

Quantum mechanics of a generalised rigid body

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ABSTRACT: We consider the quantum version of Arnold's generalisation of a rigid body in classical mechanics. Thus, we quantise the motion on an arbitrary Lie group manifold of a particle whose classical trajectories correspond to the geodesics of any one-sided-invariant metric. We show how the derivation of the spectrum of energy eigenstates can be simplified by making use of automorphisms of the Lie algebra and (for groups of Type I) by methods of harmonic analysis. We show how the method can be extended to cosets, generalising the linear rigid rotor. As examples, we consider all connected and simply-connected Lie groups up to dimension 3. This includes the universal cover of the archetypical rigid body, along with a number of new exactly-solvable models. We also discuss a possible application to the topical problem of quantising a perfect fluid.

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1 Introduction

Euler showed long ago that the classical mechanics of a free, rigid body moving about a fixed point (*e.g.* its centre of mass) simplifies, in that the motion of the angular momentum relative to the body depends (via the Euler equations) only on its own initial value and not on the position of the body in space. Moreover, the magnitude of this angular momentum is conserved.

Rigid body motion can be thought of as the geodesic motion (with respect to a one-sided-invariant metric) of a particle on the Lie group manifold $SO(3)$. In 1966, Arnold [1] showed how Euler’s simplifications generalise to geodesic motion of a particle on *any* Lie group manifold G . In a nutshell, the space of angular momenta of the rigid body is replaced by the dual of the Lie algebra, the motion in that space is governed by a set of generalised Euler-Arnold equations, and the motion is restricted to an orbit of the coadjoint

representation of G (generalising the spheres of constant magnitude of angular momentum for the rigid body).

Arnold even went so far as to generalise to infinite-dimensional Lie groups, such as the group, $\text{Diff}(M)$, of diffeomorphisms of a manifold M .¹ For the subgroup of volume-preserving diffeomorphisms, the Arnold-Euler equations describe the motion of an incompressible fluid on M ; for the particular case where M is the two-torus, Arnold obtained a celebrated result on the stability of fluid flows and thence an estimate of the unreliability of long-term weather forecasting.

The quantum mechanics of a rigid body has also been known for some time (see, *e.g.*, [3]). Here we attempt to carry out the generalisation to arbitrary G , as well as to coset spaces G/H , where H is a proper subgroup of G . We then study, as illustrative examples, all connected and simply-connected Lie groups up to dimension 3.²

It is too much to hope that these quantum-mechanical systems are all exactly solvable, because the archetypical case of an asymmetric rigid body already provides a counterexample: there we can, at best, reduce the problem of finding the energy spectrum to diagonalization of finite-dimensional matrices. But in many cases, we encounter systems that are exactly solvable, using standard results in representation theory and harmonic analysis.³ In this way, we uncover a number of new, exactly-solvable models in quantum mechanics, generalising a rigid body.⁴ We hope that some of these will turn out to be of physical interest. We also hope that our results may provide some guidance towards the solution of a problem that is of some topical interest, namely the quantisation of incompressible fluids. The quantisation of a compressible fluid has been carried out using standard methods of quantum field theory in [6, 7]. Classically, an incompressible fluid is easily obtained as a limit of a compressible fluid, by restricting to flows in which the fluid velocity is much lower than the fluid's sound speed. One cannot impose such a restriction at the quantum level.⁵ Nevertheless, a quantum incompressible fluid may exist *sui generis*, and a generalisation to infinite-dimensional Lie groups of the approach we develop here may provide a means to describe it.

The outline is as follows. In the next Section, we discuss the formalism. In §3 we quickly dispatch the 1-dimensional Lie group \mathbb{R} , where the dynamics is that of a free particle in 1-d and generalise to \mathbb{R}^n . In §4, we solve the unique non-Abelian group in 2 dimensions, which

¹Such groups do not have the usual desiderata of finite-dimensional Lie groups: the exponential map does not necessarily exist for non-compact M , and even for compact M it may not be locally-onto. Indeed $\text{Diff}(\mathbb{R})$ and $\text{Diff}(S^1)$, respectively, furnish counterexamples [2].

²Groups that are connected but not simply-connected can be described within our formalism by choosing H to be some discrete subgroup of the centre of G , the universal cover. If H is also compact, the required harmonic analysis is straightforward. For groups that are not connected, one must first construct the irreps by induction from the component connected to the identity.

³In the unsolvable cases, we find that imposing an additional symmetry on the dynamics invariably results in an exactly-solvable model. Again, this is much like the rigid body, where the cylindrically- and spherically-symmetric cases are both solvable.

⁴Refs. [4, 5] have studied the spectral properties of a restricted class of Hamiltonians (corresponding to bi-invariant metrics), using alternative path-integral and Green's function methods.

⁵To put it another way: because the sound wave modes are gapless, they cannot be integrated out to provide a low-energy, effective field theory of the incompressible modes.

is isomorphic to the group of orientation-preserving affine transformations of \mathbb{R} . There are two proper subgroups isomorphic to \mathbb{R} , up to conjugation, and we are able to formulate and solve the quantum mechanics problem on the corresponding coset space for one of these. In §5, we study the groups that arise in dimension 3, ending with our first compact example, the universal cover $SU(2)$ of the rigid body group, which is perhaps where we ought to have begun.

2 Formalism

Before we can attempt to solve the quantum mechanics of a free particle on a Lie group G or coset space G/H , we must first define it. A proper definition of quantum mechanics is still lacking (and occupies much learned discussion in the literature) and so we shall simply follow our noses, in an attempt to come up with something suitable. We begin by discussing the dynamics on a group G and then generalise to a coset space G/H .

2.1 Quantum mechanics on G

What are the necessary ingredients for the quantum mechanics of a free particle moving on a Lie group manifold G ?

Firstly, we need a Hilbert space of physical states. Every Lie group admits a unique (up to a multiplication by a constant) left (say) invariant measure μ and so an obvious candidate for the Hilbert space for the dynamics on G is the space $L^2(G, \mu)$ of (equivalence classes of) functions $f : G \rightarrow \mathbb{C}$ that are square-integrable w.r.t. μ .⁶

Secondly, we need some operators that act on the Hilbert space $L^2(G, \mu)$ and that play the rôle of observables. The obvious candidates are the group elements themselves, which have an action on $f \in L^2(G, \mu)$ by left multiplication of the argument of the function. But these are not good candidates for observables when the dynamics is that of a free particle! Indeed, consider the simplest example of free-particle motion on the Lie group \mathbb{R} , with co-ordinate $x \in \mathbb{R}$. The classical action is $\int dt \frac{1}{2} \dot{x}^2$. If we attempt to compute the 2-point correlation function of x in Fourier space, we get $\langle x(t)x(t') \rangle = \int \frac{d\omega}{\omega^2} e^{i\omega t}$; because of the pole at $\omega = 0$, the Fourier transform is undefined. In the language of physicists, the 2-point function is infra-red divergent, and so x cannot be an observable.⁷

So, we need to find some other operators to play the rôle of observables. To do so, we consider (via the exponential map), the infinitesimal version of the group action described

⁶In fact, from a physical point of view, we also wish to include states that do not belong to $L^2(G, \mu)$. These generalised eigenvectors (also known as non-normalisable modes or scattering states) can be thought of as living in the dual space to the maximal subspace of $L^2(G, \mu)$ on which the operators corresponding to observables are defined. For $G = \mathbb{R}$, for example, the generalised eigenfunctions of the operators x and ∂_x , correspond to the Dirac delta function and e^{ikx} , respectively. The corresponding eigenvalues are included automatically in the functional-analytic definition of the spectrum as the complement in \mathbb{C} of the resolvent set.

⁷A similar argument also shows that we cannot study the quantum mechanics on a Lie group G by assuming that the energy eigenstates are localised near some $g \in G$ and then using the fact that G is locally like \mathbb{R}^n to approximate the path integral as a Gaussian integral. As we shall see in explicit examples later on, the energy eigenstates correspond to wavefunctions that are completely delocalised in G .

above. This gives rise to right-invariant vector fields sending some dense subspace (say, the smooth functions of compact support) of $L^2(G, \mu)$ into $L^2(G, \mu)$ and corresponding to formally self-adjoint operators. Thus we take a subset of the observables to be the space of right-invariant vector fields on G , otherwise known as the Lie algebra, \mathfrak{g} , of G .⁸ The quantum commutators of the theory, which are defined by composition of the operators corresponding to observables, are then just given by the Lie brackets.

We can enlarge the space of observables further by passing to the universal enveloping algebra. This enables us to obtain our third and final ingredient, which is a prescription for the dynamics, in the form of a Hamiltonian. Classically, this is easy to do. A Lie group is equipped with an infinite family of metrics, namely the one-sided-invariant metrics. Picking one such metric, a Hamiltonian (which is just the kinetic energy in this free-particle case) is given by plugging the velocity vector \dot{g} corresponding to a classical trajectory $g(t) \in G$ into the metric.

Quantum mechanically, the corresponding Hamiltonian should be some operator in our space of observables. Now the one-sided-invariant metrics on a group G of dimension n are in 1-to-1 correspondence with the positive-definite, symmetric, $n \times n$, real matrices. We thus take the Hamiltonian to be the quadratic element of the universal enveloping algebra given by

$$\mathcal{H} = \sum_{i,j} \alpha_{ij} X_i^R X_j^R, \quad (2.1)$$

where α_{ij} are the entries of such a matrix and where $\{X_i^R\}$ is a basis for the Lie algebra of right-invariant vector fields. This Hamiltonian is a Laplacian operator, which is formally self-adjoint with respect to the left-invariant Haar measure. Note that it is not, in general, equal to the Laplace-Beltrami operator constructed from the metric corresponding to α_{ij} , nor is it a Casimir invariant for generic values of α_{ij} . Nevertheless, it has three desirable properties that make it suitable as a quantum mechanical Hamiltonian. Firstly, it is a natural generalisation of what one obtains for the rigid body, where α_{ij} corresponds to the inverse of the moment of inertia tensor (see, *e.g.*, [8]). Secondly, it is explicitly written in terms of other physical observables, *viz.* the right-invariant vector fields. This makes it possible to build a comprehensive physical theory from it, which relates the results of measurements of distinct observables. Thirdly, being a symmetric, elliptic differential operator, it has a self-adjoint extension to $L^2(G, \mu)$. Even better, since we can perform a change of basis, $X_i^R \rightarrow X_i'^R$, such that \mathcal{H} can be written as $\sum_i X_i'^R X_i'^R$, we have that \mathcal{H} is both essentially self-adjoint (see, *e.g.*, [9]) and non-positive [10]. This guarantees that the spectrum is a (closed) subset of $[0, \infty)$, and so the energy is bounded below.⁹

In all, \mathcal{H} is a very plausible candidate for a quantum-mechanical Hamiltonian. One could easily generalise it further, by adding, for example, a term linear in X_i^R . By analogy

⁸There is, of course, another reason for singling out \mathfrak{g} as the space of observables: it is what we get in the archetypical example of a rigid body!

⁹We remark that \mathcal{H} may even enjoy the same properties when α_{ij} is degenerate, such that $\{X_i'^R\}$ do not span \mathfrak{g} , provided that they satisfy Hörmander's condition [10], *viz.* that $\{X_i'^R\}$ and all Lie brackets formed recursively therefrom do span \mathfrak{g} .

with the rigid body, this should presumably be interpreted as corresponding to motion in a external field coupled to the generalised angular momentum, generalising a magnetic field coupling to angular momentum in the case of the rigid body.

