- i) If there is a component C which is not everywhere traceable, he finds a Hamiltonian path P in C and adds the edge which augments P to a Hamiltonian cycle;
- ii) Otherwise he does not add an edge on his turn.

To prove this, we first show the following auxilliary claim.

Claim 6. *After his move, every connected component is everywhere traceable.*

Proof. We induct on the number of edges in the graph. Hence we may assume that after his previous move, all the components were everywhere traceable. As the base case, note that the empty graph and an isolated edge are everywhere traceable, so regardless of who moves first, he will choose to add no edges and leave the graph satisfying the claim.

After Shortener's move, Prolonger is faced with a graph G. If her move did not alter the component structure of the graph process, then every component is still everywhere traceable and he will add no edges, satisfying the claim. Her move altered at most 2 components by connecting them, which produces a single component C which is not everywhere traceable. Since C was formed by joining two everywhere traceable components by an edge, we know that C contains a Hamiltonian path P. Since after her move the graph is P_k-free, we know that $|P| = |C| < k$.

Since by assumption C is not everywhere traceable, we have that $|C| > 2$ and that the endpoints u, v of P are not adjacent as C is not Hamiltonian. Since $|C| < k$, any path using the edge uv in $G \cup uv$ is contained in $V(C)$, and so has length less than k. So $G \cup uv$ is P_k-free, and Prolonger may add this edge. The component $C \cup uv$ is Hamiltonian and thus everywhere traceable. Hence after his move, Prolonger leaves every connected component everywhere traceable. □

Hence in $\mathcal{G}_{t^*}$, the total number of vertices in any two components is $\geq k$, as otherwise these two components could be joined by an edge. Since all components are Hamiltonian, every component is of size less than k and so will be complete. Hence the sum of degrees of any two disconnected vertices is at least $k-2$. Hence taking δ the minimum degree of $\mathcal{G}_{t^*}$, we have:

$$2\text{sat}_{\mathcal{G}_{-P}} \geq \min(k-2-\delta, \delta)(n-\delta-1) + \delta(\delta+1)$$

which is minimised by taking $\delta = \left\lfloor \frac{k-1}{2} \right\rfloor$, which implies that $\text{sat}_{\mathcal{G}_{-P}} \geq \frac{1}{4}n(k-2)$ as required. $\square$

In fact, the notion of ensuring that all components remain everywhere traceable almost allows for an optimal strategy for Prolonger in $\mathcal{G}(K_n; P_k)$. The only point at which Prolonger could not guarantee to leave every component everywhere traceable is when when the graph consists of a disjoint union of cliques with at most one isolated vertex. If Prolonger plays on such a graph, his move necessarily leaves a component which has the form of two cliques C_1 and C_2 joined by a single edge with $|V(C_1)| > 1$. Such a component is not everywhere traceable, as the endpoint of Prolonger's edge in C_1, u, cuts the component into two non-empty pieces. Shortener could then join another clique C_3 to u.

Suppose that $|V(C_1 \cup C_2 \cup C_3)| \geq n$. Then no additional edges can be added within the component, as any additional edge within this component causes it to have a hamiltonian path. Let the longest path within the component from u be on l vertices. Then joining l to a vertex prevents a path of length $n-l$ from being formed from this vertex on $K_n - (C_1 \cup C_2 \cup C_3)$. By use of this trick, Shortener can form a large connected component containing an induced subdivision of a star, centered on u, such that removing u disconnects all of the arms of the star in the component. Hence at the end of the game the component is the union of many cliques intersecting only at u. The sum of the size of any two cliques is then at most k. As a corollary, the average degree of the component is at most $\frac{k}{2}$, which prevents us from achieving a score of $\sim \frac{nk}{2}$, and might force the score as low as $\sim \frac{nk}{4}$ if components of this form are spanning.

4. The game $\mathcal{G}(K_n, P_4)$

We now turn to a detailed examination of the game $\mathcal{G}(K_n, P_4)$ and Theorem 2. Let us begin with the following characterisation of P_4-saturated graphs, which is easily shown by induction:

Observation 7. *A P_4-saturated graph is either a vertex-disjoint union of triangles, stars with at least four vertices and edges, or is a vertex-disjoint union of triangles and an isolated vertex (cf. Figure 1).*

This straightforward lemma leads to reasonably good bounds on the score, as we can exactly track which components could form.

