

# Symmetric polynomials in information theory: entropy and subentropy

Richard Jozsa and Graeme Mitchison

*DAMTP, Centre for Mathematical Sciences, University of Cambridge,  
Wilberforce Road, Cambridge CB3 0WA, U.K.*

## Abstract

Entropy and other fundamental quantities of information theory are customarily expressed and manipulated as functions of probabilities. Here we study the entropy  $H$  and subentropy  $Q$  as functions of the elementary symmetric polynomials in the probabilities, and reveal a series of remarkable properties. Derivatives of all orders are shown to satisfy a complete monotonicity property.  $H$  and  $Q$  themselves become multivariate Bernstein functions and we derive the density functions of their Levy-Khintchine representations. We also show that  $H$  and  $Q$  are Pick functions in each symmetric polynomial variable separately. Furthermore we see that  $H$  and the intrinsically quantum informational quantity  $Q$  become surprisingly closely related in functional form, suggesting a special significance for the symmetric polynomials in quantum information theory. Using the symmetric polynomials we also derive a series of further properties of  $H$  and  $Q$ .

## 1 Introduction

It is natural to represent the informational properties of a random variable by functions that depend on the associated probabilities only as an *unordered set*. Correspondingly, entropic functions that are used as information measures are *symmetric* functions of the probabilities. But we can incorporate this fundamental feature in a deeper mathematical way: if we use the elementary symmetric polynomials in the probabilities as primary variables, rather than the probabilities themselves, then *arbitrary* functions will automatically depend on the probabilities only as an unordered set. In this paper we will study the Shannon entropy  $H$  and subentropy  $Q$  (cf below), as functions of the symmetric polynomials in the probabilities. We will see that they both then exhibit a series of remarkable properties. Furthermore the use of symmetric polynomials as variables will reveal surprising relationships between  $H$  and the intrinsically quantum information theoretic quantity  $Q$ .

Let  $\{x_1, \dots, x_d\}$  be a probability distribution. The associated elementary symmetric polynomials are defined by

$$e_1 = \sum_j x_j, \quad e_2 = \sum_{i < j} x_i x_j, \quad e_3 = \sum_{i < j < k} x_i x_j x_k, \quad \dots$$

We will lift the probability condition  $e_1 = \sum x_j = 1$  and use  $(e_1, e_2, \dots, e_d)$  as independent variables. For non-negative  $x_j$ 's we obviously have each  $e_k \geq 0$  although not all  $d$ -tuples of

non-negative  $e_k$  values arise in this way: given any  $e_1, \dots, e_d \geq 0$  the associated  $x_j$ 's are the roots of the polynomial equation

$$p(x) = x^d - e_1 x^{d-1} \dots + (-1)^k e_k x^{d-k} \dots + (-1)^d e_d = 0 \quad (1)$$

so they may be either real non-negative or complex (in which case they must occur in complex conjugate pairs), but they cannot be real and negative. Below, when  $H$  and  $Q$  (defined initially as functions of non-negative  $x_j$ 's) are expressed as functions of the  $e_k$ 's, they are analytic functions on the associated restricted region of  $(e_1, \dots, e_d)$ -space and it will be natural and convenient to view them as functions on the full space of all non-negative real  $e_k$ 's by analytic continuation. Intriguingly, in terms of our original variables  $x_i$ , this amounts to extending the notion of probability to a certain complex domain.

The entropy  $H$  and subentropy  $Q$  are defined by the following symmetric functions of the  $x_j$ 's:

$$H(x_1, \dots, x_d) = - \sum_{i=1}^d x_i \ln x_i \quad (2)$$

$$Q(x_1, \dots, x_d) = - \sum_{i=1}^d \frac{x_i^d}{\prod_{j \neq i} (x_i - x_j)} \ln x_i. \quad (3)$$

(For coincident  $x_j$ 's the subentropy is defined to be the corresponding limit as the  $x_j$ 's become equal, which is always finite.) For  $d$ -dimensional quantum states  $\rho$  we define  $H(\rho)$  and  $Q(\rho)$  to be the above functions applied to the eigenvalues of  $\rho$ .

The subentropy function was introduced in [2, 1] where it was shown to have the following physical significance:

(a) For any quantum state  $\rho$  let  $\mathcal{E} = \{q_i; |\psi_i\rangle\}$  be an ensemble of pure states  $|\psi_i\rangle$  with density matrix  $\rho$ ; i.e.  $q_i$  are probabilities and  $\sum_i q_i |\psi_i\rangle\langle\psi_i| = \rho$ . The accessible information  $I_{\text{acc}}(\mathcal{E})$  of  $\mathcal{E}$  is defined to be the maximum amount of classical mutual information about  $i$  that can be obtained from any measurement on the pure states  $|\psi_i\rangle$ . According to Holevo's theorem [3], the von Neumann entropy  $H(\rho)$  is an attainable upper bound on  $I_{\text{acc}}(\mathcal{E})$  as  $\mathcal{E}$  ranges over all pure state ensembles with density matrix  $\rho$ . In [1] it was shown that, dually,  $Q(\rho)$  is an attainable *lower* bound (being attained for the so-called Scrooge ensemble).

(b) Let  $\mathcal{M}$  be a complete von Neumann measurement in  $d$  dimensions with associated orthonormal basis  $\{|e_i\rangle\}$ . For a quantum state  $\rho$  let  $p_i = \langle e_i | \rho | e_i \rangle$  be the measurement probabilities and let  $H_{\mathcal{M}} = H(p_1, \dots, p_d)$  be the Shannon entropy of the output distribution. If  $\langle H_{\mathcal{M}} \rangle$  denotes the Haar-uniform average over choices of measurement basis then  $Q(\rho) = \langle H_{\mathcal{M}} \rangle - (\frac{1}{2} + \dots + \frac{1}{d})$ .

Although  $H$  is a quantity that has fundamental significance in classical information theory, the subentropy  $Q$  appears to be an intrinsically quantum construct with no known natural significance in classical information theory. Nevertheless we will see that, when expressed in terms of the  $e_k$ 's as variables,  $H$  and  $Q$  become surprisingly closely related (cf for example eqs. (38) and (45) below) suggesting a special significance for the  $e_k$  variables for quantum information theory.

Elementary symmetric functions have appeared previously in quantum information theory in the guise of entanglement measures. The concurrence of a pure bipartite qubit

state [4, 5] and its generalisation [6] to arbitrary local dimension  $d$ , can be written as  $\sqrt{K}e_2(\lambda_1, \dots, \lambda_d)$ , where  $K = 2d/(d+1)$  and  $\lambda_i$  are the eigenvalues of the reduced density matrix of the state. This has been further generalised in [7] to give a family of “concurrence monotones” which are proportional to  $e_k^{1/k}$ , for  $k = 2, 3, \dots, d$ .

Our use of the elementary symmetric functions is very different: rather than interpreting these functions as measures, we assign them a primary role as the underlying *variables* for an intrinsically symmetric formulation of information measures and entropic functions. This appears to be entirely novel in information theory, and our exploration here was motivated by our earlier work [8] in which it was shown that  $\partial H/\partial e_k > 0$  for  $k \geq 2$ , as an intermediate step for developing an interpretation of Schumacher compression in terms of the geometry of Hilbert space. The monotonicity of these first derivatives was also established (by different means) in [9] for other purposes. Below we will see that, amongst other properties, similar (alternating) monotonicity conditions in fact hold for all higher order derivatives of both  $H$  and  $Q$ , and in particular the functions  $\partial H/\partial e_k$  for  $k \geq 2$ , are completely monotone functions in all their variables. Furthermore we will see that on the space  $\{(e_1, \dots, e_d) : e_1 = 1 \text{ and } e_2, \dots, e_d \geq 0\}$ , the functions  $H$  and  $Q$  are multi-variate Bernstein functions and single-variable Pick functions in each variable separately.

## 2 Contour integrals and half-axis formulae

To express  $H$  and  $Q$  as functions of the  $e_k$ 's we will use the implicit relation eq. (1) between the  $x_j$ 's and the  $e_k$ 's, together with Cauchy's integral formula of complex analysis. Writing

$$p(z) = z^d - e_1 z^{d-1} \dots + (-1)^k e_k z^{d-k} \dots + (-1)^d e_d, \quad (4)$$

we note that  $p'(z)/p(z) = \sum_{j=1}^d \frac{1}{z-x_j}$  where  $x_1, \dots, x_d$  are the roots of  $p$ , and hence

$$H(e_1, \dots, e_d) = -\frac{1}{2\pi i} \oint z \ln z \frac{p'(z)}{p(z)} dz. \quad (5)$$

Here the contour in the complex  $z$ -plane surrounds all the  $x_j$  values but excludes the origin  $z = 0$ . For the complex logarithm we always use the negative real axis as the branch cut and use the branch given by  $\ln z = \ln |z| + i \arg z$  with  $-\pi < \arg z < \pi$ .

