The Structure and Stability of Vortices in Astrophysical Discs

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To Mum, Dad, Ella, Jess and Toby—
I couldn’t have done it without you.
Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, with the exception of the algebra in Sections 7.5.1 and 7.5.2 and the order of magnitude calculations in Section 8.6.1, which were done by JCBP.

Parts of Chapters 5 to 8 make up the paper Railton and Papaloizou (2014).

A. D. Railton
April 2015
Acknowledgements

In producing this thesis, I used the \LaTeX template created and maintained by Krishna Kumar (Kumar, 2014). References were managed using the open source reference manager JabRef (JabRef Development Team, 2014). Graphs were made using gnuplot (Williams et al., 2010), MATLAB (MATLAB, 2010), IDL (Liu et al., 2013) and VisIt (Childs et al., 2012) and some diagrams with Inkscape and Gimp.

The book ‘How to Write a Better Thesis’ (Evans et al., 2011) did in fact help me write a better thesis, and I’m grateful to Ginny Haskell for giving it to me. I have also been greatly influenced by Mike McIntyre’s writing on Lucidity and Science (McIntyre, 1997) and have attempted to follow his advice. The Pomodoro technique\(^1\) took months off my submission date, while Dr Inger Mewburn’s blog ‘The Thesis Whisperer’\(^2\) made me realise that other people struggle too.

In no particular order, I would like to thank: Adrian Barker for his help getting PLUTO set up, the Maths department IT helpdesk for all the stupid questions I asked them over three and a half years, Lizzie Polgreen and Dr Nilanjana Datta for putting up with my crises, my family for their unending support and Dad for making a door so that I didn’t freeze to death while writing up. Pembroke College has been a source of support and comfort throughout my time at Cambridge and will always feel like home.

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Finally, I am indebted to the various, uncountable people who persuaded me not to quit.

\(^1\)http://pomodorotechnique.com/
\(^2\)http://thesiswhisperer.com/
Abstract

This thesis finds that vortex instabilities are not necessarily a barrier to their potential as sites for planetesimal formation.

It is challenging to build planetesimals from dust within the lifetime of a protoplanetary disc and before such bodies spiral into the central star. Collecting matter in vortices is a promising mechanism for planetesimal growth, but little is known about their stability under these conditions. We therefore aim to produce a more complete understanding of the stability of these objects.

Previous work primarily focusses on 2D vortices with elliptical streamlines, which we generalise. We investigate how non-constant vorticity and density power law profiles affect stability by applying linear perturbations to equilibrium solutions. We find that non-elliptical streamlines are associated with a shearing flow inside the vortex.

A ‘saddle point instability’ is seen for elliptical-streamline vortices with small aspect ratios and we also find that this is true in general. However, only higher aspect ratio vortices act as dust traps. For constant-density vortices with a concentrated vorticity source we find parametric instability bands at these aspect ratios. Models with a density excess show many narrow bands, though with less strongly growing modes than the constant-density solutions. This implies that dust particles attracted to a vortex core may well encounter parametric instabilities, but this does not necessarily prevent dust-trapping.

We also study the stability and lifetime of vortex models with a 2D flow in three dimensions. Producing nearly-incompressible 3D models of columnar vortices, we find that weaker vortices persist for longer times in both stratified and unstratified shearing boxes, and stratification is destabilising. The long survival time for weak, elongated vortices makes it easier for processes to create and maintain the vortex. This means that vortices with a large enough aspect ratio have a good chance of surviving and trapping dust for sufficient time in order to build planetesimals.
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Nomenclature

**Roman Symbols**

- $v_d$ dust velocity
- $v_g, v$ gas velocity
- $F_{\text{drag}}$ Epstein drag force acting on a single dust particle of mass $m_\bullet$, see equation (3.3.4), page 33
- $A(\psi)$ Bernoulli source term in equation (5.2.4) such that $A(\psi) = \frac{dF_1}{d\psi}$
- $B(\psi)$ density source term in equation (5.2.4) such that $B(\psi) = \frac{d\log\rho}{d\psi}$
- $M$ Mach number, typically $M = v_\phi/c_s$ in a disc
- $Z$ variable in the Hill equation describing vertical stability for the internal shear free case, see equation (7.5.23), page 137
- $M_{\text{disc}}$ disc mass
- $M_\odot$ mass of the Sun, $M_\odot = 1.989 \times 10^{30}$ kg
- $\bar{v}$ velocity of combined fluid, see equation (5.1.5), page 60
- $S_A$ phase function in Lagrangian WKBJ ansatz, see equation (7.1.13), page 112
- $\tilde{P}$ period round a vortex streamline, see equation (5.3.32), page 83
- $\tilde{P}_{\text{Kida}} = \frac{2\pi}{S}(\chi - 1)$, period round a streamline of a Kida vortex solution., see equation (5.3.35), page 84
- $f_\nu$ viscous force per unit volume, see equation (5.1.1), page 58
- $k$ wavenumber
- $u$ mean velocity of the dust with respect to the gas, see equation (5.1.6), page 60
- $v_d$ velocity of dust component in two-fluid equations, page 59
**Nomenclature**

\( \mathbf{v}_K \) Keplerian velocity, \( v_K = \sqrt{GM_*/r^3} \)

\( A \) scaling constant in the Bernoulli source term \( \mathcal{A}(\psi) \), see equation (5.2.5), page 66

\( B \) scaling constant in the density source term \( \mathcal{B}(\psi) \), see equation (5.2.5), page 66

\( b \) density scaling factor in vortex polytropic model, see equation (5.4.3), page 84

\( c_s \) local sound speed

\( F(\psi) \) arbitrary function that appears in the Poisson equation form of the momentum equation (5.1.29), see equation (5.1.24), page 64

\( G \) gravitational constant

\( h \) scaling constant for elliptic coordinate system, see equation (5.3.0), page 73

\( j \) specific angular momentum, \( j = r^2\Omega \) (\( l \) and \( h \) are also commonly used)

\( k_\perp \) magnitude of the horizontal wavenumber, \( k_\perp = \sqrt{k_x^2 + k_y^2} \), see equation (7.2.19), page 117

\( M_* \) mass of central star

\( m_* \) dust particle mass, page 33

\( M_\oplus \) mass of the Earth, \( M_\oplus = 5.97 \times 10^{24} \) kg

\( M_J \) Jeans mass, see equation (2.2.0), page 5

\( n \) polytropic index for density, see equation (5.4.3), page 84

\( N_1 \) The number of iterations in the first loop of vortex.f90, page 91

\( N_2 \) The number of iterations in the second, density-enhancement loop of vortex.f90, page 91

\( P \) gas pressure

\( Q \equiv c_s\Omega/\pi G\Sigma \), Toomre \( Q \) parameter, page 39

\( Q' \) the Eulerian perturbation of a quantity \( Q \), page 110

\( q_{\text{Hill}} \) buoyancy frequency in Hill equation, page 138

\( r_0 \) origin of shearing coordinates, distance to the central star., page 61

\( S = -r \frac{d\Omega}{dr} \), the shear of a background shearing flow \( \mathbf{u} = (0, -Sx, 0) \), In a Keplerian disc \( S = \frac{3}{2} \Omega \). \( q \) and \( A \) are frequently used in the literature.
$t_0$ time at which $y$ passes through its maximum value in the Fourier series expansion of particle trajectories $x$ and $y$, see equation (7.4.3), page 128

$T_s$ Stokes number, $T_s = \Omega \tau_s$

$W$ rescaled wavenumber used in vertical stability analysis, $W = \mu |k|^2$, page 135

**Greek Symbols**

$\alpha$ power law index in the Bernoulli source term $A(\psi)$, see equation (5.2.5), page 66

$\alpha_{p}$ Shakura-Sunyaev $\alpha$, a dimensionless parameter measuring the efficiency of angular momentum transport due to turbulence, see equation (2.4.26), page 21

$\tilde{\nu}$ scaled effective kinematic viscosity $\tilde{\nu} = \frac{\rho_g}{\rho_g + \rho_d} \nu$, page 69

$\beta$ power law index in the density source term $B(\psi)$, see equation (5.2.5), page 66

$\beta_P$ pressure scaling factor in vortex polytropic model, see equation (5.4.4), page 85

$\omega$ vorticity of a flow with velocity $\mathbf{v}$, $\omega = \nabla \times \mathbf{v}$, page 64

$\sigma$ stress tensor in the Navier-Stokes equations, see equation (2.4.3), page 17

$\xi$ the Lagrangian displacement, page 110

$\chi$ aspect ratio of a vortex

$\Delta Q$ the Lagrangian perturbation of a quantity $Q$, page 110

$\delta$ dimensionless diffusion parameter inside vortex, parametrises turbulence in vortices, $D = \delta c_s H$ (e.g Lyra and Lin, 2013)

$\eta$ angular elliptic coordinate, see equation (5.3.0), page 73

$\gamma$ growth rate

$\lambda$ large parameter in Lagrangian WKBJ ansatz, see equation (7.1.13), page 112

$\lambda_J$ Jeans length, see equation (2.2.0), page 5

$\mu$ Scalar quantity in vertical stability analysis $\mathbf{w}' = (\mu k_y, -\mu k_x)$, page 134

$\mu_j$ characteristic, or Floquet, exponents. See also Appendix B, page 122

$\nu$ effective kinematic viscosity, see equation (2.4.6), page 17

$\nu_t$ root mean square turbulent velocity, see equation (5.2.6), page 67
\( \Omega \) angular velocity, \( \Omega = \frac{v_\phi}{r} \)

\( \omega_m \) a measure of the total imposed vorticity in a numerical vortex solution, see equation (6.1.3), page 88

\( \Omega_{\text{GMC}} \) typical angular velocity in a GMC

\( \Omega_{\text{core}} \) angular velocity of a GMC core

\( \omega_{\text{Kida}} \) the vorticity of a Kida vortex, \( S + c = -\omega_{\text{Kida}} \), page 72

\( \Bar{\rho} \) density of combined fluid, \( \Bar{\rho} = \rho_g + \rho_d \), see equation (5.1.5), page 60

\( \Phi \) the ‘combined potential’ of the gravitational and centrifugal potentials, \( \Phi = \Phi_{gr} - \frac{1}{2} \Omega^2 r^2 \), see equation (5.1.2), page 59

\( \Phi_{gr} \) gravitational potential due to central star, \( \Phi_{gr} = \frac{GM_*}{|r|} \)

\( \psi_\sigma \) Value of the streamfunction at \((0, y_{\text{max}})\), page 142

\( \psi \) the Stokes’ streamfunction, see equation (5.1.21), page 64

\( \psi^{(ex)} \) the exterior streamfunction for the Kida solution, page 73

\( \psi^{(in)} \) the interior streamfunction for the Kida solution, page 73

\( \psi_0 \) the streamfunction of the background flow. In the shearing sheet, \( \psi_0 = Sx^2/2 \), see equation (5.1.21), page 64

\( \psi_b \) value of the streamfunction \( \psi \) evaluated on the vortex boundary at \((0, 1)\), see equation (5.2.5), page 66

\( \Gamma_d = \Sigma_d/\Sigma_g \) dust-to-gas ratio by mass

\( \rho_0 \) constant background density, page 66

\( \rho_d \) dust density

\( \rho_g \) gas density

\( \rho_m \) a measure of the total imposed mass in a numerical vortex solution, see equation (6.1.3), page 88

\( \rho_* \) dust particle density, page 33

\( \rho_{\text{max}} \) central density enhancement in an equilibrium vortex solution, page 99

\( \sigma \) coordinate determining location on a streamline, related to some fraction of the arclength, page 117
Nomenclature

Σ  total circumference of a streamline (see σ the arclength parameter), page 121
τν  viscous timescale, see Table 3.3
τcoag  coagulation or dust sticking timescale, see Table 3.3
τcollide  dust collision timescale, see Table 3.3
τdecay  orbital decay timescale, see Table 3.3
τdyn  dynamical or orbital timescale, see Table 3.3
τff  freefall time, ∼ (Gρ)^−1/2, page 5
τsc  sound crossing time, ∼ L/c_s for characteristic lengthscale L
τth  thermal timescale, see Table 3.3
τcorr  turbulent correlation time, see equation (5.2.6), page 67
τs  dust stopping time or frictional timescale, see Table 3.3
τdisc  disc lifetime, see Table 3.3
Θ  scaled ratio of horizontal and vertical wavenumbers, see equation (7.2.19), page 117
θ  angle between the perturbation wavevector \( \mathbf{k} \) and \( \mathbf{\hat{z}} \), see equation (7.2.23), page 118
τsettle  settling or sedimentation timescale, see Table 3.3
τunstable  time at which instability occurs for the 3D vortex columns in Chapter 9
ϖ  frequency while travelling round a streamline, \( \tilde{P} = 2\pi/\varpi \), page 128
\( \varrho_j \)  characteristic multipliers in Floquet theory, \( \varrho_j = e^{\mu_j \Sigma} \). See also Appendix B, page 122
ξ  radius elliptic coordinate, see equation (5.3.0), page 73
ξ_b  the bounding streamline with \( \psi = \psi_b \) is given by \( \xi = \xi_b \), page 73

Other Symbols

\{ \alpha, \beta, \rho_m \}  the three numbers that define a vortex class

Acronyms / Abbreviations

a  radius of a dust particle, see Section 3.1.1

ALMA  Atacama Large Millimeter Array
AU astronomical unit, the average distance between the Earth and the Sun, $1\text{AU} \approx 1.496 \times 10^{11} \text{m}$

GI gravitational instability

GMC giant molecular cloud


IR infrared

ISM interstellar medium

MHD magnetohydrodynamics

MMSN minimum mass solar nebula, page 10

MRI magnetorotational instability

Myr megayear, a million years, $1\text{Myr} = 10^6\text{yr}$

pc parsec, the distance at which one astronomical unit subtends an angle of one arcsecond, $1\text{pc} \approx 3.086 \times 10^{16} \text{m} \approx 2.063 \times 10^5 \text{AU}$

Re Reynolds number, a dimensionless quantity that describes the ratio of inertial forces to viscous forces, $\text{Re} \equiv UL/\nu$

RWI Rossby wave instability

SBI Subcritical baroclinic instability

SED spectral energy distribution, the distribution of flux as a function of frequency or wavelength

SGW Safronov-Goldreich-Ward mechanism of planetary formation, see Chapter 3

UV ultraviolet

YSO young stellar objects
Chapter 1

Introduction

Humanity has always wanted to know where we came from and how we came to exist. In an age where exoplanets are discovered almost daily and the likelihood of life elsewhere in the universe increases, the question of how they and our own planet came to exist is ever more salient.

The formation of planets from a dusty, gaseous disc around a young star has been investigated since the inception of the ‘nebular hypothesis’ in the 1700s. The formation of small boulders from dust in the disc and the formation of planets from large planetesimals are largely well understood processes, but getting between these two states is still an outstanding problem. The most promising route to enable us to bypass what is known as the ‘metre gap’ is by collecting matter in pressure maxima until sufficient concentration is reached and gravitational instability can take over.

Local pressure maxima at the centre of some vortices attract dust but the general stability of these structures with and without density enhancements is not really known. After all it is necessary for vortices to persist long enough in the presence of dust to build up sufficient concentrations of matter.

We therefore aim to produce a more complete understanding of the stability of vortices in protoplanetary discs.

We limit this study to the stability of equilibrium vortices to elliptical stabilities, beyond the analytical Kida case, in fluids which are incompressible, inviscid, isothermal and two-dimensional and with dust modelled as perfectly coupled to the gas. We later extend this to hydrodynamical simulations of columnar vortices in stratified shearing boxes.

The structure of the PhD thesis ‘The Structure and Stability of Vortices in Astrophysical Discs’ is as follows:

The first chapter examines the literature on protoplanetary (PP) discs as an environment for planet formation. It examines development and evolution of the PP disc and how it is closely related to star formation. Chapter 3 looks in more detail at the properties of dust in the disc and how it interacts with the gas. We consider the limitations of the Safronov–Goldreich–Ward
mechanism of planet formation and how vortices and other pressure maxima in the disc can help overcome these difficulties. It then examines the instabilities that can occur in the disc to form these vortices (and perhaps destroy them) and finally explores existing vortex models and approaches to their analysis. Chapter 4 briefly outlines our research design.

In order to perform a comprehensive study of the stability of equilibrium vortices in PP discs, it is necessary to create a large variety of stable vortex configurations to then perturb about. Chapter 5 therefore details the equations of motion in an idealised shearing sheet model of the PP disc and a well-known analytical vortex model often used as the starting point of stability investigations. We show how to extend this approach to find models that have more physically realistic configurations. Chapter 6 shows the numerical procedures in actually producing these configurations and details the results of these.

In Chapter 7 we take these new equilibrium vortex models and formulate the stability analysis of these solutions to perturbations localised on streamlines while in Chapter 8 we show how we implement the numerics of this and present the results. Chapter 9 looks towards more realistic, 3D models in the disc and uses a compressible hydrodynamic code to do so. The overall approach is evaluated and discussed, with conclusions drawn in Chapter 10.
Chapter 2

Protoplanetary discs as an environment for planet formation

The study of disc evolution has important consequences for theories of both star and planet formation. Circumstellar discs, discs around stars, perform two crucial roles in both their evolution. Firstly, they perform a dynamical role in a star’s formation by providing a disc from which the star accretes most of its mass. Secondly, in the latter part of its lifetime, the disc plays an important chemical role in processing various gas species and in the growth of dust grains into planetesimals and beyond; in this way, the circumstellar disc is characterized as a protoplanetary (PP) disc.

In this chapter we discuss the role PP discs play as an environment for planet formation. In Section 2.1 we will give an overview of the historical development of the ‘solar nebula’ disc model for planetary system formation before briefly outlining the role discs play in star formation in Section 2.2. We will review observations of PP discs and the properties we can infer from these in Section 2.3, including their lifetime, mass, size and structure and how these place some quite stringent constraints on the planet formation process. In light of these constraints we will review the structure and evolution of gas discs in Section 2.4, including a general evolutionary pathway in Section 2.4.1, before looking in detail at the equations which govern them.

2.1 Historical context

Circumstellar discs as the site of planet formation is an idea that is centuries old. The Swedish scientist and philosopher Emanuel Swedenborg is credited with first proposing the ‘nebular hypothesis’ as a model for the formation of the Solar System, suggesting that planets formed out of a nebular ‘crust’ that surrounded the Sun then broke apart (Swedenborg, 1734). Observation

\[\text{Observation} \]

The phrases *circumstellar disc* and *protoplanetary (PP) disc* are used interchangeably in this work, although not all circumstellar discs may have the potential to evolve into a planetary system.
that the solar system planets were roughly coplanar and orbit the sun in the same direction lead to the belief that the planets had a common origin in a rotating disc. Twenty years later, Kant (1755) developed this model with the idea of a nebulous cloud in slow rotation, pulled apart by gravitational forces and flattened to a spinning disc from which stars and then planets form. Laplace (1796), independent of Kant, proposed a similar model but with the planets forming before the Sun, condensing from rings of gas thrown off by the disc.

The Kant-Laplace theory dominated for the 19th Century, but a major difficulty was that the model failed to account for the angular momentum distribution observed in the Solar System; the planets contain 99% of the total angular momentum despite containing less than 1% of the mass (Woolfson, 1993). Furthermore, Maxwell showed that if all the matter in the known planets was distributed around the Sun in a disc, shearing forces (the “different rotation between the inner and outer parts of a ring”) would have prevented material condensing to individual planets without rings of hundreds of times more mass than the planets they produced.

Little progress was made on the problem until the solar nebula disc model of Safronov (1969), the first full, quantitative explanation of the formation of the Sun and Solar system which still forms the basis for modern theory. Details of this model are given in Sections 3.2 and 3.4.

2.2 Circumstellar discs and star formation

Circumstellar discs are an essential component of star formation, and indeed the creation of such discs is intimately tied up with the process for forming the central star from a giant molecular cloud (GMC).

GMCs are a type of interstellar cloud which are of a size and density such that the gas within them exists in a molecular form. This is in contrast to the majority of the interstellar medium that contains predominantly atomic gas, as the interiors of GMCs are shielded from stellar radiation by dust which allows the gas to be molecular. When part of a GMC reaches a critical size, mass or density it begins to collapse under its own gravity, creating protostars. These highly variable young stars, also known as T Tauri stars after their prototype, were discovered near molecular clouds by Joy (1945). Soon after discovering that T Tauri stars emit more IR radiation than bare photospheres should, Lynden-Bell and Pringle (1974) suggested an explanation via an accreting circumstellar disc. In this landmark work, the first full disc model calculations were done and they proposed the mechanism of angular momentum transport through the disc as a way of removing it from circumstellar material to allow it to accrete onto

\[^2^ \text{See the letters of Maxwell in the collection Harman (2002), pages 438–479.}

\[^3^ \text{This process is not very well understood as both turbulence and a magnetic field } B \text{ support the GMCs against gravity. For example, ambipolar diffusion of charged particles in the GMC's plasma could reduce the effect of } B \text{ and cause support failure.} \]
the star. The basic principles of disc evolution were later reviewed in Pringle (1981), which remains one of the best introductions to the topic.

Where do these discs come from? GMCs are not homogeneous structures, with density and velocity fields exhibiting structure across wide range of lengthscales that is characteristic of turbulence (Larson, 1981; McKee and Ostriker, 2007). Any collapsing region will therefore possess some intrinsic angular momentum. With no rotation, gas in a core could, in principle, freely collapse towards the centre forming a relatively compact protostar. However, small amounts of angular momentum prevents such radial collapse, only allowing gas to sink down to a minimum radius from the centre.

Most of the infalling matter will therefore not fall directly onto the protostar but form a disc around it (e.g Terebey et al., 1984). Random gas motions average out in favour of the direction of the cloud fragment’s net angular momentum. The stabilisation of these orbits forms a disc with thickness much smaller than its radius on a timescale $\sim 10^4 - 10^5$yr (Shu et al., 1993). These discs survive as quasi-equilibrium structures since material in the disc has specific angular momentum increasing with radius and thus is Rayleigh-stable (see Section 2.4.5).

A significant proportion of the mass of the system is initially in orbit around the protostar (Shu et al., 1987) and therefore some mechanism is required to shed angular momentum so matter can fall onto the star. Much of the final stellar mass is accreted through this disc and so discs are essential to a star’s genesis.

Consideration of the Jeans length, $\lambda_J$, of a GMC gives us a handle on the expected size of a circumstellar disc. $\lambda_J$ is the maximum size of a cloud core that is stable against gravitational collapse, with thermal pressure balancing gravity. It is found by equating the sound crossing time $\tau_{sc} \sim \lambda_J/c_s$ with the freefall timescale $\tau_{ff} \sim (G\rho)^{-1/2}$, where $\rho$ is the gas density.

$$\lambda_J = \frac{2\pi c_s}{\sqrt{4\pi G\rho}}, \quad (2.2.1)$$

which implies a Jeans mass of $M_J = \rho \lambda_J^3$. For $M_J \approx M_\odot$, where $M_\odot$ is the mass of the Sun, $c_s = \sqrt{kT/\mu m_p}$ (with $k$ the Boltzmann constant, $m_p$ the proton mass, $\mu \approx 2.3$ the mean molecular weight and $T \approx 10$K) and a number density $n \sim 10^5$cm$^{-3}$ (Shu et al., 1987) we find $\lambda_J \approx 0.1$pc, the observed size of typical molecular cloud cores (McKee and Ostriker, 2007).

With knowledge of typical angular velocities in a GMC, $\Omega_{GMC}$ (e.g. Goodman et al., 1993), we can use this $\lambda_J$ to estimate the specific angular momentum of the gas per unit mass, $j$. Material in the disc will initially circulate the protostar on eccentric orbits but will lose energy though shocks and dissipation until they travel on the minimum energy, circular orbit for a given $j$. Then, equating the angular momentum of the infalling fluid element to that of a Keplerian orbit at a distance $R$ from a central star of mass $M_*$ we can infer a disc radius $R_{\text{disc}}$ when $M_*=1M_\odot$:

$$R_{\text{disc}} = \frac{j^2}{GM_*} \approx \left(\frac{\Omega_{GMC}\lambda_J^2}{GM_*}\right)^2 \approx 10^2 - 10^4\text{AU}. \quad (2.2.2)$$
This is much larger than the radius of the protostar so some redistribution of angular momentum is required for mass to accrete further into the centre.

Discs inherit initial mass, size and chemical composition from their star formation environment, with a large variation in core accretion rates suggesting that a large diversity of initial disc masses and sizes is expected (Goodman et al., 1993). Furthermore, the evolution of a disc can be shaped by environmental effects such as stellar radiation (from the central star or other stars in the locality), gas accretion, stellar flybys or binary companions (Lodato, 2008).

### 2.3 Observational properties of discs

The most common way of detecting PP discs is via their infrared (IR) emission from hot gas in the inner regions \( r \lesssim 0.1\)AU (Hartigan et al., 1995; Najita et al., 2007), warm dust at temperatures around 100K (Kenyon and Hartmann, 1987; Chiang and Goldreich, 1997) at \( r \lesssim 1\)AU (Kenyon and Hartmann, 1995; Haisch et al., 2001; Hartmann et al., 2005) and cold (\( T \simeq 10\)K) dust in the outer disc \( r \gtrsim 50\)AU (Beckwith et al., 1990).

Discs began to be directly observed in the mid-1980s, first with dusty debris discs (Aumann et al., 1984; Smith and Terrile, 1984) and then gas-rich discs in the survey of Sargent and Beckwith (1987). Despite the difficulty of their observation due to their small size, low mass and temperature, subsequent advances in optics and telescope technology have found discs to be prevalent around young stars. There are currently 170 resolved discs known (130 discs around T Tauri stars and 40 debris discs), of which there is an online catalogue.

Observations of protoplanetary discs only came about with the invention of infrared (IR) detectors, with the first statistical survey of PP disc occurrence in star forming regions by Cohen et al. (1989) and Strom et al. (1989). It was then found by Weintraub et al. (1989) that many of these discs contained large dust grains, followed by the calculations of Beckwith et al. (1990) showing there was enough mass in these grains to form systems like our own Solar System. In the nineties, improvements in the sensitivity of optical and IR telescopes and enhancements of spatial resolution resulted in the first direct image of a PP disc (and the confirmation of a flat disc structure) by O’dell and Wen (1994). We can now even observe asymmetric and radial structure in the disc, with spiral structures in the disc around SAO 206462 (see Section 3.1.2, Figure 3.1a, Muto et al., 2012) and the possible presence of gaps opened up by protoplanets in HL Tau (Figure 2.1, Vlahakis et al., 2014).

#### 2.3.1 Classification of young stellar objects

To obtain large samples of young stars (and hence PP discs) one must observe rich star forming regions such as Ophiuchus, Taurus or Orion, between 120 – 410 pc from the Sun. Resolving

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4See Section 2.4.
5http://circumstellardisks.org/
6For the precise definition of ‘protoplanet’ and ‘planetismal’ etc., see Section 3.1.1.
2.3 Observational properties of discs

Figure 2.1 PP disc around HL Tau showing gaps possibly produced by orbiting protoplanets. HL Tau is a Class I object with a mass of around $0.33M_\odot$ and a substantial disc mass of $0.13M_\odot$ (Kwon et al., 2011), which we expect to be gravitationally unstable. It is one of the brightest known discs (Beckwith et al., 1990), which extends to around 100 AU (e.g. Lay et al., 1997). Greaves et al. (2008) claimed to find a protoplanetary candidate of around $8M_{\text{Jupiter}}$ at 75 AU, though Nero and Bjorkman (2009) suggested it could be a compact feature due to disc fragmentation. The bright rings in the disc are separated by gaps likely to have been formed by the interaction with orbiting protoplanets, which is especially interesting as the system is around $10^5$ years old, implying very rapid planet formation. (ESO/ALMA).
even large discs at these distances is difficult so statistical measures of disc occurrence and lifetime are often derived using the IR spectral energy distribution (SED) of Young Stellar Objects (YSOs; Lada and Wilking, 1984, Adams et al., 1987 and Beckwith, 1999).

YSOs are also classified by the strength of their emission lines, which are due to gas accreting onto the stellar surface. Observation of accretion is important as it is the easiest way of inferring the presence of gas in PP discs, albeit only in the innermost regions of the disc $r \lesssim 0.1\text{AU}$ (Najita et al., 2007). The classification is as follows:

- **Class 0:** The least evolved objects observed during first stages of cloud collapse (André et al., 1993), where the protostar is deeply embedded in the optically thick cloud of dust and gas. The disc forms very early on in this phase and grows extremely rapidly on a timescale of $10^4 - 10^5\text{yr}$ (Shu et al., 1993; Yorke et al., 1993; Hueso and Guillot, 2005). Most of the mass is accreted onto the object in this embedded stage (Hartmann et al., 1993).

- **Class I:** The first objects around which discs are detected, though the cold, optically thick envelope of dust and gas still dominates emission. These objects are around an order of magnitude less luminous than expected for a steady release of gravitational energy as the envelope falls onto the star. They also undergo short outbursts of activity due to periods of high accretion (Herbig, 1977). Both of these point to young discs in this embedded phase being (possibly) gravitationally unstable. Outflows and jets are also detected which can further hinder the measurement of disc properties. This phase lasts $\sim 0.5\text{Myr}$ (Evans et al., 2009).

- **Class II:** Star formation is essentially over for these objects and the envelope is completely dispersed (André and Montmerle, 1994). Emission in this case is largely from a disc, either because it is actively accreting and thus luminous or because it is being heated by stellar radiation. These are known as ‘classical T Tauri’ stars. The majority of planet formation occurs in the discs around Class I and II objects.

- **Class III:** The circumstellar material around these objects has largely disappeared so there is little accretion and thus little emission. However, some do display the signatures of an optically thin remnant debris disc. These are known as ‘weak line T Tauri’ stars.

Note that a ‘transitional disc’ is an object in an evolutionary stage between the gas-rich discs around Class II YSOs and the gas-poor ‘debris discs’ around Class III objects (Kley and Nelson, 2012 and Espaillat et al., 2014).

### 2.3.2 Disc lifetimes

The lifetime of a PP disc is a fundamental parameter, both as a timescale for the physical processes driving disc evolution and as the time available for planet formation.
We can infer the typical disc lifetime $\tau_{\text{disc}}$ from studying the fraction of stars in stellar clusters in star forming regions (assumed to have a relatively uniform age) with various disc signatures (Haisch et al., 2001; Sicilia-Aguilar et al., 2006). Disc properties that are measurable as a function of age are IR excess in the SED (Carpenter et al., 2006; Mamajek et al., 2004) (with observations of cold dust in the outer disc correlating reassuringly well with observations of inner dust discs (Andrews and Williams, 2005)), the steady decline of gas accretion rate $\dot{M}$ onto the star (Hartmann et al., 1998; Muzerolle et al., 2000; Calvet et al., 2000), measurements of dust mass (Carpenter et al., 2005; Wyatt et al., 2003) and size distribution (Kessler-Silacci et al., 2005; van Boekel et al., 2005). A review of these observations is given in Hillenbrand (2005).

There is remarkable similarity in average disc properties from region to region (Williams and Cieza, 2011). There is also good agreement between the ages of T Tauri discs and the inferred formation time of Solar System meteorites (Podosek and Cassen, 1994). However, we do find disc lifetimes are about twice as short around higher mass stars due to higher accretion rates and a stronger radiation environment (Hillenbrand, 2008).

The median disc lifetime is $2 - 3\text{Myr}$, but with a wide dispersion. Despite this, there is a fairly stringent upper limit of $\sim 10\text{Myr}$ on the lifetime of PP discs, with Strom et al. (1989) finding that $\lesssim 10\%$ of YSOs with ages in excess of $10\text{Myr}$ have discs, confirmed by more recent Spitzer surveys (Williams and Cieza, 2011). A plot of the fraction of YSOs with accretion discs vs. stellar age can be seen in Figure 2.2 which shows a steep drop-off in this fraction for protostars older than 5Myr.

The small amount of objects with an outer but no inner disc (due to photoevaporation, discussed in Section 2.3.10) show that the dissipation timescale of the entire disc once accretion stops is $\lesssim 0.5\text{Myr}$ (Skrutskie et al., 1990; Wolk and Walter, 1996; Cieza et al., 2007). This implies that the transition between Class II and Class III YSOs is very rapid and across the entire radial extent of the disc. A discussion of this process and the fate of PP discs is given in Section 2.3.10.

### 2.3.3 Disc mass

Discs are found around stars of very different masses, from brown dwarfs ($M_* < 0.08M_\odot$) to massive stars ($M_* > 8M_\odot$). In general, the most detailed information is available for discs round young objects of about $1M_\odot$. The mass of the disc, $M_{\text{disc}}$, is inferred from the disc’s dust mass.

The dust-to-gas ratio $\Sigma_d/\Sigma_g$ (which we will hereafter call $\Gamma_d$) in PP discs is initially assumed to be the canonical interstellar value of 1% (Draine, 2003), though this is expected to increase as the disc evolves. This means the dynamics of the disc are almost completely dominated by the gas while the dust component of the disc dominates the opacity and thus the emission properties of the disc. In practice this means that around 99% of the disc is functionally
Figure 2.2 The fraction of YSO in stellar clusters with accretion discs vs. stellar age, from Hillenbrand (2005). Individual clusters are treated as containing stars of the same age. It shows that the fraction of stars with discs in young stellar clusters decreases with age and after 5Myr very few stars have discs left; those that do are mostly low-mass T Tauri stars.

invisible. It also dominates the ionisation state of the disc and thus the coupling to the magnetic field. Note also that there are no strong constraints on $\Gamma_d$, which need not match the interstellar medium (ISM) (Youdin, 2010, Section 3.1.2).

The surveys of André and Montmerle (1994); Andrews and Williams (2005, 2007); Eisner and Carpenter (2006) have found disc-to-star mass ratios of

$$M_{\text{disc}}/M_*=10^{-3} - 10^1,$$

(2.3.1)

with the median $M_{\text{disc}}/M_*=0.009$ and an average disc mass $M_{\text{disc}}\approx 0.02M_{\odot}$, which is roughly the minimum mass solar nebula (MMSN)$^7$ (Kusaka et al., 1970; Weidenschilling, 1977; Hayashi, 1981). Figure 2.3 shows a plot of the variation of $M_{\text{disc}}$ with the mass of the central star.

These masses are likely to be underestimates as there will be hidden mass in large, difficult-to-detect grains (Draine, 2006). There are also not enough massive discs observed in nearby

$^7$The minimum mass solar nebula is a PP disc that contains the minimum amount of solids to build the planets of the Solar system, by no means an agreed quantity (e.g. Crida, 2009).
2.3 Observational properties of discs

Figure 2.3 Variation of $M_{\text{disc}}$ with $M_*$, from Williams and Cieza (2011). Almost all the discs with $M_* = 0.04 - 10M_\odot$ have $M_{\text{disc}}/M_* = 10^{-3} - 10^{-1}$.

star-forming regions to account for the amount of giant exoplanets (Greaves and Rice, 2010) which also implies a mass underestimate. Furthermore, we expect $M_{\text{disc}}$ to decrease with time due to both accretion onto the central star and mass loss through jets and winds (see Figure 2.4 and Section 2.3.9).

2.3.4 Disc size

Constraining the disc size requires the ability to resolve it, made even more difficult as the outer regions of discs are cool and produce weak emission. Observations in Taurus by Andrews and Williams (2007) found disc sizes of a few hundreds up to 1000AU, agreeing with the back-of-the-envelope calculation using the Jeans length in Section 2.2. We expect discs around more massive stars to be bigger (Andrews et al., 2009, 2010) and for the disc to spread viscously as the system evolves (the necessity of matter accreting onto the central star means some matter must carry the excess angular momentum outwards, see Section 2.4.6, Pringle, 1981).
Figure 2.4 The flared PP disc around the young star HH 30 in Taurus, an edge-on disc which appears as a flattened cloud of dust split into two halves by a dark region, with $H \approx 6.3$AU at 50AU (Watson and Stapelfeldt, 2004). The two red jets could be produced by the star’s magnetic field (Stapelfeldt et al., 1999) channelling gas from the disc along the rotation axis (Hubble, NASA).

2.3.5 Disc geometry and scale height

The disc scale height $H$ depends on the balance between thermal pressure in the disc and the vertical component of gravity due to the central star (derivation in Section 2.4.4). Discs flare gently, with an inner edge at some distance from the star (Kenyon and Hartmann, 1987; Beckwith, 1999) so the scale height depends on radius from the star, $H = H(r)$. Direct evidence for this comes from Hubble images of disc silhouettes in both Taurus and Orion (e.g. Burrows et al., 1996; Smith et al., 2005). For example, Figure 2.4 shows a Hubble image of the flared disc around HH 30 which has $H \approx 6.3$AU at 50AU (Watson and Stapelfeldt, 2004). By comparison, the rather more flattened disc around HK Tau B has $H \approx 3.8$AU at 50AU (Stapelfeldt et al., 1998). Most observations are from dust at large vertical heights, leading to a $H/r \approx 0.1$. $H/r = 0.05$ is a more appropriate value for the bulk of the disc mass distribution.

In Section 2.4.2 we go into more detail about the thin disc approximation that leads on from this.

2.3.6 Velocities in the disc

We saw in Section 2.2 that we expect gas and dust velocities in PP discs to be approximately Keplerian, confirmed by the observations of Hughes et al. (2011). This study, with observations
of various discs at high spectral resolution, found subsonic turbulence, near–perfect Keplerian rotation profiles and disc self–gravity not appearing to be significant (Section 3.4.1).

However, there is some evidence of deviation from Keplerian velocities, with Grady et al. (1999), Fukagawa et al. (2004) and Corder et al. (2005) finding evidence of spiral structures and Lin et al. (2006); Piétu et al. (2005) showing streaming motions along spiral arms. This was later confirmed by images from the Subaru telescope showing spiral structure in the disc around SAO 206462 (Figure 3.1a, Section 3.1.2).

2.3.7 Temperature, cooling and heating
The relatively large size of discs means that temperatures vary considerably within them, with inner regions reaching a few $10^3$K and the outer disc at a few 10K. The temperature profile is important as it sets a typical scale for the disc thickness $H$. It also drives opacity and ionization and therefore disc emission signatures.

The disc cools mainly via thermal emission from dust grains (Dullemond et al., 2007); the large surface area of the disc means this is particularly efficient and thus we can model PP discs as isothermal in $z$. Simultaneously, discs are heated via absorption of direct stellar radiation and the viscous dissipation of gravitational energy in the disc due to accretion. For most discs, radiation dominates the heating, except in the inner-most regions of the disc and in strongly accreting periods such as outbursts.

The snow line is the inner edge of the region where the temperature falls below the condensation temperature of water (Lecar et al., 2006). It moves inwards as the disc accretion rate drops with time and it lies within the planet forming region $0.1 \lesssim r \lesssim 5 \sim 50$AU for at least some fraction of the protoplanetary disc phase (Martin and Livio, 2012). The heavier silicates and metallic materials are better suited to condense at higher temperatures and thus inside the snow line we see formation of the inner, terrestrial planets made almost entirely of rock and metal. Outside the snow line, hydrogen compounds condense into relatively sticky ices which can rapidly form large, icy cores around which gas giants can form.

2.3.8 Opacity and ionisation
In PP discs, dust is the dominant opacity source and thus it dominates observational characteristics. Only in the innermost regions of the disc where $T \approx 1500$K and dust particles are destroyed do other sources dominate the opacity. At any point in the disc the total dust opacity will be a function of disc temperature, chemical composition and the size distribution of the particles (Beckwith et al., 1990).

Temperatures in the majority of the PP disc are not high enough to fully ionise the gas so at first approximation we can assume that PP discs are neutral. However, even negligible ionisation fractions suffice to couple magnetic fields dynamically to the gas (Wardle, 2007), which may be critical for particular angular momentum transport mechanisms (Section 2.4.6).
2.3.9 Accretion rates

Accretion rates are estimated from the strength of emission lines emitted as gas hits the star. This is therefore the accretion rate $\dot{M}$ as the gas hits the star and not necessarily the accretion rate at larger radii; $\dot{M}$ could differ considerably with the presence of matter sinks in the disc such as a wind or a massive planet which could act as a dam for accretion flow by opening up a gap in the disc.

During the initial collapse of part of a GMC, the accretion rate is initially very high $\dot{M} \sim 10^{-6} - 10^{-5} M_\odot/\text{yr}$ (Hartmann et al., 1993), with most of the mass accreted onto the YSO while in this embedded (Class 0) phase. This quickly drops to $\dot{M} \sim 10^{-9} - 10^{-7} M_\odot/\text{yr}$ for Solar type T Tauri stars once the infall phase is over (Nakamoto and Nakagawa, 1994; Hueso and Guillot, 2005).

2.3.10 Fate of the gas disc

The depletion of the gaseous PP disc due to accretion onto the central star is predicted to be a gradual process taking place over a few viscous timescales, $\tau_\nu$ (Section 2.4.6). However, the relative scarcity of systems transitioning from Class II to Class III YSOs suggest that the dispersal phase of PP discs is fast, $\sim 10^5$ years (Simon and Prato, 1995; Wolk and Walter, 1996). This implies that there must be other physical processes contributing to loss of gas from the disc. The most likely cause is photoevaporation, where UV or X-ray radiation, from either the central star or an ambient radiation field, heats the disc surface to the point where it becomes hot enough to escape the gravitational potential as a thermally driven wind (Clarke et al., 2001; Alexander et al., 2006a,b). This occurs for the duration of the disc’s lifetime but only becomes the dominant source of mass loss when the viscous accretion rate falls below the photoevaporation rate (Dullemond et al., 2007).

2.4 Disc structure and evolution

We have established that most knowledge about PP discs has been derived from spatially unresolved IR SEDs in nearby star forming regions. Interpreting these to extract the disc properties we have seen requires the use of theoretical models.

2.4.1 Overview of disc evolution

At this point it is instructive to give an overview of the evolution of a disc in the most general sense, from its formation shortly after the infall of a GMC to a debris disc and possible planetary system around a main sequence, adult star. A summary of this can be seen in Figure 2.5.

We have seen that discs form almost immediately after the freefall collapse of a GMC as a natural consequence of its specific angular momentum. They are unstable at early times,
2.4 Disc structure and evolution

- Massive flared disc around a Class I YSO
- Settled disc around a Class II YSO
- Photoevaporating or transitional disc
- Debris disc around a Class III YSO

Figure 2.5 Evolution of a general PP disc, based on a diagram in Williams and Cieza (2011). For details see Section 2.4.1.
accreting in bursts and with mass outflows along jets. After \( \sim 10^4 - 10^5 \) yr the envelope is accreted and a quasi-stable, flared disc exists, losing mass through accretion onto the star and photoevaporation of the outer disc (Figure 2.5a). During this and the initial phase, matter will spread outwards to \( r \gtrsim 100 \) AU (Nakamoto and Nakagawa, 1994; Hueso and Guillot, 2005) due to viscous spreading, as for accretion to occur some matter must carry angular momentum outwards (Lynden-Bell and Pringle, 1974). This spreading only stops when photoevaporation truncates the disc (Scally and Clarke, 2001).

During this protoplanetary phase, primordial dust present in the disc agglomerates into larger bodies, reducing the dust scale height \( H_d \) and the flared disc becomes flatter (Figure 2.5b). Solids from dust to planets settle into the midplane of the disc and can migrate inwards under gas drag (see Section 3.3).

The gradual process of accretion-driven gas depletion in the disc continues until the accretion rate drops below the photoevaporation rate, whereupon the disc is rapidly eroded from inside to out and we enter the transitional disc phase. With the outer disc unable to resupply the inner disc with material an inner hole is formed in the disc and accretion halts. The rest of the gas disc then dissipates. This transitional phase is very rapid, occurring on a timescale of a few \( 10^5 \) yr (Figure 2.5c).

Once the gas disc photoevaporates the \( a \lesssim 1 \mu m \) sized dust grains are blown off by radiation pressure. Slightly larger ones spiral inwards due to Poynting-Robertson effect and eventually evaporate. This leaves a gas-poor debris disc containing large grains, planetesimals and planets (Figure 2.5d). These are not always detectable and the lifetime of such a disc is poorly constrained (Wyatt, 2008).

### 2.4.2 Thin disc approximation

Modelling our PP disc requires some assumptions and approximations to be made, the first of which is the thin disc approximation. In Section 2.3.5 we found the disc scale height \( H \) to be \( H/r \lesssim 0.1 \), which allows us to treat the disc to first approximation as infinitesimally thin, or at the very least that vertical thickness of disc much less than the orbital radius. We can then introduce the small quantity \( H/r \ll 1 \) into our equations of motion. Under this approximation, most equations can then simply be integrated in the vertical direction.

This approximation follows from the fact that a disc has a large surface area and thus can cool via radiative losses efficiently. Efficient cooling implies relatively low disc pressure and temperatures which are unable to support the gas against gravity except in a geometrically thin disc (Pringle, 1981).

### 2.4.3 Gas dynamics of viscous discs

We have see that gas evolution in PP discs is driven by viscous accretion and photoevaporation while grain growth, settling and radial drift (intimately connected processes, see Section 3.3),
drive dust evolution. However, for the moment we will ignore the effect of dust altogether and consider the dynamics of disc gas.

The evolution of the gas disc can be described by the continuity and Navier-Stokes equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.4.1}\]
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} (\nabla P - \nabla \cdot \sigma) - \nabla \Phi_{gr}, \tag{2.4.2}\]

where \(\rho\) is the density, \(\mathbf{v}\) denotes the gas velocity and \(P\) the pressure distribution. The gravitational potential is denoted by \(\Phi_{gr}\) which, in most cases, is dominated by the gravitational attraction from the central star (see Section 3.4.1 for when this doesn’t apply). Therefore

\[
\Phi_{gr} = -\frac{GM_*}{R} \tag{2.4.3}\]

where \(G\) is the gravitational constant, \(M_*\) the mass of the central star and \(R\) the spherical radius (to distinguish it from the cylindrical radius \(r\)). Finally, the stress tensor \(\sigma\) describes other physical effects such as viscous forces, magnetic fields, self-gravity etc.

The geometry of the disc means we adopt a cylindrical polar coordinate system \((r, \phi, z)\) centred on the star. It is also assumed that as a first approximation the disc is axisymmetric so no fluid quantities depend on the azimuthal angle \(\phi\). We define the surface density \(\Sigma(r,t)\) as the density per unit surface area, the vertically and azimuthally averaged density under the thin disc approximation, equal to

\[
\Sigma(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho \, dz\, d\phi. \tag{2.4.4}\]

Integrating equation (2.4.1) in the vertical direction (e.g. Pringle, 1981) gives

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0, \tag{2.4.5}\]

where \(v_r\) is the radial velocity. In its simplest form, the stress tensor \(\sigma\) can be assumed to be the classical shear viscosity, whereby its only nonzero component is

\[
\sigma_{r\phi} = \rho \nu r \frac{d\Omega}{dr}, \tag{2.4.6}\]

where \(\nu\) is the kinematic viscosity, \(\Omega = v_\phi/r\) is the angular velocity and \(rd\Omega/dr\) is the rate of strain of the flow. This is a fundamental property of the flow, so much so that we define the
shear rate, $S^8$:

$$S = -r \frac{d\Omega}{dr} \tag{2.4.7}$$

so that $\sigma_{r\phi} = -\rho \nu S$. For particles in Keplerian motion, the base state we expect our gas to be in from Sections 2.2 and 2.3.6, we note that

$$\Omega = \sqrt{\frac{GM_*}{r^3}} \Rightarrow S = \frac{3}{2} \Omega. \tag{2.4.8}$$

A Keplerian flow is therefore also a *shearing* flow, with fluid closer to the star moving faster than material in the outer disc.

From the three components of equation (2.4.2) we can now derive various features of PP discs, namely the centrifugal balance from the radial component, hydrostatic equilibrium in the vertical direction and angular momentum conservation from the azimuthal component.

### 2.4.4 Vertical structure

We found in Section 2.3.3 that it is safe to make the assumption that $M_{\text{disc}} \ll M_*$ and thus we begin by neglecting the gravitational potential of the disc. This is marginal for some of the most massive discs with $M_{\text{disc}} \simeq 0.1 M_\odot$ and fails completely at very early epochs, but for now assume $M_{\text{disc}} \ll 0.1 M_\odot$ as this is a perfectly reasonable assumption during the PP disc phase. We will cover disc self-gravity in Section 3.4.1.

Therefore, considering the vertical component of equation (2.4.2), we neglect the left hand side altogether, as confining the disc to the equatorial plane in the thin disc approximation means that $v_z \ll 1$. Therefore in the $z$-direction, equation (2.4.2) reduces to

$$\frac{1}{\rho} \frac{\partial P}{\partial z} = - \frac{\partial \Phi_{\text{gr}}}{\partial z} = - \frac{GM_* z}{R^{5/2}} = - \frac{GM_* z}{(r^2 + z^2)^{3/2}}. \tag{2.4.9}$$

In the case of an isothermal disc (a reasonable assumption in the thin disc approximation $H/r \ll 1$)

$$P = c_s^2 \rho \tag{2.4.10}$$

where $c_s^2$ is the isothermal sound speed. In the thin disc approximation, $z \ll r$ and $r \sim R$ and so

$$g_z = \frac{GM_* z}{(r^2 + z^2)^{3/2}} \simeq \Omega^2 z \tag{2.4.11}$$

where $\Omega$ is the Keplerian angular velocity (from now on this will always be the case unless otherwise stated). We define $v_K = \Omega r$ to be the Keplerian (azimuthal) velocity. The differential

---

8Throughout this work we will retain both $S$ and $\Omega$, despite the simple relation between them, as they parametrise different effects; shear and rotation.
equation (2.4.9) is then easily solved to find

$$\rho = \rho_0 \exp \left( -\frac{\Omega^2 z^2}{2c_s^2} \right) = \rho_0 \exp \left( -\frac{z^2}{2H^2} \right)$$  (2.4.12)

where $\rho_0 = \frac{1}{\sqrt{2\pi}} \frac{\Sigma}{H}$ is the midplane ($z = 0$) density and we’ve found the disc scale height

$$H \equiv \frac{c_s}{\Omega}.$$  (2.4.13)

This gives the neat equivalence

$$\frac{H}{r} = \frac{c_s}{v_K} = \frac{1}{\mathcal{M}},$$  (2.4.14)

where

$$\mathcal{M} = \frac{v_\phi}{c_s} \approx \frac{v_K}{c_s},$$  (2.4.15)

is the Mach number. The expected derivation of $v_\phi$ from $v_K$ will be derived in the next Section, 2.4.5. Thus, requiring the disc to be thin is the same as requiring the disc rotation to be supersonic, which is consistent as a thin disc allows efficient cooling.

### 2.4.5 Radial balance

The radial component of equation (2.4.2) gives the radial momentum equation:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{\partial \Phi_{gr}}{\partial r}.$$  (2.4.16)

Using the thin disc approximation $H/r \ll 1$ we can deduce from equation (2.4.14) that $c_s (\equiv \Omega H) \ll v_\phi (= \Omega r)$, while the condition that accretion must occur over a long timescale (as seen in Section 2.3.9) implies that $v_r \ll c_s$. Therefore, an important ordering of velocities due to the thin disc approximation arises:

$$v_r \ll c_s \ll v_\phi.$$  (2.4.17)

Under the assumption that the gas is isothermal, the supersonic nature of $v_\phi$ means that we can neglect the first two terms in equation (2.4.16) and

$$\frac{v_\phi^2}{r} \sim \frac{c_s^2}{r^2} + \frac{GM_s}{r^2} = \frac{v_\phi^2}{\mathcal{M}^2 r^2} + \frac{GM_s}{r^2}$$  (2.4.18)

where we have also made the approximation $d\Omega/dR \approx d\Omega/dr$ (as in Section 2.4.4). Thus:

$$v_\phi = \sqrt{\frac{GM_s}{r} \left[ 1 - \mathcal{O} \left( \frac{1}{\mathcal{M}^2} \right) \right]} = v_K (1 - \eta),$$  (2.4.19)
where $v_K$ is the Keplerian velocity and $\eta \sim (H/r)^2$ is small ($\eta$ is a dimensionless measure of pressure support). This small deviation from Keplerian flow has profound effects for the radial drift of dust, discussed later in Section 3.3.4.

Therefore, to first approximation the flow is Keplerian and radial equilibrium is simply centrifugal balance. The pressure near disc midplane normally decreases outwards since discs are generally hotter and denser closer to the YSO, so $\frac{1}{\rho} \frac{d\rho}{dr}$ is generally negative and $v_\phi$ is slightly less than the Keplerian velocity.

We also note that the specific angular momentum of gas in a Keplerian disc is

$$j = rv_\phi = \sqrt{GM_\ast r} \quad (2.4.20)$$

is increasing function of radius. Thus for gas to move inwards and be accreted by the star it needs to lose angular momentum. Understanding the mechanisms that result in angular momentum loss is a central problem in accretion disc physics (Papaloizou and Lin, 1995).