2.2 Quantum mechanics on G/H

In the case of a (left, say) coset space, we can proceed in much the same way, although there is an obstacle in that we may not be able to find a G -invariant measure on G/H . Indeed, G/H admits an invariant measure iff. the restriction to H of the modular function on G coincides with the modular function on H [11].¹⁰ In this paper we assume H to be a closed Lie subgroup of G , and $\dim H < \dim G$.¹¹

If we have a measure μ , we can define a representation of G on functions in $L^2(G/H, \mu)$ by the left group action on the arguments of the functions, just as we did before. From there, we can define formally self-adjoint observables via the infinitesimal action of the exponential map. Equivalently, we may simply take the right-invariant vector fields on G and push them forward to G/H using the natural projection $\pi : G \rightarrow G/H$. This yields well-defined vector fields on G/H , since right invariance of the vector fields on G guarantees an identical outcome when pushing forward the vectors at all points in the preimage $\pi^{-1}(p) \subset G$ of a point $p \in G/H$.¹²

As before, we then assume that the quantum Hamiltonian is just an arbitrary symmetric, negative quadratic element in the universal enveloping algebra formed from these vector fields and their Lie brackets.¹³

2.3 Method of solution

We now describe our strategy for solving these quantum mechanical systems. The spectrum of energy eigenstates, for example, is given by the spectrum of a Laplacian, so we could attempt to solve this partial differential eigenvalue equation directly. We define the spectrum as the complement of the resolvent set of our Hamiltonian, and we can test for membership of this set by constructing Weyl sequences out of the solutions of the PDE. But as we shall see, these PDEs are rather uninviting in their appearance and are not

¹⁰One way to construct a measure is as follows [12]. Let G and H have dimension n and k , respectively and let $\{\omega^1, \dots, \omega^{n-k}\}$ be a set of linearly-independent left-invariant 1-forms on G whose restriction to $T_e(H)$ vanishes. The $(n-k)$ -form $\omega \equiv \omega^1 \wedge \omega^2 \dots \wedge \omega^{n-k}$ is unique up to multiplication by a constant and G/H has a G -invariant volume form Ω (and hence a G -invariant measure) iff. $d\omega = 0$. If this condition holds, ω is given by pulling back Ω using the natural projection $\pi : G \rightarrow G/H$.

¹¹In the case of finite H , its effect is to restrict the irreps appearing in the decomposition of the regular representation; consider $SU(2)/\mathbb{Z}_2 \cong SO(3)$ (§5.3) which only contains the irreps of integer spin.

¹²Proof: Let X be a vector field on G and let X_g be its value at $g \in G$. These can be defined by their action on differentiable functions from G to \mathbb{R} . Now let $g' \in G$ lie in the same coset as g , s. t. $g' = gh$, for some $h \in H$. The pushforwards of X_g and $X_{g'}$ are given by $\pi^*(X_g)f = X_gf(\pi(g))$ (where $f : G/H \rightarrow \mathbb{R}$ is differentiable) and $\pi^*(X_{g'})f = X_{g'}f(\pi(g')) = X_{g'}f(\pi(g)) = X_{gh}f(\pi(g))$. These are equal by right-invariance.

¹³Ref. [13] addresses the related question of identifying necessary conditions for an element that is quadratic in the generators of an arbitrary algebra to correspond to a quantum-mechanical Hamiltonian; one case identified is that in which the algebra corresponds to that of G -invariant vector fields on a coset space G/H admitting a G -invariant metric, *viz.* the case discussed here.

always amenable to solution by humble methods such as separation of variables, nor are the Weyl sequences easy to construct.

One approach, explored in [14], would be to express a generic function on G in terms of the simultaneous generalised eigenfunctions of one, or more, of the left-invariant vector fields.¹⁴ The left-invariant vector fields commute everywhere with the right-invariant vector fields, and by extension with the Hamiltonian. Therefore, the substitution into the PDE of a simultaneous eigenfunction of the left-invariant vector field(s) may reduce the problem to an ODE. However, the generalised eigenfunctions may themselves prove difficult to find. Even if they can be found, the necessary Weyl sequences needed to find the spectrum remain difficult to construct. Even if the spectrum can be found in this way, one is still lacking a crucial piece of physical information, namely the spectral density. This is needed to compute, *e.g.*, the thermal properties of such quantum-mechanical systems.

A more sure-fire approach is to try to exploit the group-theoretic and geometric structure underlying the dynamics in a different way. Indeed, the Hilbert space carries a highly reducible, unitary representation of G (the left regular representation) and so an obvious step is to try to reduce this representation into its unitary, irreducible representations ('unirreps', henceforth). The action of the Hamiltonian, which is an element of the universal enveloping algebra, then decomposes into actions on the irreducible subspaces, which we can attack individually.¹⁵ Moreover, this decomposition constitutes a unitary map between the Hilbert spaces of the left regular rep and the unirreps. Thus, any Weyl sequence we may construct for an eigenfunction of the Hamiltonian in the latter space (which, being of lower dimension, is often easier) gives rise to a Weyl sequence for the solution of the original PDE.

The decomposition of functions on G/H generalises the Fourier transform on \mathbb{R} and is a standard tool of harmonic analysis. Let us describe it schematically here for the simplest case, where $H = \{e\}$ and G is unimodular. Let the unirreps of G be D^λ , where λ labels the equivalence classes. Given a function f in $L^2(G, \mu_g)$ (where we put a subscript g on μ to remind ourselves that it is a measure on G), define an integral transform by

$$\tilde{f}(\lambda) \equiv \int \mu_g f(g) \overline{D}^\lambda(g), \quad (2.2)$$

where \overline{D} denotes the Hermitian conjugate of D , and the operator $\tilde{f}(\lambda)$ acts on the Hilbert space of the irrep λ . There exists a measure μ_λ on the space of equivalence classes of unirreps, such that the transform may be inverted, to obtain

$$f(g) = \int \mu_\lambda \text{tr} \left[D^\lambda(g) \tilde{f}(\lambda) \right]. \quad (2.3)$$

(As an example, for the case $G = \mathbb{R}$ with $x \in \mathbb{R}$, let the invariant measure be $\mu_x = dx$. The irreps are 1 dimensional, $D^\lambda(x) = e^{i\lambda x}$, $\lambda \in \mathbb{R}$ and the operator $\tilde{f}(\lambda)$ is just the usual

¹⁴We thank Z. Avetisyan for discussions on this point.

¹⁵This decomposition corresponds to the fact that the classical motion is confined to a coadjoint orbit of G . Each coadjoint orbit has an associated reduced phase space and mathematicians have used the presumed existence of a corresponding reduced quantum-mechanical system to find the unirreps of a variety of Lie groups. See [15] for details.

\mathbb{C} -valued Fourier transform. By the Parseval theorem, we have that the Plancherel measure is $\mu_\lambda = \frac{d\lambda}{2\pi}$.) The inversion is thus an explicit decomposition of $L^2(G, \mu_g)$ into subspaces of operators acting on irreps λ , with measure μ_λ . Notice that the decomposition involves, in general, a direct integral rather than a direct sum. For non-compact groups, some set of unirreps may have zero Plancherel measure, meaning that they do not appear in the decomposition.

At this point, a triumvirate of problems appear. The first is that, even when G is non-compact (as we shall see explicitly in the dimension two example), we will need to know all the unirreps of G to decompose the left regular reps on G/H for arbitrary H .¹⁶ The classification of the unirreps of a general Lie group is an unsolved problem¹⁷ and we will only be able to make progress in this way in cases where the irreps appearing in the decomposition are known. The second is that the decomposition is not unique for certain Lie groups, known as Type II groups; it is unclear how one should deal with these. The third is that the unirreps of non-Abelian groups are often multi-dimensional (or even infinite dimensional in the non-compact case). Thus, even if we manage to effect the decomposition, we may be left with a matrix or differential operator representing the action of the Hamiltonian whose eigenvalues we are unable to find. This is the obstruction that occurs in the rigid body case and we shall see that it arises in several other cases in dimension three.

Nevertheless, we shall see that many cases are tractable. Two simplifying observations reduce the burden. The first is that while a generic one-sided-invariant Hamiltonian has $n(n+1)/2$ arbitrary coefficients, we may be able to simplify its form, w. l. o. g., using automorphisms of the Lie algebra \mathfrak{g} . These are just the invertible linear maps from \mathfrak{g} to itself that preserve the structure constants of the algebra (in some basis) and we may use them to simplify the Hamiltonian without changing the form of the operators representing the algebra elements. Unfortunately, finding the automorphisms (which may be outer) generally requires a brute-force approach.¹⁸

The second observation is that there is a group, K , of residual automorphisms that preserve the form of the simplified Hamiltonian. If non-trivial, K provides further constraints on the form of the spectrum. Consider, for example, the maximal compact subgroup, $\overline{K} \subset K$. We can simplify the problem of finding the spectrum by decomposing the Hilbert space into irreps of \overline{K} , because Schur's lemmata then imply that the Hamiltonian \mathcal{H} acts as the identity on the irreps of \overline{K} and will not mix inequivalent irreps appearing in the decomposition (it can mix equivalent irreps, however).

¹⁶For H compact, we expect that $L^2(G/H)$ is isomorphic to the subspace of $L^2(G)$ defined by the functions that are constant within the cosets. (For H non-compact, such functions are not in $L^2(G)$.) So all the unirreps in $L^2(G/H)$ already appear in $L^2(G)$.

¹⁷Examples of general cases where the classification is known are when G is either a compact [16] or an exponential Lie group [17]. An exponential Lie group is a special case of a solvable group, for which the exponential map is a diffeomorphism.

¹⁸Some simplifying tricks may be found in [18], which classifies the automorphism groups for all Lie algebras up to dimension five. This work also makes it clear that there is no simple way to treat direct product groups, because one cannot use the automorphisms to block-diagonalize the Hamiltonian into operators acting only on the separate factors.

In fact, it is sometimes convenient to consider an even larger subgroup of automorphisms, namely those that preserve the form of the simplified Hamiltonian up to a permutation of the latter's coefficients. We call this the group of permuting automorphisms and denote it by J . Evidently, there is a homomorphism from J to the permutation group of the coefficients with kernel K and so K is a normal subgroup of J . To give an example of how J may be useful, given one energy eigenvalue, corresponding to a state carrying an irrep of J , one can find other energy eigenvalues of states in the irrep by effecting the corresponding permutations of the coefficients.

3 Dimension one and \mathbb{R}^n

In one dimension, the only Lie algebra is $[X, X] = 0$, and the corresponding (connected and simply-connected) group is isomorphic to \mathbb{R} . We might as well do $\mathbb{R}^n, n \in \mathbb{N}$, while we are at it. The non-trivial automorphisms of the algebra are all outer and consist of all invertible linear maps from \mathbb{R}^n to itself, *viz.* $GL(n, \mathbb{R})$; a transformation in $SO(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ can be used to write the most general Hamiltonian in diagonal form and then a diagonal matrix in $GL(n, \mathbb{R})$ can be used to write the Hamiltonian in the form

$$\mathcal{H} = X_1^2 + X_2^2 + \cdots + X_n^2, \quad (3.1)$$

where $\{X_1, \dots, X_n\}$ is some basis for the Lie algebra.