The case of subentropy is actually simpler: if  $g(z)$  is any function that is holomorphic within and on the contour then Cauchy's residue theorem gives (for distinct  $x_j$ 's)

$$\sum_{j=1}^d \frac{g(x_j)}{\prod_{i \neq j} (x_j - x_i)} = \frac{1}{2\pi i} \oint \frac{g(z)}{(z-x_1) \dots (z-x_d)} dz. \quad (6)$$

Setting  $g(z) = -z^d \ln z$  and comparing with eq. (3) for  $Q$  we immediately get

$$Q = -\frac{1}{2\pi i} \oint \frac{z^d \ln z}{p(z)} dz. \quad (7)$$

We now derive expressions for  $H$  and  $Q$  as real integrals on the positive real axis. These will be obtained from the contour integrals above by distorting the contour into a

keyhole contour that excludes the negative real axis, running above and below it at a small distance  $\epsilon$  between  $z = -R \pm i\epsilon$  and  $z = 0 \pm i\epsilon$ , looping around the origin  $z = 0$  in a circular arc of radius  $\epsilon$  and being completed by a circular arc of large radius  $R$ , all traversed counterclockwise. Then we will consider the limits  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$ .

Consider any integral of the form

$$\frac{1}{2\pi i} \oint h(z) \ln z dz \quad (8)$$

around the keyhole contour where  $h(z)$  generally has poles (but not branch points) inside the contour. Suppose that the sections of the contour integral on the two circular arcs each tend to zero as  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$  respectively. Then recalling that  $\arg z$  tends to  $+\pi$ , resp.  $-\pi$ , as  $z$  approaches the negative  $z$  axis from above, resp. below, and  $\ln z = \ln |z| + i \arg z$ , we see that

$$\frac{1}{2\pi i} \oint h(z) \ln z dz = \int_0^\infty h(-\tau) d\tau, \quad (9)$$

where the negative axis has been labelled as  $-\tau$  for  $\tau > 0$ .

Now consider again our contour integral formula eq. (5) for entropy:

$$H(e_1, \dots, e_d) = \frac{1}{2\pi i} \oint -z \ln z \frac{p'(z)}{p(z)} dz.$$

The integrand satisfies the desired vanishing condition for  $\epsilon \rightarrow 0$  but not for  $R \rightarrow \infty$  (the integrand diverging as  $\ln R$  on that circular arc). To remedy this, consider modifying the integrand by adding two further terms:

$$\oint \left[ -z \ln z \frac{p'(z)}{p(z)} + A \ln z + B \frac{\ln z}{z-1} \right] dz \quad (10)$$

which do not change the value of the integral (since for the two terms respectively,  $\ln z$  is holomorphic throughout the whole region inside the contour, and its residue  $\ln 1$  at  $z = 1$  is zero too). The integrand has the form  $\frac{r_1(z)}{r_2(z)} \ln z$  where  $r_1$  and  $r_2$  are polynomials of degree  $d+1$ , and using eq. (4) for  $p(z)$  we see that if we set  $A = d$  and  $B = e_1$  then the two leading coefficients of  $r_1$  become zero and the asymptotic condition on the circular arc  $R \rightarrow \infty$  will be satisfied. Thus we have

$$H = \frac{1}{2\pi i} \oint \ln z \left[ -\frac{zp'(z)}{p(z)} + \frac{e_1}{z-1} + d \right] dz$$

with the integral now satisfying both conditions on the circular arcs with  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$ .

Finally we apply eq. (9) with  $h(z) = (-zp'(z)/p(z) + e_1/(z-1) + d)$ . Introducing

$$q(\tau) = \tau^d + e_1\tau^{d-1} + \dots + e_k\tau^{d-k} + \dots + e_d, \quad (11)$$

which is the polynomial with roots  $-x_1, -x_2, \dots, -x_d$ , we get the *half-axis formula*:

$$H(e_1, \dots, e_d) = \int_0^\infty \left[ -\frac{\tau q'(\tau)}{q(\tau)} - \frac{e_1}{\tau+1} + d \right] d\tau, \quad (12)$$

with the  $e_k$  dependence appearing explicitly on substituting eq. (11) for  $q(\tau)$ . By writing the integrand as  $\tau f'(\tau)$  and integrating by parts, we obtain the further formula

$$H(e_1, \dots, e_d) = \int_0^\infty [\ln q(\tau) + (e_1 - d) \ln \tau - e_1 \ln(\tau + 1)] d\tau, \quad (13)$$

which will be useful later in section 4. This formula can also be obtained by summing the corresponding result for one variable, namely  $-x \ln x = \int_0^\infty [\ln(t+x) + (x-1) \ln t - x \ln(t+1)] dt$ . A similar formula for  $H$  is given in [9], derived there by summation from a corresponding one-variable formula. An advantage of our complex variable approach is to provide motivation for the origins of the formula beyond just starting with a quotation of the somewhat inscrutable, albeit readily verifiable, one-variable case.

For the case of subentropy we can apply the same techniques to the contour integral formula eq. (7) giving

$$Q = \frac{1}{2\pi i} \oint \ln z \left[ -\frac{z^d}{p(z)} + \frac{e_1}{z-1} + 1 \right] dz$$

with the integrand satisfying both circular arc conditions, and eq. (9) gives

$$Q(e_1, \dots, e_d) = \int_0^\infty \left[ -\frac{\tau^d}{q(\tau)} - \frac{e_1}{(\tau+1)} + 1 \right] d\tau. \quad (14)$$

It is interesting to note a tantalising similarity between the formulae for  $H$  and  $Q$  in eqs. (13) and (14) in the case of  $e_1 = 1$ :

$$\begin{aligned} H(1, e_2, \dots, e_d) &= \int_0^\infty -\ln \left[ \frac{\tau^d}{q(\tau)} \right] + \ln \left[ \frac{\tau}{\tau+1} \right] d\tau \\ Q(1, e_2, \dots, e_d) &= \int_0^\infty -\left[ \frac{\tau^d}{q(\tau)} \right] + \left[ \frac{\tau}{\tau+1} \right] d\tau \end{aligned}$$

The formulae in eqs. (12) and (14) immediately yield half-axis formulae for partial derivatives of  $H$  and  $Q$  with respect to the symmetric polynomials. For instance

$$\frac{\partial H}{\partial e_k} = \int_0^\infty \frac{\tau^{d-k}}{q(\tau)} d\tau, \quad \text{for } k \geq 2 \quad (15)$$

and

$$\frac{\partial H}{\partial e_1} = \int_0^\infty \left[ \frac{\tau^{d-1}}{q(\tau)} - \frac{1}{(\tau+1)} \right] d\tau - 1 \quad \text{for } k = 1. \quad (16)$$

### 3 Properties of H and Q

A variety of properties of  $H$  and  $Q$  follow from their half-axis formulae given above. We describe these in full in the Appendix, but we summarise here a few key results, pointing the reader to the Appendix for further details and proofs. Some of these results will be

used in our developments in the next section. In particular, we will build on the fact that the partial derivatives of  $H$  and  $Q$  have definite signs:

$$\frac{\partial H}{\partial e_k} > 0, \quad \text{for } 2 \leq k \leq d, \quad (17)$$

$$\frac{\partial Q}{\partial e_k} > 0, \quad \text{for } 2 \leq k \leq d, \quad (18)$$

$$(-1)^{m-1} \frac{\partial^m H}{\partial e_{k_1} \dots \partial e_{k_m}} > 0, \quad \text{for } m \geq 2, \quad 1 \leq k_j \leq d. \quad (19)$$

$$(-1)^{m-1} \frac{\partial^m Q}{\partial e_{k_1} \dots \partial e_{k_m}} > 0, \quad \text{for } m \geq 2, \quad 1 \leq k_j \leq d. \quad (20)$$

See Appendix, Property 7. Note that  $\partial H/\partial e_1$  and  $\partial Q/\partial e_1$  can be negative, so the restriction to  $k \geq 2$  in inequalities (17) and (18) is necessary. However the indices of the  $e$ 's in inequalities (19) and (20) can lie in the whole range  $1, \dots, d$ .

We will also use the fact that the  $m^{\text{th}}$  derivatives of  $H$  and  $Q$  depend only on the *sum* of the indices,  $\sum k_j$ . Thus for example  $\partial^2 H/\partial e_1 \partial e_5 = \partial^2 H/\partial e_2 \partial e_4 = \partial^2 H/\partial e_3^2$  since  $1 + 5 = 2 + 4 = 3 + 3$ . See Property 5.