Finally, Keplerian discs are also Rayleigh stable. Rayleigh (1917) found that a rotating flow in a non-magnetised, non-self gravitating disc is stable to infinitesimal axisymmetric hydrodynamic perturbations if and only if the specific angular momentum $j$ increases with radius:

$$\frac{dj}{dr} = \frac{d}{dr} (r^2 \Omega) > 0. \quad (2.4.21)$$

A Keplerian flow has $j \propto \sqrt{r}$ so it is predicted to be hydrodynamically stable.

### 2.4.6 The surface density evolution equation and viscous disc models

Finally, we consider the azimuthal component of equation (2.4.2) to derive an evolution equation for the surface density $\Sigma(r,t)$. For the sake of brevity, we skip the bulk of the derivation (see e.g. Ogilvie (2005) for details) and, after some algebra:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ \left( \frac{dj}{dr} \right)^{-1} \left[ \frac{\partial}{\partial r} \left( \nu \Sigma r^3 \frac{d\Omega}{dr} \right) + r(T - Sj) \right] \right\} = S \quad (2.4.22)$$

where $T(r,t)$ parametrise external torques (e.g. tides or magnetic effects on the inner disc) and $S(r,t)$ are mass sources such as jets or infall from the envelope at early times. Again, the specific angular momentum of the flow is $j = r^2 \Omega$. We define an internal torque $G(r,t)$ due to the viscosity of the fluid, parametrised via the vertically averaged kinematic viscosity $\nu$ (von Weizsäcker, 1948):

$$G = \frac{1}{2\pi} \int \int r \sigma r \phi d\phi dz = \nu \Sigma r^3 \frac{d\Omega}{dr}. \quad (2.4.23)$$

This is assumed to be positive as $\nu \geq 0$ so angular momentum transport is outwards. For Keplerian orbits and in the absence of external torques or mass sources/sinks $T = S = 0$, so

---

9 For a derivation see e.g. Pringle and King (2007).
2.4 Disc structure and evolution

Equation (2.4.22) becomes

$$\frac{\partial \Sigma}{\partial t} = 3 \frac{\partial}{\partial r} \left( \frac{r^{1/2}}{\nu} \frac{\partial}{\partial r} \left( \nu \Sigma r^{1/2} \right) \right).$$

(2.4.24)

This is a key equation in accretion disc evolution. First presented by von Weizsäcker (1944), there was a general solution found by Lüst (1952)\(^{10}\) with the first full treatment in the landmark work of Lynden-Bell and Pringle (1974).

For a constant \(\nu\) there exists an analytic solution to this diffusion equation which shows that a ring of matter will spread viscously, as is observed in PP discs (Section 2.4.1); the bulk of the mass flows inwards while a small proportion of the mass carries angular momentum outwards (Pringle, 1981).

From this equation we also get a viscous timescale

$$\tau_\nu \sim \frac{r^2}{\nu}$$

(2.4.25)

which for PP discs around Solar-type stars is around 1Myr. Thus the evolution of \(\Sigma(r,t)\) is on a timescale equivalent to the disc lifetime \(\tau_{\text{disc}}\), so, to a good first approximation, our disc is steady.

The study of steady accretion discs began with the work of Lynden-Bell (1969), applied to galactic discs. The so-called \(\alpha\)-disc model was devised by Shakura and Sunyaev (1973) to create realistic disc models while avoiding solving the viscosity problem in detail (i.e. determining the mechanisms driving it). This was done by introducing the \(\alpha\)–prescription for the disc viscosity, writing the vertically averaged turbulent viscosity throughout the entire disc as

$$\nu = \alpha_\nu c_s H.$$  

(2.4.26)

\(\alpha_\nu\) is a dimensionless parameter measuring the efficiency of angular momentum transport due to turbulence, which we give the subscript \(\nu\) to prevent confusion with an \(\alpha\) used later in this work. In reality, \(\alpha_\nu\) may vary with temperature, density and composition of disc gas and is not well constrained (Papaloizou and Lin, 1995).

Angular momentum transport is thought to be mainly driven by magnetohydrodynamic (MHD) turbulence as a result of the magnetorotational instability (Velikhov, 1959; Chandrasekhar, 1960; Balbus and Hawley, 1991; Balbus and Hawley, 1998). This instability arises as coupling a magnetic field to the disc supplies additional degrees of freedom. The presence of a poloidal magnetic field modifies the Rayleigh criterion (Section 2.4.5) such that instability requires \(d\Omega^2/dr < 0\), which a Keplerian flow satisfies. Typically, \(\alpha_\nu \simeq 10^{-2}\) for sufficiently ionized disc atmospheres, while \(\alpha_\nu \lesssim 10^{-4}\) in the MRI dead zone in the midplane (Gammie,

\(^{10}\)The solution was dedicated to Heisenberg in honour of his 50th birthday. Posterity does not tell us if Heisenberg would’ve preferred socks.
Other contributions to $\alpha_\nu$ could be from hydromagnetic winds (Blandford and Payne, 1982), external tidal forces from binary companions (Papaloizou and Pringle, 1977) or embedded protoplanets (Lin and Papaloizou, 1993), disc self-gravity (Lynden-Bell and Kalnajs, 1972; Toomre, 1964, 1981), nonlinear or transient instabilities in purely hydrodynamic flows (Lin and Papaloizou, 1980), vortices and their interactions (Adams and Watkins, 1995; Godon and Livio, 1999b) or the density waves produced by large scale vortices (Johnson and Gammie, 2005). We return to the subject of vortices in Section 3.5.

Detailed models of viscous accretion (Pringle, 1974; Lynden-Bell and Pringle, 1974; Hartmann et al., 1998) are consistent with observational constraints for disc masses, sizes and the decrease of accretion rates over time (Hartmann et al., 1998; Hueso and Guillot, 2005). They are however just first order approximations in more complex evolution.

The need for accurate models of $\Sigma(r, t)$ is crucial for the study of PP discs as average radial, vertical and velocity profiles can only be determined from resolved observations and models of the surface density.

### 2.4.7 Important timescales governing disc evolution

From the surface density evolution equation (2.4.24) we found the viscous timescale $\tau_\nu \sim r^2/\nu$. There are however other important timescales determining the evolution of PP discs. Orbital motion occurs on a timescale $\Omega^{-1} = H/c_s$ (equation (2.4.13)) so is the same as the sound crossing time – the time to achieve (vertical) hydrostatic equilibrium. We therefore define the fundamental orbital or dynamical timescale

$$\tau_{\text{dyn}} \sim \Omega^{-1}$$

which obeys the empirical relation (Heng and Kenyon, 2010)

$$\tau_{\text{dyn}} \approx 0.2 \text{ yr} \left( \frac{M_*}{M_\odot} \right)^{-1/2} \left( \frac{r}{\text{AU}} \right)^{3/2}.$$  \hspace{1cm} (2.4.28)

It is convenient to compare the sizes of other disc timescales to this quantity. For instance, the viscous timescale $\tau_\nu$ from Section 2.4.6 (the timescale on which matter diffuses through the disc under the effect of viscous torques) is related to $\tau_{\text{dyn}}$ by

$$\tau_\nu \sim \frac{r^2}{\nu} \sim \frac{r^2}{\alpha_\nu c_s H} \sim \frac{r^2}{\alpha_\nu \Omega H^2} \sim \frac{1}{\alpha_\nu} \left( \frac{H}{r} \right)^{-2} \tau_{\text{dyn}}.$$  \hspace{1cm} (2.4.29)

Thus for a thin disc with subsonic turbulence, $\alpha_\nu < 1$, we have $\tau_{\text{dyn}} \ll \tau_\nu$. Similarly, we can find a local thermal timescale on which the disc cools (Pringle, 1981), given by the ratio of the
2.5 Summary and conclusions

A summary of important physical properties deduced from the observations of PP discs can be found in Table 2.1, while one summarising the timescales found in PP disc structure and evolution can be found in Table 3.3.

<table>
<thead>
<tr>
<th>Property</th>
<th>Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>$10^{-3} - 10^{-1} M_\odot$</td>
</tr>
<tr>
<td>size</td>
<td>$10^2 - 10^4\text{AU}$</td>
</tr>
<tr>
<td>thickness</td>
<td>$H/r \ll 1$, typically $H/r \simeq 0.05$</td>
</tr>
<tr>
<td>temperature</td>
<td>10K (outer disc) – 1500K (inner regions)</td>
</tr>
<tr>
<td>accretion rate $\dot{M} (M_\odot/\text{yr})$</td>
<td>$\sim 10^{-6} - 10^{-5}$ (embedded phase), Hartmann et al. (1993) $\sim 10^{-11}$ (Brown dwarfs), Alexander and Armitage (2006) $\sim 10^{-9} - 10^{-7}$ (T-Tauri, $M_\ast \sim M_\odot$), Nakamoto and Nakagawa (1994) $\sim 10^{-5}$ (massive stars, $M_\ast &gt; 8M_\odot$), Alexander and Armitage (2006)</td>
</tr>
<tr>
<td>dust fraction, $\Gamma_d$</td>
<td>$0.01 - 0.02M_{\text{disc}}$</td>
</tr>
</tbody>
</table>

Table 2.1 Summary of observational PP disc properties given in this chapter.

thermal energy of the gas, $\Sigma c_s^2$, to the rate of viscous heating, $\nu \Sigma \Omega^2$:

$$\tau_{\text{cool}} = \tau_{\text{th}} \sim \frac{c_s^2 \Sigma}{\nu \Sigma \Omega^2} \sim \frac{c_s}{\alpha \nu \Omega r} \sim \alpha^{-1} \tau_{\text{dyn}}.$$ \hspace{1cm} (2.4.30)

We therefore find that (Lynden-Bell and Pringle, 1974; Pringle, 1981)

$$\tau_{\text{dyn}} < \tau_{\text{th}} \ll \tau_{\nu},$$ \hspace{1cm} (2.4.31)

so centrifugal balance in the radial direction (orbital timescale) and hydrostatic balance in the vertical direction (dynamical timescale) are very rapidly achieved, followed by the disc temperature evolving on a longer timescale while the evolution of $\Sigma(r, t)$ is very slow indeed. Also note that all three timescales are functions of disc radius, scaling like $r^{3/2}$ ($H/r \simeq \text{const}$). Thus the evolution of the inner disc is generally much more rapid than that in the outer disc.

2.5 Summary and conclusions

A summary of important physical properties deduced from the observations of PP discs can be found in Table 2.1, while one summarising the timescales found in PP disc structure and evolution can be found in Table 3.3.
We have found that discs are a natural consequence of star formation and, consequently, are expected to occur around most protostars. They should therefore not be thought of as particularly special nor uncommon objects. In the protoplanetary phase, they are geometrically thin structures which, to first approximation, we can treat as isothermal and 2D; the ratio $H/r \ll 1$ is a useful small quantity for scaling equations of motion.

These PP discs form rapidly ($10^4 - 10^5$ yr), persist for around 1Myr and are efficiently cleared by photoevaporation $\sim 10^5$ yr. In astronomical terms, 1Myr is not a long time\textsuperscript{11}, imposing a relatively small window for planet formation.

Gas in the disc orbits the star with approximately Keplerian velocity so we can consider PP discs to also be a *Keplerian* disc with an intrinsic shearing flow. We will find in the next chapter that the slightly sub–Keplerian orbital speed achieved by the disc gas has profound effects when we consider the role of dust.

Circumstellar discs do not just contain gas: they initially have a dust fraction $\Gamma_d \approx 0.01$ inherited from the ISM. Dust dominates the opacity and therefore most of the observational properties of discs but due to the low dust–to–gas ratio $\Gamma_d$, the dynamics of the disc are driven mainly by the gas component. At a first approximation we can therefore consider the disc to be a single fluid, an approach which produces self–consistent disc models.

Up to this point we have largely neglected the role of dust grains in PP disc evolution. Since this dust is the very matter we build planetesimals and planets out of, we will now consider it in more detail.

\textsuperscript{11}The lifetime of a $1M_\odot$ star is around 11,000Myr.
Chapter 3

The role of dust in protoplanetary discs

In the previous chapter we reviewed the observational properties of PP discs and the models frequently used to describe their structure and evolution. We now examine the role of dust in these discs; on the evolution of the disc itself, the mechanism for building $\sim 10^4\text{km}$ planets from $\sim 0.1\mu\text{m}$ dust and the severe barriers to overcome for this to occur, namely the ‘metre gap’. Finally, we will investigate how local pressure maxima in vortices offer a promising route around these problems.

In Section 3.1 we will review the role of dust in protoplanetary disc evolution and the evidence for its existence. After briefly discussing where planetesimal formation fits into a broader planet formation (both terrestrial and gas giant), in Section 3.2 we will investigate possible mechanisms of planetesimal formation in Section 3.3. In order to build planetesimals we need some mechanism of grain growth (Section 3.3.1) and to understand how it interacts with the surrounding gas (Section 3.3.2). The key effects of this aerodynamic drag and how this creates a ‘metre gap’ are discussed in Sections 3.3.3 and 3.3.4.

We discuss the Safronov–Goldreich–Ward (SGW) mechanism as a planetesimal and planet forming mechanism that can bypass the metre gap in Section 3.4, its shortfalls (Section 3.4.2) and possible routes around these problems (Section 3.4.3). The most promising of these is the role of vortices, producing both density enhancements which can locally support the SGW mechanism (Section 3.5) and trapping dust to help build planetesimals within the stringent timescales imposed (Section 3.5.1). Finally, we investigate how to produce these structures (Section 3.5.2), how long they can persist (Section 3.5.3), the various instabilities which threaten them (Section 3.5.4) and how to model them (Section 3.5.5). There is a summary of the timescales governing PP discs in in Table 3.3.
3.1 Dust in PP discs

3.1.1 A note on definitions

In the literature, dust can mean particles anything from $a \simeq 0.1\mu m$ to $a \simeq 1m$ in size. Different definitions of rocks, boulders, pebbles etc. will only serve to confuse so we will just refer their radius, $a$, instead.

Throughout, we will adopt the definition of a planetesimal\footnote{The word planetesimal was first used by Chamberlin (1900) for solid bodies that had subsequently accumulated into planets.} that was agreed at the workshop organised by Dullemond and Klahr (2006):

"A planetesimal is a solid object arising during the accumulation of planets whose internal strength is dominated by self-gravity and whose orbital dynamics is not significantly affected by gas drag. This corresponds to objects larger than approximately 1 km in the solar nebula."

The definition of a protoplanet is less well constrained, requiring that the body not only be bound by self-gravitation but also change the path of the rocks a few radii away from them. In reality this means $a \simeq 100 - 1000km$. They are also known as protoplanetary embryos. For completeness, the IAU definition of a planet is

"[...] a celestial body that: (a) is in orbit around the Sun, (b) has sufficient mass for its self-gravity to overcome rigid body forces so that it assumes a hydrostatic equilibrium (nearly round) shape, and (c) has cleared the neighbourhood around its orbit."

The planet forming region is anywhere solid and/or gas can accumulate. This excludes regions very close to the star ($\lesssim 0.1AU$) where it is too hot for solids to exist or for gravitational instability (GI) to occur. The outer limit is more difficult to constrain (the literature gives anything from 5-50AU) but since the outer disc contains less processed dust (Section 3.1.2) with longer collisions times, it is of less interest for planet formation.

3.1.2 Evidence of dust in PP discs

Discs have dust features seen in emission spectra from their optically thin outer layers heated by stellar radiation (Calvet et al., 1991; Chiang and Goldreich, 1997). As we have seen, the absorption and emission profiles of PP discs are overwhelmingly dominated by this trace dust component (Semenov et al., 2003).

The composition of dust grains in Class 0 and I YSOs is similar to the $a \simeq 0.1\mu m$ dust in the ISM, with both the models of Kruegel and Siebenmorgen (1994) and observations of Beckwith and Sargent (1991) confirming this. In discs around more evolved objects we find...
3.1 Dust in PP discs

Figure 3.1 The spiral structure observed in the disc around SAO 206462 by Muto et al. (2012) was the first clear case of spiral arms around an individual star. The disc extends to 140AU, with evidence of a dust depleted cavity at $r \lesssim 46$AU. Pérez et al. (2014) later observed signatures of an asymmetric dust concentration (Figure 3.1b) which suggests the spiral gravity waves are produced by the vortex, which is in turn induced by a planet.
The role of dust in PP discs

Figure 3.2 A large dusty vortex in the system SR21 (Pérez et al. 2014; ESO/ALMA).

large masses of dust particles \( a \simeq 0.1 - 1\text{cm} \) (Prosser et al., 1994) and signatures of dust processing and coagulation (Kessler-Silacci et al., 2007, review Natta et al., 2007). The best evidence for grain growth to \( a \simeq 1\text{cm} \) sizes is from the detection of 3.5cm dust emission in the face-on disc around TW Hya (Wilner et al., 2005). Frustratingly, the radiative inefficiency of \( a \gtrsim 1\text{cm} \) objects makes it difficult to observe when planetesimals actually form.

From observations of silhouette discs there is some evidence for the vertical stratification of grains, with the smallest \( a \simeq 0.1\mu\text{m} \) closest to the disc’s surface. There is also some radial dependence, with grains in the outer disc less processed then those closer to the star (van Boekel et al., 2004).

Reassuringly, the typical size of dust particles also appears to increase on a timescale \( \simeq \tau_{\text{disc}} \) (D’Alessio et al., 2001; Shuping et al., 2003; McCabe et al., 2003). However, there is however a large amount of variation in dust properties among different sources, with no correlation between grain size, \( a \), and stellar properties such as \( M_\star \), luminosity or age, nor with the properties of the disc such as its mass \( M_{\text{disc}} \) or accretion rate \( \dot{M} \).

Furthermore, small dust \( a \simeq 0.1\mu\text{m} - 1\text{cm} \) is seen to persist for periods \( \simeq \tau_{\text{disc}} \). This survival of a large mass of small grains to large times is a significant challenge to planet formation theory as it suggests that the process of planetesimal formation is less efficient than predicted by models, or, if forming planetesimals from small dust is as fast as predicted (see Section 3.3.1), it must be at the end of a long quiescent phase. Another option is that the
grain growth process is in fact slow (with inefficiency due to fragmentation and bouncing), but as we shall see in Section 3.3.4 this is a problem for some intermediate grain sizes due to rapid radial drift.

Finally, as predicted (e.g. Wolf and Klahr, 2002), telescopes have recently begun to resolve asymmetric dust features in discs (Andrews et al., 2011; Birnstiel et al., 2013; Casassus et al., 2013; Fukagawa et al., 2013; Isella et al., 2013; van der Marel et al., 2013; Pérez et al., 2014; Espaillat et al., 2014), possible signatures of dust trapping in large-scale vortices (see Section 3.5). Observations are largely limited to transitional discs which are not shrouded in optically thick material (Section 2.4.1) but there is no reason to believe that large scale asymmetries are limited to these late-epoch discs.

Two examples from the systems SAO206462 and SR21 can be seen in Figures 3.1 and 3.2, with the former also exhibiting spiral density waves and a cleared inner disc. This suggests a combination of a planet and induced vortex. We expect a vortex with approximately sonic velocities to produce density waves in this manner (Nelson et al., 2000), suggesting the configuration displayed in Figure 3.1c. Figure 3.2 is another recent example of an observed large vortex structure.

### 3.1.3 Importance of dust in PP discs

The evolution of dust in PP discs is governed by a plethora of physical processes such as growth via coagulation, erosion through sublimation or collisional fragmentation, radiation pressure, photoevaporation and photoionisation, interaction with magnetic fields and interactions with the gas disc. It is this last effect that is the most dominant and the process in which we are most interested.

As we saw in the last chapter, grains dominate disc emissions, therefore we need to understand the dust in order to probe the thermal and geometric structure of the disc and understand our observations. The dominant source of ionisation is the ionisation of heavy elements by high energy stellar photons or cosmic rays (Glassgold et al., 2000; Fromang et al., 2002). These heavy elements are locked up in dust grains and grain growth is an essential step in retaining these heavy elements (Weidenschilling and Cuzzi, 1993). Grains also act to shield the disc midplane from this radiation which could possibly lead to a central ‘dead zone’ where the MRI cannot operate; this is important for angular momentum transport processes (Section 2.4.6). There may however be weak hydrodynamic turbulence in the dead zone from vertical-shear instability (Nelson et al., 2013). Also note that the layered disc model of Gammie (1996) has the quiescent midplane dead zone bounded by MRI-active surface layers, so care needs to be taken establishing where our models are applicable.

As we will see, solids also play a dynamical as well as chemical role in the disc (Section 3.3.2 and hereafter), modifying properties of turbulence, introducing two-fluid instabilities (Section 3.4.2) and seeding gravitational instability in areas of particle overdensity (Section 3.4.1).
Finally, dust is the source of matter from which planets are formed. Over thirteen orders of magnitude separate $a \sim 0.1\mu m$ dust from terrestrial planets and many different areas of physics play a role throughout this process. The very existence of the Solar system and the increasing amount and diversity of exoplanet systems found$^2$ (Weidenschilling and Cuzzi, 1993; Beckwith et al., 2000) show that the planet formation process appears to be a robust one, albeit not fully understood.

3.2 Beyond planetesimal formation: creating terrestrial and gas giant planets

Modern theories of planet formation are based on the work of Safronov (1969) and can be divided into three main stages: planetesimal formation (of which this thesis is chiefly concerned), terrestrial planet formation and giant planet formation, with different physical processes dominating each one. Returning to planetesimal formation in due course, we shall look ahead to the last two stages.

Once planetesimals have been formed, they are (by definition) massive enough to be more-or-less decoupled from the gas disc. The dominant physical process governing them is therefore gravity. Terrestrial planet formation is a well–posed problem because of this, but still a challenging one due to the large number of bodies and long timescales involved. Starting from large population of planetesimals, of uncertain dimensions, there are two distinct phases.

The first is runaway growth where gravitationally focussed pairwise collisions between planetesimals results in progressively larger objects; unlike with collisions between tiny dust grains (Section 3.3.1), the gravity of the bodies in these collisions is strong enough to assume most of the mass will end up agglomerating into a larger single object. This is followed by a period of oligarchic growth where a few large protoplanets consume planetesimals in their own independent ‘feeding zones’ to make approximately earth mass cores $M \simeq M_\oplus$ within $\tau_{\text{disc}}$ (Chambers and Wetherill, 1998; Goldreich et al., 2004).

In order to accrete a large envelope of gas, giant planets have to be formed before the gas disc disperses in a time $\lesssim \tau_{\text{disc}}$. Giant planet formation has two qualitatively different, but not necessarily mutually exclusive, theories:

- **Core accretion theory** (Cameron, 1973; Pollack et al., 1996), whereby a core of ice and rock acquires a giant envelope of gas, with the icy core formed in an identical process to terrestrial planet formation. The timescale, and indeed viability of this model, depends on how quickly the core is assembled and how rapidly the gas in the envelope can cool and accrete onto the core (Papaloizou and Terquem, 2006).

\(^2\)An up-to-date list of NASA’s Kepler exoplanet discoveries can be found at http://kepler.nasa.gov/Mission/discoveries/.
• Disc instability theory (Kuiper, 1951; Cameron, 1978; Boss, 1997) has giant planets rapidly forming via the gravitational fragmentation of an unstable PP disc, similar to Safronov–Goldreich–Ward theory of planetesimal formation (Section 3.4). It requires the disc to cool rapidly on a timescale $\sim \tau_{\text{dyn}}$.

3.3 Planetesimal formation

We now go backwards in the planet formation process to look at the key processes governing the creation of planetesimals from primordial dust. The mechanism for doing so is necessarily robust, given how commonplace planets are, but finding this mechanism has proved to be elusive.

3.3.1 Grain growth

Protoplanetary discs are dust-processing factories, building planets from tiny $a \simeq 0.1 \mu m$ grains of icy, dusty particles into planetesimals, protoplanets and finally planets.

The hierarchical growth of dust grains occurs when pairwise collisions result in two particles sticking and not bouncing nor fragmenting. The likelihood of a collision involves considering detailed velocity distributions (e.g. Garaud et al., 2013; Hubbard, 2012, 2013), but once 1mm-sized particles have formed and begun to settle out of the gas there is an order of magnitude approximation for a collisional timescale (Heng and Tremaine, 2010):

$$\tau_{\text{collide}} \sim \frac{a \rho \cdot}{\Sigma d \Omega} \quad (3.3.1)$$

where $\Sigma d \simeq 0.01 \Sigma$ is the surface density of solids and $\rho_d$ is the density of the dust particles. This corresponds to $\tau_{\text{collide}} \simeq 0.005 - 0.5 \text{yr}$ for $a \simeq 1 \text{mm} - 10 \text{cm}$, while $\tau_{\text{collide}} \simeq 5 \text{yr}$ for $a \simeq 1 \text{m}$ (Heng and Kenyon, 2010). We therefore expect collisions to be frequent on astrophysical timescales.

Once a collision has taken place, it must then be determined whether it resulted in net growth (sticking), no growth at all (bouncing) or, worst of all, negative growth (fragmentation). The sticking efficiency in any given pairwise collision depends on the shape of the colliding bodies, the porosity of the grains (how compacted they are, Okuzumi et al., 2012) and the velocity of the collision (Windmark et al., 2012b) and is a field in its own right.

For small dust, growth via Brownian motion is reasonably well understood from both laboratory and numerical experiments (Blum, 2004; Henning et al., 2006), with typical relative velocities low enough for sticking. In general, sticking efficiencies tend to be high (10 – 100%) at small grain sizes $a \lesssim 1 \text{cm}$, and as relative velocities increase, the aggregates can compact and runaway growth is possible. However, for $a \gtrsim 1 \text{cm}$ sticking becomes less efficient and growth slows.
Moreover, short range van der Waals interactions only explain the sticking of individual grains at speeds \(< 1\text{ms}^{-1}\) \((\text{Dominik and Tielens, 1997})\), with grain collisions destructive at speeds in excess of \(10\text{ms}^{-1}\) \((\text{Dominik et al., 2007})\). There is also a bouncing barrier between these two limits \((\text{Güttler et al., 2010; Zsom et al., 2010, Windmark et al., 2012a})\). For \(a \gtrsim 1\text{m}\), relative velocities are in excess of \(50\text{ms}^{-1}\), above the destruction threshold \((\text{Wurm et al., 2005})\).

Thus the relative velocities of the particles regulate both collision rates and sticking properties. They are driven by the differential coupling to the disc gas with respect to particle diameter \(a\) \((\text{Weidenschilling and Cuzzi, 1993; Beckwith et al., 2000, Section 3.3.2})\). Collision speeds increase monotonically with \(a\) until decoupling occurs. Grain growth therefore cannot be fully understood without an understanding of the interaction with the gas velocity field, especially turbulence.

For example, \text{Weidenschilling (1984)} found that including turbulence initially increases coagulation rates, with aggregates quickly reaching \(a \approx 0.1 - 1\text{cm}\), before subsequent erosion, fragmentation and collisional bouncing prevents growth to larger sizes. Turbulent stirring also hinders dust settling to the midplane \((\text{Section 3.3.3})\). Furthermore, \text{Nelson and Gressel (2010)} found that MRI turbulence could seed destructive collisions between dust grains.

Coagulation models of grain growth in PP discs find the sticking process to be extremely efficient with rapid growth from \(a \approx 0.1\mu\text{m}\) to \(a \approx 1\text{mm}\) on a coagulation timescale \((\tau_{\text{coag}} \approx 10^3 - 10^4\text{yr})\) within a few AU of the star \((\text{Blum and Wurm, 2008; Zsom et al., 2010})\). This is in agreement with grain properties observed in early disc evolution \((\text{e.g. Kessler-Silacci et al., 2007})\).

One problem with these coagulation-only models is they predict similarly rapid growth beyond \(a \approx 1\text{cm}\) which is inconsistent with observations. Furthermore, the sticking process alone is so efficient that all particles with \(a \lesssim 100\mu\text{m}\) are cleared from the disc in \(\approx 10^4\text{yr}\), contrary to the presence of \(\mu\text{m}\)-sized dust observed in older discs \((\text{Dullemond and Dominik, 2005})\). Small grains must therefore be replenished by some mechanism. The continual collisional fragmentation of agglomerates \((\text{Dullemond and Dominik, 2008})\) or more inefficient sticking is a way to maintain the observed population of small dust to large times \((\text{Brauer et al., 2008; Birnstiel et al., 2011})\). However, the work of \text{Garaud et al. (2013)} finds that it is also possible to produce populations of both small and large particles by including both motion due to dust-gas interaction (midplane settling and radial drift) and stochastic motion (from Brownian motion and turbulence) without needing to overly rely on fragmentation.

The scenario whereby small particles stick to a few ‘runaway’ bodies does not work either. The collision of small grains (tightly coupled to the sub-Keplerian gas) with large solids \((\text{largely decoupled so orbiting at approximately } v_K)\) will face the full brunt of a headwind at speeds comparable to a sandblaster\(^3\) which causes rapid radial drift as well as no net growth. This is covered in more detail in Section 3.3.4.

\(^3\)Eugene Chiang is to thank for this great analogy, \text{Youdin (2010)}.
3.3 Planetesimal formation

In conclusion:

“The direct formation of kilometer-sized planetesimals cannot (yet?) be understood via sticking collisions.” – Blum and Wurm (2008)

3.3.2 Aerodynamic drag

We’ve found that considering grain growth independent of gas dynamics in the disc gives predictions inconsistent with observations. We must therefore try to understand the coupling between the gas and dust, necessary not only for grain growth but the vertical distribution and radial motion of dust and planetesimals. It is also not a one-sided arrangement; large concentrations of dust will also have an impact on the dynamics of the gas.

We will begin by considering a fundamental quantity of the gas; its mean free path $\gamma_{\text{mfp}}$. This characteristic lengthscale is the average distance travelled by a gas particle before encountering another. For the MMSN, Cuzzi et al. (1993) finds the expression

$$\gamma_{\text{mfp}} \approx \left( \frac{r}{1\text{AU}} \right)^{11/4} \text{cm.} \quad (3.3.2)$$

A good ballpark figure is $\gamma_{\text{mfp}} \approx 1\text{m}$ for typical disc conditions in the planet forming region.

This is important as the drag force exerted by the gas on a particle contained within it depends on the size of the particle $a$ compared to $\gamma_{\text{mfp}}$ (Whipple, 1972). If $a \ll \gamma_{\text{mfp}}$ then drag is the result of random collisions of gas molecules with dust particles, known as the Epstein regime (Epstein, 1924). At the other end of the scale, if $a \gg \gamma_{\text{mfp}}$ then particles feel the gas as a fluid and experience drag through the wake it creates. This is known as the Stokes regime, and drag is dependent on the Reynolds number, $\text{Re}^4$.

Consider spherical grains of radius $a$, particle density $\rho_\bullet$, particle mass $m_\bullet = 4\pi \rho_\bullet a^3/3$ and a macroscopic, fluid density $\rho_d$, moving with velocity $v_d$. Their velocity relative to that of the gas $v_g$ is therefore

$$u = v_d - v_g. \quad (3.3.3)$$

In the Epstein regime, the drag force on a particle is given by

$$F_{\text{drag}} = -\frac{m_\bullet}{\tau_s} u \quad (3.3.4)$$

where the dust friction/stopping time is

$$\tau_s = \frac{\rho_d a}{\rho_g c_s} \quad (3.3.5)$$

Gas density is denoted by $\rho_g$ and $c_s$ is again the sound speed. Though a precise value of $\tau_s$
The role of dust in PP discs depends on the size of the dust, typical conditions at the midplane, at 1 AU for 1µm dust gives \( \tau_s \approx 10 \, \text{s} \ll \tau_{\text{dyn}} \) (Papaloizou and Terquem, 2006), well coupled to the gas.

We define the Stokes number \( T_s \), a dimensionless stopping time

\[
T_s = \frac{\Omega \tau_s}{\tau_{\text{dyn}}}. \tag{3.3.6}
\]

Then, rewriting equation (3.3.5) using equation (2.4.13) for the disc scale height:

\[
a = \left( \frac{\rho_g}{\rho_d} \right) T_s H \simeq (100T_s) \, \text{cm} \Rightarrow T_s \simeq \left( \frac{a}{100\,\text{cm}} \right), \tag{3.3.7}
\]

for typical values for the MMSN at 5AU (Section 2.3.3, 2.3.5, Papaloizou and Terquem, 2006). In the Epstein regime where \( T_s \ll 1 \), the evolution of the dust particle distribution can be modelled as a pressureless fluid (Garaud and Lin, 2004).

Once \( a \geq 9\gamma_{\text{mfp}}/4 \), we move to the Stokes regime and the drag force takes the form

\[
\mathbf{F}_{\text{drag}} = -\frac{C_D}{2} \pi a^2 \rho_g u \mathbf{u}. \tag{3.3.8}
\]

The drag coefficient \( C_D \) depends on how aerodynamic the particle is. For spherical particles, as we are considering here, \( C_D \) only depends on the Reynolds number, with

\[
C_D \simeq \begin{cases} 
24\text{Re}^{-1} & \text{Re} < 1 \\
24\text{Re}^{-0.6} & 1 < \text{Re} < 800 \\
0.44 & \text{Re} > 800 
\end{cases} \tag{3.3.9}
\]

(Weidenschilling, 1977). For large spherical bodies, acceleration via gas drag scales with \( a^{-1} \) so it eventually becomes negligible once bodies of planetesimal size have formed. We find that the Stokes regime applies in the inner disc and Epstein in the outer regions, with the transition region for specific particle size, \( a \) at

\[
r_c(a) = \left( \frac{4}{9} \, \frac{a}{1\,\text{cm}} \right)^{4/11} \, \text{AU} \tag{3.3.10}
\]

(Cuzzi et al., 1993; Chavanis, 2000). In this transition region it is possible to use an interpolation between the two regimes (see Woitke and Helling, 2003 and Paardekooper, 2007). We find that the primordial \( a \approx 0.1 \mu\text{m} \) dust is effectively always in the Epstein regime while for particles \( a \approx 10 - 100\,\text{cm} \), the critical radius is \( 1.7\,\text{AU} < r_c < 3.9\,\text{AU} \). This is of particular importance when we consider the problem of radial drift in Section 3.3.4.
3.3 Planetesimal formation

3.3.3 Midplane settling

While growing, dust particles start to sediment due to the vertical component of the stellar gravitational field, creating an efficient feedback mechanism for accelerating collisional growth (e.g. Dullemond and Dominik, 2005; Schräpler and Henning, 2004).

As we’ve seen, small $a \simeq 0.1 \mu m$ grains have a large surface area to mass ratio and are therefore strongly coupled to the gas. They collide and coalesce, forming grains with a smaller surface area to mass ratio which experience greater gas drag and settle to the midplane. This is interrupted by turbulence causing some degree of vertical stirring and mixing of grains (Dullemond and Dominik, 2005).

Out of the plane of the disc, the gas is supported against gravity by a pressure gradient but this does not act on dust particles (Weidenschilling, 1977). Therefore, neglecting any turbulence in the disc for now, and considering small, tightly coupled dust particles we can assume that any relative vertical velocity in the fluid is the dust velocity, $u_z \simeq v_{z,d}$ since in the vertical direction the gas is approximately in equilibrium (Section 2.4.4).

Balancing the $z$-component of gravity due to the central star with the gas drag we find the dust particle’s terminal settling velocity to be $v_{\text{settle}} = \tau_s \Omega^2 z$ (equation (2.4.11)), implying a settling timescale of

$$\tau_{\text{settle}} = \frac{z}{v_{\text{settle}}} \simeq \frac{1}{\Omega^2 \rho_{d} a} = \frac{1}{\Omega T_s},$$

in terms of the Stoke’s number $T_s$. A more careful analysis (Youdin, 2010) gives

$$\tau_{\text{settle}} \simeq \frac{1 + 2 T_s^2}{\Omega T_s}.$$  

If $z \sim H$, we find a good first approximation at 1AU of

$$\tau_{\text{settle}} \simeq \begin{cases} 
10^5 \text{yr} & a \simeq 1 \mu m \\
10^4 \text{yr} & a \simeq 0.1 - 1 \text{cm} \\
10^3 \text{yr} & a \simeq 0.1 - 1 \text{m}, 
\end{cases}$$

uniformly short compared to the disc lifetime, $\tau_{\text{disc}}$ (Chiang and Goldreich, 1997). Note that this is strictly the minimum time to settle since vertical mixing by turbulence can hold particles aloft.

Therefore, in absence the of turbulence we expect micron-sized dust to rapidly sediment out of the upper layers of the disc. Since $\tau_{\text{settle}} \propto \rho_g$, settling will be faster at high $z$ where gas is less dense (Section 2.4.4), and as $\tau_{\text{settle}} \propto (\rho_d a)^{-1}$, any grain growth will hasten the settling process.

In reality, small scale turbulence strongly reduces sedimentation (Weidenschilling, 1980; Cuzzi et al., 1993; Dubrulle et al., 1995; Johansen and Klahr, 2005; Carballido et al., 2005;
The role of dust in PP discs

Turner et al., 2006 and Cuzzi et al., 2008) or can prevent settling occurring at all; Weidenschilling (1984) and Supulver and Lin (2000) found that particles $a \sim 0.1 - 1 \text{cm}$ were too small to settle in the presence of turbulence. Porous or fractal particles, which we would expect to find as a result of coagulation, can also slow settling (Blum and Wurm, 2008; Zsom et al., 2010).

3.3.4 Radial drift

The fact that dust particles do not experience the same pressure forces as the gas has an even more profound effect on the radial dynamics of solids (Weidenschilling, 1977). As we found in Section 2.4.5, gas in the disc orbits the star at a slightly sub-Keplerian velocity due to the radial pressure gradient, with

$$v_{\phi,g} = v_K(1 - \eta),$$

(3.3.14)

where

$$\eta \equiv -\frac{1}{r\Omega^2 \rho_g} \frac{\partial P}{\partial r} \approx \left(\frac{H}{r}\right)^2 \sim 10^{-3}$$

(3.3.15)

(Takeuchi and Lin, 2002, 2005). In the planet forming region, the $T_s \ll 1$, Epstein, limit corresponds to $a \lesssim 1 \mu m$, dust well coupled to the gas. Particles in this regime migrate slowly inwards at the same speed as the gas.

In the opposite Stokes regime limit $T_s \gg 1$ (corresponding to $a \gtrsim 1 \text{km}$), the bodies have so much inertia that they barely experience the headwind and thus only migrate inwards very slowly. Between these two limits we have $T_s = 1$, where $a \approx \gamma_{\text{mfp}} \approx 1 \text{m}$.

Following the analysis of Adachi et al. (1976) and Weidenschilling (1977) and the notation of Garaud and Lin (2004) and Youdin (2010) we find the particle drift speed

$$v_{r,d} = -\frac{2T_s \eta v_K}{1 + T_s^2},$$

(3.3.16)

so a particle with Stokes number $T_s$ experiences a headwind of

$$-\frac{\eta v_K}{1 + T_s^2}$$

(3.3.17)

This implies a headwind $-\eta v_K/2$ and maximum inward drift speed of $-\eta v_K$ for a $T_s = 1$ particle. This appears small, but the Keplerian velocity at 1AU from a $M_* = M_\odot$ star is 30kms$^{-1}$. We therefore expect $a \approx 1 \text{m}$ particles and 1AU moving at $v_K$ to experience a headwind $\sim 100 \text{ms}^{-1}$. It’s the sandblaster analogy again.

Figure 3.3 shows the inward radial drift velocity $v_{r,d}$ as a function of $T_s$ and highlights this problematic $T_s \simeq 1$ region. Note that $T_s$ corresponds to different sized particles at different distances from the star: smaller solids drift faster in the outer disc, while particles $a \simeq 1 \text{m}$ drift the fastest at 1AU.
3.3 Planetesimal formation

Figure 3.3 Dust drift velocity as a function of stopping time $T_s = \tau_s \Omega$, for two different Shakura-Sunyaev $\alpha$, from the review paper of Alexander (2008). This clearly shows the radial drift problem for $T_s \simeq 1$ particles, which will spiral into the central star in $< 100 \text{yr}$.

We therefore find a drift timescale of

$$\tau_{\text{drift}} = \frac{r}{v_{r,d}} = \frac{r(1 + T_s^2)}{2T_s \eta v_K} = \frac{1 + T_s^2}{2\eta T_s \Omega}. \quad (3.3.18)$$

While the size of the fastest migrating particles varies with $r$, the peak drift speed does not (Youdin, 2010):

$$\tau_{\text{drift, max}} \equiv \frac{r}{\max(v_{\text{drift}})} \simeq 85(r/\text{AU})\text{yr}. \quad (3.3.19)$$

In summary:

$$\tau_{\text{drift}} \sim \begin{cases} < 100 \tau_{\text{dyn}} & \text{if } a \simeq 1 \text{m } (T_s \simeq 1) \\ > \tau_{\text{disc}} & \text{if } a \lesssim 1 \mu\text{m or } a \gtrsim 1 \text{km } (T_s \ll 1 \text{ or } T_s \gg 1), \end{cases} \quad (3.3.20)$$

with the shortest drift timescale at 1AU $\lesssim 100 \text{yr}$. These particles are not safe even if the mass of dust at the midplane is sufficient to dominate the gas dynamics (so $v_{\phi,g} \simeq v_K$). In this case, collective drag on the surface of the subdisc layer then dominates and $\tau_{\text{drift}} \sim 10^3 \text{yr}$ for $a \simeq 1 \text{m}$ at 1AU (Goldreich and Ward, 1973), not much better.

Therefore, growth through the $a \simeq 0.01 - 1 \text{m}$ scale must be exceptionally rapid else solid
material will drift towards the star and be destroyed. To bypass this barrier would need dust settling and growth on a timescale $\tau_{\text{cong}}, \tau_{\text{settle}} < \tau_{\text{drift}}$ (Youdin, 2010): quite a serious constraint.

3.4 The Safronov–Goldreich–Ward mechanism

As we have found, there are two major stumbling blocks in the process of creating planetesimals from micron-sized dust. The first is that due to large relative velocities, fragmentation makes the creation of bodies $a \gtrsim 1m$ almost impossible under normal disc conditions. The second is that the headwind experienced by these same bodies around 1AU quickly scrubs off angular momentum and will rapidly deposit them into the central star. The combination of these two problems is known as the metre gap and have been a major hurdle in planet formation theory for some years.

So far we’ve made the two assumptions that dust particles are unimportant for the evolution of the gas disc and that the only important reactions between particles are pairwise collisions. However, these assumptions are only valid if the dust particles are both small and distributed uniformly throughout the disc. The Safronov–Goldreich–Ward (hereafter SGW) mechanism, proposed independently by Safronov (1969) and Goldreich and Ward (1973), has the gravitational instability (GI) of a dust-rich subdisc in the midplane as a mechanism for rapidly producing planetesimals to bypass the metre gap. It was also initially thought it could bypass the complicated and problematic process of dust sticking altogether:

"...the fate of planetary accretion no longer appears to hinge on the stickiness of the surface of dust particles" – Goldreich and Ward (1973)

though as we shall see, both processes are necessary. The mechanism can be outlined thus:

1. There is initially a well-mixed disc of gas and dust with a high Toomre parameter $Q$ (see Section 3.4.1). Effects of self-gravity have no role in disc evolution.

2. The dust settles to midplane, with some collisional growth assumed to have occurred to overcome the effect of turbulent stirring (Section 3.3.3). Inward radial drift contributes to midplane particle density in inner disc (Youdin and Shu, 2002).

3. High dust surface density $\Sigma_{d}$ and/or low velocity dispersion results in the dust-rich subdisc having $Q < 1$ and becoming gravitationally unstable. This leads to the formation of bound clumps of particles which quickly agglomerate to planetesimals, decouple from the gas disc and therefore do not suffer rapid radial drift.

We remark that this is quite inaccurately named since the precise size of particles that form this barrier depends on local disc properties. However, we will continue to use it regardless since it does apply to roughly meter-sized bodies in the planet forming region of the disc. It’s also quite a snappy name.
Initially considered to be the silver bullet of planetesimal formation, the SGW process soon ran into difficulties.

### 3.4.1 Disc self-gravity and the Toomre $Q$ parameter

As discs become more massive one can no longer neglect the effect of self-gravity when determining the gravitational potential $\Phi$. This effect was first studied by Toomre (1964) in the case of a stellar disc\(^6\) who found that as a disc becomes more massive, there is a tendency for its own self-gravity to result in the formation of overdense clumps.

A disc with surface density $\Sigma$ is unstable to the effect of self-gravity if the timescale for gravitational collapse is shorter than the timescales on which sound waves can cross a clump or shear destroys it, since both pressure forces and shear tend to resist clump formation. For axisymmetric perturbations, this can be realised as

$$\frac{c_s \Omega}{\pi G \Sigma} < 1,$$

where $Q = \frac{c_s \Omega}{\pi G \Sigma}$ is the Toomre $Q$ parameter\(^7\). Generalising to include nonaxisymmetric modes produces additional numerical factors which mean that instability sets in more easily (i.e. at a higher $Q$), with the development of spiral arms rather than axisymmetric rings. Therefore instability occurs when $Q \lesssim Q_{\text{crit}}$, where $Q_{\text{crit}} \sim O(1)$ still.

Since $H = \frac{c_s}{\Omega}$ (Section 2.4.4) and estimating the disc mass as $M_{\text{disc}} \sim \pi r^2 \Sigma$ this instability condition can be written as

$$\frac{M_{\text{disc}}}{M_*} \gtrsim \frac{H}{r}. \quad (3.4.2)$$

It is necessary to work out the enhancement we need in the gas disc density, while maintaining the same disc thickness, $H$ to get GI; this density enhancement is due to increasing the surface density of solids in the gas. We find we require a dust-to-gas ratio of around

$$\Gamma_d \simeq 20 - 100 \quad (3.4.3)$$

for GI, though the precise value of necessary enhancement remains uncertain (Youdin and Shu, 2002; Gómez and Ostriker, 2005; Johansen et al., 2009; Shi and Chiang, 2013).

Observations of Class II YSOs have found in general that their discs – protoplanetary discs – have $Q \gg 1$ (Andrews et al., 2010; Isella et al., 2009), though this is not true of early epoch discs (Section 2.3.9, Hartmann et al., 1993).

\(^6\)See Binney and Tremaine (2011) and Bertin (2000) for treatment and derivation of the fluid case and Appendix B of Chavanis (2000) for a turbulent, rotating disc.

\(^7\)For a derivation see e.g. Armitage (2010).
3.4.2 Self-excited turbulence in the dust subdisc

As we found in Sections 3.3.3 and 3.3.4, in the absence of turbulence, particles would settle to the midplane to arbitrarily high densities in times $< \tau_{\text{drift}}$, making the onset of GI trivial (Youdin, 2010). However, fully turbulent regions of the disc prevent settling to these densities (Youdin and Shu, 2002).

Furthermore, Weidenschilling (1980) showed that dust at these high densities would dominate the flow in the dust subdisc, carrying the gas at nearly Keplerian velocities. This would result in a vertical velocity shear between the midplane layer and the slower moving gas above and below it. This shear drives the Kelvin-Helmholtz instability between the layers (KHI, Kelvin, 1871; Helmholtz, 1868).

This self-excited, dust-driven turbulence results in a mixing of the dust and gas, preventing settling for reasonable values of $\Gamma_d$ (Cuzzi et al., 1993; Sekiya, 1998; Sekiya and Ishitsu, 2000). However, if $\Gamma_d$ is enhanced above the canonical, ISM value of $\sim 1\%$, GI could occur before the shearing instability disrupts the layer (Garaud and Lin, 2004), though it is unclear why this would occur.

In the absence of rotation, the hydrodynamic stability of a stratified shearing flow against KHI is usually assessed using the Richardson number (Chandrasekhar, 1961; Miles, 1961; Howard, 1961):

$$\text{Ri} \equiv g_z \frac{\partial \log \rho}{\partial z} \left( \frac{\partial v_\phi}{\partial z} \right)^{-2} = N^2 \left( \frac{\partial v_\phi}{\partial z} \right)^{-2}, \quad (3.4.4)$$

where $g_z$ is the vertical gravitational acceleration, $\rho = \rho_g + \rho_d$ and $N^2$ is the Brunt-Väisälä frequency. For a purely Cartesian flow with no rotational forces, instability occurs when $\text{Ri} < 0.25$. Naively applying this criterion to a PP disc, turbulence will disrupt the subdisc long before $Q \simeq 1$ (Supulver and Lin, 2000; Weidenschilling, 1980, 2003).

However, the fluid in PP discs is not just affected by this vertical shear due to the subdisc – rotational forces and radial shear act on it too. It is therefore not clear that $\text{Ri} < 0.25$ is sufficient for instability. For example, Gómez and Ostriker (2005) and Johansen et al. (2006a) included the Coriolis force (but not radial shear) in models of the dust subdisc and found that the onset of KHI occurs at a value of $\text{Ri}$ more than an order of magnitude greater than the traditional value of 0.25. It therefore occurs for a thicker dust subdisc, meaning GI is even more difficult to achieve, although large $\text{Ri}$ dust layers could be destabilised by sufficient cooling (Garaud and Lin, 2004). Inclusion of the radial shear (as well as Coriolis and centrifugal forces) by Chiang (2008) muddied the waters further, finding that the Richardson number alone does not determine stability, with the critical $\text{Ri}$ dependent on $\Gamma_d$. The prospects of SGW robustly bypassing the metre gap are not looking good.

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8For a derivation see Li et al. (2003) or Drazin and Reid (2004).
3.4 The Safronov–Goldreich–Ward mechanism

3.4.3 SGW despite turbulence

A couple of global mechanisms could contribute to $\Gamma_d$ enhancement; gas loss by photoevaporation (Throop and Bally, 2005) and, perversely, radial drift; the $r$-dependence of the radial drift velocity could result in the global redistribution and concentration of small dust as it drifts inwards and piles up at smaller $r$ (Youdin and Shu, 2002).

However, we conclude that globally in a PP disc it is very difficult to achieve the $\Gamma_d$ in the subdisc such that $Q < 1$. However, it is possible that locally this may not be the case. Firstly, KHI can only stir finite amounts of solids (Sekiya, 1998; Youdin and Shu, 2002). Secondly, while turbulence diffuses particles on average, it can also create intermittent particle clumps and the enhanced self-gravity of such an overdense clump could lead to GI. The two-fluid streaming instability (SI) caused by radial and azimuthal drift (Goodman and Pindor, 2000; Youdin and Goodman, 2005; Johansen and Youdin, 2007; Youdin and Johansen, 2007) is found to be a strong particle clumping mechanism, with particle overdensities generating perturbations to the drag force which sustain the instability.

Recall from Section 3.3.1 that $\tau_{\text{collide}} \propto \Sigma_d^{-1}$ so the likelihood of coagulation will increase if particle density in enhanced. Growth (and thus potential particle overdensities) can therefore be made more efficient if dust can be trapped in locally overdense regions of the disc. Consider the radial equation of motion of the gas from Section 2.4.5:

$$\frac{v_{r,g}^2}{r} = \frac{GM_*}{r} + \frac{1}{\rho_g} \frac{\partial P}{\partial r}$$  \hspace{1cm} (3.4.5)

The presence of the pressure gradient term implies that if gas has some local structure (e.g. spiral density waves, vortices), it will orbit at slightly enhanced velocities at the inner edge of pressure enhancement. This means that dust coupled to the gas in this region will experience a reduced headwind. Therefore, dust will migrate outwards towards local pressure maxima (Whipple, 1972). The real beauty of this process is that the very particles that suffer the most severe radial drift are also the most susceptible to this behaviour.

Confirming this phenomena, Rice et al. (2004, 2006b) find that spiral density waves in self-gravitating discs can lead to strong density enhancements of $a \approx 1$ m bodies in spiral arms. Similar results are found Durisen et al. (2005), with accelerated particle growth in dense rings. However, these both require the disc to be globally gravitationally unstable with $Q < 1$ – as we’ve seen, this is unlikely in the PP disc phase. Pressure maxima near the snow line (Kretke and Lin, 2007) or at the edge of the dead zone (Dzyurkevich et al., 2010) could also be suitable global locations, as could the zonal flows produced by MRI turbulence (Johansen et al., 2009) on a more intermediate scale.

The presence of a planet in a disc could produce similar behaviour, generating spiral density waves that creates a pressure maximum next to the gap opened up by that planet (Paardekooper and Mellema, 2004, 2006). These particular local pressure maxima prove to be
The role of dust in PP discs

a ‘size sorting’ mechanism in discs, as the $T_s \simeq 1$ grains tend to clump near the local maximum while well coupled, smaller dust moves with the gas and larger bodies orbit independently (Rice et al., 2006a; Garaud, 2007; Alexander and Armitage, 2007). However, this obviously requires a planet to already exist – even if a single planet can seed many others, this initial body needs to come from somewhere.

