ABELIAN QUOTIENTS OF MAPPING CLASS GROUPS OF HIGHLY CONNECTED MANIFOLDS

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Abstract. We compute the abelianisations of the mapping class groups of the manifolds $W_g^{2n} = g(S^n \times S^n)$ for $n \geq 3$ and $g \geq 5$. The answer is a direct sum of two parts. The first part arises from the action of the mapping class group on the middle homology, and takes values in the abelianisation of the automorphism group of the middle homology. The second part arises from bordism classes of mapping cylinders and takes values in the quotient of the stable homotopy groups of spheres by a certain subgroup which in many cases agrees with the image of the stable J-homomorphism. We relate its calculation to a purely homotopy theoretic problem.

1. Introduction

Let $W_g^{2n} = g(S^n \times S^n)$ denote the g-fold connected sum and choose a fixed closed disc $D^{2n} \subset W_g^{2n}$. Let $\mathrm{Diff}^+(W_g^{2n})$ be the topological group of orientation preserving diffeomorphisms of W_g^{2n}, and $\mathrm{Diff}(W_g^{2n}, D^{2n})$ be the subgroup of those diffeomorphisms which fix an open neighbourhood of the disc. Define the *mapping class groups*

$$\Gamma_{g,1}^n = \pi_0(\mathrm{Diff}(W_g^{2n}, D^{2n})) \qquad \Gamma_g^n = \pi_0(\mathrm{Diff}^+(W_g^{2n})).$$

There is a homomorphism $\gamma : \Gamma_{g,1}^n \to \Gamma_g^n$, which simply forgets that diffeomorphisms fix a disc. We will construct two abelian quotients of these groups, one coming from arithmetic properties of the intersection form of W_g^{2n}, and one coming from a cobordism theoretic construction. Together, these will give the abelianisation of either group.

Construction 1.1. Recall that Wall [Wal62] has constructed for each $(n-1)$-connected $2n$-manifold W a certain quadratic form Q_W, which we shall describe later, whose underlying bilinear form is the intersection form on $H_n(W;\mathbb{Z})$. Diffeomorphisms of the manifold act by automorphisms of this quadratic form, so there is a group homomorphism

$$\hat{f} : \Gamma_g^n \longrightarrow \mathrm{Aut}(Q_{W_g^{2n}}),$$

from which we can construct the map $f : \Gamma_g^n \to H_1(\mathrm{Aut}(Q_{W_g^{2n}}))$ to an abelian group. We will also write $\hat{f}$ for the composition $\hat{f} \circ \gamma : \Gamma_{g,1}^n \to \Gamma_g^n \to \mathrm{Aut}(Q_{W_g^{2n}})$, and similarly with f.

Construction 1.2. Let $\varphi \in \mathrm{Diff}(W_g^{2n}, D^{2n})$ be a diffeomorphism of W_g^{2n} which is the identity on a fixed disc $D^{2n} \subset W_g^{2n}$. We may form the mapping torus

$$T_\varphi = W_g^{2n} \times [0,1]/(x,0) \sim (\varphi(x),1),$$

which is a $(2n+1)$-dimensional manifold fibering over S^1, and contains an embedded $D^{2n} \times S^1$ given by the disc fixed by φ. The $(n-1)$-connected manifold obtained by surgery along this embedded $D^{2n} \times S^1$ shall be denoted T'_ϕ. This construction is

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often called an open book. By obstruction theory, a map $\tau : T'_\phi \to BO$ classifying its stable normal bundle admits a lift $\ell : T'_\phi \to BO\langle n\rangle$, unique up to homotopy, where $BO\langle n\rangle \to BO$ denotes the n-connected cover. The pair (T'_φ, ℓ) represents an element of $\Omega^{\langle n\rangle}_{2n+1}$, the cobordism theory associated to the map $BO\langle n\rangle \to BO$, and one easily verifies that the function

$$\begin{aligned} t : \Gamma^n_{g,1} &\longrightarrow \Omega^{\langle n\rangle}_{2n+1} \\ \varphi &\longmapsto [T'_\varphi, \ell] \end{aligned}$$

is a group homomorphism.

Our main theorem, proved in Sections 3–5 below, is that these two homomorphisms combine to give the maximal abelian quotient of the group $\Gamma^n_{g,1}$.

Theorem 1.3. *For all n and g (except we require $g \geq 2$ if $n = 2$) the map*

$$t \oplus f : \Gamma^n_{g,1} \longrightarrow \Omega^{\langle n\rangle}_{2n+1} \oplus H_1(\mathrm{Aut}(Q_{W^{2n}_g}))$$

is surjective, and for $n \neq 2$ and $g \geq 5$ it is the abelianisation. Furthermore, in this range

$$H_1(\mathrm{Aut}(Q_{W^{2n}_g})) \cong \begin{cases} (\mathbb{Z}/2)^2 & n \text{ even} \\ 0 & n = 1,\ 3 \text{ or } 7 \\ \mathbb{Z}/4 & \text{otherwise.} \end{cases}$$

We obtain the following table describing $H_1(\Gamma^n_{g,1};\mathbb{Z})$ for small n, using known calculations of $\Omega^{\langle 3\rangle}_* = \Omega^{\mathrm{Spin}}_*$ and $\Omega^{\langle 7\rangle}_* = \Omega^{\mathrm{String}}_*$ (see [Mil63, p. 201] for the former and [Gia71] for the latter).

Table 1. Abelianisations of $\Gamma^n_{g,1}$ for $g \geq 5$.

n	1	2	3	4	5	6	7
$H_1(\Gamma^n_{g,1};\mathbb{Z})$	0	$(\mathbb{Z}/2)^2\oplus ?$	0	$(\mathbb{Z}/2)^4$	$\mathbb{Z}/4$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/2$

In Section 7 we compare our work with that of Kreck [Kre79], who has described the groups $\Gamma^n_{g,1}$ up to extension problems. Using Theorem 1.3 we are able to resolve these extension problems when $n = 6$ or $n \equiv 5 \mod 8$, and hence give a complete description of these mapping class groups.

1.1. **The cobordism groups $\Omega^{\langle n\rangle}_{2n+1}$.** In light of Theorem 1.3, it is of interest to describe the cobordism group $\Omega^{\langle n\rangle}_{2n+1}$ in terms of more familiar objects. There is a homomorphism

$$\rho : \Omega^{\mathrm{fr}}_{2n+1} \longrightarrow \Omega^{\langle n\rangle}_{2n+1}$$

from framed cobordism obtained by simply remembering that a stably tangentially framed manifold in particular has a $BO\langle n\rangle$-structure. The cobordism theoretic interpretation of the J-homomorphism

$$J : \pi_{2n+1}(SO) \longrightarrow \pi^s_{2n+1} = \Omega^{\mathrm{fr}}_{2n+1}$$

is that it sends a map $f : S^{2n+1} \to SO$ to the stably framed manifold obtained by taking the $(2n+1)$-sphere with its usual—bounding—stable framing, and changing the framing using f. The resulting stable framing need not extend over D^{2n+2}, but the $BO\langle n\rangle$-structure does always extend (as the map $BO\langle n\rangle \to BO$ is n-connected), so $\rho \circ J$ is trivial. Thus there is an induced map

$$\rho' : \mathrm{Coker}(J)_{2n+1} \longrightarrow \Omega^{\langle n\rangle}_{2n+1}.$$

It follows from work of Stolz that this map is an isomorphism in many cases.

Theorem 1.4 (Stolz). *The map ρ' is surjective, and is an isomorphism if either*

(i) $n+1 \equiv 2 \bmod 8$ *and* $n+1 \geq 18$,

(ii) $n+1 \equiv 1 \bmod 8$ *and* $n+1 \geq 113$,

(iii) $n+1 \not\equiv 0, 1, 2, 4 \bmod 8$.

In the cases not covered by this theorem, the kernel of ρ' is at most $\mathbb{Z}/2$ if $n+1 \equiv 1, 2 \bmod 8$, and cyclic if $n+1 \equiv 0 \bmod 4$. We give more detailed information in Section 6.

1.2. **Closed manifolds.** We can also use Theorem 1.3 to calculate the abelianisation of the mapping class group Γ_g^n, of orientation preserving diffeomorphisms of the closed manifolds W_g^{2n}, because of the following result of Kreck.

Lemma 1.5 (Kreck). *The map $\gamma : \Gamma_{g,1}^n \to \Gamma_g^n$ is an isomorphism for $n \geq 3$.*

Proof. The homotopy fiber sequences

$$\mathrm{Fr}^+(W_g^{2n}) \longrightarrow B\mathrm{Diff}(W_g^{2n}, D^{2n}) \longrightarrow B\mathrm{Diff}^+(W_g^{2n})$$

and

$$SO(2n) \longrightarrow \mathrm{Fr}^+(W_g^{2n}) \longrightarrow W_g^{2n}$$

induce long exact sequence in homotopy groups, from which it is easy to see that $\Gamma_{g,1}^n \to \Gamma_g^n$ is surjective with kernel either trivial or $\mathbb{Z}/2$, as long as $n \geq 3$.

Kreck proves that the kernel is in fact trivial: Combine the discussion at the bottom of page 657 of [Kre79] with the fact that the manifolds W_g^{2n} bound the parallelisable manifolds $\natural^g(S^n \times D^{n+1})$, so the element $\Sigma_{W_g^{2n}}$ is trivial by Lemma 3b of that paper. □

1.3. **Perfection.** Recall that a group is called *perfect* if it is equal to its derived subgroup, or equivalently if its abelianisation is trivial. Table 1 shows that $\Gamma_{g,1}^n$ (or Γ_g^n, by Lemma 1.5) is perfect for $n = 1$ or $n = 3$ and $g \geq 5$, but the fact that $H_1(\mathrm{Aut}(Q_{W_g^{2n}}))$ is trivial only for $n = 1, 3, 7$ and the fact that $\Omega_{15}^{\langle 7 \rangle} \neq 0$ means that these are the only examples.

