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On Generalized Dispersion Relations

and

Meson-Nucleon Scattering

by

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Preface

The work described in sections III through IX of this thesis is original except where credit is given in the notes. The work presented was done while the author was at Trinity College, Cambridge, with the exception of parts of sections V and VI which were done while the author was at Harvard University. The author would like to take this opportunity to thank Dr. Abdus Salam for his inspiration and encouragement.

Contents

Section I: Introduction

Dispersion relations for the scattering amplitude of a relativistic field theory were introduced in a paper by Gell-Mann, Goldberger, and Thirring.[1] These authors used the causality requirement, that all observable operators commute for spacelike separations, to derive a dispersion relation for the amplitude for forward photon scattering. This relation is an expression for the real part of the amplitude in terms of an integral over the imaginary part of the amplitude. Since the imaginary part of the forward scattering amplitude is related by the optical theorem to the total cross section, the dispersion relation expresses the dispersive part of the scattering in terms of the absorption of particles out of the incident beam. Similar relations are known throughout physics, for all linear processes independent of the origin of time, in which the effect, or out-put, cannot precede the cause, or input.[2] The field theoretic causality condition in terms of commutators is just the relativistic generalization of this condition for quantum mechanics.

Karplus and Rudermann[3] were the first to apply these ideas to meson-nucleon scattering. They showed that if dispersion relations were asserted for the scattering of particles of finite mass, the relations could be used to analyse the scattering data to indicate that the sharp peak in the cross section could only be associated with a change in the sign of the real part of the amplitude and thus to a resonant behavior on the part of the phase shifts. Unfortunately they did not take into account correctly the isotopic spin of the particles. Goldberger[4] developed an argument to derive dispersion relations for the forward scattering of particles with mass from the general principles of field theory. Goldberger, Miyazawa, and Oehme[5] developed the specific relations for pion-nucleon scattering and these were used to analyse the experimental data by Anderson, Davidon, and Kruse.[6]

Goldberger's arguments turned out to be easily extended to the general angle scattering amplitude on the energy shell. This was done by Salam[7] for the case of scalar particles and Salam and Gilbert[8] for the meson-nucleon case. Similar generalizations were made by other groups[9] and were published by Capps and Takeda[10] for the general case and by Oehme[11] for the spin-flip amplitude and for the derivative of the amplitude with respect to angle in the forward direction. More rigorous derivations have been developed by Symanzik[12] and by Bogoliubov and co-workers.[13]

In this thesis we shall spend several sections developing the form of the dispersion relations for meson-nucleon scattering. We shall then make several applications of these relations. We shall apply the spin-flip relations to the resolution of the ambiguity between the Yang and the Fermi phase shifts. We shall develop a new form for dispersion relations and derive several sum rules expressing the coupling constant in terms of the phase shifts. This leads to a new evaluation of the coupling constant, and to a relation between the low energy behavior of the phase shifts and the asymptotic scattering cross section. By using the derivatives of the dispersion relations with respect to angle, equations are derived for the p-wave phase shifts. These equations are used to derive a relativistic generalization of the effective range formula for the 33-phase shift, which is shown to agree better with the high energy experimental data than does the Chew-Low relationship.[14] An approximation technique is developed to estimate the small p-phases and used to discuss the s- and d-phases.

We turn now to a general discussion of the dispersion relations for the scattering amplitude for non-forward scattering. These relations are equivalent to statements that the scattering amplitude on the energy shell is an analytic function of the energy in the upper half of the complex energy plane with a behavior at infinity no more singular than some finite power. If this is true, the real and imaginary parts of the amplitude will be related by an integral along the real axis derivable from Cauchy's integral theorem. Goldberger[4] argued that the scattering amplitude in the forward direction obeys a relation of the form:

$$\mathrm{Re}\, f(\bar{\omega}) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \bar{\omega}}\, \mathrm{Im}\, f(\omega) \tag{1.1}$$

Here ω is the laboratory energy of the meson. The general energy-shell amplitude is a function of both the energy and the angle of scattering. Gell-Mann, Goldberger, and Thirring[1] pointed out that a relation of the form

$$\mathrm{Re}\, f(\bar{\omega}, \cos\theta) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \bar{\omega}}\, \mathrm{Im}\, f(\omega, \cos\theta) \tag{1.2}$$

was not possible since such a relation could be expanded in a Legendre expansion yielding identical relations for each phase shift and thus would predict the same energy dependence at threshold for all the phases. The correct form of the relations can be found by invoking relativistic invariance. If we consider the scattering of two scalar particles which have initially four-momenta p' and q' and scatter to a state of momenta p and q (where $p^2 = p'^2 = \kappa^2$, $q^2 = q'^2 = \mu^2$, and we use a timelike metric), then the T-matrix element is, as given in appendix II,

$$T = - \langle p, q \mid j(0) \mid p' \rangle \tag{1.3}$$

The general form of this T-matrix element is a complex scalar function of three scalar variables, since the particles are real,

$$T = T(pq,\ pp',\ p'q) \tag{1.4}$$

If we impose the condition that this T-matrix element be on the energy shell, then $p + q = p' + q'$ which implies that

$$(p + q - p')^2 = q'^2 \qquad \text{or} \qquad p'q = pq - pp' + \kappa^2 \tag{1.5}$$

We eliminate one of the variables in the T-matrix element to get

$$T = T(pq, pp') \tag{1.6}$$

The advantage of a relativistic treatment becomes apparent if we observe that if the T-matrix is a smooth function of pq and pp', the phase shifts automatically have the right energy dependence. We assume that the amplitude has a Taylor expansion in pp' around the forward direction with coefficients $a_m(pq)$ which are not singular at threshold when $pq = \kappa\mu$. Then

$$T = \sum_m a_m(pq)\,(pp' - \kappa^2)^m \tag{1.7}$$

In the center-of-mass coordinates, $pp' - \kappa^2 = \mu^2\eta^2(1-\cos\theta)$ where $\mu\eta$ is the center-of-mass momentum and θ is the center-of mass angle of scattering. Then, absorbing the factors of the meson mass into the a_m 's,

$$T = \sum_m a_m(pq)\,\eta^{2m}\,(1-\cos\theta)^m \tag{1.8}$$

and if we compare this to the phase shift expansion

$$T \sim E/\eta \sum_l (2l+1)\,e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta) \tag{1.9}$$

where E is the total center of mass energy, we see that the coefficient of a $P_l(\cos\theta)$ arises from the terms in the expansion (1.8) with $m \geq l$. Therefore, using Q_{ml} for the numerical factors arising in the transition from (1.8) to (1.9),

$$E/\eta\; e^{i\delta_l}\sin\delta_l \sim \eta^{2l}\sum_{m\geq l} Q_{ml}\, a_m(pq)\,\eta^{2m-2l} \tag{1.10}$$

and we have near threshold $\delta_\ell \propto \eta^{2\ell+1}$. If we keep pp' fixed and assume the T-matrix element to be analytic in pq in the upper half pq -plane, we shall write a general dispersion relation of the form:

$$\mathrm{Re}\, T(pq, pp') = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\nu}{\nu - pq}\, \mathrm{Im}\, T(\nu, pq) \tag{1.11}$$

which is just an application of Cauchy's theorem to the contour including the real axis and a semi-circle at infinity; we naturally assume that the contribution from the semi-circle vanishes. If we assume the existence of a Taylor expansion of the amplitude in $pp' - k^2$, the relation (1.11) will act as a generating function for dispersion relations for the coefficients in this expansion. Furthermore, if we relate these coefficients to the phase shift expansion, we will get a series of relations for the phase shifts that will guarantee the correct low energy behavior for the phases.

The next problem is that the dispersion relation we have written involves the amplitude for negative, non-physical values of pq. We shall have to use some symmetry property of the amplitude to eliminate $T(-pq, pp')$ in favor of $T(pq, pp')$. We shall also use the coordinate system in which this symmetry has the simplest form to discuss the analytical properties of the amplitude and to write the dispersion relations. The symmetry we use is the crossing symmetry of Gell-Mann and Goldberger.(15) Consider the formula for the T-matrix element

$$T(p, p', q) = i \int_{+} e^{iqx} \langle p | [j(x), j(0)] | p' \rangle \, dx \tag{B.12}$$

Then for real p , p' , and q : since $j(r)$ is Hermitian,

$$T^*(p, p', q) = T(p', p, -q) \tag{1.12}$$

on the energy shell this becomes

$$T^*(pq, pp') = T(-p'q, pp') = T(-pq + pp' - k^2, pp') \tag{1.13}$$

The symmetry is obviously simpler if we change variables to $T((p+p')q, pp')$ for then

$$T^*((p+p')q, pp') = T(-(p+p')q, pp') \tag{1.14}$$

and the real part of the amplitude is even, the imaginary part is odd in the variable $(p+p')q$. This transformation is simplest in the coordinate system in which $\vec{p}+\vec{p}'=0$, which we shall call the symmetrical system. In this system the momentum of the nucleon is reversed by the collision. We let ω be the meson energy in this system, $\vec{P}$ and P_0 be the nucleon momentum and energy, and $\vec{k}$ be the meson momentum. Then

$$pp' = k^2 + 2P^2 \qquad pq = P_0\omega + P^2$$

$$\vec{P}\cdot\vec{k} = -P^2 \tag{1.15}$$

The amplitude is then a function of ω and P alone. The crossing symmetry becomes

$$T^*(\omega, P) = T(-\omega, P) \tag{1.16}$$

and we expect a dispersion relation of the form:

$$\mathrm{Re}\, T(\bar{\omega}, P) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega - \bar{\omega}} \,\mathrm{Im}\, T(\omega, P) = \frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega\, d\omega}{\omega^2 - \bar{\omega}^2} \,\mathrm{Im}\, T(\omega, P) \tag{1.17}$$

The integral still involves non-physical values of the T-matrix. Since in the symmetrical system, if θ_s is the angle of scattering,

$$\cos \theta_s = - P/k$$

from (1.15). For a real scattering to occur, we must have $k \geq P$. We have specified the nucleon recoil, thus the meson must have sufficient momentum to produce that recoil. There is a region $0 < \omega < \sqrt{\mu^2 + P^2}$ in which we shall have to get the T-matrix element by analytic continuation from its value for $\omega > \sqrt{\mu^2 + P^2}$. As we shall see, one can calculate the significant contribution from this region, a pole in the amplitude resulting from the direct absorption and emission of the meson.

The relation (1.17) can easily be written in invariant variables, since from (1.15)

$$\omega = \frac{1}{P_0} \left[pq - \frac{pp' - \kappa^2}{2} \right] \tag{1.18}$$

we have

$$\mathrm{Re}\, T(pq, pp') = \frac{2}{\pi} P \int_{\frac{pp'-\kappa^2}{2}}^{\infty} \frac{\left(\nu - \frac{pp'-\kappa^2}{2}\right) \mathrm{Im}\, T(\nu, pp')\, d\nu}{(\nu - pq)(\nu + pq - pp' + \kappa^2)} \tag{1.19}$$

and if we introduce dimensionless variables:

$z = pq/\kappa\mu$ = meson energy/ meson mass in the laboratory

and

$$\nu = \frac{p p' - \kappa^2}{\kappa \mu}$$, a dimensionless measure of the

momentum transfer, we can write the relation as:

$$\mathrm{Re}\, T(\bar{z}, \nu) = \frac{2}{\pi} P \int_{\nu/2}^{\infty} \frac{(z - \nu/2)\, dz}{(z - \bar{z})(z + \bar{z} - \nu)} \, \mathrm{Im}\, T(z, \nu) \tag{1.20}$$

If the symmetry of the amplitude had been different, if

$A^*(\omega, P) = -A(-\omega, P)$ then the relation would have been:

$$\mathrm{Re}\, A(\bar{z}, \nu) = \frac{2}{\pi} (\bar{z} - \nu/2) \, P \int_{\nu/2}^{\infty} \frac{dz}{(z - \bar{z})(z + \bar{z} - \nu)} \, \mathrm{Im}\, A(z, \nu) \tag{1.21}$$

These are the basic forms of the general dispersion relations, written in dimensionless invariant variables.

Section II:

Dispersion Relations for Scalar Particles

In this section we shall develop a heuristic derivation of the dispersion relations, repeating the generalization used by Salam[7] to the non-forward scattering case of the technique developed by Goldberger for forward scattering. We shall discuss the scattering of two scalar particles. Using the notation of section I, the T-matrix element for this scattering is, on the energy shell,

$$T(pq, pp') = i \int_{+} dx\, e^{iqx} \langle p | [j(x), j(0)] | p' \rangle \tag{2.1}$$

We shall discuss the additional term in the T-matrix arising from the commutator on a space-like surface in section III. The $+$ under the integral sign signifies that the integration is confined to the forward light-cone. The integrand must be a scalar function of invariant variables which we choose to be

$$i \langle p | [j(x), j(0)] | p' \rangle = f(pp', px, p'x, x^2) \tag{2.2}$$

Taking the complex conjugate of this last equation and using the fact that the current is Hermitian, we deduce:

$$f^{*}(pp', px, p'x, x^2) = f(pp', p'x, px, x^2) \tag{2.3}$$

In the symmetrical system this becomes a function of $\vec{P}\cdot\vec{x}$, P, x_0, and x^2, and the reality condition is:

$$f^{*}(\vec{p}\cdot\vec{x}, P, x_0, x^2) = f(-\vec{P}\cdot\vec{x}, P, x_0, x^2) \qquad (2.4)$$

We take the direction of the nucleon motion, $\vec{P}$, to be the polar axis in the x -integration. The product of the meson momentum, $\vec{k}$, and the position vector that occurs in the exponential becomes in the symmetrical system:

$$\vec{k}\cdot\vec{x} = kr\,(\cos\theta_s \cos\theta - \sin\theta_s \sin\theta \cos\varphi)$$

where

$$\cos\theta_s = -P/k \qquad\qquad \sin\theta_s = \frac{\sqrt{k^2-P^2}}{k} = \frac{Q}{k}$$

letting $Q = \sqrt{\omega^2 - P^2 - \mu^2}$ be the part of the meson momentum perpendicular to the nucleon motion. We take this square root function to be that branch of the analytic function of ω that is positive imaginary in the upper half plane and has cut lines along the real axis from $\omega = \pm\sqrt{P^2+\mu^2}$ to infinity. On the real axis above these cut lines Q is an odd function of ω, positive for $\omega > \sqrt{P^2+\mu^2}$, positive imaginary for $-\sqrt{P^2+\mu^2} < \omega < \sqrt{P^2+\mu^2}$ and negative for $\omega < -\sqrt{P^2+\mu^2}$.