3.5 Beyond SGW: Vortices

Finally, another form of localised pressure maxima are disc vortices. We expect them to be common in PP discs (Dowling and Spiegel, 1990; Abramowicz et al., 1992; Adams and Watkins, 1995), with the observations of coherent vortices in rotating, turbulent fluids existing for many decades (e.g. Hopfinger et al., 1982). We will discuss how vortices can be produced in circumstellar discs in Section 3.5.2.

Based on Descartes (1644) and the writings of Kant, von Weizsäcker (1944) and Alfven and Arrhenius (1976) were the first modern works to suggest the role of vortices in the formation of the Solar System, with the PP disc organised into a highly organised (and implausible) system of vortices where matter could interact and condense. Their role in planet formation theory was then largely ignored until Adams and Watkins (1995), Barge and Sommeria (1995) and Tanga et al. (1996) all independently proposed that the trapping of solids in vortices in PP discs enhances particle growth to make the SGW mechanism locally viable (see also Fromang et al., 2006; Bodo et al., 2007; Mamatsashvili and Rice, 2009). It has been found to be possible to create up to $a \simeq 1$km bodies in pressure maxima by just coagulation and no GI (Brauer et al., 2008), but on timescales that could be too slow ($\simeq 1000$yr, Lyra et al., 2009). Therefore, the growth of larger particles and collapse via GI should be thought of as complementary processes.

The approximately 2D nature of PP discs is of crucial importance here; in 3D, eddies are quickly damped by an energy cascade towards small scales while 2D turbulence persists without energy dissipation – instead they form larger and larger vortices until a steady, solitary vortex is formed.

Also, Barge and Sommeria (1995) found that small vortices with radial extent $R \ll H$ with typical vorticity $\Omega$ have velocity $v \sim \Omega R \ll c_s = \Omega H$ (Chavanis, 2000) and are therefore approximately incompressible, avoiding strong shocks and large density wave losses. They also appear to persist for large times after a period of vortex merging to form single, coherent structures. This growth desists when $R \simeq H$ (so $\mathcal{M} \simeq 1$), beyond which energy losses by sound waves become problematic (Barge and Sommeria, 1995).

As we saw in Section 3.1.2 and Figures 3.1 and 3.2, there is also an observational justification for studying these objects since they can both trap dust and produce spiral density

\[9\text{e.g. Jupiter’s persistent red spot (Ingersoll, 1990).}\]
waves, both of which can now be observed thanks to the resolution afforded to large arrays of telescopes such as ALMA.

### 3.5.1 Dust trapping

In a shearing flow like that in a PP disc, cyclonic (vorticity, $\omega > 0$) vortices are rapidly elongated and destroyed by the shear, while anticyclones ($\omega < 0$) persist for long times until they are ultimately destroyed by viscosity or instability. Furthermore, these two different structures have different effects on any dust particles suspended in the fluid.

In a cyclone, both centrifugal and Coriolis forces are positive and push particles outwards, while the Coriolis force in anticyclonic vortices pushes dust inwards. If the vortex rotates rapidly, the centrifugal force dominates and particles are expelled while the opposite is true if it is slowly rotating (i.e. if it is a weak vortex).

This can also be seen in terms of the pressure distribution inside the vortex; particles travel up a pressure gradient towards local maxima but are driven away from vortex cores with a central minimum. Particles do not feel the pressure gradient directly (as discussed in Section 3.3.2) - they circulate inside the vortex in epicycles with the Keplerian orbital frequency. However, the rotation frequency of a pressure–supported vortex is smaller than the Keplerian epicyclic frequency so these particles experience an headwind and spiral towards the centre of the vortex. This is entirely analogous to radial drift in the disc (Section 3.3.4).

Furthermore, the numerics of Bracco et al. (1999) and Godon and Livio (1999a,b, 2000) support this notion of particle capture, which find very efficient capture and concentration inside weak anticyclones.

The effect on potential planetesimal formation is three-fold. Firstly, the concentration of density may enhance collision rates by more than an order of magnitude (Bracco et al., 1999; Godon and Livio, 2000; Johansen et al., 2004; Klahr and Bodenheimer, 2006; Heng and Kenyon, 2010), enhancing grain growth (recall $\tau_{\text{collide}} \propto \Sigma_d^{-1}$, Section 3.3.1). Secondly, the larger $\Gamma_d$ could trigger gravitational instabilities, as is seen in Lyra et al. (2009). Thirdly, the local pressure perturbation of the self-gravitating structure will lead to density enhancement that will migrate significantly slower or not at all (Johansen et al., 2006b). Therefore the growing particles could remain trapped in vortices until they are large enough to decouple when the characteristic stopping time $T_s \gg 1$. At this point the radial drift is negligible and we avoid the catastrophic inspiral of $T_s \simeq 1$ planetesimals into the central star.

Considering these forces acting on particles in a vortex, Chavanis (2000) established the details of the dynamics of dust trapping. He found a capture timescale $\tau_{\text{capt}}$, the characteristic time for a particle of a certain size to become trapped in the closed streamlines of a vortex. In the ‘light’ particle limit $\tau_s \to 0$, where dust is perfectly coupled to the gas, $\tau_{\text{capt}} \propto (\Omega T_s)^{-1}$. These particles remain in epicycles close to the vortex boundary, or can diffuse away from vortices due to turbulent fluctuations. In addition, the capture time for primordial, well-
The role of dust in PP discs

<table>
<thead>
<tr>
<th>Particle type</th>
<th>Characteristic</th>
<th>Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>$T_s \ll 1$, $\tau_s \ll \tau_{\text{dyn}}$</td>
<td>Grains are strongly coupled to gas so follow its streamlines. Stops at epicycles close to the vortex edge.</td>
</tr>
<tr>
<td>Intermediate</td>
<td>$T_s \simeq 1$, $\tau_s \simeq \tau_{\text{dyn}}$</td>
<td>Optimal – grains can reach deeper epicycles close to the vortex core.</td>
</tr>
<tr>
<td>Heavy</td>
<td>$T_s \gg 1$, $\tau_s \gg \tau_{\text{dyn}}$</td>
<td>Can take a long time to connect with vortex epicycle and can even escape as their motion nearly unaffected by the vortex.</td>
</tr>
</tbody>
</table>

Table 3.1 Vortical dust trapping of different types of particles (Chavanis, 2000).

coupled 1µm dust is $> \tau_{\text{disc}}$. ‘Light’ dust is therefore not strongly concentrated. From this we can deduce that some degree of particle sticking and growth is necessary before vortex capture.

In the opposite ‘heavy’ particle limit $\tau_s \to \infty$, the associated timescale is $\tau_{\text{capt}} \propto \tau_s$. These heavy particles can even pass through a vortex without being captured by it as their momentum is large enough they are barely effected by the pressure perturbation. The optimal stopping time for capture is, unsurprisingly, $T_s \simeq 1$ (Chavanis, 2000; Godon and Livio, 2000; Youdin and Goodman, 2005), the same particles that settle the fastest and are most susceptible to radial drift (Section 3.3.4). Chavanis (2000) found that the capture time is minimised for vortices with aspect ratio $\chi \simeq 4$, where

$$\tau_{\text{capt}}^{\text{min}} = \frac{8}{3\Omega} = O(\tau_{\text{dyn}}).$$  (3.5.1)

Table 3.1 summarises dynamics of dust trapping for different Stokes number $T_s$.

Due to the two different aerodynamic regimes in the disc (Section 3.3.2), there are two regions in a PP disc where $\tau_{\text{capt}}$ is minimised (Chavanis, 2000). Also recall that the size of particle for which $T_s \simeq 1$ depends on location in the disc. The optimum size for trapping is 1–50cm in the planet forming region around 1AU. Building on this, Heng and Kenyon (2010) found that the annulus where vortex capture is most favourable decreases in width and strength with time. This known as ‘vortex ageing’, where more evolved discs prefer to capture smaller dust.

Considering the turbulent diffusion of particles, Chavanis (2000) finds that these particles also form the maximum dust concentrations and it is theoretically possible to get sufficient density enhancements $\Gamma_d \simeq 100$ for particles with this $T_s$, i.e. particles in the 1–50cm range. This is without the need for an initially large $\Gamma_d$, which gives it an advantage over streaming instabilities which require $\Gamma_d \simeq O(1)$ near midplane to seed GI.

Finally, also note that vortices do not have the vertical shear that causes the KHI in the
subdisc (Klahr and Bodenheimer, 2006) because of the presence of a pressure extremum. Thus sedimentation in vortices is more efficient (Lyra et al., 2009).

Therefore, at certain disc radii, for sufficiently large dust, we conclude that vortices can ‘rehabilitate’ the SGW mechanism.

3.5.2 Producing vortices in protoplanetary discs

We have established that vortices are potentially very useful for particle concentration. However, this is academic if there are no mechanisms for producing these structures in PP discs.

Taking the curl of the momentum equation (2.4.2) and defining vortensity $\omega/\rho$, we have a 2D vortensity evolution equation where, in absence of magnetic fields and negligible viscosity:

$$\frac{D}{Dt} \left( \frac{\omega}{\rho} \right) = 0 \frac{\omega}{\rho} \cdot \nabla v + \frac{1}{\rho^2} \nabla \rho \times \nabla P,$$

where $\rho$ is surface density. If the source term on the right hand side vanishes, as it does for a barotropic flow $P = P(\rho)$, any vortensity is just advected with the fluid flow. Thus total vorticity is advectively conserved and is it difficult to both create and destroy10.

Using the ideal gas equation $P = \rho R T$, with $R$ the ideal gas constant, we find that the source term is $\propto \nabla \rho \times \nabla T$. We found in the previous chapter that, in general, there is a uniform, radial, temperature distribution in the disc due to stellar irradiation (Sections 2.3.7 and 2.4.2) so we expect PP discs to be approximately locally barotropic (Adams and Watkins, 1995). Sources of vorticity will therefore arise from any asymmetries in density $\rho$.

Turbulence in the PP disc has long been proposed as a means of angular momentum transport (Pringle, 1981). In stratified PP discs there are special locations where flow is both turbulent and quasi–2D. Under these conditions, the fluid organises itself into coherent, long-lived vortices (i.e. lasting many $\tau_{\text{dyn}}$) without the need for special initial conditions (Carnevale et al., 1991; Weiss and McWilliams, 1993; Tabeling, 2002). This is because the ‘vortex stretching’ term is absent in 2D, allowing an inverse cascade of energy. Indeed, random initial velocity fields applied to a Keplerian shearing flow were found to produce large scale, long-lived anticyclones by Bracco et al. (1999); Godon and Livio (1999a,b, 2000).

Furthermore, sheared, rotating fluids support Rossby waves (Rossby, 1945; Dickinson, 1978). The Rossby Wave Instability is an edge mode instability, similar to KHI, which converts excess shear into vorticity, resulting in large scale vortices. It has a long history in astrophysical contexts (Lovelace and Hohlfeld, 1978; Toomre, 1981; Papaloizou and Pringle, 1984, 1985; Hawley, 1987) and first applied to thin accretion discs by Lovelace et al. (1999) and Li et al. (2000).

This led Varnière and Tagger (2006) to suggest that the Rossby wave instability (hereafter RWI) could be at work in the transition between MRI-active and MRI dead zones (Gammie,
to form dust-trapping disc vortices. This is supported by the work of Inaba and Barge (2006); Méheut et al. (2012b) who produced enhanced particle growth through the accumulation of dust in RWI generated, self-sustained vortices.

With a sharp enough transition region, RWI leads to the formation of multiple vortices, vortex merging and finally a single, large structure (Inaba and Barge, 2006; Li et al., 2001) which is columnar and extends throughout the vertical extent of the disc (Richard et al., 2013). The 3D simulations of barotropic, stratified discs by Méheut et al. (2010, 2012a) again found strong and persistent RWI-generated vortices with interesting meridional circulation patterns that reinforce the need to model these vortical structures in 3D (Méheut et al., 2010; Méheut et al., 2012b). Despite this, instability in three dimensions was investigated by Umurhan (2010), Méheut et al. (2012c), Lin (2012) and Lin (2014), finding that RWI is a nominally 2D instability, with negligible difference between growth rates in 2D and 3D calculations.

As touched on in Section 3.4.3, the pressure maxima at planetary gap edges are seen to excite vortices (de Val-Borro et al., 2006; Rice et al., 2006a), later explained in terms of the RWI (de Val-Borro et al., 2007; Li et al., 2009; Lyra et al., 2009; Lin and Papaloizou, 2010; Lin and Papaloizou, 2011). This is the proposed explanation for the lopsided dust distribution seen in the transitional disc around Oph IRS 48 (van der Marel et al., 2013). Zhu et al. (2014) again found that vortices of this nature behaved in an intrinsically 2D manner and were able to concentrate a wide range of grain sizes $0.02 < T_s < 20$. These gap edges could also act to filtrate and trap dust in the outer disc (Zhu et al., 2012; Pinilla et al., 2012), further increasing $\Sigma_d$ and $\Gamma_d$.

The Zombie Vortex Instability of Marcus et al. (2013), whereby vortices in stratified shear flows self-replicate, could be another source of vortex production. However, the specificity of conditions to produce it could limit its viability.

### 3.5.3 Vortex lifetime

Vortices in 2D PP discs have been found to be stable, persistent structures (Godon and Livio, 1999b; Umurhan and Regev, 2004; Johnson and Gammie, 2005), with Davis et al. (2000); Inaba and Barge (2006); Lyra et al. (2009) finding that in the absence of viscosity, 2D vortices can persist indefinitely.

However, in 3D, vortices are generally subject to hydrodynamic instabilities that may destroy them on a timescale $< \tau_v$. Close to the disc midplane the flow exhibits more 3D structure and vortices appear to be destroyed rather quickly (Shen et al., 2006). However, Barranco and Marcus (2005) found that they have a better survival time $z > 2H$ away from the midplane, where $\rho$ drops rapidly with $z$ and the fluid behaves more 2D. This also suggests the possibility that vertical stratification has a stabilising effect. However, off-midplane vortices are not observed in PP disc simulations (Chiang, 2008; Johansen et al., 2009) so producing them may be problematic.
3.5 Beyond SGW: Vortices

Investigating columnar PP disc vortices, Lithwick (2009) found that vortices live indefinitely when their vertical extent is less than their length $\Delta z \lesssim \Delta y$. In light of this, the Barranco and Marcus (2005) result can then be understood because the local scale height $H$ is reduced above midplane. It was also found that weak vortices with aspect ratios $\chi \gg 1$ can survive for long times because they are stabilised by rotation and behave as Taylor–Proudman columns (Proudman, 1916; Taylor, 1917). The Taylor–Proudman theorem states that when a solid body (in this situation a vortex at or near the midplane) is moved within a fluid that is being rotated with angular velocity $\Omega$ (with $\Omega$ large in comparison to the movement of the body) the resulting fluid velocity will be uniform along any line parallel to the axis of rotation. When considered in an isothermal, 3D box, 2D vortices would therefore act like column vortices over the height of the box.

3.5.4 Vortex instabilities

The periodic nature of circulation round a vortex means that these structures are subject to the elliptical instability. This instability is a form of parametric resonance observed when the background flow follows closed streamlines, with the instability, crucially, localised on these streamlines. Laboratory and numerical experiments both show it is a robust mechanism, leading to flow becoming highly complicated and full of small scale disorder. It is invariably 3D, with linear growth rates that scale with the shear, $S$ (Kerswell, 2002).

The form most relevant to astrophysical context was first presented in the numerical work of Pierrehumbert (1986) on an unbounded, strained vortex. This was followed by the analytical work of Bayly (1986) and Craik and Criminale (1986), who performed Floquet analysis on disturbances taking the form of Kelvin waves (Kelvin, 1880),

$$\{u(x,t), p(x,t)\} = \{\hat{u}(t), \hat{p}(t)\} \exp[i\mathbf{k}(t) \cdot \mathbf{x}]. \tag{3.5.3}$$

The periodic nature of the coefficients appearing in the differential equations governing $\{\hat{u}(t), \hat{p}(t)\}$ means this is a Floquet problem (Floquet, 1883), related to the Hill and Mathieu equations (Hill, 1886; Mathieu, 1868).

The effect of background rotation, relevant to our rotating disc, was investigated by Craik (1989), who found that anticyclonic, elliptical flows can be stable for some rotation rates. Furthermore, Miyazaki and Fukumoto (1992), found that growth rates of the elliptical instability were reduced by stable exponential stratification in the $\hat{z}$ direction, albeit not working in a rotating frame.

The elliptical instability however need not completely destroy our PP disc vortices. After vortices develop bursts of 3D turbulence in their cores, the subcritical baroclinic instability (SBI, Lesur and Papaloizou, 2010) offers a mechanism for amplifying the vortices, allowing them to survive as weaker eddies. In this way, the SBI can produce large scale vortices up to
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<table>
<thead>
<tr>
<th>Model</th>
<th>Streamlines</th>
<th>Vorticity</th>
<th>$\Omega_v$</th>
<th>Validity</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kida</td>
<td>Elliptical</td>
<td>$\frac{3\Omega(\chi^2 + 1)}{2\chi(\chi - 1)}$</td>
<td>$\frac{3\Omega}{2(\chi - 1)}$</td>
<td>$\chi &gt; 1$</td>
<td>Valid with Keplerian background shear</td>
</tr>
<tr>
<td>GNG</td>
<td>Elliptical</td>
<td>$-\frac{\sqrt{3}\Omega(\chi^2 + 1)}{\chi\sqrt{\chi^2 - 1}}$</td>
<td>$\Omega\sqrt{\frac{3}{\chi^2 - 1}}$</td>
<td>$\chi &gt; 2$</td>
<td>Zero pressure gradient along streamlines</td>
</tr>
<tr>
<td>Point</td>
<td>Circular</td>
<td>$\kappa \delta(r)$</td>
<td>$\frac{\kappa}{2\pi r}$</td>
<td>$\chi = 1$</td>
<td>Irrotational, not valid with shearing background</td>
</tr>
</tbody>
</table>

Table 3.2 Summary of vortex models

a size of order $H$ in the radial direction.

Since dust is driven into centre of vortices by drag forces, diffusion is needed to maintain a steady state over the lifetime of the disc (Klahr and Henning, 1997; Chavanis, 2000). In this context, the ratio of the relative strengths of drag and diffusion becomes important (Cuzzi et al., 1993; Dubrulle et al., 1995; Lyra and Lin, 2013). This is expressed as $T_s/\delta$, where $\delta$ is the turbulence in the vortex\(^{11}\), where decoupling of dust and gas becomes important when $T_s/\delta > 1$. Note also that this ratio increases with both $t$ and $r$, and with decreasing mass accretion rate $\dot{M}$ (Jacquet et al., 2012), so we expect the optimally coupled $T_s \simeq 1$ dust to be ejected from vortices at late epochs and far away from the central star. In addition, the presence of turbulence in vortex cores will act to diffuse out concentrations of dust, potentially leading to insufficient particle concentrations required to seed GI.

3.5.5 Existing vortex models and stability calculations

Most analyses of individual vortices in a shearing background begin with either a Kida (1981) or Goodman–Narayan–Goldreich (GNG, Goodman et al., 1987) vortex model.

Based on the rotating elliptical vortex solution of Kirchhoff (1876), and related to the Moore–Saffman vortex (see Moore and Saffman, 1971, Neu, 1984 and Saffman, 1995), the Kida solution finds an analytical solution to an elliptical, constant-density patch of uniform vorticity in a background Keplerian shearing flow, $S = 3\Omega/2$. Its streamlines are concentric ellipses of constant aspect ratio $\chi$ and its power as a model solution stems from these well-defined elliptical streamlines which allow for a degree of analytical stability analysis. It is also useful as an initial condition, with Lin and Papaloizou (2011) finding that RWI vortices excited at planetary gap edges resemble vortices formed by perturbing the disc with the Kida solution.

\(^{11}\)i.e. not $\alpha_v$, the Shakura–Sunyaev $\alpha$ from Section 2.4.6.
This model is described in detail in Section 5.3.

A similar model is the polytropic GNG solution (with their \( \varepsilon = \chi^{-1} \)) which exactly solves the compressible Euler equations with shear

\[
S = \sqrt{\frac{3(\chi - 1)}{\chi + 1}} \Omega. \tag{3.5.4}
\]

This is also an attractive model as it has a more straightforward pressure distribution than the Kida solution; its pressure gradient is zero along streamlines. (For the Kida solution this is only the case when \( \chi = 7 \).)

We discuss these two models in more detail in Sections 5.3.2 and 5.4. Their properties are summarised in Table 3.2, along with that for the classical point vortex solution (Batchelor, 2000, pages 93–96).

Lyra and Lin (2013) extended the Kida solution to include vortex-trapped dust in a transitional disc, solving for the dust distribution in steady state between gas drag (driving dust inwards in certain cases, Section 3.5.1) and diffusion, which acts to expel it. With the fluid constrained to elliptical streamlines of constant \( \chi \), like the Kida and GNG vortices, they find a Gaussian for the dust density \( \rho_d \) with standard deviation \( H_v \), the ‘dust vortex scale height’. This scale height is a function of gas scale height \( H \), \( \chi \) and \( T_s/\delta \).

They also found that it was safe to assume that dust is approximately axisymmetric along streamlines, with only small non-axisymmetric corrections. They derived a density enhancement of \( \rho_{d,\text{max}} = \Gamma_d \rho_0 (T_s/\delta + 1)^{3/2} \) and thus a constraint on the strength of turbulence, \( \delta \), in the vortex core. A similar relationship to \( T_s \) is found in a much larger scale vortex in Birnstiel et al. (2013).

With regard to analysing the linear stability of vortex models, at this point it is useful to look in detail at two previous approaches to this problem; the work of Lesur and Papaloizou (2009) (Section 3.5.5.1) and Chang and Oishi (2010) (Section 3.5.5.2).

### 3.5.5.1 The approach of Lesur and Papaloizou (2009)

With the existence, numerous methods of production and important dust-trapping property of vortices in PP discs long established, there is a need for a more in-depth understanding of the stability of these structures.

Lesur and Papaloizou (2009) approached this problem by producing simple, steady 2D vortices based on Kida vortices. A Floquet analysis of the linearised equations governing 3D perturbations in the vortex core was then performed. This was then solved numerically, with analytical expressions existing for the horizontal and vertical perturbation limits. They found a small, linearly stable regime for Kida vortices with aspect ratio \( 4 < \chi \lesssim 5.9 \).

This was then investigated in 3D for both stratified and unstratified models. Vertical stratification was found to suppress but not eliminate the elliptical instability for weak (large
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χ) vortices (see their Figure 6). They also found that results obtained from incompressible models were unaffected by introduction of moderate compressibility. In general, elliptical (and in general, parametric) instabilities were found to be always present, but often involve small radial wavelength compared to $H$ and small growth rates compared to $\tau_{\text{dyn}}$ so were hard to pinpoint numerically.

### 3.5.5.2 The approach of Chang and Oishi (2010)

This paper aimed to build upon the work of Lesur and Papaloizou (2009) by including the influence of a density gradient; the main shortcoming of the Kida (1981) solution in this context is that it applies to a constant density fluid.

Chang and Oishi (2010) perturb around both the Kida and GNG solution with an imposed density profile given by

$$\frac{\partial \log \rho}{\partial \log b} = c, \Rightarrow \rho = \rho_0 b^c,$$

(3.5.5)

where $b > 0$ is the semi-minor axis of the elliptical vortex patch and $\rho_0, c$ are constants. They find an instability, which they call the ‘Heavy Core Instability’ (HCI), for cases with $c < 0$, so vortex density gradients are destabilising for vortices with sufficiently heavy cores. Instability occurs if $\Gamma_d \gtrsim 1.2$. They therefore expect that the HCI instability will disrupt these vortices long before seeding the GI (which requires $\Gamma_d \simeq 20 - 100$, Section 3.4.1). However, it is important to note that these solutions do not correctly match the background flow as they just assumed Kida streamlines with an arbitrary density superposed.

They base their perturbation wavenumber on the Kida wavenumber of Lesur and Papaloizou (2009)

$$k = k_0 \left( \cos(\omega t + \phi_{k,0}), -\chi^{-1} \sin(\omega t + \phi_{k,0}) \right),$$

(3.5.6)

(so $k_z = 0$, considering only perturbations in the plane of the vortex), acting on streamlines

$$x = b(\cos(\omega t + \phi_0), -\chi \sin(\omega t + \phi_0)),$$

(3.5.7)

However, this form of perturbation wavenumber in equation (3.5.6) applies to the Kida case only and not the generic equilibrium. A cause for concern is their final stability equations contain explicit dependence on the phase of $k$, i.e.

$$\frac{d \tilde{\rho}}{dt} = -a(b, t) \sin(\phi_{k,0} - \phi_0) \frac{\partial \rho}{\partial b},$$

(3.5.8)

where $\tilde{\rho} \propto \rho'$, the Eulerian perturbation of $\rho$. This 2D instability has also, as yet, not been observed in any numerical simulations. They outline a potential pseudospectral numerical method for searching for this instability in Oishi and Chang (2013), but as of yet no results are forthcoming.
3.6 Summary and conclusions

A summary of important timescales for disc and dust evolution is given in Table 3.3.

In this chapter, we have found that dust dominates the observations of discs so there is plenty of evidence for its existence, general structure and evolution. There are also an increasing amount of observations of dust asymmetries in discs such as rings (Figure 2.1) and vortices (Figures 3.1, 3.2). This observational evidence reinforces the need to study these structures and suggests that dust–laden vortices could be stable, or quasi–stable, under certain conditions.

Furthermore, grain growth is relatively a well understood and robust mechanism up to about $a \simeq 1\text{cm}$ in size, where it is limited by destructive collisions and bouncing. Fragmentation is required to sustain the population of small dust grains observed at late times but also means that growth beyond $a \simeq 1\text{m}$ is next to impossible by sticking alone. At the other extreme, growth from planetesimals to terrestrial and gas giant planets is a well understood process, though still not a straightforward one.

Aerodynamic drag couples dust particles to the gas in the disc, while the action of the star’s gravity causes dust to settle vertically in the disc, a process disrupted by turbulence. Disparities between the mean angular velocity of the gas and dust due to pressure gradients causes an inward radial drift of grains and this inward motion can be very fast: $< 100\text{yr}$ for $T_s \simeq 1$ particles. This, combined with the wholly destructive collisions between $a \simeq 1\text{m}$ bodies, is a barrier to planetesimal formation known as the ‘metre gap’. There is therefore a real need for a robust mechanism for rapid growth from metre–sized objects to planetesimals.

The Safronov–Goldreich–Ward mechanism proposes that the gravitational instability of a self–gravitating disc can collapse directly from dusty gas to planetesimals. However, it appears unlikely that, globally at least, the dust–to–gas ratio $\Gamma_d$ required to seed gravitational collapse can be achieved before self–generated KHI disrupts the layer. This implies a more local approach is necessary.

Dust–trapping, anticyclonic vortices can bridge the gap between sticking processes and planetesimal collisions in the planet formation process. For certain grain sizes, capture by vortices is fast ($\simeq \tau_{\text{dyn}}$ for $T_s \simeq 1$, $a \simeq 1 – 50\text{cm}$ dust around 1AU) and the consequence of this is twofold; particles coagulate faster in the higher $\Sigma_d$ while the enhanced $\Gamma_d$ could make the SGW mechanism locally viable. Meanwhile, the initial sticking of particles to sizes $a \gtrsim 1\text{cm}$ is an indispensable step in this process as these grains are preferentially trapped within the necessary timescales. Furthermore, there are numerous ways of producing vortices in PP discs (MRI, RWI, gap edges) so we can assume they are commonplace objects. Despite this, vortices are subject to potentially destructive instabilities, so their lifetime with and without the presence of dust is unclear. Therefore, further study on the stability of these structures in Keplerian shearing flows is required.

Finally, existing vortex solutions with background shear are based around elliptical streamlines and can, to a certain extent, include density enhancements. These solutions are neces-
sarily periodic and are therefore subject to elliptical, parametric instabilities. These can be investigated using Floquet analysis of the perturbed equations, as in the work of Lesur and Papaloizou (2009) and Chang and Oishi (2010). We will attempt to build upon this work to form a more complete understanding of the stability of more general vortices.
### Table 3.3 Important timescales for PP discs.

<table>
<thead>
<tr>
<th>Timescale</th>
<th>Symbol</th>
<th>Scaling</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>dynamical, orbital, shearing</td>
<td>$\tau_{\text{dyn}}$</td>
<td>$\Omega^{-1}$</td>
<td>Time to reach centrifugal equilibrium; $\sim r^{3/2}$ for Keplerian discs. Since $\Omega^{-1} = H/c_s$, this is also the vertical sound crossing time, or the time to reach hydrostatic balance in the vertical direction. Also the growth time for MRI and gravitational instability.</td>
</tr>
<tr>
<td>thermal</td>
<td>$\tau_{\text{th}}$</td>
<td>$\frac{\tau_{\text{dyn}}}{\alpha}$</td>
<td>Time for disc to modify thermal structure; balance of $\tau_{\text{cool}}$ from cooling processes and $\tau_{\text{heat}}$ from energy release from accretion.</td>
</tr>
<tr>
<td>viscous</td>
<td>$\tau_{\nu}$</td>
<td>$\frac{r^2}{\nu} \sim \alpha^{-1} \left(\frac{H}{r}\right)^{-2}\tau_{\text{dyn}}$</td>
<td>Timescale on which matter diffuses through the disc under the effect of viscous torques. Sets scale for evolution of surface density. Around 1Myr for Solar-type stars.</td>
</tr>
<tr>
<td>disc formation</td>
<td>–</td>
<td>$\sim 10^4 - 10^5\text{yr}$</td>
<td>e.g. Shu et al. (1993)</td>
</tr>
<tr>
<td>disc lifetime</td>
<td>$\tau_{\text{disc}}$</td>
<td>few $\times 10^6\text{yr}$</td>
<td>There is large scatter in this value (e.g. Hillenbrand, 2005).</td>
</tr>
<tr>
<td>disc dispersal</td>
<td>–</td>
<td>$10^5\text{yr}$</td>
<td>Lifetime of transitional discs; time take for photoevaporation to clear a disc of gas.</td>
</tr>
<tr>
<td>coagulation timescale</td>
<td>$\tau_{\text{coag}}$</td>
<td>$\tau_{\text{coag}} \gtrsim 10^3\text{yr}$</td>
<td>Timescale for growth from $a \simeq 0.1\mu\text{m} - 1\text{mm}$ by sticking processes.</td>
</tr>
<tr>
<td>collisional timescale</td>
<td>$\tau_{\text{collide}}$</td>
<td>$\simeq 0.005 - 5\text{yr}$</td>
<td>Typical time between dust-dust collisions.</td>
</tr>
<tr>
<td>Process</td>
<td>Symbol</td>
<td>Formula</td>
<td>Description</td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>--------</td>
<td>----------------------------------------------</td>
<td>---------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Frictional dust stopping time</td>
<td>$\tau_s$</td>
<td>$\frac{\rho_d a}{\rho g c_s}$</td>
<td>Typical time for a particle with initial velocity different to the gas to stop in a frame moving with the gas velocity. $\tau_s \simeq 3s$ for 1$\mu$m particle at 1AU in disc midplane.</td>
</tr>
<tr>
<td>Dimensionless dust stopping time</td>
<td>$T_s$</td>
<td>$\tau_s \Omega$</td>
<td>Particles with $T_s \simeq 1$ are most susceptible to radial drift and vortex capture.</td>
</tr>
<tr>
<td>Sedimentation, dust settling</td>
<td>$\tau_{\text{settle}}$</td>
<td>$\frac{1 + 2T_s^2}{T_s \Omega}$</td>
<td>Time for dust to settle to midplane due to aerodynamic drag. $\tau_{\text{settle}} \sim 10^5\text{yr} (a \simeq 1\mu\text{m}) \sim 10^3\text{yr} (a \simeq 0.1 - 1\text{m})$ at 1AU from solar mass star.</td>
</tr>
<tr>
<td>Radial drift, orbital decay</td>
<td>$\tau_{\text{drift}}$</td>
<td>$\frac{1 + T_s^2}{2\eta T_s \Omega} \sim \frac{\tau_{\text{settle}}}{\eta}$</td>
<td>Typical time for particle to spiral into the central star, $\tau_{\text{drift}} \lesssim 100\tau_{\text{dyn}} (a \simeq 1\text{m})$, $&gt;\tau_{\text{disc}} (a \lesssim 1\mu\text{m}, a \gtrsim 1\text{km})$.</td>
</tr>
<tr>
<td>Dust capture timescale</td>
<td>$\tau_{\text{capt}}$</td>
<td>$\tau_{\text{capt}}^{\min} \sim \tau_{\text{dyn}}$</td>
<td>Typical time for a coherent vortex to capture dust particles.</td>
</tr>
</tbody>
</table>
Chapter 4

Research design

We have established it is challenging to produce the necessary planetesimals in protoplanetary discs, from which to build planets. It is difficult because of both timescale constraints (radial drift, a relatively short $\tau_{\text{disc}}$) and the actual physical mechanisms of growth themselves (coagulation/fragmentation and collapse via gravitational instability).

Vortices appear to be a promising solution, but despite their prevalence both in global numerical simulations and now observationally, little is known about their stability in this context. The analysis of Kida vortices in Lesur and Papaloizou (2009) is a solid base from which to generalise.

Therefore, the questions we would like to tackle in this thesis are:

(i) What happens when vortex streamlines are not elliptical, as they are in the Kida case?
(ii) How do non-constant profiles of vorticity and density affect stability?
(iii) Can the ‘heavy-core’ instability of Chang and Oishi (2010) be reproduced?
(iv) Does the stability of these more general models have any bearing in 3D?

With problems like these there are broadly two approaches; a ‘top down’ or ‘bottom up’ approach. The former involves large, global models including as much physics as possible. In this case, that could be a fluid with an $N$-body model for the dust grains, perhaps including coagulation, fragmentation, self-gravity, and radiative effects. Good examples of this approach in this context is the work of Lyra et al. (2009), with $N$-body dust, or Zhu and Stone (2014), which includes MHD effects.

On the other hand, one can isolate different features of a problem and try to understand the underlying processes governing them and how these depend on the problem’s parameters. In this way, one hopes to get a global view of the problem and to be able to come up with some generic features. This is difficult with the ‘kitchen sink’ approach, where there are a large number of parameters and a complexity limiting consideration to only a few models on a case by case basis. This work therefore predominantly falls into the second camp.
Our approach to trying to answer the above questions is to firstly formulate a set of local, equilibrium vortices in a shearing background, including more generalised vorticity and density profiles. We establish our equations of motion and general models in Chapter 5 while in Chapter 6 we demonstrate how we implement these numerically to find steady vortex solutions. Secondly, we will analyse the stability of these models to perturbations with a general wavenumber $\mathbf{k}$, with the stability analysis given in Chapter 7 and its numerical implementation in Chapter 8. In Chapter 9 we will then investigate the relative lifetimes of different columnar vortices in 3D using the hydrodynamical code PLUTO and try to gain some insight into how the instabilities manifest themselves in 3D. Finally, we draw conclusions to what this means in the context of planetesimal formation in PP discs in Chapter 10.
Chapter 5

Calculating equilibrium solutions

As highlighted in the previous chapter, a comprehensive stability analysis of a large variety of vortex configurations is necessary to investigate the instabilities that occur within them, since these could be a threat to their survival or dust-attracting ability. In order to do this, we must first develop a system for calculating equilibrium vortices which we can then investigate the stability of.

In this chapter we give the basic equations governing the fluid model we use in Section 5.1.1, its limitations and constraints. We also show how we parametrise different vortex configurations based on their vorticity and density profiles (Section 5.2). In Section 5.3 we then present the well-known analytical Kida solution which we use to verify our method and form the starting point of our stability analysis. Finally, in Section 5.4 we discuss a polytropic model that can be used to consider solutions in a non-Keplerian background flow and in a Keplerian background flow for the special value of the vortex aspect ratio $\chi = 7$.

5.1 Governing equations for well–coupled dust

We begin by considering a two–fluid model of the dust and gas circulating in a protoplanetary accretion disc. In the Epstein regime it is useful to treat dust particles collectively and describe them using a two–fluid dust and gas flow where the two fluids have different flow velocities and consequently exchange momentum through drag forces (Section 3.3.2, Cuzzi et al., 1993; Garaud and Lin, 2004).

5.1.1 Basic equations for the two–fluid model

In order to do this we must make two key assumptions. Firstly, we will assume particles are of a single size, $a$. This is so Epstein drag is constant across the fluid - else we will need a multi-fluid model and any analytic progress is impossible. Secondly, we will assume dust is collisionless, i.e. that the grains do not directly interact and consequently their size remains
Calculating equilibrium solutions

constant. As we saw in Section 3.3 these assumptions are reasonable if

1. The frequency or sticking probability of collisions is sufficiently small such that \( \tau_{\text{coag}} \gg \tau_{\text{settle}}, \tau_{\text{drift}} \), i.e. growth occurs on timescales longer than orbital changes due to interaction with the gas. This is reasonable if we consider particles with \( a \lesssim 1\text{cm} \) (Section 3.3, Table 3.3).

2. We work in a region of disc are not exposed to intense external radiation and the particles are charge neutral. We can then neglect interaction with the magnetic field and global disc radiation, i.e. there is no MRI. This is satisfied in opaque planet forming regions \( 0.1 \lesssim r \lesssim 5\text{AU} \) near the midplane (Section 2.3.8). We can therefore also neglect Lorentz forces.

3. The disc is sufficiently cold to be well below sublimation temperature. We can then neglect molecular phase changes and grains losing or gaining mass by evaporation or condensation. This is satisfied away from the snow line, which is \(< 5\text{AU} \) in at least some protoplanetary phases (Section 2.3.7).

The basic equations for the gas component are those of continuity and momentum conservation. We consider flow in a frame rotating with angular velocity \( \Omega = \Omega \hat{z} \), with \( \hat{z} \) being the unit vector in the fixed direction of rotation (here called the vertical direction - see Figure 5.1) and \( \Omega \) corresponding to the magnitude the angular velocity of Keplerian rotation at some radius. These take the form:

\[
\frac{\partial \rho_g}{\partial t} + \nabla \cdot (\rho_g v_g) = 0 \tag{5.1.1a}
\]
\[
\frac{\partial v_g}{\partial t} + (v_g \cdot \nabla)v_g + 2\Omega \times v_g + \Omega \times (\Omega \times r) = -\frac{1}{\rho_g} \nabla P - \nabla \Phi\text{gr} + \frac{f_v}{\rho_g} - \frac{\rho_d}{\rho_g} \frac{F_{\text{drag}}}{m_*} \tag{5.1.1b}
\]

Here, \( P \) is the gas pressure, \( \rho_g \) is the gas density, \( \rho_d \) the dust density, \( \Phi\text{gr} \) is the gravitational potential due to the central mass of the star, \( M_* \), and \( r \) being the position vector measured from the star. The gas velocity is \( v_g \) and \( f_v \) could be taken to be an anomalous viscous force per unit volume. This could be associated with stochastic forcing due to weak turbulence.

The drag force per unit mass acting on the gas is \(-\rho_d F_{\text{drag}}/\rho_g\), with the drag force acting on a single dust particle of mass \( m_* \) being \( F_{\text{drag}} \) (see Section 3.3.2).

Writing

\[
\Omega \times (\Omega \times r) = (\Omega \cdot r) \Omega - \Omega^2 r = -\Omega^2 r = -\nabla \left( \frac{1}{2} \Omega^2 r^2 \right) \tag{5.1.2}
\]
we incorporate the centrifugal term with the gravitational potential $\Phi_{gr}$ so $\Phi = \Phi_{gr} - \frac{1}{2}\Omega^2 r^2$. The momentum equation for the gas component is therefore:

\[
\frac{\partial \mathbf{v}_g}{\partial t} + (\mathbf{v}_g \cdot \nabla)\mathbf{v}_g + 2\Omega \times \mathbf{v}_g = -\frac{1}{\rho_g}\nabla P - \nabla \Phi + \frac{f_v}{\rho_g} - \frac{\rho_d}{\rho_g} \mathbf{F}_{\text{drag}} m_* .
\] (5.1.3)

The equations governing the evolution of the dust are that of a pressureless fluid:

\[
\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \mathbf{v}_d) = 0
\] (5.1.4a)

\[
\frac{\partial \mathbf{v}_d}{\partial t} + (\mathbf{v}_d \cdot \nabla)\mathbf{v}_d + 2\Omega \times \mathbf{v}_d = \nabla \Phi + \frac{\mathbf{F}_{\text{drag}}}{m_*}
\] (5.1.4b)

with $\mathbf{v}_d$ the velocity of the dust component. Note that the drag forces acting on the dust and gas are related such that the total momentum of the two components is conserved locally.

We saw in Section 3.3.2 that in the Epstein regime, the drag force $\mathbf{F}_{\text{drag}}$ acting on a single dust particle is given by equation (3.3.4):

\[
\mathbf{F}_{\text{drag}} = \frac{m_*}{\tau_s} (\mathbf{v}_g - \mathbf{v}_d) .
\]

This drag force is proportional to $\mathbf{v}_g - \mathbf{v}_d$, the relative velocity between the gas and dust components and the inverse of the stopping time $\tau_s = \rho_d a / \rho_g c_s$ (equation (3.3.5)).
In the limit $\tau_s \to 0$, the two fluid description reduces to that of a single combined fluid with density $\bar{\rho}$ and velocity $\bar{v}$. These are defined through

$$\bar{\rho} = \rho_g + \rho_d \quad \text{and} \quad \bar{\rho} \bar{v} = \rho_g \nu_g + \rho_d \nu_d. \quad (5.1.5)$$

The mean velocity of the dust with respect to the gas is

$$\nu = \nu_d - \nu_g. \quad (5.1.6)$$

We can then find

$$\nu_g = \bar{v} - \frac{\rho_d \nu}{\bar{\rho}}, \quad (5.1.7a)$$

$$\nu_d = \bar{v} + \frac{\rho_g \nu}{\bar{\rho}}. \quad (5.1.7b)$$

Adding the continuity equations for the gas and dust (5.1.1a and 5.1.4a) gives the continuity equation for the combined fluid in the form

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\bar{\rho} \bar{v}) = 0. \quad (5.1.8)$$

Taking a linear combination of equations (5.1.3) and (5.1.4b) that eliminates the drag force gives the momentum equation for the combined fluid. Assuming $\nu - \nu_d = \mathcal{O}(\tau_s)$ and neglecting terms of order $\tau_s^2$ and higher, this takes the form

$$\bar{\rho} \left( \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} + 2\Omega \times \bar{v} \right) = -\nabla P - \bar{\rho} \nabla \Phi + f_v. \quad (5.1.9)$$

We now use equation (5.1.4b) to find $\nu$. Consistent with neglecting contributions of order $\tau_s^2$, we may set $\nu_d = \bar{v}$ in the left hand side of equation (5.1.4b) and, recalling from equation (3.3.4) that $F_{\text{drag}} \propto \nu$, we readily obtain (correct to first order in $\tau_s$)

$$\nu = \frac{\tau_s}{\bar{\rho}} (\nabla P - f_v). \quad (5.1.10)$$

The first term on the right hand side of (5.1.10) gives rise to the drift caused by a pressure gradient which tends to lead to particles concentrating at pressure maxima. The second term that arises from the prescribed anomalous viscosity could be taken to originate from the presence of a low level of turbulence. Using equation (5.1.7b) to eliminate $\nu_d$ in equation (5.1.4a) we obtain

$$\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \bar{v}) + \nabla \cdot \left( \frac{\rho_d \rho_g \nu}{\bar{\rho}} \right) = 0. \quad (5.1.11)$$
Equations (5.1.9), (5.1.10) and (5.1.11) thus provide a description of the system, that is correct to first order in $\tau_s$, using only the mean flow variables and the dust density.

We consider vortices with small length scale, such that with reference to the sound speed in the gas, relative velocities are highly subsonic. Then we expect the gas to move incompressibly (Section 3.5). When the dust is tightly coupled to the gas as $\tau_s \to 0$ this will also move incompressibly.

In this limit, with $u$ is set to zero and we see that each of $\overline{\rho}$, $\rho_g$, and $\rho_d$ satisfy the same continuity equation. For a fluid moving incompressibly we have

$$\nabla \cdot \mathbf{v} = 0$$

(5.1.12)

and hence

$$\frac{\partial \overline{\rho}}{\partial t} + \mathbf{v} \cdot \nabla \overline{\rho} = 0.$$  (5.1.13)

The equations governing an incompressible gas and dust in the limit $\tau_s \to 0$ are seen to be (5.1.9), (5.1.12) and (5.1.13). These are seen to be identical to those for a single incompressible fluid with a variable density that is conserved on fluid elements.

### 5.1.3 Steady state solutions in the inviscid limit

In a steady state for which viscous forces may be neglected, the equation of motion (5.1.9) reduces to

$$\mathbf{v} \cdot \nabla \mathbf{v} + 2 \Omega \times \mathbf{v} = -\nabla P + \nabla \Phi.$$  (5.1.14)

We also make two further simplifications when considering a PP disc. Firstly, we will use the thin disc approximation from Section 2.4.2, so to first approximation we consider 2D, isothermal state. Secondly, we impose a low disc mass $M_{\text{disc}}$ compared to the mass of the star $M_*$, such that

$$\frac{M_{\text{disc}}}{M_*} < \frac{H}{r} \ll 1$$  (5.1.15)

so that it can be considered to be non-self-gravitating ($Q > 1$, Sections 2.3.3 and 3.4.1).

### 5.1.4 The shearing sheet approximation

In order to consider local steady state solutions within a 2D Keplerian disc in detail, we adopt a local shearing box with origin centred on a point of interest and rotating with its Keplerian angular velocity $\Omega(r_0) = \Omega_0$, where $r_0$ is the distance to the central star, as shown in Figure 5.2. Its symmetry can be seen in Figure 5.3. This is a commonly used local model of a differentially rotating disc, first developed by Goldreich and Lynden-Bell (1965) for the study of galactic discs. A discussion of how it is implemented numerically is given by Hawley et al. (1995) and its limitations in Regev and Umurhan (2008). Considering an arbitrary reference point with
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Figure 5.2 The shearing sheet model with \( x = r - r_0 \) and \( y = r_0 [\phi - (\phi_0 + \Omega_0 t)] \).

Figure 5.3 The shearing sheet is invariant to rotations by \( \pi \) around the \( z \)-axis; the sheet doesn’t know the value or sign of \( r_0 \), only \( \Omega_0 \) and \( S_0 \).

cylindrical coordinate \((r_0, \phi_0 + \Omega_0 t, 0)\), we use this as the origin of a local Cartesian coordinate system:

\[
\begin{align*}
    x &= r - r_0 \\
    y &= r_0 [\phi - (\phi_0 + \Omega_0 t)] \\
    z &= z
\end{align*}
\]  

(5.1.16a) 

(5.1.16b) 

(5.1.16c)

Orbital motion is represented locally as uniform rotation \( (\Omega_0) \) plus a linear shear flow \( -S_0 x \mathbf{e}_y \), where, as established in Section 2.4.3, the shear rate is given by equation (2.4.7):

\[
S(r) = -r \frac{d\Omega}{dr},
\]

\( S_0 = S(r_0) \) and in the Keplerian case \( S = 3/2\Omega \). The length scale associated with each dimension of the box can be taken to be the vertical scale height, which, in the thin disc approximation, is assumed to be \( H \ll r_0 \).
5.1.5 The effective potential

We want to find the effective potential \( \Phi \) (i.e. the combined centrifugal and gravitational \( \Phi_{gr} \) potentials) in this rotating frame as a function of our new variables. Applying our disc mass assumption from Section 5.1.3, the point mass gravitational potential in cylindrical coordinates around a central star of mass \( M_* \) is

\[
\Phi_{gr}(r, z) = -\frac{GM_*}{\sqrt{r^2 + z^2}}, \tag{5.1.17}
\]

which is both axisymmetric and symmetric about the \( z = 0 \) plane. Additionally, the angular velocity for a body moving in Keplerian motion at radius \( r \) is \( \Omega = (GM_*/r^3)^{1/2} \). Using \( r = r_0 + x \) we expand our effective potential \( \Phi = \Phi_{gr}(r, z) - \frac{1}{2} \Omega_0^2 r^2 \)

to second order in \( x \) and \( z \):

\[
\Phi = \Phi_{gr}(r_0, 0) + x \left. \frac{\partial \Phi_{gr}}{\partial r} \right|_{(r_0, 0)} + \frac{1}{2} x^2 \left. \frac{\partial^2 \Phi_{gr}}{\partial r^2} \right|_{(r_0, 0)} + \frac{1}{2} z^2 \left. \frac{\partial^2 \Phi_{gr}}{\partial z^2} \right|_{(r_0, 0)} - \frac{1}{2} \Omega_0^2 (r_0^2 + 2r_0x + x^2) + \cdots \tag{5.1.18}
\]

where \( \left. \partial \Phi_{gr}/\partial z \right|_{(r_0, 0)} = 0 \) due to the symmetry of \( \Phi_{gr} \) in the \( z \)-direction. The two terms \( \Phi_{gr}(r_0, 0) \) and \( \frac{1}{2} \Omega_0^2 r_0^2 \) are constant so can be ignored and we note that

\[
\left. \frac{1}{r_0} \frac{\partial \Phi_{gr}}{\partial r} \right|_{(r_0, 0)} = \frac{1}{r_0} \frac{GM_* r_0}{r_0} = \frac{GM_*}{r_0} = \Omega_0^2, \tag{5.1.19}
\]

so the \( x \) coefficients cancel. Hence our effective potential is

\[
\Phi = \frac{1}{2} x^2 \left[ \frac{\partial}{\partial r} (r \Omega^2) - \Omega^2 \right] \bigg|_{(r_0, 0)} + \frac{1}{2} z^2 \left[ \frac{GM_*}{r^3} \right] \bigg|_{(r_0, 0)} = \frac{1}{2} x^2 [ -2\Omega_0 S_0 ] + \frac{1}{2} z^2 \left[ \Omega_0^2 \right] = -\Omega_0 S_0 x^2 - \frac{1}{2} \Omega_0^2 z^2 = -\frac{1}{2} \Omega_0^2 (3x^2 - z^2), \tag{5.1.20}
\]

where we used equation (2.4.8) to eliminate \( S_0 \). From now on we will drop the subscript ‘0’ from \( \Omega \) and \( S \) since inside the shearing box these will only serve to confuse.

In the case of a 2D disc the fluid state variables are independent of \( z \) and the \( z \)-dependence of the effective potential \( \Phi \) is ignored. Although our final solutions will therefore not depend on \( z \), they may apply to horizontal planes of an isothermal disc for which hydrostatic equilibrium
Calculating equilibrium solutions

holds in the vertical direction (Lesur and Papaloizou, 2009). As we saw in Section 2.4.4, given the form for the effective potential $\Phi$ in equation (5.1.20) in the isothermal, $P = c_s^2 \rho$ case,

$$\rho \propto \exp \left( -\frac{\Omega^2 z^2}{2c_s^2} \right) = \exp \left( -\frac{z^2}{2H^2} \right).$$

(5.1.21)

The factor $\exp \left[ -\frac{z^2}{(2H^2)} \right]$ may be applied to the two dimensional solutions for $\rho$ and $P$ obtained from Equations (5.1.12)-(5.1.14). Then, when we restore the $z$-dependence to $\Phi$, hydrostatic equilibrium will hold in the $z$-direction.

The description of local solutions in vertical hydrostatic equilibrium has been found numerically to be applicable to vortices generated by the RWI (Section 3.5.2, e.g. Lin, 2012). Note too that in the limit of zero stopping time $\tau_s \to 0$ considered above, dust is frozen into the fluid so midplane settling (Section 3.3.3) does not occur.

5.1.6 The Stokes’ streamfunction

We look for solutions of equations (5.1.12)–(5.1.14) in the case of a 2D disc where the fluid state variables are independent of $z$. In order to satisfy the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ we introduce a Stokes streamfunction $\psi = \psi(x, y)$ such that

$$\mathbf{v} = \nabla \times (\psi \hat{z}) = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right).$$

(5.1.22)

The vorticity of the flow is therefore

$$\omega = \nabla \times \mathbf{v} = -\nabla^2 \psi \hat{z}. \quad \text{(5.1.23)}$$

For the undisturbed background Keplerian flow $\mathbf{v}_0 = (0, -Sx, 0)$, we have the streamfunction of the background, $\psi_0$:

$$\psi_0 = \frac{1}{2} Sx^2. \quad \text{(5.1.24)}$$

5.1.7 Working form of the momentum equation

For a steady state, equation (5.1.13) becomes

$$\mathbf{v} \cdot \nabla \rho = 0. \quad \text{(5.1.25)}$$

For a two-dimensional flow this implies that density is constant along streamlines and thus is a function of $\psi$ alone: $\rho = \rho(\psi)$.