The unirreps of the Abelian group \mathbb{R}^n are one-dimensional, with action on \mathbb{C} given by multiplication by $e^{i\vec{k}\cdot\vec{x}}$, with $\vec{k} \in \mathbb{R}^n$. The decomposition of the left regular representation (in which the Lie algebra acts as $X_i = \frac{\partial}{\partial x_i}$) into irreps is achieved by means of the Fourier transform

$$f(\vec{x}) = \int \frac{d^n \vec{k}}{(2\pi)^n} e^{i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k}). \quad (3.2)$$

The energy eigenvalues are thus given by $E = \vec{k} \cdot \vec{k} \geq 0$, with Plancherel measure $\frac{d^n \vec{k}}{(2\pi)^n}$.

In this case, then, we are easily able to find the spectrum without recourse to the group, $O(n, \mathbb{R})$, of residual automorphisms of (3.1). We could, of course, also decompose the left regular rep into irreps of the group $O(n, \mathbb{R})$, by passing to hyperspherical polar coordinates, whereupon the decomposition reduces to that of functions on the hypersphere. This is hardly a simplification, since the only invariant subspace of both \mathbb{R}^n and $O(n, \mathbb{R})$ is spanned by the constant function. There is one simplifying consequence, however, which is that the invariant subspaces of \mathbb{R}^n that are carried into one another by the action of $O(n, \mathbb{R})$, namely all those with equal values of $\vec{k} \cdot \vec{k}$, must be degenerate in energy, as indeed they are.

4 Dimension two

We now illustrate the decomposition of the left regular rep on $f \in L^2(G, \mu)$ into irreps, and the subsequent derivation of the spectrum, in the case where G is a Lie group of dimension 2. There is a unique non-Abelian 2-d Lie algebra (up to isomorphism), with Lie bracket

$[X, Y] = Y$ in some basis $\{X, Y\}$. The corresponding connected and simply-connected group is the group of orientation-preserving affine transformations of the real line. We denote this by $G = \text{Aff}(\mathbb{R})^+$, also known as the ‘ $ax + b$ group’ with $a > 0$. This group is non-unimodular and so we must be careful to pick the left-invariant Haar measure for μ .

4.1 $\text{Aff}(\mathbb{R})^+$ group properties and irreps

$\text{Aff}(\mathbb{R})^+$ is isomorphic to the group of matrices

$$\text{Aff}(\mathbb{R})^+ = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\} \quad (4.1)$$

under matrix multiplication. Its algebra is isomorphic to

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \equiv \{\alpha X + \beta Y\},$$

where, as above, $\{X, Y\}$ form a basis with the canonical Lie bracket $[X, Y] = Y$. The left- and right-invariant Haar measures are proportional to $\frac{da db}{a^2}$ and $\frac{da db}{a}$, respectively.

The right-invariant vector fields are

$$X^R = -a\partial_a - b\partial_b, \quad Y^R = -\partial_b, \quad (4.2)$$

and the left-invariant vector fields are

$$X^L = a\partial_a, \quad Y^L = a\partial_b. \quad (4.3)$$

Each of these yields the desired algebra $[X, Y] = Y$, with the Lie brackets of left-invariant fields with right-invariant fields vanishing.

The group is both non-Abelian and non-compact and has 2 infinite-dimensional unirreps

$$D^\pm(s, t; \frac{a}{b}) = e^{\pm 2\pi i b s} \delta(as - t)t, \quad (4.4)$$

which act on functions $\phi(t) \in L^2(\mathbb{R}^+, \frac{dt}{t})$ thus:

$$\int_0^\infty \frac{dt}{t} D^\pm(s, t; \frac{a}{b}) \phi(t) = e^{\pm 2\pi i b s} \phi(as). \quad (4.5)$$

There is also a family of 1 dimensional unirreps,

$$D^c(\frac{a}{b}) = a^{2\pi i c}, \quad (4.6)$$

labelled by $c \in \mathbb{R}$, which act by multiplication on the Hilbert space \mathbb{C} [19].

4.2 Decomposition of the left regular representation into irreps

By analogy with the similarity transform that block diagonalizes a finite dimensional reducible matrix representation, we seek a unitary transformation from $L^2(G, \mu)$ to functions on the unitary dual of G , which consists of equivalence classes of G irreps, and is equipped with the concomitant Plancherel measure. In picking an explicit basis for coordinates on the group manifold and its unitary dual, we will forgo many of the subtleties of harmonic analysis and instead derive our results in explicit examples by appealing to properties of the Euclidean Fourier transform. We caution the reader that similar arguments may not be applicable in the general case.

Following (2.2), we first define integral transforms¹⁹²⁰

$$\begin{aligned}\tilde{f}^p(s, t) &= \int \frac{dad b}{a^2} \overline{D}^p(s, t; \frac{a}{b}) \sqrt{s} f(\frac{a}{b}), \\ \tilde{f}^c &= \int \frac{dad b}{a^2} \overline{D}^c(\frac{a}{b}) f(\frac{a}{b}),\end{aligned}$$

where $p \in \{\pm\}$ and $\overline{D}(s, t) \equiv D^*(t, s)$ denotes the Hermitian conjugate of D . Note that the desired group transformations of the \tilde{f} s are fixed by the properties of the representation matrices $D(g)$ and the Haar measure μ_g . Schematically, the action $T(g')$ of the group element g' is:

$$\begin{aligned}T(g') \tilde{f}^p(s, t) &= \int \mu_g \overline{D}^p(s, t; g) \sqrt{s} f((g')^{-1}g) \\ &= \int \frac{du}{u} \overline{D}^p(u, t; g') \int \mu_{(g')^{-1}g} \overline{D}^p(s, u; (g')^{-1}g) \sqrt{s} f((g')^{-1}g) \\ &= \int \frac{du}{u} \tilde{f}^p(s, u) \overline{D}^p(u, t; g'),\end{aligned}$$

and thus the \tilde{f}^p (and similarly the \tilde{f}^c) indeed transform under an irrep of G .

It is then the inversion of the above integral transforms — the reconstruction of the original $f(\frac{a}{b})$ from the $\tilde{f}^p(s, t)$ and \tilde{f}^c — that determines the weights with which each irrep appears in the left regular representation. To this end we observe that, using the counting measure $\sum_{p=\pm}$ over the two infinite dimensional irreps, it is possible to obtain a resolution of the identity on the group coordinates:²¹

$$\sum_{p=\pm} \int \frac{ds dt}{st} D^p(s, t; \frac{a'}{b'}) \sqrt{t} \overline{D}^p(t, s; \frac{a}{b}) \sqrt{t} = \delta(a - a') \delta(b - b') a^2,$$

¹⁹The integral transform in (2.2) is an operator on the Hilbert space of irreps; here we transform the ‘operator’s matrix elements’, with ‘indices’ s, t and measures $ds/s, dt/t$ instead.

²⁰The extra factor of \sqrt{t} arises because of a subtlety in the harmonic analysis of this and other non-unimodular groups. For more details, see [20, 21].

²¹The equivalent relation on the irreps is just the Schur orthogonality relation

$$\int \frac{dad b}{a^2} \overline{D}^p(t, s; \frac{a}{b}) \sqrt{t} D^{p'}(s', t'; \frac{a}{b}) \sqrt{t'} = \delta^{pp'} \delta(s - s') \delta(t - t') st.$$

and we can therefore write the inversion as

$$f\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \sum_{p=\pm} \int \frac{dsdt}{st} D^p(s, t; \begin{smallmatrix} a \\ b \end{smallmatrix}) \sqrt{t} \tilde{f}^p(t, s). \quad (4.7)$$

$\sum_{p=\pm}$ is the Plancherel measure on the irreps, and is notably zero on the 1-d c irreps, which do not feature in the decomposition of the left regular representation. Again, we explicitly compute the group action on our formula. Under left translation by $(g')^{-1}$

$$\begin{aligned} f((g')^{-1}g) &= \sum_{p=\pm} \int \frac{dsdt}{st} D^p(s, t; (g')^{-1}g) \sqrt{t} \tilde{f}^p(t, s) \\ &= \sum_{p=\pm} \int \frac{dsdt}{st} \frac{du}{u} D^p(s, u; (g')^{-1}) D^p(u, t; g) \sqrt{t} \tilde{f}^p(t, s) \\ &= \sum_{p=\pm} \int \frac{dudt}{ut} D^p(u, t; g) \sqrt{t} \left(\int \frac{ds}{s} \tilde{f}^p(t, s) D^p(s, u; g') \right), \end{aligned}$$

it is clear that the $\tilde{f}^p(s, t)$ which comprise $f\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ transform under the irrep action (4.5).

4.3 The spectrum of the left regular representation

Following the rubric of §2, the quantum Hamiltonian is given by $\mathcal{H} = a_{ij} X_i^R X_j^R$, where $X_1^R \equiv X^R$ and $X_2^R \equiv Y^R$ are the right-invariant vector fields defined in (4.2). To reduce clutter, we drop the R superscripts in what follows.

To find the automorphisms of the algebra, consider an arbitrary, invertible, linear transformation $M \in GL(\dim G, \mathbb{R})$ of the Lie algebra \mathfrak{g} , and require that $[M\xi, M\eta] = M[\xi, \eta]$, $\forall \xi, \eta \in \mathfrak{g}$, *i.e.*, it leaves the structure constants (namely the relation $[X, Y] = Y$, and the other trivial Lie brackets) invariant. Thus, let $M_{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the invariance of $[X_i, X_j] = f_{ij}^k X_k$, we obtain the set of constraints $M_{ip} M_{jq} f_{ij}^k (M^{-1})_{rk} = f_{pq}^r$, where $f_{12}^2 = -f_{21}^2 = 1$ and all others are zero, and we sum over repeated indices. This yields two non-trivial equations, $(ad - bc)b = 0$ (implying $b = 0$, since M is non-singular), and $a = 1$, leaving

$$M = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, d \neq 0. \quad (4.8)$$

Under conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, these take the same form as (4.1) without the restriction that d be positive: the group of automorphisms is therefore isomorphic to $\text{Aff}(\mathbb{R})$. For general $\mathcal{H} = \alpha X^2 + \beta\{X, Y\} + \gamma Y^2$, there exists a transformation $M = \begin{pmatrix} 1 & 0 \\ \frac{\beta}{\sqrt{\alpha\gamma - \beta^2}} & \frac{\gamma}{\sqrt{\alpha\gamma - \beta^2}} \end{pmatrix}$ to put the Hamiltonian in the simplified form $\mathcal{H} = A(X^2 + Y^2)$, $A > 0$, which we will assume hereafter.

We are now in a position to solve for the spectrum of the left regular representation, which is the set of eigenvalues $-E$ of the Hamiltonian²²

$$\mathcal{H}f = A((a\partial_a + b\partial_b)^2 + (\partial_b)^2)f = -Ef.$$

²²We write $-E$ as we have used the mathematicians' convention of skew-Hermitian group generators, leading to a negative definite spectrum. We expect $E \geq 0$.