Corollary 1.6. *For $g \geq 5$, the groups $\Gamma_{g,1}^n$ and Γ_g^n are perfect if and only if n is 1 or 3.*

If we denote by $\mathring{W}_{g,1}^{2n}$ the complement of the chosen disc D^{2n} in W_g^{2n}, then the group $\mathrm{Diff}(W_g^{2n}, D^{2n})$ is isomorphic to $\mathrm{Diff}_c(\mathring{W}_{g,1}^{2n})$, the group of compactly supported diffeomorphisms of $\mathring{W}_{g,1}^{2n}$. Thurston [Thu74] has proved that for any manifold M without boundary the identity component $\mathrm{Diff}_c(M)_0^\delta$, considered as a discrete group, is perfect (in fact, it is simple). Thus the extension of discrete groups

$$1 \longrightarrow \mathrm{Diff}_c(\mathring{W}_{g,1}^{2n})_0^\delta \longrightarrow \mathrm{Diff}(W_g^{2n}, D^{2n})^\delta \longrightarrow \Gamma_{g,1}^n \longrightarrow 1$$

shows that the discrete group $\mathrm{Diff}(W_g^{2n}, D^{2n})^\delta$ is perfect if and only $\Gamma_{g,1}^n$ is, and more generally that the abelianisation of the discrete group $\mathrm{Diff}(W_g^{2n}, D^{2n})^\delta$ is also described by Theorem 1.3. Similarly, the abelianisation of the discrete group $\mathrm{Diff}^+(W_g^{2n})^\delta$ is also described by Theorem 1.3. See [Nar14] for more information about the homology of $\mathrm{Diff}(W_g^{2n}, D^{2n})^\delta$.

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2. Wall's quadratic form

The fibration $S^n \to BO(n) \to BO(n+1)$ gives a long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_{n+1}(BO(n+1)) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\tau} \pi_n(BO(n)) \xrightarrow{s} \pi_n(BO(n+1)) \longrightarrow 0,$$

and we let $\Lambda_n = \mathrm{Im}(\partial) \subset \mathbb{Z}$. The map τ sends 1 to the map which classifies the tangent bundle of the n-sphere, so Λ_n is trivial if n is even, $\mathbb{Z}$ if $n = 1$, 3 or 7, and $2\mathbb{Z}$ otherwise, by the Hopf invariant 1 theorem. The data $((-1)^n, \Lambda_n)$ is a *form parameter* in the sense of Bak [Bak69, Bak81].

Suppose that $n \geq 4$, and let W be an $(n-1)$-connected $2n$-manifold which is stably parallelisable. We will describe how to associate to it a non-degenerate quadratic form Q_W having form parameter $((-1)^n, \Lambda_n)$, following Wall [Wal62]. The $\mathbb{Z}$-module

$$\pi_n(W) \cong H_n(W; \mathbb{Z})$$

has a $(-1)^n$-symmetric bilinear form

$$\lambda : H_n(W; \mathbb{Z}) \otimes H_n(W; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by the intersection form, which is non-degenerate by Poincaré duality. If $x = [f] \in \pi_n(W)$, then by a theorem of Haefliger [Hae61] as $n \geq 4$ we may represent it uniquely up to isotopy by an embedding $f : S^n \hookrightarrow W$, which has an n-dimensional normal bundle which is stable trivial. This represents an element

$$\alpha(x) \in \mathbb{Z}/\Lambda_n = \mathrm{Ker}\left(\pi_n(BO(n)) \xrightarrow{s} \pi_n(BO(n+1))\right),$$

and Wall has shown that this satisfies

(i) $\alpha(a \cdot x) = a^2 \cdot \alpha(x)$, for $a \in \mathbb{Z}$,
(ii) $\alpha(x + y) = \alpha(x) + \alpha(y) + \lambda(x, y)$, where $\lambda(x, y)$ is reduced modulo Λ_n.

Thus the data $(\pi_n(W), \lambda, \alpha)$ is a quadratic form with form parameter $((-1)^n, \Lambda_n)$.

Remark 2.1. This construction above does not quite work for $n \leq 3$, as Haefliger's theorem does not apply, but we can proceed anyway. When $n = 1$ or 3 we have $\mathbb{Z}/\Lambda_n = \{0\}$ and so a quadratic form with parameter $(-1, \Lambda_n)$ should be a module with skew-symmetric bilinear form. We take $H_n(W_g; \mathbb{Z})$ with its intersection form.

When $n = 2$ we have $\mathbb{Z}/\Lambda_2 = \mathbb{Z}$ and so a quadratic form with parameter $(1, \Lambda_2)$ should be an even symmetric bilinear form. The intersection form on $H_2(W_g; \mathbb{Z})$ *is* even, so we can take this.

By construction, it is clear that if $\varphi : W_0 \to W_1$ is a diffeomorphism then $\varphi_* : H_n(W_0; \mathbb{Z}) \to H_n(W_1; \mathbb{Z})$ is a morphism of quadratic forms. The most elementary quadratic form is the *hyperbolic form*

$$H = \left(\mathbb{Z}^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}; \alpha(e) = \alpha(f) = 0\right).$$

The manifold $W_g^{2n} = g(S^n \times S^n)$ has associated quadratic form $H^{\oplus g}$, the direct sum of g copies of the hyperbolic form, and so we have a homomorphism

$$\hat{f} : \Gamma_g^n \longrightarrow \mathrm{Aut}(H^{\oplus g}).$$

Kreck [Kre79] has shown that this map is surjective for $n \geq 3$, Wall [Wal64] has shown it is surjective for $n = 2$ as long as $g \geq 5$, and it is well-known to be surjective for $n = 1$ and all g. We obtain an abelian quotient

$$f : \Gamma_g^n \longrightarrow H_1(\mathrm{Aut}(H^{\oplus g}); \mathbb{Z}). \qquad (2.1)$$

Proposition 2.2. *There are isomorphisms*

$$H_1(\mathrm{Aut}(H^{\oplus g});\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^2 & n \text{ even} \\ 0 & n = 1,\ 3 \text{ or } 7 \\ \mathbb{Z}/4 & \text{otherwise.} \end{cases}$$

as long as $g \geq 5$.

Proof. By Charney's stability theorem [Cha87] for the homology of automorphism groups of quadratic forms over a PID, the group $H_1(\mathrm{Aut}(H^{\oplus g});\mathbb{Z})$ is independent of g as long as $g \geq 5$. In fact, the statement in [Cha87] claims this only for $g \geq 6$, but using the slightly improved connectivity for the necessary poset / simplicial complex which is established in [GRW14a, Theorem 3.2] this can be improved to $g \geq 5$ (the poset $HU_g = HU(H^{\oplus g})$ of [Cha87] is the face poset of the simplicial complex $K^a(H^{\oplus g})$ of [GRW14a], so they have homeomorphic geometric realisations).

If n is even, then $\mathrm{Aut}(H^{\oplus g}) = \mathrm{O}_{g,g}(\mathbb{Z})$ is the indefinite orthogonal group over the integers. This is a subgroup of $\mathrm{O}_{g,g}(\mathbb{R})$, which has maximal compact subgroup $\mathrm{O}_g(\mathbb{R}) \times \mathrm{O}_g(\mathbb{R})$; the determinants of these two factors provides a surjective homomorphism $a : \mathrm{O}_{g,g}(\mathbb{Z}) \to (\mathbb{Z}/2)^2$. In [GHS09, Theorem 1.7] it is shown that a certain index 4 normal subgroup $\widetilde{\mathrm{SO}}^+_{g,g}(\mathbb{Z})$ of $\mathrm{O}_{g,g}(\mathbb{Z})$, for the definition of which we refer to that paper, has trivial abelianisation. Thus the homomorphism a is the abelianisation.

If $n = 1$, 3 or 7 then a quadratic form with parameter $(-1, \Lambda_n)$ is nothing but an antisymmetric bilinear form, so $\mathrm{Aut}(H^{\oplus g}) = \mathrm{Sp}_{2g}(\mathbb{Z})$ is the symplectic group over the integers. This is well-known to have trivial abelianisation, as long as $g \geq 3$.

For the remaining odd n, $\mathrm{Aut}(H^{\oplus g}) = \mathrm{Sp}^q_{2g}(\mathbb{Z}) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ is the subgroup of those symplectic matrices which stabilise the quadratic form $\alpha(e_i) = \alpha(f_i) = 0$. The abelianisation of this group has been computed in [JM90, Theorem 1.1] to be $\mathbb{Z}/4$ as long as $g \geq 3$. □

Remark 2.3. The argument above can be used to strengthen the "only if" part of Corollary 1.6: for $n \neq 1, 3$, the mapping class groups are not perfect for *any* $g \geq 1$.

For $n = 7$ this is in fact the case for $g \geq 0$, as the generator of $\Omega_{15}^{\langle 7 \rangle} = \mathbb{Z}/2$ can be hit by a diffeomorphism supported inside a disc. For $n \neq 7$ we argue as follows. The matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ defines an element of $\mathrm{Aut}(H)$ for all n, and is easily seen to be realised by an element of $\Gamma^n_{1,1}$. For n even this maps to a non-trivial element of $(\mathbb{Z}/2)^2$, and for n odd apart from $1, 3, 7$, it follows from the formula [JM90, p. 147] that it maps to the order-two element of $\mathbb{Z}/4$.

3. Low-dimensional cases

The cases $n < 3$ of Theorem 1.3 require special treatment, so let us dispense with them first. We will then focus on the generic case $n \geq 3$.