This cannot be determined if $\tau_s = 0$ as it is then an invariant that must be input externally. However, when $\tau_s \neq 0$ but small and slow diffusive processes are included, this is no longer the case and it may be considered to be the result of evolutionary processes taking place on a long
5.2 Functions specifying the vorticity and density profiles

We begin by separating the solution corresponding to an undisturbed Keplerian background flow for which $\psi = \psi_0$. We write the streamfunction $\psi$ as the sum of a contribution from the time scale. This is discussed in detail in Section 5.2.2.

Using the identity

$$\mathbf{v} \cdot \nabla \mathbf{v} = \omega \times \mathbf{v} + \nabla \left( \frac{1}{2} |\mathbf{v}|^2 \right),$$

(5.1.26)

the steady momentum equation (5.1.14) can be written as

$$(2\Omega + \omega) \times \mathbf{v} = -\frac{P}{\rho^2} \nabla \rho - \nabla \left( \frac{P}{\rho} + \Phi + \frac{1}{2} |\mathbf{v}|^2 \right).$$

(5.1.27)

Expressing quantities in terms of $\psi$, equation (5.1.27) becomes

$$\left( -\nabla^2 \psi + 2\Omega + \frac{P}{\rho^2} \frac{d\rho}{d\psi} \right) \nabla \psi = -\nabla \left( \frac{P}{\rho} + \Phi + \frac{1}{2} |\nabla \psi|^2 \right) \equiv -\nabla F,$$

(5.1.28)

where we have now dropped the overbars for clarity. Since we are ignoring the $z$–dependence of $\Phi$ in equation (5.1.20) the combined potential in the shearing sheet is

$$\Phi = -\frac{3}{2} \Omega^2 x^2.$$

(5.1.29)

As both sides of equation (5.1.28) are proportional to $\nabla \psi$, it follows that $F = F(\psi)$. Similar to the density, in the absence of diffusive processes the arbitrary function $F(\psi)$ has to be input externally (see Section 5.2.2). In the absence of a density gradient, the derivative of $F(\psi)$ represents a conserved vorticity.

Equation (5.1.28) can then be written as a second order partial differential equation for the streamfunction $\psi$ thus:

$$\nabla^2 \psi = \frac{dF}{d\psi} + 2\Omega + \frac{P}{\rho^2} \frac{d\rho}{d\psi}.$$

(5.1.30)

Once $F(\psi)$ is specified, the pressure is expressed in terms of the streamfunction through the relation

$$\frac{P}{\rho} = -\Phi - \frac{1}{2} |\nabla \psi|^2 + F(\psi).$$

(5.1.31)

Solutions of equation (5.1.30) corresponding to local vortices with central dust concentrations may be sought once the arbitrary functions $\rho(\psi)$ and $F(\psi)$ and appropriate boundary conditions are specified. In this context, note that after making an appropriate adjustment to $F$, equation (5.1.28) is invariant to adding an arbitrary constant to $P$. 

5.2 Functions specifying the vorticity and density profiles

We begin by separating the solution corresponding to an undisturbed Keplerian background flow for which $\psi = \psi_0$. We write the streamfunction $\psi$ as the sum of a contribution from the
Calculating equilibrium solutions

background flow and a deviation $\psi_1$ that will correspond to a superposed vortex.

$$\psi = \psi_0 + \psi_1 = \frac{3}{4} \Omega x^2 + \psi_1.$$  \hspace{1cm} (5.2.1)

Consider the function $F(\psi)$ for the background flow:

$$F_0 = \frac{P}{\rho} \left| \frac{\Phi}{\rho} + \frac{1}{2} |\nabla \psi|^2 \right|$$

$$= \frac{P}{\rho} \left[ -\frac{3}{2} \Omega x^2 + \frac{1}{2} \cdot \frac{9}{4} \Omega^2 x^2 \right]$$

$$= \frac{P}{\rho} \left[-\frac{\Omega \psi^2}{2}\right].$$  \hspace{1cm} (5.2.2)

We now choose an arbitrary additive constant for $P$ such that the pressure is zero for the pure background flow (Section 5.1.7). Thus we set

$$F = -\frac{\Omega \psi^2}{2} + F_1,$$  \hspace{1cm} (5.2.3)

where $F_1$ vanishes for the background flow. Equation (5.1.30) then yields

$$\nabla^2 \psi_1 = \frac{dF_1}{d\psi} + \frac{P}{\rho} \frac{d\rho}{d\psi} = A(\psi) + \frac{P}{\rho} B(\psi),$$  \hspace{1cm} (5.2.4)

where

$$B(\psi) = \frac{d \log \rho}{d\psi}.$$  \hspace{1cm} (5.2.5)

5.2.1 The arbitrary source terms

We now denote $A(\psi)$ as the Bernoulli source term and $B(\psi)$ as the density source in the Poisson equation for the background flow, equation (5.2.4). These are both functions of the overall streamfunction $\psi$. As we are in the $\tau_s = 0$ limit, these functions are invariants that have to be specified (Section 5.2.2).

Physically these profiles are expected to be determined by prior evolution governed by effective viscosity, particle diffusion and friction, and hence they are constrained when they are included. However, as weak turbulence is likely to have an important role (Lyra and Lin, 2013), finding forms of $A(\psi)$ and $B(\psi)$ from evolutionary calculations of gas/dust systems that can be used to characterise steady state solutions with adequate resolution is not practical, particularly if local stability is to be considered.

We therefore consider specifications of $A(\psi)$ and $B(\psi)$ that enable a large class of steady state solutions with varying vorticity and density profiles to be considered. The Bernoulli and density sources are superposed on horizontal planes, on which there is a uniform background
Keplerian flow with density $\rho_0$. They are both non-zero only on streamlines that circulate interior to some bounding streamline $\psi_b$.

When we solve these equations numerically, we will specify the point where this streamline crosses the $y$–axis. The arbitrary unit of length is chosen so that this point is at $(0, 1)$, while the unit of time is chosen so that $\Omega = 1$ and we set $\rho_0 = 1$. From now on, the ignorable $z$–coordinate is suppressed.

We adopt power law functions for our sources of the form

$$A(\psi) = A|\psi - \psi_b|^\alpha \tag{5.2.6a}$$

$$\rho(\psi) - \rho_0 = B|\psi - \psi_b|^\beta. \tag{5.2.6b}$$

which will provide us with solutions covering a wide range of aspect ratios with a variety of vorticity and density profiles by varying $A$, $B$, $\alpha$ and $\beta$.

The functions $A(\psi)$ and $\rho(\psi) - \rho_0$ are set to be zero on streamlines exterior to those with $\psi = \psi_b$. The constants $A$ and $B$ are chosen to scale the total vorticity and relative mass excesses associated with the Bernoulli and mass sources respectively, while $\alpha$ and $\beta$ are constant indices. Note that $A > 0$ as this gives rise to an anticyclonic vortex and $B > 0$ to give a mass-loaded vortex. We specify $\rho(\psi)$ instead of $B(\psi)$ as this enables a more meaningful scaling later (Section 6.1.2).

When $\alpha = \beta = B = 0$ we obtain the analytical Kida solution (Section 3.5.5, Kida, 1981; Lesur and Papaloizou, 2009) which is a useful test case.

### 5.2.2 Constraints on the functions $F(\psi)$ and $\rho(\psi)$

When $\tau_s \rightarrow 0$, the functions $F(\psi)$ and $\rho(\psi)$ have to be input externally. However, when frictional processes are included, these functions are expected to be determined by the evolutionary history of the system; as we are studying steady state solutions we do not study this in this work.

However, we can show that there are two constraints on every streamline which must be satisfied and which in principle allow $F(\psi)$ and $\rho(\psi)$ to be determined if the stopping time $\tau_s$ (equation (3.3.5)) and the viscous force $f_\nu$ (equation (5.1.1b)) are known.

#### 5.2.2.1 Constraints on every streamline arising from dust diffusion

We begin with equation (5.1.11) for the dust density in the form

$$\frac{\partial \rho_d}{\partial t} + \nabla \cdot (\rho_d \mathbf{v}) + \nabla \cdot \left( \frac{\rho_d \rho_d \mathbf{u}}{\bar{\rho}} \right) = 0.$$
The dust-to-mass ratio, $\Gamma_d$, defined in Section 2.3.3 is related to $\rho_g$, $\rho_d$ and $\bar{\rho}$ thus:

$$
\rho_d = \Gamma_d \bar{\rho} \tag{5.2.7a}
$$

$$
\rho_g = (1 - \Gamma_d) \bar{\rho}. \tag{5.2.7b}
$$

Using equation (5.1.8) to eliminate the gradients of $\bar{\rho}$ and equation (5.1.10) to eliminate the relative velocity $\mathbf{u}$ we obtain the following equation for $\Gamma_d$:

$$
\bar{\rho} \left( \frac{\partial \Gamma_d}{\partial t} + \mathbf{v} \cdot \nabla \Gamma_d \right) + \nabla \cdot \left[ \Gamma_d (1 - \Gamma_d) \tau_s (\nabla P - f_\nu) \right] = 0. \tag{5.2.8}
$$

When turbulence is present there will be a stochastic component to $\mathbf{u}$ which is expected to lead to dust diffusion. Assuming this is due to the action of $f_\nu$, we model its effect over long times by introducing a diffusion term for $\Gamma_d$. Thus we replace equation (5.2.8) with

$$
\bar{\rho} \left( \frac{\partial \Gamma_d}{\partial t} + \mathbf{v} \cdot \nabla \Gamma_d \right) + \nabla \cdot \left[ \Gamma_d (1 - \Gamma_d) \tau_s \nabla P - \bar{\rho} D \nabla \Gamma_d \right] = 0. \tag{5.2.9}
$$

where the diffusion coefficient is given in terms of the root mean square of the turbulent velocity $\nu_t$ and the turbulent correlation time $\tau_{corr}$

$$
D = \langle \nu_t^2 \rangle \tau_{corr}. \tag{5.2.10}
$$

Equation (5.2.9) can be interpreted as an advection-diffusion equation for $\Gamma_d$. Note that when there is dependence on $z$ we could introduce an anisotropic diffusion coefficient to allow diffusion in the vertical direction to independently balance the tendency for the dust to settle due to the vertical pressure gradient (e.g. Dubrulle et al., 1995; Fromang et al., 2006, and references therein). This would remove derivatives with respect to $z$ from the steady state form of equation (5.2.9).

Using the mass conservation equation for the average density $\bar{\rho}$ (equation (5.1.8)) and repeated use of equation (5.2.7b), the steady state form of (5.2.9) is:

$$
\nabla \cdot \left[ \rho_d \mathbf{v} + \frac{\rho_g}{\bar{\rho}} \left( 1 - \frac{\rho_g}{\bar{\rho}} \right) \tau_s \nabla P + \bar{\rho} D \nabla \left( \frac{\rho_g}{\bar{\rho}} \right) \right] = 0. \tag{5.2.11}
$$

We integrate it over a cylindrical volume for which the boundary of the cylinder in the $(x, y)$ plane is a streamline. An arbitrary interval may be taken in the $z$–direction as there is no effective variation in $z$.

As the vertical surface is composed of streamlines, there is no contribution from the term involving the combined fluid velocity $\mathbf{\bar{v}} = \nabla \times \mathbf{\psi} \hat{z}$. With the $z$ integration performed along a
line of unit length, the resulting surface integral becomes the following contour integral:

\[
0 = \int_V \nabla \cdot \left[ \rho_g \mathbf{v} + \frac{\rho_g}{\bar{\rho}} \left( 1 - \frac{\rho_g}{\bar{\rho}} \right) \tau_s \nabla P + \bar{\rho} D \nabla \left( \frac{\rho_g}{\bar{\rho}} \right) \right] dV
\]

\[
= \oint_S \left[ \frac{\rho_g}{\bar{\rho}} \left( 1 - \frac{\rho_g}{\bar{\rho}} \right) \tau_s \nabla P + \bar{\rho} D \nabla \left( \frac{\rho_g}{\bar{\rho}} \right) \right] \cdot \frac{\nabla \psi}{|\nabla \psi|} ds
\]

\[
= \oint_S \left[ \left( 1 - \frac{\rho_g}{\bar{\rho}} \right) \tau_s \nabla P - \bar{D} \nabla \rho \right] \cdot \frac{\nabla \psi}{|\nabla \psi|} ds. \quad (5.2.12)
\]

To reach the last line we have used the fact that \( \rho_g \) does not vary in horizontal planes and \( \bar{\rho} = \bar{\rho}(\psi) \) so is constant round the line integral. Therefore, \( \rho_g/\bar{\rho} \) may be removed as a factor. Here \( ds \) is the line element on a streamline and we have used the fact that to lowest order in \( \tau_s \) and \( D \), the integral around any streamline can be calculated using solutions for \( \tau_s = 0 \).

Equation (5.2.12) can be viewed as expressing the balance between the rate of accumulation of dust driven towards a pressure maximum by the pressure gradient and outward diffusion. It gives a relation between \( D \) and other equilibrium quantities.

In addition, by considering the signs of the two terms comprising equation (5.2.12) we can see that this condition is consistent with there being a density maximum at the centre of the vortex if there is also a pressure maximum there and an inconsistency if there is a central pressure minimum. When the pressure has a saddle point at the centre of the vortex then a more detailed calculation is needed.

Also note that if \( D \) is assumed to be a function of \( \psi \) alone, it may be taken outside the integral and calculated directly. Alternatively, if \( D \) is specified in terms of other quantities, the constraint could be regarded as specifying the function \( d\bar{\rho}(\psi)/d\psi \). This can be seen by writing equation (5.2.12) in the form

\[
\frac{d\bar{\rho}}{d\psi} \oint_S D |\nabla \psi| ds = \oint_S \left( 1 - \frac{\rho_g}{\bar{\rho}} \right) \tau_s \nabla P \cdot \frac{\nabla \psi}{|\nabla \psi|} ds. \quad (5.2.13)
\]

However, as both the form of \( \psi \) and the right hand side of equation (5.2.13) depend on the choice of \( d\bar{\rho}(\psi)/d\psi \), there is an implicit dependence of these quantities on \( d\bar{\rho}(\psi)/d\psi \).

### 5.2.2.2 A constraint on every streamline arising from viscous diffusion

We return to the two-fluid momentum equation (5.1.9):

\[
\bar{\rho} \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + 2\Omega \times \mathbf{v} \right) = -\nabla P - \bar{\rho} \nabla \Phi + \mathbf{f}_v.
\]

We shall adopt the variable-density incompressible limit. It is possible to add correction terms \( \propto \tau_s \) to this, which would lead to additional contributions to the discussion below. For simplicity these will be neglected so that discussion applies to vortices with dust frozen into the fluid. Furthermore, for convenience we shall drop the overbars in equation (5.1.9). Assuming
Calculating equilibrium solutions

the standard form (Batchelor, 2000), the $i$ component of the viscous force per unit mass $f_\nu$ is taken to be

$$f_{\nu,i} = \frac{\partial}{\partial x_j} \left[ \rho g \nu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right], \quad (5.2.14)$$

where $\nu$ is the effective kinematic viscosity that we assume is independent of $z$ (or can be vertically averaged due to the thin disc approximation, Section 2.4.2). Defining the scaled viscosity $\tilde{\nu}$:

$$f_{\nu,i} = \frac{\partial}{\partial x_j} \left[ \rho \tilde{\nu} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right], \quad (5.2.15)$$

where, recalling equation (5.1.5), $\rho = \rho_g + \rho_d$:

$$\tilde{\nu} = \frac{\rho_g}{\rho_g + \rho_d} \nu. \quad (5.2.16)$$

Taking the scalar product of equation (5.1.9) with $v$ and assuming a steady state we obtain

$$\nabla \cdot \left[ v \left( \rho \Phi + \frac{1}{2} \rho |v|^2 + P \right) \right] = v \cdot f_\nu. \quad (5.2.17)$$

Now integrate this over a cylindrical volume, $V$, with a horizontal cross-sectional area $A$ that is bounded by a fixed streamline labelled by $\psi$. As we consider any steady states for which any $z$-dependence of the density can be factored out (see Section 5.1.5) we assume independence of $z$ so, again, the height of this cylinder is arbitrary.

Since the velocity is parallel to the bounding surfaces (by definition of $\psi$), the use of the divergence theorem shows that the left hand side of equation (5.2.17) gives no contribution to the integral. Thus we obtain

$$0 = \int_V v \cdot f_\nu \, dA = \int_S \rho \tilde{\nu} v_i \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) e_j |dS| - \frac{1}{2} \int_V \rho \tilde{\nu} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \, dA, \quad (5.2.18)$$

where $e_j$ is the unit vector in the $j$ direction. Taking the height of the cylinder to be of unit length, as before, we may write $dV \to dA$ and $|dS| \to |ds|$ where $ds$ is the line element parallel to the bounding streamline.

The first term on the right hand side of equation (5.2.18) represents the work done by viscous stresses on the bounding streamline. The second term represents the interior rate of energy dissipation.

We can removed $v_i$ from the constraint (5.2.18) and instead express it in terms of the
streamfunction since
\[ v = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right). \tag{5.1.22} \]

In doing so, we allow the scaled kinematic viscosity \( \bar{\nu} \) to vary with position; such a dependence could occur through a dependence on \( \rho \), in which case we could consider \( \nu \) as a function of \( \psi \). The use of the divergence theorem, integration by parts and moving around some terms results in:

\[
\oint_C \rho \bar{\nu} \left( \nabla^2 \psi \right) |\nabla \psi| |ds| = \int_A \left\{ \rho \bar{\nu} \left[ (\nabla^2 \psi)^2 + 2 \nabla \psi \cdot \nabla (\nabla^2 \psi) \right] + \nabla (\rho \bar{\nu}) \cdot \nabla \left( |\nabla \psi|^2 \right) \right\} dA \tag{5.2.19}
\]

This condition is satisfied for the flow associated with an analytic Kida vortex (see Section 5.3) when \( \bar{\nu} \) is constant, as long as \( A \) is interior to the vortex. However, as the vorticity is discontinuous at the vortex boundary in this case, it cannot be satisfied more generally if it is assumed that \( \rho \) and \( \bar{\nu} \) are smoothly varying because the left hand side is discontinuous as the boundary is passed through.

For general vortices, the forms of \( \bar{\nu} \) and \( \psi \) can be seen to be connected by equation (5.2.19).

### 5.2.2.3 The case without dust

In this case the Poisson equation (5.1.30) is simply

\[ \nabla^2 \psi = -\omega(\psi) = 2\Omega + \frac{dF}{d\psi}. \tag{5.2.20} \]

Integrating the left hand side of equation (5.2.19) by parts again and using

\[
\frac{\partial \psi}{\partial x_i} \frac{d \omega}{d \psi} = -\frac{\partial \psi}{\partial x_i} \frac{d}{d \psi} (\nabla^2 \psi)
\Rightarrow \frac{d \omega}{d \psi} \nabla \psi = -\nabla \left( \nabla^2 \psi \right) \tag{5.2.21}
\]

we find that the constraint due to viscous diffusion, equation (5.2.19), simplifies to

\[
\int_A \rho \bar{\nu} |\nabla \psi|^2 \frac{d \omega}{d \psi} dA = \int_A \nabla (\rho \bar{\nu}) \cdot \left[ \nabla \left( |\nabla \psi|^2 \right) - \nabla \psi (\nabla^2 \psi) \right] dA \tag{5.2.22}
\]

This can be used to specify \( d\omega/d\psi \) in terms of \( \psi \). Recalling that the integration is over an area \( A \) enclosed by the streamline labelled by \( \psi \),

\[
dA = \left| \frac{ds |d\psi}{|\nabla \psi|} \right|. \tag{5.2.23}
\]
so after differentiating equation (5.2.22) with respect to $\psi$ we find

$$\frac{d\omega}{d\psi} \oint_C \rho \bar{\nu} |\nabla \psi| dA = \oint_C \nabla (\rho \bar{\nu}) \cdot \left[ \nabla \left( |\nabla \psi|^2 \right) - \nabla \psi (\nabla^2 \psi) \right] \left| \frac{ds}{|\nabla \psi|} \right|. \quad (5.2.24)$$

This is now of the form of a constraint on a closed streamline. This indicates that variations in vorticity require variations in $\bar{\nu}$ and that a smoothly varying $\bar{\nu}$ is associated with a smooth vorticity profile. If a steady state vortex exists in a viscous fluid produced by e.g. RWI, the vorticity profile is constrained by the form of the viscosity which also has to be consistent with the RWI.

### 5.3 An analytic solution: the Kida vortex

There exists an exact solution of the incompressible Euler equations (5.1.14), (5.1.25) consisting of an elliptical vortex patch of constant vorticity $\omega_{Kida}$ that matches to a Keplerian shearing flow at large distances. The streamlines are concentric elliptical epicycles (Kida, 1981, Section 3.5.5). The contents of this section are not new work, but are useful to have in one place.
5.3 An analytic solution: the Kida vortex

5.3.1 Calculating the streamfunction $\psi$

As established in Section 5.1.6, the Stokes’ streamfunction of a background Keplerian shearing flow $\mathbf{v}_0 = (0, -Sx, 0)$ is $\psi_0 = \frac{1}{2}Sx^2$ (equation (5.1.24)).

Following the approach of Section 5.2 we split our streamfunction into the sum of its background and superposed vortex parts:

$$\psi = \psi_0 + \psi_1 = \frac{1}{2}Sx^2 + \psi_1.$$  \hspace{1cm} (5.3.1)

This is a constant density solution, so $B(\psi) = 0$ and the Poisson equation (5.2.4) reduces to

$$\nabla^2 \psi_1 = \frac{dF_1}{d\psi} = A(\psi) = \begin{cases} c & \text{inside vortex} \\ 0 & \text{outside vortex}, \end{cases}$$  \hspace{1cm} (5.3.2)

where $c$ is a constant obeying $S + c = -\omega_{\text{Kida}}$ and $\omega_{\text{Kida}}$ is the total vorticity inside the vortex patch. The vorticity profile takes the form of a ‘top hat’ shape, as in Figure 5.4. We let $\psi^{(\text{in})}$ and $\psi^{(\text{ex})}$ be the overall potentials interior and exterior to the vortex respectively:

$$\nabla^2 \psi^{(\text{in})} = S + c$$  \hspace{1cm} (5.3.3a)\hspace{2cm}$$\nabla^2 \psi^{(\text{ex})} = S.$$  \hspace{1cm} (5.3.3b)

Inside the vortex we look for a solution with elliptical streamlines of constant aspect ratio, so we try a solution of the form

$$\psi^{(\text{in})} = A_1x^2 + A_2y^2 \Rightarrow 2(A_1 + A_2) = S + c$$  \hspace{1cm} (5.3.4)

In order to make progress it is now convenient to switch to an elliptical coordinate system\(^1\). In this system the $x$– and $y$–coordinates are parameterised by a radius-like unit $\xi$ and angular-like unit $\eta$ thus:

$$x = h \sinh \xi \sin \eta$$  \hspace{1cm} (5.3.5a)\hspace{2cm}$$y = h \cosh \xi \cos \eta.$$  \hspace{1cm} (5.3.5b)

The scaling quantity $h$ is the focus of some specified ellipse. How these two parameters map to the Cartesian system can be seen in Figure 5.5. Note that this coordinate system is a system of confocal ellipses of different aspect ratios, $\chi$, whereas the Kida solution contains streamlines of the same $\chi$ throughout.

We get around this problem by scaling to the bounding streamline. We define the bounding streamlines $\psi = \psi_b$ of the vortex to pass through $(x, y) = (0, 1)$ with $\xi = \xi_b$. The semi-minor

\(^1\)For more details see Appendix A
Figure 5.5 The elliptic coordinate system. The constant $\eta$ curves are in green, the constant $\xi$ curves are the concentric ellipses. The boundary of the Kida vortex is given by $\xi = \xi_b$. 
and semi-major axes $a$ and $b$ are therefore
\[ a = h \sinh \xi_b \]
\[ b = h \cosh \xi_b = 1 \] (5.3.6)
and the aspect ratio $\chi$ is
\[ \chi = \frac{b}{a} = \frac{1}{\tanh \xi_b} \] (5.3.7)
with $\chi > 1$ everywhere. The boundary between interior and exterior solutions is at
\[ \xi_b = \tanh^{-1} \left( \frac{1}{\chi} \right) = \frac{1}{2} \log \left( \frac{\chi + 1}{\chi - 1} \right), \quad \chi > 1. \] (5.3.8)
Using standard hyperbolic identities,
\[ (h \sinh \xi_b, h \cosh \xi_b) = \left( \frac{h}{\sqrt{\chi^2 - 1}}, \frac{h\chi}{\sqrt{\chi^2 - 1}} \right) \] (5.3.9)
and we define
\[ \tilde{h} = \frac{h}{\sqrt{\chi^2 - 1}}. \] (5.3.10)
Our adapted elliptical coordinate system inside the vortex, for $\tilde{h} \in (0, \chi^{-1}]$ is therefore
\[ x = \tilde{h} \sin \eta \] (5.3.11a)
\[ y = \tilde{h}\chi \cos \eta. \] (5.3.11b)
Outside the vortex, $\xi > \xi_b$, we use the standard elliptical coordinates given by equation (5.3.5).

We will now consider the external solution then match $\psi^{(in)}$ and $\psi^{(ex)}$ on the boundary $\xi = \xi_b$ to determine our unknowns $A_1$ and $A_2$ given by equation (5.3.4). Outside the vortex, $\psi_1$ obeys Laplace’s equation and since
\[- \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \psi_1^{(ex)} = \frac{1}{h^2 \left( \sinh^2 \xi + \sin^2 \eta \right)} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \psi_1^{(ex)} = \frac{\chi^2 - 1}{h^2 \left( \sinh \xi + \sin^2 \eta \right)} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \psi_1^{(ex)} = 0 \]
(see Appendix A) we are left with
\[- \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \psi_1^{(ex)} = 0. \]
The boundary conditions at infinity are given by requiring the flow tends to the background
shear for large \( x \) and \( y \), so

\[
\frac{\partial \psi^{(ex)}_1}{\partial x} \to 0, \quad \frac{\partial \psi^{(ex)}_1}{\partial y} \to 0 \quad \text{as} \quad x, y \to \infty
\]  

(5.3.12)

in the new coordinate system (again, see Appendix A) become

\[
\frac{\partial \psi^{(ex)}_1}{\partial \eta} \to 0, \quad \frac{\partial \psi^{(ex)}_1}{\partial \xi} \to 0 \quad \text{as} \quad \xi \to \infty.
\]

(5.3.13)

Thus the solution of our Lagrangian for \( \psi^{(ex)}_1 \) plus the background term and an arbitrary constant \( d \) gives our overall exterior potential

\[
\psi^{(ex)} = \psi_0 + \psi^{(ex)}_1 = \frac{1}{2} S x^2 + \alpha_0 \xi + \sum_{m>0} \alpha_m e^{-m\xi} \cos (m\eta) + d.
\]

(5.3.14)

We now need to match the interior and exterior solutions on the boundary of the ellipse, insisting that \( \psi \) and the tangential velocity \( \frac{\partial \psi}{\partial \xi} \) are continuous there. On the boundary \( \xi = \xi_b \) and

\[
A_1 x^2 + A_2 y^2 = A_1 h^2 \sinh^2 \xi_b \sin^2 \eta + A_2 h^2 \cosh^2 \xi_b \cos^2 \eta = E.
\]

For an elliptical streamline on the boundary, we need constant \( E \):

\[
A_1 h^2 \sinh^2 \xi_b = A_2 h^2 \cosh^2 \xi_b
\]

\[
\Rightarrow \quad \tanh \xi_b = \sqrt{\frac{A_2}{A_1}} = \frac{1}{\chi}.
\]

(5.3.15)

The exterior solution should also be constant on \( \xi = \xi_b \) since we want \( \xi_b \) to be a streamline. Note that this requires \( \alpha_m = 0 \) for \( m \geq 3 \) thus

\[
\psi^{(ex)}_{|\xi=\xi_b} = \frac{1}{2} S h^2 \sinh^2 \xi_b \sin^2 \eta + \alpha_0 \xi_b + \alpha_2 e^{-2\xi_b} \cos 2\eta + d
\]

\[
= \frac{1}{4} S h^2 \sinh^2 \xi_b (1 - \cos 2\eta) + \alpha_0 \xi_b + \alpha_2 e^{-2\xi_b} \cos 2\eta + d
\]

\[
\Rightarrow \quad \alpha_2 = \frac{1}{4} S h^2 e^{2\xi_b} \sinh^2 \xi_b.
\]

(5.3.16)

We also require the continuity of \( \frac{\partial \psi}{\partial \xi} \) on \( \xi = \xi_b \):

\[
\frac{\partial \psi^{(ex)}}{\partial \xi}_{|\xi=\xi_b} = 2 h^2 \cosh \xi_b \sinh \xi_b \left[ A_1 \sin^2 \eta + A_2 \cos^2 \eta \right]
\]

\[
= h^2 \cosh \xi_b \sinh \xi_b [(A_1 + A_2) + (A_2 - A_1) \cos 2\eta]
\]

\[
\frac{\partial \psi^{(in)}}{\partial \xi}_{|\xi=\xi_b} = \frac{1}{2} S h^2 \cosh \xi_b \sinh \xi_b (1 - \cos 2\eta) + \alpha_0 - \frac{1}{2} S h^2 \sinh^2 \xi_b \cos 2\eta
\]
Comparing the coefficients of $\cos 2\eta$ we find

$$\frac{2}{S}(A_1 - A_2) = 1 + \tanh \xi_b.$$  \hfill (5.3.17)

Therefore, with equations (5.3.4) and (5.3.15), we find:

$$c = \frac{S(\chi + 1)}{\chi(\chi - 1)}$$  \hfill (5.3.18)

$$\omega_{\text{Kida}} = \frac{S(\chi^2 + 1)}{\chi(1 - \chi)}.$$  \hfill (5.3.19)

This shows that (as expected), the aspect ratio, $\chi$, of a Kida vortex is a function of the background shear $S$ and the vorticity of the patch $\omega_{\text{Kida}} = -(S + c)$. The opposite signs of $\omega_{\text{Kida}}$ and $S$ shows that only anticyclonic vortices of this constant-vorticity type occur in Keplerian flows. There are a range of pressure profiles associated with different $\chi$, as can be seen in Figure 5.6 which are calculated explicitly in Section 5.3.2. As we have found in Sections 3.4.3 and 5.2.2.1, this is particularly important when considering dusty gases as particles tend to drift towards pressure maxima.

We can also find the values of $\alpha_0$, $A_1$, $A_2$ and $d$:

$$\alpha_0 = \frac{Sh^2}{2(\chi - 1)^2}$$ \hfill (5.3.20a)

$$A_1 = \frac{S\chi^2}{2\chi(\chi - 1)}$$ \hfill (5.3.20b)

$$A_2 = \frac{S}{2\chi(\chi - 1)}$$ \hfill (5.3.20c)

$$d = \frac{Sh^2(1 - 2\xi_b)}{4(\chi - 1)^2}.$$ \hfill (5.3.20d)

and therefore finally we arrive at our complete solution for $\psi$:

$$\psi^{(in)} = \frac{S}{2\chi(\chi - 1)} \left( \chi^2 x^2 + y^2 \right)$$ \hfill (5.3.21a)

$$\psi^{(ex)} = \frac{Sh^2}{4} \sinh^2 \xi_b (1 - \cos 2\eta) + \frac{Sh^2}{4(\chi^2 - 1)} e^{2(\xi_b - \xi) \cos 2\eta}$$

$$+ \frac{Sh^2 \xi_b}{2(\chi - 1)^2} + \frac{Sh^2(1 - 2\xi_b)}{4(\chi - 1)^2}.$$ \hfill (5.3.21b)
5.3.2 The pressure distribution

5.3.2.1 The interior solution

To find the pressure distribution inside the vortex we apply equation (5.1.31) to the streamfunction. Thus for \( \psi^{(\text{in})} \) we find (dropping the superscripts)

\[
\nabla \psi = (2A_1x, 2A_2y) \quad \Rightarrow \quad \frac{1}{2} |\nabla \psi|^2 = 2(A_1^2x^2 + A_2^2y^2),
\]

so

\[
\frac{P}{\rho_0} = \frac{2}{3} S^2 x^2 - \frac{1}{2} |\nabla \psi|^2 + (c - \frac{S}{3}) \psi + \frac{P_0}{\rho_0}.
\]

The constant \( P_0/\rho_0 \) term can be taken to be zero without any loss of generality. Substituting for \( \psi \) and \( c \) using equation (5.3.4):

\[
\frac{P}{\rho_0} = \left[ \frac{2}{3} S^2 + 2A_1A_2 - \frac{4}{3} SA_1 \right] x^2 + \left[ 2A_1A_2 - \frac{4}{3} SA_2 \right] y^2,
\]

so \( P/\rho_0 \) is a quadratic form and we expect the pressure contours to be either elliptical or hyperbolic. We make the substitutions

\[
\alpha_p = \frac{2}{3} S^2 + 2A_1A_2 - \frac{4}{3} SA_1 = \frac{S^2(7 - 4\chi)}{6(\chi - 1)^2} \quad (5.3.23a)
\]

\[
\beta_p = 2A_1A_2 - \frac{4}{3} SA_2 = \frac{S^2(4 - \chi)}{6\chi(\chi - 1)^2} \quad (5.3.23b)
\]

such that

\[
\frac{P}{\rho_0} = \alpha_p x^2 + \beta_p y^2. \quad (5.3.24)
\]

5.3.2.2 Pressure gradients inside the vortex

It is useful to consider gradients of \( P \) in the \( \tilde{h} \)– and \( \eta \)–directions. Using the adapted elliptical coordinates described in equation (5.3.5) we find that

\[
P = \rho_0 \tilde{h} S^2 \frac{1}{6(\chi - 1)^2} \left[ (7 - 4\chi) \sin^2 \eta + \chi(4 - \chi) \cos^2 \eta \right] \quad (5.3.25a)
\]

\[
\frac{\partial P}{\partial \tilde{h}} \bigg|_\eta = \rho_0 \tilde{h} S^2 \frac{1}{3(\chi - 1)^2} \left[ (\chi - 1)(\chi - 7) \sin^2 \eta + \chi(4 - \chi) \right] \quad (5.3.25b)
\]

\[
\frac{\partial P}{\partial \eta} \bigg|_\tilde{h} = \rho_0 \tilde{h} S^2 \left( \frac{\sin \eta \cos \eta}{\chi - 7} \right) \quad (5.3.25c)
\]
Considering equation (5.3.25b), let \( a = \sin^2 \eta \) and

\[
 f(\chi, a) = (\chi - 1)(\chi - 7)a + \chi(4 - \chi),
\]

(5.3.26)

where \( f(\chi, 0) = \chi(4 - \chi) \) and \( f(\chi, 1) = 7 - 4\chi \). For \( a \neq 1 \),

\[
 f(\chi, a) = \left[ 9 \left( a - \frac{1}{2} \right)^2 + \frac{7}{4} \right] - (1 - a) \left[ \chi + \frac{2a(2a - 1)}{1 - a} \right]^2
\]

(5.3.27)

so for \( a \in [0, 1) \) this is a parabola with a maximum point. We are interested in the behaviour of \( f(\chi, a) \) as \( a = \sin^2 \eta \) varies as this will indicate whether the core of a Kida vortex contains a minimum, maximum or saddle point in its pressure distribution. The positive root (since \( \chi > 1 \)) of \( f(\chi, a) = 0 \) occurs at

\[
 \chi^* = -\frac{2(2a - 1) + \sqrt{9 \left( a - \frac{1}{2} \right)^2 + \frac{7}{4}}}{1 - a}.
\]

(5.3.28)

\( \chi^* \) decreases monotonically from \( \chi^* = 4 \) in the range \( a \in [0, 1) \), with \( \chi^* \to 7/4 \) as \( a \to 1 \). Therefore, as can be seen in Figure 5.7, \( f(\chi, a) = f(\chi, \sin^2 \eta) \) will be negative \( \forall \eta \) if \( \chi > 4 \) and positive \( \forall \eta \) if \( \chi < 7/4 \).

Since \( \tilde{h} \) increases outwards from the vortex centre, we find that we have:

- pressure minima \( 1 < \chi < 7/4 \)
- saddle \( 7/4 < \chi < 4 \)
- pressure maxima \( \chi > 4 \).

The first case is of the least astrophysical interest as dust is drawn up pressure gradients to pressure maxima (Sections 3.4.3 and 3.5) so these vortices will not attract dust to their core (e.g. Chavanis, 2000). The \( \chi \to 1 \) limit is also that of the infinitely strong vortex, so we expect that the creation of such vortices is also difficult in practice. Correspondingly, the weaker vortices with \( \chi > 4 \) are of the most interest in a planetesimal formation context, while the transition cases \( 7/4 < \chi < 4 \) prove to exhibit interesting stability behaviour (see Section 3.4.3).

The pressure distributions for a variety of Kida vortices of different aspect ratios can be found in Figure 5.6, where the minima, maxima and saddle regions can be seen.

Considering equation (5.3.25c) for the pressure gradient around a streamline, we find it is zero for an entire streamline when \( \chi = 7 \). This special case has particular importance when we look at the parametric model in Section 5.4. It also means that the frozen–in dust distribution is axisymmetric around streamlines inside the vortex for Kida vortices when \( \chi = 7 \) (Lyra and Lin, 2013).
Calculating equilibrium solutions

(a) $\chi = 3/2$

(b) $\chi = 13/8$

(c) $\chi = 7/4$

(d) $\chi = 2$

(e) $\chi = 3$

(f) $\chi = 4$
5.3 An analytic solution: the Kida vortex

Figure 5.6 The pressure distributions of Kida vortices with a range of aspect ratios. As the aspect ratio increases (or, the amount of additional vorticity inside the vortex patch decreases), a pressure maximum appears inside the vortex.

In Figure 5.6a we see a pressure minimum at the centre of the vortex, which by Figure 5.6c has moved to a transitive state where the pressure has no $x$–dependence in the vortex core. In Figure 5.6d we have a saddle point, with hyperbolic pressure contours inside the vortex, while we see $y$–independence for $P$ in Figure 5.6f, the second transitory case. For $\chi > 4$ there is a pressure maximum, as can be seen in Figures 5.6g and 5.6h. This is important as dust is drawn towards local pressure maxima.
Calculating equilibrium solutions

Figure 5.7 The behaviour of \( f(\chi, a) = (\chi - 1)(\chi - 7)a + \chi(4 - \chi) \), where \( a = \sin^2 \eta \). \( f(\chi, a) \) governs the Kida pressure distribution in the ‘radial’ direction \( \tilde{h} \). When \( \chi > 4 \), \( f < 0 \) and there is a pressure maximum, while for \( 1 < \chi < 7/4 \), \( f > 0 \) and the core contains a dust–rejecting pressure minima. There is a saddle point between these two regions.

5.3.2.3 The exterior solution

For completeness, here the pressure distribution obeys

\[
\frac{P}{\rho_0} = \frac{2}{3} S^2 x^2 - \frac{1}{2} |\nabla \psi|^2 - \frac{1}{3} S \psi \tag{5.3.29}
\]

with \( \psi = \psi^{(ex)} \). In order to work out the \( \frac{1}{2} |\nabla \psi|^2 \) term we need

\[
|\nabla \psi|^2 = \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 = \frac{1}{\tilde{h}^2 (\sinh^2 \xi + \sin^2 \eta)} \left[ \left( \frac{\partial \psi}{\partial \eta} \right)^2 + \left( \frac{\partial \psi}{\partial \xi} \right)^2 \right].
\]
An analytic solution: the Kida vortex

(see Appendix A). Thus, for $\psi^{(ex)}$

\[
\left( \frac{\partial \psi}{\partial \xi} \right)^2 = \frac{S^2 h^4}{4} \left[ \sinh 2\xi \sin^2 \eta \right. \\
\left. - \frac{e^{2(\xi - \xi)}}{\chi^2 - 1} \cos 2\eta + \frac{1}{(\chi - 1)^2} \right]^2 = \frac{S^2 h^4}{4} D^2_\xi \tag{5.3.30a}
\]

\[
\left( \frac{\partial \psi}{\partial \eta} \right)^2 = \frac{S^2 h^4}{4} \sin^2 2\eta \left[ \sinh^2 \xi - \frac{e^{2(\xi - \xi)}}{\chi^2 - 1} \right]^2 = \frac{S^2 h^4}{4} D^2_\eta \tag{5.3.30b}
\]

and

\[
\frac{P}{\rho_0} = \frac{2}{3} S^2 h^2 \sinh^2 \xi \sin^2 \eta - \frac{S^2 h^2}{8(\sinh^2 \xi + \sin^2 \eta)} D^2_\xi + D^2_\eta - \frac{1}{3} S \psi^{(ex)} \tag{5.3.31}
\]

5.3.3 Period round a streamline

It is useful to investigate the internal shear within the vortices by calculating the period

\[
\tilde{P} = \oint \frac{d\sigma}{|\nabla \psi|} \tag{5.3.32}
\]

around streamlines, where $\sigma$ is our arclength (see Section 6.3 for results and Section 8.1.1.3 for how it is calculated from the gridded data). Kida vortices have no internal shear so we expect a constant period throughout the vortex patch. Parametrising our streamline using the adapted elliptical coordinates of Section 5.3.1, equation (5.3.5), we have $(x, y) = (\tilde{h} \sin \eta, \tilde{h} \chi \cos \eta)$ and therefore our arclength element is

\[
d\sigma = \sqrt{(dx)^2 + (dy)^2} = \tilde{h} \sqrt{\cos^2 \eta + \chi^2 \sin^2 \eta} \, d\eta \tag{5.3.33}
\]

while equation (5.3.21a) implies that

\[
|\nabla \psi| = \frac{S}{\chi(\chi - 1)} \sqrt{\chi^4 x^2 + y^2} \tag{5.3.34}
\]

so substituting for the elliptical coordinates again, equation (5.3.32) yields

\[
\tilde{P}_{\text{Kida}} = \oint \frac{d\sigma}{|\nabla \psi|} = \int_0^{2\pi} \frac{\tilde{h} \sqrt{\cos^2 \eta + \chi^2 \sin^2 \eta}}{S h} \, d\eta \\
= \int_0^{2\pi} \frac{\chi - 1}{S} \, d\eta \\
= \frac{2\pi}{S}(\chi - 1), \tag{5.3.35}
\]
Calculating equilibrium solutions

which is indeed independent of streamline choice inside the vortex.

5.4 The polytropic model

It is possible to consider different values of $S$ (i.e. a non–Keplerian background flow) while retaining the potential $\Phi$ appropriate for a Keplerian disc. This requires the pressure gradient to be non-zero in the background flow and consequently enables us to consider situations where the vortex is centred on a background where there is pressure extremum. This is of interest as dust is expected to accumulate at the centre of a ring where there is a pressure maximum (Whipple, 1972). The Rossby wave instability can also result in vortices forming at such locations (see e.g. Méheut et al., 2010, 2012b).

For example, if we set

$$S = \sqrt{\frac{3(\chi - 1)}{\chi + 1}} \Omega \quad (5.4.1)$$

then we have (to within a constant in the vortex core) that

$$\frac{P}{\rho} = -\frac{S\psi}{\chi(\chi - 1)} + F(\psi). \quad (5.4.2)$$

This makes $P/\rho$ a function of $\psi$ alone; it will be a linear function of $\psi$ provided that $F(\psi)$ is. However, when such a solution is matched to an exterior function (see Lesur and Papaloizou, 2009) the background flow will correspond to one with $v_y = -Sx$. From equation 5.4.1, this corresponds to the Keplerian case strictly only when $\chi = 7$. For other values of $S$, consideration of the exterior Kida solution implies that there is an implied background pressure maximum at the co-orbital radius for $\chi > 7$ and a background pressure minimum (which will not attract dust) at $\chi < 7$.

This turns out to be useful for constructing a vortex model with non-uniform density. This will have a stream function of the form in equation 5.3.21a inside the vortex core with $S$ given by equation (5.4.1). The density will take the form

$$\rho = \left[1 - \frac{b(\psi - \psi_b)}{\psi_b}\right]^n \quad (5.4.3)$$

inside the vortex, where $\psi_b$ is the stream function on the core boundary and the background density is taken to be unity. The quantities $b$ and $n$ are constants determining the profile and magnitude of the density excess above the background. At the vortex centre $\rho = (1 + b)^n$ while at the boundary $\rho = 1$, the background value.

Following the standard form taken by the pressure in polytropic models $P \propto \rho^{1+1/n}$ (Chan-
drasekhar, 1939), the pressure is assumed to take the form

\[ P = \frac{\beta_P \psi b^{1+1/n}}{b(n+1)}, \tag{5.4.4} \]

where \( \beta_P \) is a constant determined such that the equilibrium conditions apply. Substituting for \( \nabla^2 \psi \) and \( \rho \) in equation (5.1.30) yields:

\[ \frac{S(\chi^2 + 1)}{\chi(\chi - 1)} = \frac{dF(\psi)}{d\psi} + 2\Omega - \frac{n\beta_P}{n+1} \tag{5.4.5} \]

while substituting for \( P \), \( \rho \) and \( \psi \) in equation (5.4.2) gives

\[ \frac{\beta_P \psi b}{b(n+1)} \left[ 1 - \frac{b(\psi - \psi_b)}{\psi_b} \right] = -\frac{S\psi}{\psi(\psi - 1)} + F(\psi). \tag{5.4.6} \]

Taking the derivative of equation (5.4.6) with respect to \( \psi \) and eliminating \( dF/d\psi \) from equation (5.4.5) results in an expression for \( \beta_P \):

\[ \beta_P = \Omega \left( 2 - \chi \sqrt{\frac{3}{\chi^2 - 1}} \right). \tag{5.4.7} \]

The parameter \( n > 0 \) can be specified arbitrarily, while \( b \) can then be chose to scale the density excess above the background in the centre of the vortex provided \( \chi > 2 \) (\( \beta_P \) disappears when \( \chi = 2 \).) Note that this is related to the GNG solution of Goodman et al. (1987) with their variable \( \varepsilon = \chi^{-1} \).

### 5.5 Summary and conclusions

In this chapter, starting from the two–fluid equations we formulate the equations of motion for a dusty gas in a 2D, Keplerian, shearing flow. In the incompressible limit we consider the case where the friction/ stopping time \( \tau_s \to 0 \), so the dust and gas are tightly coupled. We can then formulate a Poisson equation for the Stokes’ streamfunction for a vortex patch in the local shearing sheet approximation.

The source term of this Poisson equation can be split into a vorticity and density contribution, for which we choose to adopt power law forms of \( \psi \). This will give us the freedom to investigate the stability of a large variety of vortex solutions, with and without mass.

We also detail the analytical Kida solution, which will provide a good test case for subsequent numerical work. The polytropic solution of Goodman et al. (1987), although not generally true for a Keplerian shearing flow, is useful in future stability analysis.

In the next chapter we will discuss finding numerical solutions to these equations and the results of this.
In Chapter 5 we described our mathematical model for our set of equilibrium solutions. We now need a method of calculating these. In this chapter we present how we solved for these solutions numerically and tested their validity and convergence.

In Section 6.1 we show how we set up the problem, presenting the various parametrisations and scalings used. Section 6.2 examines the numerical routines used to iteratively solve the Poisson equation with discussion of the problems found, while Section 6.3 demonstrates a selection of equilibrium solutions, with discussion of the importance of their pressure distributions and internal shear.

6.1 Setting up the problem

Since analytical solutions to the Poisson equation (5.2.4) only occur for the constant-density, constant-vorticity Kida case we must integrate this equation numerically to produce different vortex solutions. We will organise them into different aspect ratio solutions within different vortex classes.

6.1.1 Parametrising the vortex class

We need a way of prescribing different combinations of different $A(\psi)$ and $B(\psi)$. We do this through four numbers:

\[
\begin{align*}
\alpha & : \text{ the vorticity power law index, equation (5.2.6a)} \\
\beta & : \text{ the density power law index, equation (5.2.6b)} \\
\rho_m & : \text{ a measure of the total mass added} \\
\omega_m & : \text{ a measure of the total vorticity added}
\end{align*}
\]
Calculating equilibrium solutions: numerical approach and results

We will refer to the first three numbers \( \{\alpha, \beta, \rho_m\} \) as a vortex class. We produce different aspect ratio, \( \chi \), solutions inside this vortex class by varying \( \omega_m \).

Another secondary quantity is \( \rho_{\text{max}} \), the central density enhancement, given by

\[
\rho_{\text{max}} = \max_{\psi} [\rho(\psi) - \rho_0].
\]  
(6.1.1)

Its relationship to \( \chi \) is shown in Section 6.3.3.

6.1.2 Scalings

As mentioned in Section 5.2.1, we chose our arbitrary units of length, time and mass such that we have:

- a length scale such that bounding streamline passes though \((x, y) = (0, 1)\)
- a time scale such that \(\Omega = 1\)
- a mass scale such that \(\rho_0 = 1\).

We need to scale both the Bernoulli and density sources in the momentum equation (5.2.4) and we do this via the two quantities \(\omega_m\) and \(\rho_m\). We define \(\omega_m\) as

\[
\pi\omega_m = \int A(\psi) dS,
\]  
(6.1.2)

where the integral is taken over the total area inside the vortex core, i.e. inside the bounding streamline that passes through \((0, 1)\). This quantity is a measure of the total vorticity inside the vortex. In order to rescale the Bernoulli source (equation (5.2.6a)), we recalculated \(A\) after every iteration to ensure that \(\pi\omega_m\) remained fixed. This procedure was iterated until there was no discernible difference in successive streamfunction and pressure contours; the global streamline pattern will then have converged. For vortices with \(B = 0\) this generally required about 50 iterations.

Comparison with the analytic solution for the Kida vortex case (Section 5.3) enabled our numerical procedure to be checked.

In a similar way we scale the mass added to the vortex by imposing a fixed value of \(\rho_m\), where

\[
\rho_m = \int [\rho(\psi) - \rho_0] dS,
\]  
(6.1.3)

where again integration is over the vortex core. Note that scaling additional mass in this way (as to be consistent with the treatment of total vorticity) results in different strength vortices having different central mass enhancements, \(\rho_{\text{max}}\); for the same \(\rho_m\), more elongated structures will have a greater \(\rho_{\text{max}}\) (Section 6.3.3).
6.2 Calculating equilibrium vortex solutions using vortex.f90

The program vortex.f90 is designed to take the four descriptors \( \{\alpha, \beta, \rho_m, \omega_m\} \) and a grid size \([n, m]\) and return the equilibrium vortex solution. The general procedure for this can be seen in the flow chart in Figure 6.1. Due to the extra computation it takes to calculate the pressure distribution at each iteration, initially an equilibrium solution is found with zero density perturbation \( \forall \beta, B \) (i.e. the first loop in the flow chart). Only once this has been found is density added to the solution, if \( B \neq 0 \). In order that the addition of this mass does not perturb the new solution too far away from the previous one at each iteration, mass is added gradually to enable convergence (see Section 6.2.4 for details).

We perform our calculations on a 512 \( \times \) 2048 grid that covers the vortex core. The \( x \)- and \( y \)-coordinates are both in the range \([-1, 1]\). This grid was chosen to ensure enough resolution to undertake a local stability analysis (see Chapters 7 and 8). Very similar results were obtained when the grid resolution was reduced by a factor of two. We have different resolutions in the \( x \)- and \( y \)-directions because of the geometry of our vortex; it always has a semi-major axis of length 1 along the \( y \)-axis.

An unavoidable result of this is that for elongated vortices there are significantly fewer grid points resolving the core on the \( x \)-axis than on the \( y \)-axis. Consequently, in the latter stability analysis we usually use a streamline that passes through \((0, 0.85)\), where the streamline is still contains a large number of grid points.

6.2.1 Calculating the Poisson integral

We initially solve a reduced version of the Poisson equation with \( B = 0 \) that is applicable to the uniform density case, namely

\[
\nabla^2 \psi_1 = A(\psi).
\]

We start by defining the Bernoulli source term \( A \) to be a nonzero constant inside the unit circle passing through \((0, 1)\) and zero outside this. We then solve equation (6.2.1) using the 2D Green’s Function, obtaining

\[
\psi_1(r) = \frac{1}{2\pi} \int \int \log |r - r'| A(\psi) d^2r'.
\]

Calculation of the Green’s function \( \frac{1}{2\pi} \log |r - r'| \) is an \( O(n^2m^2) \) operation. Therefore, to cut down computational time, the Green’s function is only calculated for \( x \in [0, 1], y \in [0, 1] \). The symmetry of the solution is then exploited to find \( \psi \) for the other three quadrants.