We write the T-matrix element in the symmetrical system:

$$T(\omega, P) = \int_0^\infty r^2 dr \int_r^\infty e^{i\omega x_0}\, dx_0 \int \sin\theta\, d\theta\, d\varphi \qquad (2.5)$$

$$\times\ e^{iQr\sin\theta\cos\varphi}\, e^{i\vec{P}\cdot\vec{x}}\, f(\vec{P}\cdot\vec{x}, P, x_0, x^2)$$

The azimuthal integration may be done to yield a Bessel function:

$$= 2\pi \int_0^\infty r^2 dr \int_r^\infty e^{i\omega x_0} dx_0 \int_{-1}^{1} d\cos\theta \; J_0(Q r \sin\theta) \times e^{iPr\cos\theta} f(\cos\theta, P, x_0, r) \tag{2.6}$$

As a consequence of the symmetry condition (2.4) and the fact that $J_0(x)$ is an even function, the integrand of the $\cos\theta$ integration is real and is

$$H(\cos\theta, P, r, x_0) = 2\pi\left[\cos(Pr\cos\theta)\,\mathrm{Re}\, f(\cos\theta, P, r, x_0) - \sin(Pr\cos\theta)\,\mathrm{Im}\, f(\cos\theta, P, r, x_0)\right] \tag{2.7}$$

The same argument used by Goldberger to justify the dispersion relations for forward scattering may be applied to this general T-matrix element. The real and imaginary parts of the T-matrix element are:

$$\mathrm{Re}\, T(\omega, P) = \int_0^\infty r^2 dr \int_r^\infty \cos(\omega x_0) dx_0 \int_{-1}^{1} d\cos\theta \; J_0(Qr\sin\theta)\, H(\cos\theta, r, x_0, P)$$

$$\mathrm{Im}\, T(\omega, P) = \int_0^\infty r^2 dr \int_r^\infty \sin(\omega x_0) dx_0 \int_{-1}^{1} d\cos\theta \; J_0(Qr\sin\theta)\, H(\cos\theta, r, x_0, P) \tag{2.8}$$

We assume that $\mathrm{Im}\, T(\omega, P)$ behaves as some finite power of ω at infinity. This assumption is connected to the commutator of the currents being only a finite derivative of a delta-function on the light cone. For simplicity we shall assume that the integral

$$= 2\pi \int_0^\infty r^2 dr \int_r^\infty e^{i\omega x_0} dx_0 \int_{-1}^1 d\cos\theta \; J_0\,(Qr\sin\theta)$$

$$\times\, e^{iPr\cos\theta}\; f\,(\cos\theta, P, x_0, r) \tag{2.6}$$

As a consequence of the symmetry condition (2.4) and the fact that $J_0(x)$ is an even function, the integrand of the $\cos\theta$ integration is real and is

$$H\,(\cos\theta, P, r, x_0) =$$

$$2\pi \left[\, \cos\,(Pr\cos\theta)\; \mathrm{Re}\; f\,(\cos\theta, P, r, x_0) \right. \tag{2.7}$$

$$\left. - \sin\,(Pr\cos\theta)\; \mathrm{Im}\; f\,(\cos\theta, P, r, x_0) \,\right]$$

The same argument used by Goldberger to justify the dispersion relations for forward scattering may be applied to this general T-matrix element. The real and imaginary parts of the T-matrix element are:

$$\mathrm{Re}\; T\,(\omega, P) = \int_0^\infty r^2 dr \int_r^\infty \cos(\omega x_0) dx_0 \int_{-1}^1 d\cos\theta\; J_0\,(Qr\sin\theta)\;\; H\,(\cos\theta, r, x_0, P)$$

$$\mathrm{Im}\; T\,(\omega, P) = \int_0^\infty r^2 dr \int_r^\infty \sin(\omega x_0) dx_0 \int_{-1}^1 d\cos\theta\; J_0\,(Qr\sin\theta)\;\; H\,(\cos\theta, r, x_0, P) \tag{2.8}$$

We assume that $\mathrm{Im}\; T(\omega, P)$ behaves as some finite power of ω at infinity. This assumption is connected to the commutator of the currents being only a finite derivative of a delta-function on the light cone. For simplicity we shall assume that the integral

$$P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\bar{\omega}}\,\mathrm{Im}\,T(\omega,P) \tag{2.9}$$

converges. If this does not converge, we would consider integrals of the form

$$P\int_{-\infty}^{\infty}\frac{d\omega}{\prod_{m=1}^{\ell}(\omega-\omega_m)}\,\mathrm{Im}\,T(\omega,P)$$

In these integrals the principal value is to be taken at the explicit singularity in the denominator. The contour passes above any other singularities of $\mathrm{Im}\,T(\omega,P)$. We assume that we can interchange the order of integration in (2.8) and (2.9), performing the ω integration before the space-time integration in (2.8). Then since

$$\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\bar{\omega}}\sin(\omega x_0)\,J_0(Qr\sin\theta)$$
$$=\cos(\bar{\omega}x_0)\,J_0(\bar{Q}r\sin\theta) \tag{2.10}$$

we have

$$\mathrm{Re}\,T(\bar{\omega},P)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\bar{\omega}}\,\mathrm{Im}\,T(\omega,P) \tag{2.11}$$

We derive (2.10) by considering the integral

$$P\int_{-\infty}^{\infty}\frac{d\omega}{\omega-\bar{\omega}}\,e^{i\omega x_0}\,J_0(Qr\sin\theta) \tag{2.12}$$

The contour of integration may be closed at infinity in the upper half ω -plane. $e^{i\omega x_0}$ is a decreasing exponential in the upper half plane since x_0 is positive. Although $J_0(Qr\sin\theta)$ behaves as an increasing exponential for imaginary argument, $x_0 \geqslant r\sin\theta$ since the integral is restricted to the forward light cone, and the integrand is at worst bounded at $i\infty$ on the light cone. This is the result of the causality requirement, that the integral is restricted to the interior of the forward light cone. We could use several powers of ω in the denominator to take care of the possibility that the contribution from the light cone itself is significant in (2.8). Having closed the contour, we may apply Cauchy's theorem since the integrand is analytic and has no other poles than those produced by the denominator. Thus (2.12) is

$$\pi i\, e^{i\bar{\omega}x_0}\, J_0\left(\sqrt{\bar{\omega}^2-p^2-\mu^2}\; r\sin\theta\right)$$

and taking the imaginary part of (2.12) yields (2.10).

We have glossed over one major difficulty: the expression (2.6) for the T-matrix element only defines the T-matrix in the physical region. For the range of values of ω which make Q imaginary, the integral appears to be undefined. This is the non-physical region $|\omega| < \sqrt{\mu^2+p^2}$ in which the incoming meson cannot undergo a real scattering process. In this region $J_0(Qr\sin\theta)$ behaves as an increasing exponential. We assume that the behavior of the rest of the integrand is such that the integral is actually convergent in this region. Since the difficulty arises when the spatial integral is extended to infinity, this assumption is reasonable because we do not expect anything of physical importance to depend on contributions to the commutator in (2.1) at spatial infinity. That is, if we consider the function

$$T_R(\omega, P) = \int_0^R r^2 dr \int_r^\infty e^{i\omega x_0} dx_0 \int d\cos\theta \, J_0(Qr\sin\theta) \, H(\cos\theta, P, r, x_0).$$

The passage to the limit $R \to \infty$ is not important in the physical region. This integral is defined for all real ω and describes an analytic function with the correct symmetry properties whose limit on the real axis in the physical region approximates as closely as we wish the scattering amplitude. We shall further discuss the contributions from the non-physical region and the bound state term in section III.

If the T-matrix element obeys a relation of the form (2.11) it is clearly the limit on the real axis from above of a function analytic in the upper half plane. We shall assume this to be the case and investigate briefly the possible forms of the dispersion relations. Consider, as an example, the case of a function, analytic and obeying a symmetry condition of the form $f^*(\omega) = f(-\omega)$ in the symmetrical system and bounded at infinity on the real line and in the upper half plane. Then the function $f(\omega)/\omega^2 - \bar{\omega}^2$ may be integrated over a contour running along the real axis and closed by a semi-circle at infinity. The contribution from the semi-circle vanishes, and Cauchy's theorem yields:

$$\frac{f(\bar\omega)}{2\bar\omega} - \frac{f(-\bar\omega)}{2\bar\omega} = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - \bar\omega^2} f(\omega)$$

where the principal value refers to the poles at $\omega = \pm\bar\omega$ and the contour passes above any singularities of $f(\omega)$. Then the symmetries of the function, that the real part is even and the imaginary part odd in ω on the real axis, make the real part cancel on the left hand side and the imaginary part cancel under the integral sign leaving:

$$\operatorname{Im} f(\omega) = -\frac{2}{\pi} P \int_0^{\infty} \frac{d\omega}{\omega^2 - \bar\omega^2} \operatorname{Re} f(\omega) \tag{2.13}$$

In order to derive a relation containing the imaginary part of the amplitude under the integral, the function considered in Cauchy's theorem must have an odd function of ω multiplying $f(\omega)$. For example $\frac{\omega f(\omega)}{(\omega^2 - \bar\omega^2)(\omega^2 - \bar\omega'^2)}$ will vanish rapidly enough at infinity for Cauchy's theorem to be applied. Then the symmetries of the function pick out the relation

$$\operatorname{Re} f(\bar\omega) - \operatorname{Re} f(\bar\omega') = \frac{2}{\pi}(\bar\omega^2 - \bar\omega'^2) P \int_0^{\infty} \frac{\omega\, d\omega}{(\omega^2 - \bar\omega^2)(\omega^2 - \bar\omega'^2)} \operatorname{Im} f(\omega) \tag{2.14}$$

Thus for a bounded analytic function, the relation describing the odd part as an integral over the even part is simpler than the conjugate relation. This last integral (2.14) can also be thought of as a "difference" integral, as having been obtained by the subtraction of two dispersion relations for the even part in terms of the odd part of the form

$$\mathrm{Re}\, f(\bar{\omega}) = \frac{2}{\pi}\, P \int_0^\infty \frac{\omega\, d\omega}{\omega^2 - \bar{\omega}^2}\, \mathrm{Im}\, f(\omega) \tag{2.15}$$

for two different values of $\bar{\omega}$. Such simpler relations apply to functions vanishing at infinity. For functions behaving as a first power of ω at infinity, both the real and the imaginary parts obey difference relations. The real part would be related to the imaginary part by (2.14) while the conjugate relation would be:

$$\frac{\mathrm{Im}\, f(\bar{\omega})}{\bar{\omega}} - \frac{\mathrm{Im}\, f(\bar{\omega}')}{\bar{\omega}'} = \frac{2}{\pi}\,(\bar{\omega}^2 - \bar{\omega}'^2)\, P \int_0^\infty \frac{d\omega}{(\omega^2 - \bar{\omega}^2)(\omega^2 - \bar{\omega}'^2)}\, \mathrm{Re}\, f(\omega) \tag{2.16}$$

For functions behaving as ω^2 at infinity, the integral over the even part may be simpler than that over the odd part. This way of reading off the dispersion relations using the symmetries and the boundedness properties of the functions yields, in some cases, stronger forms of the relations than would be expected from the consideration of functions like $f(\omega)/\omega - \bar{\omega}$ alone.

Section III

Dispersion Relations for Particles with Spin

In treating the scattering of pions and nucleons, the new facets are the introduction of the spin and isotopic spin of the nucleons and the isotopic spin of the mesons. The initial state of the system shall contain a nucleon of four-momentum p', spin λ', and a meson of four-momentum q', isotopic spin β. The T-matrix element for the transition to a final state p, λ, q, and α is, from appendix II:

$$T_{\alpha\beta}(p,p';q) = i\int_{+} dx\, e^{iqx} \langle p\lambda | [j_\alpha(x), j_\beta(0)] | p'\lambda' \rangle \tag{3.1}$$

$$+ \; i\int_{\sigma=0} d\sigma_\mu\, e^{iqx} \overleftrightarrow{\partial_\mu} \langle p\lambda | [\varphi_\alpha(x), j_\beta(0)] | p'\lambda' \rangle$$

The nucleon isotopic spin is taken into consideration by treating this T-matrix element as a two-by-two isotopic spin matrix. This structure can be reduced to simpler forms by utilizing the invariance properties of the T-matrix. The isotopic spin behavior can be separated out easily by a method of Goldberger's.[5] $T_{\alpha\beta}(p,p';q)$ is a tensor in isotopic spin space and can be decomposed into symmetric and antisymmetric parts:

$$T_{\alpha\beta}(p,p';q) = \delta_{\alpha\beta}\, T^{e}(p,p';q) + \tfrac{1}{2}[\tau_\alpha, \tau_\beta]\, T^{o}(p,p';q) \tag{3.2}$$

The nucleon spin can be taken into account since the general structure of this T-matrix element is

$$\bar{u}_\lambda(p) \quad O(p, p', q) \quad u_{\lambda'}(p')$$

in which $O(p, p', q)$ is some matrix depending on the γ's and on p, p', and q. The $u(p)$ are solutions of the free particle Dirac equation with the renormalized mass κ. Since the T-matrix is invariant, the only independent functions of the spinors are $\bar{u}(p)\, u(p')$ and $\bar{u}(p)\, \gamma q\, u(p')$, for all other scalar invariant combinations of the gamma's and the momentum vectors can be reduced to these. No terms in γ_5 or in $\gamma_5 \gamma q$ enter since the T-matrix element for this scattering process is a scalar. The general form of the amplitude on the energy shell is then

$$T(p, p'; q) = \bar{u}(p)\, u(p')\, f(pp'; pq) + \bar{u}(p)\gamma q\, u(p')\, g(pp'; pq) \tag{3.3}$$

The crossing relation will yield the symmetry properties of the functions f^e, f^o, g^e, and g^o. From the general expression (3.1), one deduces by taking complex conjugates:

$$T_{\alpha\beta}(p, \lambda, p', \lambda', q)^* = T_{\alpha\beta}(p', \lambda', p, \lambda, -q) \tag{3.4}$$

Comparison of this crossing relation with the isotopic spin decomposition (3.2) shows that

$$\begin{aligned} T^e(p, p', q)^* &= T^e(p', p, -q) \\ T^o(p, p', q)^* &= -T^o(p', p, -q) \end{aligned} \tag{3.5}$$

and a further comparison with the spinor decomposition yields

$$
\begin{aligned}
f^e(pp', pq)^* &= f^e(pp', -p'q) \\
g^e(pp', pq)^* &= -g^e(pp', -p'q) \\
f^o(pp', pq)^* &= -f^o(pp', -p'q) \\
g^o(pp', pq)^* &= g^o(pp', -p'q)
\end{aligned}
\tag{3.6}
$$

These symmetry properties govern the ultimate form of the dispersion relations that we shall write for these four functions.