3.1. $\mathbf{n = 1.}$ The relevant bordism group is $\Omega_3^{\langle 1 \rangle}$, third oriented bordism, which is well-known to be zero. Thus the first part of Theorem 1.3 states that $\Gamma^1_{g,1}$ surjects onto the trivial group, which is certainly true, and the second part states that the abelianisation of $\Gamma^1_{g,1}$ is zero as long as $g \geq 5$. This is [Pow78, Theorem 1] (which in fact only requires $g \geq 3$).

3.2. $\mathbf{n = 2.}$ The relevant bordism group is $\Omega_5^{\langle 2 \rangle}$, fifth Spin bordism, which is zero by the results of [Mil63, p. 201]. Thus in this case Theorem 1.3 just says that the map $f : \Gamma^2_{g,1} \to H_1(\mathrm{Aut}(Q_{W^4_g}))$ is surjective for $g \geq 2$. But the homomorphism $\hat{f} : \Gamma^2_{g,1} \to \mathrm{Aut}(Q_{W^4_g})$ is already surjective in this case, by [Wal64, Theorem 2]. Though we do not require it for our results, Kreck [Kre79, Theorem 1] has shown that for $g \geq 2$ the kernel of the surjective map $\Gamma_g^2 \to \mathrm{Aut}(Q_{W_g^4})$ is precisely the subgroup of those diffeomorphisms pseudoisotopic to the identity.

4. Nontriviality of the mapping torus construction

Using the stabilisation maps we have a homomorphism

$$d : \Gamma_{0,1}^n \longrightarrow \Gamma_{g,1}^n$$

for any g, and the group $\Gamma_{0,1}^n$—the mapping class group of the sphere relative to a disc—is isomorphic to the group Θ_{2n+1} of exotic $(2n+1)$-spheres via the clutching construction.

Lemma 4.1.

(i) The image of Θ_{2n+1} in $\Gamma_{g,1}^n$ is central.

(ii) The composition $\Theta_{2n+1} \overset{d}{\to} \Gamma_{g,1}^n \overset{t}{\to} \Omega_{2n+1}^{\langle n \rangle}$ is surjective, so in particular t is surjective.

(iii) The composition $\Theta_{2n+1} \overset{d}{\to} \Gamma_{g,1}^n \overset{\hat{f}}{\to} \mathrm{Aut}(H^{\oplus g})$ is trivial.

Proof. Let f be a diffeomorphism of W_g fixing a neighbourhood U of D^{2n}, and g be a diffeomorphism supported in a disc disjoint from the marked one. Then g is isotopic to a diffeomorphism g' supported in U but still disjoint from D^{2n}, and now g' commutes with f. Thus $\mathrm{Im}(d) \subset \Gamma_{g,1}^n$ is central.

The map $t \circ d$ sends an exotic $(2n+1)$-sphere to its $BO\langle n \rangle$-bordism class (such an exotic sphere has a canonical $BO\langle n \rangle$-structure by virtue of being highly-connected), so we must show that any $(2n+1)$-dimensional manifold with $BO\langle n \rangle$-structure (W, ℓ_W) is cobordant to a n-connected manifold (as it is then $2n$-connected by Poincaré duality).

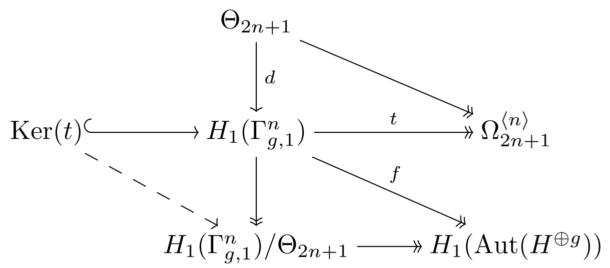
This follows from the methods of Kervaire and Milnor, specifically [KM63, Theorem 6.6]. They work with manifolds which are stably parallelisable, but this is only used in two ways: to show that homotopy classes of dimension $* \leq n$ can be represented by framed embeddings, and to show that the trace of the surgery is stably parallelisable. A $BO\langle n \rangle$-structure still allows one to represent homotopy classes of dimension $* \leq n$ by framed embeddings, and a $BO\langle n \rangle$-structure can be induced on the trace of the surgery, too.

Finally, a mapping class in the image of d is supported in a small disc, and so acts trivially on the homology of W_g, so $\hat{f} \circ d$ is trivial. □

This lemma has the following implication regarding the kernel of the mapping torus construction t.

Corollary 4.2. *The kernel of the homomorphism $t : H_1(\Gamma_{g,1}^n) \to \Omega_{2n+1}^{\langle n \rangle}$ has cardinality at least 4 if $n \neq 1$, 3 or 7 and $g \geq 5$.*

Proof. Consider the commutative diagram



where the middle row is exact. Diagram chasing shows that the dashed arrow is surjective, and so $\mathrm{Ker}(t) \to H_1(\Gamma^n_{g,1}) \to H_1(\mathrm{Aut}(H^{\oplus g}))$ is surjective. The target has cardinality 4 in these cases by Proposition 2.2. □

5. A refinement of the mapping torus construction

From now on we suppose that $n \geq 3$. The proof of the remainder of Theorem 1.3 uses two more involved theorems proved recently by the authors, which concern not the mapping class groups but the entire diffeomorphism groups of the manifolds W_g^{2n}. There are continuous homomorphisms

$$\mathrm{Diff}(W_g^{2n}, D^{2n}) \longrightarrow \mathrm{Diff}(W_{g+1}^{2n}, D^{2n})$$

given by connect-sum with W_1^{2n} inside the marked disc, and extending diffeomorphisms by the identity. In [GRW14a, Theorem 1.2] we showed that for $n \geq 3$ the maps on classifying spaces

$$B\mathrm{Diff}(W_g^{2n}, D^{2n}) \longrightarrow B\mathrm{Diff}(W_{g+1}^{2n}, D^{2n})$$

induce homology isomorphisms in degrees $2* \leq g-3$. In particular, as long as $g \geq 5$ they induce isomorphisms on first homology. The map $H_1(B\mathrm{Diff}(W_g^{2n}, D^{2n}); \mathbb{Z}) \to H_1(\Gamma^n_{g,1}; \mathbb{Z})$ is also an isomorphism, which shows that the stabilisation map

$$H_1(\Gamma^n_{g,1}; \mathbb{Z}) \longrightarrow H_1(\Gamma^n_{g+1,1}; \mathbb{Z})$$

is an isomorphism for $g \geq 5$.

Secondly, we showed how to identify the stable homology, that is, the homology of $\mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g^{2n}, D^{2n})$, as follows. Let $\theta_n : BO(2n)\langle n\rangle \to BO(2n)$ denote the n-connected cover, and $\theta_n^*\gamma_{2n}$ denote the pullback of the tautological $2n$-dimensional vector bundle. Write $\mathbf{MT}\theta_n$ for the Thom spectrum of the virtual bundle $-\theta_n^*\gamma_{2n} \to BO(2n)\langle n\rangle$. Parametrised Pontrjagin–Thom theory provides maps

$$\alpha_g : B\mathrm{Diff}(W_g^{2n}, D^{2n}) \longrightarrow \Omega_0^\infty \mathbf{MT}\theta_n$$

which assemble to a map $\alpha_\infty : \mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g^{2n}, D^{2n}) \to \Omega_0^\infty \mathbf{MT}\theta_n$ which we show in [GRW14b, Theorem 1.1] induces an isomorphism on homology as long as $n \geq 3$. Given these two theorems, we are reduced to calculating $H_1(\Omega_0^\infty \mathbf{MT}\theta_n)$.

Recall that $\mathbf{MT}\theta_n = \mathbf{Th}(-\theta_n^*\gamma_{2n} \to BO(2n)\langle n\rangle)$. Let us write $\mathbf{MO}\langle n\rangle$ for the Thom spectrum[1] of the tautological bundle over $BO\langle n\rangle$, so the stabilisation map induces a spectrum map

$$s : \mathbf{MT}\theta_n \longrightarrow \Sigma^{-2n}\mathbf{MO}\langle n\rangle.$$

Lemma 5.1. *The composition*

$$H_1(\Gamma^n_{g,1}) \xleftarrow{\sim} H_1(B\mathrm{Diff}(W_g^{2n}, D^{2n})) \xrightarrow{\alpha_g} H_1(\Omega_0^\infty \mathbf{MT}\theta_n) \cong \pi_1(\mathbf{MT}\theta_n) \xrightarrow{s_*} \Omega_{2n+1}^{\langle n\rangle}$$

agrees with the mapping torus construction t.

Proof. Both apply the Pontryagin–Thom construction to the mapping torus. □

[1]Some authors denote the $(n-1)$-connected cover of a space X by $X\langle n\rangle$, and so write $\mathbf{MO}\langle n\rangle$ for the Thom spectrum associated to the $(n-1)$-connected cover of BO. We emphasise that our notation is different.

5.1. **A long exact sequence in stable homotopy.** Let us write $\mathbf{F}_n$ for the homotopy fibre of the spectrum map s. There is a commutative diagram

$$\begin{array}{ccccc} SO/SO(2n) & \longrightarrow & BO(2n)\langle n\rangle & \longrightarrow & BO(2n) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & BO\langle n\rangle & \longrightarrow & BO, \end{array} \tag{5.1}$$

where both squares are homotopy pullback. The left square induces a map of homotopy cofibres $\Sigma(SO/SO(2n)) \to (BO\langle n\rangle)/(BO(2n)\langle n\rangle)$ which we see from the Serre spectral sequence to be $(3n+1)$-connected. The whole diagram maps to BO, and may be Thomified. The map of cofibres, desuspended $(2n+1)$ times, gives a map

$$\Sigma^{-2n}SO/SO(2n) \longrightarrow \mathbf{F}_n,$$

which is n-connected.