This PDE looks hard to solve (in particular, it cannot be solved by the method of separation of variables using the coordinates (a, b)), but its solution is made simple by using the group structure of (4.7) and integrating by parts:²³

$$\sum_{p=\pm} \int \frac{dsdt}{st} D^p(s, t; \frac{a}{b}) \sqrt{t} \left[A(s^2 \frac{d^2}{ds^2} + s \frac{d}{ds} - 4\pi^2 s^2) + E \right] \tilde{f}^p(t, s) = 0.$$

The resulting ODE is a modified Bessel equation, whose solutions are well-known [22]. We select from these only the generalised eigenfunctions which are the limit of the Weyl sequences constructed in Appendix A (which in turn yields generalised eigenfunctions of $L^2(\text{Aff}(\mathbb{R})^+, \frac{da db}{a^2})$ after recomposition). The eigenvalues and eigenfunctions are therefore

$$E \in [0, \infty); \quad \tilde{f}^p(t, s) = \tilde{K} \sqrt{\frac{E}{A}} (2\pi s) g(t), \quad g(t) \in L^2(\mathbb{R}^+, \frac{dt}{t}) \text{ arbitrary.} \quad (4.9)$$

$\tilde{K}_\nu(x)$ is a modified Bessel function of imaginary order ν , behaving like $\sin(\nu \log x)$ near the origin, but exponentially decaying as $x \rightarrow +\infty$ [22, 10.45]. For $g(t)$, we can choose a basis of generalised functions, such as $\sqrt{t} e^{2\pi i \lambda t}$, $\lambda \in \mathbb{R}$. Thus λ (together with $p \in \{\pm\}$) parametrises a set of degenerate generalised eigenfunctions.

Out of curiosity, and as a final check, we can recompose our solutions to the Laplacian in terms of the group coordinates a and b . Let, say, $\tilde{f}^+(t, s) = \tilde{K}_\nu(2\pi s) \sqrt{t} e^{2\pi i \lambda t}$ and $\tilde{f}^-(t, s) = 0$, where $\nu = \sqrt{\frac{E}{A}}$ and $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} f(\frac{a}{b}) &= \int \frac{dsdt}{st} D^+(s, t; \frac{a}{b}) \sqrt{t} \tilde{K}_\nu(2\pi s) \sqrt{t} e^{2\pi i \lambda t}, \\ &= a \int ds e^{2\pi i (b + \lambda a) s} \tilde{K}_\nu(2\pi s), \\ &\propto a (1 - i(b + \lambda a))^{-1+i\nu} \frac{1}{F} \left((1 - F)^{2i\nu} - (1 + F)^{2i\nu} \right) \text{ where } F(\lambda) = \sqrt{\frac{i(b + \lambda a) + 1}{i(b + \lambda a) - 1}}. \end{aligned}$$

We have explicitly checked that this is a generalised eigenfunction of the Laplacian, with eigenvalue $-A\nu^2 = -E$.

²³One way of doing this would be to integrate over the delta function in $D^p(s, t; \frac{a}{b})$ in the decomposition (4.7)

$$f(\frac{a}{b}) = \sum_{p=\pm} \int \frac{dt}{\sqrt{t}} e^{2\pi i p \frac{bt}{a}} \tilde{f}^p(t, \frac{t}{a}),$$

whence, for example,

$$\begin{aligned} a \partial_a f(\frac{a}{b}) &= \sum_{p=\pm} \int \frac{dt}{\sqrt{t}} e^{2\pi i p \frac{bt}{a}} \left[-2\pi i p \frac{bt}{a} \tilde{f}^p(t, \frac{t}{a}) - \frac{t}{a} \tilde{f}^{p(0,1)}(t, \frac{t}{a}) \right] \\ &= \sum_{p=\pm} \int \frac{dsdt}{st} D^p(s, t; \frac{a}{b}) \sqrt{t} \left[-2\pi i p b s - s \frac{d}{ds} \right] \tilde{f}^p(t, s), \end{aligned}$$

$\tilde{f}^{p(0,1)}$ denoting a derivative with respect to the second argument.

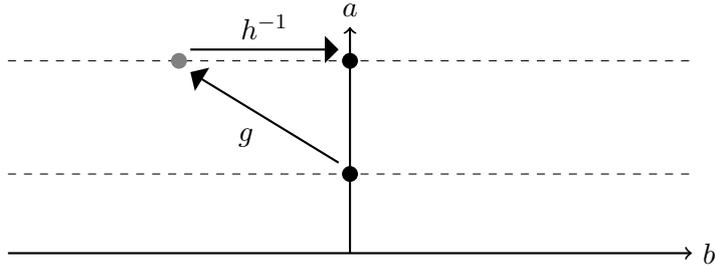


Figure 1: The coset structure on G/H for $\text{Aff}(\mathbb{R})^+$. All points of equal a belong to the same coset, indicated by the dashed lines. We choose points along the a axis (such as the black circles) to form our coset representative. The group action on the coset representative is left multiplication by $g \in G$, followed by the projection of the result onto its coset representative by right multiplication by h^{-1} .

4.4 The spectrum of the coset representation

We now consider quantum mechanics on the quotients of $\text{Aff}(\mathbb{R})^+$ by subgroups isomorphic to \mathbb{R} . There are two such subgroups up to conjugation, namely $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$, for which suitable coset representatives are $G/H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right\}$, illustrated in Fig. 1, and $H' = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right\}$, for which suitable coset representatives are $G/H' = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$.²⁴ Then the group actions on G/H and G/H' , obtained by multiplying on the left by $\begin{pmatrix} g_a & g_b \\ 0 & 1 \end{pmatrix} \in G$, are $a \mapsto g_a a$ for G/H and $b \mapsto g_a b + g_b$ for G/H' .

To formulate quantum mechanics we require a G -invariant measure. The modular function on G is $\Delta(a, b) = 1/a$. Both H and H' are unimodular, so only H (with $a = 1$) yields an invariant measure on G/H , given by $\frac{da}{a}$.

Imbued with a group action, the functions $f(\Sigma) \in L^2(G/H, d\mu)$ form a representation of G , which may similarly be decomposed into irreps, following the same logic as before. We first define an integral transform of functions in this space

$$\tilde{f}^c = \int \frac{da}{a} \overline{D}^c \begin{pmatrix} a \\ 0 \end{pmatrix} f(a).$$

Now the appropriate decomposition of the identity is

$$\int dc D^c \begin{pmatrix} a' \\ 0 \end{pmatrix} \overline{D}^c \begin{pmatrix} a \\ 0 \end{pmatrix} = a \delta(a - a'),$$

²⁴Note that, as expected from the arguments of §2, the pushforwards of right-invariant fields, given by $\pi^*(X^R) = -a\partial_a$, $\pi^*(Y^R) = 0$ and $\pi'^*(X^R) = -b\partial_b$, $\pi'^*(Y^R) = -\partial_b$, are well-defined on G/H and G/H' , respectively, but that the pushforward $\pi'^*(Y^L) = a\partial_b$ is not.

implying the inversion theorem

$$f(a) = \int \mathrm{d}c D^c \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) \tilde{f}^c.$$

From this we read off that the coset representation on $L^2(G/H, \frac{\mathrm{d}a}{a})$ decomposes into a direct integral of the 1-d irreps labelled by c , with Plancherel measure $\mathrm{d}c$.

Note that the irreps appearing in the coset rep on G/H are distinct from those appearing in the left regular rep on G (where the Plancherel measure is given by the counting measure on the irreps \pm). As we remarked earlier, this can only happen because H is non-compact.

The most general Hamiltonian is $\mathcal{H} = A[\pi^*(X^R)]^2$ and the eigenvalue equation reduces to:

$$[A(a^2 \partial_a^2 + a \partial_a) + E]f(a) = \int \mathrm{d}c D^c \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) [-A(2\pi c)^2 + E] \tilde{f}^c = 0,$$

with a spectrum $E = A(2\pi c)^2 \in [0, \infty)$ and uniform measure on c .

4.5 Solution of the PDEs via simultaneous eigenfunctions

It is useful to compare these results with what is obtained via solution of the PDEs via simultaneous (generalised) eigenfunctions. Indeed, the PDEs above can be reduced to second-order ODEs by substituting a functional form that is simultaneously an eigenfunction of a vector field that commutes with the Hamiltonian. This is most simply illustrated in the case of the right regular representation (with the right Haar measure $\frac{\mathrm{d}a \mathrm{d}b}{a}$), where the Hamiltonian \mathcal{H} is formed from the left-invariant fields $X^L = a \partial_a$ and $Y^L = a \partial_b$:

$$\mathcal{H} = A [(a \partial_a)^2 + (a \partial_b)^2]. \quad (4.10)$$

The Hamiltonian commutes by construction with $Y^R = \partial_b$, whose eigenfunctions are proportional to $e^{i\lambda b}$, $\lambda \in \mathbb{R}$. Substitution of $f \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) = e^{i\lambda b} g(a)$ into $\mathcal{H}f = -Ef$ reduces the problem to an ODE

$$a^2 g'' + a g' + \left(\frac{E}{A} - a^2 \lambda^2 \right) g = 0, \quad (4.11)$$

with solutions for $E \geq 0$ of

$$e^{i\lambda b} \tilde{K} \sqrt{E/A} (|\lambda| a). \quad (4.12)$$

The spectrum is identical to that of the unitarily equivalent left regular representation, and so these generalised eigenfunctions are demonstrably complete by comparison with the results obtained via the Fourier transform above. The left regular representation's own PDE also simplifies to an ODE upon substitution of $f \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) = a^q g(b)$, which is a simultaneous generalised eigenfunction of $X^L = a \partial_a$. The generalised eigenfunctions that are obtained by solving the resulting ODE are not the same as those obtained previously in §4.3; this should come as no surprise, given that the generalized eigenfunctions corresponding to a given eigenvalue are infinitely degenerate.

5 Dimension three

Up to isomorphism, there are nine classes, I–IX, of 3-d Lie algebras (two of which contain infinitely many algebras), which were first catalogued by Bianchi [23, 24]. Bianchi I is isomorphic to \mathbb{R}^3 , *cf.* §3. Bianchi II–VII all take the form of a semi-direct product group $\mathbb{R}^2 \rtimes \mathbb{R}$ and can be treated in a unified way; we treat the Heisenberg group (Bianchi II) in detail below, in analogue to our treatment of $\text{Aff}(\mathbb{R})^+$ in §4. Within the treatment of the Heisenberg case, references are made to Tables 1 and 2, from which the reader may substitute in the corresponding results for his or her favourite Bianchi algebra II–VII.

Many of these results come from [14] (though our conventions differ by minus signs), in which the unitary duals are obtained by induction via the Mackey machine and the Plancherel measures are derived. Ref. [14] also contains a result on the spectrum of a symmetric quadratic form of (left- or right-)invariant vectors, which would correspond to the spectrum of our quantum-mechanical Hamiltonian. Unfortunately, the result given is incorrect. The erroneous result is that (in our sign convention) the spectrum is given by $[-c, \infty)$, where $c < 0$ is the constant scalar Riemann curvature of the metric. The result is obtained using a result of Donnelly [25], also shown in [26], that applies only to the Laplace-Beltrami operator on manifolds whose sectional curvatures are all non-positive. Unfortunately, neither is the operator in question the Laplace-Beltrami operator, nor are the sectional curvatures of the Bianchi groups all non-positive. Indeed, it is easy to furnish a counterexample: the Bianchi II group has Haar measure $dx dy dz$ in our co-ordinates and the Hamiltonian is purely derivative. Thus, the constant function on the group is a generalised eigenfunction, and the spectrum includes $0 < -c$. Many further counterexamples appear below.