We add on the background shear \( \psi_0 = \frac{1}{2} S x^2 \) to \( \psi_1 \) to get the total streamfunction \( \psi \). Using the value of \( \psi \) at \((0, 1)\) we find the coordinates of the bounding streamline. Inside this boundary we then rescale the Bernoulli source as indicated in Section 6.1.2 and equations (6.1.2) and (5.2.6a). and apply the Green’s function to obtain an updated solution for \( \psi_1 \). This procedure is
Calculating equilibrium solutions: numerical approach and results

Figure 6.1 Program design for vortex.f90. The two main loops are controlled by the variables $N_1$ and $N_2$, which typically took the values 40 and 20 respectively. The boxes coloured pink are procedures expanded in detail in the text. Restarted runs already have the correct equilibrium vorticity solution found so the first loop is bypassed when the restart option is set.
6.2 Calculating equilibrium vortex solutions using vortex.f90

repeated for \( N_1 \) iterations; usually a \( N_1 = 40 \) was sufficient to obtain a density-free equilibrium vortex solution.

6.2.2 Calculating the pressure distribution

We now have the total streamfunction for the flow \( \psi = \psi_0 + \psi_1 \) for a constant density equilibrium. To find solutions with increased density in the central parts of the vortex core the pressure distribution has to be calculated at each iteration. This is found from equation (5.1.31) which yields

\[
\frac{P}{\rho} = \frac{2}{3} S^2 x^2 - \frac{1}{2} |\nabla \psi|^2 - \frac{1}{3} S (\psi - \psi_b) + \int_{\psi_b}^{\psi} A(\psi') \, d\psi',
\]

(6.2.3)

where \( \psi_b \) is the value of the streamfunction on the boundary of the vortex. The terms \( \partial \psi / \partial x \) and \( \partial \psi / \partial y \) are calculated using first order finite differences. We use the trapezium rule to approximate this integral for each \( \psi \) on grid points inside the vortex

\[
\int_{\psi_b}^{\psi} A(\psi') \, d\psi' \approx h \left[ \frac{A(\psi_b) + A(\psi)}{2} + \sum_{i=1}^{n-1} A(\psi_i) \right]
\]

(6.2.4)

where

\[
\begin{align*}
    h &= \frac{\psi - \psi_b}{n} \\
    \psi_i &= \psi_b + ih.
\end{align*}
\]

(6.2.5a, 6.2.5b)

Throughout our calculations, \( n = 100 \).

6.2.3 Calculating \( B(\psi) \) and the total source term

With a form for \( \psi \) we use equation (5.2.6b) to find \( \rho(\psi) \). After calculating \( \rho = |\psi - \psi_b|^2 \) we scale the mass added by this contribution using equation (6.1.3). We set \( \rho = 1 \) and \( \nabla \rho = 0 \) outside the vortex and prescribe \( d\rho / d\psi \) inside the vortex exactly. We can then assemble the density source \( B(\psi) = d \log \rho / d\psi \) and hence the total source term

\[
A(\psi) + \frac{P}{\rho} B(\psi).
\]

(6.2.6)

We can now integrate this using the 2D Green’s function of the form

\[
\psi_1(\mathbf{r}) = \frac{1}{2\pi} \iint \log |\mathbf{r} - \mathbf{r}'| \left[ A(\psi) + \frac{P}{\rho} B(\psi) \right] \, d^2 \mathbf{r}'
\]

(6.2.7)

where the \( P/\rho \) distribution is calculated from the previous iteration via equation (6.2.3). This loop is repeated \( N_2 \) times for each \( \rho_m \) (which is itself increased incrementally to aid convergence – see Section 6.2.4). Typically \( N_2 = 20 \).
6.2.4 Convergence

Convergence for models with constant density was straightforward, although resolution issues arise for a fixed grid when $\alpha$ becomes too large due to the peaking of the vorticity profile in the centre of the vortex. To avoid this we used $\alpha \leq 4$.

For models with enhanced density inside the vortex core, convergence was more difficult and required a starting model close to the final one. We began by imposing only small increases to either, or both of $\beta$ and $\rho_m$ from the values appropriate to an existing equilibrium solution. These changes moved them in the direction of our target parameters. This procedure was especially necessary for the cores of vortices with non-Kida, $\alpha \neq 0$, vorticity profiles. Having done this, a new form of $\psi_1$ was obtained as before by iterating the Green’s function solution $N_2 \geq 10$ times. At this point additional small increments of the order of 1% were made to $\rho_m$ and the process repeated until the target values were attained. The Green’s function solution could then be iterated to convergence.

Convergence was tested for in a systematic way by observing the percentage change in the value of the source term $A(\psi)$ at the origin. Once the change in $A(\psi)$ was beneath a fixed value ($10^{-4}$) between iterations, convergence was deemed to have occurred. For solutions that didn’t converge (e.g. because too much mass was added in one go), one saw large and often periodic variations in this value. There was a ceiling imposed on $N_2$ to terminate such cases.

6.3 Results

We begin by reproducing the streamlines for Kida vortices. Since analytical expressions exist for the streamfunction and pressure distributions (Section 5.3) our numerical procedure can be validated.

The aspect ratio, $\chi$, of a vortex is a good way to parametrize its relative strength. Weak vortices will be sheared out more by the background flow and will therefore have a large aspect ratio, while strong vortices have a small aspect ratio. We are also interested in cases for which the resulting pressure distribution has a maximum at the centre of the vortex as this location will then attract dust (Section 3.5, Whipple, 1972). Recall that while $\{\alpha, \beta, \rho_m\}$ are fixed, $\omega_m$ is varied to produce vortices with different $\chi$.

The aspect ratio for each vortex stated in the figures is measured by evaluating the ratio of the largest $y$– and $x$–coordinates for the streamline passing through $(0, 0.85)$. The boundary itself was not used as in the stability calculations in Chapters 7 and 8, calculating certain derivatives on the boundary proved problematic (due to e.g. discontinuities in the vorticity profile). Calculating derivatives on a streamline a little away from the boundary solved these problems, so the $\chi$ calculations in this chapter are consistent with this streamline choice.

\[\text{A Kida vortex with } \chi = 5 \text{ would have } A(\psi) = 0.45 \text{ inside the vortex patch. } \alpha > 0 \text{ solutions would have larger values.}\]
We also investigate the internal shear within the vortices by calculating the period around streamlines (see Section 5.3.3 and equation (5.3.32)). Kida vortices have no internal shear so a constant period throughout the vortex patch, $\tilde{P}_{\text{Kida}} = 2\pi(\chi - 1)/S$. The results for the Kida vortex are in good agreement with analytic expectation (see Figure 6.2a). We plot $\tilde{P}$ against the value of the positive $y$–coordinate $y_{\text{max}}$ where each streamline crosses the $y$–axis.

Note that the periods we plot were obtained by locating coordinate extrema from the results of numerical integrations of fluid particles moving around streamlines stored on a relatively coarse temporal grid. This results in some low level jitter at a relative level $\sim 10^{-3}$. This is also a measure of the departure from the constant value of the period $\tilde{P}_{\text{Kida}}$ obtained numerically in the Kida vortex case.

For each equilibrium solution displayed we will show the vorticity/density distribution, the $\psi$ contours, the pressure distribution and $\tilde{P}$ against $y_{\text{max}}$.

### 6.3.1 Solutions with constant density

In Figure 6.2 we illustrate results for constant-density vortices with the same value of $\omega_m$. We increase the power law index $\alpha$ of the Bernoulli source term ($A(\psi)$, equation (5.2.6a)) over the range $0 \leq \alpha \leq 4$, with $\alpha = 0$ corresponding to the Kida vortex. This results in an increased concentration of vorticity in the centre of the vortex.

Vortices with larger $\alpha$ but equivalent $\omega_m$ have characteristics of stronger Kida vortices as they are sheared less by the surrounding flow and accordingly have smaller $\chi$. This process also decreases the pressure maximum (or removes it altogether), making these vortices less susceptible to dust collection.

Apart from the case for $\alpha = 4$ in Figure 6.2d, there is a pressure maximum at the centre of the vortex patch, with a saddle point in the latter case. We find that the transition from central saddle point to maximum occurs at smaller aspect ratios as $\alpha$ increases and the vorticity distribution becomes more centrally concentrated.

In addition, streamlines for vortices with centrally concentrated vorticity sources tend to become pinched towards the $y$–axis near the boundary, compared to the Kida case. This leads to a relative increase in circulation period. Another result of this is that the $\chi$ of streamlines near the centre of the vortex are smaller than that of boundary streamlines. This needs to be considered when plotting stability curves against aspect ratio in subsequent chapters. This effect is demonstrated in Figures 6.3 and 6.4.

The introduction of a nonzero $\alpha$ also introduces a shear inside the vortex. A fluid parcel on a streamline in a Kida vortex core will move round with the same period as all other fluid parcels and thus there is no internal shear in the vortex. However, the introduction of a non-uniform vorticity profile destroys this configuration and we see the development of a nonlinear relationship between the period around a streamline and its position in the vortex core. This effect becomes more noticeable as $\alpha$ increases.
Calculating equilibrium solutions: numerical approach and results

Figure 6.2 (Continued on next page.)
6.3 Results

Figure 6.2 Vortices with \( \omega_m = 0.09 \) and density equal to the background value. From left to right we have: (i) the vorticity distribution, (ii) the \( \psi \) distribution, (iii) the pressure distribution and (iv) a plot of the period \( \tilde{P} \) around a streamline against the value of the positive \( y \)-coordinate where it intersects the \( y \)-axis. Increasing \( \alpha \) from 0 to 4 results in a vortex that is stronger (i.e., sheared less by the background flow) and accordingly has a smaller aspect ratio. Note that all these vortices, except the one illustrated in the bottom panels, have a pressure maximum at the vortex centre. The pressure distribution in the latter vortex has a saddle point. There is also significant shear, as indicated by the variation in the period \( \tilde{P} \) to circulate around a streamline.
Figure 6.3 The streamlines of the vortex \( \{4, 0, 0\} \) with \( \omega_m = 0.0183 \) \((\chi_{\text{Kida}} = 10)\) where one can see significant variation of streamline aspect ratio with position in the vortex.

A consequence of this is that disturbances that overlap adjacent streamlines are expected to shear out. Therefore localised disturbances might be expected to have different behaviour to that exhibited in a Kida vortex and thus the local stability may be different.
6.3 Results

Figure 6.4 The pair of \((\omega_m, \chi_{\text{Kida}})\) describes each vortex, where \(\omega_m\) is given by equation (6.1.2) and \(\chi_{\text{Kida}}\) is the Kida aspect ratio you would achieve for a vortex patch with \(\omega_m\) added. Here we can see the significant variation in the aspect ratio for streamlines within the same vortex when \(\alpha = 4\) caused by the ‘pinching’ of streamlines close the boundary. Recall that in the Kida case these lines will be horizontal as the streamlines are ellipses. Figure 6.3 shows the streamlines of the vortex with \(\omega_m = 0.018333\) where one can clearly see this ‘pinching’. 

(a) The variation of \(\chi\) with \(y_{\text{max}}\) for different strength vortices with \(\alpha = 1\). 

(b) The variation of \(\chi\) with \(y_{\text{max}}\) for different strength vortices with \(\alpha = 4\).
6.3.2 Solutions with a central density enhancement

In our models the dust is completely coupled to the gas (i.e. the $\tau_s \to 0$ limit in a two fluid model), so density increases above the background value model a concentration of dust.

In Figure 6.5 we give results for vortices for which the Bernoulli vorticity source $A(\psi)$ is identical to that of the Kida case. A density enhancement is imposed through the application of a scaled density profile extending over the whole vortex. Results are given for models for which the central density ranges from between four and 16 times the background value $\rho_0 = 1$.

Results for vortices with variable Bernoulli sources of the same form as those illustrated in Figure 6.2 together with a density enhancement are given in Figure 6.6.

In all these cases the solutions were made by taking the mass–free versions of the vortex class as our initial solution (e.g. if we wanted to make vortices with $\{0, 1, 1\}$ we would start from $\{0, 0, 0\}$), then changing $\beta$ to its intended value and adding small increments to $\rho_m$, as detailed in Section 6.2.4.

Figures 6.5 and 6.6 show that increasing $\rho_m$ results in a higher pressure maximum at the centre of the resulting vortex. However, and somewhat surprisingly, varying this parameter has very little impact on either the form of the streamlines or the shear profile within the vortex. In fact, increasing $\rho_m$ for a vortex with the Kida, $\alpha = 0$, Bernoulli source reduces the variation of $P$ inside the vortex. The reason for this is that while the pressure profile varies by a large amount, $P/\rho$ varies by very little and this is the quantity that appears in equation (5.2.4).

Figure 6.7 also demonstrates that the transitional behaviour seen in the Kida case in Figure 5.6 also appears with different Bernoulli and density profiles. In Figure 6.7a we see a strong pressure maximum but as we increase $\omega_m$ and produce stronger vortices we see the aspect ratio decrease and the pressure distribution going through a transitional phase with a saddle point (Figure 6.7c) until in Figure 6.7d have a vortex exhibiting a pressure minima at its centre.

We encountered significant difficulties in finding solutions with significant central density enhancements ($\rho_{\text{max}} > 10$). Typically the largest we could consistently get to converge had $\rho_{\text{max}} \approx 4$, as can be seen in Figure 6.8. Similarly, it was difficult to get solutions with $\beta > 1$ to converge successfully.

6.3.3 Relationship between the central density enhancement, $\rho_{\text{max}}$, and aspect ratio, $\chi$

A consequence of the way we have chosen to scale the density added to these vortices (i.e. using equation (6.1.3)) is that the central density enhancement has a dependence on the vortex aspect ratio $\chi$. Figures 6.8a–6.8c show how the central density enhancement $\rho_{\text{max}}$ varies with vortex class and $\chi$. Figure 6.9 shows the variation of $\rho$ viewed as a slice through the $y$–axis against aspect ratio $\chi$ for class $\{0, 1, 0.1\}$. The act of adding a fixed amount of mass to each vortex
across a class has resulted in significant variation in the density enhancement at the centre of the vortex. In the limit $\chi \to \infty$, the vortex becomes a segment of an infinitesimally thin circular ring. In this case, finite mass implies infinite central density which is consistent with the previous observation.

### 6.4 Summary and conclusions

In this chapter, we have parametrised each vortex solution using four numbers, with $\alpha$ and $\omega_m$ controlling the vorticity profile and $\beta$ and $\rho_m$ the density profile. The variable $\alpha$ controls the steepness of the vorticity profile, with $\alpha = 0$ corresponding to the flat, Kida case, while $\omega_m$ parametrises the amount of vorticity in the patch and therefore its strength. Accordingly, increasing vortex strength $\omega_m$ is related to a decreasing $\chi$.

Similarly, the parameter $\beta$ regulates the steepness of the density profile while additional mass in the vortex is scaled by $\rho_m$. In practice, we are limited to $\beta = 1$ for convergence to occur.

We solve the Poisson equation (5.2.4) iteratively for the streamfunction $\psi$ using the 2D Green’s function. Initially this is done with no density enhancement, regardless of the final desired set of parameters $\{\alpha, \beta, \rho_m, \omega_m\}$. Then, for solutions with a central density enhancement, mass is added gradually in order that the solution does not deviate too much between iterations and convergence can occur. We systematically tested convergence by observing the change in $A(\psi)$ at the origin, as well as changes to the streamline shape.

Despite this, there was some difficulty in getting very large $\rho_{\text{max}}$ solutions to converge. This limitation on density enhancements is a numerical issue and we later use the analytic polytropic solution to argue that there is no reason to expect such a limitation in reality (Section 8.4). Convergence is worse when there is a density enhancement as the right hand side of equation (5.2.4) involves higher derivatives of $\psi$ through the pressure terms; such derivatives are prone to amplifying errors. Trying to improve the scheme would probably involve moving to an iterative implicit scheme which would be difficult to implement at the required resolution and with no guarantee of success. We leave this approach for future investigation.

We also note that the recent paper of Raettig et al. (2015)\(^2\) finds that dust strongly coupled to the fluid (i.e. $T_s \ll 1$, which is under consideration here since we are working in the $\tau_s = 0$ limit) is not strongly concentrated in 2D vortices. Instead of the very strong localised concentrations of particles which occur in the $T_s \simeq 1$ case, small grains are instead spread over the entire vortex. Thus being limited to a power law index of $\beta = 1$ may not be too unreasonable. The $T_s \simeq 1$ case certainly needs more study, with a different approach to the one presented here.

Finally, we find that constant density solutions, for the same total vorticity inside the

\(^2\)Uploaded to ArXiv while writing up.
Calculating equilibrium solutions: numerical approach and results

(a) \( \{\alpha, \beta, \rho_m\} = \{0, 1, 0.1\} \), with \( \chi = 4.93 \).

(b) \( \{0, 1, 0.3\} \), with \( \chi = 4.84 \).

Figure 6.5 (Continued on next page.)
6.4 Summary and conclusions

Figure 6.5 Vortices with a Kida vorticity profile for the Bernoulli source term, with $\omega_m = 0.09$ and nonzero density enhancement parameter $\rho_m$. The central densities in these vortices range between 0.3 and 6 times the background value. Note substantial changes in the pressure distribution but relatively little variation in the streamlines. The aspect ratio, $\chi$, calculated at $(0, 0.85)$. 

(c) $\{0, 1.0, 0.5\}$, with $\chi = 4.80$. 

$\rho(\psi) - \rho_0$ 

$\tilde{\rho}_{kida} = 16.01$ 

Model 

$\tilde{P}_{kida}$
Calculating equilibrium solutions: numerical approach and results

Figure 6.6 Vortices with a non-Kida Bernoulli vorticity source, with \( \omega_m = 0.09 \) and nonzero density enhancement parameter \( \rho_m \). The vorticity profiles in these vortices are non-uniform, resulting in significant variation of the period for circulating around internal streamlines and hence significant internal shear.
vortex patch, behave like stronger vortices when $\alpha$ is increased and the vorticity profile is more strongly peaked. There is also significant deviation from the constant period of circulation observed for Kida vortices, with $\alpha \neq 0$ producing internal shear between vortex streamlines. However, adding mass does not have the same effect and can even reduce internal shear. We also observe transitional behaviour in the pressure distribution inside vortices as in the Kida case. This has generic consequences for the ‘saddle point instability’ detailed in the next chapter.

We now move to test the linear stability of these solutions.
Calculating equilibrium solutions: numerical approach and results

(a) \( \{\alpha, \beta, \rho_m, \omega_m\} = \{0.25, 1, 0.5, 0.05\} \), resulting \( \chi = 6.05 \)

(b) \( \{0.25, 1, 0.5, 0.16\} \), resulting \( \chi = 3.48 \)

(c) \( \{0.25, 1, 0.5, 0.3\} \), resulting \( \chi = 2.59 \)

Figure 6.7 (Continued on next page.)
(d) $\{0.25, 1.0, 1.13\}$, resulting $\chi = 1.77$.

Figure 6.7 Different aspect ratio vortices, with non-uniform but smooth vorticity profiles, produced by varying $\omega_m$. The parameters determining the density and vorticity profiles are fixed, apart from the scaling of the Bernoulli source, which is controlled by $\omega_m$. Quite different behaviour to that seen for the Kida case as illustrated in Figure 5.6 is seen.
Calculating equilibrium solutions: numerical approach and results

(a) The variation of central density enhancement $\rho_{\text{max}}$ with $\chi$ for vortex classes with $\alpha = 0$. The relationship between $\rho_{\text{max}}$ and $\chi$ in all cases is linear.

(b) The variation in central density enhancement $\rho_{\text{max}}$ with $\chi$ for vortex classes with $\alpha = 0.25$. Note that equivalent vortices with $\alpha = 0$ (comparing with Figure 6.8a) have lower $\rho_{\text{max}}$ than when $\alpha = 0.25$ as $\alpha \neq 0$ solutions concentrate mass more.

Figure 6.8 (Continued on next page.)
The variation in central density enhancement $\rho_{\text{max}}$ with $\chi$ for vortex classes with $\alpha = 0.5$. Again, $\rho_{\text{max}}$ is greater for these vortices with $\alpha = 0.5$ than vortices with $\alpha < 0.5$ for the same aspect ratio $\chi$.

**Figure 6.8** Variation in central density enhancement $\rho_{\text{max}}$ against $\chi$ for various $\alpha$ values. Again, the three numbers in curly brackets related to $\{\alpha, \beta, \rho_{\text{m}}\}$. 
Figure 6.9 The variation of $\rho$ with $\chi$ along the $y$-axis for the vortex class $\{0, 1, 0.1\}$. This plot is made from combining vertical slices through the density distribution along the $y$-axis for the entire range of aspect ratios. There is significant variation in the height of the central density maximum from the most circular vortices $\chi \approx 1$ where $\rho_{\text{max}} \approx 1$ (which are have a larger area therefore less density piled into the middle) to the elongated vortex where $\chi = 14$ and $\rho_{\text{max}} \approx 1.8$. 
Chapter 7

Stability Analysis

In the previous two chapters we set up our model and calculated a large set of equilibrium solutions. In this chapter we formulate the stability analysis of these solutions to perturbations localised on streamlines.

In Section 7.1 we consider making both Eulerian and Lagrangian perturbations to our equilibrium solutions in the most general form. When perturbations can be associated with a time–independent wavenumber they can lead to exponentially growing modes, as is found for the known instabilities of the Kida vortex, which we generalise in Section 7.2. Various parametrisations are made in Section 7.2.2.1 and 7.2.2.3 which leads to a working form of the (Eulerian) stability equations which we can integrate in Section 7.2.2.4. The periodic nature of the equilibrium solutions means that Floquet theory can be used to determine the growth rates of instability, and this is detailed in Section 7.2.3.

We investigate the horizontal \( (k_z \to \infty) \) limit of the stability calculations in Section 7.3, followed by a discussion of the ‘saddle point’ and parametric instabilities that we expect to occur in Sections 7.3.2 and 7.3.3, respectively.

The generic case when the period of circulation around a streamline in a vortex is not constant is explored in Section 7.4 with a wavenumber with magnitude that ultimately increases linearly with time and its application discussed.

Finally we consider the vertical stability (i.e. perturbations that have \( k_z = 0 \)) of solutions in Section 7.5 for vortices with and without internal shear. The polytropic model of Section 5.4 is used to investigate the latter case.

7.1 Perturbation analysis

We now have a large variety of equilibrium solutions and we are interested in their linear stability. The introduction of shear in the vortex core via the Bernoulli source \( A(\psi) \) (see equation (5.2.6a)) and a density profile via \( B(\psi) \) (see equation (5.2.6b)) provides a lot of new solutions to investigate the stability profiles of. The stability of the Kida vortex is explored in
detail in Lesur and Papaloizou (2009), so this solution will again be used as a test case of our method.

It is useful to consider both the Lagrangian and Eulerian formulations of the linear stability problem as they are found to be convenient for different purposes. Throughout this work we use \( Q' \) to denote the Eulerian perturbation of \( Q \), the perturbation of \( Q \) at a fixed point in space. \( \Delta Q \) meanwhile denotes the Lagrangian perturbation of \( Q \), the perturbation of a quantity as experienced by a fluid element, taking into account its displacement by a distance \( \xi \). These perturbations are related by

\[
\Delta Q = Q' + \xi \cdot \nabla Q, \tag{7.1.1}
\]

to first order in \( \xi \) (Lynden-Bell and Ostriker, 1967).

### 7.1.1 Eulerian formalism

The time-dependent momentum equation in the rotating frame (see equation (5.1.9)) in the case where viscous forces may be neglected is

\[
\frac{Dv}{Dt} + 2\Omega \times v = \frac{1}{\rho} \nabla P - \nabla \Phi, \tag{7.1.2}
\]

where we have dropped the overbars for clarity and \( \Phi \) is the combined potential (Section 5.1.5, equation (5.1.29)).

We begin by making the Eulerian perturbations:

\[
v \mapsto v_0 + v'
\]

\[
P \mapsto P_0 + P'
\]

\[
\rho \mapsto \rho_0 + \rho'
\]

where \( \{v_0, P_0, \rho_0\} \) are the equilibrium quantities from solving our steady equation of motion (5.2.4). Perturbing equation (7.1.2) and linearising yields the linearised momentum equation in component form:

\[
\frac{Dv_x'}{Dt} + v' \cdot \nabla v_{0,x} - 2\Omega v_y' = -\frac{1}{\rho_0} \frac{\partial P'}{\partial x} + \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial x} \tag{7.1.4a}
\]

\[
\frac{Dv_y'}{Dt} + v' \cdot \nabla v_{0,y} + 2\Omega v_x' = -\frac{1}{\rho_0} \frac{\partial P'}{\partial y} + \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial y} \tag{7.1.4b}
\]

\[
\frac{Dv_z'}{Dt} = -\frac{1}{\rho_0} \frac{\partial P'}{\partial z} \tag{7.1.4c}
\]

where in these linearised equations the convective (or Lagrangian) derivative \( D/Dt = \partial/\partial t + \)
\( \mathbf{v} \nabla \) is replaced by the expression

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla.
\]  

(7.1.5)

Using this expression for \( D/Dt \) in the linearised equations has second order corrections to the original equations.

Due to the local nature of these perturbations, we use Cowling’s approximation (Cowling, 1941) to neglect the variation of the gravitational potential. Note that in the \( z \)-direction we have hydrostatic balance as \( \mathbf{v}_0 = (v_{0,x}, v_{0,y}, 0) = \nabla \times (\psi_0 \mathbf{\hat{z}}) \). We also have the continuity equation from the two-fluid averaged version of equation (5.1.8)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]  

(7.1.6)

so taking the Eulerian perturbation and linearising:

\[
\frac{D\rho'}{Dt} + \rho' \nabla \cdot \mathbf{v}_0 + \mathbf{v}' \cdot \nabla \rho_0 + \rho \nabla \cdot \mathbf{v}' = 0
\]

\[
\Rightarrow \quad \frac{D\rho'}{Dt} + \mathbf{v}' \cdot \nabla \rho_0 = 0
\]  

(7.1.7)

where we use the fact that \( \mathbf{v}_0 \) is the curl or a vector \( \mathbf{v}_0 = \nabla \times (\psi_0 \mathbf{\hat{z}}) \) and \( \nabla \cdot \mathbf{v}' = 0 \) since we are insisting on incompressibility everywhere.

These equations (7.1.4, 7.1.7) expressed in terms of Eulerian variations were found to be simpler to use when analysing vertical stability (see Section 7.5). They also are solved numerically when considering vortex stability in Chapter 8 for the same reason.

### 7.1.2 Lagrangian formalism

While the Eulerian formulation is convenient in some contexts the Lagrangian formalism is useful for some aspects, such as the analytic discussion of the saddle point instability (Section 7.3.2).

Firstly, the Lagrangian derivative (equation (7.1.5)) commutes with the Lagrangian perturbation so that

\[
\frac{D}{Dt} (\Delta Q) = \Delta \left( \frac{DQ}{Dt} \right),
\]  

(7.1.8)

as shown by Lynden-Bell and Ostriker (1967). The displacement \( \xi \) is the Lagrangian variation of the fluid element position vector such that

\[
\Delta \mathbf{r} = \xi
\]  

(7.1.9)
and
\[ \Delta v = \frac{D\xi}{Dt}. \]  
(7.1.10)

Taking the Lagrangian perturbation of equation (7.1.2) we obtain
\[ \frac{D^2\xi}{Dt^2} + 2\Omega \times \frac{D\xi}{Dt} = \Delta F \]  
(7.1.11a)
with \( F = -\nabla P/\rho - \nabla \Phi \) and
\[ \Delta F = -\frac{\nabla P'}{\rho} + \frac{\rho'}{\rho^2} \nabla P - \xi \cdot \left( \frac{\nabla P}{\rho} + \nabla \Phi \right), \]  
(7.1.11b)

where we have again used Cowling’s approximation and equation (7.1.1). Considering the continuity equation (7.1.6) and incompressibility
\[ \frac{D\rho}{Dt} = 0, \]
which implies that \( \rho \) is advected with the fluid and hence
\[ \Delta \rho = \rho' + \xi \cdot \nabla \rho_0 = 0. \]  
(7.1.12)

Equations (7.1.11, 7.1.12), together with the incompressibility condition \( \nabla \cdot \xi = 0 \) lead to a system of equations for the horizontal components of \( \xi \) that is fourth order in time (see Section 7.2.1).

7.1.3 Local analysis

We consider perturbations that are localised on streamlines. To do this we assume that any perturbation quantity is of the form
\[ \Delta Q = \Delta Q_0 \exp(i \lambda S_A). \]  
(7.1.13)

Here we are adopting the WKBJ\(^1\) ansatz with amplitude factor \( \Delta Q_0 \) (the WKBJ envelope), phase function \( S_A \) and constant \( \lambda \) taken to be a large parameter. Discussion of this approach in a variety of contexts can be found in Lifschitz and Hameiri (1991), Sipp and Jacquin (2000) and Papaloizou (2005). The effective wavenumber is
\[ k = \lambda \nabla S_A \]  
(7.1.14)

which has a large magnitude. The amplitude factor \( \Delta Q_0 \) is assumed to be a function of time and slowly varying relative to \( S_A \) while we have rapid variation of the complex phase \( \lambda S_A \).

\(^1\)See the work of Wentzel (1926), Kramers (1926), Brillouin (1926) and Jeffreys (1925).
The standard assumption of local analysis is that only the variation of this phase needs to be considered when taking space derivatives. The only exception is when variation is along streamlines since we require that \( v_0 \cdot \nabla S_A = 0 \) and so we need to consider the \( \mathbf{v}_0 \cdot \nabla (\Delta Q_0) \) contribution. Due to this rapid variation of the complex phase \( \lambda S_A \) we perform a WKBJ analysis where the state variables \( \Delta Q_0 \) are expanded in inverse powers of \( \lambda \), substituting equation (7.1.13) into equation (7.1.11), recalling that \( \xi = \Delta \mathbf{r} \):

\[
\left[ \frac{1}{\lambda^2} \frac{D^2(\Delta r_0)}{Dt^2} + \frac{i}{\lambda} \left( 2 \frac{DS_A D(\Delta r_0)}{Dt} + \frac{D^2 S_A}{Dt^2} \Delta r_0 \right) - \left( \frac{DS_A}{Dt} \right)^2 \Delta r_0 \right] \\
+ 2 \Omega \times \left[ \frac{1}{\lambda^2} \frac{D(\Delta r_0)}{Dt} + \frac{i}{\lambda} \frac{DS_A}{Dt} \Delta r_0 \right] = \frac{1}{\lambda^2} \Delta F_0 \tag{7.1.15}
\]

where we have divided through by \( \lambda^2 \) and cancelled the common factor \( \exp(i\lambda S_A) \). In the term \( \Delta F \), consider \( \nabla P' \):

\[
P' = \Delta P - \xi \cdot \nabla P \\
\nabla P' = i\lambda \nabla S_A P' + \nabla (\Delta P_0 - \Delta r_0 \cdot \nabla P) \exp(i\lambda S_A). \tag{7.1.16}
\]

Thus the lowest order term on the RHS of equation (7.1.15) is \( \lambda^{-2} \nabla P' \) which is \( O(\lambda^{-1}) \); the rest of the terms are \( O(\lambda^{-2}) \). Therefore, to lowest order, equation (7.1.15) gives

\[
\frac{DS_A}{Dt} = 0. \tag{7.1.17}
\]

When working to the next order only the variation of the phase \( S_A \) needs to be considered when taking spatial derivatives, apart from when considering expressions involving \( D/Dt \) as this annihilates \( S_A \). In this case, the contribution \( \mathbf{v} \cdot \nabla (\Delta Q_0) \) needs to be retained. Therefore, equation (7.1.11), together with equation (7.1.12) yield

\[
\frac{D^2 \xi}{Dt^2} + 2 \Omega \times \frac{D \xi}{Dt} = -\frac{ik P'}{\rho} - \frac{\xi \cdot \nabla P}{\rho^2} + \xi \cdot \nabla (\mathbf{v} \cdot \nabla \mathbf{v} + 2 \Omega \times \mathbf{v}), \tag{7.1.18}
\]

while the incompressibility condition \( \nabla \cdot \xi = 0 \) gives

\[
k \cdot \xi = 0. \tag{7.1.19}
\]

We can also find an equation for the time evolution of the wavenumber \( k \) using equations (7.1.14) and (7.1.19):
\[
\frac{Dk_i}{Dt} = \lambda \left[ \frac{\partial}{\partial t} \frac{\partial S_A}{\partial x_i} + v_0,j \frac{\partial^2 S_A}{\partial x_i \partial x_j} \right] = \lambda \left[ -\frac{\partial v_0,j}{\partial x_i} \frac{\partial S_A}{\partial x_j} + v_0,j \frac{\partial^2 S_A}{\partial x_i \partial x_j} \right] = -\lambda \frac{\partial v_0,j}{\partial x_i} \frac{\partial S_A}{\partial x_j} \Rightarrow \frac{Dk_i}{Dt} = -k_j \nabla v_0,j
\] (7.1.20)

We can eliminate \( P' \) using the vertical component of equation (7.1.18) (see Section 7.2.1 and equation (7.2.4)). Then equations (7.1.18)-(7.1.20) give a complete system for the evolution of \( \xi \) and \( k \) as an initial value problem. Evolution consists of advection of data along streamlines so it is possible to consider disturbances localised on individual streamlines (Papaloizou, 2005). In general one could start with an arbitrary initial \( S_A \) and then \( k \) would depend on time.

### 7.2 Time–independent wavenumber

An initial route into this problem is to look for solutions for which \( S_A \) is independent of time and a function of quantities conserved on unperturbed streamlines so \( v_0 \cdot \nabla S_A = 0 \). Then in an Eulerian viewpoint, \( k = \lambda \nabla S_A \) is fixed for all time and we only have to solve for \( \xi \).

Our vortex solutions are two-dimensional with an initial state independent of \( z \) (see Section 5.1.5) so we take

\[
S_A = g(\psi) + \frac{k_z z}{\lambda}.
\] (7.2.1)

Here \( \psi \) is our unperturbed streamfunction, \( g \) an arbitrary function and \( k_z \) the constant vertical wavenumber. With this form of \( S_A \), our wavenumber is

\[
k = \lambda \frac{dg}{d\psi} \nabla \psi + k_z \hat{z}.
\] (7.2.2)

For now we’ll assume that \( k_z \neq 0 \) which is appropriate to the physically realistic case where perturbations are localised in \( z \). The \( k_z = 0 \) case will be dealt with separately in Section 7.5.

This form of \( k \) is not the most general solution of equation (7.1.17), except fortuitously in the case where the velocity is linear in the coordinates (as in the Kida case, see Section 5.3). For other solutions we would expect the magnitude of the wavenumber to ultimately increase linearly with time. However, in that situation we expect that although there may be temporary amplification, the system may not ultimately show growth of linear perturbations exponentially
with time; this situation is well known in the context of the shearing box (Goldreich and Lynden-Bell, 1965 and Appendix C). This is looked at in more detail in Section 7.4.

### 7.2.1 Lagrangian form

When $k$ takes the form dictated by equation (7.2.2) the perturbed momentum equation 7.1.18:

$$\frac{D^2 \xi_i}{Dt^2} + 2\epsilon_{ij3} \Omega \frac{D \xi_j}{Dt} + \frac{ik_i P'}{\rho} = H_i = \left[ -\xi \cdot \nabla \rho \nabla P + \xi \cdot \nabla (v \cdot \nabla v + 2\Omega \times v) \right]_i.$$  

(7.2.3)

Considering the $z$-component of this expression:

$$\frac{D^2 \xi_z}{Dt^2} = -\frac{ik_z P'}{\rho},$$  

(7.2.4)

and, noting that $k_z$ is constant, we rearrange the incompressibility condition (equation (7.1.19)) to find

$$\xi_z = -\frac{k_x \xi_x + k_y \xi_y}{k_z} = -\frac{k_j \xi_j}{k_z}.$$  

(7.2.5)

In this last expression, and throughout this section, summation is over $j = 1, 2$ only. Then

$$\frac{i P'}{\rho} = -\frac{1}{k_z} \frac{D^2 \xi_z}{Dt^2} = \frac{1}{k_z} \frac{D^2}{Dt^2} \left( \frac{k_j \xi_j}{k_z} \right) = \frac{1}{k_z^2} \left[ \frac{k_j D^2 \xi_j}{Dt^2} + 2 \frac{Dk_j}{Dt} \frac{D \xi_j}{Dt} + \xi_j \frac{D^2 k_j}{Dt^2} \right]$$  

(7.2.6)

Substituting this expression into equation (7.2.3) we arrive at

$$\left( \delta_{ij} + \frac{k_j k_l}{k_z^2} \right) \frac{D^2 \xi_i}{Dt^2} + 2 \left( \epsilon_{ij3} \Omega + \frac{k_i}{k_z^2} \frac{Dk_j}{Dt} \right) \frac{D \xi_j}{Dt} + \frac{k_j \xi_j D^2 k_j}{k_z^2 \frac{D^2}{Dt^2}} = H_i$$  

(7.2.7)

with

$$\frac{Dk_i}{Dt} = -k_j \frac{\partial v_{i,j}}{\partial x_i}$$  

(7.2.8a)

$$\frac{D^2 k_i}{Dt^2} = k_l \frac{\partial^2 v_{i,j}}{\partial x_j \partial x_i} - k_j \frac{\partial v_{i,j}}{\partial x_j} \frac{\partial^2 v_{0,j}}{\partial x_j \partial x_l}$$  

(7.2.8b)

using equation (7.1.20). Neglecting the vertical stratification in this calculation can be justified if it is assumed that $k_z^2/(k_x^2 + k_y^2)$ is large. Else, the modes can be assumed to be localised in the vicinity of the midplane where the vertical stratification is least.

Though the presence of the arbitrary function $g(\psi)$ may seem a cause for concern, since the derivatives in equation (7.2.7) correspond to advection round a streamline and $dg/d\psi$ is constant on streamlines it acts as a multiplicative constant scaling the magnitude of the
wavenumber:
\[ k_x^2 + k_y^2 = \lambda^2 \left( \frac{dg}{d\psi} \right)^2 |\nabla \psi|^2 = \Lambda^2 |\nabla \psi|^2, \]  
where \( \Lambda \) is another large constant. We return to the use of these equations in Section 7.3.

### 7.2.2 Eulerian form

We begin by remarking that with the Lagrangian perturbations given in the form of equation (7.1.13)
\[ \Delta Q = \Delta \tilde{Q} \exp(i\lambda S_A). \]
the corresponding Eulerian perturbation is
\[ Q' = \Delta Q - \xi \cdot \nabla Q = \Delta Q - \Delta r \cdot \nabla Q \]
\[ = \tilde{Q}' \exp(i\lambda S_A). \]  
(7.2.10)

Our WKBJ approximation still holds as any gradients of \( \Delta \tilde{Q} - \Delta \tilde{r} \cdot \nabla Q \) will be an order in \( \lambda \) lower than the gradient of \( \exp(i\lambda S_A) \).

We again aim to eliminate \( v'_z \) and \( P' \) to produce a matrix equation in our remaining perturbed quantities \( \mathbf{x}' = (v'_x, v'_y, \rho')^T \) that we can integrate. Applying the incompressibility condition \( \nabla \cdot \mathbf{v}' = 0 \) when \( S_A \) takes the form described by equation (7.2.2) gives:
\[ i\lambda \frac{dg}{d\psi} \left( v'_x \frac{\partial \psi}{\partial x} + v'_y \frac{\partial \psi}{\partial y} \right) + ik_z v'_z = 0. \]  
(7.2.11)

Rearranging to eliminate \( v'_z \):
\[ v'_z = -\lambda \frac{dg}{k_z d\psi} \mathbf{v}' \cdot \nabla \psi. \]  
(7.2.12)

We substitute this into equation (7.1.4c) to get an expression for \( P' \)
\[ \frac{Dv'_z}{Dt} = \frac{D}{Dt} \left( \frac{\lambda}{k_z} \frac{dg}{d\psi} \mathbf{v}' \cdot \nabla \psi \right) = -\frac{ik_z P'}{\rho_0} \]  
(7.2.13)

\[ \Rightarrow P' = -\frac{i\rho_0 \lambda}{k_z} \frac{D}{Dt} \left( \mathbf{v}' \cdot \nabla \psi \right). \]  
(7.2.14)

Then eliminating \( P' \) from equations (7.1.4a,b) using the highest order term in equation (7.1.16) so \( \nabla P' = i\lambda \nabla P' S_A \) and

\[ \frac{Dv'_i}{Dt} + \mathcal{M}_{ij} v'_j - \frac{i\lambda}{k_z} \frac{dg}{d\psi} \frac{D}{Dt} \left( \mathbf{v}' \cdot \nabla \psi \right) \left[ i\lambda \frac{\partial \psi}{\partial x_i} \frac{dg}{d\psi} \right] = \frac{\rho'}{\rho_0} \frac{\partial P_0}{\partial x_i}, \]
\[ \frac{Dv'_i}{Dt} + \mathcal{M}_{ij} v'_j + \frac{\lambda^2}{k_z^2} \left( \frac{dg}{d\psi} \right)^2 \frac{D}{Dt} \left( \mathbf{v}' \cdot \nabla \psi \right) \frac{\partial \psi}{\partial x_i} = \frac{\rho'}{\rho_0} \frac{\partial P_0}{\partial x_i}. \]  
(7.2.15)
where the $2 \times 2$ matrix $\mathcal{M}_{ij}$ is given by

$$
\mathcal{M} = \left( \begin{array}{cc}
\frac{\partial v_{0,x}}{\partial x} & \frac{\partial v_{0,x}}{\partial y} - 2\Omega \\
\frac{\partial v_{0,y}}{\partial x} + 2\Omega & -\frac{\partial v_{0,y}}{\partial y}
\end{array} \right)
$$

(7.2.16)

and summation is over $i = 1, 2$, $j = 1, 2$ only. We can expand the $\frac{D}{Dt}(v' \cdot \nabla \psi)$ term to collect terms of the form $\frac{Dv'_j}{Dt}$, so:

$$
\frac{Dv'_j}{Dt} + \mathcal{M}_{ij} v'_i + \lambda^2 \left( \frac{dg}{d\psi} \right)^2 \left[ \frac{Dv'_j}{Dt} \frac{\partial \psi}{\partial x_i} + v'_j \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right] = \rho' \frac{\partial P_0}{\partial x_i} \\
\Rightarrow \left[ \delta_{ij} + \lambda^2 \left( \frac{dg}{d\psi} \right)^2 \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \right] \frac{Dv'_j}{Dt} = \\
- \left[ \mathcal{M}_{ij} + \lambda^2 \left( \frac{dg}{d\psi} \right)^2 \frac{\partial \psi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} \right] v'_i + \frac{1}{\rho_0^2} \frac{\partial P_0}{\partial x_i} \rho'.
$$

(7.2.17)

The density perturbation $\rho'$ cannot be simplified beyond equation (7.1.7):

$$
\frac{D\rho'}{Dt} = - \frac{\partial P_0}{\partial x_i} v'_i,
$$

(7.2.18)

recalling that $D/Dt = \partial/\partial t + \mathbf{v}_0 \cdot \nabla$.

### 7.2.2.1 Parametrisation via $\Theta$ and $\theta$

At this point it is now convenient to introduce a variable $\Theta$, a scaled ratio of horizontal and vertical wavenumbers:

$$
\Theta \propto \frac{k_x^2 + k_y^2}{k_z^2} = \frac{k_{\perp}^2|_{\sigma=0}}{k_z^2}.
$$

(7.2.19)

Without loss of generality, we have set our horizontal wavenumber to be evaluated at zero arclength $\sigma = 0$ which corresponds to the point on the positive $y$ axis where the chosen streamline crosses, as in Figure 7.1. Also, note that $k_{\perp} = \sqrt{k_x^2 + k_y^2}$ is the relevant quantity over $k_x$ and $k_y$ since any perturbations along streamlines result in no deformation to the streamline and hence will produce no instability.

Since $k = \lambda \nabla S_A$ (equation 7.1.14) and $S_A$ depends only on $\psi$ in the time–independent case, there is only one parameter to be chosen here; choosing $k_{\perp}$ makes the most sense as it is independent of coordinate system. As

$$
k_{\perp} = \nabla_{\perp} (\lambda g) = \lambda \frac{dg}{d\psi} \nabla_{\perp} \psi
$$

(7.2.20)

we find that

$$
k_{\perp}^2|_{\sigma=0} = \lambda^2 \left( \frac{dg}{d\psi} \right)^2 v_0^2|_{\sigma=0}.
$$

(7.2.21)
Figure 7.1 The coordinate $\sigma$ determining the location on a streamline and total arclength $\Sigma$ for an arbitrary streamline.

We therefore define $\Theta$ by

$$\Theta = \left[ \frac{1}{v_0^2} \frac{k_0^2}{k_z^2} \right]_{\sigma=0} = \left( \frac{dg}{d\psi} \right)^2.$$  \hspace{1cm} (7.2.22)

The angle between $k = (k_\perp, k_z)$ and the vertical $\hat{z}$ pointing out of the plane of the disc/vortex is given by

$$\tan \theta = \left. \frac{k_\perp}{k_z} \right|_{\sigma=0} = \sqrt{\Theta} \cdot v_0|_{\sigma=0}$$  \hspace{1cm} (7.2.23)

and is one of the variables in the final stability plots (see e.g. Section 8.1.2). Its geometric meaning can be seen in Figure 7.3. Note that $k_\perp$ is, by definition, perpendicular to streamlines, or $k_\perp \cdot v_0 = 0$.

7.2.2.2 Matrix form of the stability equations

Using equation (7.2.22), the stability equation (7.2.17) can now be written

$$\left( \delta_{ij} + \Theta \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \right) \frac{Dv_j'}{Dt} = - \left( M_{ij} + \Theta \frac{\partial \psi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} v_0, k \right) v_j' + \frac{1}{\rho_0} \frac{\partial P_0}{\partial x_i} \rho'.$$  \hspace{1cm} (7.2.24)
Figure 7.2 The orientation of the wavevector \( \mathbf{k} \) in relation to the plane of the disc; \( \hat{z} \) is out of the plane of the disc and vortex. In (a) we have the horizontal stability limit \( k_z \to \infty \), whereas in (b) we show the vertical stability limit \( k_z = 0 \).

Note that this problem only depends on the direction of the wavevector \( \mathbf{k} \) and not its magnitude so is scale–independent. We will write equation (7.2.24) in matrix form as

\[
A_{ij} \frac{Dv'_j}{Dt} = B_{ij}v'_j + C_i \rho' \tag{7.2.25a}
\]

where

\[
A_{ij} = \delta_{ij} + \Theta \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \tag{7.2.25b}
\]

\[
B_{ij} = -M_{ij} - \Theta \frac{\partial \psi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} v_{0,k} \tag{7.2.25c}
\]

\[
C_i = \frac{1}{\rho_0^2} \frac{\partial P_0}{\partial x_i} \tag{7.2.25d}
\]

This will eventually be solved, alongside equation (7.2.18) as an initial value problem, detailed in the following sections. A quick calculation shows that

\[
\det(A) = \Theta |\nabla \psi|^2 + 1 \geq 1 \tag{7.2.26}
\]

so \( A \) is invertible.
7.2.2.3 Parametrisation via arclength $\sigma$

In general, time $t$ is not the most convenient choice of parameter for conducting our stability analysis; instead we chose arclength $\sigma$.

\[
|v_0| = \frac{D\sigma}{Dt} \quad \Rightarrow \quad \frac{D}{Dt} = |v_0| \frac{D}{D\sigma}
\]

Including the matrix form of our continuity equation:

\[
\begin{align*}
\frac{D\rho'}{Dt} &= D_i v'_i \quad (7.2.27a) \\
D_i &= -\frac{\partial \rho_0}{\partial x_i} \quad (7.2.27b)
\end{align*}
\]

the final form of our equation,

\[
\frac{Dx'}{D\sigma} = \frac{1}{|v_0(\sigma, \psi)|} \Pi(\sigma, \psi)x'
\]  
(7.2.28a)

where

\[
\Pi = \begin{pmatrix} A^{-1}B & A^{-1}C \\ D & 0 \end{pmatrix}
\]

is a $3 \times 3$ matrix, $A$, $B$ are $2 \times 2$ matrices, $C$, $D$ are 2D vectors and $x' = (v'_x, v'_y, \rho')^T$. 

**Figure 7.3** The geometric realisation of the parameter $\theta$ in terms of angle between the unit vector $\hat{z}$ (out of the plane of the disc and vortex) and wavevector $k$. 

Stability Analysis
7.2.2.4 The working form of equation (7.2.28)

Expanding $v_0$ as $v_0 = \nabla \times (\psi \hat{z}) = (\partial \psi / \partial y, -\partial \psi / \partial x)$ we find that

$$A^{-1} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

$$a_1 = \frac{1}{1 + \Theta |\nabla \psi|^2} \left[ 1 + \Theta \left( \frac{\partial \psi}{\partial y} \right)^2 \right]$$

$$a_2 = a_3 = -\frac{\Theta}{1 + \Theta |\nabla \psi|^2} \frac{\partial \psi \partial \psi}{\partial x \partial y}$$

$$a_4 = \frac{1}{1 + \Theta |\nabla \psi|^2} \left[ 1 + \Theta \left( \frac{\partial \psi}{\partial x} \right)^2 \right]$$

and

$$B = -\left( M + \Theta \frac{\partial \psi}{\partial x_i} \frac{\partial^2 \psi}{\partial x_j \partial x_k} v_{0,k} \right)$$

$$= -\begin{pmatrix} \frac{\partial^2 \psi}{\partial x \partial y} + \Theta \frac{\partial \psi}{\partial x} B_1 & \frac{\partial^2 \psi}{\partial y^2} - 2\Omega + \Theta \frac{\partial \psi}{\partial x} B_2 \\ 2\Omega - \frac{\partial^2 \psi}{\partial x^2} + \Theta \frac{\partial \psi}{\partial y} B_1 & \frac{\partial^2 \psi}{\partial x \partial y} + \Theta \frac{\partial \psi}{\partial y} B_2 \end{pmatrix},$$

where

$$B_1 = \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \psi}{\partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial \psi}{\partial x}$$

$$B_2 = \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial \psi}{\partial y} - \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \psi}{\partial x}.$$

The vectors $C = \frac{1}{\rho_0^2} \frac{\partial P_0}{\partial x_i}$ and $D = -\frac{\partial \rho_0}{\partial x_i}$ have no simplified versions.

7.2.3 Use of Floquet theory

Recall equation (7.2.28):

$$\frac{Dx'}{D\sigma} = \frac{1}{|v_0(\sigma, \psi)|} \Pi(\sigma, \psi) x'$$

where $x' = (v'_{x}, v'_{y}, \rho')^T$ and $\sigma$ is the arclength as measure round a vortical streamline from the point where the streamline crosses the positive $y$ axis. We let $\Sigma$ be the total circumference of a chosen streamline. Around any particular streamline all the matrix elements $\Pi_{ij}$ will be $\Sigma$–periodic, so $\Pi(\sigma) = \Pi(\sigma + \Sigma)$. We can analyse this sort of periodic system using Floquet theory (see e.g. Whittaker and Watson, 1996, and Appendix B).

We have three perturbed quantities, $v'_{x}$, $v'_{y}$ and $\rho'$ so consider $n = 3$ vectors $x'_{k}$, all solutions
to equation (7.2.28). Let $X'_{ki}$ be the $i^{th}$ element of $x'_k$ so

$$X' = ([x'_1], [x'_2], [x'_3])$$

$$\Rightarrow \frac{DX'_{ki}}{D\sigma} = \frac{1}{|v_0|} \Pi_{ij} X'_{kj}.$$ 

Our approach is to make $X'_{(\sigma = 0)}$ the principal fundamental matrix\(^2\) so

$$X'_{(\sigma = 0)} = \mathbb{I}$$

and integrate equation (7.2.28) to find $X'_{(\sigma = \Sigma)}$. This is equivalent to integrating from an initial state of $x'_{1} = (v'_x, v'_y, \rho')^T = (1, 0, 0)^T$ then $x'_{2} = (0, 1, 0)^T$ and $x'_{3} = (0, 0, 1)^T$ so any initial perturbation can be reconstructed. The eigenvalues of the matrix $X'_{(\sigma = \Sigma)}$ are the three characteristic multipliers $\varrho_j$ for the matrix $\Pi$. The corresponding characteristic (or Floquet) exponents are $\mu_j \in \mathbb{C}$ satisfying

$$\varrho_j = e^{\mu_j \Sigma}.$$ 

Floquet theory shows that the entire solution is stable if all the characteristic multipliers satisfy $|\varrho_j| \leq 1$. For a given vortex solution, once we have calculated the three eigenvalues $\varrho_j \in \mathbb{C}$ we convert to the exponential complex form with modulus $|\varrho_j|$ and argument $\theta_j$:

$$\varrho_j = |\varrho_j|e^{i\theta_j} = e^{s_j \Sigma} e^{i\theta_j} = e^{s_j \tilde{P} e^{i\theta_j}}.$$ 

We define the growth factor per orbit, $s$ and temporal growth rate, $\gamma$ by taking the maximum $|\varrho_j|$, with

$$s = \frac{1}{\Sigma} \max_j \{|\varrho_j|\}$$

$$\gamma = \frac{1}{\tilde{P}} \max_j \{|\varrho_j|\}$$

More details of how this is implemented numerically are detailed in Chapter 8.