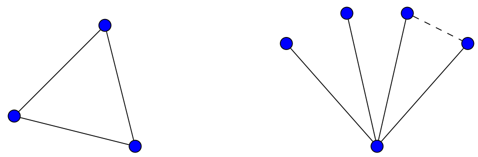

Figure 1: Maximal components of a P_4-saturated graph

Proof of Theorem 2. The upper bound is demonstrated by considering the following strategy for Shortener. She will:

i) extend a $K_{1,2}$ to a $K_{1,3}$ if possible, otherwise

ii) draw an isolated edge if possible, otherwise

iii) extend a star by attaching the central vertex to an isolated vertex if possible, otherwise

iv) extend a $K_{1,2}$ to a K_3.

Claim 8. *After Prolonger's move, there is at most one $K_{1,2}$ component. Shortener will not complete the $K_{1,2}$ to a K_3, unless this makes the graph P_4-saturated. After Shortener's move, there is at most one $K_{1,2}$ component. If there is a $K_{1,2}$ component, Prolonger will extend it to a K_3 and make the graph P_4-saturated.*

Proof. We proceed by induction.

1. Suppose that G_i has a $K_{1,2}$ component after Prolonger's move.
   1) If there is an isolated vertex in G_i, Shortener will extend the $K_{1,2}$ to a $K_{1,3}$. Hence there is no $K_{1,2}$ component in G_{i+1}, and there can be at most one after Prolonger's next move to G_{i+2}.
   2) If there is no isolated vertex in G_i, Shortener will extend the $K_{1,2}$ to a K_3. Two components of size > 1 cannot be joined without creating a P_4. Hence no further components can be joined or extended, and the graph is P_4-saturated.
2. Suppose that G_i has no $K_{1,2}$ component after Prolonger's move.
   1) If Shortener creates a $K_{1,2}$ component, then G_i contained exactly one isolated vertex. Hence all components are now of size > 1, so Prolonger can only complete the $K_{1,2}$ to a K_3.
   2) Otherwise there are no $K_{1,2}$ components in G_{i+1}, and Prolonger can produce at most one $K_{1,2}$ in G_{i+2}.

This finishes the proof. □

By Claim 8, until t^*, Shortener has ensured that the graph is a vertex disjoint union of stars. Let there be λ components in G_{t^*}. Since there is at most 1 triangle, the score is bounded above by $n+1-\lambda$, with $n-\lambda$ moves producing non-trivial components (i.e. creating isolated edges) or extending stars. To prevent her from making a new non-trivial component by case (ii) of her strategy, Prolonger must make a $K_{1,2}$, which occurs at most once for each component of $\mathcal{G}_{t^*}$. Hence at most λ of Shortener's moves fail to make a non-trivial component. Hence there are at least $\frac{1}{2}(n-\lambda)-\lambda$ components. So $\lambda \geq \frac{1}{5}n$, and the score is at most $\frac{4}{5}n+1$.

The lower bound is demonstrated by considering the following strategy for Prolonger. He will:

i) complete a triangle component if possible, otherwise

ii) complete a $K_{1,2}$ component if possible, otherwise

iii) extend a star component if possible, otherwise

iv) draw an isolated edge.

Note that Prolonger is forced to play an isolated edge only as the first move or after Shortener completes a triangle. We say that a move *uses k new vertices* if the number of isolated vertices is reduced by k as a result of that move. Note that at the end of the game either there are *no* isolated vertices or there is exactly one and the remaining vertices are covered by triangles, which entails $n-1 > \frac{4}{5}n - \frac{14}{5}$ edges have been produced. Hence we can assume that over the course of the game n vertices are used.

We first claim that if Prolonger creates a $K_{1,2}$ component in $\mathcal{G}_i$, at most 2 isolated vertices are used between $\mathcal{G}_i$ and $\mathcal{G}_{i+2}$. If Shortener plays elsewhere, Prolonger will extend the $K_{1,2}$ to a K_3. If Shortener extends the $K_{1,2}$ to a K_3, Prolonger can make an arbitrary move. If Shortener extends the $K_{1,2}$ to a $K_{1,3}$ then Prolonger can extend that to a $K_{1,4}$. In all cases at most two new vertices are used.