$SL(2, \mathbb{R})$ (Bianchi VIII) and $SU(2)$ (Bianchi IX) are discussed separately, in §§5.2–5.3.

5.1 Bianchi II–VII

Denote by H the Heisenberg group formed by the set of unit determinant upper triangular 3×3 real matrices

$$H = \left\{ \begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

under matrix multiplication. Note that the group composition law is a semi-direct product of those of $(x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}$: $(x, y, z) \circ (x', y', z') = (x + x', y + y' + zx', z + z')$ (Table 1a). The corresponding algebra is then:

$$\left\{ \begin{pmatrix} 0 & \gamma & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{R} \right\} \equiv \alpha X + \beta Y + \gamma Z,$$

with the Lie bracket $[Z, X] = Y$. The group is unimodular and has Haar measure $dx dy dz$ (the other Bianchi groups are not unimodular and in general we use the left-invariant Haar measure $\det F(-z) dx dy dz$, see Table 1a).

H has a continuum of inequivalent infinite-dimensional irreps labelled by $k \in \mathbb{R}/\{0\}$

$$D^k(s, t; \frac{x}{y}, \frac{z}{z}) = e^{ik(xs+y)}\delta(s+z-t), \quad (5.1)$$

which act on functions $\phi(t) \in L^2(\mathbb{R}, dt)$ (Table 1b). There are also one dimensional irreps, labelled by $(\mu, \nu) \in \mathbb{R}^2$,

$$D^{\mu, \nu}(\frac{x}{y}, \frac{z}{z}) = e^{i(\mu z + \nu x)}, \quad (5.2)$$

which act by multiplication on the Hilbert space \mathbb{C} [19].

As before, to decompose $L^2(H)$ we require completeness relations on both the irrep (or unitary dual) coordinates, and the group coordinates. The former comes from the Schur orthogonality relation:

$$\int dx dy dz \overline{D^k}(t, s; \frac{x}{y}, \frac{z}{z}) D^{k'}(s', t'; \frac{x}{y}, \frac{z}{z}) = \delta(k - k') \delta(s - s') \delta(t - t') \frac{4\pi^2}{|k|}.$$

The $\frac{4\pi^2}{|k|}$ suggests that in the latter we should use the Plancherel measure $\frac{|k|}{4\pi^2}$ (Table 1c)

$$\int dk ds dt \frac{|k|}{4\pi^2} D^k(s, t; \frac{x'}{y'}, \frac{z'}{z'}) \overline{D^k}(t, s; \frac{x}{y}, \frac{z}{z}) = \delta(x - x') \delta(y - y') \delta(z - z'),$$

to achieve the correct resolution of the identity w.r.t the Haar measure $dx dy dz$. Note again that the 1d (μ, ν) irreps do not feature in the decomposition of $L^2(H)$, which takes the form:²⁵

$$\begin{aligned} \tilde{f}^k(s, t) &= \int dx dy dz \overline{D^k}(s, t; \frac{x}{y}, \frac{z}{z}) f(\frac{x}{y}), \\ f(\frac{x}{y}) &= \int dk ds dt \frac{|k|}{4\pi^2} D^k(s, t; \frac{x}{y}, \frac{z}{z}) \tilde{f}^k(t, s). \end{aligned}$$

The Hamiltonian in the left regular rep is a quadratic form $a^{ij} X_i X_j$ of the Lie algebra components $X_j = (X, Y, Z)^T$, which may be simplified by the automorphism M_{ij} , where

$$M_{ij} = \begin{pmatrix} a & 0 & c \\ d & aj - cg & f \\ g & 0 & j \end{pmatrix}, \quad a, c, d, f, g, j \in \mathbb{R}, \quad aj - cg \neq 0.$$

Without loss of generality, we may then write $\mathcal{H} = X^2 + AY^2 + Z^2$ (Table 1d,f). All automorphisms M_{ij} of the algebra induce an action (also an automorphism) on the group coordinates $e^{\alpha_i X_i} \rightarrow e^{\alpha_i M_{ji} X_j}$, and, through the D matrices, also an action on the irrep coordinates, whereby unirreps are mixed. Unlike $\text{Aff}(\mathbb{R})^+$, there is a non-trivial group of

²⁵Generally,

$$\begin{aligned} \tilde{f}^{k(,k_2)}(s, t) &= \int dx dy dz \det F(-z) \overline{D}^{k(,k_2)}(s, t; \frac{x}{y}, \frac{z}{z}) \det F(\frac{1}{2}s) f(\frac{x}{y}), \\ f(\frac{x}{y}) &= \sum_{k_2} \int dk ds dt \frac{\nu(k, k_2)}{4\pi^2} D^{k(,k_2)}(s, t; \frac{x}{y}, \frac{z}{z}) \det F(\frac{1}{2}t) \tilde{f}^{k(,k_2)}(t, s), \end{aligned}$$

$\nu(k, k_2)$ being the Plancherel measure of Table 1c.

residual automorphisms, locally isomorphic to $SO(2, \mathbb{R})$, that preserve the simplified form of the Heisenberg Hamiltonian, viz. (Table 1e),

$$\left\{ \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}, \theta \in [0, 2\pi).$$

These only mix unirreps with identical spectra, identifying a possible degeneracy. These are the pairs of irreps labelled by k and $-k$, as we will see presently.

The actions of the Lie algebra elements are $X^R f\left(\frac{x}{y}, \frac{z}{y}\right) = \partial_x f\left(\frac{x}{y}, \frac{z}{y}\right)$, $Y^R f\left(\frac{x}{y}, \frac{z}{y}\right) = \partial_y f\left(\frac{x}{y}, \frac{z}{y}\right)$ and $Z^R f\left(\frac{x}{y}, \frac{z}{y}\right) = (\partial_z + x\partial_y) f\left(\frac{x}{y}, \frac{z}{y}\right)$. This yields an eigenvalue equation:

$$\begin{aligned} [\mathcal{H} + E]f &= [\partial_x^2 + A\partial_y^2 + (\partial_z + x\partial_y)^2 + E]f \\ &= \int dk ds dt \frac{|k|}{4\pi^2} D^k(s, t; \frac{x}{y}, \frac{z}{y}) \left[\frac{d^2}{ds^2} + E - k^2(A + s^2) \right] \tilde{f}^k(t, s) \\ &= 0. \end{aligned}$$

The ODE (Table 2a), which is just the Schrödinger equation for the simple harmonic oscillator, is solved by parabolic cylinder functions [22], of which the normalisable ones reduce to (Table 2b,c):

$$E = (2n + 1)|k| + k^2 A, n \in \mathbb{Z}^+;$$

$$\tilde{f}^k(t, s) = e^{-\frac{1}{2}|k|s^2} H_n(\sqrt{|k|}s) g(t), g(t) \in L^2(\mathbb{R}, dt) \text{ arbitrary.}$$

H_n are the Hermite polynomials, as seen in the wavefunction of the harmonic oscillator. Summing over the possible values of k , the spectrum of the left regular representation is $E \in [0, \infty)$.

The spectra of the left regular representations of the other Bianchi groups are obtained in same way. When decomposed into their actions on the irreps, the Laplacians give second order ODEs in the form of a Schrödinger equation, $f'' + (\epsilon - V(s))f = 0$ for some positive potential $V(s)$, and where $\epsilon \propto E$. Only the ODEs of Bianchi II, III and V are analytically solvable, but we can still find the spectrum of each Bianchi group as follows.²⁶ The potentials of Bianchi IV, VI ($q < 0$) and VII tend to zero as one of either $s \rightarrow \infty$ or $s \rightarrow -\infty$; the scattering theory of such one sided potentials predicts a continuum of solutions for $\epsilon > 0$ [28], and their degeneracies are given by the respective Plancherel measure. For Bianchi VI ($q > 0$), $V(s) \rightarrow \infty$ as $s \rightarrow \pm\infty$, and we expect a infinite discrete set of ‘bound’ solutions, the energies of which may be approximated by the WKB method, and are seen to tend smoothly to zero as $k \rightarrow 0$.²⁷ Thus for all Bianchi groups, the set of all eigenvalues in the left regular representation is a continuum $E \geq 0$ and we know the density of these states in all but one case (Bianchi VI, $q > 0$).

²⁶In all cases, insisting that the Hamiltonian commute with one of the right-invariant vector fields results in an exactly-solvable model.

²⁷The nature of the WKB method as an asymptotic expansion of the exact energy eigenvalues means that, as the first term in the expansion for Bianchi VI ($q > 0$) is an elliptic integral, the true values *cannot* be expressed in terms of simple functions.

5.2 Bianchi VIII — $SL(2, \mathbb{R})$

The Bianchi VIII algebra is semi-simple and isomorphic to the Lie algebra of the group $SL(2, \mathbb{R})$. The latter group has fundamental group $\pi_1(SL(2, \mathbb{R})) = \mathbb{Z}$ and its universal cover is peculiar in that it is not a matrix Lie group, having no finite-dimensional faithful representations. The unirreps are given and the Plancherel theorem derived in [29] and we simply quote the results here, in the notation of [30]. The irreps are given by the principal series $C_q^{(\tau)}$, with $0 \leq \tau < 1$ and $q > \frac{1}{4}$, the two discrete series D_l^\pm , with $l > 0$, and those in the exceptional domain, $E_q^{(\tau)}$ with $0 \leq \tau < 1$ and $\tau(1 - \tau) < q \leq \frac{1}{4}$. Denoting $\sigma = \sqrt{q - \frac{1}{4}} > 0$ the Plancherel decomposition is

$$f(g) = \int_0^\infty d\sigma \int_0^1 d\tau \operatorname{Re} \tanh[\pi(\sigma + i\tau)] \operatorname{tr}[(C_q^{(\tau)}(g))^* \tilde{f}(\sigma, \tau)] + \sum_{\pm} \int_{\frac{1}{2}}^\infty dl (l - \frac{1}{2}) \operatorname{tr}[(D_l^\pm(g))^* \tilde{f}^\pm(l)]. \quad (5.3)$$

Note that only the l irreps with $l > \frac{1}{2}$ appear.

Unfortunately the spectrum is not obtainable analytically, in general, as we can see by focussing our attention on the irreps with $\tau \in \{0, \frac{1}{2}\}$ (which are, in fact, the principal series irreps appearing in the Plancherel measure on $SL(2, \mathbb{R})$). The Lie algebra ($[X - Y, H] = -2(X + Y)$, $[H, X + Y] = 2(X - Y)$ and $[X + Y, X - Y] = -2H$, written in the $SU(1, 1)$ basis) admits $SO(2, 1)$ automorphisms, where the $SO(2)$ subgroup is $\{H, X + Y\}$. This simplifies the Hamiltonian to $\mathcal{H} = A(X - Y)^2 + B(X + Y)^2 + CH^2$, which on the principal irreps have action

$$(A + e + g \cos(4\phi))f'' - g(i\sigma + 2) \sin(4\phi)f' + (1 + \sigma^2)(g \cos(4\phi) - e)f = -Ef$$

where $e = \frac{1}{2}(B + C)$, $g = \frac{1}{2}(B - C)$ and $f(\phi) \in L^2(S^1, d\phi)$ and is odd in ϕ if $\tau = \frac{1}{2}$, and even otherwise. $\phi \in (-\pi, \pi]$. When the ODE is put in normal form, the first order term in the WKB approximation to E reduces to an elliptic integral, implying that the ('all-order') exact result is not a simple function of exponentials, logarithms and powers. However, in the symmetric limit $g \rightarrow 0$ (*i.e.* $B \rightarrow C$), the ODE and spectrum for these irreps tend to those of a free particle on S^1 .