We shall now discuss the second term of (3.1). This term

$$
i \int_{\sigma=0} d\sigma_\mu \, e^{iqx} \overleftrightarrow{\partial}_\mu \langle p | [\phi_\alpha(x), j_\beta(0)] | p' \rangle
$$

involves the commutator of the fields on a spacelike surface. It can always be evaluated using the canonical commutation relations. If we make the assumption that $j_\beta(0)$ does not contain $\dot{\phi}(x)$ the only part of the commutator that appears is $[\dot{\phi}_\alpha(x), j_\beta(0)]$. If this is so, then $j_\beta(0)$ does not contain the spatial derivatives of $\phi(y)$ since the current must depend on the fields in an invariant manner. The commutator will then reduce to a scalar operator multiplied by a three-dimensional delta-function, and this term will not depend on the meson energy-momentum q. In this case the commutator is also a multiple of $\delta_{\alpha\beta}$. This follows from the fact that this surface integral term in the T-matrix element is essentially the second variational derivative of the interaction

Lagrangian with respect to the meson field:

$$\frac{\delta}{\delta \varphi_\alpha(r)} \frac{\delta}{\delta \varphi_\beta(0)} \mathcal{L}_I$$

This derivative is symmetric under the combined interchange of the isotopic spin indices and the coordinates of the meson field operators. If the final contribution is a delta-function, corresponding to a q -independent term, it is even in coordinate space and thus even in the isotopic indices. If the contribution is a first derivative of a delta-function, corresponding to a term linear in q , it would be odd in the isotopic indices. Similar arguments hold for higher momentum dependences. In the case at hand, we expect the current to depend on the meson field through terms such as $\delta\mu^2 \varphi_\beta(0)$, a mass renormalization, and $(\lambda + \delta\lambda)\varphi^3$, the direct meson-meson interaction and renormalization, and possibly a meson pair term. The surface term is then of the form

$$\delta_{\alpha\beta} \; \bar{u}_\lambda(p) \; u_{\lambda'}(p') \; \Lambda(pp')$$

If we use the crossing relation, we see that $\Lambda(pp')$ is a real function. Thus it contributes only to $\mathrm{Re}\, f^e$ and is, as far as the dispersion relations are concerned, a constant term in f^e . Even if energy dependent terms were produced by this surface integral, they would clearly be analytic functions of the meson energy with a pole at infinity; therefore they would only upset the boundedness properties of the functions. This Λ -term is in general part of the renormalization of the T-matrix. It contains an infinite multiple of a delta-function in the forward direction that is part of a mass renormalization. This delta-function does not enter the dispersion relations, however, because we always consider the T-matrix in the forward direction to be the limit of the T-matrix for small angle scattering. The $\delta\lambda$ meson-meson renormalization term is not a delta-function and would enter at all angles.

Now we decompose the integrand of the space-time integral in the T-matrix element in analogy to the decomposition in momentum space that we have just made. Consider first the even isotopic index part of the T-matrix by considering the integrand for $T_{\alpha\alpha}$. The integrand may be rewritten:

$$i \langle p\lambda | [j_\alpha(x), j_\alpha(0)] | p'\lambda' \rangle = \bar{u}_\lambda(p)\, u_{\lambda'}(p')\, F(px, p'x, x^2, pp') \tag{3.7}$$

$$+ \bar{u}_\lambda(p)\, i\gamma\partial\, u_{\lambda'}(p')\; G(px, p'x, x^2, pp')$$

The causality condition requires that

$$F(px, p'x, x^2, pp') = 0 \qquad x^2 < 0$$

and

$$\bar{u}(p)\, i\gamma\partial\, u(p')\; G(px, p'x, x^2, pp') = 0 \qquad x^2 < 0$$

By taking the complex conjugate of the matrix element we deduce:

$$F(px, p'x, x^2, pp')^* = F(p'x, px, x^2, pp')$$

The second term in (3.7) must be integrated by parts to bring it into the form (3.3). This integration yields

$$\bar{u}(p)\,\gamma q\, u(p') \int_0^\infty dx_0\, d^3x\, e^{iqx}\, G(px, p'x, x^2, pp')$$

$$- i\,\bar{u}(p)\,\gamma_0\, u(p') \lim_{\sigma\to\infty} \int_\sigma d^3x\, e^{iqx}\, G(px, p'x, x^2, pp') \qquad (3.8)$$

$$+ i\,\bar{u}(p)\,\gamma_0\, u(p') \int_{\sigma=0} d^3x\, e^{iqx}\, G(px, p'x, x^2, pp')$$

Terms arising from the boundaries at spatial infinity have been dropped since the fields are assumed to be localized. The second and third terms have a structure that is different from the first. They both must vanish since the T-matrix must be independent of the orientation of these arbitrary spacelike surfaces. If the spacelike surfaces are flat, these terms have the form

$$\bar{u}(p)\,\gamma\eta\, u(p')\, h(p, p', q, \eta) \qquad (3.9)$$

in which η is the normal to the spacelike surface. There is no scalar function formed from the scalar products of the vectors p , p' , q , and η which will make (3.9) independent of η . The first term in (3.8) has been written as an integral over the half space $x_0 > 0$. The function $G(px, p'x, x^2, pp')$ must vanish outside the light-cone, however, since otherwise the integral would depend on the spacelike surface passing through the origin. Now each of the functions in momentum space can be written as a Fourier transform over the forward light cone.

$$f^e(pp', pq) = \int_+ dx\, e^{iqx} F(px, p'x, x^2, pp')$$
$$g^e(pp', pq) = \int_+ dx\, e^{iqx} G(px, p'x, x^2, pp') \tag{3.10}$$

The crossing relation yields for G :

$$G(px, p'x, x^2, pp')^* = - G(p'x, px, x^2, pp')$$

Similar results hold for the odd isotopic index functions f^o and g^o . The argument of section II can be applied to (3.10) to yield generalized dispersion relations.

We shall show that with the choice of phases that have been used in defining the four functions f^e , f^o , g^e , and g^o , the imaginary parts of these functions correspond to energy-momentum conserving intermediate states. We expand the commutator in the expression for the amplitude (3.1) over a complete set of intermediate states. Denoting these states by $|n, i\rangle$ in which n_μ is the eigenvalue of the total energy-momentum four-vector and i represents all the other quantum numbers required to specify the state, using the translational invariance of the theory (B.3), and writing j_α for $j_\alpha(0)$, we have:

$$T_{\alpha\beta}(p, p'; q) = i \int_+ dx\, e^{iqx} \sum_{n,i} \Big\{ \langle p| j_\alpha |n,i\rangle \langle n,i| j_\beta |p'\rangle\, e^{ix(p-n)}$$
$$- \langle p| j_\beta |n,i\rangle \langle n,i| j_\alpha |p'\rangle\, e^{ix(n-p')} \Big\} \tag{3.11}$$

The quantity

$$\sum_i \langle p | j_\alpha | n, i \rangle \langle n, i | j_\beta | p' \rangle \tag{3.12}$$

is invariant and may be written as the sum of two independent terms

$$= \bar{u}(p)\, u(p') \quad H_{\alpha\beta}(pn, p'n, n^2, pp')$$

$$+ \bar{u}(p)\, \gamma n \; u(p') \quad J_{\alpha\beta}(pn, p'n, n^2, pp')$$

Taking the complex conjugate of (3.12), we have:

$$H_{\alpha\beta}(pn, p'n, n^2, pp')^* = H_{\beta\alpha}(p'n, pn, n^2, pp')$$

$$J_{\alpha\beta}(pn, p'n, n^2, pp')^* = J_{\beta\alpha}(p'n, pn, n^2, pp') \tag{3.13}$$

The isotopic spin dependence of $H_{\alpha\beta}$ and $J_{\alpha\beta}$ can be separated out in the same way it was for the T-matrix element as a whole. Introducing H^e and J^e for the coefficients of $\delta_{\alpha\beta}$ and H^o and J^o for the coefficients of $\frac{1}{2}[\tau_\alpha, \tau_\beta]$, we get four functions which have the following symmetries:

$$H^e(pn, p'n, n^2, pp')^* = H^e(p'n, pn, n^2, pp')$$

$$H^o(pn, p'n, n^2, pp')^* = H^o(p'n, pn, n^2, pp') \tag{3.14}$$

$$J^e(pn, p'n, n^2, pp')^* = J^e(p'n, pn, n^2, pp')$$

$$J^o(pn, p'n, n^2, pp')^* = J^o(p'n, pn, n^2, pp')$$

which symmetry happens to be the same for all these functions. We insert this decomposition into (3.11) and convert the γ_m into a derivative. This yields for the even isotopic index term:

$$T^e = \bar{u}(p)u(p')\, i \int_+ dx\, e^{iqx} \int_+ \frac{dn}{(2\pi)^4} H^e(pn, p'n, n^2, pp') \times \left[e^{ix(p-n)} - e^{ix(n-p')} \right]$$
$$+ \bar{u}(p)\gamma^\nu u(p')\, i \int_+ dx\, e^{iqx} \int_+ \frac{dn}{(2\pi)^4} (i\partial_\nu) J^e_\ell(pn, p'n, n^2, pp') \times \left[e^{ix(p-n)} + e^{ix(n-p')} \right] \tag{3.15}$$

We compare this to (3.7) to yield the relations

$$F^e = i \int_+ \frac{dn}{(2\pi)^4} H(pn, p'n, n^2, pp') \left[e^{ix(p-n)} - e^{ix(n-p')} \right]$$
$$G^e = i \int_+ \frac{dn}{(2\pi)^4} J(pn, p'n, n^2, pp') \left[e^{ix(p-n)} - e^{ix(n-p')} \right] \tag{3.16}$$

The space-time integrals have been written as restricted to the forward light-cone. This is unnecessary since the integrands in (3.10) vanish outside the light-cone. Extending the integrals to cover the half space $x_0 \geq 0$ and reversing the order of the coordinate and the intermediate momentum integrations in (3.10) and (3.16) permits us to do the coordinate integrals. The spatial integral yields a three-dimensional delta-function providing for momentum conservation in the intermediate state. The time integration yields an energy denominator containing an infinitesimal imaginary part arising from the Abelian convention. Thus

$$f^{e}(pp';pq) = \int_{+} \frac{dn}{(2\pi)^4}\, H^{e}(pn, p'n, n^2, pp') \times \left\{ \frac{(2\pi)^3\, \delta^3(q+p-n)}{n_0 - p_0 - q_0 - i\varepsilon} + \frac{(2\pi)^3\, \delta^3(q-p'+n)}{n_0 - p_0' + q_0 + i\varepsilon} \right\}$$

$$g^{e}(pp';pq) = \int_{+} \frac{dn}{(2\pi)^4}\, J^{e}(pn, p'n, n^2, pp') \times \left\{ \frac{(2\pi)^3\, \delta^3(q+p-n)}{n_0 - p_0 - q_0 - i\varepsilon} - \frac{(2\pi)^3\, \delta^3(q-p'+n)}{n_0 - p_0' + q_0 + i\varepsilon} \right\} \tag{3.17}$$

The separation of the functions f^e and g^e into real and imaginary parts is an invariant one, as is the division into absorptive and dispersive parts. By absorptive and dispersive parts we mean respectively the terms containing a delta-function and those terms containing a principal part function arising from the energy denominators in (3.17). Specializing to the symmetrical system, we can show that the imaginary parts of $\mathcal{H}^e$ and $\mathcal{J}^e$ will vanish in the integration. If in the symmetrical system we use Cartesian coordinates and take the z-direction to be the direction of nucleon motion and the xz-plane to be the plane of scattering we have

$$p = P_0,\ 0,\ 0,\ P \qquad p' = P_0,\ 0,\ 0,\ -P$$

$$q = \omega,\ Q,\ 0,\ -P \tag{3.18}$$

then

$$\vec{p} + \vec{q} = Q,\ 0,\ 0 \qquad \vec{q} - \vec{p}' = Q,\ 0,\ 0$$

The n_z part of the n -integration is restricted by the delta-functions to $m_z = 0$. Therefore $pm = P_0 m_0 - P m_z$ becomes identical to $p'm = P_0 m_0 + P m_z$. Comparison with (3.14) shows that in this case only the real parts of H^e and $\bar{J}^e$ appear in the integrands. We conclude that in the symmetrical system the imaginary parts of f^e and of g^e are identical to the absorptive parts of f^e and g^e, and similarly for the real and dispersive parts. Since these decompositions are invariant, this is true in any system. For the odd isotopic index parts of the T-matrix element the terms arising from the second term in the commutator enter with the opposite sign, we define a new $\bar{H}^o$ to absorb the term in κJ arising when γm becomes a derivative.

$$T^o = \bar{u}(p)\, u(p')\, i \int_+ dx\, e^{iqx} \int_+ \frac{dn}{(2\pi)^4}\, \bar{H}^o(pn, p'n, n^2, pp')$$
$$\times \left[e^{ix(p-n)} + e^{ix(n-p')} \right] \tag{3.19}$$
$$+ \bar{u}(p)\, \gamma^\nu u(p')\, i \int_+ dx\, e^{iqx} (i\partial_\nu) \int_+ \frac{dn}{(2\pi)^4}\, J^o(pn, p'n, n^2, pp')$$
$$\times \left[e^{ix(p-n)} - e^{ix(n-p')} \right]$$

As before, the coordinate integrals may be extended and performed. Then

$$f^o(pp', pq) = \int_+ \frac{dn}{(2\pi)^4}\, \bar{H}^o(pn, p'n, n^2, pp')$$
$$\times \left\{ \frac{(2\pi)^3 \delta^3(p+q-n)}{n_0 - p_0 - q_0 - i\varepsilon} - \frac{(2\pi)^3 \delta^3(q-p'+n)}{n_0 - p_0' + q_0 + i\varepsilon} \right\} \tag{3.20}$$
$$g^o(pp', pq) = \int_+ \frac{dn}{(2\pi)^4}\, J^o(pn, p'n, n^2, pp')$$
$$\times \left\{ \frac{(2\pi)^3 \delta^3(p+q-n)}{n_0 - p_0 - q_0 - i\varepsilon} + \frac{(2\pi)^3 \delta^3(q-p'+n)}{n_0 - p_0' + q_0 + i\varepsilon} \right\}$$

Again in the symmetrical system only the real parts of $\bar{H}^{\circ}$ and J° enter, and the imaginary parts of f° and g° correspond to the absorptive parts of the amplitudes, or to the existence of energy-momentum conserving intermediate states in the physical region.