A surprisingly close functional relationship between  $H$  and  $Q$ , not so apparent from the defining eqs. (2) and (3), appears in the equalities

$$H = e_1 + \sum_{k=1}^d k e_k \frac{\partial H}{\partial e_k};$$

$$Q = e_1 + \sum_{k=1}^d e_k \frac{\partial H}{\partial e_k},$$

and

$$-\frac{\partial Q}{\partial e_k} = \frac{\partial^2 H}{\partial e_l \partial e_m},$$

for any  $k, l, m$  with  $k = l + m$  and  $l, m \geq 1$ . See Appendix, Properties 1 and 4.

One can also obtain some bounds, for instance

$$H - Q \geq \sum_{k=2}^d \frac{d^{k-1} e_k}{\binom{d-1}{k-1} e_1^{k-1}}.$$

See Property 3. If the  $e_k$ 's are the symmetric polynomials in the eigenvalues of a state  $\rho$ , then this is a lower bound on the range of the accessible information obtainable from pure state ensembles with density matrix  $\rho$ . Even the first term is meaningful, for it says

$$H - Q \geq \frac{d e_2}{(d-1) e_1},$$

implying that the range is non-zero when the state  $\rho$  is not a pure state.

Finally, we mention the following upper bound for  $H$  (there is a corresponding one for  $Q$ ), which depends only on  $e_1$  and  $e_2$ :

$$H(e_1, e_2, \dots, e_d) \leq -(d-1)a \ln a - b \ln b,$$

where  $a$  and  $b$  are roots of  $(n-1)a + b = e_1$  and  $\binom{n-1}{2}a^2 + (n-1)ab = e_2$ . See Property 11. A similar type of bound (i.e. depending only on  $e_1$  and  $e_2$ ) was obtained by Hellmund and Uhlmann [10]. However, the above bound is optimal amongst all bounds that depend only on  $e_1$  and  $e_2$ , as it is always attained by an actual assignment  $(x_1, \dots, x_d) = (a, \dots, a, b)$  satisfying the given  $e_1$  and  $e_2$  values.

## 4 H and Q as Bernstein and Pick functions

A function  $f(s)$  is said to be *completely monotone* if  $(-1)^k d^k f / ds^k \geq 0$ , for  $s \in [0, \infty)$  and  $k \geq 0$ . We extend this definition to several variables by requiring that

$$(-1)^k \partial^k f / \partial s_{i_1} \dots \partial s_{i_k} \geq 0, \quad \text{for } s_{i_j} \geq 0, \quad k \geq 0. \quad (21)$$

Bernstein's theorem [11, 13, 12] asserts that a function  $f(s)$  of one variable is completely monotone iff  $f(s)$  is the Laplace transform of a non-negative density  $\nu$ , viz.

$$f(s) = \int_0^\infty e^{-st} \nu(t) dt.$$

Here the density  $\nu$  is generally supported on the half *closed* interval  $[0, \infty)$  [12], and may involve point weights (e.g. the completely monotone  $f(s) = 1$  has  $\nu(t) = \delta(t)$ ). If  $b$  is the weight of the point  $t = 0$  then the Bernstein representation may also be written as

$$f(s) = b + \int_{0^+}^\infty e^{-st} \nu(t) dt.$$

In our applications we will always have  $b = 0$ .

The Bernstein representation theorem can be generalised to multivariate completely monotone functions as follows (cf. Theorem 4.2.1 in [14], stated there without proof).

**Lemma 1.**  $f(s_1, \dots, s_m)$  is completely monotone iff  $f(s_1, \dots, s_m) = \mathcal{L}[\xi(t_1, \dots, t_m)]$  where  $\xi$  is a non-negative density for  $t_i \geq 0$ ,  $1 \leq i \leq m$ . Explicitly the condition is

$$f(s_1, \dots, s_m) = \int_0^\infty e^{-(s_1 t_1 + \dots + s_m t_m)} \xi(t_1, \dots, t_m) dt_1 \dots dt_m. \quad (22)$$

**Proof** Consider first the case of two variables. Since  $f(s_1, s_2)$  is completely monotone in the first variable, Bernstein's theorem implies that  $f(s_1, s_2) = \mathcal{L}[\nu(t_1, s_2)]$ , the Laplace transform applying only to the first variable. Next note that eq. (21) implies that for each  $k$ ,  $(-1)^k \partial^k f / \partial s_2^k$  is completely monotone in  $s_1$  so it is the Laplace transform of a non-negative function. But we already have  $(-1)^k \partial^k f / \partial s_2^k = \mathcal{L}[(-1)^k \partial^k \nu(t_1, s_2) / \partial s_2^k]$  so by uniqueness of the Laplace transform,  $(-1)^k \partial^k \nu(t_1, s_2) / \partial s_2^k$  must be non-negative for all  $s_2 \geq 0$ . Thus  $\nu(t_1, s_2)$  is completely monotone in  $s_2$  and so itself must be the Laplace

transform (for the single variables  $t_2$  and  $s_2$ ) of some non-negative  $\xi(t_1, t_2)$ . Finally then  $f(s_1, s_2) = \mathcal{L}[\nu(t_1, s_2)] = \mathcal{L}[\xi(t_1, t_2)]$ . This argument readily extends to any number of variables.  $\square$

Equations (17) - (20) (see Property 7) can be interpreted as asserting that each first derivative  $\partial H/\partial e_k$  and  $\partial Q/\partial e_k$ , for  $2 \leq k \leq d$ , is completely monotone in the multivariate sense. But what about  $H$  and  $Q$  themselves? Note first that the variable  $e_1$  has a distinguished role since  $\partial H/\partial e_1$  and  $\partial Q/\partial e_1$  are not generally non-negative. Furthermore  $H$  itself is not generally non-negative for  $e_1 > 1$  (e.g.  $H(x, 0, \dots, 0) = -x \ln x < 0$  for  $x > 1$ ). Thus let us set  $e_1 = 1$  and introduce the positive cone  $\mathcal{E}^+$  defined by  $e_k \geq 0$ ,  $2 \leq k \leq d$  (though, as pointed in the Introduction, only part of this cone corresponds to real, positive  $x_i$ ). We know  $H = 0$  when  $e_1 = 1$  and  $e_2 = \dots = e_d = 0$ , since this corresponds to one of the underlying probabilities being 1 and the others 0. But then every point in  $\mathcal{E}^+$  can be reached by moving along its coordinate axes independently, and it follows from  $\partial H/\partial e_k > 0$  for  $k \geq 2$ , that  $H$  must be positive everywhere in  $\mathcal{E}^+$ . By Property 4, a similar conclusion applies to  $Q$ .

Although non-negative,  $H(1, e_2, \dots, e_d)$  is not completely monotone, since both it and its first derivatives are non-negative: there is no change of sign between the function and its first derivative, as the definition requires. Similarly,  $Q(1, e_2, \dots, e_d)$  is not completely monotone. However, a function that is non-negative on the positive cone and has completely monotone first derivatives is said to be a *Bernstein* function and from Property 7 we have

**Proposition 1.**  $H(1, e_2, \dots, e_d)$  and  $Q(1, e_2, \dots, e_d)$  are multivariate Bernstein functions.

There is an extensive literature on Bernstein functions [12]. Any such function has a *Levy-Khintchine representation*, which in one variable takes the form

$$f(s) = a + bs + \int_0^\infty (1 - e^{-st})\mu(t) dt, \quad (23)$$

for some non-negative kernel  $\mu$  supported on  $(0, \infty)$ , satisfying  $\int_0^\infty \min(1, t)\mu(t) dt < \infty$ . This can be obtained [12] by integrating Bernstein's Laplace transform representation of  $f'(s)$  mentioned above. The constants  $a$  and  $b$  can be identified [12] as  $a = f(0)$  and  $b = \lim_{s \rightarrow \infty} f(s)/s$ . In our applications and extensions below we will always have  $a = b = 0$ .