We may therefore rewrite the long exact sequence in stable homotopy for the map $s : \mathbf{MT}\theta_n \to \Sigma^{-2n}\mathbf{MO}\langle n\rangle$ in the following way,

$$\begin{array}{ccccccc} & & \cdots & \xrightarrow{s_*} & \pi_{2n+2}(\mathbf{MO}\langle n\rangle) & & \\ & & & \swarrow^{\partial_*} & & & \\ \pi^s_{2n+1}(SO/SO(2n)) & \longrightarrow & \pi_1(\mathbf{MT}\theta_n) & \xrightarrow{s_*} & \pi_{2n+1}(\mathbf{MO}\langle n\rangle) & & \\ & & & \swarrow & & & \\ \pi^s_{2n}(SO/SO(2n)) & \longrightarrow & \pi_0(\mathbf{MT}\theta_n) & \xrightarrow{s_*} & \pi_{2n}(\mathbf{MO}\langle n\rangle) & \longrightarrow & 0, \end{array} \tag{5.2}$$

as $SO/SO(2n)$ is $(2n-1)$-connected, so $\pi^s_{2n-1}(SO/SO(2n)) = 0$. By this connectivity property, the group $\pi^s_{2n+1}(SO/SO(2n))$ is in the range of the Freudenthal suspension theorem as long as $2n+1 \leq 2(2n-1)$ i.e. $n \geq 2$, as is $\pi^s_{2n}(SO/SO(2n))$. Thus to compute these groups we may as well compute the associated unstable homotopy groups of $SO/SO(2n)$. As $SO(2n+m) \to SO$ is $(2n+m-1)$-connencted, the homotopy groups of $SO/SO(2n)$ agree with those of the Stiefel manifold $V_{2n+m,m}$ of m-frames in $\mathbb{R}^{2n+m}$ in degrees $* \leq 2n+m-2$.

Now Paechter [Pae56] has computed the homotopy groups of Stiefel manifolds in a range of degrees, which along with the discussion above gives the following.

Lemma 5.2 (Paechter)**.** *For $n \geq 2$, $\pi^s_{2n}(SO/SO(2n)) \cong \mathbb{Z}$, and $\pi^s_{2n+1}(SO/SO(2n))$ is isomorphic to $\mathbb{Z}/4$ when n is odd and to $(\mathbb{Z}/2)^2$ when n is even.*

It will not be quite enough for us to know these as abstract groups, we shall need to know a little about their behaviour under the Hurewicz map.

Lemma 5.3. *Suppose that $n \geq 2$.*

(i) The group $H^{2n+1}(SO/SO(2n);\mathbb{F}_2)$ is 1-dimensional, generated by a class x_{2n+1} which maps under

$$\partial^* : H^{2n+1}(SO/SO(2n);\mathbb{F}_2) \longrightarrow H^{2n+2}(\mathbf{MO}\langle n\rangle;\mathbb{F}_2)$$

to $w_{2n+2}\cdot u$, where $u \in H^0(\mathbf{MO}\langle n\rangle;\mathbb{F}_2)$ is the Thom class.

(ii) The Hurewicz map $\pi^s_{2n+1}(SO/SO(2n)) \to H_{2n+1}(SO/SO(2n);\mathbb{F}_2)$ is surjective.

(iii) The pullback of the Euler class along $SO/SO(2n) \to BSO(2n)$ gives twice a generator of $H^{2n}(SO/SO(2n);\mathbb{Z}) \cong \mathbb{Z}$.

Proof. For (i), we consider the Serre spectral sequence for the fibre sequence

$$SO/SO(2n) \longrightarrow BSO(2n) \longrightarrow BSO.$$

The $\mathbb{F}_2$-cohomology of BSO is the polynomial algebra on the Stiefel–Whitney classes $w_i \in H^i(BSO;\mathbb{F}_2)$ for $i \geq 2$, and the $\mathbb{F}_2$-cohomology of $BSO(2n)$ is the polynomial algebra on the Stiefel–Whitney classes $w_i \in H^i(BSO;\mathbb{F}_2)$ for $2 \leq i \leq 2n$. Thus for each $i \geq 2n$ there must be a class

$$x_i \in H^i(SO/SO(2n);\mathbb{F}_2)$$

which transgresses to w_{i+1}, and $H^*(SO/SO(2n);\mathbb{F}_2)$ is isomorphic as a vector space to the exterior algebra on the classes x_i. (This could also be computed using the Eilenberg–Moore spectral sequence.) As we have assumed that $n \geq 2$, it follows that in degrees $2n \leq i \leq 2n+2$ the ith cohomology of $SO/SO(2n)$ is 1-dimensional and is generated by x_i.

The $(3n+1)$-connected map $\Sigma(SO/SO(2n)) \to (BO\langle n\rangle)/(BO(2n)\langle n\rangle)$ induces an isomorphism $H_{i-1}(SO/SO(2n)) \cong H_i(BO\langle n\rangle, BO(2n)\langle n\rangle)$ when $i \leq 3n$, under which the transgression in the Serre spectral sequence for the fibre sequence

$$SO/SO(2n) \longrightarrow BO(2n)\langle n\rangle \longrightarrow BO\langle n\rangle$$

may be identified with the connecting homomorphism in the long exact sequence for homology of the pair $(BO\langle n\rangle, BO(2n)\langle n\rangle)$. This long exact sequence is isomorphic, via the Thom isomorphism, with the long exact sequence for the cofibre sequence

$$\mathbf{F}_n \longrightarrow \mathbf{MT}\theta_n \longrightarrow \Sigma^{-2n}\mathbf{MO}\langle n\rangle.$$

Hence x_{2n+1} maps to $w_{2n+2} \cdot u$ under the connecting homomorphism.

For (ii), recall that the Hurewicz map for a $(k-1)$-connected space with $k \geq 2$ is an isomorphism in degree k and a surjection in degree $(k+1)$. Hence

$$\mathbb{Z} \cong \pi^s_{2n}(SO/SO(2n)) \longrightarrow H_{2n}(SO/SO(2n);\mathbb{Z}) \tag{5.3}$$

is an isomorphism and

$$\pi^s_{2n+1}(SO/SO(2n)) \longrightarrow H_{2n+1}(SO/SO(2n);\mathbb{Z}) \tag{5.4}$$

is a surjection. By (5.3) multiplication by 2 on $H_{2n}(SO/SO(2n);\mathbb{Z})$ is an injection: it then follows from the Bockstein exact sequence that

$$H_{2n+1}(SO/SO(2n);\mathbb{Z}) \longrightarrow H_{2n+1}(SO/SO(2n);\mathbb{F}_2)$$

is a surjection, which combined with (5.4) gives the result.

For (iii), observe that the map

$$S^{2n} = SO(2n+1)/SO(2n) \longrightarrow SO/SO(2n)$$

is $2n$-connected, so induces an injection on $H^{2n}(-;\mathbb{Z})$ (and both spaces have $2n$th cohomology $\mathbb{Z}$). Now $S^{2n} \to SO/SO(2n) \to BSO(2n)$ classifies the tangent bundle of S^{2n}, which has Euler number 2, so the pullback of the Euler class to $SO/SO(2n)$ is not divisible by more than 2; on the other hand, the Euler class reduces to w_{2n} modulo 2, which vanishes on $SO/SO(2n)$. Hence it is divisible by precisely 2. □

We now analyse the long exact sequence (5.2) in low degrees.

Lemma 5.4. *The map $\pi^s_{2n}(SO/SO(2n)) \to \pi_0(\mathbf{MT}\theta_n)$ is injective.*

Proof. Under the Thom isomorphism, the Euler class gives a map $E : \mathbf{MT}\theta_n \to \mathbf{H}\mathbb{Z}$, and the composition

$$\Sigma^{-2n}(SO/SO(2n)) \longrightarrow \mathbf{MT}\theta_n \xrightarrow{E} \mathbf{H}\mathbb{Z}$$

is twice a generator of $H^{2n}(SO/SO(2n);\mathbb{Z}) \cong \mathbb{Z}$ by Lemma 5.3 (iii). The claim follows by taking π_0 of this composition. □

The long exact sequence (5.2) thus simplifies to

$$\cdots \longrightarrow \Omega^{\langle n\rangle}_{2n+2} \xrightarrow{\partial_*} \pi^s_{2n+1}(SO/SO(2n)) \longrightarrow \pi_1(\mathbf{MT}\theta_n) \xrightarrow{s_*} \Omega^{\langle n\rangle}_{2n+1} \longrightarrow 0,$$

and we recall that under the isomorphism

$$H_1(\Gamma^n_{g,1}) \xleftarrow{\sim} H_1(B\mathrm{Diff}(W^{2n}_g, D^{2n})) \xrightarrow{\sim} H_1(\Omega^\infty_0 \mathbf{MT}\theta_n) \cong \pi_1(\mathbf{MT}\theta_n)$$

the map s_* coincides with the map t.