In the non-physical region, the delta-function of energy that is contained in the absorptive or imaginary part of the amplitude cannot always be fulfilled. We know the energy spectrum available to the intermediate states. The state of lowest energy is, by the conservation of heavy particles, the one nucleon state. The contribution from this state, the bound state term, may be separated out and leads to a pole in the amplitude and to the introduction of the coupling constant into the dispersion relations. The rest of the intermediate states have rest masses greater than one nucleon and one meson

$$m^2 \geqslant (\kappa + \mu)^2$$

If we assume that the delta-function of momentum may be removed before the analytic continuation is performed, the energy-conserving delta-function is of the quantity

$$\sqrt{m^2 + (\vec{p} + \vec{q})^2} - p_0 - q_0$$

and can only be fulfilled if $m^2 = (p+q)^2$ or

$$pq \geqslant \kappa\mu$$

This is the threshold for forward scattering and is below the threshold for scattering with a given momentum transfer. In invariant variables the threshold for finite angle scattering occurs at

$$pq \geq \frac{pp' - \kappa^2}{2} + \sqrt{\frac{pp' - \kappa^2}{2}} \sqrt{\frac{pp' - \kappa^2}{2} + m^2}$$

or

$$z \geq x/2 + \sqrt{x/2 + \kappa/m} \sqrt{x/2 + m/\kappa}$$

The lower limit for the integrals over the imaginary part found from the symmetry condition in section I is $pq \geq \frac{pp' - \kappa^2}{2}$. or $z > x/2$ and so the lower limit of the integral is the larger of $x/2$ or 1 . Since we are ultimately interested in expansions of the amplitude around $x = 0$, we shall write the lower limit as 1 , after the bound state contribution has been removed.

The Bound State Contribution

We now consider the one-nucleon intermediate state in the expansion of the T-matrix element over a complete set of intermediate states (3.11). This is the state of lowest energy to enter since by the law of conservation of heavy particles all the intermediate states must contain one more nucleon than antinucleons. This term in the summation is, letting $\bar{p}$ be the energy momentum of the intermediate nucleon:

$$\frac{1}{(2\pi)^3}\int \frac{d^3\bar{p}}{2\bar{p}_0}\Big[\frac{\langle p|j_\alpha|\bar{p}\rangle\langle\bar{p}|j_\beta|p'\rangle(2\pi)^3\delta^3(p+q-\bar{p})}{\bar{p}_0-p_0-q_0-i\varepsilon} \tag{3.21}$$

$$+\frac{\langle p|j_\beta|\bar{p}\rangle\langle\bar{p}|j_\alpha|p'\rangle(2\pi)^3\delta^3(p'-q-\bar{p})}{\bar{p}_0-p_0'+q_0+i\varepsilon}\Big]$$

If the numerator is assumed to be an analytic function of $\vec{q}$ after the integration has removed the delta-function it may be continued into the non-physical region where $\vec{q}$ is imaginary. The vanishing of the denominator will correspond to a pole in the T-matrix element. In the symmetrical system the pole occurs when

$$\omega = \pm\frac{p^2+m^2/2}{P_0} - i\varepsilon$$

The matrix element $\langle p|j_\alpha|\bar{p}\rangle$ defines the interaction of an external meson field with a real nucleon. Since the matrix element is an invariant and since the meson field is pseudo-scalar, the general form for the matrix element is

$$\bar{u}(p)\, i\gamma_5\tau_\alpha\, u(\bar{p})\quad f(p\bar{p})$$

Since j_α is Hermitian, $f(p\bar{p})$ is a real scalar function. We shall determine the bound state contribution by computing it as a delta-function contribution to the imaginary part of the amplitude. The absorptive part of the one nucleon term is

Introducing a delta-function to express the fact that the nucleon is real and performing the sum over the spin variables of the intermediate nucleon, see appendix I, yields

$$\pi \int d\bar{p}\; \delta(\bar{p}^2 - \kappa^2)\, f(p\bar{p})\, f(p'\bar{p})$$

$$\times \Big\{ \delta(\bar{p}-p-q)\, \bar{u}(p)\, i\gamma_5 \tau_\alpha (\bar{\not p}+\kappa)\, i\gamma_5 \tau_\beta\, u(p')$$

$$- \delta(\bar{p}-p'+q)\, \bar{u}(p)\, i\gamma_5 \tau_\beta (\bar{\not p}+\kappa)\, i\gamma_5 \tau_\alpha\, u(p') \Big\}$$

or, since $p(p+q) = p'(p+q)$ on the energy shell

$$= \pi f^2(\kappa^2 - \mu^2) \big[\bar{u}(p)\tau_\alpha \tau_\beta\, \gamma q\, u(p')\, \delta(2pq+\mu^2)$$

$$+ \bar{u}(p)\tau_\beta \tau_\alpha\, \gamma q\, u(p')\, \delta(2p'q - \mu^2) \big]$$

The bound state term contributes only to the relations for g^e and g^o. This is a consequence of the meson field's being pseudo-scalar. If the meson were scalar there would be large bound state contributions to the functions f^e and f^o as well as to the g^e and g^o. Extracting the coefficients of the even and odd isotopic index parts, we have for the bound state contribution to the imaginary part of the g-functions

$$\operatorname{Im} g^e \approx \frac{\pi}{2} f^2(\kappa^2 - \mu^2/2)\left[\delta(pq + \mu^2/2) + \delta(p'q - \mu^2/2)\right]$$

$$(3.22)$$

$$\operatorname{Im} g^o \approx \frac{\pi}{2} f^2(\kappa^2 - \mu^2/2)\left[\delta(pq + \mu^2/2) - \delta(p'q - \mu^2/2)\right]$$

The function $f^2(\kappa^2 - \mu^2/2)$ is related to the coupling constant. If the meson-nucleon vertex is renormalized not for the scattering of a nucleon by a static external field but for a non-physical process: the emission of a real meson by a real nucleon, $f(\kappa^2 - \mu^2/2)$ will be G, the renormalized unrationalized pseudo-scalar coupling constant. This way of defining the coupling constant was originally introduced in a calculation by Watson and Lepore.[16]

The Dispersion Relations

Now we write out the dispersion relations based on the symmetries (3.6) and the bound state contributions (3.22). Again introducing dimensionless variables

$$z = pq/\kappa\mu \qquad x = \frac{pp' - \kappa^3}{\kappa\mu} \qquad b = \mu/2\kappa$$

we have

$$\mathrm{Re}\, f^0(x, \bar{z}) = (\bar{z} - x/2)\, P \int_1^\infty \frac{dz}{(z - \bar{z})(z + \bar{z} - x)}\, \mathrm{Im}\, f^0(x, z)$$

$$\mathrm{Re}\, g^0(x, \bar{z}) = \frac{2}{\pi}\, P \int_1^\infty \frac{(z - x/2)\, dz}{(z - \bar{z})(z + \bar{z} - x)}\, \mathrm{Im}\, g^0(x, z) + \frac{b + x/2}{(\bar{z} + b)(\bar{z} - b - x)}\, G^2/\kappa\mu$$

$$\mathrm{Re}\, g^e(x, \bar{z}) = \frac{2}{\pi}(\bar{z} - x/2)\, P \int_1^\infty \frac{dz}{(z - \bar{z})(z + \bar{z} - x)}\, \mathrm{Im}\, g^e(x, z) - \frac{\bar{z} - x/2}{(\bar{z} + b)(\bar{z} - b - x)}\, G^2/\kappa\mu \tag{3.25}$$

$$\mathrm{Re}\, f^e(\nu, \bar{z}) = \frac{2}{\pi}\, P \int_1^\infty \frac{(z - \nu/2)\, dz}{(z - \bar{z})(z + \bar{z} - \nu)}\, \mathrm{Im}\, f^e(\nu, z) \qquad + \Lambda(\nu)$$

$$\mathrm{Re}\, f^e(\nu, \bar{z}) - \mathrm{Re}\, f^e(\nu, \bar{z}') =$$

$$\frac{2}{\pi}(\bar{z} - \bar{z}')(\bar{z} + \bar{z}' - \nu)\, P \int_1^\infty \frac{(z - \nu/2)\, dz \quad \mathrm{Im}\, f^e(\nu, z)}{(z - \bar{z})(z + \bar{z} - \nu)(z - \bar{z}')(z + \bar{z}' - \nu)}$$

The relation for f^e is written both as a single relation involving the Λ -term and as a difference to eliminate the Λ -term. This second way of writing the relation for may also be looked upon as reflecting the statement that since f^e behaves as a constant at high energies, the appropriate quantity to consider is $f^e(\omega, P) / \omega^2 - \bar{\omega}^2$ in the symmetrical system. This would be an analytic function vanishing at infinity in the complex plane. In the forward direction the relations (3.23) become

$$\mathrm{Re}\, f^o(\bar{z}) = \frac{2}{\pi}\, \bar{z}\, P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2}\, \mathrm{Im}\, f^o(z)$$

$$\mathrm{Re}\, g^o(\bar{z}) = \frac{2}{\pi}\, P \int_1^\infty \frac{z\, dz}{z^2 - \bar{z}^2}\, \mathrm{Im}\, g^o(z) \quad + \frac{\nu G^2/\kappa M}{\bar{z}^2 - \nu^2}$$

(3.24)

$$\mathrm{Re}\, f^e(\bar{z}) - \mathrm{Re}\, f^e(\bar{z}') = \frac{2}{\pi}(\bar{z}^2 - \bar{z}'^2)\, P \int_1^\infty \frac{z\, dz \quad \mathrm{Im}\, f^e(z)}{(z^2 - \bar{z}^2)(z^2 - \bar{z}'^2)}$$

$$\mathrm{Re}\, g^e(\bar{z}) = \frac{2}{\pi}\, \bar{z}\, P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2}\, \mathrm{Im}\, g^e(z) \quad - \frac{\bar{z}\, G^2/\kappa M}{\bar{z}^2 - \nu^2}$$

We have written the simplest relations possible, making the strongest assumptions about the high energy behavior consistent with a constant term in f^e .

The Direct and Spin-Flip Amplitudes

The functions f^e , f^o , g^e , and g^o that have been introduced must now be related to the more usual spin-flip and non-spin-flip amplitudes. The connection between the separation into even and odd isotopic index dependence and the usual 1/2 and 3/2 isotopic spin dependence has been given by Goldberger, Miazawa, and Oehme.[5] Since the projection operators for the isotopic spin 1/2 and 3/2 states are $\frac{1}{3}\tau_\alpha\tau_\beta$ and $\delta_{\alpha\beta} - \frac{1}{3}\tau_\alpha\tau_\beta$ respectively, the connections are that

$$A^e = \frac{1}{3}(A^1 + 2A^3)$$

$$A^o = \frac{1}{3}(A^1 - A^3) \tag{3.25}$$

Where $A^{e,o}$ is any amplitude relating to the even or odd isotopic index dependence and A^1 , A^3 are the amplitudes for the 1/2 and 3/2 isotopic spin states. The nucleon spin dependence may be obtained by rewriting the quantities

$$\bar{u}(p)\,u(p') \qquad \text{and} \qquad \bar{u}(p)\,\gamma q\,u(p')$$

in the center-of-mass system. The specialization of these invariant forms to any system is easily performed using the relation

$$u(p) = \left[1 + \gamma_5 \frac{\vec{\sigma}\cdot\vec{p}}{p_0 + \kappa} \right] h_\lambda \sqrt{p_0 + \kappa}$$

in which h_λ is $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ depending on the spin of the nucleon. In the center-of-mass system, suppressing the spin indices in favor of a two-by-two matrix notation, we have

$$\bar{u}(p) \left[f + \gamma q \, g \right] u(p')$$

$$= (p_0 + \kappa)\left[f + (p_0 + q_0 - \kappa) g \right] + \frac{\sigma\cdot\vec{p}\,\sigma\cdot\vec{p}'}{p_0 + \kappa}\left[(p_0 + q_0 + \kappa) g - f \right] \tag{3.25}$$

in the laboratory system we would have

$$= \sqrt{2\kappa}\sqrt{p_{0_L}' + \kappa}\left[f + q_{0_L} g - g \frac{\vec{\sigma}\cdot\vec{p}_L{}'\ \vec{\sigma}\cdot\vec{q}_L}{p_{0_L} + \kappa} \right]$$

and in the symmetrical system

$$= 2P_0 f + 2\kappa\omega g - 2i\,\sigma\cdot\vec{P}\times\vec{k}\, g\,(P_0 + \kappa)$$

The expression (3.25) is the one of interest. Introducing symbols for the various normalization factors that have arisen, we shall use

E	for the total center-of-mass energy
$\mu\eta$	for the center-of-mass momentum
$\beta = p_0 + \kappa$	in the center-of-mass

and $\alpha = E + \kappa$

We write the invariant T-matrix element in terms of center-of-mass variables, and $\mathcal{D}$, for direct, and $\mathcal{S}$, for spin-flip, amplitudes.

$$T = D + i\,\vec{\sigma}\cdot\hat{n}\,\sin\theta\,\frac{\mu^2\eta^2}{\beta}\,S \tag{3.26}$$

$\hat{n}$ is a unit vector in the direction of $\vec{p}\times\vec{p}'$,

θ is the angle of scattering. Then from (3.25) and (3.26)

$$S = \alpha g - f$$
$$D = \kappa\mu\left[2z - \tfrac{\alpha}{\beta}x\right] f + \left[2\kappa + \frac{\kappa\mu x}{\beta}\right] g \tag{3.27}$$

since $x = \frac{\mu}{\kappa}\,\eta^2(1-\cos\theta)$ in center-of-mass. We also express the invariant functions in terms of the direct and spin-flip amplitudes

$$f = \frac{\alpha D - \kappa\mu\left[2z - \tfrac{\alpha}{\beta}x\right] S}{2\beta E}$$
$$g = \frac{D + \left[2\kappa + \frac{\kappa\mu x}{\beta}\right] S}{2\beta E} \tag{3.28}$$

and we record the algebraic identities

$$2\beta E = 2\kappa\,(\alpha + \mu z)$$
$$E = \sqrt{\kappa^2 + \mu^2 + 2\kappa\mu z} \tag{3.29}$$
$$\eta\,E/\kappa = \sqrt{z^2-1} \quad = \text{laboratory momentum.}$$