As with Lemma 1 and Bernstein's theorem, the Levy-Khintchine representation may also be extended to multivariate functions. We illustrate this by deriving the Levy-Khintchine representation for  $H(1, e_2, \dots, e_d)$ . Recollecting eq. (15), we have

$$\frac{\partial H(1, e_2, \dots, e_d)}{\partial e_k} = \int_0^\infty \frac{\tau^{d-k}}{\tau^d + \tau^{d-1} + e_2\tau^{d-2} + \dots + e_d} d\tau.$$

Now the integrand for any given  $\tau$  can be written as a Laplace transform of the variable  $e_d$ , by using  $\int_0^\infty \exp(-at) dt = 1/a$  to obtain:

$$\frac{\tau^{d-k}}{\tau^d + \tau^{d-1} + e_2\tau^{d-2} + \dots + e_d} = \int_0^\infty \tau^{d-k} e^{-e_d t_d} e^{-(\tau^d + \tau^{d-1} + \dots + e_{d-1}\tau)t_d} dt_d.$$

Next we write the integrand above as the Laplace transform with respect to the variable



$e_{d-1}$  to give the double transform:

$$\frac{\tau^{d-k}}{\tau^d + \tau^{d-1} + e_2 \tau^{d-2} + \dots e_d} = \int_0^\infty \tau^{d-k} e^{-(e_{d-1} t_{d-1} + e_d t_d)} e^{-(\tau^d + \tau^{d-1} + \dots e_{d-2} \tau^2) t_d} \delta(t_{d-1} - \tau t_d) dt_{d-1} dt_d.$$

Continuing in this way we find

$$\frac{\tau^{d-k}}{\tau^d + \tau^{d-1} + \dots e_d} = \int_0^\infty \tau^{d-k} e^{-\sum_{i=2}^d e_i t_i} e^{-(\tau^d + \tau^{d-1}) t_d} \prod_{i=1}^{d-2} \delta(t_{d-i} - \tau^i t_d) dt_2 \dots dt_d.$$

Integrating over  $\tau$  now gives

$$\frac{\partial H(1, e_2, \dots, e_d)}{\partial e_k} = \mathcal{L}[\phi_k(t_2, \dots, t_d)], \quad (24)$$

where

$$\phi_k(t_2, \dots, t_d) = \int_0^\infty \tau^{d-k} e^{-(\tau^d + \tau^{d-1}) t_d} \prod_{i=1}^{d-2} \delta(t_{d-i} - \tau^i t_d) d\tau. \quad (25)$$

Now let us pause and consider the consequences of the Laplace transforms (24). We can differentiate under the integral to obtain

$$\frac{\partial^2 H(1, e_2, \dots, e_d)}{\partial e_j \partial e_k} = \mathcal{L}[t_j \phi_k(t_2, \dots, t_d)] = \mathcal{L}[t_k \phi_j(t_2, \dots, t_d)], \quad (26)$$

which, by the uniqueness of the Laplace transform, implies

$$t_j \phi_k(t_2, \dots, t_d) = t_k \phi_j(t_2, \dots, t_d), \quad (27)$$

for  $2 \leq j, k \leq d$ . This means we can define a function  $\mu$  by

$$\mu(t_2, \dots, t_d) = \frac{\phi_2(t_2, \dots, t_d)}{t_2} = \dots = \frac{\phi_d(t_2, \dots, t_d)}{t_d}. \quad (28)$$

Now let us integrate the equations (24) to obtain

$$\begin{aligned} H(1, e_2, 0, \dots, 0) - H(1, 0, 0, \dots, 0) &= \int_0^\infty \frac{1 - e^{-e_2 t_2}}{t_2} \phi_2(t_2, \dots, t_d) dt_2 \dots dt_d \\ &= \int_0^\infty (1 - e^{-e_2 t_2}) \mu(t_2, \dots, t_d) dt_2 \dots dt_d, \\ H(1, e_2, e_3, \dots, 0) - H(1, e_2, 0, \dots, 0) &= \int_0^\infty e^{-e_2 t_2} \frac{1 - e^{-e_3 t_3}}{t_3} \phi_3, \\ &= \int_0^\infty \left( e^{-e_2 t_2} - e^{-(e_2 t_2 + e_3 t_3)} \right) \mu(t_2, \dots, t_d) dt_2 \dots dt_d. \end{aligned}$$

Continuing in this way to  $H(1, \dots, e_{d-1}, e_d) - H(1, \dots, e_{d-1}, 0)$ , then adding all the equations, and using  $H(1, 0, 0, \dots, 0) = 0$ , we get

$$H(1, e_2, e_3, \dots, e_d) = \int_0^\infty \left(1 - e^{-\sum_{i=2}^d e_i t_i}\right) \mu(t_2, \dots, t_d) dt_2 \dots dt_d. \quad (29)$$

If we evaluate  $\phi_k$  from eq. (25) we can use eq. (28) to give  $\mu$  explicitly. Let us first, however, make a further deduction from the Laplace transforms (24). Property 5 says that higher derivatives depend only on the sums of indices. This implies further relations besides eq. (27), and these entail relations between the variables  $t_i$ . For example,  $\partial^2 H / \partial e_2 \partial e_5 = \partial^2 H / \partial e_3 \partial e_4$  implies  $t_2 \phi_5 = t_3 \phi_4$ , which eq. (28) allows us to write as  $t_2 t_5 \mu = t_3 t_4 \mu$ , or  $t_2 t_5 = t_3 t_4$ . This tells us that the kernel  $\mu$  of the Levy-Khintchine representation we seek lies in some variety within the positive cone  $\mathbb{R}_+^{d-2}$  defined by the  $t_i$ , and we can see this also from the product of delta-functions in eq. (25), since this product is only non-vanishing when there is a value of  $\tau$  which simultaneously satisfies  $t_{d-i} = \tau^i t_d$  for  $1 \leq i \leq d-2$ . We now characterise that variety:

**Lemma 2.** *Property 5 implies  $d-3$  independent relations amongst the  $d-1$  variables  $t_2, \dots, t_d$ , namely*

$$t_d t_i = t_{d-1} t_{i+1} \quad \text{for } 2 \leq i \leq d-2. \quad (30)$$

*These are also the relations obtained by eliminating  $\tau$  from*

$$t_{d-i} = \tau^i t_d \quad \text{for } 1 \leq i \leq d-2, \quad (31)$$

*which must be satisfied for the non-vanishing of the product of delta-functions in eq. (25). These  $d-3$  relations define a 2D surface in the space of the  $t_i$ .*

**Proof** The relations following from Property 5 have the form  $t_i t_j = t_k t_l$ , for  $i+j = k+l$ . To show that any such relation can be obtained from eqs. (30), note that we can combine two of them to get  $(t_d t_i)(t_{d-1} t_j) = (t_{d-1} t_{i+1})(t_d t_{j+1})$ , which implies  $t_i t_j = t_{i+1} t_{j-1}$ . Iterating gives  $t_i t_j = t_{i+1} t_{j-1} = t_{i+2} t_{j-2} = \dots = t_k t_{j-(k-i)}$ , which is a general relation of the form we seek. It is easy to see that eqs. (30) are equivalent to eqs. (31) with  $\tau = t_{d-1}/t_d$ .  $\square$

**Theorem 1.** *The kernel  $\mu$  in eq. (29), the Levy-Khintchine representation of  $H(1, e_2, \dots, e_d)$ , is given by*

$$\mu(t_2, \dots, t_d) = \frac{1}{t_d^2} \exp\left(-t_d \left(r^d + r^{d-1}\right)\right) \prod_{i=2}^{d-2} \delta(t_i - r t_{i+1}), \quad (32)$$

*where  $r = t_{d-1}/t_d$ . The corresponding kernel for  $Q(1, e_2, \dots, e_d)$ , is given by*

$$\mu(t_2, \dots, t_d) = \frac{r^d}{t_d} \exp\left(-t_d \left(r^d + r^{d-1}\right)\right) \prod_{i=2}^{d-2} \delta(t_i - r t_{i+1}). \quad (33)$$

*The product of delta-functions restricts the function to the surface defined by eq. (30) in Lemma 2.*

**Proof** Write

$$\alpha(t_2, \dots, t_d) = 1 - e^{-\sum_{i=2}^d e_i t_i}$$

and

$$\beta(t_{d-1}, t_d) = \frac{1}{t_d^2} \exp\left(-t_d \left( (t_{d-1}/t_d)^d + (t_{d-1}/t_d)^{d-1} \right)\right).$$

If we define  $\mu$  by eqs. (28) and (25) with  $k = d$  and substitute it into eq. (29), we get

$$H(1, e_2, \dots, e_d) = \int_0^\infty \alpha(\tau^{d-2} t_d, \dots, \tau t_d, t_d) \beta(\tau t_d, t_d) t_d d\tau dt_d. \quad (34)$$

Then replacing the variable  $\tau$  by  $t_{d-1}$  via  $\tau = t_{d-1}/t_d$ , and writing

$$A(t_{d-1}, t_d) = \alpha((t_{d-1}/t_d)^{d-3} t_{d-1}, \dots, (t_{d-1}/t_d) t_{d-1}, t_{d-1}, t_d), \quad (35)$$

we get

$$H(1, e_2, \dots, e_d) = \int_0^\infty A(t_{d-1}, t_d) \beta(t_{d-1}, t_d) dt_{d-1} dt_d \quad (36)$$

which is equivalent to inserting the claimed formula eq. (32) for  $\mu$  into eq. (29), and integrating out the delta functions.