According to this theory, if some internal mode with a natural oscillation frequency dependent on $k$ is described by either the Eulerian perturbation equation (7.2.28) or the Lagrangian form in equation (7.2.7), unstable bands of exponential growth are expected as $k$ is varied to allow resonances of this frequency with the frequency of motion around the streamline.

\(^2\)For relevant Floquet theory definitions and theorems please see Appendix B.
7.3 Horizontal stability

We will begin by specialising to the case when $k_z \gg \sqrt{k_x^2 + k_y^2}$, what is known as the horizontal instability case. In this limit, motion occurs in uncoupled horizontal planes which will enable us to investigate the instability described in Section 7.2.3.

It can be seen from the definition of $\Theta$ in equation (7.2.22) that in the $k_z \to \infty$ limit, $\Theta \to 0$. Therefore, the Eulerian form of the stability equations (7.2.24) become a system of equations with constant coefficients for the components of $x'$.

In the $k_z \to \infty$ limit we find that the Lagrangian form of the stability equations equation (7.2.7) reduces to

$$\delta_{ij} \frac{D^2 \xi_i}{Dt^2} + 2\epsilon_{i3j} \Omega \frac{D \xi_j}{Dt} = H_i$$

(7.3.1)

or

$$\frac{D^2 \xi}{Dt^2} + 2\Omega \times \frac{D \xi}{Dt} = -\frac{\xi \cdot \nabla \rho}{\rho^2} \nabla P + \xi \cdot \nabla (v \cdot \nabla v + 2\Omega \times v).$$

(7.3.2)

Note that the $\nabla \cdot \xi = 0$ condition yields $\xi_z \to 0$ so $\xi = (\xi_x, \xi_y)$. Since the steady state equation of motion (equation (5.1.14)) is

$$v \cdot \nabla v + 2\Omega \times v = -\frac{\nabla P}{\rho} - \nabla \Phi$$

we find that the $\xi \cdot \nabla \rho$ contributions disappear:

$$\frac{D^2 \xi}{Dt^2} + 2\Omega \times \frac{D \xi}{Dt} = -\frac{\xi \cdot \nabla \rho}{\rho^2} \nabla P - \xi \cdot \nabla \left( \frac{\nabla P}{\rho} + \nabla \Phi \right)$$

$$= -\frac{\xi \cdot \nabla \rho}{\rho^2} \nabla P - \left( \frac{\xi}{\rho} \cdot \nabla \right) \nabla P + \xi \cdot \nabla \rho \nabla P - (\xi \cdot \nabla) \nabla \Phi$$

(7.3.3)

When $\rho$ is constant and $P$ and $\Phi$ are quadratic in $x$ and $y$, equation (7.3.3) becomes an equation with constant coefficients. As this is always the case arbitrarily close to the centre of any regular vortex where there is a stagnation point there are some generic consequences (see Section 7.3.2).

7.3.1 Kida horizontal limit

Applying the system of equations given in Section 7.2.2.4 in this horizontal case gives some insight into the saddle point instability which appears in the next Section 7.3.2. Remember that in this horizontal limit, $\Theta = 0$ and in the absence of a density gradient the matrix $D = 0$. Therefore $\rho'$ is a constant which, without loss of generality, we can set to be zero and ignore. (This also saves us having to find the matrix $C$.)

Firstly, recall equation (5.3.21a) for the streamfunction of the Kida vortex with aspect ratio
\( \psi = \frac{S}{2\chi(\chi - 1)} \left( \chi^2 x^2 + y^2 \right) \)

With a little algebra we therefore find the forms of the matrices \( A \) and \( B \), using \( 2\Omega = 4S/3 \) to find the final form of \( B \):

\[
A_{\text{Kida}} = \mathbb{I} \\
B_{\text{Kida}} = \frac{S}{\chi - 1} \begin{pmatrix} 0 & (2\chi + 1)(2\chi - 3) \\ 4 - \chi & \frac{3\chi}{3} \\ \frac{3\chi}{0} & 0 \end{pmatrix} \tag{7.3.5}
\]

so

\[
\frac{Dv'}{Dt} = B_{\text{Kida}}v'. \tag{7.3.6}
\]

We found in Section 5.3.3 that the arclength element \( d\sigma \) in a Kida vortex is given by

\[
d\sigma = b\sqrt{\cos^2 \eta + \chi^2 \sin^2 \eta} \, d\eta \tag{7.3.7}
\]

where \( \eta \) is the one of the elliptical coordinates and \( b \) is the semi-minor axes of a chosen streamline. We remark that

\[
|v_0| = |\nabla \psi| = \frac{Sb}{\chi - 1} \sqrt{\cos^2 \eta + \chi^2 \sin^2 \eta} = \frac{S}{\chi - 1} \frac{D\sigma}{D\eta} \tag{7.3.8}
\]

so performing the change of variables from \( t \) to \( \eta \) (similar that that detailed in Section 7.2.2.3) we have

\[
\frac{D}{Dt} = |v_0| \frac{D}{D\sigma} = \frac{S}{\chi - 1} \frac{D}{D\eta} \tag{7.3.9a}
\]

and finally

\[
\frac{Dv'_x}{D\eta} = \frac{(2\chi + 1)(2\chi - 3)}{3\chi} v'_y \tag{7.3.9a}
\]

\[
\frac{Dv'_y}{D\eta} = 4 - \chi \frac{3}{3} v'_x. \tag{7.3.9b}
\]

this system describes horizontal epicyclic oscillations with frequency

\[
\kappa_{\text{Kida}}^2 = -\frac{(2\chi + 1)(2\chi - 3)(4 - \chi)}{9\chi}. \tag{7.3.10}
\]

As shown in Figure 7.4, the epicyclic frequency \( \kappa_{\text{Kida}}^2 \) is only negative in the range \( 3/2 < \chi < 4 \) corresponding to exponential growth in \( v' \) and thus instability. In this range the pressure distribution is in a transitional saddle point regime between a minima \( \chi < 3/2 \) and maxima for \( \chi > 4 \) (see Figure 5.6). There is a generalisation of this instability seen in more generic
7.3 Horizontal stability

Figure 7.4 Epicyclic frequency for the Kida vortex in the horizontal $k_z \to \infty$ limit. $\kappa_{\text{Kida}}$ is only real outside the range $3/2 \leq \chi \leq 4$, corresponding to the location of the saddle point instability.

7.3.2 The saddle point instability

In the horizontal limit we solve equation (7.3.3) by setting

$$\xi = \xi_0 \exp i \gamma_h t,$$  

(7.3.11)

where $\xi_0$ is a constant vector, and finding an algebraic expression for $\gamma_h$. The right hand side of equation (7.3.3) is

$$- \left( \frac{1}{\rho} \frac{\partial^2 P}{\partial x_i \partial x_j} + \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) \xi_j = E_{ij} \xi_j$$  

(7.3.12)

where we note that the matrix $E$ is symmetric. Therefore equation (7.3.3) becomes

$$- \gamma_h^2 \xi_i + 2i \Omega \gamma_h \epsilon_{ij} \xi_j = E_{ij}.$$  

(7.3.13)

The two components of equation (7.3.13) are then

$$(\gamma_h^2 + E_{11}) \xi_x + (2i \Omega \gamma_h + E_{12}) \xi_y = 0$$  

(7.3.14a)

$$(-2i \Omega \gamma_h + E_{21}) \xi_x + (\gamma_h^2 + E_{11}) \xi_y = 0$$  

(7.3.14b)
which implies

\[(\gamma^2_h + E_{11})(\gamma^2_h + E_{22}) = -(2i\Omega\gamma_h + E_{12})(2i\Omega\gamma_h - E_{21})\]
\[
\gamma^4_h + \gamma_h^2(E_{11} + E_{22}) + E_{11}E_{22} = 4\Omega^2\gamma_h^2 + E_{12}E_{21} - 2i\Omega\gamma_h(E_{12} - E_{21}). \tag{7.3.15}
\]

We exploit the symmetry of \(E\) in equation (7.3.15) to cancel the imaginary term to leave us with

\[\Rightarrow \quad \gamma^4_h + \left[\text{Tr}(E) - 4\Omega^2\right] \gamma^2_h + \text{det}(E) = 0, \tag{7.3.16}\]

where ‘Tr’ and ‘det’ denote the trace and determinant of \(E\) respectively.

We have instability if \(\kappa\) has at least one complex root. A sufficient condition for this to occur is \(\text{det}(E) < 0\). In the limit approaching the vortex centre \(|r| \to 0\), the elements of \(E\) are

\[E_{ij} = -\frac{1}{\varrho} \frac{\partial^2}{\partial x_i \partial x_j} \left(P - S\varrho x^2\right), \tag{7.3.17}\]

recalling the form of the effective potential given in equation (5.1.20). The \(\text{det}(E) < 0\) condition is then equivalent to \(P - S\varrho x^2\) having a saddle point at the centre. This will occur when the \(P\) has a saddle point that appears as a maximum along the \(x\)-coordinate and an minimum along the \(y\)-coordinate line, as shown in Figure 7.5. These occur in the analytic solution for Kida vortices with aspect ratio \(\frac{3}{2} < \chi < 4\), as detailed in Section 7.3.1. Saddle points of this type are generically associated with instability in all cases, independent of density or vorticity profile (Section 8.2.2).
7.3.3 The parametric instability

Equation (7.3.3) applies in the limit $k_z \to \infty$ and is an equation with constant coefficients when $\rho$ is constant and both $P$ and $\Phi$ are quadratic in $x$ and $y$. This is always the case for any streamline in a Kida vortex core. However, in more general cases, $P$, $\rho$ and hence $E$ will be represented by a power series in $x^2$ and $y^2$. In this case, $E$ will be periodic with period $\tilde{P}/2$, one half of that associated with circulating round a streamline (see equation (5.3.32)). Then equation (7.3.3) will have coefficients that are periodic in time and parametric instability becomes possible (Section 3.5.4).

Following the discussion in Papaloizou (2005), Appendix B, close to the vortex centre the time–dependence can be treated as a perturbation and the parametric instability can be derived analytically, albeit in terms of unknown coefficients.

The parametric instability is first expected to occur when the epicyclic oscillation period is equal to the period $\tilde{P}$. Higher order bands are expected to be generated when the ratio of epicyclic oscillation period to $\tilde{P}$ is $1/2$, $1/3$, $\ldots$. For a vortex with a core like the Kida solution, these resonances occur when $\chi = 4.65$, 5.89 and 7.32 respectively (Lesur and Papaloizou, 2009).

7.4 Time–dependent wavenumber

We now consider a more general $k$ that is a function of time $t$.

7.4.1 The general form of $S_A$

The general form of $S_A$ is obtained from the solution of equation (7.1.17):

\[ \frac{DS_A}{Dt} = 0. \]

For the case where the background flow is independent of $z$ we can write

\[ S_A = S_\bot(x, y, t) + \frac{k_z z}{\lambda}. \]  \hspace{1cm} (7.4.1a)

where $S_\bot(x, y)$ satisfies

\[ \frac{\partial S_\bot}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial S_\bot}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial S_\bot}{\partial y} = 0. \]  \hspace{1cm} (7.4.1b)

The general solution of equation (7.4.1b) requires $S_\bot$ to be a function of only quantities that are invariants of the particle trajectories. These trajectories are found by solving

\[ \frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \] \hspace{1cm} (7.4.2a)

\[ \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}. \] \hspace{1cm} (7.4.2b)
The solutions for $x$ and $y$ define streamlines that are periodic in time and on which $\psi$ is constant, as per the definition of the streamfunction. The period is $\tilde{P} = 2\pi/\varpi$, as in Section 6.3, where in general $\varpi$ will be a function of $\psi$. We can therefore express $x$ and $y$ as Fourier series:

\begin{align}
  x &= \sum_{n=-\infty}^{\infty} x_n(\psi) \exp(in\phi) \quad \text{(7.4.3a)} \\
  y &= \sum_{n=-\infty}^{\infty} y_n(\psi) \exp(in\phi) \quad \text{(7.4.3b)}
\end{align}

with

\[ \phi = \varpi(t-t_0). \quad \text{(7.4.3c)} \]

Without loss of generality we will take $t_0$ to be the time at which $y$ passes through its maximum value. Equation (7.4.1b) states that $S_\perp$ is constant on an orbit defining a streamline. The general solution to this is that $S_\perp$ is an arbitrary function of $\psi$ and $t_0$, $S_\perp = S_\perp(\psi, t_0)$. As the orbits are by definition periodic this function should too be periodic in $t_0$ with period $\tilde{P} = 2\pi/\varpi$. Therefore $S_\perp$ can also be written as a Fourier series:

\[ S_\perp = \sum_{n=-\infty}^{\infty} S_n(\psi) \exp(-in\varpi t_0) = \sum_{n=-\infty}^{\infty} S_n(\psi) \exp(in[\phi - \varpi t]). \quad \text{(7.4.4)} \]

We can now find the time–dependent wavenumber $k = \lambda \nabla S_A$ where

\[ \nabla S_\perp = \frac{\partial S_\perp}{\partial \psi} \nabla \psi + \frac{\partial S_\perp}{\partial \phi} \nabla \phi = \frac{\partial S_\perp}{\partial \psi} \nabla \psi + \frac{\partial S_\perp}{\partial \phi} \varpi \nabla t. \quad \text{(7.4.5)} \]

We removed the explicit $t$ dependence using equation (7.4.2):

\[ \nabla t = \left( \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y} \right) = \left( \varpi \frac{\partial y}{\partial \psi}, -\varpi \frac{\partial x}{\partial \psi} \right), \quad \text{(7.4.6)} \]

arriving at

\begin{align}
  k_x &= \lambda \left( \frac{\partial \psi}{\partial x} \frac{\partial S_\perp}{\partial \psi} + \varpi \frac{\partial y}{\partial \psi} \frac{\partial S_\perp}{\partial \phi} \right) \quad \text{(7.4.7a)} \\
  k_y &= \lambda \left( \frac{\partial \psi}{\partial y} \frac{\partial S_\perp}{\partial \psi} - \varpi \frac{\partial x}{\partial \psi} \frac{\partial S_\perp}{\partial \phi} \right) \quad \text{(7.4.7b)}
\end{align}

Quantities are either expressed as functions of $r = (x, y)$ or with $(\phi, \psi)$ as independent variables. Transforming between these two representations is more straightforward than would initially appear as

\[ \mathbf{u} = \frac{D\mathbf{r}}{Dt} = \frac{d\phi}{dt} \frac{\partial \mathbf{r}}{\partial \phi} = \varpi \frac{\partial \mathbf{r}}{\partial \phi}. \quad \text{(7.4.8)} \]
7.4 Time–dependent wavenumber

and the Jacobian is

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = \det \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial y}{\partial \psi} & -\frac{\partial x}{\partial \psi} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{bmatrix} = 2\omega. \quad (7.4.9)$$

In addition, if only the $n = 0$ term is considered in equation (7.4.4) then $S_\perp = S_\perp(\psi) \Rightarrow \partial S_\perp/\partial \phi = 0$ and we recover the time–independent wavenumber equation (7.2.1) from Section 7.2:

$$S_A = g(\psi) + \frac{k_z z}{\lambda}.$$

7.4.2 Wavenumber increasing with time

We rewrite equations (7.4.7) for the components of $k$ so the time dependence is more explicit. If terms with $n \neq 0$ occur and $d\omega/d\psi = 0$ (the condition for no internal shear inside the vortex, see Section 5.3.3) then the wavenumber is expected to depend on $t$ as well as the coordinates. Considering the form of $S_\perp$ in equation (7.4.4) we see that

$$\frac{\partial S_\perp}{\partial \psi} = \frac{\partial S_\perp}{\partial \psi} |_\omega + \frac{d\omega}{d\psi} \frac{\partial S_\perp}{\partial \omega}$$

$$= \frac{\partial S_\perp}{\partial \psi} |_\omega - t \frac{d\omega}{d\psi} \frac{\partial S_\perp}{\partial \phi}, \quad (7.4.10)$$

where $\partial S_\perp/\partial \psi |_\omega$ results in the production of $\partial S_n/\partial \psi$ in the Fourier series expansion of $S_\perp$. Therefore, substituting for $\partial S_\perp/\partial \psi$ in equations (7.4.7a,b) we find

$$k_x = \lambda \left[ \frac{\partial \psi}{\partial x} \frac{\partial S_\perp}{\partial \psi} |_\omega - \left( \frac{d\omega}{d\psi} t - \omega \frac{\partial y}{\partial \psi} \right) \frac{\partial S_\perp}{\partial \phi} \right] \quad (7.4.11a)$$

$$k_y = \lambda \left[ \frac{\partial \psi}{\partial y} \frac{\partial S_\perp}{\partial \psi} |_\omega - \left( \frac{d\omega}{d\psi} t + \omega \frac{\partial x}{\partial \psi} \right) \frac{\partial S_\perp}{\partial \phi} \right] \quad (7.4.11b)$$

In the limit $t \to \infty$ we have

$$k_x^2 + k_y^2 \sim \lambda^2 \left( \frac{d\omega}{d\psi} \right)^2 \left( \frac{\partial S_\perp}{\partial \phi} \right)^2 |\psi|^2 t^2 \quad (7.4.12)$$

which is a product of $|\psi|^2 t^2$ and a factor that is constant on a streamline, indicating that the magnitude of the wavenumber increases to arbitrary large values at all points on it. We expect the system to be stable to these perturbations at large $t$, behaving like the shearing waves detailed in Goldreich and Lynden-Bell (1965). In Appendix C we demonstrate this is the case for a vortex with infinite aspect ratio ($y_{\text{max}} \to \infty$ with $x_{\text{max}}$ finite). It is stable at large $t$ to shearing waves, with the $k_y \neq 0$ modes in particular displaying transient growth.

In short, although some transient growth will occur, ultimately the ‘shearing–up’ of the
waves will lead to stability.

7.4.3 A note on the Kida wavenumber form

The form of the wavenumber given by equation (7.2.2) in Section 7.2 differs from that adopted in the analysis of the Kida solution in Lesur and Papaloizou (2009), which uses a special form of $k$ unique to the Kida streamlines.

Following from equations (7.1.17), (7.1.20) and

$$v_0 = \frac{S}{\chi - 1} \left( \frac{y}{\chi}, -\chi x \right),$$

we find that

$$\frac{Dk_x}{Dt} = -k_x \frac{\partial v_x}{\partial x} - k_y \frac{\partial v_y}{\partial x} = \frac{S\chi k_y}{\chi - 1} \quad (7.4.13a)$$

$$\frac{Dk_y}{Dt} = -k_x \frac{\partial v_x}{\partial y} - k_y \frac{\partial v_y}{\partial y} = -\frac{Sk_x}{\chi(\chi - 1)} \quad (7.4.13b)$$

$$\Rightarrow \frac{D^2k_x}{Dt^2} = -\frac{S^2}{(\chi - 1)^2} k_x. \quad (7.4.13c)$$

Without loss of generality we chose

$$k_x = k_0 \chi \sin \zeta \cos \phi(t) \quad (7.4.14)$$

where

$$\phi(t) = \frac{S}{\chi - 1} (t - \bar{t}) \quad (7.4.15)$$

and $\zeta, \bar{t}$ are integration constants. Substituting equation (7.4.14) into equation (7.4.13a) we find the form of $k_y$. With $k_z = k_0 \cos \zeta$ we recover an equivalent form of the $k$ used in Lesur and Papaloizou (2009):

$$k(t) = k_0 (\chi \sin \zeta \cos \phi(t), -\sin \zeta \sin \phi(t), \cos \zeta) \quad (7.4.16)$$

It is important to note that equation (7.4.16) only works when the velocity components are linear functions of $x$ and $y$. In spite of this restriction, Chang and Oishi (2010) used it in the problem of stability of a vortex core with a density gradient where this condition would not be expected to be self-consistently satisfied. Thus an assessment of the situation that occurs when this is not the case should be carried out.

From the definitions of $\Theta$ (equation (7.2.22)) and $\theta$ (equation (7.2.23)) we have that the horizontal $k_z \rightarrow \infty$ limit corresponds to $\Theta$, $\theta$, $\zeta = 0$ while the vertical limit $k_z = 0$ has $\Theta \rightarrow \infty$ and $\theta$, $\zeta = \pi/2$. Thus $\zeta$ and $\theta$ are equivalent and from now on we will use $\theta$ instead of the
redundant $\zeta$:

$$k(t) = k_0(\chi \sin \theta \cos \phi(t), -\sin \theta \sin \phi(t), \cos \theta). \quad (7.4.17)$$

For a Kida vortex in a Keplerian background recall that the period to circulate round streamlines (equation (5.3.35)) is

$$\tilde{P}_{\text{Kida}} = \frac{2\pi}{S}(\chi - 1)$$

which is independent of $\psi$ and thus there is no internal shear. Thus $\varpi_{\text{Kida}}$ is a constant;

$$\varpi_{\text{Kida}} = \frac{2\pi}{\tilde{P}_{\text{Kida}}} = \frac{S}{\chi - 1} \Rightarrow \frac{\partial}{\partial \psi}\varpi_{\text{Kida}} = 0, \quad (7.4.18)$$

which is why the terms $\propto t$ in equation (7.4.11) are absent.

For completeness’ sake, we would like to find the form of $S_A(\psi, t)$ to which this $k$ relates. Recall that the parametric representation of an ellipse given in Section 5.3 has

$$(x, y) = (b \sin \eta, b \chi \cos \eta)$$

For reasons that will become clear shortly, transform the elliptical coordinate $\eta$ thus:

$$\eta = \frac{\pi}{2} + \phi(t), \quad (7.4.19)$$

so

$$(x, y) = (b \cos \phi(t), -b \chi \sin \phi(t)). \quad (7.4.20)$$

Given that $b$ is constant on the ellipse around which the fluid parcel moves, $b = b(\psi)$. Eliminating $\phi$ we find that

$$\frac{x^2}{b^2} + \frac{y^2}{b^2 \chi^2} = 1$$

$$\Rightarrow b^2 \chi^2 = \chi^2 x^2 + y^2 = \frac{2\chi(\chi - 1)}{S} \psi = \frac{2\chi}{\varpi_{\text{Kida}}} \psi$$

$$\Rightarrow b = \sqrt{\frac{2}{\chi \varpi_{\text{Kida}}}} \psi^{1/2}, \quad (7.4.21)$$

where we used the expression for the Kida streamfunction, equation (5.3.21a) and the definition of $\varpi_{\text{Kida}}$, equation (7.4.18). Then we have

$$x = \sqrt{\frac{2}{\chi \varpi_{\text{Kida}}}} \psi^{1/2} \cos[\phi(t)] \quad (7.4.22a)$$

$$y = -\chi \sqrt{\frac{2}{\chi \varpi_{\text{Kida}}}} \psi^{1/2} \sin[\phi(t)], \quad (7.4.22b)$$

which can be consistency checked against equation (7.4.2). Comparing with the general form
of the particle trajectories given in equation (7.4.3) we see that in this form only terms with
\( n = \pm 1 \) are present. Taking the ansatz
\[
S_\perp = S_{\perp,0} \psi^{1/2} f(\phi) \tag{7.4.23}
\]
for some currently unknown \( f(\phi) \), applying equation (7.4.11) we find
\[
k_\perp = \frac{1}{2} \lambda S_{\perp,0} \varpi_{\text{Kida}} \sqrt{\frac{2}{\chi \varpi_{\text{Kida}}} \left( \chi \left[ f \cos \phi - f' \sin \phi \right], -\left[ f \sin \phi + f' \cos \phi \right] \right). \tag{7.4.24}
\]
Comparing this to the form of equation (7.4.17), if we set \( f(\phi) = 1 \) we can match coefficients:
\[
k_0 \sin \theta = \frac{1}{2} S_{\perp,0} \lambda \sqrt{\frac{2 \varpi_{\text{Kida}}}{\chi}} \tag{7.4.25}
\]
\[
k_0 \cos \theta = k_z \tag{7.4.26}
\]
We therefore reproduce the wavenumber of Lesur and Papaloizou (2009) (their equation (16),
with elliptical coordinates defined with semi-major axis in the \( x \)-direction).

Although this time–dependent wavenumber was derived from terms with \( n = \pm 1 \), and the
time–independent form is derived from adopting \( n = 0 \), you end up with the same equations
governing the stability of the Kida vortex regardless (these are given in Section 7.4.3.1). This
is because the Kida stability equations are invariant to a shift in the origin of time (it doesn’t
depend where on the streamline you start) and are thus independent of \( \bar{t} \). We may specify
\((k_x, k_y) \propto \nabla \psi \) and exactly the same equations are recovered.

### 7.4.3.1 General stability equations for the Kida vortex

Following the approach of Section 7.3.1 , the linearised equations for the Kida vortex with
aspect ratio \( \chi \) are:
\[
\frac{Dv'}{D\eta} = \Pi_{\text{Kida}}(\eta, \psi)v' \tag{7.4.27}
\]
where \( \Pi_{\text{Kida}} = A_{\text{Kida}}^{-1} B_{\text{Kida}} \) and
\[
\Theta_{\text{Kida}} = \frac{S^2}{(\chi - 1)^2} \Theta \tag{7.4.28a}
\]
\[
A_{\text{Kida}}^{-1} = \frac{1}{1 + \Theta_{\text{Kida}}(\chi^2 \sin^2 \eta + \cos^2 \eta)} \begin{pmatrix}
1 + \Theta_{\text{Kida}} \cos^2 \eta & -\Theta_{\text{Kida}} \chi \sin \eta \cos \eta \\
-\Theta_{\text{Kida}} \chi \sin \eta \cos \eta & 1 + \Theta_{\text{Kida}} \chi^2 \sin^2 \eta
\end{pmatrix}, \tag{7.4.28b}
\]
and

\[
B_{\text{Kida}} = \begin{pmatrix}
\Theta_{\text{Kida}} \sin \eta \cos \eta & \frac{(2\chi + 1)(2\chi - 3)}{3\chi} + \Theta_{\text{Kida}} \chi \sin^2 \eta \\
\frac{4 - \chi}{3} - \Theta_{\text{Kida}} \chi \cos^2 \eta & \Theta_{\text{Kida}} \sin \eta \cos \eta
\end{pmatrix}.
\tag{7.4.28c}
\]

Note that these equations are independent of semi-minor axis \(b\), i.e. independent of streamline choice. This is because \(\tilde{P}_{\text{Kida}}\) is the same on all streamlines.

### 7.4.4 Generic use of the time-independent wavenumber

In the generic case, \(\varpi\) is not the same on different streamlines. However, as discussed in Section 7.4.2, if a wavenumber increases linearly with time only temporary exponential growth is expected, with perturbations ultimately subject to at most weaker-than-exponential growth with time. This requires nonlinear analysis to determine the outcome. This can shown to be the case for the systems considered here. Furthermore, the time-dependent wavenumber in the \(t \to \infty\) limit is equivalent to the vertical stability limit \(k_z = 0\) (see the next section, 7.5).

Accordingly, we can extend the linear stability analysis for the Kida vortex (Lesur and Papaloizou, 2009) to more general cases by adopting the time-independent wavenumber \(n = 0\) and using the form for \(S_A\) given by equation (7.2.1).

The form given by equation (7.4.17) however should only be reserved for cases where \(\varpi\) is constant.

There is also an analogy with the situation that occurs in differentially rotating discs for which the fluid elements orbit on circles. Time-independent wavenumber modes here correspond to axisymmetric \((m = 0)\) modes in the disc while modes with wavenumbers that increase with time correspond to non-axisymmetric modes \((m \neq 0)\).

### 7.5 Vertical stability

We return to consider vertical stability (see Figure 7.2) for which \(k_z = 0\): the vertical velocity perturbation is zero and we only consider perturbations in the plane of the vortex. From the discussion in Section 7.4.2 and equation (7.4.12) we expect that

\[
\lim_{t \to \infty} \frac{k_z^2}{k_x^2 + k_y^2} = 0
\]

for choices of wavenumber that increase linearly with time. Our \(\Theta\) parametrisation used in Section 7.2.2 has \(\Theta \to \infty\) as \(t \to \infty\).

We will start from the perturbed momentum and continuity equations (7.1.4, 7.1.7), in the
Eulerian formalism. The incompressibility condition \( \nabla \cdot \mathbf{v}' = 0 \) gives \( k \cdot \mathbf{v}' = 0 \) so

\[
k_x v'_x + k_y v'_y = 0 \tag{7.5.1a}
\]

and we can set

\[
\mathbf{v}' = (\mu k_y, -\mu k_x) \tag{7.5.1b}
\]

for some scalar \( \mu \). The linearised form of the perturbed continuity equation is then

\[
\frac{D\rho'}{Dt} = -\mathbf{v}' \cdot \nabla \rho = -\mathbf{v}' \cdot \nabla \psi \frac{d\rho}{d\psi}
= -\mu (k_y, -k_x) \cdot \left( \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right) \frac{d\rho}{d\psi}
= \mu \left( k_x \frac{\partial \psi}{\partial x} - k_y \frac{\partial \psi}{\partial y} \right) \frac{d\rho}{d\psi}
= \mu \mathbf{k} \cdot \mathbf{v}_0 \frac{d\rho}{d\psi} \tag{7.5.2}
\]

Starting from equations (7.1.4a,b) we have

\[
\frac{D\mathbf{v}'}{Dt} + \mathbf{v}' \frac{\partial \mathbf{v}_i}{\partial x_j} - 2\epsilon_{ij3} \Omega \mathbf{v}'_j = -\frac{1}{\rho_0} \frac{\partial P'}{\partial x_i} + \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial x_i} \tag{7.5.3}
\]

and will use

\[
v_i = \epsilon_{ij3} \frac{\partial \psi}{\partial x_j} \tag{7.5.4a}
\]

\[
v'_i = \mu \epsilon_{ij3} k_j \tag{7.5.4b}
\]

\[
\frac{\partial P'}{\partial x_i} = i k_i P' \tag{7.5.4c}
\]

with this last expression from equation (7.1.16). Equation (7.5.3) then becomes

\[
\epsilon_{ij3} \frac{D(\mu k_j)}{Dt} + \mu \epsilon_{jk3} k_k \frac{\partial}{\partial x_j} \left( \epsilon_{i3} \frac{\partial \psi}{\partial x_3} \right) - 2\epsilon_{ij3} \Omega \mu \epsilon_{jk3} k_k = -\frac{1}{\rho_0} (i k_i P') + \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial x_i}. \tag{7.5.5}
\]

Using both

\[
\epsilon_{jk3} \epsilon_{i3} = \delta_{ij} \delta_{lk} - \delta_{ik} \delta_{jl}
\]

\[
\epsilon_{i3} \epsilon_{jk3} = \delta_{ik}
\]

equation (7.5.5) becomes

\[
\epsilon_{ij3} \frac{D(\mu k_j)}{Dt} + \mu \left[ k_j \frac{\partial^2 \psi}{\partial x_i \partial x_j} - k_i \frac{\partial^2 \psi}{\partial x_j^2} \right] - 2\mu \Omega k_i = -\frac{i k_i P'}{\rho_0} + \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial x_i}. \tag{7.5.6}
\]
We then take the dot product of equation (7.5.6) with \( v' \) to eliminate \( P' \):

\[
\epsilon_{ij3} \frac{D(\mu k_j)}{Dt} (\epsilon_{ik3}k_k) + \mu k_j \frac{\partial^2 \psi}{\partial x_i \partial x_j} (\epsilon_{ik3}k_k) = \frac{\rho'}{\rho_0^2} \frac{\partial P_0}{\partial x_i} (\epsilon_{ij3}k_j)
\]

\[
\Rightarrow k_j \frac{D(\mu k_j)}{Dt} - \mu k_j k_k \frac{\partial \psi}{\partial x_j} (\epsilon_{ik3}k_k) = \frac{\rho'}{\rho_0^2} \epsilon_{3ij}k_j \frac{\partial P_0}{\partial x_i} k_j
\]

\[
\Rightarrow k_j \frac{D(\mu k_j)}{Dt} - \mu k_j k_k \frac{\partial \psi}{\partial x_j} = \frac{\rho'}{\rho_0^2} (\nabla P \times k) \cdot \hat{z}
\]

\[
\Rightarrow k_j \frac{D(\mu k_j)}{Dt} + \mu k_j \frac{Dk_j}{Dt} = \frac{\rho'}{\rho_0^2} (\nabla P \times k) \cdot \hat{z},
\]

using the expression for the time evolution of \( k \) (equation (7.1.20)) in the last step. Finally, then

\[
\Rightarrow \frac{D}{Dt} (\mu |k|^2) = \frac{\rho'}{\rho_0^2} (\nabla P \times k) \cdot \hat{z}
\]

where we’ve cancelled the superfluous \( \mu \) term, since we’re assuming \( \mu \neq 0 \), and \( \hat{z} \) is the unit vector in the \( z \)-direction.

The two equations (7.5.7,7.5.2) provide a pair of first order differential equations to integrate around streamlines. In the general case, with \( k \) taking the form in equation (7.4.11), the wavenumber \( k \) is not a periodic function of time so these equations do not lead to a Floquet problem. It is also worth noting that although \( k \) grows linearly with time, its form in equation (7.4.11) implies that \( k \cdot v_0 \) remains bounded as the terms proportional to \( t \) cancel.

### 7.5.1 Vertical stability for the general vortex with internal shear

Considering the variable

\[
s = t^{-1}, \quad \Rightarrow \quad \frac{dt}{ds} = -s^{-2}
\]

and let \( W = \mu |k|^2 \). Rescaling both equations (7.5.7 and 7.5.2) we find

\[
\frac{D\rho'}{Ds} = -W \frac{k \cdot v_0}{|k|^2} \frac{d\rho}{d\psi} s^{-2}
\]

\[
\frac{DW}{Ds} = -\frac{\rho'}{\rho_0^2} (\nabla P \times \hat{k}) \cdot \hat{z} |k|^2 s^{-2}
\]

We are interested in the asymptotic behaviour of this equation. Vortex solutions with internal shear have \( d\varpi/d\psi \neq 0 \) so, provided \( \partial S_\perp / \partial \phi \neq 0 \) we have \( k \sim t \sim s^{-1} \). Since \( k \cdot v_0 \) is bounded (Section 7.5), along with the background values \( \nabla P, \rho_0 \) and \( d\rho/d\psi \) we define the coefficients

\[
\mathcal{I} = k \cdot v_0 \frac{d\rho}{d\psi}
\]

\[
\mathcal{J} = \frac{(\nabla P \times \hat{k}) \cdot \hat{z}}{\rho_0^2}
\]
and equations (7.5.9 and 7.5.10) have the asymptotic forms

\[
\frac{D\rho'}{Ds} \sim -|k|^{-2}s^{-2}W \sim -IW \quad (7.5.12)
\]

\[
\frac{DW}{Ds} \sim -J|k|s^{-2}\rho' \sim -Js^{-3}\rho'. \quad (7.5.13)
\]

Then

\[
\frac{D^2\rho'}{Ds^2} \sim -I\frac{DW}{Ds} \sim \left[IJ\right]s^{-3}\rho', \quad (7.5.14)
\]

For small \(s\) (i.e. large time \(t\)), the bracketed coefficient on the right hand side will be large. Applying WKBJ analysis to equation (7.5.14) we find

\[
\rho'(s) \sim s^{3/4}\exp\left[\pm 2IJs^{-1/2}\right]. \quad (7.5.15)
\]

Therefore, for some \(2|IJ| = \mathcal{K} > 0\) we find

\[
\rho'(t) \sim t^{-3/4}\exp\left[\mathcal{K}t^{1/2}\right]. \quad (7.5.16)
\]

The growth rate of \(\rho'\) is \(\sim t^{-1/2} \to 0\) as \(t \to \infty\) so there can be no exponentially growing solutions that apply at large times. Weaker growth could occur, however.

### 7.5.2 Vertical stability for vortices with no internal shear

We now consider the vertical stability of the polytropic model (Section 5.4), where vortex has the Kida streamlines given by equation (5.3.21a) and non-constant \(\rho = \rho(\psi)\). Note that we also have \(P = P(\psi)\) and no internal shear, \(d\varpi(\psi)/d\psi = 0\). We therefore adopt the Kida form of the wavenumber given by equation (7.4.17) with \(\zeta = \theta = 0\), corresponding to \(k_z = 0\) without encountering the problem of wavenumber increasing linearly with time:

\[
k(t) = k_0(\chi \cos \phi(t), -\sin \phi(t), 0). \quad (7.5.17)
\]

The coordinates on the streamline can be specified as indicated by equation (7.4.22b) so

\[
x = \sqrt{\frac{2(\chi - 1)}{S\chi}} \psi^{1/2} \sin \phi
\]

\[
y = \sqrt{\frac{2\chi(\chi - 1)}{S}} \psi^{1/2} \cos \phi.
\]
With the Kida velocity field given by
\begin{align}
v_0 = v_{\text{Kida}} &= \frac{S}{\chi - 1} \left( \frac{y}{\chi} - \chi x \right) \\
&= \sqrt{\frac{2S}{\chi(\chi - 1)}} \psi^{1/2} (\cos \phi(t), -\chi \sin \phi(t)), \quad (7.5.18)
\end{align}
and defining
\begin{align}
\Gamma &= \sqrt{\frac{2S\chi}{\chi - 1}} \quad (7.5.19)
\end{align}
we find that \( k \cdot v_0 = \Gamma k_0 \psi^{1/2} \) and equation (7.5.2) becomes
\begin{align}
\frac{D\rho'}{Dt} = \mu \Gamma \psi^{1/2} k_0 \frac{d\rho}{d\psi}. \quad (7.5.20)
\end{align}
Meanwhile, tackling equation (7.5.7) we have
\begin{align}
|k|^2 &= k_0^2 \left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \\
\nabla P \times k &= -(k \times \nabla \psi) \frac{dP}{d\psi} = -\Gamma k_0 \psi^{1/2} \frac{dP}{d\psi} \hat{z}
\end{align}
so the equation for the time evolution of \( \mu |k|^2 \) is
\begin{align}
\frac{D}{Dt} \left( \mu |k|^2 \right) = \frac{D}{Dt} \left[ \mu k_0^2 \left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \right] = -\frac{dP}{d\psi} \Gamma k_0 \psi^{1/2} \frac{dP}{d\psi} \frac{d\rho}{d\psi}. \quad (7.5.21)
\end{align}
Then, scaling equation (7.5.20):
\begin{align}
\left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \frac{D\rho'}{Dt} = \mu k_0 \left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \Gamma \psi^{1/2} \frac{d\rho}{d\psi}
\Rightarrow \frac{D}{Dt} \left[ \left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \frac{D\rho'}{Dt} \right] = -\frac{\rho'}{\rho^2} \Gamma k_0 \psi^{1/2} \frac{dP}{d\psi} \frac{d\rho}{d\psi} \quad (7.5.22)
\end{align}
using the time-independence of \( \psi \) and \( v \cdot \nabla \psi = 0 \). Note that the pre-factor of \( \rho' \) on the right hand side of this equation is constant on streamlines. If we now set
\begin{align}
Z = \left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \frac{D\rho'}{Dt} \quad (7.5.23)
\end{align}
then (recalling \( D\rho/Dt = 0 \)) we find that \( Z \) satisfies a form of the Hill’s equation (Hill, 1886; Whittaker and Watson, 1996) that can be written in the form
\begin{align}
\frac{D^2 Z}{Dt^2} = -\frac{\Gamma^2 \psi}{\left( \sin^2 \phi + \chi^2 \cos^2 \phi \right) \rho^2} \frac{dP}{d\psi} \frac{d\rho}{d\psi} Z = -\frac{q_{\text{Hill}}}{\sin^2 \phi + \chi^2 \cos^2 \phi} Z, \quad (7.5.24)
\end{align}
where the quantity \(q_{\text{Hill}}\) is

\[
q_{\text{Hill}} = \frac{\Gamma^2 \psi}{\rho^2 dP d\psi d\rho} = \frac{2S\chi\psi}{(\chi - 1)\rho^2 d\psi d\rho}.
\]  

(7.5.25)

This equation (7.5.24) can be interpreted as describing the evolution of a gravity wave with a time–dependent wavenumber, with \(q_{\text{Hill}}\) the square of the radial buoyancy or Brunt-Väisälä frequency. The possibility of parametric instability is expected when this frequency is large enough to be comparable to the vortex frequency \(\varpi\). Solutions to this are discussed in Section 8.4.

### 7.6 Summary and conclusions

In this chapter we have generalised the stability analysis applied to core of Kida vortex in Lesur and Papaloizou (2009) so that it can be extended to the more general vortex models calculated in Chapter 6. We consider both Lagrangian and Eulerian approaches to perturbation analysis. The former more useful for drawing analytical conclusions (such as behaviour in the horizontal stability limit in Section 7.3 and the vertical limit in Section 7.5), while the latter is more useful for future numerical calculations.

The special Kida case is a useful one as the stability problem becomes separable for a specific choice of time–dependent wavenumber (Section 7.4.3). Although this does not apply in general, it is a useful test of the numerical work performed in the next chapter and as a starting point of our perturbation analysis. Furthermore, we show the parity between the time–dependent and –independent wavenumber form for this special, shear–free vortex solution. Our treatment of a more general \(S_A\) is therefore consistent with Lesur and Papaloizou (2009), despite apparently very different forms for the wavenumber \(k\).

In the general vortex case, where we may have a non–constant vorticity profile, density profile, or both, it is still possible to look for modes localised on streamlines. From an Eulerian viewpoint, these modes can have either a time–independent wavenumber (Section 7.2) or a time–dependent one (Section 7.4).

However, when the circulation period is not constant on streamlines (i.e. the vortex contains internal shear), the time–dependent wavenumber ultimately increases linearly with time. These cannot be associated with conventional, exponentially growing linear modes, a situation familiar in shearing box calculations (Goldreich and Lynden-Bell, 1965). However, time–independent wavenumber modes can be exponentially growing; for only these can the Kida analysis be extended to more general vortices, contrary to the analysis of Chang and Oishi (2010). We therefore justify extending the use of time–independent wavenumber to cases other than the Kida vortex.

In the horizontal limit \((\theta = 0,\ \text{perturbations out of the vortex plane})\), we find a general
‘saddle point instability’, where a streamline’s epicyclic frequency is negative. This occurs when the pressure distribution has a saddle point and this instability occurs independent of density or vorticity profile. In the vertical limit ($\theta = \pi/2$, perturbations in the vortex plane), we find that vortices with Kida streamlines (where a time-dependent wavenumber applies) will be stable. However, this is not true in general and so we expect to find some exponential growth in this region. We need to move to numerical methods to establish behaviour inbetween these two limits.
Chapter 8

Numerical treatment of stability analysis

In this chapter we demonstrate how we implemented the numerics required for the stability analysis described in the previous chapter. Two different approaches were used for determining the growth rate of instability for the equilibrium vortex solutions detailed in Chapters 5 and 6.

The first method is detailed in Sections 8.1, based on extracting the various derivatives needed in equation (7.2.28) directly from the numerical vortex solutions in Chapter 6. We show the results of this approach in Sections 8.1.3. This method only gives good results for small \( \theta \lesssim 30^\circ \) (high \( k_z \)) and no central density enhancement (Section 8.1.4). In order to overcome these shortcomings, we formulate a second approach (Section 8.2), built around calculating the matrix coefficients from an analytical fit of \( \psi \). The results of this are presented in Section 8.2.2 and we discuss how the two numerical approaches support each other.

The point vortex model and its role as a limiting case are discussed in Section 8.3, while in the penultimate section we investigate the stability of the polytropic model. Finally, we present the results of our stability analysis and discuss why the instabilities of the type found here may not prevent a significant dust accumulation in vortices with a large aspect ratio.

8.1 First numerical approach

Our first method for finding growth rates in the \((\chi, \theta)\) plane (where \( \theta \) is the angle between the vertical \( \hat{z} \) and the wavevector \( k \), see Section 7.2.2.1 and Figure 7.3) for a vortex with class \( \{\alpha, \beta, \rho_m \} \) involves first picking a streamline inside the vortex and finding the coordinates of that contour. We then calculate the matrix elements in the perturbation equation (7.2.28) along that streamline. These two procedures are done in the program arclength.f90, detailed in Section 8.1.1.
Then, for each value in a set of $\theta$, we integrate each Floquet equation round a streamline and calculate the eigenvalues of the resulting $3 \times 3$ matrix to find the growth rates. This is done using a program called int_gr.f90 and is detailed in Section 8.1.2. Finally, we plot the resulting stability curve in the $(\chi, \theta)$ plane.

### 8.1.1 arclength.f90: Finding quantities along streamlines

In Chapter 6 we describe our method for producing equilibrium vortex solutions from the four parameters $\{\alpha, \beta, \rho_m, \omega_m\}$. From these variables we produce a grid of values for the stream-function $\psi$, the pressure distribution $P$, the vorticity source $A(\psi)$ and the density distribution $\rho(\psi)$. From these fields we need to find the various derivatives found in the matrix perturbation equation (7.2.28), namely

$$\frac{\partial \psi}{\partial x_i}, \frac{\partial^2 \psi}{\partial x_i \partial x_j}, \frac{\partial P}{\partial x_i} \text{ and } \frac{\partial \rho}{\partial x_i} \{i, j = 1, 2\}.$$

#### 8.1.1.1 Calculation of the streamline contour

Given a position $y_{\text{max}} \in (0, 1)$ on the $y$–axis we then calculate the streamline that passes through this point. The typical value of $y_{\text{max}}$ chosen was $y_{\text{max}} = 0.85$ for two reasons: it is far enough away from the bounding streamline that passes through $(0, 1)$, avoiding any potential boundary issues and it is far away enough from the origin that the resulting streamline passes through enough grid points to be of an appropriate resolution. This reduces the amount of interpolation inside our integration program. The effect of changing $y_{\text{max}}$ is investigated in Section 8.2.2.

From this position $(0, y_{\text{max}})$, we find the value of the streamfunction at this point, $\psi_\sigma$. We now follow this streamline around one revolution of the vortex, finding the coordinates of this curve and then evaluate the quantities $\partial \psi/\partial x_i$, $\partial P/\partial x_i$ etc. on this contour. We calculate the fields of all the required derivatives over the original grid using the following finite difference
schemes:

\[
\begin{align*}
    f_x(x, y) &= \frac{1}{12h_x} \left[ -f(x + 2h_x, y) + 8f(x + h_x, y) - 8f(x - h_x, y) + f(x + 2h_x, y) \right] + O(h_x^2), \\
    f_{xx}(x, y) &= \frac{1}{h_x^2} \left[ f(x + h_x, y) - 2f(x, y) + f(x - h_x, y) \right] + O(h_x^2), \\
    f_{yy}(x, y) &= \frac{1}{h_y^2} \left[ f(x, y + h_y) - 2f(x, y) + f(x, y - h_y) \right] + O(h_y^2), \\
    f_{xy}(x, y) &= \frac{1}{h_x h_y} \left[ f(x + h_x, y + h_y) - f(x + h_x, y - h_y) - f(x - h_x, y + h_y) + f(x - h_x, y - h_y) \right] + O(h_x h_y),
\end{align*}
\]

where \( h_x \) is the grid spacing in the \( x \)-direction, \( h_y \) is the grid spacing in the \( y \)-direction.

However, the equilibrium vortex solutions created in Chapter 6 are expressed over a rectangular grid (typically of \( 512 \times 2048 \) points) and not a grid that follows the contours (like an elliptical grid in the Kida case). We therefore need to calculate the coordinates of the chosen streamline.

For \( |y| \in [y_{max}/2, y_{max}] \), the curve of the streamline contour is tightest (especially for large \( \chi \) vortices) so we find the contour here by fixing \( x \) then interpolating between grid points in the \( y \)-direction. As in Figure 8.1, if we have some \( x = X \) where we want to find the \( y \)-coordinate of the streamline \( \psi = \psi_\sigma \) we begin by iterating in the \( y \)-direction to find the two grid points either side of the streamline. We then use a simple linear interpolation to find the \( y \)-coordinate \( Y \) where the streamline crosses \( x = X \):

\[
Y = y_n + \frac{y_{n+1} - y_n}{\psi(X, y_{n+1}) - \psi(X, y_n)} \times (\psi_\sigma - \psi(X, y_n)),
\]

where \( \psi(X, y_n) \) is the value of the streamfunction at grid point \( (X, y_n) \). Evaluating quantity \( Q \) (which could be \( \partial \psi / \partial x_i \), \( \partial P / \partial x_i \), etc.) at the point \( (X, Y) \) also requires linear interpolation:

\[
Q(X, Y) = Q(X, y_n) + \frac{Q(X, y_{n+1}) - Q(X, y_n)}{y_{n+1} - y_n} \times (Y - y_n),
\]

where again \( Q(X, y_n) \) is the value of \( Q \) at grid point \( (X, y_n) \). In the range \( |y| < y_{max}/2 \), we do an entirely similar calculation, fixing \( y = Y \) instead of \( x \) due to the steep gradient of the contour in this region. This helps achieve a relatively even spacing of points around the streamline, important for avoiding large jumps in our arc length variable which could lead to inaccurate interpolation.

Once we have the coordinates of a complete contour, we calculate the arc length \( \sigma \) around
Figure 8.1 Calculating a point on the contour $\psi = \psi_\sigma$ while fixing $x = X$ or $y = Y$. 
the vortex streamline using the simple distance norm:

\[ \sigma_{n+1} = \sigma_n + \sqrt{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}. \]  

(8.1.4)

This was deemed accurate enough due to the grid size; for the Kida vortex, a streamline passing through \((0, 0.85)\) had approximately 400 points for \(\chi = 10\) and approximately 3500 points for \(\chi \approx 1\).

Finally, we have arrays of all our derivative quantities \(Q\) as functions of \(\sigma_n\). Therefore, we can proceed to integrate the matrix stability equation (7.2.28) for different \(\theta\) (Section 8.1.2).

### 8.1.1.2 Calculation of aspect ratio

The aspect ratio for the chosen streamline is calculated using the formula

\[ \chi = \frac{y_{\text{max}}}{x_{\text{max}}}, \]  

(8.1.5)

where \(x_{\text{max}}\) is the maximum extent of the streamline along the positive \(x\)-axis. Note that \(\chi\) can vary throughout the vortex (Section 6.3.1). Unless otherwise stated, the value \(\chi\) for a given vortex will be the value at \(y_{\text{max}} = 0.85\).

### 8.1.1.3 Calculation of the period

Recall from Section 5.3.3 a vortex’s internal shear can be visualised using the period around streamlines, calculated using equation (5.3.32):

\[ \tilde{P} = \oint \frac{d\sigma}{|\nabla \psi|}. \]

For a set of \(y_{\text{max}} \in (0, 1)\), for each \(y_{\text{max}}\) we calculate \(1/|\nabla \psi|\) as a function of the arclength using the approach outlined above. By definition of the streamfunction this shouldn’t be zero anywhere so singularities won’t be a problem. With \(f(\sigma) = 1/|\nabla \psi|\) and \(N\) points \(\{\sigma_n : n = 1, N\}\) around our streamline contour, we calculate the period:

\[ \tilde{P} = \oint \frac{d\sigma}{|\nabla \psi|} = \sum_{n=1}^{N-1} \int_{\sigma_n}^{\sigma_{n+1}} f(\sigma) d\sigma \approx \sum_{n=1}^{N-1} (\sigma_{n+1} - \sigma_n) \left[\frac{f(\sigma_n) + f(\sigma_{n+1})}{2}\right]. \]  

(8.1.6)

This enables us to produce plots of \(\tilde{P}\) vs. \(y_{\text{max}}\) for our equilibrium solutions, as in Chapter 6.

The method was verified using the analytical Kida solution, which has period constant with \(\chi\) (equation 5.3.35):

\[ \tilde{P}_{\text{KIDA}} = \frac{2\pi}{S} (\chi - 1). \]
8.1.2 **int_gr.f90**: Integrating solutions along streamlines to find growth rates

This program takes the various derivatives found along streamlines by `arclength.f90`, performs a numerical integration of equation (7.2.28) for a range of $\theta$ (via the related variable $\Theta$, see equation (7.2.23)), and find the resulting growth rate of instability. Following the Floquet approach detailed in Section 7.2.3, we approach this as an initial value problem (IVP) in arclength $\sigma$, solving for a matrix of three perturbation vectors $x'_i = (v'_x, v'_y, \rho')^T$ such that

$$X' = ([x'_1], [x'_2], [x'_3]).$$

The initial condition is set to be $X'(\sigma = 0) = I$, so that $x'_1 = (1, 0, 0)^T$ etc., spanning the entire space of perturbations. Then, equation (7.2.28) is integrated using the Bulirsch–Stoer algorithm (Stoer and Bulirsch, 2002) once around the vortex to find $X'(\sigma = \Sigma)$. This is done using various routines from ‘Numerical Recipes’ (Press et al., 1993), namely **ODEINT**, **MMID**, **RAN1** and **BSSTEP**. Linear interpolation is again used to supply the value of any quantity for any arclength value in the range $[0, \Sigma]$, as required by the integration routine. Care has to be taken to ‘wrap around’ the coefficients outside $0 \leq \sigma \leq \Sigma$, since they are all periodic functions, with period $\Sigma$.