Note that if Prolonger can create a $K_{1,2}$ component when creating $\mathcal{G}_i$ but does not, then he must extend a $K_{1,2}$ into a K_3. Hence at most 2 new vertices are used between $\mathcal{G}_{i-2}$ and $\mathcal{G}_i$.

If Prolonger cannot create a K_3 or $K_{1,2}$ component then either there are no isolated edges in $\mathcal{G}_{i-1}$, or there are no isolated vertices in $\mathcal{G}_{i-1}$. Hence Shortener uses at most one isolated vertex from $\mathcal{G}_{i-2}$ or $\mathcal{G}_{i-1}$ is P_4-saturated. Prolonger uses 2 new vertices only if he adds an isolated edge to form $\mathcal{G}_i$, which requires that Shortener completed a triangle into $\mathcal{G}_{i-1}$ and used no new vertices. Hence either 1 new vertex is used to end the game or at most 2 new vertices are used between $\mathcal{G}_{i-2}$ and $\mathcal{G}_i$.

Note that with this strategy of Prolonger when $\mathcal{G}_i$ is created from $\mathcal{G}_{i-2}$ we never use 4 new vertices. Furthermore, we use 3 new vertices only if in $\mathcal{G}_{i-2}$ there was no $K_{1,2}$ component and in $\mathcal{G}_i$ there is. As a consequence, no two consecutive pairs of moves by Shortener and then Prolonger both use 3 new vertices. If Prolonger moves first, then his first move consumes two new vertices. If Shortener makes the last move last then her move may consume two new vertices. If there are an odd number of pairs of moves by Shortener and then Prolonger we may have 1 more pair using 3 new vertices than 2. It is plainly seen that these three possibilities all consume more vertices than $\frac{5}{4}$ of the number of moves, and so reduce the final score from $\frac{5}{4}n$. So we have:

$$\frac{5}{4}\left(\mathrm{sat}_{\mathcal{G}}(P_4; K_n) - 1 - 1 - 2\right) \geq n - 2 - 2 - 3$$

Where the terms correspond to losing 2 free vertices to 1 move and 3 free vertices to 2 moves respectively. Hence $\mathrm{sat}_{\mathcal{G}}(P_4; K_n) \geq \frac{4}{5}n - \frac{14}{5}$, which completes the proof of the lower bound. □

5. The game $\mathcal{G}(K_n, P_5)$

We now turn to a detailed examination of the game $\mathcal{G}(K_n, P_5)$ and Theorem 3. Denote a double star with k pendant edges at one end of the central edge and l at the other by $D_{k,l}$. Denote a triangle with k pendant edges at one vertex by T_k (cf. Figure 2). As in the case of the P_4-saturation game, we start by characterising the P_5-saturated graphs, which is easily shown by induction:

Observation 9. *A P_5-saturated graph is* either *a vertex-disjoint union of copies of K_4, $T_{\geq 0}$, $D_{k,l}$ and at most one isolated edge, or is a vertex-disjoint union of one isolated vertex and copies of K_4.*

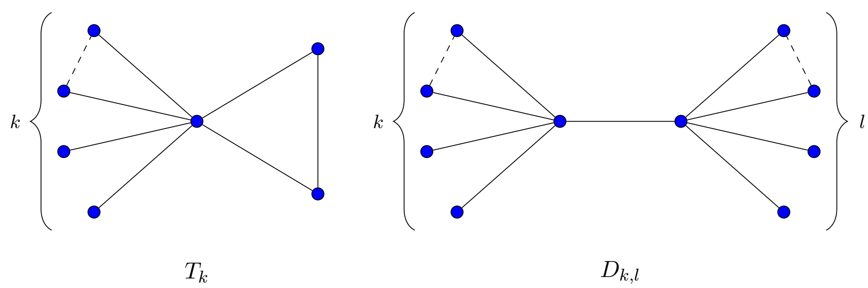

Figure 2: T_k and $D_{k,l}$

Proof of Theorem 3. The upper bound is demonstrated by considering the following strategy for Shortener. She will:

i) extend a P_4 to a $D_{1,2}$ or extend a $K_{1,3}$ to a $D_{1,2}$ or extend a T_1 to a T_2 if possible, otherwise

ii) extend an isolated edge to a $K_{1,2}$ if possible, otherwise

iii) extend a component of 5 or more vertices by attaching to it an isolated vertex if possible, otherwise

iv) draw an isolated edge if possible, otherwise

v) play arbitrarily.