For  $Q$  we follow a similar argument at all the steps. Starting from

$$\frac{\partial Q}{\partial e_k} = \int_0^\infty \frac{\tau^{2d-k}}{q(\tau)^2} d\tau,$$

and using  $1/(s+a)^2 = \mathcal{L}[te^{-at}]$ , we obtain in place of eq. (36)

$$Q(1, e_2, \dots, e_d) = \int_0^\infty A(t_{d-1}, t_d) \beta(t_{d-1}, t_d) (t_{d-1}/t_d)^d t_d dt_{d-1} dt_d. \quad \square$$

$H(1, e_2, \dots, e_d)$  and  $Q(1, e_2, \dots, e_d)$ , regarded as functions of each single  $e_k$  separately, can be seen to belong to a further special class, the *complete Bernstein functions*, where  $\mu$  in eq. (23) (for functions of one variable) is itself completely monotone. Equivalently [12], such functions can be characterised as non-negative valued functions that are operator monotone functions, or Pick functions. A function  $f$  is operator monotone if  $A \leq B$  implies  $f(A) \leq f(B)$  for any self-adjoint matrices  $A, B$ . A Pick function is a real-valued function on  $[0, \infty)$  that possesses an analytic continuation on  $\mathbb{C} \setminus (-\infty, 0)$  that maps the upper half plane into itself. It is known [16] that  $f$  is operator monotone iff  $f$  is a Pick function.

**Proposition 2.**  *$H(1, e_2, \dots, e_d)$  and  $Q(1, e_2, \dots, e_d)$  are Pick functions in each of their variables separately (with all other variables being set to any fixed positive real values).*

**Proof** For the case of  $H$ , setting  $e_1 = 1$  in eq. (12) we see that the integrand has the form  $r_1(\tau)/r_2(\tau)$  with  $r_2(\tau) = q(\tau)(\tau + 1)$  having all roots on the negative real axis and  $r_1$  being of degree two less than  $r_2$  for all complex values of  $e_2, \dots, e_d$ . Hence if any  $e_k$  is extended to  $\mathbb{C} \setminus (0, \infty)$ , the integral remains finite and is a holomorphic function of the

chosen  $e_k$ . Furthermore, setting  $e_1 = 1$  in the alternative formula eq. (13) (and all other variables apart from  $e_k$  being set to positive real values) we get

$$H(1, e_2, \dots, e_d) = \int_0^\infty [\ln(\tau^d + \tau^{d-1} + e_2\tau^{d-2} + \dots + e_d) + (1-d)\ln\tau - \ln(\tau+1)] d\tau.$$

so the upper half plane will be preserved by  $H$ , since if  $\Im(z)$  denotes the imaginary part of  $z$ , we have  $\Im(z) > 0$  implies  $\arg(z) > 0$  i.e.  $\Im(\ln z) > 0$  and  $\Im(H) > 0$ .

For the case of  $Q$ , setting  $e_1 = 1$  in eq. (14) we get

$$Q(1, e_2, \dots, e_d) = \int_0^\infty \left[ \frac{-\tau^d}{(\tau^d + \tau^{d-1} + e_2\tau^{d-2} + \dots + e_d)} - \frac{1}{(\tau+1)} + 1 \right] d\tau.$$

Then extending  $e_k$  to the complex plane (with other variables remaining real), the imaginary part of the integrand will be  $\Im(e_k)\tau^{2d-k}/|q|^2$  so the upper half plane will be preserved by  $Q$ .  $\square$

There is yet another way of looking at Property 7. If  $f$  is a function whose first derivatives are completely monotone, then  $e^{-f}$  is completely monotone [13]. This is easy to check by repeated differentiation of  $e^{-f}$ . It follows that  $e^{-H(1, e_2, \dots, e_d)}$  is the Laplace transform of a non-negative function  $\mu(t_2, \dots, t_d)$ , and since  $H(1, 0, \dots, 0) = 0$ ,  $\mu$  is a probability density. Actually, we can say more than this, since  $e^{-H(1, e_2, \dots, e_d)} = (e^{-H/m})^m = \mathcal{L}[\nu^{*m}]$ , where  $e^{-H/m} = \mathcal{L}[\nu]$ . This means that, for any integer  $m$ ,  $\mu$  is the  $m$ -fold convolution of a measure  $\nu$ . This property is called *infinitely divisibility* [13], and is possessed by many fundamental statistical distributions, such as the Gaussian. Thus we know that  $e^{-H(1, e_2, \dots, e_d)}$  is the Laplace transform of an infinitely divisible function, and, since all the above remarks apply to  $Q$ , the same is true of  $e^{-Q(1, e_2, \dots, e_d)}$ .

Finally, we observe that the Levy-Khintchine representations for  $H$  and  $Q$  are multivariate versions of eq. (23) where the constants  $a$  and  $b$  are both zero. They are therefore what Audenaert [15] calls *bare* Bernstein functions. He shows (in the single variable case) that such functions satisfy various properties, including a type of matrix trace inequality. It would be interesting to know if this result, and also the operator monotone property, can be extended to the multivariate case. Ways of extending the concept of operator monotone functions to many variables have been explored [18, 19].

## 5 Concluding remarks

We have noted in the Introduction that informational properties of a random variable should be independent of the labelling of its outcomes and hence be represented by symmetric functions of the probabilities ( $x_j$ 's). Then on passing to the elementary symmetric polynomials ( $e_k$ 's) as variables, we have seen that the resulting functional forms of the entropy and subentropy exhibit a series of novel structural properties. A further benefit of the  $e_k$ 's is that derivatives and integrals with respect to these variables automatically preserve symmetry with respect to the  $x_j$ 's (in contrast to say  $\partial f(x_1, \dots, x_d)/\partial x_j$  for a symmetric function  $f$ ) and hence we have the benefit of the full power of calculus within the "informationally meaningful" regime of constructions that are symmetric in the probabilities.

A further intriguing feature of the  $e_k$  variables is that the entropy and subentropy become functionally surprisingly closely related. As subentropy is an intrinsically *quantum* informational construct (whereas the entropy function features fundamentally in both classical and quantum information theory) this suggests that the  $e_k$  variables may offer a special advantage for studying the extra intricacies of quantum over classical information theory. Such an advantage was also suggested in [8] where the monotonicity property of the first derivatives  $\partial H/\partial e_k$  for  $k \geq 2$  was used to provide a new interpretation of Schumacher compression of quantum information (which reduced to a trivial statement for the special case of classical information compression). But subentropy aside, the novel properties we have seen for  $H(e_1, \dots, e_d)$  itself suggest a role for the  $e_k$  variables already in just classical information theory.

Complex numbers have repeatedly played a key role in our constructions. In fact, they arise in three distinct ways:

Firstly, although non-negative  $x_j$ 's map to non-negative  $e_k$ 's, the image of this map is only a subset of  $\mathbb{R}_+^d$ , and it is natural to analytically extend the entropy functions to the full positive cone  $\mathbb{R}_+^d$  of all  $d$ -tuples of non-negative  $e_k$ 's. But then, mapping back to the space of  $x_j$ 's amounts to allowing probabilities to become complex, occurring always as complex conjugate pairs, although never being real and negative.

Secondly, we note that our half-axis formulae for entropy and subentropy were transparently derived starting from complex contour integral expressions, exploiting the basic relation eq. (1) between the  $x_j$ 's and  $e_k$ 's, together with properties of the complex logarithm function. In standard information theory and thermodynamics, the logarithm function serves to endow entropy with a fundamental extensionality property for composite independent systems, via  $\ln ab = \ln a + \ln b$ , whereas for us, in the context of the key-hole contour integral, it plays an entirely different role: the *complex* logarithm function serves to provide a constant discontinuity of  $2\pi$  in its imaginary part along its branch cut  $(-\infty, 0)$ .

Thirdly, complex numbers feature again in the notion of Pick functions and the mapping properties of  $H(1, e_2, \dots, e_d)$  and  $Q(1, e_2, \dots, e_d)$ , when each  $e_k$  for  $k \geq 2$  is allowed to vary over the upper half complex plane.