We perform the above integration for each of the three initial values of $x'$, for a set of values of $\Theta = \tan^2 \theta$ corresponding to equally spaced $\theta \in \{0, 85^\circ\}$. Once the matrix $X'(\sigma = \Sigma)$ has been calculated for each $\theta$, we then find its eigenvalues using the Numerical Recipes routines **ELMHESS** and **HQR**. The first of these routines converts $X'(\sigma = \Sigma)$ into an upper Hessenberg matrix, while the second finds the eigenvalues of this matrix. It returns the three characteristic Floquet multipliers, $\varrho_j$, in the form

$$\varrho_j = \Re(\varrho_j) + \Im(\varrho_j)i,$$

which, according to equation (7.2.33b), we can extract our final growth rate $\gamma$:

$$\gamma = \frac{1}{P} \max_j \{\log |\varrho_j|\}. \quad (8.1.7)$$

Repeating this for all the $\omega_m$ in our vortex class $\{\alpha, \beta, \rho_m\}$, we plot $\gamma$ as a heat plot against the axes $\chi$ (calculated as in Section 8.1.1.2) and $\theta$.

8.1.3 Results from first approach

Stability plots for vortices with variable vorticity profile and constant density were, for the most part, of acceptable quality. The results of these can be seen in Figure 8.2. They are plotted
for the regions $1 < \chi \lesssim 10$, $0^\circ \leq \theta < 70^\circ$ and with growth rates in the range $10^{-4}\Omega < \gamma < 1\Omega$. We show $\alpha$ in the range $0 \leq \alpha \leq 4$ (see Figure 6.2 for the equilibrium distributions of these).

As discussed, the chosen streamline passes through $(0, 0.85)$, unless otherwise stated.

Figure 8.2a shows the Kida case which we can compare to Figure 8.3, produced using analytic derivatives for the matrix coefficients. The main band emerging from $3/2 < \chi < 4$ (the so called 'saddle point' instability) is reproduced, as is the band emerging from $\theta = 0^\circ$, $\chi \approx 5.9$. However, there is a spurious, phantom band above this main one. This phantom band persists throughout the stability curves for vortices with constant density. Furthermore, when we extend the plots to the full range of $\theta \in [0^\circ, 90^\circ]$, above $\theta \approx 70^\circ$ the factor $\Theta = \tan^2 \theta$ appearing in the matrix coefficients becomes very large. This magnifies interpolation errors, especially for vortices with $\chi > 5$. The result of this is a broad, spurious band for large $\theta$, as shown in Figure 8.5. For these reasons we restrict $\theta < 30^\circ$ for this first method. This range is sufficient however for a meaningful comparison with the second method detailed later.

Increasing $\alpha$ leads to more bands appearing for $\chi \gtrsim 4$, but for the $\{\alpha = 2, 4\}$ cases it is difficult to know what are real resonant bands and what are spurious. Noise from small aspect ratio cases, $\chi \lesssim 2$, starts to creep in due to convergence issues in the original equilibrium solutions.

There are real issues when a central density enhancement is introduced; typical output is shown in Figure 8.4. The main saddle point instability band is picked up but everything else was obscured by noise that no amount of filtering could remove. Therefore a new approach needed to be found.

8.1.4 Summary of the difficulties with this approach

In Figures 8.5 and 8.6 we show the various problems that occur using this method. We find the following problems in the regions marked in Figure 8.6:

- **Region (1):** Affected by convergence issues with the original equilibrium solutions. In order to produce solutions in the region of $\chi = 1$, increasingly large $\omega_m$ need to be added to existing solutions. This makes both convergence and producing many solutions in this region difficult.

- **Region (2):** Interpolation in `arclength.f90` and `int_gr.f90` results in errors magnified for large $\theta/\Theta$.

- **Region (3):** The fixed grid over which the equilibrium solutions are generated creates resolution difficulties for large $\chi$ vortices which contain less points. We also have the appearance of ‘phantom’ resonance band(s) at $\theta \approx 40^\circ$.

Furthermore, increasing the resolution of these plot in the $\chi$–direction is very costly as it requires calculating further equilibrium solutions; even starting from nearby solutions this
Figure 8.2 Results of the first approach, finding stability plots in the \((\chi, \theta)\) plane for vortex solutions with no density. The region due the saddle point instability appears on the left hand side as a wide band \(3.2 < \chi < 4\) which shrinks in width and travels to smaller \(\chi\) before disappearing.
Figure 8.3 Stability plot for the Kida vortex, produced using analytic coefficients. This is in agreement with the plot in Lesur and Papaloizou (2009) (their Figure 3). Note the logscale along the $x$-axis; since using analytic expressions for the matrix coefficients we are not confined by the number of high quality equilibrium solutions we can create so can extend the $x$-axis arbitrarily far.
Numerical treatment of stability analysis

Figure 8.4 Results of the first approach, finding stability plots in the \((\chi, \theta)\) plane for vortex solutions with a central density enhancement. Despite picking up the saddle point instability \(1.5 \lesssim \chi \lesssim 4\), all other bands are obscured by noise.

8.2 Second numerical approach

Most of the problems with the first approach occur because of interpolation in both contouring and the calculation of the derivatives in \texttt{arclength.f90}, to the detriment of the stability calculations. Therefore, instead of interpolating to find quantities along streamlines, we produced polynomial fits of the streamfunction \(\psi\).

8.2.1 Implementation

The fits of \(\psi\) from the grid of values produced for each equilibrium solution are calculated using the \texttt{polyfitweighted2.m}, MATLAB code of Rogers (2007). This finds a least-squares fit of the 2D data \(\psi(x, y)\) with an \(n\)th order polynomial and weighting \(w(x, y)\). The original data was weighted by either 0 or 1 depending on whether a point was outside or inside the vortex boundary, respectively. Due to the symmetry of the solutions we fit polynomials in \(x^2\) and \(y^2\),
8.2 Second numerical approach

Figure 8.5 Stability plot for \( \{0.25, 0, 0\} \) using the first method, for full range of \( \theta \). The topmost region, circled in blue, is a spurious horizontal band, caused by the large \( \theta/\Theta \) factor that appears in the matrix coefficients, magnifying resolution and interpolation errors. The lower highlighted region is a phantom resonance band. Note that there is nevertheless good agreement with the Kida results shown in Figure 8.3 for \( \theta \lesssim 30^\circ \).

Trying quadratic, cubic and quartic fits. For example, the form in the quadratic case would be:

\[
\psi(x^2, y^2) = a_{00} + a_{10} x^2 + a_{01} y^2 + a_{20} x^4 + a_{11} x^2 y^2 + a_{02} y^4.
\]

In order to enforce \( \psi = 0 \) at the centre of the vortex, without loss of generality we set \( a_{00} = 0 \). We successfully found quadratic and cubic fits, but the extra degrees of freedom associated with the quartic fits meant they usually failed to converge. We used the cubic fit, checking their validity against the quadratic fits, the results of the previous approach for \( \theta < 30^\circ \) and the growth rates from the analytic Kida solution. The coefficients \( a_{10}, a_{01} \) etc. were well behaved and varied in a predictable way so could be straightforwardly interpolated as functions of \( \chi \) and \( y_{\max} \). This allowed for any amount of interpolation between \( \chi \), eliminating the problem of calculating models ab initio at high resolution along the \( \chi \)-axis.

With these fits found, the location on the streamline as the integration proceeds is specified
Figure 8.6 An infographic showing the problem areas in the \((\chi, \theta)\) stability diagram while using the first numerical approach. Full explanation is given in Section 8.1.4.

by solving the equations

\[
\frac{Dx}{Dt} = \frac{\partial \psi}{\partial y}, \tag{8.2.1}
\]

\[
\frac{Dy}{Dt} = -\frac{\partial \psi}{\partial x}, \tag{8.2.2}
\]

with the pressure field calculated using equation (6.2.3):

\[
P = \frac{2}{3} S^2 x^2 - \frac{1}{2} |\nabla \psi|^2 - \frac{1}{3} S (\psi - \psi_b) + \int_{\psi_b}^{\psi} A(\psi') d\psi',
\]

and the density distribution using equation (5.4.3):

\[
\rho = \left[1 - \frac{b(\psi - \psi_b)}{\psi_b}\right]^n.
\]

At the vortex centre \(\psi = 0\) so \(\rho_{\text{max}} = (1 + b)^n\). With the central \(\rho_{\text{max}}\) taken from the gridded density data and \(n = 1\) we can reconstruct \(\rho(\psi)\). When \(n = 1\), the different expressions for \(\rho\) given in equations (5.2.6b) and (5.4.3) are equivalent, with \(n \equiv \beta\).

The various derivatives are then calculated by differentiating the analytic expressions, therefore avoiding the interpolation errors introduced in the first method. Integration is done using
an analogous, Bulirsch–Stoer method as in the first approach and we also use the same \( \theta \) to parametrise \( k \). After integration, the growth rate of any instability present is obtained by solving the additional equation

\[
\frac{D\gamma_2}{Dt} = \log |v'|.
\]  

(8.2.3)

For a system with growth rate \( \gamma \), we expect that ultimately \( \gamma_2 \to \gamma t^2/2 \) so we determine \( \gamma \) by making a parabolic fit to \( \gamma_2 \) at large times. We found that integrations running for 1000 circulations around streamlines could detect growth rates down to \( \gamma \sim 10^{-4}\Omega \).

In order to resolve fine parametric bands, regions of instability often require high resolution in the \((\chi, \theta)\) phase space; typically for a model \( \{\alpha, \beta, \rho_n\} \) we require a \( 300 \times 300 \) grid. This method does allow for a significant increase in resolution in the \( \chi \)–direction over the first approach (where we were limited to around 50 vortex models).

### 8.2.2 Results from second approach

We present the results for the constant–density cases in Figure 8.7. The Kida solution case in Figure 8.7a agrees with that of Figure 8.3 both in terms of structure and magnitude of growth rate. Similarly we can see the same structures (provided \( \theta \lesssim 30^\circ \)) between the nonzero \( \alpha \) cases shown in Figure 8.2 and Figures 8.7b–8.7e.

In Figure 8.7 we show results for streamlines with \( y_{\text{max}} = 0.85 \) and \( \alpha = \{0, 0.25, 0.5, 1.0, 2.0, 4.0\} \). The associated maximum growth rate plot can be seen in Figure 8.8. As \( \alpha \) increases, the instability band originating from \( \chi \approx 4.65 \) widens and moves to smaller values of \( \chi \), while the small-\( \chi \) region associated with the saddle point instability shrinks and eventually disappears, as can be clearly seen in the maximum growth rate plot in Figure 8.8b. This is due to vortices with these steep vorticity profiles no longer containing the pressure distribution saddle points upon which this instability depends. Several additional instability bands appear at larger values of \( \chi \).

In Figure 8.7f, the deviation of the \( \alpha = 4 \) case from the smaller \( \alpha \) cases preceding it, the results from the first approach (Figure 8.2f) and the limiting point vortex case (Figure 8.15) is most probably due to the extreme streamline shape near the boundary of these vortices. As was observed in Section 6.3.1, a vorticity profile this steep results in severely ‘pinched’ streamlines near the \( y \)–axis. Therefore, the fits of \( \psi \) in \( x^2 \) and \( y^2 \) will be less accurate near the boundary of such vortices (we would, however, expect them to still be good near the vortex core).

In order to study the effects of introducing a variable vorticity profile in the core, in Figure 8.9 we illustrate the stability of vortices with \( \alpha = 0.25 \). The stability of motion on streamlines with \( y_{\text{max}} = \{0.5, 0.85, 0.95\} \) is shown in Figures 8.9a, 8.9b and 8.9c respectively. As indicated in Section 7.3.3, moving outwards from the vortex centre, we expect parametric instability to occur for \( \chi \approx 4.65 \). This is visible for the \( y_{\text{max}} = 0.5 \) streamline, where a narrow
Numerical treatment of stability analysis

Figure 8.7 (Continued over the next two pages.)

(a) Stability plot for vortex class \( \{\alpha, \beta, \rho_m\} = \{0, 0, 0\} \)

(b) Stability plot for vortex class \( \{0.25, 0, 0\} \)
Figure 8.7 (Continued on next page.)

(c) Stability plot for vortex class \{0.5, 0, 0\}

(d) Stability plot for vortex class \{1, 0, 0\}
Figure 8.7 Results of the second approach finding stability plots in the $(\chi, \theta)$ plane for vortex solutions with uniform density. Plots of the maximum growth rates for these solutions can be seen in Figure 8.8. These plots were produced using IDL (Liu et al., 2013).
Figure 8.8 Plot of maximum growth rate against $\chi$ for vortex solutions with uniform density, including comparison with the point vortex case discussed in Section 8.3. The solutions tend to match the point vortex at large $\chi$ as we would expect. Figure 8.8a shows this over the whole range of $\chi$ while 8.8b restricts it to the maximum growth rate of the strong saddle point instability, showing its migration to smaller $\chi$ while remaining the same strength. Note that this instability does not exist for the $\alpha = 2$ solution nor the point vortex, the limit of large $\alpha$. 
Numerical treatment of stability analysis

Figure 8.9 The stability plot for vortex class \{0.25, 0, 0\} for different $y_{\text{max}}$, i.e. different streamlines in the same vortex solution.
vertical instability band appears at this location. As $y_{\text{max}}$ is increased (i.e. moving away from the vortex centre), this band is seen to broaden. Simultaneously, the instability region at large $\chi$ extends towards the $\chi$–axis, intersecting this axis at smaller values of $\chi$. In addition, the region associated with the saddle point instability (Section 7.3.2) shifts towards smaller values of $\chi$.

The stability of vortices with $\alpha = 0$, $\beta = 1$ and varying $\rho_m$ are illustrated in Figure 8.11, while results for vortices with $\alpha = \{0.25, 0.5\}$ and a central density enhancement are demonstrated in Figure 8.13. For these calculations, recall that $\chi$ was varied by changing $\omega_m$ while $\rho_m$ was chosen such that a fixed mass per unit length was added to the vortex for all $\chi$. This
Figure 8.11 Resulting stability plots when a central density enhancement is added to a Kida vorticity profile. The maximum growth rate plots for these solutions can be seen in Figure 8.12. The small ‘islands’ of instability are a result of both insufficient resolution to pick up the whole band and IDL’s plotting algorithm.
results in the central density $\rho_{\text{max}}$ increasing with $\chi$ (see Section 6.3.3), with central densities ranging from 3 to 12 times the background level at large values of $\chi$.

In Figure 8.11, we can see the effect of introducing a small density excess on the stability of a Kida vortex. For the case with $\rho_m = 2$, Figure 8.10 shows the results for streamlines with $y_{\text{max}} = \{0.5, 0.67, 0.85, 0.95\}$. In the case of $y_{\text{max}} = 0.5$, where departures from the Kida–like quadratic streamfunction are smaller, a parametric instability band is seen to emanate from the expected location $\chi \approx 4.65$. This appears to connect with the band originating from large $\chi$ and $\theta$ in the Kida case (Figure 8.7a), leaving a region that is very weakly unstable (or possibly stable) between them. Bands with these features seem to a be common feature in these calculations, requiring time–consuming calculations at high resolution to locate them. (This renders the first method we adopted inapplicable.) In this case, even with the much improved second numerical approach, we found it impractical to resolve instabilities with growth rates $\gamma < 10^{-4}\Omega$.

When the streamline at $y_{\text{max}} = 0.85$ is considered, the band originating from $\chi \approx 4.85$ has broadened to produce and unstable region with growth rate $\gamma \sim 0.05\Omega$ for $4 < \chi < 5$. Two additional narrow bands appear at larger $\chi$ with characteristic growth rates $\gamma \sim 0.01\Omega$. The instability band that appears between $2 < \chi < 4$ is similar to that seen in the models with large $\alpha$ and no density excess.
Figure 8.13 The stability of non–Kida vorticity profiles with a central density enhancement. In both cases, $\rho_{\text{max}} \approx 2.5$ when $\chi = 8$. The associated plots for the maximum growth rate can be found in Figure 8.14.
8.3 Point vortex stability

We also examined a point vortex model. This adopts the streamfunction

$$\psi = K \log |r| + \frac{3}{4} \Omega^2 x^2$$  \hspace{1cm} (8.3.1)

which corresponds to a Bernoulli source localised at the vortex centre/box origin (see e.g. Batchelor, 2000). We consider it here as it can be thought of as the limiting case of large $\alpha$. The aspect ratio of the streamline at $y_{\text{max}} = 0.85$ is fixed by an appropriate choice of the constant

![Diagram](image_url)
Numerical treatment of stability analysis

Figure 8.15 Stability plot for the point vortex model with streamfunction \( \psi = K \log |r| + \frac{3}{4} \Omega^2 x^2 \), the limiting case for large \( \alpha \). Note its similarities to Figure 8.7e. The maximum growth rate for each \( \chi \) is compared to the other constant density solutions in Figure 8.8a.

\( K \). The plot is produced by varying \( K \) for a given \( y_{\text{max}} \); requiring that \( \psi(x_{\text{max}},0) = \psi(0,y_{\text{max}}) \) in equation (8.3.1), and given \( \chi = y_{\text{max}}/x_{\text{max}} \) we find that

\[
K = \frac{3 \Omega^2 y_{\text{max}}^2}{4 \chi^2 \log \chi}.
\]  

Thus \( K = K(\chi) \) provides a streamline with a specified \( \chi \) that passes through a given \( y_{\text{max}} \).

The stability properties are obtained using the same method as for the other models, with the resulting stability plot is given in Figure 8.15. Growth rates for \( \chi \lesssim 5 \) are \( \gamma \sim 0.05\Omega \).

The point vortex is the limiting case of a constant density vortex with infinite \( \alpha \). The relationship is apparent comparing the large-\( \chi \) solutions for the constant density, \( \alpha = 2 \) case shown in Figure 8.7e with the point vortex results in Figure 8.15. When \( \alpha \) is large and the vorticity source strongly peaked, a streamline close to the boundary experiences a potential very similar to the point vortex, resulting in a similar shape and stability characteristics. We note that the failure of the polynomial fits in \( x^2 \) and \( y^2 \) to describe \( \psi \) near the boundary in the \( \alpha = 4 \) case (Figure 8.7f) prevent it from exhibiting these same features; they are however
seen in for $\theta \lesssim 30^\circ$ in the results of the first approach (Figure 8.2f).

Although these models (and those detailed in Figure 8.7) do not have any density excess, they are relevant to streamlines outside a high-density core so are of generic significance for vortices accumulating dust in their cores.

### 8.4 Polytropic model stability

In Figure 8.16 we show the stability of the polytropic models detailed in Section 5.4. We use $n = 1$ (so there is a direct relationship to the earlier power law solutions – see Section 8.2.1) and central densities of 3 and 6 times the background value of $\rho$. The results are for streamlines with $y_{\text{max}} = 0.95$ as these show the most detail. As can be seen in Figure 8.17, a larger central density enhancement does not result in any substantial increase in the maximum growth rate, although there is the loss of a stable ‘gap’ as $\rho_{\text{max}}$ is increased.

Recall that for these models, vortices associated with values of $\chi \neq 7$ have to be considered to be immersed in a non–Keplerian background flow. The results are qualitatively similar to models with a central density excess in a Keplerian background flow, with more instability bands appearing as $\rho_{\text{max}}$ is increased. The characteristic growth rates at $\chi \approx 7$ are $\gamma \sim 0.01 \Omega$. Similarly, the results of the constant density polytropic model shown in Figure 8.16a are qualitatively similar to that of the Kida vortex shown in Figure 8.7a, with the strong saddle point instability region shifted to smaller values of $\chi$.

We also consider the vertical stability of the polytropic model, as discussed in Section 7.5.2. Note that this model has no internal shear and so in this limit, assuming exponentially growing modes, we can proceed analytically. Solving equation (7.5.24) leads to a standard Floquet problem. $q_{\text{Hill}}$ is constant on streamlines and can be interpreted as the square of a radial buoyancy/Brunt-Väisälä frequency and is what the period round the streamline resonates with. Note that $q_{\text{Hill}} = q_{\text{Hill}}(b, n)$, where $\rho_{\text{max}} = (1 + b)^n$.

We consider the case when $\chi = 7$ as this is associated with the Keplerian background case (see equation (5.4.1)). The growth rate is plotted as a function of $q_{\text{Hill}}/\Omega^2$ in Figure 8.18. The maximum growth rate of $\gamma \sim 0.12 \Omega$ occurs for $q_{\text{Hill}}/\Omega^2 = 1.5$, which is around an order of magnitude greater than the growth rates seen in Figure 8.16 at this value of $\chi$. Thus modes with $k_z = 0$ (as in Chang and Oishi, 2010) may dominate in this case. However, as noted in Section 7.2.1, the neglect of vertical stratification is only justified if $k_z^2/k_\perp^2$ is large so this is likely to only be valid close to the midplane where the vertical stratification is least.

### 8.5 Summary

In this chapter we took the analytical work of Chapter 7 and attempted to find the exponential growth rate, $\gamma$, for the various vortex solutions found in Chapter 6.
(a) Stability plot for polytrope solution no central density enhancement ($\rho_{\text{max}} = 1$).

(b) Stability plot for polytrope solution with $n = 1$ and $\rho_{\text{max}} = 3$.

Figure 8.16 (Continued on next page.)
(c) Stability plot for polytrope solution with $n = 1$ and $\rho_{\text{max}} = 6$.

**Figure 8.16** Polytrope solutions for $\rho_{\text{max}} = \{0, 3, 6\}$. A plot of the maximum growth rates for each aspect ratio are shown in Figure 8.17.
Figure 8.17 Plot of maximum growth rate against $\chi$ for the polytropic vortex solutions, with the Kida case illustrated for comparison. Note that the polytropic solutions do not exist below $\chi = 2$ for $\rho_{\text{max}} > 1$ (see Section 5.4 for details). There is some low-level noise due to the difficulty of resolving the fine resonance bands for large $\chi$.

Figure 8.18 Growth rate of the parametric instability of the polytropic model for $\chi = 7$ and $k_z = 0$. 
Our first approach involved taking the gridded data for $\psi$, $P$ and $\rho$, producing fields of derivatives $\partial \psi / \partial x_i$ etc. using a finite difference regime, finding a streamline contour via linear interpolation then calculating the values of these derivatives along this streamline as functions of arclength $\sigma$. With these we then performed a Bulirsch–Stoer integration once round the vortex streamline for each desired value of $\theta$. Finally, we extracted the associated instability growth rate by exploiting Floquet theory.

As shown in Figure 8.6, there were significant problems with this method. The need to calculate new equilibrium vortex models to increase the resolution in the $\chi$–direction made this approach computationally expensive, especially in the small–$\chi$ region. Errors introduced by interpolation at various points in arclength.f90 and int_gr.f90 also caused significant problems at large $\theta$ and for vortex solutions with a central density enhancement. Results for $\theta \lesssim 30^\circ$ were however reliable and allowed comparison with the second method and the analytical Kida case.

The second approach revolved around avoiding the repeated use of interpolation. This was done by producing quadratic and cubic fits in $\{x^2, y^2\}$ for each vortex solution’s streamfunction, $\psi$. These coefficients could then be expressed as functions of $\chi$, increasing the resolution along this axis. The various quantities needed in the matrix coefficients of equation (7.2.28) were then directly calculated around the chosen streamline contour and integration was again done using Bulirsch–Stoer. The viability of this method was tested against the result shown in Lesur and Papaloizou (2009) for the Kida vortex (Figure 8.3), the constant density results of the first approach and what we knew analytically in the horizontal, $k_z \to \infty$ limit from Chapter 7.

Using this second approach, we have generalised the Kida vortices’ strong ‘saddle point’ instability (Section 7.3.2) to the centre of general vortices. Apart from having a strong instability, vortices of this type are of less interest as the associated saddle point in the pressure distribution means they will not attract dust (Section 3.5).

We have also shown how parametric instability bands should appear, moving outwards from the vortex centre, and consistent with the numerical results varying $y_{\text{max}}$ in Section 8.2.2. Additionally, vortices with a concentrated vorticity source and no density excess have strong instability bands at all aspect ratios with growth rates $\gamma \sim 0.05 \Omega$. This is also seen in the limiting case of the point vortex (Section 8.3). Meanwhile, models with a density excess can show many narrow instability bands, though those with flatter (small $\alpha$) vorticity profiles show less strongly growing modes with $\gamma \sim 0.01 \Omega$. In general, we find that including a non-zero $\rho_{\text{max}}$ does not lead to an increase in $\max_{\theta} \gamma$, but it does reduce the size of any completely stable gap or eliminates it completely (see Figure 8.14).

We also investigated the stability of the dust–laden polytropic model (Sections 5.4 and 8.4) with $\chi = 7$ to modes localised on streamlines, adopting a time–dependent wavenumber for the vertical stability case with $k_z = 0$. This latter form is the case considered by Chang and Oishi (2010). Note that this analysis of the polytropic model is possible as it is a special case with no
internal shear that matches onto the background Keplerian shearing flow only when $\chi = 7$. We found that when $k_z = 0$ (Figure 8.18), modes could occur with growth rates $\gamma \sim 0.1\Omega$, around an order of magnitude greater than when a general wavevector is considered (Figure 8.16). However, these results would be affected by shear inside the vortex if an attempt is made to generalise them to other vortices. This model has no internal shear and so is special; if shear is included, this may shear out disturbances before they can grow.

Even with the weak internal shear $\sim 0.01\Omega$ shown by vortices in Figure 6.5, with putative growth rates $\gamma \sim 0.1\Omega$, we might expect temporary growth for 10 growth times, or a temporary amplification factor $\sim 10^4$. A nonlinear analysis would be required to resolve the outcome in this case. Also, as noted in Section 7.2.1, these polytropic $k_z = 0$ modes are also likely to be affected by vertical stratification unless we are considering streamlines very close to the disc midplane. We will investigate this further in the next chapter.

8.6 Discussion and conclusions

We have seen that dust particles attracted from the outer disc to a vortex core with high aspect ratio, $\chi$, may well encounter parametric instabilities with characteristic growth rates of a few $10^{-2}\Omega$ up to $0.1\Omega$. This is the case even outside any high density core and so it is important to assess potential consequences for dust accumulation in vortices.

8.6.1 Inhibiting parametric instability

The instabilities we have found are both parametric and local so they can be inhibited by either dissipative effects or effects that disrupt the periodicity of the circulated motion. The magnitude of the dissipative effects is uncertain (for example it is dependent on location in the PP disc) so we will just consider the second effect.

Firstly, the accumulation of dust in vortices may occur rapidly (e.g. Lyra et al., 2009; Méheut et al., 2012b), such that dust particle motion departs significantly from being periodic. This may also be the case for gas motion (Méheut et al., 2012b). For these reasons, parametric instability may not have been seen in simulations of dust trapping up to now.

Secondly, if we suppose this parametric instability is present, it is likely to lead to some low-level turbulence. This is indicated by work of both Lesur and Papaloizou (2010) and Lyra and Klahr (2011) who find that such instabilities do not have a strongly disruptive effect on large aspect ratio vortices produced by the subcritical baroclinic instability (SBI). Therefore, although parametric instability may act to cause a vortex to ultimately decay, it may be successfully maintained if there is some mechanism to generate it such as the SBI or RWI. We have just considered steady vortex solutions not supported or generated by any particular mechanism.

For Kida vortices, a strong, exponentially growing and potentially rapid vortex-destroying
instability only occurs due to the saddle point instability for $3/2 < \chi < 4$. These vortices do not attract dust due to their unfavourable pressure gradient. However, for larger aspect ratios, we might expect there to be a balance between inward flow due to the mean pressure gradient and turbulent diffusion (e.g. Lyra and Lin, 2013). We will do a rough, back-of-the-envelope calculation for this. The inflow rate for small particles driven by the pressure gradient is

$$|v| \sim \frac{|\nabla P| \tau_s}{\rho}, \quad (8.6.1)$$

(e.g. Papaloizou and Terquem, 2006). Supposing a vortex has lengthscale $L_v$ in the minor axis direction (so $\chi \sim L_v^{-1}$) we estimate that

$$P \sim \rho \Omega^2 L_v^2 \quad (8.6.2)$$

$$|v| \sim \Omega^2 L_v \tau_s. \quad (8.6.3)$$

As the unstable modes are local, the wavelength should be $|k| \ll L_v$. For the purposes of this crude approximation we will take $\pi |k|^{-1} = L_v/10$. An estimate of the associated diffusion coefficient based on dimensional scaling is

$$D = \frac{\gamma}{|k|^2} \quad (8.6.4)$$

where $\gamma$ is our parametric growth rate. Balancing pressure–driven inflow against diffusion we obtain

$$\frac{|\nabla \rho|}{\rho} = \frac{1}{L_\rho} \sim \frac{|v|}{D} \sim \frac{(10\pi \Omega)^2 \tau_s}{\gamma L_v}, \quad (8.6.5)$$

where $L_\rho$ is the density lengthscale. Hence $L_\rho \sim f L_v$ where

$$\frac{L_\rho}{L_v} = f \sim \frac{\gamma/\Omega}{100\pi^2 \Omega \tau_s}. \quad (8.6.6)$$

Significant dust concentrations become possible once $f < 1$. For $\gamma = 0.1\Omega$ this becomes equivalent to $T_s = \Omega \tau_s \gtrsim 10^{-4}$ with characteristic inflow time $(\Omega^2 \tau_s)^{-1}$. Recall from Section 3.5.1 that the most favourably trapped dust has $T_s \simeq 1$.

Although this estimate is highly uncertain, it does indicate that the existence of parametric instabilities do not necessarily prevent the possibility of dust accumulation in vortices.

### 8.6.2 Including dust back-reaction

Recall that in this study we consider a fluid of perfectly coupled dust and gas, $\tau_s = 0$ (Section 5.1.2). This becomes less valid when $\Gamma_d$ becomes large, or when large particles not in the Epstein regime are trapped. Fu et al. (2014)\(^1\) finds that dust back-reaction becomes important

\(^1\)Published in parallel to Railton and Papaloizou (2014).
for the dynamics of the flow when $\Gamma_d \gtrsim 1$, i.e. when our $\rho_{\text{max}} \gtrsim 1$. This is because one cannot neglect momentum transfer when there are large particle overdensities. They show that there is a dynamic instability when both dust feedback is included and $\Gamma_d$ exceeds unity, leading to shorter vortex lifetimes for vortices with higher $\Gamma_d$ and larger particle sizes (i.e. larger $T_s \propto \tau_s$). We do not see evidence of this as we have worked in the $\tau_s = 0$ regime and could not produce any very high $\Gamma_d$ equilibrium solutions.

A subsequent study by Raettig et al. (2015)\(^2\) on the feedback of dust on vortical flow finds that particle trapping occurs in 2D, shearing sheet solutions for all $T_s$ and initial $\Gamma_d$. Furthermore, it is possible to produce a density enhancement from an initial $\Gamma_d = 10^{-2}$ to $\Gamma_d = 100$ with $T_s = 1$ particles, enough to trigger the streaming instability and possibly GI. Initially elliptical streamlines are bent to more complex motions when dust back-reaction is include (i.e. when $\tau_s > 0$) and there are larger resulting $\Gamma_d$ when $\tau_s > 0$. However, very small, well-coupled dust $T_s \simeq 0.01$ is harder to capture and the deviation from Kida streamlines is smaller, preventing particles from accumulating too densely (Lyra and Lin, 2013).

Furthermore, they find that dust concentrations are very localised in the optimal $T_s = 1$ case (less so for $T_s \ll 1$ but this dust is harder to capture anyway, Section 3.5.1), meaning that any vortex instability would be over a few streamlines instead of the entire vortex. We, with our tightly (perfectly!) coupled dust, are considering $T_s \ll 1$ particles and have a density contrast spread more evenly across the vortex patch. Therefore we have more streamlines affected by instability, as is shown when we vary $y_{\text{max}}$ in Figures 8.9 and 8.10.

Finally, one further complication is that when $\Gamma_d$ in a vortex becomes high, some consideration should be given to dust coagulation/fragmentation (Testi et al., 2014).

### 8.6.3 Conclusions

In conclusion, we find that all vortices have some instability somewhere. There are not stable gaps (like in the Kida case for $4 < \chi \lesssim 4.85$) when all streamlines $y_{\text{max}} \in [0, 1]$ are considered\(^3\). Dust particles attracted from the outer disc to a vortex core with high aspect ratio, $\chi$, may well encounter parametric instabilities with characteristic growth rates 0.01–0.1$\Omega$. We also find that, broadly speaking, the larger the aspect ratio, the more stable the vortex.

Small dust grains trapped by vortices end up in a more or less smoothed profile over the vortex core (Lyra and Lin, 2013; Raettig et al., 2015). Therefore, most or all of the internal streamlines will have some parametric instability at some $k$. However, this may not be sufficient to disrupt the vortex, as argued in Section 8.6.1.

If we consider larger grains (i.e. $T_s = 1$, more preferentially captured by vortices), concentrations of mass are more localised on a few streamlines, either piled into the centre or in a

\(^2\)Published after write up was started.

\(^3\)Although not considered in this work, this includes streamlines outside the core. See Lesur and Papaloizou (2009) for more details.
ring-like structure. This implies that only a few streamlines will be subject to potentially disruptive parametric instabilities, or we are in a high $\beta$ regime where the majority of additional mass is piled into the centre. Recall that including mass decreases the instances of stable ‘gaps’ in $\chi$, though since instability occurs in bands, there will still be some $k$ that are stable. We would need further non–linear analysis, with dust considered as separate fluid (ideally with a range of sizes and including coagulation/fragmentation/bouncing effects) to determine if these concentrations are sufficient to disrupt the entire vortex. We leave this to further work.

We also conclude that the situation is not as bad as presented in Chang and Oishi (2010). Their stability analysis started from solutions that never correctly matched the background flow as they assumed a Kida solution with arbitrary density superposed – as we showed in Section 5.4 this is only possible in some cases when $\chi = 7$. Furthermore, they assume a form of the wavenumber that can only be applied to the Kida case only, not a generic equilibrium, as shown in Section 7.4.3. If we assumed $k_z = 0$ and use the Kida wavenumber; since it increases with time thus we do not get instabilities growing exponentially with time (Section 7.4.2). Only by considering $k_z \neq 0$ and a time–independent wavenumber do we get exponentially growing modes.

We note that in a PP disc, the generation of vortices will compete with weak instability. Lin (2014) finds that the RWI produces columnar vortices from an initial radial density bump in a time $\mathcal{O}(10\Omega^{-1})$. As these results are linear we need to make a nonlinear study to determine the outcome of the competition of these two processes – will the destruction time be longer than a few 10s of orbits? We investigate this in the next chapter.

It is worth remembering that we do not need every vortex capable of trapping dust to persist for long times in order to be useful for planetesimal formation. Planetesimals captured and grown in one vortex (which it may or may not destroy) could easily be captured by another vortex, up to the point where they become large enough to decouple from the flow. Vortices are not difficult to make in PP discs so we’d expect that any vortex would exist alongside (or even help generate) neighbouring structures.

Furthermore, we have also ignored the fact that vortices are not just generated and then left to their own devices; they can be sustained by the very mechanisms that made them, such as the RWI. There is a subtle interplay between generation, dust trapping and the speed of these two processes, the structure of trapped dust, whether vortices are sustained or not and the instabilities they are subject to. The flip side of the statement that ‘all vortices have some instability somewhere’ is that there are also going to regions and circumstances where vortices are stable, or stable enough to quickly grow some dust grains before it is disrupted. A dust profile does not always cause a vortex to be completely unstable and in this context that may well be good enough that vortices are still useful and promising sites for planetesimal formation.
Chapter 9

A study of the stability of vortex models with a 2D flow to 3D perturbations

The work up to this point has been concerned with the linear stability of steady vortices in 2D. In this chapter we aim to move beyond equilibrium vortex solutions, using the PLUTO code (detailed in Section 9.1) to investigate possible nonlinear behaviour in 2D (Section 9.2), the form of 3D vortices with and without the presence of vertical gravity (Section 9.3), the lifetime of such vortices under different conditions and for a range of aspect ratios and finally how instability, if it occurs, manifests itself.

9.1 The PLUTO code

In this section we used the compressible hydrodynamical code PLUTO (Mignone et al., 2007). We chose to use this code as it is widely used, well–documented and would allow us to investigate the stability of vortices in two– and three–dimensions.

PLUTO is a finite-volume/finite-difference, shock capturing code which implements both Newtonian hydrodynamics and ideal MHD. Computations are done using double-precision arithmetic and we use the static grid version of the code. It can allow for initial data to be read in from previous calculations and has a shearing box module already built in (Hawley et al., 1995; Balbus, 2003; Regev and Umurhan, 2008). Additional physics, such as special relativity and non-ideal effects like viscosity, resistivity and thermal conduction and cooling can also be included. It also has a series of built in test cases for verifying it is working correctly (Section 9.1.3).

Throughout this chapter, we will be working in a shearing box of dimension $L = (L_x, L_y, L_z)$ with the number of grid points given by $N = (N_x, N_y, N_z)$. We also use a scaled time where
Ω = 1 so time is in terms of the dynamical timescale \( \tau_{\text{dyn}} = \Omega^{-1} \).

9.1.1 Grid

We use a uniform grid with spacing \( \Delta x = L_x/N_x \) (similar in the \( y \)– and \( z \)–directions), corresponding to grid points

\[
\begin{align*}
  x[i] &= -\frac{L_x}{2} + \Delta x(i - 1), \quad i = \{1, N_x\} \quad \text{(9.1.1a)} \\
  y[j] &= -\frac{L_y}{2} + \Delta y(j - 1), \quad j = \{1, N_y\} \quad \text{(9.1.1b)} \\
  z[k] &= \Delta z(k - 1), \quad k = \{1, N_z\} \quad \text{(unless otherwise stated) \ (9.1.1c)}
\end{align*}
\]

where e.g. \( x[i] \) defines the left hand side/bottom of the grid cell. Variables are defined at the cell centre.

9.1.2 Boundary conditions

A variety of boundary conditions are applied to the boxes in this chapter, namely periodic, shearing, outflow and reflective conditions.

Periodic boundary conditions in \( y \)–direction (also used in the \( z \)–direction in unstratified boxes) take the form:

\[
q(x, y, z, t) = q(x, y + L_y, z, t) \quad \text{(9.1.2)}
\]

The shearing boundary conditions always apply along the \( x \)–boundaries, with adjacent boxes sliding past each other with relative velocity \( u = |SL_x| \):

\[
\begin{align*}
  q(x, y, z, t) &= q(x \pm L_x, y \mp ut, z, t) \\
  v_y(x, y, z, t) &= v_y(x \pm L_x, y \mp ut, z, t) \pm u
\end{align*}\]

where \( q \) represents all other quantities aside from \( v_y \).

In our 3D stratified simulations, we also sometimes use an outflow boundary condition (zero gradient across the boundary) at the top of the box \( z = L_z \):

\[
\frac{\partial q}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \text{(9.1.4)}
\]

while at the midplane \( z = 0 \) we use a reflective boundary condition:

\[
v = (v_x, v_y, v_z) \rightarrow (v_x, v_y, -v_z), \quad q \rightarrow q. \quad \text{(9.1.5)}
\]

Again, \( q \) represents all other quantities not explicitly stated.
9.2 2D models

(a) Steady case for $v' = 0$ and magnetic field $B = 0$.  (b) Unsteady case for $v'_y \sim 10^{-3}$ Gaussian noise and constant $B = B_0 \hat{z}$.

Figure 9.1 Testing the PLUTO shearing box implementation. As expected we produce a steady shearing box when $v' = 0$ is imposed (for zero or constant $B$) and get MRI turbulence when a constant magnetic field is imposed and $v' \neq 0$. In these cases, $L = (0.25, 1, 0.25)$ and $c_s = 1.14$.

9.1.3 Initial tests

As verification that we were using the code correctly we performed a few initial tests. PLUTO provides a variety of test cases with the code. The results of a couple of these can be seen in Figure 9.1.

The first (Figure 9.1a) is a 3D empty box run containing just the background shearing flow with all magnetic terms set to zero and no random perturbations added. This is to check that the code maintains background shear flow correctly; as you can see this is the case as we only see the shearing background flow for large times.

In Figure 9.1b we reproduce the MRI in a shearing box by imposing a constant, vertical magnetic field $B = B_0 \hat{z}$ and Gaussian noise in the $y$-direction of magnitude $\left| v'_y \right| \sim 10^{-3}$ (Section 9.3.1 for how perturbations are implemented). $B_0$, in units of $v_0 \sqrt{4 \pi \rho_0} = SL_x \sqrt{4 \pi \rho_0}$, is given by $B_0 = c_s \sqrt{2(\gamma \beta)^{-1}}$, where $c_s = 1.14$, the adiabatic index $\gamma = 1$ and $\beta = 10000$.

Using the various functions available in the VisIt software (Childs et al., 2012, also used to produce the two figures in Figure 9.1 and many others in this chapter) we also checked that vorticity was conserved.
Stability of vortices to 3D perturbations

Table 9.1 Initial 2D runs, testing box size and resolution.

<table>
<thead>
<tr>
<th>Run</th>
<th>$L_x$</th>
<th>$L_y$</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N/L$</th>
<th>$\chi$</th>
<th>$c_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>2</td>
<td>4</td>
<td>128</td>
<td>128</td>
<td>–</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>1b</td>
<td>2</td>
<td>4</td>
<td>128</td>
<td>256</td>
<td>64</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1c</td>
<td>4</td>
<td>4</td>
<td>256</td>
<td>256</td>
<td>64</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1d</td>
<td>4</td>
<td>8</td>
<td>256</td>
<td>512</td>
<td>64</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1e</td>
<td>4</td>
<td>12</td>
<td>256</td>
<td>768</td>
<td>64</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1f</td>
<td>8</td>
<td>4</td>
<td>512</td>
<td>256</td>
<td>64</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1g</td>
<td>2</td>
<td>4</td>
<td>256</td>
<td>512</td>
<td>128</td>
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<td>*</td>
</tr>
<tr>
<td>1h</td>
<td>4</td>
<td>4</td>
<td>512</td>
<td>512</td>
<td>128</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1i</td>
<td>4</td>
<td>8</td>
<td>512</td>
<td>1024</td>
<td>128</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1j</td>
<td>4</td>
<td>12</td>
<td>512</td>
<td>1536</td>
<td>128</td>
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<td>*</td>
</tr>
<tr>
<td>1k</td>
<td>2</td>
<td>4</td>
<td>512</td>
<td>1024</td>
<td>256</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1l</td>
<td>4</td>
<td>4</td>
<td>1024</td>
<td>1024</td>
<td>256</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>2(a−l)</td>
<td>as Run 1</td>
<td>8</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2D models

Once we had convinced ourselves that the shearing box module was working correctly, we tested the affect of different box dimensions and resolutions on an imposed Kida vortex in a 2D shearing flow. This is in the $xy$–plane, so there is no stratification or $z$–dependence.

We considered four different box dimensions ($L_x, L_y$) and four resolutions $N_x \times N_y$, summarised in Table 9.1. In all cases the imposed vortex lies within $|x| \leq \chi^{-1}, |y| \leq 1$.

The vortices were created by inputting the Kida velocity field

$$v = (v_x, v_y) = \left(\frac{3\Omega y}{2\chi(\chi - 1)} - \frac{3\Omega \chi x}{2(\chi - 1)}\right)$$

inside an ellipse with boundary passing through $(0, 1) = (0.5L_x, 0.75L_y)$. As in previous the analysis, we work in a regime where $\Omega = 1$. The $z$–component of vorticity is then calculated by the code using a simple difference scheme:

$$\text{vortz}[i][j][k] = 0.5 \times (vy[i+1][j][k] - vy[i-1][j][k]) / dx[i] - 0.5 \times (vx[i][j+1][k] - vx[i][j-1][k]) / dy[j]$$

where $\text{vortz}[i][j][k]$ represents the vorticity $\omega = \partial_x v_y - \partial_y v_x$ at the centre of the grid cell defined by $x[i], y[j], z[k]$ (Section 9.1.1). The $x$– and $y$–velocities $v_x, v_y$ are defined similarly by $vx[i][j][k]$ and $vy[i][j][k]$. Note that in this 2D case, $k=0$ is constant.

This initial vorticity profile has a discontinuity at the boundary, where numerical truncation

\footnote{The only source of perturbation is therefore from numerical truncation and rounding errors.}

\footnote{Documentation can be found at http://visitusers.org/index.php?title=Main_Page}
errors are significant. This can be seen in Figure 9.2a which shows the numerical representation of the vorticity at the first timestep, $t = 0$.

We use an initial columnar Kida vortex with $\chi = 5$ inside the vortex core, since Lesur and Papaloizou (2009) found a small linearly stable region around this aspect ratio. This allows for a straightforward comparison of resolution, box dimension and other model conditions. We ignore the perturbation of the background shearing flow outside the core because the analytical solution is large scale and does not satisfy the boundary conditions, leading to numerical difficulties. Instead we let the solution relax to a steady solution.

We note that we are looking at an incompressible regime with a compressible code; this is achieved by specifying a large $c_s$ in an isothermal equation of state $P = c_s^2 \rho$. We can estimate how close we are to the incompressible regime by comparing $c_s$ with $\frac{1}{2} S L_x$ (the shearing velocity at the initial vortex boundary is $-S/\chi e_y$). For incompressibility to be reasonable, we expect this to be small compared to $c_s$. This is certainly true in these 2D cases where when $L_x = 4$ and $c_s = 5$, $c_s$ is more than three times as large. We discuss this in more detail for the 3D cases in Section 9.3.6.

### 9.2.1 General evolution of imposed $\chi = 5$ Kida vortex

Here we evolve a simple imposed vortex patch in a 2D $(L_x, L_y) = (4, 4)$ shearing box with uniform initial density. Snapshots of the vorticity profile for a $N_x \times N_y = 512 \times 512$ run are shown in Figure 9.2 after a range of simulation times to illustrate the general evolution we see.

Frame 9.2a is the initial state, a simple anticyclonic vortex patch. The high vorticity around the vortex boundary is caused by the finite difference method used to calculate the vorticity. The dynamical relaxation of the patch causes it to shear out in frame 9.2b, splitting into three distinct anticyclonic vortices in frame 9.2c due to phase where the central vorticity is temporarily increased in magnitude. The symmetry of the shearing box means we get rotational symmetry with respect to rotation through $180^\circ$ around the origin. These extra vortices are both smaller and weaker in higher resolution runs.

The two outer vortices are dragged to the top of the box, loop back round (as we have periodic boundary conditions at the $x$-boundaries) before getting absorbed by the main vortex in frame 9.2d. The patch quickly tends to an equilibrium state as seen in frame 9.2e; an anticyclonic patch lying on top of the rest of the background shearing flow ($\omega = -1.5$), which does not change much between $t \simeq 100 - 400 \Omega^{-1}$.

In Figure 9.3a we see the vorticity profile (compared to the initial solution) at $t = 88\Omega^{-1}$ along both the $x$- and $y$-axes. There has been some smoothing from the sharp-profiled ‘top hat’ of the Kida solution. A time series for the vorticity profile along the $y$-direction in Figure 9.3b, showing the effect of diffusion with time. As expected the strength of the vortex decreases with $t$ under the effect of diffusion. This increases the aspect ratio since weaker vortices are more susceptible to the background shear. This process is slower at higher resolution, again,
Figure 9.2 The time evolution of vorticity for Run 1h, an imposed Kida vortex with $\chi = 5$, $c_s = 5$, $L = (4, 4)$ and $N = (512, 512)$. See Section 9.2.1 in the main text for details.
as expected.

Finally, Figure 9.3c shows the $\chi$–profile through the vortex for the just-settled equilibrium (i.e. once the initial transient behaviour has died down), calculated from the maximum $x$ and $y$ extent of the closed contours. There has already been some numerical diffusion by this point so only the innermost streamlines have $\chi$ close to the initial value $\chi = 5$. Recall from Chapter 6 that vortices with this $\chi$ profile contain internal shear.

In Figure 9.6 we reproduce the $\chi$–profiles of the $\alpha = 1$, constant–density equilibrium solutions of Chapter 6, with the profiles of Figures 9.3c and 9.5c superposed. We find qualitatively similar profiles compared to these equilibrium solutions, demonstrating that the choice of Bernoulli source term $A(\psi)$ as a power law, given in equation (5.2.6a), is reasonable. There is a quantitative disparity between initial profile (i.e. $\omega_m$) and final $\chi$–profile due to the discontinuity in the boundary in the PLUTO solutions removing vorticity from the system.

Also note from Figures 9.2 and 9.3a that just outside the core we have vorticity $\omega > -1.5$, as seen in Lesur and Papaloizou (2010).

### 9.2.2 General evolution of imposed $\chi = 8$ Kida vortex

This is similar to Run 1, except a weaker, $\chi = 8$ vortex is imposed. More structure is seen inside the vortex core, and it persists for longer than the $\chi = 5$ case (Figure 9.4). However, this structure is only transient and a stable, smoothly-profiled core is still produced. In Figure 9.5 we again see the vorticity profile for the stable vortex (9.5a) and its time series (9.5b), which is qualitatively similar to the stronger $\chi = 5$ case. There is a somewhat smoother $\chi$–profile (9.5c).

### 9.2.3 Effect of box dimension

Stretching the shearing box in the $y$–direction (while keeping all other parameters the same) causes pairs of vortices to travel further as they interact and loop past each other, as can be seen in Figure 9.7. The only real difference between these and boxes with shorter $L_y$ is that the relaxation time to a steady state is longer.

Figures 9.8a and 9.9a are plots of the minimum vorticity in the box, $\omega_{\text{min}}^3$, found in the box, related to the strength of the resulting anticyclonic vortex. Box dimension has a nonzero but not hugely significant effect on the value of $\omega_{\text{min}}$. The vorticity dip around $t = 10\Omega^{-1}$ is due to the shearing out of the initial imposed vortex patch which piles up the majority of the vorticity in a narrow, central structure (for example, see Figure 9.2b). Typically the vortices relax to a steady state around $t \simeq 40\Omega^{-1}$ and then persist until $t = 200\Omega^{-1}$ and beyond.

---

3For most cases this is at the centre of the box. Exceptions are when the final vortex straddles the periodic $y$–boundary and when strong, initially low $\chi$ vortices split into two or more vortices.
Figure 9.3 Vorticity and $\chi$–profiles for the $\chi = 5$ vortex in a $L = (4, 4), 256^2$ box. We find uniformly lower $\chi$ for higher resolution cases due to decreased numerical diffusion.
Figure 9.4 The time evolution of vorticity for Run 2h (see Table 9.1), an imposed Kida vortex with $\chi = 8$, showing the more complex structure that forms in the core. See Section 9.2.2 in the main text for details.
Stability of vortices to 3D perturbations

Figure 9.5 Vorticity and $\chi$ profiles for the $\chi = 8$ vortex in a $(L_x, L_y) = (4, 4)$, $256^2$ box.
Figure 9.6 Reproduction of Figure 6.4a with the $\chi$–profiles of Figures 9.3c and 9.5c superposed. The lines show the various $\chi$–profiles for equilibrium solutions with $\alpha = 1$, while the numbers in brackets are $(\omega_m, \chi_k)$, where $\chi_k$ is the expected Kida aspect ratio for that $\omega_m$.

Figure 9.7 Run 1i, $L = (4, 8)$, $N = (512, 1024)$, showing smaller, extra vortices travelling away from the central disturbance (9.7a) then the final structure (9.7b). Similar behaviour is seen in the larger boxes $L = (4, 12)$ but are not seen at all at the lowest resolution $N/L = 64$. 
Figure 9.8 Effects of box size and resolution of time evolution on an imposed $\chi = 5$ Kida vortex. Figure 9.8a shows less variation of $\omega_{\text{min}}$ with $L_y$ than $L_x$ and an increase in relaxation time with an increase of $L_y$. Figure 9.8b shows broadly similar behaviour for different resolutions, with the exception of the lowest resolution case.
Figure 9.9 Effects of box size and resolution of time evolution on an imposed $\chi = 8$ Kida vortex. The conclusions are the same as for the $\chi = 5$ case in Figure 9.8.
9.2.4 Effect of resolution

Resolution has a more profound effect on vortex evolution than choice of box dimension, as can be seen in both Figures 9.8b and 9.9b. We looked at three different resolutions, \( N/L = \{64, 128, 256\} \) for most of the box dimensions, as can be seen in Table 9.1.