Lemma 5.5. *If $n \neq 3$ or 7 then the map $\partial_* : \Omega^{\langle n\rangle}_{2n+2} \to \pi^s_{2n+1}(SO/SO(2n))$ is zero.*

Proof. By Corollary 4.2 the kernel of the map t, and hence s_*, has cardinality at least 4, and so the kernel of $s_* : \pi_1(\mathbf{MT}\theta_n) \to \Omega^{\langle n\rangle}_{2n+1}$ also has cardinality at least 4. On the other hand, the exact sequence and Lemma 5.2 shows that it has cardinality at most 4, hence it has cardinality precisely 4, so ∂ is zero. □

Lemma 5.6. *If $n = 3$ or 7 then $\partial_* : \Omega^{\langle n\rangle}_{2n+2} \to \pi^s_{2n+1}(SO/SO(2n))$ is surjective (so s_* is injective).*

Proof. Consider the diagram

$$\begin{array}{ccc}
\pi_{2n+2}(\mathbf{MO}\langle n\rangle) & \xrightarrow{\partial_*} & \pi^s_{2n+1}(SO/SO(2n)) \cong \mathbb{Z}/4 \\
\downarrow h & & \downarrow h \\
H_{2n+2}(\mathbf{MO}\langle n\rangle; \mathbb{F}_2) & \longrightarrow & H_{2n+1}(SO/SO(2n); \mathbb{F}_2) \cong \mathbb{Z}/2,
\end{array}$$

where h denotes the Hurewicz map, and the surjectivity on the right is by Lemma 5.3 (ii). By Lemma 5.3 (i), the isomorphism $H_{2n+1}(SO/SO(2n); \mathbb{F}_2) \cong \mathbb{Z}/2$ is given by evaluating against the class $x_{2n+1} \in H^{2n+1}(SO/SO(2n); \mathbb{F}_2)$, which under ∂^* corresponds to $w_{2n+2} \cdot u$. Thus the composition $\partial_* \circ h$ can be identified with the functional $\Omega^{\langle n\rangle}_{2n+2} \to \mathbb{Z}/2$ given by $[W^{2n+2}] \mapsto \langle [W], w_{2n+2}(TW)\rangle$.

The manifolds $[\mathbb{H}P^2] \in \Omega^{\langle 3\rangle}_8$ and $[\mathbb{O}P^2] \in \Omega^{\langle 7\rangle}_{16}$ have Euler characteristic 3, so non-trivial top Stiefel–Whitney class. Thus $\partial_* \circ h = h \circ \partial_*$ is surjective in these cases, but it follows that ∂_* must then be surjective. □

5.2. **Proof of Theorem 1.3.** For the surjectivity part of the statement, we have already explained how Kreck's result ([Kre79]) implies the surjectivity of the homomorphism $f : \Gamma^n_{g,1} \to H_1(\mathrm{Aut}(Q_{W^{2n}_g}))$. To see surjectivity of $t \oplus f$ it suffices to see that the restriction $t|_{\mathrm{Ker}(f)} : \mathrm{Ker}(f) \to \Omega^{\langle n\rangle}_{2n+1}$ is surjective, but that follows from Lemma 4.1.

It remains to see that the induced map $t \oplus f : H_1(\Gamma^n_{g,1}) \to \Omega^{\langle n\rangle}_{2n+1} \oplus H_1(\mathrm{Aut}(Q_{W^{2n}_g}))$ is injective for $n \neq 2$ and $g \geq 5$. For $n = 3$ or 7, the second summand vanishes, so it suffices to prove that t is injective, which we did in Lemma 5.6. In the remaining cases, Lemma 5.5 gives a short exact sequence fitting into the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi^s_{2n+1}(SO/SO(2n)) & \hookrightarrow & \pi_1(\mathbf{MT}\theta_n) & \xrightarrow{t} & \Omega^{\langle n\rangle}_{2n+1} & \longrightarrow 0 \\
& \downarrow & & \| & & & \\
& H_1(\mathrm{Aut}(Q_{W_g})) & \xleftarrow{f} & H_1(\Gamma^n_{g,1}). & & &
\end{array}$$

By the proof of Corollary 4.2, the map $\pi^s_{2n+1}(SO/SO(2n)) \to H_1(\mathrm{Aut}(Q_{W_g}))$ is an isomorphism, giving a splitting of the exact sequence in the top row of the diagram. This proves Theorem 1.3 in these cases.

6. A filtration of the sphere spectrum

In this section we shall describe and study a filtration of the sphere spectrum, and a resulting filtration of the stable homotopy groups of spheres. This plays a role in computing the cobordism groups $\Omega_{2n+1}^{\langle n\rangle}$ in terms of the stable homotopy groups of spheres and the J-homomorphism.

Recall that $BO\langle n\rangle \to BO$ denotes the n-connected cover, and there is an associated Thom spectrum $\mathbf{MO}\langle n\rangle$. Thus $\mathbf{MO}\langle 0\rangle = \mathbf{MO}$, the spectrum representing unoriented cobordism theory, $\mathbf{MO}\langle 1\rangle = \mathbf{MSO}$, $\mathbf{MO}\langle 2\rangle = \mathbf{MO}\langle 3\rangle = \mathbf{MSpin}$, etc. There are maps

$$\mathbf{MO} = \mathbf{MO}\langle 0\rangle \longleftarrow \mathbf{MO}\langle 1\rangle \longleftarrow \mathbf{MO}\langle 2\rangle \longleftarrow \mathbf{MO}\langle 3\rangle \longleftarrow \cdots$$

with inverse limit $\mathbf{S}$, the sphere spectrum. We write $\iota_n : \mathbf{S} \to \mathbf{MO}\langle n\rangle$, and define a filtration of the stable homotopy groups of spheres by

$$F^n\pi_k(\mathbf{S}) = \mathrm{Ker}\left(\pi_k(\iota_n) : \pi_k(\mathbf{S}) \to \pi_k(\mathbf{MO}\langle n\rangle)\right).$$

Let us write $\overline{\mathbf{MO}\langle n\rangle}$ for the homotopy cofibre of $\mathbf{S} \to \mathbf{MO}\langle n\rangle$.

Lemma 6.1.

(i) $F^n\pi_k(\mathbf{S}) = 0$ *for* $k < n$.
(ii) $F^n\pi_k(\mathbf{S})$ *contains the image of* $J : \pi_k(O) \to \pi_k(\mathbf{S})$ *for* $k \geq n$.
(iii) $F^n\pi_k(\mathbf{S})$ *is equal to the image of* $J : \pi_k(O) \to \pi_k(\mathbf{S})$ *for* $2n \geq k \geq n$.

Proof. The spectrum $\overline{\mathbf{MO}\langle n\rangle}$ is n-connected, and so $\pi_k(\mathbf{S}) \to \pi_k(\mathbf{MO}\langle n\rangle)$ is injective for $k < n$; this establishes (i).

In the cobordism-theoretic interpretation of the homotopy groups of spheres, $J(\alpha : S^k \to O)$ is given by the manifold S^k with the framing given by twisting the standard (bounding) framing of S^k using α to obtain a new framing ξ_α.

$$\begin{array}{ccccc} S^k & \xrightarrow{\xi_\alpha} & EO & \longrightarrow & BO\langle n\rangle \\ \downarrow & & & & \downarrow \\ D^{k+1} & & \longrightarrow & & BO \end{array}$$

While this framing cannot necessarily be extended to D^{k+1}, the associated $BO\langle n\rangle$-structure can be extended as long as $k \geq n$, as the right-hand map is n-co-connected; this establishes (ii).

Let (M^k, ξ) be a framed cobordism class representing an element of $F^n\pi_k(\mathbf{S})$, so considered as a $BO\langle n\rangle$-manifold M bounds a $BO\langle n\rangle$-manifold W. Now $k+1 \leq 2n+1$ so (similarly to the proof of Lemma 4.1) by the techniques of [KM63, Theorems 5.5 and 6.6] we may perform surgery on the interior of W to obtain a new $BO\langle n\rangle$-manifold W' which is $\lfloor k/2\rfloor$-connected, with the same framed boundary M. We may then find a handle structure on W' having no handles of index between 1 and $\lfloor k/2\rfloor$, and so $W' \setminus D^{k+1}$ is a $BO\langle n\rangle$-cobordism from M to S^k which may be obtained from M by attaching handles of index at most $k - \lfloor k/2\rfloor \leq n$. As $EO \to BO\langle n\rangle$ is n-connected, it follows that the framing ξ on M may be extended to $W' \setminus D^{k+1}$, and so (M, ξ) is framed cobordant to (S^k, ζ) for some framing ζ of the sphere. But those cobordism classes represented by spheres with some framing are precisely the image of the J-homomorphism; this establishes (iii). □

We wish to understand the group $\Omega_{2n+1}^{\langle n\rangle} = \pi_{2n+1}(\mathbf{MO}\langle n\rangle)$, which is related to $F^n\pi_{2n+1}(\mathbf{S})$ and lies just outside of the range treated in Lemma 6.1 (iii). However, the groups $F^n\pi_k(\mathbf{S})$ for $k > 2n$ have been studied, though not quite expressed in this form, by Stolz [Sto85]. Let us explain his technique.

For $n \geq 2$ the universal (virtual) bundle $\gamma\langle n\rangle$ over $BO\langle n\rangle$ is Spin, and so the Thom spectrum $\mathbf{MO}\langle n\rangle$ has a KO-theory Thom class, λ_n. There is thus a KO-theory class $\gamma\langle n\rangle \cdot \lambda_n \in KO^0(\mathbf{MO}\langle n\rangle)$, which we represent by a map $\alpha_n : \mathbf{MO}\langle n\rangle \to \mathbf{ko}$ to the connective KO-theory spectrum. As the bundle $\gamma\langle n\rangle \in KO^0(BO\langle n\rangle)$ becomes trivial when restricted to a point, the class lifts to $\gamma\langle n\rangle \in KO^0(BO\langle n\rangle, *)$, and α_n factors through a map $\alpha'_n : \overline{\mathbf{MO}\langle n\rangle} \to \mathbf{ko}$, and as $\overline{\mathbf{MO}\langle n\rangle}$ is n-connected this lifts further to a map

$$\overline{\alpha}_n : \overline{\mathbf{MO}\langle n\rangle} \longrightarrow \mathbf{ko}\langle n\rangle.$$

Stolz defines $A[n+1]$ to be the homotopy fibre of $\overline{\alpha}_n$. Under the Thom isomorphism we have

$$H^*(\overline{\mathbf{MO}\langle n\rangle}) \cong H^*(BO\langle n\rangle, *) = H^*(\Omega^\infty(\mathbf{ko}\langle n\rangle), *),$$

and using the known cohomology of $BO\langle n\rangle$ and $\mathbf{ko}\langle n\rangle$ as modules over the Steenrod algebra Stolz establishes the following.