All of the above suggests that some features of information theory may acquire a special simplicity if formulated in an enlarged mathematical setting that includes the elementary symmetric polynomials and complex variables. Yet, paradoxically, there is no obvious probabilistic interpretation of the underlying ingredients. Extending probabilities  $x_k$  into the complex domain has no immediately apparent meaning; nor can one interpret the elementary symmetric polynomials as probabilistic objects, except perhaps in trivial ways, e.g. regarding  $k!e_k$  as the probability that  $k$  samples of a random variable with probabilities  $\{x_1, \dots, x_d\}$  yield no repeated outcome. This is not the sort of meaning we seek; discovering the elusive deeper meaning is a fascinating puzzle.

## 6 Appendix: properties of $H$ and $Q$

Here we derive various relations and inequalities from the half-axis formulae. Any proofs that are omitted follow straightforwardly from the half-axis formulae. First, we have

**Property 1.**

$$H = e_1 + \sum_{k=1}^d k e_k \frac{\partial H}{\partial e_k}; \quad (37)$$

$$Q = e_1 + \sum_{k=1}^d e_k \frac{\partial H}{\partial e_k}. \quad (38)$$

□

(Here and below we use □ to mark the end of statements of Properties as well as the end of Proofs.) It is striking how similar  $H$  and  $Q$  appear in this formulation, compared to the very different-looking expressions eq. (2) and eq. (3). We can make eq. (37) look even closer to eq. (38) by introducing the variables  $f_k = e_k^{1/k}$  to get

$$H = f_1 + \sum_{k=1}^d f_k \frac{\partial H}{\partial f_k}. \quad (39)$$

Note that there is an analogous expression for  $H$  in terms of derivatives with respect to the  $x_k$ , obtained directly from eq. (2):

$$H = e_1 + \sum_{k=1}^d x_k \frac{\partial H}{\partial x_k}. \quad (40)$$

Since the integrand in eq. (15) is positive, it follows that  $\partial H / \partial e_k \geq 0$  for  $k \geq 2$  [9]. However, one can show something a little stronger, since by Bernoulli's inequality  $q \leq (\tau + e_1/d)^d$  and hence

$$\int_0^\infty \frac{\tau^{d-k}}{q} d\tau \geq \int_0^\infty \frac{\tau^{d-k}}{(\tau + e_1/d)^d} d\tau = c_{d,k},$$

where

$$c_{d,k} = \frac{d^{k-1}}{(d-k+1) \binom{d-1}{k-2} e_1^{k-1}}. \quad (41)$$

Thus we get the positive lower bound

**Property 2.**

$$\frac{\partial H}{\partial e_k} \geq c_{d,k} > 0 \quad \text{for } k \geq 2. \quad (42)$$

□

This was proved in [8], by a more complicated argument. Combining this with eq. (37) and eq. (38), we get

$$H - Q = \sum_{k=2}^d (k-1) e_k \frac{\partial H}{\partial e_k} \geq \sum_{k=2}^d \frac{d^{k-1} (k-1) e_k}{(d-k+1) \binom{d-1}{k-2} e_1^{k-1}},$$

or

**Property 3.**

$$H - Q \geq \sum_{k=2}^d \frac{d^{k-1} e_k}{\binom{d-1}{k-1} e_1^{k-1}}. \quad (43)$$

□

One also has:

$$\frac{\partial H}{\partial e_1} \leq -1 \quad \text{if } e_1 \geq 1. \quad (44)$$

A further type of connection between  $H$  and  $Q$  is given by

**Property 4.** For any  $k, l, m$  with  $k = l + m$  and  $l, m \geq 1$  (so  $k \geq 2$ ) we have

$$-\frac{\partial Q}{\partial e_k} = \frac{\partial^2 H}{\partial e_l \partial e_m} \quad (45)$$

□

Note that this easy consequence of the half-axis formulae does not arise directly from differentiation of eq. (38).

**Property 5.** The  $m^{\text{th}}$  derivative  $\partial^m H / \partial e_{i_1} \dots \partial e_{i_m}$  as a function of  $(e_1, \dots, e_d)$  depends only on the sum of indices  $i_1 + \dots + i_m$ , and the same holds for  $Q(e_1, \dots, e_d)$  too. □

**Property 6.** Consider the  $m^{\text{th}}$  derivative  $\partial^m H / \partial e_{i_1} \dots \partial e_{i_m}$  for  $H(e_1, \dots, e_d)$  with  $d$  variables. Introduce the entropy function  $\tilde{H}(\tilde{x}_1, \dots, \tilde{x}_{dm})$  with  $dm$  variables and corresponding elementary symmetric polynomials  $\tilde{e}_1, \dots, \tilde{e}_{dm}$ . Then for any  $(e_1, \dots, e_d)$  arising from roots  $x_1, \dots, x_d$ , any  $m^{\text{th}}$  derivative of  $H$  can be expressed as a *first* derivative of  $\tilde{H}$ :

$$(-1)^{m-1} \frac{\partial^m H}{\partial e_{i_1} \dots \partial e_{i_m}}(e_1, \dots, e_d) = \frac{\partial \tilde{H}}{\partial \tilde{e}_K}(\tilde{e}_1, \dots, \tilde{e}_{md}) \quad (46)$$

where  $K = i_1 + \dots + i_m$  and the RHS is evaluated at the point  $(\tilde{e}_1, \dots, \tilde{e}_{md})$  being the elementary symmetric polynomial values for the  $md$   $\tilde{x}_j$ 's

$$(\tilde{x}_1, \dots, \tilde{x}_{dm}) = (x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_d, \dots, x_d)$$

having each  $x_i$  repeated  $m$  times. □

Each time we differentiate eq. (15) there is a switch in sign. Combining this with Properties 2, 6 and 4 gives:

**Property 7.** For  $m \geq 2$  we have

$$(-1)^{m-1} \frac{\partial^m H}{\partial e_{i_1} \dots \partial e_{i_m}} \geq c_{md,i} > 0 \quad \text{for all } i_1, \dots, i_m \geq 1, \quad (47)$$

$$(-1)^{m-1} \frac{\partial^m Q}{\partial e_{i_1} \dots \partial e_{i_m}} \geq c_{(m+1)d,i} > 0 \quad \text{for all } i_1, \dots, i_m \geq 1, \quad (48)$$

where  $i = \sum_l i_l$  and  $c_{d,k}$  is given by eq. (41). For first derivatives we have

$$\frac{\partial H}{\partial e_i} \geq c_{d,i} > 0 \text{ and } \frac{\partial Q}{\partial e_i} \geq c_{2d,i} > 0 \text{ for } i \geq 2.$$

□

Note that the above now subsumes Property 2.

**Property 8.** For  $\theta \geq 1$

$$Q(\theta x) \leq \theta Q(x), \tag{49}$$

$$Q(\theta e) \leq \theta Q(e), \tag{50}$$

$$H(\theta x) \leq \theta H(x), \tag{51}$$

$$H(\theta e) \geq \theta H(e), \tag{52}$$

$$Q(\theta x) - \theta Q(x) = H(\theta x) - \theta H(x) \tag{53}$$

where  $Q(x)$  and  $Q(e)$  denote the subentropy  $Q$  regarded as a function of the  $x_k$ 's and  $e_k$ 's respectively (and similarly for  $H$ ). □

**Proof** By definition

$$\begin{aligned} \frac{1}{\theta} Q(\theta x) &= - \sum_k \frac{x_k^d \ln(\theta x_k)}{\prod_{i \neq k} (x_k - x_i)} = Q(x) - \sum_k \frac{x_k^d}{\prod_{i \neq k} (x_k - x_i)} \ln \theta \\ &= Q(x) - e_1 \ln \theta, \end{aligned}$$

(where the last equality follows since  $\sum_k \frac{x_k^d}{\prod_{i \neq k} (x_k - x_i)}$  is a rational symmetric function, homogeneous of degree one, with all singularities removable, so it must be a multiple of  $e_1$ .) Since  $e_1 \ln \theta \geq 0$  for  $\theta \geq 1$ , this proves inequality (49). Inequality (51) is immediate from eq. (2), and eq. (53) follows.

To prove inequality (50), introduce  $\xi = q(\tau) - \tau^d = e_1 \tau^{d-1} + e_2 \tau^{d-2} + \dots + e_d$  and then eq. (14) directly gives

$$\theta Q(e) - Q(\theta e) = \int_0^\infty \theta \left[ -\frac{\tau^d}{\tau^d + \xi} + 1 \right] - \left[ -\frac{\tau^d}{\tau^d + \theta \xi} + 1 \right] d\tau.$$

After a little algebra, the integrand simplifies to  $\frac{(\theta-1)\theta\xi^2}{q(\tau)(\tau^d + \theta\xi)}$  which is positive for all  $\tau > 0$  and  $\theta > 1$ , and inequality (50) follows. Similarly, from eq. (12) we obtain  $\theta H(e) - H(\theta e) = \theta(\theta-1) \int_0^\infty \frac{\xi(\tau q'(\tau) - dq(\tau))}{q(\tau)(\tau^d + \theta\xi)}$ , which is negative for all  $\tau > 0$  and  $\theta > 1$  since  $\tau q'(\tau) - dq(\tau) \leq 0$ . This gives inequality (52). □

Our next result applies to bipartite systems. Although the linear operation of forming marginals of a joint probability distribution does not translate naturally into an operation on the corresponding symmetric polynomials, we still have the following result in that context.