In the forward direction (3.28) reduces to

$$f = \frac{\alpha D - 2 z k m S}{2\beta E}$$

(3.30)

$$g = \frac{D + 2 k S}{2\beta E}$$

Using these and identities (3.29) the relations in the forward direction (3.24) can be combined to yield Goldberger's Relations

$$\mathrm{Re}\, D^{o}(\bar{z}) = \frac{2}{\pi}\bar{z}\, P\int_{1}^{\infty}\frac{dz}{z^2-\bar{z}^2}\,\mathrm{Im}\, D^{o}(z) + 2\bar{z}\nu G^2/\bar{z}^2-\nu^2$$

(3.31)

$$\mathrm{Re}\, D^{e}(\bar{z}) - \mathrm{Re}\, D^{e}(\bar{z}') = \frac{2}{\pi}(\bar{z}^2-\bar{z}'^2)\, P\int_{1}^{\infty}\frac{z\,dz\,\mathrm{Im}\, D^{e}(z)}{(z^2-\bar{z}^2)(z^2-\bar{z}'^2)}$$

$$+ \frac{(\bar{z}^2-\bar{z}'^2)\,2\nu^2 G^2}{(\bar{z}^2-\nu^2)(\bar{z}'^2-\nu^2)}$$

The invariant T-matrix that we have been using obeys the relation

$$\mathrm{Im}\, T = \frac{1}{2}\, T\, T^{+}$$

in the forward direction. Thus the total cross section is related to the imaginary part of the direct amplitude by

$$\frac{1}{2\mu\eta E}\,\mathrm{Im}\, D = \sigma \qquad \text{(A.11)}$$

since $4\mu\eta E$ is the incident flux times the incident nucleon density with our normalization. The relation between the even and odd isotopic index cross sections, the 1/2 and 3/2 isotopic spin cross sections, and the total cross sections for positive and negative mesons from protons (σ^+, σ^-) are,[5]

$$\frac{1}{2\mu\eta E}\,\mathrm{Im}\,D^e = \sigma^e = \frac{1}{3}(\sigma^1 + 2\sigma^3) = \frac{1}{2}(\sigma^+ + \sigma^-)$$

$$\frac{1}{2\mu\eta E}\,\mathrm{Im}\,D^o = \sigma^o = \frac{1}{3}(\sigma^1 - \sigma^3) = \frac{1}{2}(\sigma^- - \sigma^+)$$

In order to work with the dispersion relations, we want the phase shift expansions of the direct and spin-flip amplitudes. Since both parity and total angular momentum are good quantum numbers for pion-nucleon scattering, the orbital angular momentum is also a good quantum number, since for each j there are only two values of ℓ available. We introduce the usual projection operators

$$(\ell+1)\,P_\ell(\cos\theta) - i\,\sigma\cdot\hat{n}\,P_\ell'(\cos\theta) \quad \text{for} \quad j = \ell + 1/2$$

and (3.31)

$$\ell\,P_\ell(\cos\theta) + i\,\sigma\cdot\hat{n}\,P_\ell'(\cos\theta) \quad \text{for} \quad j = \ell - 1/2$$

The isotopic spin dependence has already been separated out. If we compare the expansion of the T-matrix element in terms of the projection operators (3.31) with the unitarity condition for the T-matrix in the region in which only one-meson intermediate states contribute, the coefficients of the expansion can be determined.

The unitarity condition in this energy range is

$$- \langle p, q + | j | p' \rangle + \langle p | j | p' q' + \rangle =$$

$$\frac{i}{(2\pi)^6} \int \frac{d^3\bar{p}\; d^3\bar{q}}{2\bar{p}_0 \;\; 2\bar{q}_0} \langle p | j | \bar{p}, \bar{q} \rangle \langle \bar{p}, \bar{q} | j | p' \rangle (2\pi)^4 \delta(p + q - \bar{p} - \bar{q})$$

Where $\bar{p}$ and $\bar{q}$ are the four-momenta of the intermediate nucleon and meson. The expansions for the direct and spin-flip amplitudes are then

$$D = \frac{8\pi E}{p\mu} \sum_{\ell} \left(\ell a_{\ell -} + (\ell + 1) a_{\ell +} \right) P_\ell (\cos\theta)$$

$$S = \frac{8\pi E\beta}{p^3\mu^3} \sum_{\ell} \left(a_{\ell -} - a_{\ell +} \right) P_\ell' (\cos\theta) \qquad (3.32)$$

where $P_\ell' = \frac{d}{d\cos\theta} P_\ell(\cos\theta)$. These expansions are for each isotopic spin state separately. The quantities $a_{\ell -}$ and $a_{\ell +}$ are

$$a_{\ell -} = e^{i\delta} \sin\delta \qquad \text{for} \quad j = \ell - 1/2$$

$$\qquad (3.33)$$

$$a_{\ell +} = e^{i\delta} \sin\delta \qquad \text{for} \quad j = \ell + 1/2$$

in the energy range in which there is no inelastic scattering. We shall use the usual notation for the s- and p-waves, a_1 and a_3 referring to $e^{i\delta}\sin\delta$ for the $T = \frac{1}{2}$ and $T = 3/2$ s-waves, $a_{2T, 2J}$ for the p-waves. We shall also use a superscript '0' to denote the scattering lengths, taking the low energy phase shifts to be of the form $\delta \sim \delta^{\circ} \eta$ for s-waves and $\delta \sim \delta^{\circ} \eta^3$ for the p-waves. Thus the s- and p-wave terms are, in the forward direction:

$$D^e = \frac{8\pi E}{3\eta\mu}\left[a_1 + 2a_3 + a_{11} + 2a_{13} + 2a_{31} + 4a_{33}\right]$$

$$D^o = \frac{8\pi E}{3\eta\mu}\left[a_1 - a_3 + a_{11} + 2a_{13} - a_{31} - 2a_{33}\right] \qquad (3.34)$$

$$S^e = \frac{8\pi E\beta}{3\eta^3\mu^3}\left[a_{11} - a_{13} + 2a_{31} - 2a_{33}\right]$$

$$S^o = \frac{8\pi E\beta}{3\eta^3\mu^3}\left[a_{11} - a_{13} - a_{31} + a_{33}\right]$$

The High-Energy Behavior

There is very little evidence as to the behavior of the scattering amplitudes at high energies, apart from the fact that the dispersion relations agree quite well with experiment and thus could be interpreted as justifying the assumptions about the convergence of the integrals involved. Experimentally the cross sections are known only to a couple of Bev. The measurements of Cool, Piccioni, and Clark(17) are that for π^- at 1.9 Bev and for π^+ at 1.8 Bev the cross sections are $\sigma^- = 31.3 \pm 1.6$ mb and $\sigma^+ = 31.7 \pm 2.4$ mb. The measurement of σ^- at 4.4 Bev by Bandtel, Bostick, Moyer, Wallace, and Wilkner(18) at Brookhaven is $\sigma^- = 30 \pm 5$ mb, but there is a measurement made by Maenchen et al.(19) at Berkeley to the effect that $\sigma^- = 19.7 \pm 3.4$ mb at 4.5 Bev. This is not very adequate experimental information and the energies are low compared to the energies entering the question of asymptotic behavior. We shall assume that the cross sections for positive and negative mesons on protons become constant and equal at high energies and take the value 30 mb. That is

$$\sigma^{+} + \sigma^{-} \longrightarrow \bar{\sigma} = 60 \text{ mb} = 3/\mu^{2}$$

and (3.35)

$$\sigma^{-} - \sigma^{+} \longrightarrow 0$$

For the imaginary part of the direct amplitude:

$$\mathrm{Im}\, D^{e} = k\mu\sqrt{z^{2}-1}\,(\sigma^{+}+\sigma^{-}) \longrightarrow k\mu z\bar{\sigma}$$

$$\mathrm{Im}\, D^{o} \longrightarrow k\mu z(\sigma^{-}-\sigma^{+})$$

There is some evidence, from Haberschaim's work,[20] that $\mathrm{Im}\, D^{o}$ does obey the dispersion relation (3.30). This requires $\mathrm{Im}\, D^{o}$ to behave as $z^{1-\alpha}$ for positive α, and the derivation of the dispersion relation then requires $\mathrm{Re}\, D^{o}$ to behave also no more singularly than $z^{1-\alpha}$. Using the asymptotic forms for the functions in (3.30)

$$2\beta E \longrightarrow 2k\mu z$$

$$\alpha \longrightarrow \sqrt{2k\mu z}$$

the high energy behavior of our amplitudes in the forward direction, which is the only one that we shall discuss, is

$$g \rightarrow \frac{1}{2k\mu z} D + \frac{1}{\mu z} S$$

$$f \rightarrow \frac{1}{\sqrt{2k\mu z}} D - S$$

If the spin-flip amplitudes are not significant at high energies, we have

$$\operatorname{Im} g^e \rightarrow 1$$

$$\operatorname{Im} f^e \rightarrow \sqrt{z}$$

and we have gained in the degree of convergence making this relativistic separation.

It is reasonable for the cross sections to approach a constant at high energies. This is the behavior that would be expected if the interaction were governed by a fixed range. At high energies one would expect the interaction to be mainly inelastic: all particles striking within the interaction range being absorbed. This argument is the standard argument for a black sphere: that the phase shift for each $\ell < \eta a$ where a is the range of the interaction is large and imaginary, since each angular momentum state in the incident beam that represents a particle striking within the interaction range is completely absorbed. The total cross section is thus $2\pi a^2$, twice geometric, but there is no reason for the 1/2 and 3/2 isotopic spin states to have the same interaction range. We can further inquire about the real part of the amplitude. For all states $\ell < \eta a$ the amplitude is pure imaginary, but there will be contributions to the real part from the states of $\ell \geq \eta a$. A rough estimate of these contributions can be obtained by assuming that the $\ell = \eta a$ state produces its maximum real part $\delta_\ell \sim \pi/4$, and all the other higher phases are zero. Then

$$\mathcal{R}e\, D^e \sim \frac{8\pi E}{3\mu\eta}\, 2\eta a \sim E a$$

The real part would increase as $\sqrt{z}$ for high energies. That this may be an overestimate can be seen from the dispersion relations. Since $\mathcal{I}m\, D^e(z)$ and $\mathcal{R}e\, D^e(z)$ increase no faster than z at infinity, the relation for D^e, (3.31), is valid. This relation requires $\mathcal{R}e\, D^e$ to become constant at infinity if, as we have assumed, the cross section becomes constant at a few Bev, or if $\lim_{z\to\infty} [\mathcal{I}m\, D^e - \kappa\mu\bar{\sigma}\xi] \to 0$. To show that this is the case we compute the value of $\mathcal{R}e\, D^e$ at infinity. The relation (3.31) may be modified to make it more convergent to facilitate the passage to the limit. Since (see (5.13)), using the symbol ξ for $\sqrt{z^2-1}$:

$$\frac{2}{\pi} P \int_1^\infty \frac{z\,dz}{(z^2-1)(z^2-\bar{z}^2)}\,\xi = 0 \qquad \bar{z} > 1$$

The limiting value of $\mathcal{I}m\, D^e = \kappa\mu\bar{\sigma}\xi$ may be subtracted from the integrand without changing the value of the integral in (3.31). Then

$$\mathrm{Re}\, D^e(\infty) - \mathrm{Re}\, D^e(1) = 2 b^2 G^2 - \frac{2}{\pi} \int_1^\infty \frac{z\,dz}{z^2-1} \left[\mathrm{Im}\, D^e(z) - \kappa \mu \bar{\sigma}\, \xi \right]$$

Inserting the values for the cross sections collected by Anderson, Davidon, and Kruse,[6] and using $\bar{\sigma} = 3/\mu^2$, $f^2 = b^2 G^2/4\pi$,

$$\mathrm{Re}\, D^e(\infty) = 8\pi f^2 + 12.3 + \mathrm{Re}\, D^e(1)$$

Orear's[21] values for the s-wave scattering lengths make

$\mathrm{Re}\, D^e(1)$ = -3.9 and a coupling constant f^2 = 0.082 yields

$\mathrm{Re}\, D^e(\infty)$ = 10.5 which is of the order $2\kappa/\mu$ which is of the order of unity since the amplitude has been defined with an extra factor of 2κ .

We can apply the same argument to the spin-flip amplitudes. For the phases representing partial waves that are totally absorbed, there is no contribution to the spin-flip amplitude. Again we estimate the contribution for $\ell \simeq \eta a$ by assuming the contribution to be a maximum for this ℓ . Then since $P_\ell'(1) = \frac{\ell(\ell+1)}{2}$

$$S \propto \frac{\beta E}{\eta^3} (\eta a)^2 \sim a^2 \sqrt{z}$$

Since this argument overestimates the contribution to $\mathrm{Re}\, D^e$, it is also likely to overestimate S^e . Assuming that both the real and imaginary parts of S^e behave as $\sqrt{z}$ and D^e and D^o behave as we have previously supposed, we have for the relativistic amplitudes:

$$\mathrm{Re}\, g^e \sim 1/\sqrt{z} \qquad \mathrm{Im}\, g^e \sim 1$$

$$\mathrm{Re}\, f^e \sim \sqrt{z} \qquad \mathrm{Im}\, f^e \sim \sqrt{z}$$

and

$$g^{\circ} \sim 1/\sqrt{z} \qquad\qquad f^{\circ} \sim \sqrt{z}$$

All four of the relations (3.24) will hold as written.