Claim 10. *Given this strategy by Shortener, in any graph G_t, there is either at most one component of size four and at most one isolated edge or, there are no components of size four and at most two isolated edges.*

Proof. We proceed inductively; clearly the condition holds after Prolonger's first move. Suppose it holds after Prolonger moves to $\mathcal{G}_i$. We split into cases according to the existence of isolated vertices.

Suppose first that there are no isolated vertices in $\mathcal{G}_i$. Then the only way to create new 4-vertex components is to join two isolated edges to create a P_4. But inductively if $\mathcal{G}_i$ contains a 4-vertex component then there is at most one isolated edge, so Shortener cannot make a new 4-vertex component. Otherwise there are ≤ 2 isolated edges, so at most one 4-vertex component can be produced. Hence the condition of the lemma is satisfied both in $\mathcal{G}_{i+1}$ and $\mathcal{G}_{i+2}$, and hence after Prolonger's next move.

Suppose alternatively there is an isolated vertex in $\mathcal{G}_i$. We claim that either there is at most either a single P_4, $K_{1,3}$ or T_1 and at most one isolated edge, or there are no components of size 4 and at most 2 isolated edges.

If there is a P_4, $K_{1,3}$ or T_1 in $\mathcal{G}_i$ then Shortener will extend it. If there are two isolated edges in $\mathcal{G}_i$ then Shortener will extend one of them to a $K_{1,2}$. If $\mathcal{G}_i$ contains no P_4, $K_{1,3}$, T_1 or isolated edge then Shortener will unify a $\geq$ 5-vertex component with an isolated vertex; otherwise Shortener will not produce a component of size 4. So in $\mathcal{G}_{i+1}$ either there is no component of size 4, and there is at most one isolated edge, or there is one of P_4, $K_{1,3}$ or T_1, no isolated edges and no isolated vertices.

In the former case, in $\mathcal{G}_{i+2}$ Prolonger can create at most one new 4-vertex component, or at most one isolated edge; in the latter case, Prolonger can create in $\mathcal{G}_{i+2}$ no new 4-vertex components and no isolated edges. So the condition is satisfied both in $\mathcal{G}_{i+1}$ and $\mathcal{G}_{i+2}$, and thus also after Prolonger's next move.

Hence the claim holds after every move by Prolonger or Shortener. □

By Claim 10, there is at most one K_4. By Lemma 9 the number of edges does not exceed the number of vertices in any other component. Hence the game score is at most $n + 2$, showing the claimed upper bound.

Before outlining the argument for the lower bound, we define some additional notation. Let us call a component *trivial* if it consists of an isolated vertex. Let us call a non-trivial component *standalone* if it can not be connected to another non-trivial component without completing a P_5. Note that if a component is not standalone it must have a vertex which is not the endpoint of an induced P_3. From Lemma 9, the only P_5 free components which have a vertex which is not the endpoint of an induced P_3 are stars. Hence any other component may only be joined to an isolated vertex, as otherwise a P_5 will neccessarily appear.

The lower bound is demonstrated by considering the following strategy for Prolonger. He will:

(i) complete a triangle in a $D_{1,2}$ component to make it a T_2 or in a $K_{1,3}$ component to make it a T_1, or, if not possible

(ii) complete a triangle in a component without a triangle, or, if not possible

(iii) connect two isolated edges to form a P_4, or, if not possible

(iv) complete a $K_{1,2}$ component, or, if not possible

(v) draw an isolated edge, or, if not possible

(vi) play arbitrarily.