Theorem 6.2 (Stolz [Sto85])**.** *The spectrum $A[n+1]$ is $(2n+1)$-connected, and*

$$\pi_{2n+2}(A[n+1]) = \begin{cases} \mathbb{Z} & n+1 \equiv 0, 4 \bmod 8 \\ \mathbb{Z}/2 & n+1 \equiv 1, 2 \bmod 8 \\ 0 & \textit{otherwise.} \end{cases}$$

Let us write $J : \pi_k(O) \to \pi_k(\mathbf{S})$ for the J-homomorphism. For $\alpha \in \pi_k(O)$, $J(\alpha)$ is given by the stably framed manifold obtained by changing the bounding framing on S^k using α. As explained in the proof of Lemma 6.1 (ii), the associated $BO\langle n\rangle$-structure extends canonically over D^{k+1} as long as $k \geq n$, which gives a map

$$\overline{J} : \pi_k(O) \longrightarrow \pi_{k+1}(\overline{\mathbf{MO}\langle n\rangle})$$

such that $\partial \circ \overline{J} = J$. The composition

$$\pi_k(O) \xrightarrow{\overline{J}} \pi_{k+1}(\overline{\mathbf{MO}\langle n\rangle}) \xrightarrow{\overline{\alpha}_n} \pi_{k+1}(\mathbf{ko}\langle n\rangle)$$

is an isomorphism (cf. [Sto85, Lemma 3.7]), and it follows from the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \pi_{2n+2}(\mathbf{MO}\langle n\rangle) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \pi_{2n+2}(A[n+1]) & \longrightarrow & \pi_{2n+2}(\overline{\mathbf{MO}\langle n\rangle}) & \xrightarrow{\overline{\alpha}_n} & \pi_{2n+2}(\mathbf{ko}\langle n\rangle) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\partial} & \nwarrow{\scriptstyle\overline{J}} & \| & & \\
 & & & & \pi_{2n+1}(\mathbf{S}) & \xleftarrow{J} & \pi_{2n+1}(O) & & \\
 & & & & \downarrow & & & & \\
 & & & & \pi_{2n+1}(\mathbf{MO}\langle n\rangle) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

that there is an exact sequence

$$\pi_{2n+2}(A[n+1]) \xrightarrow{\sigma} \mathrm{Coker}(J)_{2n+1} \longrightarrow \pi_{2n+1}(\mathbf{MO}\langle n\rangle) \longrightarrow 0.$$

Hence, given the description of $\pi_{2n+2}(A[n+1])$ in Theorem 6.2, it follows that the quotient $F^n\pi_{2n+1}(\mathbf{S})/\mathrm{Im}(J)$ is cyclic. Stolz finds various conditions under which the quotient $F^n\pi_{2n+1}(\mathbf{S})/\mathrm{Im}(J)$ is in fact trivial, i.e. the map σ is zero.

Theorem 6.3 (Stolz [Sto85]). *If either*

(i) $n+1 \equiv 2 \bmod 8$ *and* $n+1 \geq 18$,
(ii) $n+1 \equiv 1 \bmod 8$ *and* $n+1 \geq 113$,
(iii) $n+1 \not\equiv 0, 1, 2, 4 \bmod 8$,

then $F^n\pi_{2n+1}(\mathbf{S}) = \mathrm{Im}(J)_{2n+1}$.

If $n+1 = 4\ell$ *then* $F^n\pi_{2n+1}(\mathbf{S})/\mathrm{Im}(J)$ *is generated by the exotic sphere* Σ *which is the boundary of the manifold obtained by plumbing together two copies of the linear* 4ℓ*-dimensional disc bundle over* $S^{4\ell}$ *having trivial Euler class and representing a generator of* $\pi_{4\ell}(BO) \cong \mathbb{Z}$.

Proof. By [Sto85, Theorem B (i) and (ii)], in the case $\pi_{2n+2}(A[n+1]) = \mathbb{Z}/2$, these map to zero in $\mathrm{Coker}(J)_{2n+1}$ under the conditions given in the statement of the proposition.

It follows from [Sto85, Lemma 10.3] that when $n+1 = 4\ell$ a generator of $\mathbb{Z} = \pi_{2n+2}(A[n+1])$ in $\pi_{2n+2}(\overline{\mathbf{MO}\langle n\rangle})$ is given by the class of the plumbing described in the statement of the proposition. □

In the case $n+1 = 4\ell$, it seems to be a difficult problem to obtain any information about the order, or indeed the nontriviality, of $[\Sigma] \in \mathrm{Coker}(J)_{8\ell-1}$. All calculations we have attempted are consistent with the following conjecture.

Conjecture A. $[\Sigma] = 0 \in \mathrm{Coker}(J)_{8\ell-1}$.

This conjecture would imply that the map σ is zero in these cases too, and so $\Omega_{8\ell-1}^{\langle 4\ell-1\rangle} \cong \mathrm{Coker}(J)_{8\ell-1}$. The most promising approach to this conjecture seems to be as follows. By the discussion above, the map

$$\mathrm{Coker}(J)_{2n+1} \longrightarrow \pi_{2n+1}(\mathbf{MO}\langle n+1\rangle)$$

is an isomorphism, so Conjecture A is equivalent to

Conjecture B. The map $\pi_{8\ell-1}(\mathbf{MO}\langle 4\ell\rangle) \to \pi_{8\ell-1}(\mathbf{MO}\langle 4\ell-1\rangle)$ is injective.

For example, when $\ell = 1$ this asks if $\Omega_7^{\mathrm{String}} \to \Omega_7^{\mathrm{Spin}}$ is injective, which it is as both groups are zero. When $\ell = 2$ this asks if $\mathrm{Coker}(J)_{15} \to \Omega_{15}^{\mathrm{String}}$ is injective, which it is as $\pi_{15}(\mathbf{MO}\langle 8\rangle) = \mathrm{Coker}(J)_{15} = \mathbb{Z}/2$, $\Omega_{15}^{\mathrm{String}} = \mathbb{Z}/2$, and generators of either group may be represented by an exotic sphere [KM63, Gia71].

Corollary 6.4. *If* n *satisfies one of the conditions of Theorem 6.3 then the cobordism group* $\Omega_{2n+1}^{\langle n\rangle}$ *occurring in Theorem 1.3 is isomorphic to* $\mathrm{Coker}(J)_{2n+1}$.

7. Relation to the work of Kreck

Kreck has given [Kre79] a description of the mapping class groups of $(n-1)$-connected $2n$-manifolds, up to two extension problems. Applied to our situation, he gives extensions [Kre79, Proposition 3]

$$1 \longrightarrow \mathcal{I}_{g,1}^n \longrightarrow \Gamma_{g,1}^n \xrightarrow{\hat{f}} \mathrm{Aut}(Q_{W_g^{2n}}) \longrightarrow 1$$

and

$$1 \longrightarrow \Theta_{2n+1} \longrightarrow \mathcal{I}_{g,1}^n \xrightarrow{\chi} \mathrm{Hom}(H_n(W_g), S\pi_n(SO(n))) \longrightarrow 1$$

where $S\pi_n(SO(n)) = \mathrm{Im}(\pi_n(SO(n)) \to \pi_n(SO(n+1)))$. These groups are given, for $n \geq 3$, by Table 2 (except that $S\pi_6(SO(6)) = 0$).

Table 2. The groups $S\pi_n(SO(n))$, except that $S\pi_6(SO(6)) = 0$.

$n \bmod 8$	0	1	2	3	4	5	6	7
$S\pi_n(SO(n))$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$

The map χ may be described as follows: to a diffeomorphism $\varphi : W_{g,1}^{2n} \to W_{g,1}^{2n}$ which acts as the identity on homology, and a class $x \in H_n(W_g^{2n};\mathbb{Z}) \cong \pi_n(W_{g,1}^{2n})$ represented by an embedding $x : S^n \hookrightarrow W_{g,1}^{2n}$, the sphere $\varphi \circ x$ is isotopic to x and so by the isotopy extension theorem we may suppose that $\varphi \circ x = x$. Then $\varepsilon^1 \oplus \nu_x \cong \varepsilon^{n+1}$ and the differential $D\varphi|_{x(S^n)}$ gives an automorphism of this bundle, corresponding to a map $\chi(\varphi)(x) : S^n \to SO(n+1)$. It can be checked that this map lies in $S\pi_n(SO(n))$.

It is generally difficult to understand the structure of these extensions (for example, whether they are non-trivial). To our knowledge the only case in which complete information is known is Γ_1^3, due to Krylov [Kry03]. Crowley [Cro11] has also been able to solve the extension problem for

$$1 \longrightarrow \mathrm{Hom}(H_n(W_g), S\pi_n(SO(n))) \longrightarrow \Gamma_{g,1}^n/\Theta_{2n+1} \longrightarrow \mathrm{Aut}(Q_{W_g^{2n}}) \longrightarrow 1$$

for $n = 3$ and 7. An immediate consequence of our Theorem 1.3 is as follows.

Corollary 7.1. *For $g \geq 5$ the kernel of the composition $\Theta_{2n+1} \to \Gamma_{g,1}^n \to \Omega_{2n+1}^{\langle n \rangle}$ is generated by commutators of elements of $\Gamma_{g,1}^n$. In particular, this is true for the subgroup $bP_{2n+2} < \Theta_{2n+1} < \Gamma_{g,1}^n$.* □

Our results can also be used to shed light on some of these extension problems, especially for those n such that $S\pi_n(SO(n)) = 0$.