5.3 Bianchi IX — $SU(2)$

Finally, we turn to the archetypical rigid body, starting our discussion with the more familiar case of $SO(3)$, before addressing its universal cover, $SU(2)$. Examples abound in nature of systems whose configuration is described by an arbitrary rotation, and whose Hamiltonian is also rotation invariant. Indeed, via spinning tops and the rotational spectroscopy of gases, both the classical and quantum dynamics of the rigid body have been well studied. We calculate the spectrum of the quantum mechanical rigid body using the formalism of the previous Sections, and show it to be consistent with known results.

The $SO(3)$ group coordinates are typically given in terms of Euler angles α, β, γ (we use here the z-y-z 'active' convention [31]), where $\alpha, \gamma \in [0, 2\pi]$, $\beta \in [0, \pi]$. The Haar

measure is $\frac{d\alpha d(\cos\beta)d\gamma}{8\pi^2}$. The unirrep matrices are the Wigner D-matrices $D_{mk}^j(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma})$, labelled by $j \in \{0, 1, 2, \dots\}$, where $m, k \in \{-j, -j+1, \dots, j\}$ are the indices of the resulting $(2j+1) \times (2j+1)$ matrix. The explicit form of D_{mk}^j , and the composition of two sets of Euler angles are, in general, complicated. We refer the reader to [31], from which we quote the following results.

The Wigner D-matrices form, respectively, a Schur orthogonality relation on the irreps, and a completeness relation on the group coordinates:

$$\int \frac{d\alpha d\cos\beta d\gamma}{8\pi^2} \overline{D}_{km}^j(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) D_{m',k'}^{j'}(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) = \frac{1}{2j+1} \delta^{jj'} \delta^{mm'} \delta^{kk'},$$

$$\sum_{j,m,k} (2j+1) D_{mk}^j(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) \overline{D}_{km}^{j'}(\frac{\alpha'}{\beta'}, \frac{\gamma'}{\gamma'}) = 8\pi^2 \delta(\alpha - \alpha') \delta(\cos\beta - \cos\beta') \delta(\gamma - \gamma'),$$

where $\sum_{j,m,k} = \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} \sum_{k=-j}^{+j}$. Using the same notation, one may decompose the left regular representation via the decomposition of functions $f \in L^2(SO(3))$:

$$f(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) = \sum_{j,m,k} (2j+1) D_{mk}^j(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) \tilde{f}_{mk}^j,$$

$$\tilde{f}_{mk}^j = \int \frac{d\alpha d\cos\beta d\gamma}{8\pi^2} \overline{D}_{mk}^j(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}) f(\frac{\alpha}{\beta}, \frac{\gamma}{\gamma}).$$

Note that, for a given value of j , there are $(2j+1)$ vectors \tilde{f}_{mk}^j (labelled by $k \in \{-j, -j+1, \dots, j\}$) in the decomposition of f which, under the group action, transform under the j th irrep of $SO(3)$. This is a manifestation of a general result for compact groups, the Peter-Weyl theorem [16, 32], where an irrep of dimension d features d times in the Hilbert space of the left regular representation. The Plancherel measure is thus $2j+1$ times the counting measure on $j \in \{0, 1, 2, \dots\}$. These $2j+1$ copies are degenerate in energy in the rotationally-invariant systems we consider, but are easily observed, *e.g.* in the rotation spectra of molecules in the presence of an electric field [8].

Having obtained the decomposition of the regular representation, we again proceed to construct the Hamiltonian. The structure constants of the Lie algebra are ϵ_{ijk} and the automorphism group is immediately seen to be $SO(3)$. We may thus pick the special orthogonal transformation necessary to diagonalize the real symmetric matrix of the Hamiltonian coefficients, leaving $\mathcal{H} = AJ_1^2 + BJ_2^2 + CJ_3^2$ for some (positive) eigenvalues A, B and C .²⁸ The residual automorphisms are the group of order 4 generated by $\{e^{\pi J_i}\}$, isomorphic to the Klein group, whose effect is to change the sign of two of the three J_i . The Klein group is Abelian, with 4, 1-d irreps. The permuting automorphisms are the group of order 24 generated by $\{e^{\frac{\pi}{2} J_i}\}$, isomorphic to the symmetry group of a cube, with 2 1-d, 1 2-d, and 2 3-d irreps.

The action of \mathcal{H} on the left regular representation is via the self-adjoint right-invariant

²⁸In other words, we can pick principal axes with respect to which the moment of inertia tensor is diagonal.

vectors

$$\begin{aligned} J_1^R &= (\cot \beta \cos \alpha \partial_\alpha + \sin \alpha \partial_\beta - \frac{\cos \alpha}{\sin \beta} \partial_\gamma), \\ J_2^R &= (\cot \beta \sin \alpha \partial_\alpha - \cos \alpha \partial_\beta - \frac{\sin \alpha}{\sin \beta} \partial_\gamma), \\ J_3^R &= -\partial_\alpha. \end{aligned}$$

Making use of the properties

$$\begin{aligned} J_3^R D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &= im D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \text{ and} \\ (J_1^R \pm iJ_2^R) D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} &= i\sqrt{(j \pm m)(j \mp m + 1)} D_{m \mp 1, k}^j \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \end{aligned}$$

we find the action of \mathcal{H} on the irreps to be

$$\mathcal{H} f \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = - \sum_{j, m, m', k} (2j + 1) D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} [AJ_1^{j^2} + BJ_2^{j^2} + CJ_3^{j^2}]_{mm'} \tilde{f}_{m'k}^j,$$

where J_i^j is the canonical form of the angular momentum generator in the j th matrix rep. Finding the spectrum of the left regular rep then amounts to diagonalizing a real symmetric matrix, whose elements are linear combinations of A , B and C , for each j . This diagonalization is simplified by first decomposing with respect to the Klein group of residual automorphisms. Thus, the 3-d $j = 1$ irrep decomposes into one copy of each of the non-trivial irreps of the Klein group and the spectrum is found to be $E = \frac{A+B}{2}, \frac{B+C}{2}$, or $\frac{C+A}{2}$, with a 3-fold degeneracy of each.²⁹ For $j = 2$, the 5-d irrep decomposes into one copy of each of the non-trivial irreps of the Klein group and two copies of the trivial irrep. By Schur's lemmata, the latter two can mix and finding the spectrum requires us to diagonalize a 2×2 matrix. Unfortunately, no closed form is available for the spectrum at arbitrary j . However, thus computed, these energies (and indeed the degeneracies, measured by the strength of spectral features) are precisely those observed in molecular microwave spectra [8, 33], where A , B and C are reciprocals of twice the principal moments of inertia of the molecule.

The group $SO(3)$ furnishes us also with the archetype of a coset representation. This is seen, for example, in a linear molecule (such as CO or CO₂, but not H₂O) where the system's configurations under a rotation about a given axis are equivalent. The orientation of the linear molecule is then instead given by the two angles of $SO(3)/SO(2) \cong S^2$.

We choose α and β to be the two remaining coordinates, whereas γ parametrises the coset $SO(2)$. We can make a closed, $SO(3)$ -invariant volume form on the group, namely $(\cos \gamma \sin \beta d\alpha - \sin \gamma d\beta) \wedge (-\sin \gamma \sin \beta d\alpha - \cos \gamma d\beta) = -\sin \beta d\alpha \wedge d\beta$, from two left-invariant one forms. Let the measure be $\frac{\sin \beta d\alpha d\beta}{4\pi}$.

The matrices $D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}$ are proportional to the spin-weighted spherical harmonics on S^2 , parametrised by spherical polar coordinates (β, α) .³⁰ For each k , they form a complete

²⁹The $j = 1$ irrep 'decomposes' into one of the 3-d irreps of the cube group of permuting automorphisms and so we find that we can obtain all of the energy eigenvalues from any one by permutations of A , B , and C .

³⁰In the simplest case $k = 0$, $D_{m0}^l \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = \sqrt{\frac{4\pi}{2l+1}} Y_{l,m}(\beta, \alpha)$, the usual spherical harmonic functions.

set of functions on S^2 , leading to the relation

$$\sum_{j,m,k} D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \bar{D}_{km}^j \begin{pmatrix} \alpha' \\ \beta' \\ 0 \end{pmatrix} = 4\pi \delta(\cos \beta - \cos \beta') \delta(\alpha - \alpha').$$

The coset representation decomposes thus:

$$f \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \sum_{j,m,k} D_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \tilde{f}_{km}^j,$$

$$\tilde{f}_{mk}^j = \int \frac{d \cos \beta d\alpha}{4\pi} \bar{D}_{mk}^j \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} f \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Note that the extra degeneracy of the higher dimension irreps is gone. The coset representation contains each j irrep exactly once and the Plancherel measure is now just the counting measure on $j \in \{0, 1, 2, \dots\}$.

The projection to the coset space is given by $\pi \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, which gives $\pi^*(\partial_\alpha) = \partial_\alpha$, $\pi^*(\partial_\beta) = \partial_\beta$ and $\pi^*(\partial_\gamma) = 0$. The linear rigid rotor Hamiltonian is given by

$$\mathcal{H} = A((J_1^R)^2 + (J_2^R)^2 + (J_3^R)^2), \quad (5.4)$$

where J_i^R are now the pushforwards to S^2 of the right-invariant vector fields on $SO(3)$. We find

$$\mathcal{H} = A \left[\partial_\beta^2 + \cot \beta \partial_\beta + \frac{1}{\sin^2 \beta} \partial_\alpha^2 \right]. \quad (5.5)$$

The spectrum is that of the Laplacian in spherical polar coordinates. This can equally be seen via the decomposition, which contains a piece

$$\mathcal{H} f \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \supset \sum_{j,m} -[Aj(j+1)] D_{m0}^j \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \tilde{f}_{0m}^j,$$

which simply multiplies each vector \tilde{f}_{0m}^j by $Aj(j+1)$. The action on the $\tilde{f}_{km}^j, k \neq 0$ is more complicated, but does not yield eigenfunctions. Thus the possible states have energy $Aj(j+1)$ and degeneracy $2j+1$ for each $j \in \{0, 1, 2, \dots\}$, as seen in the heat capacity of molecular hydrogen.

The results for the case of $SU(2)$ are analogous and easily obtained. The irreps of $SU(2)$ contain, in addition to those of $SO(3)$, the irreps with $j \in \{\frac{1}{2}, \frac{3}{2}, \dots\}$. The Plancherel measure is $2j+1$ times the counting measure on $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, by Peter-Weyl. The groups of residual and permuting automorphisms are the preimages under the projection map $\pi : SU(2) \rightarrow SO(3)$ of the corresponding groups for $SO(3)$. The residual group of automorphisms of $SU(2)$ is isomorphic to the (non-Abelian) quaternion group of order 8 [34] and features, in addition to the 4 1-d irreps of the Klein group, a 2-d irrep.³¹ The $j = \frac{1}{2}$ irrep of $SU(2)$ decomposes into the 2-d irrep. The energy eigenvalue $\frac{A+B+C}{4}$ is 4-fold degenerate, with a 2-fold degeneracy coming from rotational invariance via the Plancherel measure and another 2-fold degeneracy that can be understood as a Kramers degeneracy arising from time-reversal invariance.