It is interesting to observe that the behavior of $\mathrm{Re}\, S^{e}$ as $\sqrt{z}$ arises naturally out of the dispersion relations for $g^{e}(z)$ if we make the assumption that $\mathrm{Im}\, S^{e}(z)$ goes to zero at high energies. The relation is at high energies

$$\frac{\mathrm{Re}\, D^{e}(\bar{z})}{2K\mu\bar{z}} + \frac{1}{\mu\bar{z}}\,\mathrm{Re}\, S^{e}(\bar{z}) =$$

$$-\frac{1}{\bar{z}}\, G^{2}/K\mu + \frac{2}{\pi}\,\bar{z}\,P\int_{1}^{\infty}\frac{dz}{z^{2}-\bar{z}^{2}}\,\frac{\mathrm{Im}\, D^{e}}{2\beta E} \tag{3.37}$$

$$+\frac{2}{\pi}\,2K\bar{z}\,P\int_{1}^{\infty}\frac{dz}{z^{2}-\bar{z}^{2}}\,\mathrm{Im}\, S^{e}/2\beta E$$

The assumption we have made about $\mathrm{Im}\, S^{e}$ implies that the last term behaves as $1/\bar{z}$ at high energies. $\mathrm{Re}\, D^{e}(z)$ becomes a constant and the only parts of $\mathrm{Re}\, S^{e}$ that are not constant in the limit must arise from

$$\mathrm{Re}\, S^{e}(z) \sim \frac{2}{\pi}\,\bar{z}^{2}\mu\int_{1}^{\infty}\frac{dz}{z^{2}-\bar{z}^{2}}\,\mathrm{Im}\, D^{e}/2\beta E$$

Now

$$\frac{1}{2\beta E} = \frac{1}{2K[\mu z + K + E]} = \frac{\mu z + K - E}{2K\mu^{2}(z^{2}-1)}$$

If we insert this in the integral, the term in z/z^2-1 can be performed using the dispersion relation for $D^e(z)$. It is part of the $\mathrm{Re}\, D^e(\bar{z})$ term on the left hand side of (3.37) and terms of the order unity. We are left with

$$\frac{1}{2K\mu} \doteq \frac{2}{\pi} \bar{z}^2 \, P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \, \frac{K - E}{z^2 - 1} \, \mathrm{Im}\, D^e(z) \tag{3.38}$$

The most singular behavior arises from the term in the total energy E. To evaluate the behavior of this term, we observe that $E = \sqrt{K^2 + \mu^2 + 2K\mu z}$ is an analytic function of z in the upper half plane, real along the real axis from $z = -\lambda = -\frac{K^2+\mu^2}{2K\mu}$ to infinity and positive imaginary along the negative real axis from $-\infty$ to $-\lambda$. Then $E D^e$ is an analytic function behaving as $z^{3/2}$ at infinity and an application of Cauchy's theorem to

$$\frac{2}{\pi i} \int_{-\infty}^{\infty} \frac{dz}{(z^2 - \bar{z}^2)(z^2 - 1)} \, E(z)\, D^e(z)$$

where the contour passes beneath the poles at $z = \pm 1$ and $z = \pm \bar{z}$ and may be closed at infinity, yields, on taking the real part and using the symmetries of D^e, for $\bar{z} > \lambda$:

$$\frac{2}{\pi} P \int_0^\infty \frac{dz}{(z^2 - \bar{z}^2)(z^2 - 1)} \, E \, \mathrm{Im}\, D^e(z)$$

$$- \frac{2}{\pi} P \int_0^\lambda \frac{dz}{(z^2 - \bar{z}^2)(z^2 - 1)} \sqrt{K^2 + \mu^2 - 2K\mu z} \, \mathrm{Im}\, D^e(z)$$

$$+ \frac{2}{\pi} P \int_\lambda^\infty \frac{dz}{(z^2 - \bar{z}^2)(z^2 - 1)} \sqrt{2K\mu z - K^2 - \mu^2} \, \mathrm{Re}\, D^e(z) =$$

$$= \frac{1}{\bar{z}^2 - 1} \quad \frac{1}{\bar{z}} \quad \bar{E} \, \mathrm{Re} \, D^e(z) \quad - \quad \frac{\sqrt{2\kappa\mu\bar{z} - \kappa^2 - \mu^2}}{\bar{z}(\bar{z}^2 - 1)} \quad \mathrm{Im} \, D^e(\bar{z})$$

$$- \quad \frac{2\kappa}{\bar{z}^2 - 1} \quad \mathrm{Re} \, D^e(1)$$

In the region $0 < z < 1$, $\mathrm{Im}\, D^e$ is just the delta-function that yields the bound state term. The first integral then becomes the desired integral. The other terms on the left hand side behave as $1/z^2$; the bound state term arising from the first integral, the second integral since it has a finite upper limit of integration, and the third because $\mathrm{Re}\, D^e$ becomes a constant at high energies. On the right hand side the term in $\mathrm{Im}\, D^e$ is the largest at infinity. The contribution to $\mathrm{Re}\, S^e$ is then

$$\mathrm{Re}\, S^e \sim \frac{1}{\sqrt{2\kappa\mu z}} \quad \mathrm{Im}\, D^e(z) \quad \sim \quad \sqrt{\frac{\kappa\mu}{2}} \quad \sigma \quad \sqrt{z} \qquad (3.39)$$

The term in κ in (3.38) produces a term in $\mathrm{Re}\, S^e$ proportional to $\log \bar{z}$. Since

$$\frac{2}{\pi} \, P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \quad 1/\xi \quad = \quad \frac{1}{\pi \bar{z} \xi} \quad \ln \left(\frac{\bar{z} - \xi}{\bar{z} + \xi} \right)$$

If we subtract the limiting value of $\mathrm{Im}\, D^e$ under the integral we get a rapidly convergent integral

$$\lim_{\bar{z} \to \infty} \quad \frac{\kappa}{2\kappa\mu} \quad \frac{2}{\pi} \quad \bar{z}^2 P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \quad \frac{\mathrm{Im}\, D^e}{z^2 - 1} \quad =$$

$$\lim_{\bar{z} \to \infty} \left[\quad \frac{\bar{z}^2}{\pi\mu} \quad P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \quad \frac{1}{z^2 - 1} \quad (\mathrm{Im}\, D^e - \kappa\mu\sigma\xi) \right.$$

$$\left. + \quad \frac{\bar{z}^2}{2\pi\mu} \quad \frac{\kappa\mu\sigma}{\bar{z}\xi} \quad \ln \frac{\bar{z} - \xi}{\bar{z} + \xi} \quad \right]$$

$$= \quad O(1) \quad - \quad \frac{\kappa\sigma}{\pi} \ln \bar{z}$$

and a term in $\mathrm{Re}\, S^e$ of the form:

$$\mathrm{Re}\, S^e \sim \frac{k}{\pi}\, \sigma \ln z$$

These terms are not terribly significant since there is no reason for $\mathrm{Im}\, S^e$ to go to zero as we have supposed. If $\mathrm{Im}\, S^e$ were bounded at infinity, it would produce an additional term in $\log z$ in $\mathrm{Re}\, S^e$ which could cancel the term we have found. A more singular behavior of $\mathrm{Im}\, S^e$ would be necessary to eliminate the $\sqrt{z}$ dependence of $\mathrm{Re}\, S^e$, thus we are left with the conclusion that S^e must be unbounded at infinity, if the total cross section remains finite at infinity, and we shall assume that S^e behaves as $\sqrt{z}$.

Section IV: The Yang-Fermi Ambiguity

The first important application of the dispersion relations was made by Anderson, Davidon, and Kruse.[6] They used the dispersion relations for the non-spin-flip forward scattering amplitude to check the experimental phases against integrated total cross sections. These relations determine the sign of the real part of the forward scattering amplitude. Thus Anderson, Davidon, and Kruse were able to resolve the ambiguity in the phases as to the overall sign of the phase shifts and as to whether or not the real part of the amplitude changed sign at the first maximum in the scattering cross section. They observed that the amplitude did change sign, and thus concluded that δ_{33} went through 90° at the maximum. Actually, all that was shown was that there was no cusp in the energy dependence of the phases. The alternative Yang phase shifts are also chosen to reproduce the real part of the forward scattering amplitude. In fact, any two sets of phase shifts that predict the same angular distributions will predict the same absolute magnitude of the real part of the forward scattering amplitude.

If the $T = \frac{1}{2}$ phase shifts are assumed to be negligible in the low energy region, the Yang phase shifts are simply related to the Fermi phase shifts by[22]

$$\delta_{31}' - \delta_{33}' = \delta_{33} - \delta_{31}$$

$$e^{2i\delta_{31}'} + 2e^{2i\delta_{33}'} = e^{2i\delta_{31}} + 2e^{2i\delta_{33}} \tag{4.1}$$

in which the primes refer to the Yang phases, the unprimed phases are the Fermi ones. The second equation is just the statement that the forward scattering amplitude is unchanged. The change in sign of the difference between δ_{33} and δ_{31} has a profound effect on the spin-flip amplitudes. Qualitatively, the Yang phases have a large δ_{31}' which goes through a resonance at a low energy. We shall show that the Yang phase shifts are not consistent with the dispersion relations for the relativistic spin-flip amplitudes g^e and g^o. These relations determine the sign of the real part of the spin-flip amplitude in terms of a large coupling constant term and relatively small integrals over the imaginary parts of the spin-flip amplitudes.

In the forward direction, we combine the relations for g^e and g^o given by (3.24) to obtain a relation containing only $\mathrm{Re}\, g^3$ on the left hand side.

$$\mathrm{Re}\, g^3(\bar{z}) = \mathrm{Re}\, g^e - \mathrm{Re}\, g^o = -\frac{G^2/\kappa\mu}{\bar{z}-\nu} + \frac{2}{\pi} P \int_1^\infty \frac{dz}{z^2-\bar{z}^2}\left(\bar{z}\,\mathrm{Im}\, g^e(z) - z\,\mathrm{Im}\, g^o(z)\right) \tag{4.2}$$

Now using the expression for $g^3(z)$ in terms of the direct and spin-flip amplitudes (3.30) and the phase shift expansions for D and S (3.32)

$$g(z) = \frac{1}{2\beta E} D + \frac{\kappa}{\beta E} S$$

$$= \frac{4\pi}{\beta\mu\eta} \sum_0^\infty \left(l a_{l-} + (l+1) a_{l+}\right) + \frac{8\pi\kappa}{\mu^3\eta^3} \sum_1^\infty \left(a_{l-} - a_{l+}\right) \frac{l(l+1)}{2} \tag{4.3}$$

The term in D is of order $\frac{\mu^2\eta^2}{2\beta\kappa}$ the term in S. This coefficient is $O(\mu^2/4\kappa^2)$ below the resonance and only 9% at one Bev. Under the integrals the terms in D can be dropped also. For g^e at threshold

$$\mathrm{Re}\, g^e(1) = -G^2/\kappa\mu + \frac{2}{\pi} \int_1^\infty \frac{dz}{z^2-1}\, \mathrm{Im}\, D^e/2\beta E$$

$$+ 2\kappa/\pi \int_1^\infty \frac{dz}{z^2-1}\, \mathrm{Im}\, S/\beta E$$

Using the total cross sections collected by Anderson, Davidon, and Kruse,(6) the integral over the direct amplitude may be evaluated:

$$\frac{2}{\pi} \int_1^\infty \frac{dz}{z^2-1} \frac{\mathrm{Im}\, D^e}{2\beta E} = \frac{\mu}{\pi} \int_1^\infty \frac{dz}{z^2-1} \frac{\eta}{\beta} (\sigma^+ + \sigma^-) = 0.2/\mu^2$$

The coupling constant term in the relation for $\mathrm{Re}\, g^e$ at threshold is $G^2/\kappa\mu \sim 28/\mu^2$, so the direct amplitude has only a 1% contribution. For the odd isotopic index term at threshold:

$$\mathrm{Re}\, g^o(1) = i\, G^2/\kappa\mu + \frac{2}{\pi} \int_1^\infty \frac{z\, dz}{z^2-1}\, \mathrm{Im}\, D^o/2\beta E$$

$$+ \frac{2}{\pi} \kappa \int_1^{\infty} \frac{z\,dz}{z^2-1} \; \mathrm{Im}\, S^{\circ}/\beta E$$

and the integral is

$$\frac{1}{\pi} \int_1^{\infty} \frac{z\,dz}{z^2-1} \; \frac{\mathrm{Im}\, D^{\circ}}{\beta E} = \frac{\mu}{\pi} \int_1^{\infty} \frac{z\,dz}{z^2-1} \; \frac{\eta}{\beta} (\sigma^- - \sigma^+) = -0.11/\mu^2$$

while the coupling constant term is $h\, G^2/\kappa\mu \sim 2.1/\mu^2$. Here the contribution of the direct amplitude is 5% of the coupling constant term, but in the relation for $\mathrm{Re}\, g^3$ when the relations for g^e and g° are subtracted, the total contribution from the non-spin-flip amplitudes is 1% of the coupling constant term and thus is negligible. Then

$$\frac{\kappa}{\beta E} \mathrm{Re}\, S^3(\bar z) = - \frac{G^2/\kappa\mu}{\bar z - h} + \frac{2}{3\pi} P \int_1^{\infty} \frac{z + 2\bar z}{z^2 - \bar z^2} \; \frac{\kappa}{\beta E} \; \mathrm{Im}\, S^3(z)\, dz$$

$$- \frac{2}{3\pi} P \int_1^{\infty} \frac{dz}{z + \bar z} \; \frac{\kappa}{\beta E} \; \mathrm{Im}\, S^1(z) \tag{4.4}$$

We shall drop the d-phases and the higher phase shifts under the integrals. These phases are small at low energies, $<$ 300 Mev., and at high energies we expect the spin-flip scattering to become small. In any case the contributions from the higher phases would be the same for the Yang case as for the Fermi case. On the left hand side of the relation (4.4) for the spin-flip amplitudes we shall drop the d-phases. They are small, and do not enter when the relation is specialized to threshold. Then

$$\frac{1}{\eta^3}\left(\sin\delta_{31}\cos\delta_{31} - \sin\delta_{33}\cos\delta_{33}\right) =$$

$$-\frac{2f^2}{\bar{z}-\nu} + \frac{2}{3\pi}\,P\int_1^\infty \frac{z+2\bar{z}}{z^2-\bar{z}^2}\,dz\left(\frac{\sin^2\delta_{31}-\sin^2\delta_{33}}{\eta^3}\right) \quad (4.5)$$

$$+\frac{2}{3\pi}\int_1^\infty \frac{dz}{z+\bar{z}}\left(\frac{\sin^2\delta_{11}-\sin^2\delta_{13}}{\eta^3}\right)$$

Where f^2 is the rationalized, renormalized pseudovector coupling constant: $f^2 = \lambda^2 g^2 = \left(\frac{\mu}{2K}\right)^2 G^2/4\pi$.