Theorem 7.2. *The map*

$$t \times \hat{f} : \Gamma_{g,1}^6 \longrightarrow \Omega_{13}^{\langle 6 \rangle} \times \mathrm{O}_{g,g}(\mathbb{Z})$$

is an isomorphism, and $\Omega_{13}^{\langle 6 \rangle} = \Omega_{13}^{\mathrm{String}} \cong \mathbb{Z}/3$.

Proof. We have $S\pi_6(SO(6)) = 0$, and so the two extensions reduce to

$$1 \longrightarrow \Theta_{13} \longrightarrow \Gamma_{g,1}^6 \xrightarrow{\hat{f}} \mathrm{O}_{g,g}(\mathbb{Z}) \longrightarrow 1.$$

The group bP_{14} is trivial ([KM63]) and so $\Theta_{13} \cong \mathrm{Cok}(J)_{13}$, which is $\mathbb{Z}/3$, and is isomorphic to $\Omega_{13}^{\langle 6 \rangle}$ by Theorem 1.4. Thus $\Theta_{13} \to \Gamma_{g,1}^6 \xrightarrow{t} \Omega_{13}^{\langle 6 \rangle}$ is an isomorphism, which shows that the extension is trivial. □

When $n \equiv 5 \bmod 8$, the other case in which $S\pi_n(SO(n)) = 0$, we also solve the extension problem left open by Kreck.

Theorem 7.3. *If $n \equiv 5 \bmod 8$ then there is a central extension*

$$1 \longrightarrow bP_{2n+2} \longrightarrow \Gamma_{g,1}^n \xrightarrow{t \times \hat{f}} \Omega_{2n+1}^{\langle n \rangle} \times \mathrm{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow 1,$$

where we write $\mathrm{Sp}_{2g}^q(\mathbb{Z}) \leq \mathrm{Sp}_{2g}(\mathbb{Z})$ for the subgroup of those automorphisms of the symplectic space $\mathbb{Z}^{2g}$ which preserve the standard quadratic function. We have $\Omega_{2n+1}^{\langle n \rangle} \cong \mathrm{Cok}(J)_{2n+1}$, and if $g \geq 5$ then the subgroup $bP_{2n+2} \leq \Gamma_{g,1}^n$ is generated by commutators.

Proof. We have $S\pi_n(SO(n)) = 0$ and so Kreck's exact sequences reduce to

$$1 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \xrightarrow{\hat{f}} \mathrm{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow 1.$$

Furthermore, in this dimension the Kervaire–Milnor [KM63] exact sequence is

$$1 \longrightarrow bP_{2n+2} \longrightarrow \Theta_{2n+1} \longrightarrow \mathrm{Coker}(J)_{2n+1} \longrightarrow 1 \qquad (7.1)$$

and $n + 1 \equiv 6 \bmod 8$ so by Theorem 1.4 the map $\mathrm{Cok}(J)_{2n+1} \to \Omega_{2n+1}^{\langle n \rangle}$ is an isomorphism. Thus the kernel of $t \times \hat{f}$ is precisely the subgroup $bP_{2n+2} \leq \Theta_{2n+1} \leq \Gamma_{g,1}^n$. Furthermore, we know $t \times \hat{f}$ induces an isomorphism on abelianisations for $g \geq 5$, so bP_{2n+2} consists of commutators. □

Finally, we determine the extension in Theorem 7.3. It can be pulled back to a central extension

$$1 \longrightarrow bP_{2n+2} \longrightarrow E(g,n) \longrightarrow \mathrm{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow 1, \tag{7.2}$$

with $E(g,n) = \mathrm{Ker}(t : \Gamma^n_{g,1} \to \Omega^{\langle n\rangle}_{2n+1})$. Brumfiel [Bru68, Bru69, Bru70] has constructed a splitting of (7.1) (at least for for $n \neq 2^k - 2$, which is satisfied as we are supposing that $n \equiv 5 \bmod 8$). Any splitting $s : \mathrm{Cok}(J)_{2n+1} \to \Theta_{2n+1}$ gives rise to a composition

$$\mathrm{Cok}(J)_{2n+1} \xrightarrow{s} \Theta_{2n+1} \longrightarrow \Gamma^n_{g,1} \xrightarrow{t} \Omega^{\langle n\rangle}_{2n+1}$$

which is an isomorphism. As Θ_{2n+1} lies in the centre of $\Gamma^{2n}_{g,1}$, we obtain a splitting

$$\Gamma^n_{g,1} \cong E(g,n) \times \mathrm{Cok}(J)_{2n+1}, \tag{7.3}$$

giving the following improvement to Corollary 7.1: For $g \geq 5$ the subgroup $bP_{2n+2} < \Theta_{2n+1} < \Gamma^n_{g,1}$ is generated by commutators of elements from the subgroup $E(g,n) < \Gamma^n_{g,1}$. Hence bP_{2n+2} vanishes in the abelianisation of $E(g,n)$, and in fact we may deduce that the group homomorphism $E(g,n) \to \mathrm{Sp}^q_{2g}(\mathbb{Z})$ induces an isomorphism of abelianisations. We shall use this fact to determine the class of the extension (7.2) in Theorem 7.7 below.

Lemma 7.4. *The homomorphism $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Sp}^q_{2g}(\mathbb{Z})$ which sends the generator to the matrix*

$$X_g = \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

admits a lift to $E(g,n)$.

For $g \geq 5$ the resulting homomorphisms $\mathbb{Z}/4\mathbb{Z} \to E(g,n) \to \mathrm{Sp}^q_{2g}(\mathbb{Z})$ both induce isomorphisms in $H_1(-;\mathbb{Z})$ and hence in the torsion subgroups of $H^2(-;\mathbb{Z})$.

Proof. Using the standard embedding $W^{2n}_{1,0} = S^n \times S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ it is easy to lift the matrix X_1 to a diffeomorphism of $W^{2n}_{1,0}$, namely the restriction of the linear map $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \mapsto (-y_1, y_2, \dots, y_{n+1}, x_1, \dots, x_{n+1})$. We obtain an order-four element $z' \in \Gamma^n_{1,0}$, which by Lemma 1.5 lifts to an order-four element $z'' \in \Gamma^n_{1,1}$ with $\hat{f}(z'') = X_1 \in \mathrm{Sp}^q_2(\mathbb{Z})$. This may be stabilised to an order-four element $z''_g \in \Gamma^n_{g,1}$ with $\hat{f}(z''_g) = X_g \in \mathrm{Sp}^q_{2g}(\mathbb{Z})$. The element z''_g may not lie in the subgroup $E(g,n) = \mathrm{Ker}(t)$, but we may use the splitting (7.3) to project it to an element $z_g \in E(g,n)$ with $z_g^4 = 1$. Since $\hat{f}(z_g) = X_g \in \mathrm{Sp}^q_{2g}(\mathbb{Z})$ this gives the required lift.

For the claim about $H_1(-;\mathbb{Z})$, we have already seen that $E(g,n) \to \mathrm{Sp}^q_{2g}(\mathbb{Z})$ induces an isomorphism of abelianisations for $g \geq 5$. For $\mathbb{Z}/4\mathbb{Z}$, it follows from the formula in [JM90, p. 147] that the composition $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Sp}^q_{2g}(\mathbb{Z}) \to H_1(\mathrm{Sp}^q_{2g}(\mathbb{Z});\mathbb{Z})$ is an isomorphism (this only requires $g \geq 3$). ☐

By the second part of Lemma 7.4, the maps $\mathbb{Z}/4 \to E(g,n) \to \mathrm{Sp}^q_{2g}(\mathbb{Z})$ induce isomorphisms on torsion subgroups of $H^2(-;\mathbb{Z})$, and hence give compatible splittings

$$\begin{aligned} H^2(\mathrm{Sp}^q_{2g}(\mathbb{Z});\mathbb{Z}) &\xrightarrow{\sim} H^2(\mathbb{Z}/4;\mathbb{Z}) \oplus \mathrm{Hom}(H_2(\mathrm{Sp}^q_{2g}(\mathbb{Z});\mathbb{Z}),\mathbb{Z}) \\ H^2(E(g,n);\mathbb{Z}) &\xrightarrow{\sim} H^2(\mathbb{Z}/4;\mathbb{Z}) \oplus \mathrm{Hom}(H_2(E(g,n);\mathbb{Z}),\mathbb{Z}) \end{aligned} \tag{7.4}$$

of the universal coefficient sequences.

The space $B\mathrm{Sp}^q_{2g}(\mathbb{Z})$ carries a local coefficient system $\mathcal{L} = E\mathrm{Sp}^q_{2g}(\mathbb{Z}) \times_{\mathrm{Sp}^q_{2g}(\mathbb{Z})} \mathbb{Z}^{2g}$, given by the tautological action of $\mathrm{Sp}^q_{2g}(\mathbb{Z})$ on $\mathbb{Z}^{2g}$. As this group preserves the standard symplectic form, there is a map $\omega : \mathcal{L} \otimes \mathcal{L} \to \mathbb{Z}$ restricting to a non-singular skew-symmetric form on each fibre.

Lemma 7.5. *For $g \geq 5$ there is a unique class $\mu \in H^2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z})$ with the following properties.*

(i) For any closed oriented surface S and map $f : S \to B\mathrm{Sp}_{2g}^q(\mathbb{Z})$, the signature of the symmetric form

$$\langle -,- \rangle : H^1(S; f^*\mathcal{L}) \otimes H^1(S; f^*\mathcal{L}) \xrightarrow{\cup} H^2(S; f^*\mathcal{L} \otimes f^*\mathcal{L}) \xrightarrow{\omega} H^2(S;\mathbb{Z}) \xrightarrow{[S]} \mathbb{Z}$$

agrees with $8(f^*\mu)[S]$.