Claim 11. *Given this strategy for Prolonger, the set of star components after his move may be: empty; or one isolated edge; or one $K_{1,2}$. After Shortener's move, the set of non-trivial stars components may be: empty, $K_{1,2}$, $K_{1,3}$, $K_{1,2}$ and an isolated edge, two isolated edges, or one isolated edge.*

Proof. We induct on the number of moves. The result holds trivially for G_0 and G_1. If the condition holds after Prolonger's move, it can easily be checked that Shortener's move can only produce sets of stars as stated in the lemma. After Shortener's move, Prolonger will:

(i) complete a K_3 from a $K_{1,2}$ component, or produce a T_1 in the $K_{1,3}$ component, both of which are standalone;

(ii) if not possible, he will complete a P_4 from two isolated edges, which is standalone;

(iii) if not possible, he will complete a $K_{1,2}$ component from one isolated edge;

(iv) if not possible, he will draw an isolated edge;

(v) otherwise, he will play arbitrarily but his move will not extend a star into a larger star (otherwise he could have completed a triangle in it), so his edge will be a part of a standalone component.

In all cases the set of non-trivial star components after Prolonger's move is as described in the claim. □

Claim 12. *Given this strategy for Prolonger, in $\mathcal{G}_{t^*}$ all standalone components will contain a triangle. The set of non-trivial star components will consist of an isolated vertex or an isolated edge.*

Proof. We claim by induction that after Prolonger's move, the components of size greater than one without a triangle will be empty or consist of one component which will be either an isolated edge, $K_{1,2}$ or P_4. Clearly this holds for $\mathcal{G}_0$ and $\mathcal{G}_1$. Suppose it holds for after Prolonger's move to $\mathcal{G}_i$. By claim 11 there is at most one star component in $\mathcal{G}_i$, so if Shortener connects two components one of them is an isolated vertex. So in $\mathcal{G}_{i+1}$ the set of non-trivial components without a triangle will be empty, a K_2, a $K_{1,2}$ or P_4 or be one of the preceding and an isolated edge or be a $K_{1,3}$ or a $D_{1,2}$. In each case, to form $\mathcal{G}_{i+2}$ Prolonger will:

(i) complete a triangle in them to create a T_2 component or a T_1 component or a K_3 component or

(ii) connect two isolated edges to form a P_4 component or

(iii) connect an isolated edge to an isolated vertex to form a $K_{1,2}$ component or

(iv) create an isolated edge or

(v) else there is at most one non-trivial component without a triangle which can only be an isolated edge and he can play arbitrarily

so the set of non-trivial components without a triangle in $\mathcal{G}_{i+2}$ consists of an isolated edge, a $K_{1,2}$ or a P_4.

Hence Shortener cannot create $D_{k,l}$ components with both $k, l \geq 2$. By Lemma 11 there is at most one star component in $\mathcal{G}_i$, so the component would have to be formed via a $D_{1,2}$ or a $K_{1,3}$ component, which are immediately completed into a T_2 or T_1 component by Prolonger. Hence at the end of the game the non-trivial components without a triangle will be an isolated vertex or an isolated edge, since the other components cannot be a $D_{k,l}$ with $k, l \geq 2$ in G_i and thus contain a triangle by Lemma 10. □

So by Lemma 12 all components in $\mathcal{G}_{t^*}$ will contain a triangle except for at most one isolated edge or isolated vertex. Hence the number of edges in these components is greater or equal to the number of vertices. Hence $\text{sat}_{\mathcal{G}}(K_n, P_5) \geq n - 1$.

□

6. Game of avoiding all trees on k vertices

Recall that $\mathcal{T}_k$ is defined to be the family of all trees on k vertices. Consider the game $\mathcal{G}(K_n, \mathcal{T}_k)$. Clearly, the condition that G is $\mathcal{T}_k$-free is equivalent to requireing that all connected components of G have less than k vertices. Hence being $\mathcal{T}_k$-saturated implies that all components will be cliques of size at most $k-1$ with any two components having total size at least k.

Proof of Theorem 4. Suppose G is $\mathcal{T}_k$-saturated. Then $e(G)$ is a convex quadratic function of the clique sizes, and so is maximised when all but one clique is of size $k-1$. The upper bounds follow immediately.

To demonstrate the lower bounds, suppose that Prolonger chooses two components with the greatest total number of vertices such that this number is at most $k-1$ and connects them by an edge.