The total vorticity between different box dimensions and resolutions is the same (to 2 s.f.) so this is not due to any substantial effects at the vortex core boundary, as initially suspected. Instead, adding more grid points reveals more structure that takes vorticity away from the more crudely resolved extra vortices in the lowest resolution runs. As can be seen in Figures 9.8b and 9.9b, in both \( \chi = 5, 8 \) cases, moving to the highest \( N/L = 256 \) resolution caused very little deviation from the \( N/L = 128 \) case.

We conclude that for the purposes of creating a stable solution to use to investigate the stability in 3D, a \( 2 \times 4 \) box of \( 128 \times 256 \) grid points is sufficient, with a subset of the calculations being run at higher resolution to double check. Increasing the box dimension serves to only increase the relaxation time, while increasing the resolution resolves finer structures which have to be absorbed by the main disturbance before a stable configuration settles down, with little effect on the final structure.

9.2.5 Stability of solutions

For a series of \( L = (2, 4) \) boxes (with \( N = (128, 256) \) for \( \chi < 10 \) and \( N = (192, 256) \) for \( \chi \geq 10 \)) we evolved vortices of different strengths, starting with an initial Kida configuration (see Figure 9.2a).

There was Gaussian noise added with \( v \sim 10^{-3} \) to see how perturbations behave in 2D, following the same procedure we will use in 3D (for how this is implemented, see Section 9.3.1).

In all cases stable structures were formed which persisted for \( t = 400\Omega^{-1} \), as can be seen in Figure 9.10a. We compare \( \omega_{\text{min}} \) with the initial vorticity minimum (i.e. the Kida vorticity) in Figure 9.10b, which shows slightly weaker vortices produced than expected from the original Kida vorticity. This is partially due to the effect of numerical diffusion (which also causes the slow reduction of this ratio with time) and also the boundary effects seen in Figure 9.2a which act to remove some initial vorticity from the system.

We expect these solutions to be stable; this corresponds to the vertical stability limit detailed in Section 7.5 and shown in the various plots of Chapter 8 for large \( \theta \).

9.3 3D models

Here we expand the calculations to include the \( z \)-direction. Moving to three dimensions requires some decisions about the form the imposed vortex will take, how to implement stratification (Section 9.3.1) and the boundary conditions at the two \( z \)-boundaries (Section 9.3.2).
Figure 9.10 Results for the 2D $L = (2, 4)$, $N = (128, 256)$ box with $\nu' \sim 10^{-3}$, showing $\omega_{\text{min}}$ against time. For all cases the final configuration is stable with a steady decrease in $\omega_{\text{min}}$ due to numerical diffusion.
Figure 9.11 A vorticity contour plot showing the initial imposed vorticity for a $\chi = 3.5$ Kida vortex with $v' \sim 0.1$.

### 9.3.1 Initial conditions

Random noise is added to the shearing boxes by generating additive white Gaussian noise with zero mean and standard deviation of 1, some $AGWN()$. This white noise is generated using the Box-Muller transform (Box and Muller, 1958; Pike, 1965) which transforms two uniformly distributed random numbers $(U_1, U_2) \in (0, 1]$ to $(Z_0, Z_1)$, independent random variables with a standard normal distribution

$$Z_0 = R \cos \Theta = \sqrt{-2 \log (U_1) \cos (2\pi U_2)} \quad (9.3.1a)$$

$$Z_1 = R \sin \Theta = \sqrt{-2 \log (U_1) \sin (2\pi U_2)}. \quad (9.3.1b)$$

The function $AGWN()$ then returns $AGWN() = Z_0$. The uniform distributions $U_1, U_2$ are generated using the ratio of $\text{rand()}$ and $\text{RAND\_MAX}$ from the standard C library:

$$U_i = \frac{\text{rand}()}{\text{RAND\_MAX}}, \quad (9.3.2)$$

where $\text{rand}()$ returns a random number between 0 and $\text{RAND\_MAX}$. If 0 was returned by $\text{rand}()$, the ratio was calculated again until a nonzero result was returned.

We then add multiples of this noise to both the velocity and density distributions at $t = 0$. If $v_0, \rho_0$ are the velocity and density distributions of the initial imposed vortex structure then
### 9.3 3D models

**Figure 9.12** Initial unstratified box tests, testing effect of 3D and adding a Gaussian density perturbation. This shows that the 3D calculations remain 2D if no perturbation (or a very small one) is introduced. Also, instability sets in after a time that increases with aspect ratio $\chi$. Done for a box with $L = (4, 4, 4)$, $128^3$, $c_s = 5$.

$v \rightarrow v_0 + v'$ and $\rho \rightarrow \rho_0 + \rho'$, where we use the notation

$$v'_i \sim 10^{-3} \quad \Rightarrow \quad v'_i = 10^{-3} \times AGWN(), \quad i = \{1, 2, 3\} \quad (9.3.3a)$$

$$|\rho'| \sim 0.1 \quad \Rightarrow \quad \rho' = |0.1 \times AGWN()|, \quad (9.3.3b)$$

In the density case, the perturbation added is always positive to ensure that $\rho < 0$ never occurs. The velocity perturbation remains at $|v'| \sim 10^{-3}$ throughout as any larger perturbation serves to quickly overwhelm the higher $\chi$ (weaker) imposed vortices and prevent them from relaxing to an initial steady state. Changing the magnitude of $\rho'$ served to perturb the box to greater or lesser extent without preventing the weaker vortices from relaxing to an initial steady state.

The initial vortex solution takes the form of the 2D case and is thus independent of $z$, i.e. it is a columnar vortex patch. Attempts to initialise the solution with a vortex not extending over the full height of the box led to it quickly stretching to the full box height. We therefore decided to initialise our vortex over the full height of the box: a typical initial vortex can be seen in Figure 9.11. These columnar vortices relax to pseudo-stable (Section 9.3.5) structures in an analogous way to the 2D case in Sections 9.2.1 and 9.2.2.

Moving to 3D we ran some initial unstratified tests over a $L = (4, 4, 4)$, $128^3$ box, with and
without the additional density perturbation $\rho'$, which served to perturb the system. The results of this are shown in Figure 9.12, demonstrating that under no or negligible perturbations, our 3D calculations effectively remain 2D.

These initial test runs proved to be under-resolved in the $x$-direction ($L_x = 4, N_x = 128$) so we reduced the $x$ extent to $L_x = 2$ with no effect on the onset of any instability. For high aspect ratio vortices $\chi > 10$, $N_x$ was increased from 128 to 192 to ensure that the solution relaxed to a coherent vortex and not a ‘strip’ of vorticity along the $y$-axis.

9.3.2 Boundary conditions

For unstratified, $g_z = 0$ cases we impose periodic boundary conditions at the $z$–boundaries (Section 9.3.2). This is the same approach as Lesur and Papaloizou (2009). In stratified cases we use half-boxes modelling $z \geq 0$ with a reflective boundary at the midplane $z = 0$ and an outflow condition at $z = L_z$ (see Section 9.3.4). This is equivalent to imposing reflection symmetry with respect to the midplane.

In both cases, there are shearing boundary conditions at the $x$–boundaries and periodic boundary conditions at the $y$–boundaries, unchanged from the 2D case.

9.3.3 Stratification

In the stratified cases, vertical gravity is imposed over the bottom 80% of the box where the midplane $z = 0$ was the lower $z$–boundary:

$$g_z = \begin{cases} -\Omega^2 z, & 0 \leq z \leq 0.8L_z \\ 0, & z > 0.8L_z. \end{cases} \quad (9.3.4)$$

This introduces a small low–density buffer zone for $0.8L_z < z < L_z$ which is found to make the upper boundary consistent with periodic boundary conditions in $z$ and to maintain its stability. Stone et al. (1996) found this approach to be effective in vertically stratified shearing box simulations. Also, this is in line with the approach of Lesur and Papaloizou (2009).

We also undertook some additional tests changing how $g_z$ was implemented. Moving the boundary to $z = 0.9L_z$ had a negligible effect, as did smoothing $g_z$ in the region $0.8L_z < 0.9L_z$:

$$g_z = \begin{cases} -\Omega^2 z, & 0 \leq z \leq 0.8L_z \\ -\Omega^2 z \left( \frac{0.9L_z - z}{0.1L_z} \right), & 0.8 \leq z \leq 0.9L_z \\ 0, & z \geq 0.9L_z. \end{cases} \quad (9.3.5)$$

For the stratified cases, the vortex column in the region where $g_z = 0$ behaves differently on the onset of instability. The column remains largely vertical here, while below the $g_z$ transition it

\footnote{The middle 80\% was stratified in the few full-box simulations of Section 9.3.4.}
shears along the $y$–axis as in Figure 9.16e. However, as can be seen in Figure 9.13, the height of the buffer zone does not play an important role in the stability. There is little difference in the instability onset time $\tau_{\text{unstable}}$ (Section 9.3.5) between the three forms of $g_z$ used. Similarly, only imposing a perturbation over the bottom half of the box $z < 0.5L_z$ increased the time until the vortex broke up (unsurprising since the box was less perturbed) but had no effect on how the instability manifested itself.

We note that since $H \equiv c_s/\Omega$ (see Section 2.4.4 and equation (2.4.12) for $\rho(z)$), and since we use $\Omega = 1$ throughout, effectively $H \equiv c_s$.

9.3.4 Use of halfboxes for stratified models

We expect that the symmetry around the midplane $z = 0$ means we need to only consider the positive $z$ domain. In order to confirm this we compared the use of a full stratified box $(-3 \leq z \leq 3)$ with half boxes of two different dimensions, $0 \leq z \leq 3$ and $0 \leq z \leq 6$, with equal resolutions across each. The full box had period boundary conditions in $z$ while the half boxes had an outflow condition at $z = L_z$ and a reflective condition at the midplane.

The results of this can be seen in Figure 9.14, where we see little quantitative difference between them. For this reason we will henceforth use the half box with $0 \leq z \leq 3$ for our stratified runs. We also tested the required resolution in the $z$–direction and found that $N_z = 32$ was sufficient for $L_z = 3$ for our stratified half-boxes (Section 9.3.4).

Figure 9.18 shows the density stratification at late times (after onset of instability) for two different $\rho'$, with otherwise identical initial conditions. As we expect for isothermal simulations, we get a resulting hydrostatic equilibrium with $\rho(z) \propto \exp(-z^2/2H^2)$ (where we have scaled each with the midplane value from the simulation). This profile is independent of the strength of the imposed vortex. We have a stably stratified system with $N^2 = 0$, where $N^2$ is the buoyancy or Brunt-Väisälä frequency

$$N^2 \equiv -g_z \left[ \frac{\partial \log \rho}{\partial z} - \frac{1}{\gamma} \frac{\partial \log P}{\partial z} \right]. \quad (9.3.6)$$

Since a strict isothermal equation of state is used everywhere, the adiabatic index $\gamma = 1$. However, there are vertical restoring forces towards the midplane with frequency $\sim \Omega$ which may play a role in the stratified cases.

9.3.5 Pseudo–stable solutions and the onset of instability

We call the columnar vortices formed after the initial Kida vortex column relaxes pseudo–stable and they invariably have a period of stability in both stratified and unstratified boxes before being destroyed by instability.

Looking at the time series for $\omega_{\text{min}}$ (as in Figure 9.12) as well as eyeballing the 3D evolution of the vorticity distribution (Figures 9.15 and 9.16) it is clear how instability manifests itself.
(a) Effect of different stratification on an imposed $\chi = 5$ vortex. The models with sharp transitions at $z = 0.8, 0.9L_z$ both went unstable at $\tau_{\text{unstable}} = 123\Omega^{-1}$.

(b) Effect of different stratification on an imposed $\chi = 8$ vortex. Like the $\chi = 5$ case there was very good agreement between models.

**Figure 9.13** Effect of different stratification regimes on the onset of instability.
In unstratified boxes, where there is no preferred value of $z$, the unstable mode has a particular $k_z$ as can be seen in Figure 9.15, as expected from the work in Chapter 8.

In stratified cases the onset of instability is profoundly different. The vortex column begins to shear in the $yz$–plane before splitting into smaller structures and becoming incoherent as in Figure 9.16.

We sought a consistent diagnostic for the onset of instability in the stratified case. Comparing the time series of $\omega_{\text{min}}$ with 3D visualisations of the vorticity showed the stable columnar vortices going unstable (unsurprisingly) with a net decrease and an increasing amount of unsteadiness in $\omega_{\text{min}}$. This behaviour can be seen in Figure 9.17, where $\omega_{\text{min}}$ (purple line at the top of the plot) of a 3D steady vortex slowly increases with time (under the action of numerical diffusion) before dropping when instability occurs. However, it was sometimes difficult to self consistently pinpoint an instability time using this measure, as can be seen from comparing Figures 9.12 and 9.14.

Therefore, considering the volume averages $\langle v_x^2 \rangle$, $\langle v_y^2 \rangle$, $\langle v_z^2 \rangle$, together with $\omega_{\text{min}}$ and the 3D contours of $\omega_z$, we found that instability kicked in close to the point where there was often a

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5The behaviour of $\omega_{\text{max}}$ proved less useful as this mainly reflected the background shear.
local minimum in $\langle v_y^2 \rangle$. Some care needed to be taken to pick out the correct minimum but overall this was a robust method for pinpointing the onset of instability for the stratified cases. We define this time to be $\tau_{\text{unstable}}$.

Note that this local minima in $\langle v_y^2 \rangle$ also corresponds to the beginning of an increase in $\langle v_z^2 \rangle$; this is expected since the pseudo-stable solutions largely move in parallel $xy$–planes (Section 5.1.5) so any large scale vertical motions are associated with instability.

The different form of the onset of instability in the unstratified cases meant that this method did not work; instead we pinpointed the time when the vortex column first broke up in 3D vorticity contour plots.
9.3 3D models

Figure 9.15 Onset of instability for the unstratified $|v'| \sim 10^{-3}$, $\chi = 3.5$ case. Instability occurs in a qualitatively different way to the stratified cases displayed in Figure 9.16, with no preferred value of $z$ in this unstratified case. The $z$-dependence indicates an unstable mode with a particular $k_z$ as expected from the linear stability theory (Chapter 7). The variable $\langle v_y^2 \rangle$ was not a useful diagnostic in unstratified cases so determining the instability timescale was a case of observing the 3D contours over time. Note that in these plots we have imposed a ceiling on the value of $\omega_z$ displayed so the columnar structure can be seen.
Stability of vortices to 3D perturbations

(a) $\rho' \sim 0.1$, $\chi = 8$, $t = 80\Omega^{-1}$, steady columnar vortex.

(b) $t = 112\Omega^{-1}$, vortex is sheared in the $yz$-plane, close to the onset of instability.

(c) $t = 200\Omega^{-1}$, the columnar vortex is disrupted though some structure still remains.

(d) $\rho' \sim 0.5$, $\chi = 12$, $t = 40\Omega^{-1}$, steady columnar vortex.

(e) $t = 120\Omega^{-1}$, vortex is sheared in the $yz$-plane, close to the onset of instability.

(f) $t = 200\Omega^{-1}$, the columnar vortex is disrupted

Figure 9.16 The onset of instability for the $\rho' \sim 0.1$, $\chi = 8$ stratified case (a)–(c) (compare with Figure 9.17) and the $\rho' \sim 0.5$, $\chi = 12$ case (d)–(f).
9.3.6 Effect of $c_s$

Increasing $c_s$ has a stabilising effect in stratified boxes, as demonstrated in Figure 9.19. Since disc scale height $H \equiv c_s/\Omega$, increasing $c_s$ is equivalent to increasing $H$ and thus the ratio of the $y$-extent of these vortices to $H$ causes stabilisation (Lithwick, 2009). Since we are, for the most part, close to the incompressible limit, we do not expect much dependence on $c_s$.

Density waves are excited at a distance from the centre of the vortex where the background shearing flow $|−Sx e_y| = c_s$ and propagate out of the box (Lesur and Papaloizou, 2010). Therefore, as we reduce $c_s \lesssim 1$ this sonic boundary moves inside the box.

In models with $c_s = 1$ we do find density waves crossing the box. Furthermore, we do not find a stable columnar vortex structure found for the entire range of $z$. Instead, a much shorter column is formed over the top 30% of the box, with turbulence around the midplane. Columnar structures, as seen in the more incompressible cases with $c_s > 2.5$, do not last long, but a few coherent vortices do persist in this top region for long times $t \gtrsim 200\Omega^{-1}$. This is shown in Figure 9.20.

9.3.7 Effect of initial vortex strength on vortex lifetime

Here we investigate how the strength of the imposed vortex (parametrised in terms of the Kida aspect ratio $\chi$ of the imposed flow) affects the lifetime of the resulting vortex. This is done for density perturbations $\rho' \sim \{0.1, 0.5\}$. As discussed in Section 9.3.3, we use the box $L = \{2, 4, 3\}$ with $z \geq 0$ and

$$g_z = \begin{cases} -\Omega^2 z & 0 \leq z \leq 2.4 \\ 0 & z > 2.4. \end{cases}$$

(9.3.7)

We plot the resulting $\omega_{\text{min}}$ as a function of $t$ for different imposed vorticity patches for both $\rho' \sim 0.1$ and $\rho' \sim 0.5$ models in Figures 9.21 and 9.22 respectively. Using the procedure described in Section 9.3.5 we found the time $\tau_{\text{unstable}}$ at which each of these solutions first became unstable. The results of $\tau_{\text{unstable}}$ as a function of $\chi$ are plotted in Figure 9.24 for various stratified and unstratified cases.

There are few cases where resolution is increased from $N = (128, 256, 32)$ to $N = (192, 384, 32)$ and the resulting $\tau_{\text{unstable}}$ are also shown in Figure 9.24. For $\chi = \{5, 8, 10\}$, $\tau_{\text{unstable}}$ increases with resolution, while for $\chi = \{15, 20\}$ it decreases relative to the lower resolution models. This latter effect is most likely due to the increased resolution picking up weaker instabilities. However, note that we still observe the same trend of $\tau_{\text{unstable}}$ increasing with $\chi$. 
Figure 9.17 Determining the onset of instability using $\langle v_y^2 \rangle$ for the $\rho' \sim 0.1$ stratified case. The variables $\omega_{\text{min}}$, $\langle v_x^2 \rangle$, $\langle v_y^2 \rangle$ and $\langle v_z^2 \rangle$ have been scaled to the same scale and smoothed using the Bézier smoothing algorithm in gnuplot (Williams et al., 2010). For these, and all cases with $\chi \gtrsim 3$, the system has settled to a stable columnar vortex by $t \simeq 30 \Omega^{-1}$. The onset time of instability is denoted $\tau_{\text{unstable}}$. 
Figure 9.18 Density in stratified boxes for different $\rho'$ in a $L = (2, 4, 3)$ box with $g_z$ given by equation (9.3.4), demonstrating hydrostatic equilibrium. The density becomes constant for $z > 0.8L_z$, inside the buffer zone where vertical gravity is zero.

Figure 9.19 Investigating the different effects of $c_s$ on $\chi = 5$ vortices. Increasing $c_s$ improves lifetime of columnar vortex. The coloured bars show the stable regions for each model. When $c_s = 2.5$, there are a pair of vortices till $t \simeq 100$ when it finally becomes a single stable structure. These runs were performed in the standard $L = (2, 4, 3) N = (128, 256, 32)$ box.
Stability of vortices to 3D perturbations

(a) $t = 60\Omega^{-1}$, showing the approximately steady vortex (split over the $y$-axis) that exists in the top $\approx 30\%$ of the stratified box.

(b) $t = 150\Omega^{-1}$, showing the subsequent vortices that persist in the box for $t > 200\Omega^{-1}$.

**Figure 9.20** The time evolution of vorticity for a compressible stratified model with $c_s = 1$ and $\chi = 8$. This shows a slice of the $xy$-plane at $z = L_z$ for clarity.
Figure 9.21 Results for the stratified box with $L = (2, 4, 3)$ $N = (128, 256, 32)$, $c_s = 5$ and $\rho' \sim 0.1$, showing $\omega_{\text{min}}$ against time. Note that the vortices $2 \leq \chi \leq 3$ do not reach a stable configuration before breaking up, while the larger aspect ratios $\chi < 15$ survive until $t \simeq 100\Omega^{-1}$ before going unstable. The $\chi = 15, 20$ cases survive beyond $t = 200\Omega^{-1}$. 
Figure 9.22 Results for the stratified box with $L = (2,4,3) \ N = (128,256,32)$, $c_s = 5$ and $\rho' \sim 0.5$, showing $\omega_{\text{min}}$ against time. The key in Figure 9.21 also applies to this plot. Again, vortices with $2 \leq \chi \leq 3$ do not reach a stable configuration before breaking up, though under this stronger perturbation all vortices with $3 < \chi \leq 20$ go unstable $t < 150\Omega^{-1}$; an earlier time compared to when $\rho' \sim 0.1$ (Figure 9.24).
Figure 9.23 Results for unstratified models in a box with $L = (2, 4, 3)\ N = (128, 256, 32), \ c_s = 5$ and $\rho' \sim 0.1$. We find that $\chi = \{2, 2.5\}$ never form stable solutions, $\chi = \{3, 3.5\}$ do and go unstable $\tau_{\text{unstable}} < 200$, $\chi = 4$ goes unstable $\tau_{\text{unstable}} \simeq 228$ and all weaker vortices remain stable for at least $300\Omega^{-1}$. This is broadly consistent with the 2D models in Section 9.2.5.
Figure 9.24 Instability time as a function of χ for stratified boxes which shows that weaker (higher χ) vortices persist for longer times. In the larger perturbation ρ’ ∼ 0.5 case there is a similar trend to the ρ’ ∼ 0.1 case, but instability sets in more quickly. The χ = 20 results could be a result of under resolution, or too large a perturbation with respect to ρ’, v’.

The dashed line is for a box of half the normal height for the ρ’ ∼ 0.1 case. The few unstable unstratified cases are shown in green; like the stratified cases the very strong vortices χ ≲ 2.5 never settled to coherent structures, while χ > 5 were stable for t > 300 so are not plotted.
9.4 Summary and conclusions

From the preliminary work on 2D models in the shearing box, we find that the use of power law prescriptions for the Bernoulli source in Chapters 5 and 6 is justified as vortices naturally relax to these forms. Numerical diffusion decreases the strength of these vortices to below expected for the imposed vorticity profile, an effect that persists in 3D. The 2D models are also stable, as we expect from the vertical stability limit discussed in Chapter 7 and 8.

Moving to three dimensions, we confirm that weaker vortices persist for longer times (Figure 9.24). We expect that this is due to the narrower (and weaker) instability bands we see at larger $\chi$ in Chapter 8. Also, larger $\chi$ vortices are also not subject to the saddle point instability (Section 7.3.2), which occurs $\chi \lesssim 4$.

As can be seen in Figures 9.23 and 9.24, stratification is destabilising. This is because vertical restoring forces have frequency $\sim \Omega$ which matches to the vortex turnover frequency to destabilise the system. Reassuringly, the choice of buffer zone does not play an important role in the stability and vortices subject to larger perturbations break up sooner (Figures 9.21, 9.22 and 9.24).

Increasing resolution picks up weaker instabilities, leading to a small reduction in vortex lifetime (Figure 9.24). However, this does not prevent the RWI generating vortices (Méheut et al., 2012a) since these structures still have a long survival time, which makes it easier for competing processes to create and maintain the vortex. Lin (2014) finds that the RWI produces columnar vortices from an initial radial density bump in a few 10s of orbits, i.e. $\mathcal{O}(10\Omega^{-1})$. Thus vortices with large enough aspect ratios – so that destruction time significantly exceeds $\mathcal{O}(10\Omega^{-1})$ – have a good chance of surviving.

For $c_s > 1$ we find that increasing the sound speed has a small stabilising effect in stratified boxes. This because it effectively moves us closer towards a strictly incompressible regime. For small $c_s \lesssim 1$ we find that the sonic boundary moves inside the shearing box and we find density waves crossing the box. There are no steady structures in this regime but a few coherent and long-lived vortices do persist in the top 30% of the box.

As we are for the most part nearly incompressible, we do not expect (nor see) much density change in the radial direction\(^6\). This is because subsonic velocities do not disturb the hydrostatic equilibrium and thus the equilibrium density profile is approximately maintained. To get density enhancements we would have to consider the dust separately, which we leave to further work. Despite this, the fact that compressibility has been included while our models are not far from incompressibility makes this work more physically realistic then considering a strictly incompressible regime, as in Lesur and Papaloizou (2009, 2010).

We have created 3D analogues of the 2D, $\alpha \neq 0$ vortices devised in Chapters 5 and 6 in stratified boxes. The columnar and overwhelmingly planar nature of the flow reinforces our

\(^6\)Since the model can extend over many vertical scale heights, we do however see that $\rho$ has a $z$-dependence, as shown in Figure 9.18.
use of 2D models for this work’s stability analysis. Despite the instability bands observed in Chapter 8, such as those seen in Figures 8.7 and 8.8, the weaker vortices ($\chi \gtrsim 5$) can have long lifetimes, persisting for more than 120 orbits, after taking around 30 orbits to create a stable structure. This implies that although growth rates of around $10^{-1.5} \approx 0.03\Omega$ are sufficient to eventually disrupt the vortices, there is still a decent-sized window where they are approximately stable. We have also only considered vortices without any mechanisms sustaining them. This bodes well for the potential of these structures to capture dust and provide a place for it to coalesce and grow.

The fact that these 3D models do not have a significant density structure reminiscent of a profile of trapped dust does not detract from this as similarly sized instability growth rates were found in both vortices with and without a central density enhancement. Also, as discussed in Section 8.6, any trapped grains are likely to be localised on a few streamlines, limiting the effect on the whole vortex.
Chapter 10

Conclusions and further work

This study aimed to produce a more complete understanding of the stability of vortices in protoplanetary discs, chiefly whether parametric vortex instabilities could hinder these structures being sites for dust trapping and eventual planetesimal formation. Vortices are thought to be the most promising route around the planet formation ‘metre gap’ and thus any threat to the lifetime of these structures has ramifications for the formation theory of planetesimals and thus planets.

10.1 Main results

10.1.1 Stable vortex structures

In Chapters 5 and 6 we developed a framework for producing equilibrium vortex solutions in the shearing sheet for a range of parameters. Starting from the Kida solution with uniformly elliptical streamlines, we make models of vortices with streamlines that were not elliptical, corresponding to a flow with some internal shear. We find that adding mass does not alter the shear profile nor the shape of streamlines to any great degree, which is surprising. It was also difficult to construct equilibrium solutions with very large $\rho$-enhancements (greater than about four times the background), due to our constraint that dust is perfectly coupled to the gas and is a smooth function of $\psi$. We would expect less well-coupled dust grains to form more localised structures and not be smoothed over the whole vortex (e.g. Raettig et al., 2015). However, in the small-grain limit we consider this lack of large $\rho$-enhancement to be physically realistic.

The use of power law prescriptions for the Bernoulli vorticity source is reinforced by the 2D models in Chapter 9, which produced smooth power law profiles once an initial Kida solution relaxed. The work in this chapter also supported our use of 2D models for the stability analysis, as the 3D flow still largely behaved in a planar, 2D manner.

These models could potentially be of use as the starting points for other simulations,
removing the restriction to just Kida vortices.

10.1.2  Generic vortex stability – vortices are probably ‘stable enough’

In this thesis we sought to answer the following questions posed in Chapter 4:

(i)  What happens when vortex streamlines are not elliptical, as they are in the Kida case?  
Non–uniformly elliptical or non–elliptical streamlines introduced an internal shear. We 
find that a nonzero $\alpha$ produces a new instability band around $\chi \approx 4.85$ which increases 
in width as the internal shear increases. There we also find a general band of instability 
attributed to the saddle point of the pressure distribution, the width of which shrank as 
streamlines became less uniformly elliptical. The magnitude of instability remains similar 
to Kida, with growth rates $\simeq 0.1\Omega$ for $\chi \lesssim 5$ and $\simeq 0.05\Omega$ for $\chi \gtrsim 5$.

(ii)  How do non–constant profiles of vorticity and density affect stability?  
Results in the loss 
of a completely stable band (as in the Kida case $4 \leq \chi \lesssim 4.85$) but there are no substantial 
changes to the magnitude of the maximum growth rate for any $\chi$. Parametric instability 
bands move outwards from vortex centre, and weaker vortices have lower growth rates. 
This is confirmed by 3D work in stratified and unstratified cases.

When there is no central density enhancement we get strong instability bands at all aspect 
ratios. The limit of this is the point vortex case which reinforces our findings. When 
there is a density enhancement, we get many narrow, weaker instability bands, again at 
all $\chi$. Adding additional mass using the polytropic model (which only matches correctly 
to the background at $\chi = 7$ but is still a useful testing ground) confirms that there is no 
dramatic jump in growth rate when mass is added, with maximum growth rates for each 
$\chi \gtrsim 5$ around 0.05$\Omega$.

(iii)  Can the ‘heavy–core’ instability of Chang and Oishi (2010) be reproduced?  We found no 
evidence of this instability and caution against applying the Kida wavenumber of Lesur 
and Papaloizou (2009) to general vortex solutions. Our wavenumber analysis in Chapter 7 
shows that the use of this time–dependent wavenumber doesn’t work in the generic case 
as they cannot be associated with exponentially growing linear modes except in both the 
special Kida case and the polytropic solution with $\chi = 7$. The only possibility for them 
are Kida-like cases with $\chi = 7$ for which growth rates $\simeq 0.05\Omega$ were obtained.

Furthermore, the two nonlinear studies of Fu et al. (2014); Raettig et al. (2015) do not 
support their suggestion that vortices with significant dust concentrations cannot be 
sustained.

(iv)  Does the stability of these more general models have any bearing in 3D? Yes, it reinforced 
the trend that weaker vortices with $\chi \gtrsim 5$ were subject to weaker instability and therefore
10.2 Directions for future work

10.2.1 Back reaction of dust

In this thesis we considered dust that was perfectly coupled to the fluid. We found it difficult to produce equilibrium solutions that had high central dust enhancements, which as Lyra and Lin (2013) and Raettig et al. (2015) show, has some grounding in reality. However, the particles most susceptible to both radial drift and vortex dust trapping are $T_s = 1$ particles. A promising line of enquiry is to consider the stability of vortices with captured $T_s = 1$ particles, which could be modelled either as a second pressureless fluid or via an $N$–body regime. Furthermore, having dust partially decoupled from the flow would not limit us to smooth profiles of dust on streamlines, although finding equilibrium solutions of dust not spread uniformly inside a vortex may well be infeasible and thus a different approach would be necessary. We would also then perhaps see additional instabilities from dust–gas coupling, namely the streaming instability (Youdin and Goodman, 2005).

10.2.2 Relationship between instability and other processes

We have so far only considered parametric instability in isolation, though in reality there will be a balance between competing processes both generating and sustaining vortices. It would also be interesting to investigate vortices close to the end of their ‘useful’ life, when a high $\Gamma_d$ results in local gravitational instability. We would also expect to see some sort of cycle of particle concentration and vortex destruction, with recently grown particles being kicked out of, or destroying, their original vortices then being captured by neighbouring ones. Considering vortices not as isolated structures but as part of some sort of network would also be interesting.
10.2.3 More 3D realism

Finally, it would be instructive to consider the effect of different vertical structure in our naive 3D models, such as the presence of a thin dusty subdisc (perhaps formed of $N$-body dust) inside a thicker gas structure. How and where columnar vortices could survive in this regime is important to this problem.

There is also the perennial astrophysical question of ‘What about magnetic fields?’ to address. Lyra and Klahr (2011) find that only at low ionisation can vortices be produced and sustained by the baroclinic instability. If a magnetic field is included, these vortices only survive until the MRI develops in the box, whereupon the strain of the turbulence destroys it. However, Fromang and Nelson (2005); Lyra and Mac Low (2012) suggest that vortices may occur when magnetic fields are present and so stability studies where magnetic fields are included are still of interest.
Appendix A

Elliptic coordinates

In 2D we take

$$ r = h \sinh \xi \sin \eta i + h \cosh \xi \cos \eta j $$

(A.0.1)

so

$$ e_\xi = \frac{\partial r}{\partial \xi} = h \cosh \xi \sin \eta i + h \sinh \xi \cos \eta j $$

(A.0.2a)

$$ e_\eta = \frac{\partial r}{\partial \eta} = h \sinh \xi \cos \eta h \cosh \xi \sin \eta j $$

(A.0.2b)

$$ \Rightarrow |e_\xi| = |e_\eta| = h \sqrt{\sinh^2 \xi + \sin^2 \eta} $$

(A.0.2c)

$|e_\xi| = h_\xi$ and $|e_\eta| = h_\eta$ are the two scale factors of this coordinate system ($h_z = 1$ is the third), from which all the vector calculus quantities can be calculated using the general curvilinear coordinate formula (see e.g. Riley et al. (2006)).

For working out quantities like $d/dx$ and $d/dy$ there is a useful trick. With

$$ x = h \sinh \xi \sin \eta $$

(A.0.3)

$$ y = h \cosh \xi \cos \eta $$

(A.0.4)

we can formulate the following

$$ h^2 = \frac{x^2}{\sinh^2 \xi} + \frac{y^2}{\cosh^2 \xi} $$

(A.0.5a)

$$ h^2 = \frac{x^2}{\sin^2 \eta} - \frac{y^2}{\cos^2 \eta} $$

(A.0.5b)
Then we apply $\frac{\partial}{\partial x}|_y$ and $\frac{\partial}{\partial y}|_x$ to both equations (A.0.5a) and (A.0.5b), rearrange and find

$$\frac{\partial \xi}{\partial x}|_y = \frac{\cosh \xi \sin \eta}{h \Gamma}$$  \hfill (A.0.6a)

$$\frac{\partial \xi}{\partial y}|_x = \frac{\sinh \xi \cos \eta}{h \Gamma}$$  \hfill (A.0.6b)

$$\frac{\partial \eta}{\partial x}|_y = \frac{\sinh \xi \cos \eta}{h \Gamma}$$  \hfill (A.0.6c)

$$\frac{\partial \eta}{\partial y}|_x = -\frac{\cosh \xi \sin \eta}{h \Gamma}$$  \hfill (A.0.6d)

where

$$\Gamma = \sinh^2 \xi + \sin^2 \eta.$$  \hfill (A.0.7)

Thus the chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\cosh \xi \sin \eta}{h \Gamma} \frac{\partial}{\partial \xi} + \frac{\sinh \xi \cos \eta}{h \Gamma} \frac{\partial}{\partial \eta}$$  \hfill (A.0.8a)

$$\frac{\partial}{\partial y} = \frac{\sinh \xi \cos \eta}{h \Gamma} \frac{\partial}{\partial \xi} - \frac{\cosh \xi \sin \eta}{h \Gamma} \frac{\partial}{\partial \eta}.$$  \hfill (A.0.8b)

We can then happily calculate quantities such as $\mathbf{u} = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$. 
Appendix B

Floquet theory as solution of an initial value problem

The stability analysis we do on our equilibrium vortex solutions requires solving a series of linear differential equations with periodic coefficients. This calls for Floquet Theory (Floquet, 1883).

We have a problem of the form

$$x' = A(t)x \quad \text{(B.0.1)}$$

where $A(t)$ is a $n \times n$ periodic matrix with period $T$ so

$$A(t + T) = A(t) \quad \forall t. \quad \text{(B.0.2)}$$

Then $x$ need not be periodic but must be of the form $e^{\mu t}p(t)$ where $p(t)$ is also $T$-periodic so

$$p(t + T) = p(t) \quad \forall t. \quad \text{(B.0.3)}$$

Let $x^1(t), \ldots x^n(t)$ be $n$ solutions of (B.0.1). Furthermore, let

$$X(t) = \left( \begin{bmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{bmatrix} \right) \quad \text{(B.0.4)}$$

such that $X(t)$ is an $n \times n$ solution of $X' = A(t)X$.

**Definition 1** (Fundamental Matrix). If our $x^1(t), \ldots x^n(t)$ are linearly independent then $X(t)$ is non-singular and is called a fundamental matrix.

**Definition 2** (Principle Fundamental Matrix). If $X(t_0) = I$, where $t_0$ is the initial time, then $X(t)$ is called the principle fundamental matrix.

**Lemma 1.** If $X(t)$ is a fundamental matrix then so is $Y(t) = X(t)B$, where $B$ a non-singular
**constant matrix.**

**Proof.** $X(t)$ and $B$ are non-singular so their inverses exist.

$$Y^{-1}(t) = B^{-1}X^{-1}(t)$$

so $Y$ is non-singular too. Then

$$Y' = XB = AXB = AY$$

\[ \square \]

**Lemma 2.** *Let the Wronskian $W(t)$ of $X(t)$ be the determinant of $X(t)$. Then*

$$W(t) = W(t_0) \exp \left[ \int_{t_0}^{t} \text{tr} (A(s)) \, ds \right]. \quad (B.0.5)$$

**Proof.** Let $t_0$ be some time. Expanding $X(t)$ as a Taylor series:

$$X(t) = X(t_0) + (t - t_0)X'(t_0) + O\left((t - t_0)^2\right)$$

$$= X(t_0) + (t - t_0)A(t_0)X(t_0) + O\left((t - t_0)^2\right)$$

$$= [I + (t - t_0)A(t_0)]X(t_0) + O\left((t - t_0)^2\right).$$

Then

$$\det(X(t)) = \det[I + (t - t_0)A(t_0)] \det X(t_0)$$

$$\Rightarrow W(t) = \det[I + (t - t_0)A(t_0)] W(t_0)$$

Since $\det(I + \epsilon C) = 1 + \epsilon \text{tr}(C) + O(\epsilon^2)$,

$$W(t) = W(t_0) \left[1 + (t - t_0)\text{tr}(A(t_0))\right]$$

$$= W(t_0) + (t - t_0)W'(t_0) + O\left((t - t_0)^2\right)$$

from taking the Taylor series. We have made no assumption about $t_0$ so

$$W''(t_0) = W(t_0)\text{tr}(A(t_0))$$

$$\Rightarrow W'(t) = W(t)\text{tr}(A(t_0))$$

$$\Rightarrow W(t) = W(t_0) \exp \left[ \int_{t_0}^{t} \text{tr} (A(s)) \, ds \right]$$

\[ \square \]

**Theorem 1.** *If $X(t)$ is a fundamental matrix then:*
1. $\mathbf{X}(t+T)$ is fundamental

2. $\exists$ a non-singular and constant matrix $\mathbf{B}$ s.t. $\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B}$

3. $\det(\mathbf{B}) = \exp \left[ \int_0^T \text{tr} (\mathbf{A}(s)) \, ds \right]$.

**Proof.**

1. Let $\mathbf{Y}(t) = \mathbf{X}(t+T)$. Then

   $$\mathbf{Y}'(t) = \mathbf{X}'(t+T) = \mathbf{A}(t+T)\mathbf{X}(t+T)$$
   $$= \mathbf{A}(t)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{Y}(t)$$
   $$\Rightarrow \mathbf{X}(t+T) \text{ is fundamental.}$$

2. Let $\mathbf{B}(t) = \mathbf{X}^{-1}(t)\mathbf{Y}(t)$. Then

   $$\mathbf{Y}(t) = \left[ \mathbf{X}(t)\mathbf{X}^{-1}(t) \right] \mathbf{Y}(t) = \mathbf{X}(t)\mathbf{B}(t).$$

   Let $\mathbf{B}_0 = \mathbf{B}(t_0)$. By Lemma 1, $\mathbf{Y}_0(t) = \mathbf{X}(t)\mathbf{B}_0$ is a fundamental matrix. Since both $\mathbf{Y}_0(t)$ and $\mathbf{Y}(t)$ are solutions of equation (B.0.1), by the uniqueness of the solution $\mathbf{Y}_0(t) = \mathbf{Y}(t) \forall t$.

   Thus $\mathbf{B}_0 = \mathbf{B}(t_0)$ and $\mathbf{B}$ is time-independent.

3. Lemma 2 gives

   $$W(t) = W(t_0) \exp \left[ \int_{t_0}^t \text{tr} (\mathbf{A}(s)) \, ds \right]$$
   $$\Rightarrow W(t+T) = W(t_0) \exp \left[ \int_{t_0}^t \text{tr} (\mathbf{A}(s)) \, ds + \int_t^{t+T} \text{tr} (\mathbf{A}(s)) \, ds \right]$$
   $$= W(t) \exp \left[ \int_t^{t+T} \text{tr} (\mathbf{A}(s)) \, ds \right]$$
   $$= W(t) \exp \left[ \int_0^T \text{tr} (\mathbf{A}(s)) \, ds \right] \quad \text{(B.0.6)}$$

   where the last line used the periodicity of $\mathbf{A}$. Also

   $$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B}$$
   $$\det (\mathbf{X}(t+T)) = \det (\mathbf{X}(t)) \det \mathbf{B}$$
   $$\Rightarrow W(t+T) = W(t) \det \mathbf{B}$$

   $$\Rightarrow \det \mathbf{B} = \exp \left[ \int_0^T \text{tr} (\mathbf{A}(s)) \, ds \right] \quad \text{(B.0.7)}$$
Definition 3 (Characteristic Multipliers). The eigenvalues of $B$, $\varrho_1, \ldots, \varrho_n$ are the characteristic multipliers for equation (B.0.1).

Definition 4 (Characteristic Exponents). The characteristic (or Floquet) exponents are $\mu_1, \ldots, \mu_n \in \mathbb{C}$ satisfying

\[
\varrho_1 = e^{\mu_1 T}, \ldots, \varrho_n = e^{\mu_n T} \tag{B.0.8}
\]

The characteristic multipliers $\varrho_1, \ldots, \varrho_n$ of $B = X(T)$ with $X(0) = I$ satisfy

\[
\det B = \varrho_1 \varrho_2 \ldots \varrho_n = \exp \left[ \int_0^T \text{tr} \left( A(s) \right) ds \right] \tag{B.0.9}
\]

The characteristic exponents are not unique since if $\varrho_j = e^{\mu_j T}$ then

\[
\varrho_j = \exp \left[ \left( \mu_j + \frac{2\pi in}{T} \right) T \right], \quad n \in \mathbb{Z} \tag{B.0.10}
\]

Lemma 3. The characteristic multipliers $\varrho_j$ are an intrinsic property of equation (B.0.1) and do not depend on the choice of fundamental matrix.

Proof. Suppose $\hat{X}(t)$ is another fundamental matrix. Then

\[
\hat{X}(t + T) = \hat{X}(t) \hat{B}.
\]

By Lemma 1, since $X(t)$ and $\hat{X}(t)$ are fundamental matrices, $\exists$ a constant non-singular matrix $C$ such that

\[
\hat{X}(t) = X(t) C
\]

\[
\Rightarrow \hat{X}(t + T) = X(t + T) C
\]

\[
\Rightarrow \hat{X}(t) \hat{B} = X(t) BC
\]

\[
\Rightarrow X(t) CB = X(t) BC
\]

\[
\Rightarrow C \hat{B} = BC
\]

\[
\Rightarrow B = C \hat{B} C^{-1} \tag{B.0.11}
\]

Thus the eigenvalues of $B$ and $\hat{B}$ are the same.

Theorem 2. Let $\varrho$ be a characteristic multiplier and let $\mu$ be the corresponding characteristic exponent so that $\varrho = e^{\mu T}$. Then $\exists$ a solution $x(t)$ of equation (B.0.1) such that

1. $x(t + T) = \varrho x(t)$

2. $\exists$ a $T$-periodic solution $p(t)$ such that $x(t) = e^{\mu t} p(t)$. 

$\square$
Proof. 1. Let \( b \) be an eigenvector of \( B \) corresponding to eigenvalue \( \varrho \):

\[
Bb = \varrho b.
\]

Let \( x(t) = X(t)b \). Then \( x' = Ax \).

\[
x(t + T) = X(t + T)b \\
= X(t)Bb \\
= \varrho X(t)b \\
= \varrho x(t)
\]

2. Let \( p(t) = x(t)e^{-\mu t} \). We need to show that \( p(t) \) is \( T \)-periodic.

\[
p(t + T) = x(t + T)e^{-\mu(t+T)} \\
= \varrho x(t)e^{-\mu t}e^{-\mu T} \\
= \frac{\varrho}{e^{\mu T}}x(t)e^{-\mu t} \\
= x(t)e^{-\mu t} \\
= p(t)
\]

Note that:

\[
x_j(t + T) = \varrho_j x_j(t) \\
x_j(t + NT) = \varrho_j^N x_j(t)
\]

Baring that in mind, each characteristic multiplier falls into one of the following categories:

1. If \( |\varrho| < 1 \) then \( \Re(\mu) < 0 \) and so \( x(t) \xrightarrow{t \to \infty} 0 \)

2. If \( |\varrho| = 1 \) then \( \Re(\mu) = 0 \) and so we have a psuedo-periodic solution. If \( \varrho = \pm 1 \) then the solution is periodic with period \( T \).

3. \( |\varrho| > 1 \) then \( \Re(\mu) > 0 \) and so \( x(t) \xrightarrow{t \to \infty} \infty \)

The entire solution is stable if all characteristic multipliers satisfy \( |\varrho_j| \leq 1 \).
Appendix C

Growth rates in the shearing sheet for $k \propto t$

This derivation is based on the lecture notes\(^1\) of Prof. Gordon Ogilvie, who gave the clearest and most concise explanation I could find.

Consider a 3D homogeneous, inviscid\(^2\), incompressible fluid, unbounded or periodic in $x, y, z$ and rotating in a frame with $\Omega = \Omega \hat{z}$, in the shearing box of Goldreich and Lynden-Bell (1965). The Navier-Stokes equations governing this are

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + 2\Omega \times u = -\frac{1}{\rho} \nabla P - \nabla \Phi$$

\hspace{4cm} (C.0.1a)

$$\nabla \cdot u = 0$$

\hspace{4cm} (C.0.1b)

Neglecting vertical gravity we have

$$\Phi = -\Omega S x^2,$$

\hspace{4cm} (C.0.2)

where $S$ is the shear rate. The basic state of the shearing box is given by $u = -Sx \hat{y}$ and $P = \text{const}$. Let $v$ be the velocity relative to the shearing sheet, $P'$ be the pressure relative to the background and $\Psi = P'/\rho$, where $\rho$ is constant. Then

$$\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + v \cdot \nabla \right) v_x = -2\Omega v_y = -\frac{\partial \Psi}{\partial x}$$

\hspace{4cm} (C.0.3a)

$$\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + v \cdot \nabla \right) v_y + (2\Omega - S)v_x = -\frac{\partial \Psi}{\partial y}$$

\hspace{4cm} (C.0.3b)

$$\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} + v \cdot \nabla \right) v_z = -\frac{\partial \Psi}{\partial z}$$

\hspace{4cm} (C.0.3c)

\(^1\)http://www.damtp.cam.ac.uk/user/gio10/dad.html

\(^2\)The contribution due to $\nu \nabla^2 u$ can be included and dealt with via an integrating factor.
Consider plane waves with time-dependent wavenumber \( k(t) \):

\[
\begin{align*}
\mathbf{v}(x,t) &= \tilde{v}(t) \exp[i k(t) \cdot x] \\
\Psi(x,t) &= \tilde{\Psi}(t) \exp[i k(t) \cdot x]
\end{align*}
\]  

(C.0.4)  

(C.0.5)

Then

\[
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} \right) \mathbf{v} = \left\{ \frac{d\tilde{v}}{dt} + i \left( \frac{dk}{dt} \cdot x - Sxk_y \right) \tilde{v} \right\} \exp[i k(t) \cdot x].
\]  

(C.0.6)

Choosing \( dk/dt = S k_y \hat{x} \) means that this reduces to

\[
\left( \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} \right) \mathbf{v} = \frac{d\tilde{v}}{dt} \exp[i k(t) \cdot x]
\]  

(C.0.7)

and \( \mathbf{v} \cdot \nabla \mathbf{v} = 0 \) since incompressibility gives \( i \mathbf{k} \cdot \tilde{v} = 0 \). Equation (C.0.3) then becomes

\[
\begin{align*}
\frac{d\tilde{v}_x}{dt} &= -2\Omega \tilde{v}_y = -ik_x \tilde{\Psi} \\
\frac{d\tilde{v}_y}{dt} + (2\Omega - S) \tilde{v}_x &= -ik_y \tilde{\Psi} \\
\frac{d\tilde{v}_z}{dt} &= -ik_z \tilde{\Psi}
\end{align*}
\]  

(C.0.8a)  

(C.0.8b)  

(C.0.8c)

with

\[
i \mathbf{k} \cdot \tilde{v} = 0
\]  

(C.0.8d)

and

\[
\mathbf{k} = (k_{x,0} + S k_y t, k_y, k_z),
\]  

(C.0.8e)

where \( k_{x,0}, k_y, k_z \) are constants. Eliminating \( \tilde{\Psi}, \tilde{v}_y \) and \( \tilde{v}_z \) results in the second order ODE:

\[
\frac{d^2}{dt^2} \left( k^2 \tilde{v}_x \right) + \kappa^2 k_z^2 \tilde{v}_x = 0,
\]  

(C.0.9)

with \( k^2 = (k_{x,0}^2 + k_y^2 + k_z^2) + S^2 k_y^2 t^2 \) and the epicyclic frequency \( \kappa^2 = 2\Omega (2\Omega - S) \). In Keplerian discs, \( \kappa^2 = \Omega^2 > 0 \), which is what we are concerned with in this work.

If \( k_y = 0 \), we have axisymmetric disturbances, \( \mathbf{k} \) is constant and the equation has constant coefficients. The system will therefore oscillate with period \( \kappa k_z / k \).

If \( k_y \neq 0 \) we have sheared, non-axisymmetric waves and the solutions to equation (C.0.9) are Legendre functions. As \( t \to \infty \), this equation acts like

\[
\frac{d^2}{dt^2} \left( t^2 \tilde{v}_x \right) + a^2 \tilde{v}_x = 0, \quad a^2 = \frac{\kappa^2 k_z^2}{S^2 k_y^2}.
\]  

(C.0.10)
This is a Cauchy-Euler equation so we look for solutions of the form $\tilde{v}_x = t^r$. This gives

$$(r + 1)(r + 2) + a^2 = 0,$$  \hspace{1cm} (C.0.11)

which has roots

$$r = -3 \pm \sqrt{1 - 4a^2} = -\frac{3}{2} \pm \sigma$$  \hspace{1cm} (C.0.12)

Since we are interested in growth rates at large $t$, we are only concerned about the positive root, $|\tilde{v}_x| \propto t^{\sigma - 3/2}$. Equation (C.0.8d) implies that

$$\tilde{v}_y, \tilde{v}_z \sim t^{\sigma - 1/2}$$  \hspace{1cm} (C.0.13)

while equation (C.0.8c) shows $\tilde{\Psi} \sim t^{\sigma - 3/2}$. Therefore, at late times, $|\tilde{v}| \sim t^{\sigma - 1/2}$. The following cases arise:

(i) If $1 - 4a^2 < 0$, or

$$\kappa^2 > \frac{S^2k_y^2}{4k_z^2}$$  \hspace{1cm} (C.0.14)

then $\sigma$ is imaginary and $|\tilde{v}| \sim t^{-1/2} \to 0$ as $t \to \infty$.

(ii) If $0 < 1 - 4a^2 < \frac{1}{4}$, or

$$0 < \kappa^2 < \frac{S^2k_y^2}{4k_z^2}$$  \hspace{1cm} (C.0.15)

then $0 < \sigma^2 < \frac{1}{4}$ and $|\tilde{v}| \sim t^{\sigma - 1/2} \to 0$ as $t \to \infty$.

(iii) If $1 - 4a^2 > \frac{1}{4}$, $\kappa^2 < 0$ and $|\tilde{v}| \sim t^{\sigma - 1/2} \to \infty$ as $t \to \infty$.

We find that flows that are Rayleigh-stable (Rayleigh, 1917) with $\kappa^2 > 0$, such as Keplerian rotation, are also stable at large $t$ to shearing waves.
# Appendix D

## PLUTO runs

Table D.1 The variables governing different PLUTO runs. Typically, $t = 200 \Omega^{-1}$.

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<th>$L_z$</th>
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2D boxes in line with 3D versions

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**Unstratified 3D boxes – testing perturbations**

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**Stratified boxes, full**

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**Stratified half-boxes, $z \geq 0$**

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Stratified short half-boxes

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High resolution boxes

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Low $c_s$, looking for density waves

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Checking stratified interface, perturbations only for $z \leq 0.5L_z$, compare to run 219

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Table D.1 – Continued from previous page

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Unstratified boxes

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Checking stratified interface, perturbations for $z \leq L_z$, compare to runs 219, 232

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Investigating low $c_s$, stratified cases, compare to runs 219, 220

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Investigating a high resolution case – is there a higher $\chi_{crit}$, above which all columns are stable?

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Bibliography


Chamberlin, T. C. (1900). An attempt to test the nebular hypothesis by the relations of masses and momenta. *Journal of Geology*, 8:58–73.