(ii) μ is in the kernel of the map $H^2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) \to H^2(\mathbb{Z}/4\mathbb{Z};\mathbb{Z})$, where $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Sp}_{2g}^q(\mathbb{Z})$ is the homomorphism from Lemma 7.4.

Proof. We first claim that the indicated symmetric form $\langle -,- \rangle$ is non-degenerate (modulo torsion) and even, and hence (cf. [MH73, Ch. II Theorem 5.1]) has signature divisible by 8. It is non-degenerate modulo torsion because, under the identification $(f^*\mathcal{L})^\vee \cong f^*\mathcal{L}$ given by ω, the adjoint to $\langle -,- \rangle$ is given by

$$H^1(S; (f^*\mathcal{L})^\vee) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_1(S; f^*\mathcal{L}), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H^1(S; f^*\mathcal{L}), \mathbb{Z}).$$

Here, the first map comes from the Universal Coefficient Theorem with local coefficients (cf. [Spa66, p. 283]) and is an isomorphism modulo torsion, and the second map is given by precomposing with the Poincaré duality isomorphism $- \cap [S] : H^1(S; f^*\mathcal{L}) \to H_1(S; f^*\mathcal{L})$.

To prove that it is even, we may reduce the form modulo 2 and show that $\langle x, x\rangle \equiv 0 \bmod 2$ for any $x \in H^1(S; f^*\mathcal{L})$. In order to compute this, we choose a triangulation $S \approx |K|$ and let $\varphi \in C^1(K; f^*\mathcal{L})$ be a simplicial cocycle, which assigns to each 1-simplex $[v_0, v_1] \in K$ a section $\varphi_{[v_0,v_1]}$ of the coefficient system $f^*\mathcal{L}$ over $[v_0, v_1]$. Then by the Alexander–Whitney formula we have

$$\omega(\varphi \cup \varphi)([v_0, v_1, v_2]) = \omega_{v_1}(\varphi_{[v_0,v_1]}, \varphi_{[v_1,v_2]}),$$

where $\omega_{v_1}(-,-)$ is the bilinear form on $f^*\mathcal{L}$ over the point v_1, and so

$$\langle \varphi, \varphi \rangle \equiv \sum_{[v_0,v_1,v_2] \in K} \omega_{v_1}(\varphi_{[v_0,v_1]}, \varphi_{[v_1,v_2]}) \mod 2,$$

where the sum is taken over all 2-simplices of K (note that the choice of ordering of the vertices of the 2-simplex does not affect this formula, as we are working modulo 2 and $\omega_{v_1}(a, a) = 0$ by skew-symmetry). As φ is a cocycle we have $\varphi_{[v_0,v_1]} + \varphi_{[v_1,v_2]} = \varphi_{[v_0,v_2]}$, and so using the quadratic refinement $q_{v_1}(-)$ associated to the bilinear form $\omega_{v_1}(-,-)$ reduced modulo 2 we obtain

$$q_{v_1}(\varphi_{[v_0,v_2]}) = q_{v_1}(\varphi_{[v_0,v_1]}) + q_{v_1}(\varphi_{[v_1,v_2]}) + \omega_{v_1}(\varphi_{[v_0,v_1]}, \varphi_{[v_1,v_2]}) \mod 2.$$

Hence

$$\langle \varphi, \varphi \rangle \equiv \sum_{[v_0,v_1,v_2] \in K} q_{v_1}(\varphi_{[v_0,v_2]}) + q_{v_1}(\varphi_{[v_0,v_1]}) + q_{v_1}(\varphi_{[v_1,v_2]}) \mod 2,$$

but as each 1-simplex is the face of precisely two 2-simplices, this sum is zero. Hence the form $\langle -,- \rangle$ is even as claimed, so has signature divisible by 8.

Now note that the signature of this form only depends on the oriented cobordism class of $f : S \to B\mathrm{Sp}_{2g}^q(\mathbb{Z})$, or in other words on the homology class $f_*([S])$. Hence there is a homomorphism

$$s = \mathrm{sign}/8 : H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

By the universal coefficient theorem, this proves the existence of a μ satisfying (i) and determines it uniquely up to adding any torsion element. The splitting (7.4) of $H^2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z})$ and (ii) uniquely determines the torsion summand. □

Definition 7.6. Let $\mathbb{Z} \to E_g \to \mathrm{Sp}_{2g}^q(\mathbb{Z})$ be the central extension classified by the class μ. For $d \in \mathbb{Z}$, let $\mathbb{Z}/d\mathbb{Z} \to E_{g,d} \to \mathrm{Sp}_{2g}^q(\mathbb{Z})$ be the extension with $E_{g,d} = E_g/d\mathbb{Z}$, i.e. the extension classified by the image of μ in cohomology modulo d.

Theorem 7.7. *Let $n \equiv 5 \bmod 8$ and $g \geq 5$. The homomorphism $E(g,n) \to \mathrm{Sp}_{2g}^q(\mathbb{Z})$ obtained by restricting $\hat{f}$ lifts to an isomorphism*

$$\begin{array}{ccc} E(g,n) & \xrightarrow{\cong} & E_{g,|bP_{2n+2}|} \\ \downarrow & & \downarrow \\ \mathrm{Sp}_{2g}^q(\mathbb{Z}) & = & \mathrm{Sp}_{2g}^q(\mathbb{Z}). \end{array} \tag{7.5}$$

Proof. To produce a lift of $\hat{f}$, it will suffice to prove that the pullback of μ to $H^2(E(g,n);\mathbb{Z})$ is divisible by $|bP_{2n+2}|$. By the splittings (7.4) and the characterisation of μ in Lemma 7.5, in order to do this it is enough to show that the map

$$H_2(E(g,n);\mathbb{Z}) \xrightarrow{i_*} H_2(\Gamma_{g,1};\mathbb{Z}) \xrightarrow{\hat{f}_*} H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) \xrightarrow{\mathrm{sign}} \mathbb{Z}$$

is divisible by $8 \cdot |bP_{2n+2}|$, where $i : E(g,n) \to \Gamma_{g,1}^n$ is the inclusion and the map sign sends a second homology class to the signature of the symmetric form described in Lemma 7.5 (i).

Firstly, by the splitting (7.3) and the Künneth theorem, the map i_* is an isomorphism modulo torsion, so the divisibility of the map $\mathrm{sign} \circ \hat{f}_* \circ i_*$ is the same as that of $\mathrm{sign} \circ \hat{f}_*$. Secondly, the composition

$$H_2(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Z}) \longrightarrow H_2(\Gamma_{g,1};\mathbb{Z}) \xrightarrow{\hat{f}_*} H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) \xrightarrow{\mathrm{sign}} \mathbb{Z}$$

has the first map surjective and sends a smooth bundle $E \to S$ with fibres $W_{g,1}$ and base a closed oriented surface to the signature of $H^1(S; H^n(W_{g,1};\mathbb{Z}))$ and so by [CHS57] to the signature of the total space E (with $S \times D^{2n}$ glued in to make it a closed manifold). This total space defines a class in $\Omega_{2n+2}^{\langle n \rangle} = \Omega_{2n+2}^{\langle n+1 \rangle}$, and is thus cobordant to a closed smooth manifold which is framed away from a point. By [MK60, p. 457] the signature of such a manifold is divisible by $8 \cdot |bP_{2n+2}|$, as required.

We have constructed the map $E(g,n) \to E_{g,|bP_{2n+2}|}$ making the square (7.5) commute, and it remains to prove that the induced map of kernels $bP_{2n+2} \to \mathbb{Z}/|bP_{2n+2}|$ is an isomorphism. To see this, we consider the induced map of Serre spectral sequences, and in particular the commutative square

$$\begin{array}{ccc} H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) = E_{2,0}^2 & \xrightarrow{d^2} & E_{0,1}^2 = H_1(bP_{2n+2};\mathbb{Z}) = bP_{2n+2} \\ \| & & \downarrow \\ H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) = E_{2,0}^2 & \xrightarrow{d^2} & E_{0,1}^2 = H_1(\mathbb{Z}/|bP_{2n+2}|;\mathbb{Z}) = \mathbb{Z}/|bP_{2n+2}|. \end{array}$$

The lower horizontal map is identified with

$$H_2(\mathrm{Sp}_{2g}^q(\mathbb{Z});\mathbb{Z}) \xrightarrow{s} \mathbb{Z} \longrightarrow \mathbb{Z}/|bP_{2n+2}|$$

so is surjective if $s = \mathrm{sign}/8$ is indivisible. In this case the right-hand vertical map must also be surjective, and hence must be an isomorphism as both groups are cyclic of the same order.

To show that s is indivisible, consider the maps

$$\mathrm{Sp}_{2g}(\mathbb{Z},2) \longrightarrow \mathrm{Sp}_{2g}^q(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$$

from the level 2 congruence subgroup and to the full symplectic group. Meyer has shown that the signature map $\mathrm{sign} : H_2(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Z}) = \mathbb{Z} \to \mathbb{Z}$, defined as in

Lemma 7.5 (i), has image $4\mathbb{Z}$ as long as $g \geq 3$ [Mey73, Satz 2]. Putman has shown that $H_2(\mathrm{Sp}_{2g}(\mathbb{Z}, 2); \mathbb{Z}) \to H_2(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Z})$ has image $2\mathbb{Z}$ [Put12, Theorem F] as long as $g \geq 4$. Thus the signature map restricted to the level 2 congruence subgroup has image $8\mathbb{Z}$, so in particular it hits $8 \in \mathbb{Z}$ for $g \geq 4$, and so the signature map restricted to $\mathrm{Sp}_{2g}^q(\mathbb{Z})$ does too; hence s hits $1 \in \mathbb{Z}$. □

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