Claim 13. *After Prolonger's move, yielding $\mathcal{G}_i$, either (1) there exists at most one connected component $C_i \subseteq \mathcal{G}_i$ with $1 < |V(C_i)| < k-1$, or (2) there is an isolated edge, a connected component of size $k-2$ and connected components of size $k-1$.*

Proof. The conditions of (1) hold in $\mathcal{G}_0 = E_n$ and $\mathcal{G}_1 = K_2 \cup (n-2)K_1$. We proceed inductively, and split the analysis of Shortener's move into two cases:

a) Shortener connects two isolated vertices to make an isolated edge.

b) Shortener does not form an isolated edge, so either no components are changed in size or C_i is joined to an isolated vertex u.

If Shortener has formed an isolated edge uv, then if $|C_i| \leq k-3$, Prolonger joins it to uv to satisfy the conditions of (1), with $V(C_{i+2}) = V(C_i) \cup \{u, v\}$. If instead there is an isolated vertex v and $|C_i| = k-2$, then Prolonger joins it to v to satisfy the conditions of (1). Otherwise no component can be extended and the conditions of (2) are satisfied for the rest of the game.

Suppose Shortener has not formed an isolated edge. Then she must have added an edge uv which either extended C_i or left the component structure unchanged. If there was a set C_i, we say $C_{i+1} = C_i \cup (uv)$ if either u or v were vertices in C_i, otherwise we take $C_{i+1} = C_i$. Note that in either case $|V(C_{i+1})| \leq |V(C_i)| + 1 \leq k-1$.

If there are no isolated vertices then no component can be extended and the conditions of (1) are satisfied for the rest of the game. If there is an isolated vertex w and C_{i+1} exists, with $|C_{i+1}| \leq k-2$, Prolonger joins C_{i+1} to w satisfying the conditions of (1). If there is no set C_i or $|C_{i+1}| = k-1$, then if if there are two isolated vertices Prolonger joins them to form C_{i+2} satisfying the conditions of (1). If not, then no component can be extended and the conditions of (1) are satisfied for the rest of the game. □

Hence if $n \not\equiv 1 \mod (k-1)$ the conditions of (2) cannot hold, and since G is $\mathcal{T}_k$-saturated at the end of the game there cannot be a component of size $\leq k-2$ and an isolated vertex. Hence there are $\left\lfloor \frac{n}{k-1} \right\rfloor$ K_{k-1}'s and one further clique, which saturates the upper bound.

If $n \equiv 1 \mod (k-1)$ and $k \geq 3$, then the conditions of (2) could hold, in which case precisely $k-2$ edges are lost from removing a vertex from a K_{k-1} and 1 is gained from an isolated edge. Hence the bound is $k-3$ below the upper bound. □

7. Forbidding the graph $K_{1,k+1}$

Instead of forbidding the family of all trees T_{k+2}, we consider merely forbidding the graph $K_{1,k+1}$. Trivially this corresponds to requiring that in the graph process, $\Delta(\mathcal{G}_t) \leq k$. From this, we immediately see that in a $K_{1,k+1}$-saturated graph G we have that $\{v \in G : d(v) < k\}$ must form a clique in G, as otherwise we could add an edge without producing a $K_{1,k+1}$. Hence we have that the score $\text{sat}_{\mathcal{G}}(K_n, K_{1,k+1}) \geq \left\lfloor \frac{1}{2}nk - \left(\frac{k-1}{2}\right)^2 \right\rfloor$. This lower bound can be improved somewhat.

Proof of Theorem 5. The upper bound follows trivially from the fact that $\Delta(G) \leq k$ in any $K_{1,k+1}$-saturated graph G. Let Prolonger have the following strategy: Given a graph $\mathcal{G}_i$ by

Shortener, she adds the least edge in $\bar{\mathcal{G}}_i$, where the edges uv of $\bar{\mathcal{G}}_i$ are ordered lexicographically by the minimum degree of u and v and then by the maximal degree. Note first that he will attempt to add edges between vertices of degree $\delta(\mathcal{G}_i)$. If Prolonger is unable to find such an edge, then the vertices of degree $\delta(\mathcal{G}_i)$ must form a clique, and hence there are at most $\delta(\mathcal{G}_i)+1 \leq k+1$ of them. These final $\leq k+1$ vertices may require their degrees to be increased by adding edges to vertices of degree greater than $\delta(\mathcal{G}_i)$.