Let  $\mathbb{R}_+^n = \{(\tau_1, \dots, \tau_n) : \tau_i \geq 0, i = 1, \dots, n\}$  be the positive cone for  $n$  real variables. Let  $\phi_n(\tau_1, \dots, \tau_n)$  on  $\mathbb{R}_+^n$  be a family of symmetric functions which we may also view as



functions  $\phi_n(e_1, \dots, e_n)$  of the corresponding symmetric polynomials. Let  $A$ ,  $B$  and  $AB$  be systems to which we associate variables  $(x_1, \dots, x_m) \in \mathbb{R}_+^m$ ,  $(x_1, \dots, x_n) \in \mathbb{R}_+^n$  and  $(x_{1,1}, \dots, x_{m,n}) \in \mathbb{R}_+^{mn}$  respectively. We will write  $\phi_{mn}(AB)$  for  $\phi_{mn}(x_{1,1}, \dots, x_{m,n})$  and  $\phi_m(A)$  (resp.  $\phi_n(B)$ ) for  $\phi_m$  (resp.  $\phi_n$ ) evaluated on the *marginal* variable values  $x_i = \sum_j x_{i,j}$  (resp.  $x_j = \sum_i x_{i,j}$ ). For these sets of variables let the corresponding symmetric polynomials (all constructed from the  $x_{i,j}$ 's) be denoted by  $e_k^{AB}$ ,  $e_k^A$  and  $e_k^B$ , so that the symmetric functions  $\phi_{mn}(AB)$ ,  $\phi_m(A)$  and  $\phi_n(B)$  may alternatively be viewed as functions of these symmetric polynomials.

**Lemma 3.** *Suppose  $\phi_n$  is a family of symmetric functions as above, satisfying the following two properties for all  $n$  (when taken as functions of the symmetric polynomial variables):*

*(Extendability):  $\phi_{n+1}(e_1, e_2, \dots, e_n, 0) = \phi_n(e_1, e_2, \dots, e_n)$ ;*

*(Monotonicity):  $\partial\phi_n/\partial e_k \geq 0$ ,  $2 \leq k \leq n$ .*

*Then  $\phi_m(A) \leq \phi_{mn}(AB)$ .*

**Proof** First note that  $\phi_k(e_1, \dots, e_k) \leq \phi_n(e_1, \dots, e_k, e_{k+1}, \dots, e_n)$ , for any  $k < n$ . This follows from extendability which gives  $\phi_n(e_1, \dots, e_k, 0, \dots, 0) = \phi_k(e_1, \dots, e_k)$ , and then monotonicity as we increase the last  $n - k$  coordinates from  $(0, \dots, 0)$  to  $(e_{k+1}, \dots, e_n)$ .

Next observe that the symmetric polynomial variables satisfy  $e_1^A = e_1^{AB}$ , and  $e_k^A \leq e_k^{AB}$ , for  $k > 1$ , since every product of  $x_{i,j}$  variables of the joint system that appears in  $e_k^A$  also appears in  $e_k^{AB}$ . Thus

$$\begin{aligned} \phi_m(A) = \phi_m(e_1^A, \dots, e_m^A) &\leq \phi_m(e_1^{AB}, \dots, e_m^{AB}) \\ &\leq \phi_{mn}(e_1^{AB}, \dots, e_m^{AB}, e_{m+1}^{AB}, \dots, e_{mn}^{AB}) = \phi_{mn}(AB). \quad \square \end{aligned}$$

**Property 9.**

$$\begin{aligned} H(A) &\leq H(AB), \\ Q(A) &\leq Q(AB), \\ (-1)^{m-1} \frac{\partial^m H(A)}{\partial e_{i_1} \dots \partial e_{i_m}} &\leq (-1)^{m-1} \frac{\partial^m H(AB)}{\partial e_{i_1} \dots \partial e_{i_m}} \text{ for } m \geq 1, \\ (-1)^{m-1} \frac{\partial^m Q(A)}{\partial e_{i_1} \dots \partial e_{i_m}} &\leq (-1)^{m-1} \frac{\partial^m Q(AB)}{\partial e_{i_1} \dots \partial e_{i_m}} \text{ for } m \geq 1. \end{aligned}$$

□

This follows from Lemma 3 and Property 7.  $H$  and  $Q$  and their derivatives are extendable because  $e_{k+1} = \dots = e_n = 0$  implies that  $n - k$  of the  $x_i$  are zero, and both  $H$  and  $Q$  are clearly extendable as functions of the  $x_i$ . The first inequality in Property 9 is just the well known non-negativity of conditional Shannon information and the second inequality was proved also in [17] by different means.

The next result is curious in that it starts from a property on  $(e_1, \dots, e_d)$ -space and deduces one on  $(x_1, \dots, x_d)$ -space.

**Lemma 4.** *Let  $f(x_1, \dots, x_d)$  be a symmetric function that satisfies  $\partial f/\partial e_q \geq 0$  for  $q \geq 2$  (when represented as a function of the  $e_k$ 's). Then  $f$  is Schur concave in the  $x_j$  variables.*

**Proof** Write  $f_j = \partial f / \partial x_j$ . Viewing eq. (1) as implicitly defining  $x_i = x_i(e_1, \dots, e_d)$  we get (cf eqs. (11) and (12) in [8])  $\partial x_j / \partial e_k = (-1)^{k+1} \sum_{j=1}^d x_j^{d-k} / \prod_{i \neq j} (x_j - x_i)$ . Then the chain rule gives

$$\frac{\partial f}{\partial e_k} = (-1)^{k+1} \sum_{j=1}^d \frac{x_j^{d-k} f_j}{\prod_{i \neq j} (x_j - x_i)}.$$

Now let  $\hat{e}_i$  denote the elementary symmetric functions of the  $d-2$  variables  $\{x_3, \dots, x_d\}$ , with  $\hat{e}_0 = 1$ . Then

$$\begin{aligned} \sum_{q=2}^d \hat{e}_{q-2} \frac{\partial f}{\partial e_q} &= - \sum_{j=1}^d f_j \frac{[x_j^{d-2} - \hat{e}_1 x_j^{d-3} + \dots + (-1)^{d-2} \hat{e}_{d-2}]}{\prod_{i \neq j} (x_j - x_i)} \\ &= - \sum_{j=1}^d f_j \frac{(x_j - x_3) \dots (x_j - x_d)}{\prod_{i \neq j} (x_j - x_i)} = - \frac{(f_1 - f_2)}{(x_1 - x_2)}. \end{aligned}$$

However,  $\partial f / \partial e_q \geq 0$  for  $q \geq 2$  implies  $\sum_{q=2}^d \hat{e}_{q-2} \partial f / \partial e_q \geq 0$ , so  $(f_1 - f_2) / (x_1 - x_2) \leq 0$ , which is equivalent to Schur concavity in the  $x_j$  variables.  $\square$

**Property 10.**  $H(x_1, \dots, x_d)$  and  $Q(x_1, \dots, x_d)$  are Schur concave.  $\square$

Schur concavity of  $Q$  was also shown by different means in [17].

Finally we make a connection with a result proved by Hellmund and Uhlmann [10]. They proved an upper bound on the von Neumann entropy of a state that specializes, for diagonal matrices, to give

$$H(e_1, e_2, \dots, e_d) \leq -e_1 \log e_1 + \log d \sqrt{\frac{2de_2}{d-1}}. \quad (54)$$

We can obtain similar bounds depending only on  $e_1$  and  $e_2$ , for both  $H$  and  $Q$ , using the following result.