³¹The permuting group is of order 48 and has, in addition to the irreps of the cube group, 2 2-d irreps and 1 4-d irrep.

6 Discussion

We have given a formulation of the quantum mechanics of a particle on a Lie group manifold G , whose classical motions follow geodesics of an arbitrary one-sided-invariant metric. We have also shown how this formulation can sometimes be extended to motion on a coset space G/H . In many cases, the system turns out to be exactly solvable using standard methods of harmonic analysis and representation theory; we have also described the various obstacles to doing so.

We have studied all Lie algebras up to dimension three as examples. The prototypical example of a rigid body is somewhat atypical, since it is the only case in $d \leq 3$ that is simply-connected and also compact, with finite-dimensional unitary, irreducible representations. In non-compact cases, the spectrum is continuous and (at least in the cases where we are able to compute it) takes the simple form $E \geq 0$. This does not mean, of course, that the physics of these different systems is the same. Indeed, what is important for the physics is not just the spectrum of eigenvalues, but also the degeneracy of those eigenvalues — the density of states, in physicists' language. The degeneracies, given by the Plancherel measure in each case, are markedly different and we expect that they will lead to markedly different physics.

It would be of interest to try to reproduce our results using path integral methods.³² We expect this to be non-trivial, even in the case where G is compact, where the quantum mechanics on G/H can be described by the non-linear sigma model in 0+1-d with target space G/H . Indeed, just as for the quantum mechanics of a free particle on \mathbb{R}^n (where the generalised eigenfunctions $e^{ik \cdot x}$ correspond to a uniform probability density for finding the particle anywhere in \mathbb{R}^n), the energy eigenstates for general G/H are delocalised (in the sense that the moduli-squared of the generalised eigenfunctions do not tend to 0 at infinity) in all cases where we have obtained their explicit form. This is in accord with our earlier observation that attempting to evaluate the path integral by assuming that the particle is localised leads to a contradiction. So, evaluation of the path integral will need to take account of global considerations.

Whilst these exactly-solvable, free-particle models may seem somewhat esoteric, one should bear in mind that the notion of dynamics on a manifold with invariance under a Lie group action is a rather general one in physics. We anticipate that the exactly-solvable models provide a useful starting point for doing quantum-mechanical perturbation theory for dynamical systems of this type, generalising the use of the harmonic oscillator and free particle. Indeed, the rigid body can be used as the starting point for a perturbative study centrifugal distortions of polyatomic molecules [8].

The quantum mechanical systems of a rigid body and a free particle, corresponding to $G = SO(3)$ and $G = \mathbb{R}^n$ are, of course, ubiquitous in physics and chemistry. Do there exist dynamical systems to which we can apply the results for other G ? To give just one example, it has been suggested that the dynamics of a certain ellipsoidal vortex solution in a fluid can, under certain restrictions, be modelled as a system moving on $SO(2, 1)$ [36].

³²For the rigid body case, see [35].

The most interesting application, from our point of view, is to the quantisation of a perfect, incompressible fluid moving on a manifold M . This corresponds to $G = \text{SDiff}(M)$, which is infinite dimensional. Two ways to proceed suggest themselves. The first is to consider an M which admits a complex structure, for which one can consider a simplified toy model of a fluid, in which one insists that the fluid maps be not just smooth, but rather be analytic. $\text{SDiff}(M)$ is then replaced by the group of (volume-preserving) conformal holomorphisms of M , which is finite-dimensional and can be quantised using the methods described here. These simplified systems (which we propose to call ‘analytic fluids’) may present a useful intermediate step on the road to quantising a genuine incompressible fluid.

The second approach is to try to define the infinite-dimensional group $\text{SDiff}(M)$, for some M , as a limit of a sequence of finite-dimensional groups. For the case where M is either a 2-sphere or a 2-torus (which bore so much fruit for Arnold at the classical level), it has been observed [37, 38] that the structure constants of the Lie algebra of $\text{SDiff}(M)$ can be obtained (in a specific basis) as a convergent sequence of those of $SU(N)$, as $N \rightarrow \infty$. Unfortunately, it is known [39] that these algebras are not isomorphic, so it is unclear whether this observation can be used in a mathematically-sound way.

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A Weyl sequences for the $\text{Aff}(\mathbb{R})^+$ group

We wish to show that the spectrum of the differential operator $L = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - x^2$, in the function space $L^2(\frac{dx}{x}, \mathbb{R}^+)$, is indeed $E \in [0, \infty)$, as stated in §4.3. To do so, for each possible eigenvalue $E \equiv \nu^2$ we construct a Weyl sequence out of the corresponding generalised eigenfunction $\tilde{K}_\nu(x) \equiv K_{i\nu}(x)$. A Weyl sequence is a set of functions $\{w_n : w_n \in L^2(\frac{dx}{x}, \mathbb{R}^+), \forall n \in \mathbb{N}\}$, all of which have unit L^2 norm ($\|w_n\| = 1$), and for which $\|(L + \nu^2)w_n\| \rightarrow 0$ as $n \rightarrow \infty$. It will suffice to show that

$$w_n = A_n x^{\frac{1}{n}} \tilde{K}_\nu(x), \tag{A.1}$$

(where A_n is the appropriate prefactor to ensure $\|w_n\| = 1, \forall n \in \mathbb{N}$) satisfies these requirements for any $\nu \geq 0$, and hence $E \in [0, \infty)$ by exhaustion.

To compute the norm of the w_n , we need to evaluate the integrals

$$F(\lambda, \nu) = \int_0^\infty dx x^{-\lambda} |\tilde{K}_\nu(x)|^2.$$

The properties of $K_\mu(x)$ are such that it is real when μ and x are both real and positive [22, 10.45]. Moreover, $K_\mu(x)$, when $x \neq 0$, is entire in μ . We may therefore assert, using

Schwartz's reflection principle, that $K_\mu(x)^* = K_{\mu^*}(x)$ for real x . Thus $\tilde{K}_\nu(x)^* = \tilde{K}_{-\nu}(x)$, and

$$F(\lambda, \nu) = \int_0^\infty dx x^{-\lambda} K_{i\nu}(x) K_{-i\nu}(x) = \frac{1}{4} \sqrt{\pi} \frac{\Gamma(\frac{1}{2} - \frac{\lambda}{2})}{\Gamma(1 - \frac{\lambda}{2})} \Gamma(\frac{1}{2} - \frac{\lambda}{2} + i\nu) \Gamma(\frac{1}{2} - \frac{\lambda}{2} - i\nu) \text{ where } \lambda < 1, \quad (\text{A.2})$$

from [40, 6.576.4]. The choice $A_n^2 = F(1 - \frac{2}{n}, \nu)^{-1}$ guarantees $\|w_n\| = 1$. To evaluate $\|(L + \nu^2)w_n\|$, first use integration by parts and the Bessel function's ODE to write the following integrals in terms of F :

$$\begin{aligned} \int_0^\infty dx x^{-\lambda} [K'_{i\nu}(x) K_{-i\nu}(x) + K_{i\nu}(x) K'_{-i\nu}(x)] &= \lambda F(\lambda + 1, \nu) \text{ where } \lambda < 0, \\ \int_0^\infty dx x^{-\lambda} K'_{i\nu}(x) K'_{-i\nu}(x) &= [\nu^2 + \frac{1}{2}(\lambda + 1)^2] F(\lambda + 2, \nu) - F(\lambda, \nu) \text{ where } \lambda < -1. \end{aligned}$$

Then, by writing

$$A_n^{-1}(L + \nu^2)w_n = \frac{2}{n} x^{1+\frac{1}{n}} K'_{i\nu}(x) + \frac{1}{n^2} x^{\frac{1}{n}} K_{i\nu}(x), \quad (\text{A.3})$$

we obtain

$$\begin{aligned} \|(L + \nu^2)w_n\| &= A_n^2 \frac{4}{n^2} \int_0^\infty dx x^{1+\frac{2}{n}} K'_{i\nu}(x) K'_{-i\nu}(x) \\ &\quad + A_n^2 \frac{2}{n^3} \int_0^\infty dx x^{\frac{2}{n}} [K'_{i\nu}(x) K_{-i\nu}(x) + K_{i\nu}(x) K'_{-i\nu}(x)] \\ &\quad + A_n^2 \frac{1}{n^4} \int_0^\infty dx x^{-1+\frac{2}{n}} K_{i\nu}(x) K_{-i\nu}(x) \\ &= \frac{4\nu^2}{n^2} + \frac{5}{n^4} - \frac{4}{n^2} \frac{F(-1 - \frac{2}{n}, \nu)}{F(1 - \frac{2}{n}, \nu)}. \end{aligned} \quad (\text{A.4})$$

The ratio of gamma functions implicit in the last line simplifies to

$$\|(L + \nu^2)w_n\| = \frac{4\nu^2 n^3 + 5n + 2}{(2+n)n^4} \xrightarrow{n \rightarrow \infty} 0. \quad (\text{A.5})$$

We have therefore found a Weyl sequence for every eigenvalue $E \in [0, \infty)$.

References

- [1] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, *Ann. Inst. Fourier* **16** (1966) 316–361.
- [2] B. Khesin and R. Wendt, *The geometry of infinite-dimensional groups*, vol. 51 of *A Series of Modern Surveys in Mathematics*. Springer-Verlag, 2009.
- [3] H. B. G. Casimir, *Rotation of a Rigid Body in Quantum Mechanics*. J. B. Wolters, 1931.
- [4] M. S. Marinov and M. V. Terentyev, *Dynamics on the group manifold and path integral*, *Fortschritte der Physik* **27** (1979), no. 11-12 511–545.
- [5] N. Krausz and M. S. Marinov, *Exact evolution operator on noncompact group manifolds*, *Journal of Mathematical Physics* **41** (2000), no. 8.