Consider the graph process given that Prolonger is following this strategy. Let t_i be least such that $\delta(\mathcal{G}_{t_i}) \geq i$ and Shortener has just played. Let $g_i = \sum_v \max(d_{\mathcal{G}_{t_i}}(v) - (i+1), 0)$, if t_i exists, and $g_i = 0$ otherwise. Suppose that t_i exists and that in $\mathcal{G}_{t_i}$ there are λ_i vertices of degree $> i$. Then after at most $\left\lfloor \frac{1}{2}(n-k-1-\lambda_i) \right\rfloor$ moves by Prolonger there are $\leq k+1$ vertices of degree i, and after at most another $k+1$ moves by Prolonger there are no vertices of degree i, unless the game has ended. So we know that the t_{i+1} exists unless the game ends, which requires that at least $n-k$ vertices have degree k. So if:

$$k(n-k) \geq (i+1)(n-(k+1)) + (g_i - \lambda_i) + (n-k-1-\lambda_i) + 3(k+1)$$

Then even if every vertex used by Shortener and all of the $k+1$ last minimum degree vertices have to be paired with a higher degree vertex by Prolonger, there will not be $n-(k+1)$ vertices of degree k, and so the game will not have ended before Prolonger has increased the degree of each degree i vertex. In this case t_{i+1} exists and:

$$g_{i+1} \leq (g_i - \lambda_i) + (n-k-1-\lambda_i) + 3(k+1).$$

Note that $\lambda_i \geq g_i/(k-i)$, as any vertex contributes at most $k-i$ to g_i. Define:

$$f_0 = 0, \quad f_{i+1} = f_i + (n+2k+2) - 2f_i/(k-i).$$

We have that g_{i+1} is decreasing in λ_i and for all $i \leq k-2$, f_{i+1} is increasing in f_i. Hence:

$$g_i \leq g_{i-1} + (n+2k+2) - 2\lambda_{i-1} \leq f_i$$

for all $i \leq k-2$. So if $f_i \leq (k-i)(n-k)$, then we have that g_{i-1} is small enough that t_i exists, and so the minimum degree at the end of the game will be at least i. By induction, we have that $f_i = i(n+2k+2)\frac{k-i}{k-1}$. Hence to show that t_i exists for all $i \leq k-2$ it suffices that:

$$i(n+2k+2)\frac{k-i}{k-1} \leq (k-i)(n-k) \Leftrightarrow i \leq (k-1)\frac{n-k}{n+2k+2}$$

holds for each $i \leq k-2$. So for $n \geq 3k^2 - 3k - 4$, we have t_i exists for all $i \leq k-2$, and so the minimum degree of the saturated graph is at least $k-2$. Hence we have that $\text{sat}_{\mathcal{G}}(K_n, K_{1,k+1}) \geq \frac{1}{2}\left(kn - 2(k-1)\right)$ as required. □

8. Concluding Remarks

There remain many interesting open problems, mainly the resolution of the triangle saturation game. Given a graph G which is not a tree, providing effective bounds on the $\mathcal{G}(K_n, G)$ would be highly desirable. In our results, we show that a careful analysis of the maximal components is of substantive use, and that the supply of low degree vertices controls the ability of both players to enforce conditions on the game. However, our results are strongly predicated on finding explicit strategies; analyses of the $\mathcal{F}$-saturation game which were not of this form would be remarkable.

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[1] C. Biró, P. Horn, and D. J. Wildstrom. An upper bound on the extremal version of Hajnal's triangle-free game. *ArXiv e-prints*, September 2014.

[2] Béla Bollobas. *Extremal Graph Theory*. Dover Publications, Incorporated, 2004.

[3] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Mathematica Academiae Scientiarum Hungarica*, 10(3-4):337–356, 1959.

[4] Z. Füredi, D. Reimer, and A. Seress. Hajnal's triangle-free game and extremal graph problems. *Congressus Numerantium*, 82:123–128, 1991.

[5] D. B. West. The f-saturation game (2009) and game saturation number (2011). `http://www.math.uiuc.edu/~west/regs/fsatgame.html`.