Dropping the $T = \frac{1}{2}$ p-waves and introducing δ° for the scattering lengths, $\delta \sim \delta^\circ \eta^3$ for p-waves, we have at threshold

$$\delta^\circ_{31} - \delta^\circ_{33} = -\frac{2f^2}{1-\nu} + \frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\left(\frac{\sin^2\delta_{31}-\sin^2\delta_{33}}{\eta^3}\right) \quad (4.6)$$

We shall show analytically that the Yang phases are not a possible set. Using Haberschaim's[20] value for f^2 , $f^2 = 0.082$, the coupling constant term is -0.18. The difference in scattering lengths on the left hand side of equation (4.6) is, from experiment $\sim$ -0.25. Thus in the Fermi case, the contribution of the integral is less than one-half that of the coupling constant term and of the same sign. In order for the Yang phase shifts defined by (4.1) to be a possible alternative set, the right hand side of equation (4.6) must change sign when the Yang shifts are substituted for the Fermi shifts under the integral, because the difference in scattering lengths that occurs on the left hand side is defined to change sign. The coupling constant term remains fixed, however, so the integral must change sign and increase in magnitude six fold since the Fermi integral is

$$\frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\left(\frac{\sin^2\delta_{31}-\sin^2\delta_{33}}{\eta^3}\right) \sim -0.25+0.18 \sim 0.07 \qquad (4.7)$$

and the Yang integral must be

$$\frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\left(\frac{\sin^2\delta_{31}'-\sin^2\delta_{33}}{\eta^3}\right) \sim 0.25+0.18 \sim 0.43$$

The forward scattering amplitudes are equal, hence

$$\sin^2\delta_{31}' + 2\sin^2\delta_{33}' = \sin^2\delta_{31} + 2\sin^2\delta_{33}$$

and since the Fermi $\sin^2\delta_{31}$ is negligible,

$$\sin^2\delta_{31}' < 2\sin^2\delta_{33}$$

The positive contribution of the integral in (4.6) is maximized if only the δ_{31}' phase is considered and the δ_{33}' dropped

$$\frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\left(\frac{\sin^2\delta_{31}'-\sin^2\delta_{33}'}{\eta^3}\right) < \frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\,\frac{\sin^2\delta_{31}'}{\eta^3} < 2\,\frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\,\frac{\sin^2\delta_{33}}{\eta^3}$$

The integral over $\sin^2\delta_{33}$, is just the integral in (4.7), neglecting $\sin^2\delta_{31}$, and so

$$\frac{2}{3\pi}\int_1^\infty \frac{z+2}{z^2-1}\,dz\ \frac{\sin^2\delta_{31}' - \sin^2\delta_{33}'}{\eta^3} < 0.14$$

less than a third of the size necessary for the Yang phase shifts to be a possible fit. This procedure overestimates the contribution of the integral in the Yang case. The term in $\sin^2\delta_{33}'$ is quite large and a more accurate calculation yields even worse agreement. Such a calculation was performed by Screaton, integrating the Anderson phase shifts(23) to check the Fermi and Yang fits, and was published by Gilbert and Screaton.(24) Another resolution of the Yang-Fermi ambiguity was published by Davidon and Goldberger.(25)

Section V: Sum Rules for Meson-Nucleon Scattering

We shall now derive several interesting sum rules for the coupling constant in terms of integrals over phase shifts. The first is an approximate sum rule that relates an integral over the δ_{33} phase to the coupling constant. The others will be exact rules relating the coupling constant to integrals over the real parts of the amplitudes. The first sum rule is approximate because in order to derive it we must make too stringent assumptions about the high energy behavior of the theory. We assume a theory in which the scattering goes to zero at high energies. That is, we assume some kind of cut-off, but we shall not specify this cut-off. Then the matrix elements in the sum over intermediate states in the part of $f^e(z)$ that arises from a commutator of currents would vanish for high energies both on and off the energy shell, and we would expect a relation for $f^e(z)$ in the forward direction of the form

$$\mathrm{Re}\, f^e(\bar{z}) = \Lambda + \frac{2}{\pi} P \int_1^\infty \frac{z\,dz}{z^2 - \bar{z}^2}\, \mathrm{Im}\, f^e(z) \tag{5.1}$$

Similarly we could repeat Goldberger's derivation for the forward scattering amplitude and obtain for the direct amplitude a relation of the form

$$\mathrm{Re}\, D^e(\bar{z}) = 2\kappa\Lambda + \frac{2}{\pi} P \int_1^\infty \frac{z\,dz}{z^2-\bar{z}^2}\, \mathrm{Im}\, D^e(z) - \frac{2\mu^2 G^2}{\bar{z}^2-\mu^2} \tag{5.2}$$

We retain the Λ -term in both these relations since there is no reason, even in a cut-off theory, to forbid a possible meson-meson interaction; besides, a finite constant must occur in these equations if the theory is to have repulsive s-waves. Another way to approach what we are doing is simply to assume these two relations as the simplest possible relations and ask about the behavior of a theory whose scattering amplitude obeys them; a theory in which the total cross section vanishes faster than $1/z$ at high energies. We now eliminate the constant term in Λ from these two equations. Since $D = 2\kappa f + 2\kappa m z g$ a relation involving only g^e is obtained:

$$\bar{z}\, \mathrm{Re}\, g^e(\bar{z}) = -\frac{\mu^2 G^2/\kappa m}{\bar{z}^2-\mu^2} + \frac{2}{\pi} P \int_1^\infty \frac{z^2\,dz}{z^2-\bar{z}^2}\, \mathrm{Im}\, g^e(z) \tag{5.3}$$

This relation differs from the equation previously obtained for $g^e(z)$, equation (3.24), only by the expression

$$0 = \frac{2}{\pi}\int_1^\infty dz\, \mathrm{Im}\, g^e(z) + G^2/\kappa m \tag{5.4}$$

which is also the value of relation (5.3) for $\bar{z} = 0$. In terms of the direct and spin-flip amplitudes this becomes

$$G^2/_{K\mu} = -\frac{2}{\pi}\int_1^\infty dz\,\frac{\ln D^e(z)}{2\beta E} - \frac{2}{\pi}\int_1^\infty \frac{2K}{2\beta E}\,dz\,\ln S^e(z) \qquad (5.5)$$

and in terms of cross sections

$$G^2/_{2K^2\mu} = -\frac{\mu}{2\pi K}\int_1^\infty dz\,\frac{\eta}{\beta}\,(\sigma^+ + \sigma^-) - \frac{2}{\pi}\int_1^\infty \frac{dz}{2\beta E}\,\ln S^e(z)$$

Now the term arising from the direct amplitude is of order $\mu^2/_{K^2}$ in a cut-off theory. In a theory in which the cross section is constant at high energies, the term in D^e is divergent. We can estimate the contribution of this term by imposing a sharp cut-off and evaluating the integral using the experimental cross sections. We evaluate numerically

$$\frac{\mu}{\pi}\int_1^{z_c} dz\,\frac{\eta}{\beta}\,(\sigma^+ + \sigma^-)$$

For a cut-off z_c = 7.7, or 930 Mev, the contribution of the integral is 6% of the coupling constant term for g^2 = 15. For z_c = 10, the integral is 8%, and for z_c = 19, 2.5 Bev laboratory energy, the integral is 20% of the coupling constant term. Since this term is small and thus its effect is relatively insensitive to the cut-off, we drop it.

Dropping the higher phases in the spin-flip amplitude, we have, writing out the p-wave terms:

$$G^2/_{\kappa\mu} = -\frac{2}{\pi}\int_1^\infty dz\,\frac{\ln D^e(z)}{2\beta E} - \frac{2}{\pi}\int_1^\infty \frac{2\kappa}{2\beta E}\,dz\,\ln S^e(z) \qquad (5.5)$$

and in terms of cross sections

$$G^2/_{2\kappa^2\mu} = -\frac{\mu}{2\pi\kappa}\int_1^\infty dz\,\frac{\eta}{\beta}\,(\sigma^+ + \sigma^-) - \frac{2}{\pi}\int_1^\infty \frac{dz}{2\beta E}\,\ln S^e(z)$$

Now the term arising from the direct amplitude is of order $\mu^2/_{\kappa^2}$ in a cut-off theory. In a theory in which the cross section is constant at high energies, the term in D^e is divergent. We can estimate the contribution of this term by imposing a sharp cut-off and evaluating the integral using the experimental cross sections. We evaluate numerically

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Dropping the higher phases in the spin-flip amplitude, we have, writing out the p-wave terms:

$$\frac{3\pi}{4}\,\frac{\mu^2}{K^2}\,g^2 = 3\pi f^2 = \int_1^\infty \frac{dz}{\eta^3}\left[2\sin^2\delta_{33} - 2\sin^2\delta_{31} + \sin^2\delta_{13} - \sin^2\delta_{11}\right] \tag{5.6}$$

The coupling constant term is rather large compared to the integral of a phase shift unless the phase shift is resonant or very large at a low energy. From the way the phases enter in (5.6) we can make some qualitative observations about the theory. Only two of the four p-phases, the two $P_{3/2}$ phases, enter with the correct sign. If either of the $P_{1/2}$ phases is large, it gives a negative contribution making it still more difficult to fulfill (5.6). The Yang shifts would give a negative integral and so are ruled out in a theory in which this sum rule holds. Since the 33-phase shift enters with twice the weight of the 13-phase, it is easier, in terms of a slowly varying phase shift, to fulfill the equation with a resonance in the 33-state than in the 13-state.

In line with these observations, we shall estimate the position of the resonance, dropping all but the 33-phase shift, Then (5.6) reduces to

$$\frac{3\pi}{8}\,\frac{\mu^2}{K^2}\,g^2 = \frac{3\pi}{2} f^2 = \int_1^\infty dz\,\frac{\sin^2\delta_{33}}{\eta^3} \tag{5.7}$$

If we approximate the integral by neglecting the contribution to the integrand while δ_{33} increases from zero at threshold to 90°

and assume that δ_{33} = 90° from the resonance out to infinity, we have the integral $\int_{z_r}^{\infty} dz/\eta^3$ which can easily be estimated numerically and will yield an estimate for z_r which will approximate the position of the resonance. The integral is

z_r	=	4	3	2.8	2.5	2	1.7
$\int_{z_r}^{\infty} dz/\eta^3$	=	0.27	0.36	0.39	0.44	0.62	0.79

The value of the coupling constant term in (5.7) is, for g^2 = 15

$$\frac{3\pi}{8} g^2 \, \mu^2/\kappa^2 = 0.39$$

Thus this estimate for z_r yields 2.8 or 250 Mev laboratory energy. This estimate should be too large since the high energy contributions to the integral have been overestimated. The contribution to the integral for $z_r > 19$ or energies greater than 2.5 Bev is 0.1 but any reasonable phase would have become small or have been lost in the other phases at such energies. If we had tried to fit (5.6) using only the 13-phase shift, the coupling constant term would be twice as large and z_r would have to be 1.7 or 98 Mev. Such behavior for the 13-phase shift is in violation of the effective range formulae,[14] which require a negative phase shift for the 13-state while this resonant behavior would require a large positive shift, and of experiment.

Returning to (5.7), we shall evaluate this expression using the effective range formula of Chew and Low[26]

$$\frac{\eta^3}{\omega^*} \cot \delta_{33} = 8.05 - 3.8\,\omega^*$$

in which $\omega^* = E - \kappa$. The integral is then

$$\int_1^\infty \frac{dz}{\eta^3} \sin^2\delta_{33} = 0.37$$

corresponding to $g^2 = 14.1$ or $f^2 = 0.079$. We shall use formula (5.7) to evaluate integrals of slowly varying functions of z multiplied by $\frac{\sin^2\delta_{33}}{\eta^3}$. We shall use our knowledge from experiment, that the 33-state is dominant and has a resonance, to estimate quantities that can be related to the integral in (5.7). The resonance behavior given by an effective range relationship or by experiment makes the function $\frac{\sin^2\delta_{33}}{\eta^3}$ quite sharply peaked, so we can estimate integrals of the form

$$\int_1^\infty h(z) \frac{\sin^2\delta_{33}}{\eta^3}\, dz \simeq h(z_r) \frac{3\pi}{8} \frac{\mu^2}{\kappa^2} g^2 \tag{5.8}$$

where z_r is the value of z near resonance. As an example of this use of the sum rule, we compute the difference in the 3/2 isotopic spin p-wave scattering lengths that arose in the discussion of the Yang-Fermi ambiguity. We had equation (4.6)

$$\delta_{31}^0 - \delta_{33}^0 = -2f^2/1-\mu - \frac{2}{3\pi} \int_1^\infty \frac{z+2}{z^2-1}\, dz \frac{\sin^2\delta_{33}}{\eta^3}$$

The integral is

$$\frac{2}{3\pi} \int_1^\infty \frac{z+2}{z^2-1}\, dz \frac{\sin^2\delta_{33}}{\eta^3} \simeq \frac{1}{4} \frac{z_r+2}{z_r^2-1} \mu^2/\kappa^2\, g^2$$

and if we take the position of the resonance to be $z_r = 2.4$, or 195 Mev, and $g^2 = 15$, we get -0.08 for the integral and

$$\delta_{31}^{0} - \delta_{33}^{0} \approx -0.18 - 0.08 = -0.26$$

in good agreement with an integration of the experimental phases shifts.[24] Another application is the estimation of the p-wave contributions to the difference between the s-wave scattering lengths. If the Goldberger relation (3.31) for the odd isotopic index direct amplitude is specialized to threshold and only the 33-phase retained under the integral, we get

$$\delta_1^{0} - \delta_3^{0} = 3\lambda g^2 \mu/_{\kappa+\mu} - \frac{4}{\pi}\frac{1}{\kappa+\mu}\int_1^\infty \frac{dz}{z^2-1}\, E/_{\eta}\, \sin^2\delta_{33}$$

Since $z^2 - 1 = \frac{\eta^2 E^2}{\kappa^2}$, the integral is estimated to be

$$\int_1^\infty \frac{dz}{z^2-1}\, E/_{\eta}\, \sin^2\delta_{33} \approx \frac{3\pi}{8}\, \mu^2/_{E_r}\, g^2$$

and then

$$\delta_1^{0} - \delta_3^{0} = \frac{\mu g^2}{\kappa+\mu}\left[3\lambda - 3/_2\, \mu/_{E_r}\right]$$

where E_r is the value of E at resonance. Again taking $z_r = 2.4$ and $g^2 = 15$, E_r becomes 8.8 and the result is

$$\delta_1^{0} - \delta_3^{0} = 0.436 - 0.332 = 0.10$$

In this example the contribution from the integral over the direct amplitude is overestimated by making this δ_{33} -approximation. This will be discussed again in section VIII.