- [6] S. Endlich, A. Nicolis, R. Rattazzi, and J. Wang, *The quantum mechanics of perfect fluids*, *JHEP* **1104** (2011) 102, [[arXiv:1011.6396](https://arxiv.org/abs/1011.6396)].
- [7] B. Gripiaios and D. Sutherland, *Quantum field theory of fluids*, *Phys. Rev. Lett.* **114** (2015), no. 7 071601, [[arXiv:1406.4422](https://arxiv.org/abs/1406.4422)].
- [8] W. Gordy and R. Cook, *Mircrowave molecular spectra*. Wiley, 1984.
- [9] A. F. M. Ter Elst and D. W. Robinson, *Elliptic operators on lie groups*, *Acta Applicandae Mathematica* **44** no. 1 133–150.
- [10] L. Hörmander, *Hypoelliptic second order differential equations*, *Acta Math.* **119** (1967) 147–171.
- [11] N. J. Vilenkin and A. U. Klimyk, *Simplest Lie groups, special functions and integral transforms*, vol. I of *Representation of Lie groups and special functions*. Springer, 1991.
- [12] L. A. Santaló, *Integral geometry and geometric probability*, vol. 1 of *Encyclopedia of mathematics and its applications*. Addison-Wesley, 1976.
- [13] A. Y. Morozov, A. Perelomov, A. Rosly, M. Shifman, and A. Turbiner, *Quasi-exactly-solvable quantal problems: One-dimensional analogue of rational conformal field theories*, *International Journal of Modern Physics A* **05** (1990), no. 04 803–832, [<http://www.worldscientific.com/doi/pdf/10.1142/S0217751X90000374>].
- [14] Z. Avetisyan and R. Verch, *Explicit harmonic and spectral analysis in Bianchi IVII-type cosmologies*, *Class. Quant. Grav.* **30** (2013) 155006, [[arXiv:1212.6180](https://arxiv.org/abs/1212.6180)].
- [15] A. A. Kirillov, *Lectures on the orbit method*, *Graduate Studies in Mathematics* **64** (2004).
- [16] F. Peter and H. Weyl, *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe*, *Math. Ann.* **97** (1927) 737–755.
- [17] B. N. Currey, *Explicit orbital parameters and the plancherel measure for exponential lie groups*, *Pacific J. Math.* **219** (2005) 97–138.
- [18] D. J. Fisher, R. J. Gray, and P. E. Hydon, *Automorphisms of real Lie algebras of dimension five or less*, *J. Phys. A* **46** (2013), no. 22 225204, [[arXiv:1303.3376](https://arxiv.org/abs/1303.3376)].
- [19] A. A. Kirillov, *Introduction to the theory of representations and noncommutative harmonic analysis*, *Encyclopaedia of Mathematical Sciences* **22** (1994) 1.
- [20] I. Khalil, *Sur l’analyse harmonique du groupe affine de la droite*, *Studia Mathematica* **51** (1974), no. 2 139–167.
- [21] M. Duflo and C. C. Moore, *On the regular representation of a nonunimodular locally compact group*, *Journal of Functional Analysis* **21** (1976), no. 2 209 – 243.
- [22] NIST, “Digital library of mathematical functions.” <http://dlmf.nist.gov/>.
- [23] L. Bianchi, *Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti*, *Soc. Ital. Sci. Mem. di Mat.* **11** (1898) 267.
- [24] L. Bianchi, *On the three-dimensional spaces which admit a continuous group of motions*, *Gen. Rel. Grav.* **33** (2001) 2171–2253.
- [25] H. Donnelly, *On the essential spectrum of a complete riemannian manifold*, *Topology* **20** (1981) 1–14.
- [26] H. P. McKean, *An upper bound to the spectrum of δ on a manifold of negative curvature*, *J. Differential Geom.* **4** (1970), no. 3 359–366.

- [27] A. Harvey, *Automorphisms of the Bianchi model Lie groups*, *J. Math. Phys.* **20** (1979) 251.
- [28] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Advanced Mathematical Methods for Scientists and Engineers. Springer, 1999.
- [29] L. Pukanszky, *The Plancherel formula for the universal covering group of $SL(\mathbb{R}, 2)$* , *Math. Ann.* **156** (1964) 96–143.
- [30] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, *Annals of Mathematics* **48** (1947), no. 3 pp. 568–640.
- [31] L. C. Biedenharn, J. D. Louck, and P. A. Carruthers, *Angular momentum in quantum physics: theory and application*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.
- [32] A. Barut and R. Raczka, *Theory of group representations and applications*. World Scientific, 1986.
- [33] H. W. Kroto, *Molecular rotation spectra*. Dover phoenix editions. Dover Publications, 2003.
- [34] J. S. Dowker and D. F. Pettengill, *The quantum mechanics of the ideal asymmetric top with spin (frozen mixmaster universe)*, *J. Phys. A* **7** (1974) 1527.
- [35] L. Schulman, *A path integral for spin*, *Phys. Rev.* **176** (Dec, 1968) 1558–1569.
- [36] P. J. M. S. P. Meacham and G. R. Flierl, *Hamiltonian moment reduction for describing vortices in shear*, *Phys. Fluids* **9** (1997) 2310.
- [37] J. Goldstone, *Unpublished*, .
- [38] J. Hoppe, *Quantum theory of a relativistic surface*, *Elem. Part. Res. J. (Kyoto)*.
- [39] M. Bordemann, J. Hoppe, P. Schaller, and M. Schlichenmaier, *$GL(\infty)$ and geometric quantization*, *Commun. Math. Phys.* **138** (1991) 209–244.
- [40] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*. Elsevier Science, 2014.

Bianchi	A.k.a	(a) $F(z)$	(b) \tilde{k}_0 , cross-section	
II	Heisenberg	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} 0 \\ k \end{pmatrix}, k \in \mathbb{R}/\{0\}$	
III	$\text{Aff}(\mathbb{R}) \times \mathbb{R}$	$\begin{pmatrix} e^z & 0 \\ 0 & 1 \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} k_2 \\ k \end{pmatrix}, (k, k_2) \in \mathbb{R} \times \{-1, +1\}$	
IV		$\begin{pmatrix} e^z & 0 \\ ze^z & e^z \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} k \\ k_2 \end{pmatrix}, (k, k_2) \in \mathbb{R}_{+0} \times \{-1, +1\}$	
V		$\begin{pmatrix} e^z & 0 \\ 0 & e^z \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} \cos k \\ \sin k \end{pmatrix}, k \in \mathbb{R}/2\pi\mathbb{Z}$	
VI $q \in (-1, 0)$		$\begin{pmatrix} e^z & 0 \\ 0 & e^{-qz} \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} \cos k \\ \sin k \end{pmatrix}, k \in \mathbb{R}/2\pi\mathbb{Z}$	
VI $q \in (0, 1)$	<i>e.g.</i> Poincaré		$\tilde{k}_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{k_2} \begin{pmatrix} 1 \\ k \end{pmatrix}, (k, k_2) \in \mathbb{R}_{+0} \times \{0, 1, 2, 3\}$	
VII $p \in \mathbb{R}$	<i>e.g.</i> Euclidean	$e^{pz} \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}$	$\tilde{k}_0 = \begin{pmatrix} k \\ 0 \end{pmatrix}, k \in (-e^{\pi p}, -1] \cup [1, e^{\pi p})$	
Bi.	(c) Plancherel	(d) Automorphs	(e) Residual automorphs	(f) \mathcal{H}
II	$ k $	$\begin{pmatrix} a & 0 & c \\ d & aj - cg & f \\ g & 0 & j \end{pmatrix}$	$\left\{ \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$	$X^2 + AY^2 + Z^2$
III	1	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$X^2 + Y^2 + \alpha\{X, Y\} + AZ^2$
IV	$1 + k$	$\begin{pmatrix} a & 0 & c \\ d & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$A(X^2 + Y^2) + BZ^2$
V	1	$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$	$X^2 + Y^2 + AZ^2$
VI	$\begin{cases} \cos^2 k - q \sin^2 k \\ q^{k_2 \bmod 2} \end{cases}$	$\begin{pmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$X^2 + Y^2 + \alpha\{X, Y\} + AZ^2$
VII	$ k $	$\begin{pmatrix} a & b & c \\ -b & a & f \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$X^2 + AY^2 + BZ^2$

Table 1: The properties of the Bianchi groups II–VII. Results (a)–(d) from [14].

(a) $F(z)$ defines the group composition $(x, y, z) \circ (x', y', z') = (A_x, A_y, z + z')$, where $\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + F(z) \begin{pmatrix} x' \\ y' \end{pmatrix}$.

(b) The irreps of the group, labelled by k and sometimes also the discrete k_2 . $D^{k, (k_2)}(s, t; \frac{x}{y}) = \exp(i\tilde{k}_0^T F(s)\tilde{x})\delta(s + z - t)$, where $\tilde{x} = (\frac{x}{y})$.

(c) The Plancherel measure on the irreps. From the k dependent part of $|\frac{\partial F^T(s)\tilde{k}}{\partial s \partial k}|$.

(d) The automorphisms M_{ij} of the Lie algebra, basis $X_j = (X, Y, Z)^T$, where $[M\xi, M\eta] = M[\xi, \eta], \forall \xi, \eta \in \mathfrak{g}$. All real values of the parameters are permitted, provided the matrix is non-singular. See also [27].

(e) The subgroup of automorphs that leave the simplified Hamiltonian invariant.

(f) The simplified Hamiltonian.

Bianchi	(a) ODE	DLMF
II	$f'' + (E - A(k^2 + s^2))f = 0$	12.2.4
III	$Af'' + (E - e^{2s} - k^2 - 2\alpha k_2 k e^s)f = 0$	13.14
IV	$Bf'' + (E - Ae^{2s}(1 + k^2 + 2k_2 k s + k^2 s^2))f = 0$	-
V	$Af'' + (E - e^{2s})f = 0$	10.45
VI $q \in (-1, 0)$	$Af'' + (E - \cos^2 k e^{2s} - \sin^2 k e^{-2qs} - 2\alpha \cos k \sin k e^{(1-q)s})f = 0$	-
VI $q \in (0, 1)$	$Af'' + (E - e^{2s} - k^2 e^{-2qs} - 2\alpha k e^{(1-q)s})f = 0$	-
VII $p \in \mathbb{R}$	$Bf'' + (E - k^2 e^{2ps}(\cos^2 s + A \sin^2 s))f = 0$	-

Bianchi	(b) Eigenvalues	(c) Solutions
II	$E = (2n+1) k + k^2 A, n \in \mathbb{Z}^+$	$e^{-\frac{1}{2} k s^2} H_n(\sqrt{ k }s)$
III	$E \geq k_1^2$, and, $(n + \frac{1}{2})\sqrt{A} + \sqrt{k^2 - E} = -\alpha k_2 k, n \in \mathbb{Z}^+$	$e^{\frac{1}{2}s} W_{-\alpha k_1 k_2 A^{-\frac{1}{2}}, A^{-\frac{1}{2}} i \sqrt{E - k_1^2}}(2A^{-\frac{1}{2}} e^s),$ and $e^{\frac{1}{2}s} W_{-\alpha k_1 k_2 A^{-\frac{1}{2}}, A^{-\frac{1}{2}} \sqrt{k_1^2 - E}}(2A^{-\frac{1}{2}} e^s),$ $e^{\frac{1}{2}s} M_{-\alpha k_1 k_2 A^{-\frac{1}{2}}, A^{-\frac{1}{2}} \sqrt{k_1^2 - E}}(2A^{-\frac{1}{2}} e^s)$
IV	$E \geq 0$	-
V	$E \geq 0$	$\tilde{K}_{\frac{E}{A}}(A^{-\frac{1}{2}} e^s)$
VI $q \in (-1, 0)$	$E \geq 0$	-
VI $q \in (0, 1)$	E discrete	-
VII $p \in \mathbb{R}$	$E \geq 0$	-

Table 2:

- (a) The ODEs coming from the action of $[\mathcal{H} + E]$ on the irreps. $f' \equiv \frac{d}{ds} f$ etc.
- (b) The values of E for the irrep $k, (k_2)$.
- (c) The analytic solutions $f(s)$ of the ODE (where possible) in the notation of the corresponding DLMF reference. $\tilde{f}^{k, (k_2)}(s, t) = f(s)g(t)$, where $g(t) \in L^2(\mathbb{R}, dt)$ is arbitrary.