**Lemma 5.** *If  $e_1$  and  $e_2$  are specified then all the  $e_k$ 's are simultaneously maximised at the unique assignment of  $\{x_1, \dots, x_d\}$  values of the form  $\{a, \dots, a, b\}$  with  $a < b$ , that is consistent with the given  $e_1$  and  $e_2$  values. More precisely, let  $e_k$  be the elementary symmetric functions for the set  $\{x_1, \dots, x_d\}$  and  $\tilde{e}_k$  be those for the set  $\{a, \dots, a, b\}$ , where  $a$  is repeated  $n-1$  times. Choose  $a$  and  $b$  so that  $\tilde{e}_1 = e_1$  and  $\tilde{e}_2 = e_2$ , i.e. so that*

$$(d-1)a + b = e_1, \quad (55)$$

$$\binom{d-1}{2} a^2 + (d-1)ab = e_2. \quad (56)$$

Choose the root

$$a = \frac{(d-1)e_1 - \sqrt{(d-1)^2 e_1^2 - 2e_2 d(d-1)}}{d(d-1)}. \quad (57)$$

Then  $\tilde{e}_k \geq e_k$  for  $3 \leq k \leq d$  and this maximum is unique over all unordered sets of  $x_i$  values realising the given  $e_1$  and  $e_2$  values.

Before giving the proof we introduce some terminology to facilitate our argument. Let us say that a pair of non-negative  $e_1$  and  $e_2$  values is *realisable* (with  $d$  variables) if there exist  $(x_1, \dots, x_d)$  with  $x_i \geq 0$  that have these values as  $e_1(x_1, \dots, x_d)$  and  $e_2(x_1, \dots, x_d)$ . Not all non-negative pairs are realisable and in fact, Maclaurin's inequalities for symmetric polynomials imply that  $\{e_1, e_2\}$  is realisable iff  $(d-1)e_1^2 \geq 2de_2$ . Let us also say that the values of  $d$  variables  $x_1, \dots, x_d$  comprise a *canonical set* if they consist of a value  $a$  repeated  $d-1$  times and one  $b$ , with  $0 \leq a \leq b$ . Note that any realisable pair of  $e_1$  and  $e_2$  values determines a unique corresponding canonical set via eqs. (55), (56) and (57).

**Proof of lemma 5** Our proof will proceed by induction on  $d$  at each successive  $k$  level. Note that for each  $k$ ,  $e_k(x_1, \dots, x_d)$  is defined only for  $d \geq k$ . For  $k \geq 3$  and  $d \geq k$  introduce the induction hypothesis  $F(k, d)$  which is the following statement: "if  $e_1$  and  $e_2$  is any realisable pair for  $d$  variables  $(x_1, \dots, x_d)$  then, over all sets of values for  $\{x_1, \dots, x_d\}$  that realise  $e_1$  and  $e_2$ ,  $e_k(x_1, \dots, x_d)$  is maximised uniquely on the corresponding canonical set  $\{a, \dots, a, b\}$ ."

For the induction step, suppose we have established  $F(k, d)$  at:

- (i) all full levels below level  $K$  i.e. for  $(k, d)$  with  $3 \leq k \leq K-1$  and all  $d \geq k$ ; and
- (ii) at level  $K$  up to  $d = D-1$  i.e. for  $(k, d)$  with  $k = K$  and  $K \leq d \leq D-1$ .

We aim to prove  $F(K, D)$ . Suppose on the contrary that (with  $e_1$  and  $e_2$  specified),  $e_K$  for  $d = D$  is maximised by some  $\xi_1, \dots, \xi_D$  that is not a canonical set. Then one can remove one of the  $\xi$ 's, which we can assume to be  $\xi_1$ , with the remaining  $\xi$ 's still being a non-canonical set. Now, for all  $k \geq 1$  we have (with  $e_0 = 1$ )

$$e_k(x_1, \dots, x_D) = e_k(x_2, \dots, x_D) + x_1 e_{k-1}(x_2, \dots, x_D), \quad (58)$$

and since the inductive hypothesis holds at  $(k, d) = (K, D-1)$  and  $(K-1, D-1)$ , the non-canonical assignment  $\xi_2, \dots, \xi_D$  does not maximise either  $e_K(x_2, \dots, x_D)$  or  $e_{K-1}(x_2, \dots, x_D)$ , subject to  $e_1$  and  $e_2$  being fixed at the values  $e_1(\xi_2, \dots, \xi_D)$  and  $e_2(\xi_2, \dots, \xi_D)$ , respectively. (For the case of level  $K = 3$ ,  $e_{K-1} = e_2$  which has been fixed and our argument proceeds as below by using only the  $e_K$  term and not both the  $e_K$  and  $e_{K-1}$  terms, as may be easily checked.) Thus we can find  $\eta_2, \dots, \eta_D$  such that  $e_K(\eta_2, \dots, \eta_D) > e_K(\xi_2, \dots, \xi_D)$  and  $e_{K-1}(\eta_2, \dots, \eta_D) > e_{K-1}(\xi_2, \dots, \xi_D)$ , respectively, while  $e_1$  and  $e_2$  (on these  $D-1$  elements) are fixed. Then returning to the full set of  $D$  variables we have

$$\begin{aligned} e_2(\xi_1, \eta_2, \dots, \eta_D) &= e_2(\eta_2, \dots, \eta_D) + \xi_1 e_1(\eta_2, \dots, \eta_D) \\ &= e_2(\xi_2, \dots, \xi_D) + \xi_1 e_1(\xi_2, \dots, \xi_D) \\ &= e_2(\xi_1, \dots, \xi_D), \end{aligned}$$

and similarly  $e_1(\xi_1, \eta_2, \dots, \eta_D) = e_1(\xi_1, \dots, \xi_D)$ . Thus we have found a set of values for the  $D$  variables such that  $e_1$  and  $e_2$  are fixed and

$$\begin{aligned} e_K(\xi_1, \eta_2, \dots, \eta_D) &= e_K(\eta_2, \dots, \eta_D) + \xi_1 e_{K-1}(\eta_2, \dots, \eta_D) \\ &> e_K(\xi_2, \dots, \xi_D) + \xi_1 e_{K-1}(\xi_2, \dots, \xi_D) \\ &= e_K(\xi_1, \dots, \xi_D). \end{aligned}$$

This contradicts the assumption that  $e_K$  is maximised by a non-canonical set.

Finally we need to establish the initial steps of the induction viz. the cases of  $F(k, d)$  for  $k = d$  and  $k, d \geq 3$ . For  $k = d = 3$  we wish to maximise  $x_1 x_2 x_3$  subject to  $x_1 + x_2 + x_3 = e_1$

and  $x_1x_2 + x_1x_3 + x_2x_3 = e_2$  where  $e_1$  and  $e_2$  are any realisable pair i.e.  $e_1^2 \geq 3e_2 > 0$ . By squaring the  $e_1$  condition, the two constraints may be equivalently written as  $x_1 + x_2 + x_3 = e_1$  and  $x_1^2 + x_2^2 + x_3^2 = e_1^2 - 2e_2$  which has a simple geometrical interpretation as the intersection of a plane and a sphere, restricted to the positive octant. Applying Lagrange multipliers we see that each of  $x_1, x_2$  and  $x_3$  satisfies the same quadratic equation, allowing at most two different values for them. Thus the maximising point  $(x_1, x_2, x_3)$  lies either at a boundary point of the octant or at a point of the form  $(a, a, b)$  or  $(a, b, b)$  with  $a \leq b$  (and the  $a$  and  $b$  values differ in these two cases). Any boundary point has one of the  $x_i$ 's being zero so  $e_3 = 0$  there and hence it cannot be the maximum. A straightforward calculation shows that the form  $(a, a, b)$  rather than  $(a, b, b)$  maximises  $e_3$ , with  $a$  and  $b$  then given by eqs. (55), (56) and (57). For the cases  $d = k$  with  $d, k \geq 4$ , eq. (58) takes the simpler form  $e_k(x_1, \dots, x_k) = x_1 e_{k-1}(x_2, \dots, x_k)$  and the argument proceeds as given previously after eq. (58), now requiring only the already established case  $F(k-1, d-1)$ .  $\square$

Using  $\partial H / \partial e_k > 0$  and  $\partial Q / \partial e_k > 0$ , Lemma 5 tells us that  $H$  and  $Q$  can only increase when the  $x_k$  are replaced by the set  $\{a, \dots, a, b\}$ . The basic definitions of  $H$  and  $Q$ , eqs. (2) and (3), then give the following bounds:

**Property 11.**

$$H(e_1, e_2, \dots, e_d) \leq -(d-1)a \log a - b \log b, \quad (59)$$

$$Q(e_1, e_2, \dots, e_d) \leq \frac{d! a^2 \log a}{2(a-b)} - \frac{b^d \log b}{(b-1)^{d-1}}, \quad (60)$$

where  $a$  and  $b$ , depending only on  $e_1$  and  $e_2$ , are given by eqs. (57) and (55).  $\square$

These are the tightest possible bounds that depend only on  $e_1$  and  $e_2$  since they are attained by a particular corresponding assignment of  $x_j$ 's. In particular, inequality (59) is tighter than (54).

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