Another approach to this sum rule is to derive it by assuming that the real part of $g^e(z)$ goes to zero at infinity faster than $1/z$. If $\lim_{z\to\infty} z\,\mathrm{Re}\, g^e(z) \to 0$ the relation (3.24) for $g^e(z)$ would imply that

$$\frac{2}{\pi}\int_1^\infty dz\, \mathrm{Im}\, g^e(z) = -\,G^2/K\mu$$

on taking the limit of the relation as z goes to infinity. Of course this condition on $\mathrm{Re}\, g^e(z)$ obtains in the case that we have considered where relations (5.1) and (5.2) hold. Then

$$\lim_{z\to\infty} \mathrm{Re}\, D^e(z) \to 2K\Lambda$$

and

$$\lim_{z\to\infty} \mathrm{Re}\, f^e(z) \to \Lambda$$

therefore

$$\lim z\,g^e(z) = \lim_{z\to\infty}\left[\frac{1}{2K\mu} D^e - \frac{1}{\mu} f^e\right] \to 0$$

In order to derive a similar sum rule based on $g^o(z)$ we would have to show that $\lim_{z\to\infty} z^2\mathrm{Re}\, g^o(z) \to 0$. This would require a stronger cut-off than has been required for the even amplitude. The static theory sum rules that correspond to (5.6) and to the one that could be derived for $g^o(z)$ have been studied by Cini and Fubini.[27] The simultaneous existence of both sum rules requires the small p-phase shifts to have large contributions at energies near one Bev, since the two sum rules cannot simultaneously be satisfied with a resonant 33-phase shift alone.

We shall now derive another sum rule relating an integral of the real part of the T-matrix element to the coupling constant. This sum rule is exact in the sense that no very stringent assumptions about the behavior of the functions at high energies have to be made. In the process of deriving this sum rule, we shall develop a new set of relations that will connect the imaginary part of an amplitude to an integral of the real part in the physical region and to a bound state term. These new relations differ from the conjugate Hilbert transforms that relate an integral of the real part of an amplitude to the imaginary part in that the integrals are restricted to the experimentally known region above threshold. Consider as an example the function $g^e(z)$. We have previously written a relation for $g^e(z)$:

$$\mathrm{Re}\, g^e(\bar{z}) = \frac{2}{\pi} \bar{z}\, P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2}\, \mathrm{Im}\, g^e(z) - \frac{\bar{z}}{\bar{z}^2 - \mu^2} \, G^2/K\mu$$

Which corresponds to the following properties of the function:

1) $g^e(z)$ is analytic in the upper half of the complex z -plane.

2) $\mathrm{Im}\, g^e(z)$ is zero on the real axis for $-1 < z < 1$ with the exception of a delta-function contribution

$$\pi/2 \; G^2/K\mu \left(\delta(z-\mu) + \delta(z+\mu) \right)$$

3) the real part of $g^e(z)$ is an odd function on the real axis, the imaginary part of $g^e(z)$ is an even function on the real axis, and

4) $g^e(z)$ is less than z at infinity in the complex plane.

Now we consider the function $g^{e}(z)/\sqrt{z^2-1}$. By $\sqrt{z^2-1}$ we mean that branch of the function that is analytic in the upper half plane, positive for z real, $z >$ 1, negative for z real, $z <$ -1, and positive imaginary for z real, -1 < z < 1. The imaginary part of this function is positive in the upper half plane, and the function has no zeros above the real axis. This function is just the dimensionless laboratory momentum of the incoming meson. Now the function $\frac{g^{e}(z)}{\sqrt{z^2-1}}$ will have the following properties:

1) it is analytic in the upper half plane,

2) it goes to zero at infinity,

3) the limit of this function onto the real axis from above has an even real part and an odd imaginary part, and

4) between z = -1 and z = 1 on the real axis the real part of the function is zero with the exception of two delta-function singularities, the real part of this function being $\mathrm{Im}\, g^{e}(z)/\sqrt{1-z^2}$ in this region.

From the boundedness and symmetry properties we can write:

$$\mathrm{Im}\left[\frac{g^{e}(\bar{z})}{\sqrt{\bar{z}^2-1}}\right] = -\frac{2}{\pi}\,\bar{z}\,P\int_0^\infty \frac{dz}{z^2-\bar{z}^2}\;\mathrm{Re}\left[\frac{g^{e}(z)}{\sqrt{z^2-1}}\right]$$

where $\mathrm{Im}[\;]$ and $\mathrm{Re}[\;]$ mean the imaginary and real parts of the limit of the analytic function inside the brackets on the real axis from above. No trouble arises when the contour of integration is brought to the real axis. The integration is thought of, at first, as including semi-circles above the points $z = \pm 1$. The contribution of these semi-circles vanishes as the radius shrinks to zero, since the singularity in $\mathrm{Re}\left[g^e/\sqrt{z^2-1} \right]$ at $z = \pm 1$ is integrable. Using the relation between the real part of $g^e/\sqrt{z^2-1}$ and the imaginary part of g^e in the non-physical region:

$$\mathrm{Im}\left[\frac{g^e(\bar{z})}{\sqrt{\bar{z}^2-1}} \right] = -\frac{2}{\pi}\,\bar{z}\, P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\, g^e(z)}{\sqrt{z^2-1}} - \frac{2}{\pi}\,\bar{z}\, P\int_0^1 \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Im}\, g^e(z)}{\sqrt{1-z^2}} \tag{5.9}$$

$$= -\frac{2}{\pi}\,\bar{z}\, P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\frac{\mathrm{Re}\, g^e(z)}{\sqrt{z^2-1}} + \frac{\bar{z}}{\bar{z}^2-\mu^2}\, G^2/K\mu \; 1/\sqrt{1-\mu^2}$$

The left hand side of this relation is

$$\frac{\mathrm{Im}\, g^e(\bar{z})}{\sqrt{\bar{z}^2-1}} \qquad \text{for} \qquad \bar{z} > 1$$

and

$$-\frac{\mathrm{Re}\, g^e(\bar{z})}{\sqrt{1-\bar{z}^2}} \qquad \text{for} \qquad \bar{z} < 1$$

There is an infinite discontinuity in (5.9) at the point $\bar{z} = 1$, since $\mathrm{Im}\, g^e(z)$ goes to zero as η at threshold while $\mathrm{Re}\, g^e(z)$ is a constant in the neighborhood of the threshold. A sum rule is obtained by multiplying this relation by $\bar{z}$ and taking the limit as $\bar{z}$ goes to infinity. Then we have, dropping the μ^2 in $\sqrt{1-\mu^2}$:

$$\mathrm{Im}\, g^e(\infty) = G^2/K\mu + \frac{2}{\pi}\int_1^\infty dz\, \frac{\mathrm{Re}\, g^e(z)}{\sqrt{z^2-1}} \tag{5.10}$$

We expect this to converge, since we have argued that at high energies $\mathrm{Re}\, D^e \sim 1$, $\mathrm{Im}\, D^e \sim z$ and S is of the order of $\sqrt{z}$. The term in $\mathrm{Im}\, g^e(\infty)$ is simply related to the high energy cross sections. From (3.35)

$$\lim_{z\to\infty} \mathrm{Im}\, D^e(z) = z \kappa \mu \bar{\sigma}$$

and

$$\lim_{z\to\infty} \mathrm{Im}\, g^e(z) = \frac{\kappa \mu \bar{\sigma}}{2\kappa\mu} = \bar{\sigma}/2$$

since the contribution from S^e to g^e behaves as $1/\sqrt{z}$. If we assume that the cross sections remain constant above 1.5 Bev, then $\bar{\sigma} = 3/\mu^2$ and this term is about 5% of the coupling constant term in (5.10).

Before exploiting this sum rule we record relations for the other amplitudes. Introduce $\xi = \sqrt{z^2-1}$, the dimensionless laboratory momentum. Then, dropping ν^2 in $\sqrt{1-\nu^2}$; some of the other possible relations are

$$\mathrm{Im}\left[g^o(\bar{z})/\bar{\xi} \right] = -\frac{2}{\pi} P \int_1^\infty \frac{z\, dz}{z^2-\bar{z}^2}\, \mathrm{Re}\, g^o(z)/\xi - \frac{\nu}{\bar{z}^2-\nu^2}\, G^2/\kappa\mu$$

$$\mathrm{Im}\left[\frac{D^o(\bar{z})}{\bar{\xi}} \right] = -\frac{2}{\pi} P \int_1^\infty \frac{dz}{z^2-\bar{z}^2}\, \mathrm{Re}\, D^o/\xi - \frac{\bar{z}}{\bar{z}^2-\nu^2}\, 2\nu G^2 \tag{5.11}$$

$$\mathrm{Im}\left[D^e(\bar{z})/\bar{\xi} \right] - \mathrm{Im}\left[D^e(\bar{z}')/\bar{\xi}' \right] = -2\nu^2 G^2 \left[\frac{\bar{z}^2-\bar{z}'^2}{(\bar{z}^2-\nu^2)(\bar{z}'^2-\nu^2)} \right]$$

$$- \frac{2}{\pi}(\bar{z}^2-\bar{z}'^2)\, P \int_1^\infty \frac{z\, dz}{(z^2-\bar{z}^2)(z^2-\bar{z}'^2)}\, \mathrm{Re}\, D^e/\xi$$

The relations for the direct amplitudes can be rewritten in terms of cross sections. Since $\mathrm{Im}\, D^{o} = \xi\, \kappa\mu\, (\sigma^{-} - \sigma^{+})$ and $\mathrm{Im}\, D^{e} = \xi\, \kappa\mu\, (\sigma^{+} + \sigma^{-})$

$$\sigma^{-} - \sigma^{+} = -b\, G^2/\kappa\mu\, \bar{z} - \frac{2}{\pi}\, \bar{z}/\kappa\mu\; P \int_1^{\infty} \frac{dz}{z^2 - \bar{z}^2}\, \mathrm{Re}\, D^{o}/\xi$$

$$(\sigma^{+} + \sigma^{-})_{\bar{z}} - (\sigma^{+} + \sigma^{-})_{\bar{z}'} = (\bar{z}^2 - \bar{z}'^2) \left\{ - \frac{2 b^2 G^2}{\kappa\mu\, \bar{z}^2 \bar{z}'^2} \right. \tag{5.12}$$

$$\left. - \frac{2}{\pi \kappa\mu}\, P \int_1^{\infty} \frac{z\, dz}{(z^2 - \bar{z}^2)(z^2 - \bar{z}'^2)}\, \mathrm{Re}\, D^{e}(z)/\xi \right\}$$

The integrals over the real parts of the amplitudes would be difficult to handle near threshold since the integrand is so singular there. The integrands can be modified in order to remove either the principal part singularity or the singularity at the threshold. Since ξ has the analytic properties listed earlier, the following integrals are easily derived from Cauchy's theorem

$$\frac{2}{\pi}\, P \int_1^{\infty} \frac{z\, dz}{z^2 - \bar{z}^2}\, 1/\xi = \begin{cases} 0 & \bar{z} > 1 \\ \dfrac{1}{\sqrt{1 - \bar{z}^2}} & \bar{z} < 1 \end{cases}$$

$$P \int_1^{\infty} \frac{z\, dz}{(z^2 - \bar{z}^2)(z^2 - \bar{z}'^2)}\, 1/\xi = 0 \qquad \bar{z}, \bar{z}' > 1 \tag{5.13}$$

Using these integrals, the integrals in (5.12) could be modified for easier computation.

$$P \int_1^{\infty} \frac{dz}{z^2 - \bar{z}^2}\, \frac{\mathrm{Re}\, D^{o}(z)}{\xi} = \int_1^{\infty} \frac{dz}{z^2 - \bar{z}^2}\, \frac{1}{\xi} \left[\mathrm{Re}\, D^{o}(z) - \frac{z}{\bar{z}}\, \mathrm{Re}\, D^{o}(\bar{z}) \right]$$

for example, and for the relation for the even isotopic index amplitude

$$(\sigma^+ + \sigma^-)_{\bar{z}} - (\sigma^+ + \sigma^-)_1 = \bar{\xi}^2 \Big\{ -\, 2 r^2 G^2 / \kappa\mu\, \bar{z}^2$$

$$- \frac{2}{\pi \kappa \mu}\, P \int_1^\infty \frac{z\, dz}{z^2 - \bar{z}^2} \left[\frac{\mathrm{Re}\, D^e(z) - \mathrm{Re}\, D^e(1)}{\xi^3} \right] \Big\}$$

Returning to the relation (5.10), dropping the total cross section term at infinity, and rewriting the g^e -amplitude in terms of the direct and spin-flip amplitudes, we have:

$$- G^2/\kappa\mu = \frac{2}{\pi} \int_1^\infty \frac{dz}{2\beta E}\, \mathrm{Re}\, D^e/\xi + \frac{2}{\pi}\, 2\kappa \int_1^\infty dz\ \mathrm{Re}\, S^e/2\beta E \xi$$

The term containing $\mathrm{Re}\, D^e$ is of order μ^2/κ^2 of the second term. Dropping all but the p-waves in the spin-flip amplitude and assuming the scattering to be completely elastic we have:

$$\frac{3\pi}{2}\, \mu^2/\kappa^2\, g^2 = 6\pi f^2 = \int_1^\infty \frac{dz}{\xi \eta^3}$$

$$\times \left(2 \sin 2\delta_{33} - 2 \sin 2\delta_{31} + \sin \delta_{13} - \sin \delta_{11} \right)$$

This relation may be used to determine the coupling constant. The contributions of the small p-waves are important since the integral is over $\sin 2\delta$ rather than $\sin^2 \delta$. Using Anderson's[19] "machine fit" phase shifts, which have a positive δ_{13}, one gets

$$f^2 = 0.090$$

Using the same phases but reversing the sign of δ_{13} one would have

$$f^2 = 0.084$$

Using the values for the small phases calculated in section VII with Anderson's δ_{33} :

$$f^2 = 0.089$$

Using the Chew and Low effective range formula for δ_{33} (0.235 slope at threshold) along with the p-waves of section VII

$$f^2 = 0.084$$

The variation in the coupling constant is not excessive considering the large variation in the small p-waves between these evaluations. If we take the term at infinity in (5.10) into account this lowers the coupling constant by

$$f^2 - \frac{\mu^3 \sigma}{32\pi K}$$

which is 0.0045 if σ is $3/\mu^2$. This leaves for the Anderson phases

$$f^2 = 0.086 \tag{5.14}$$

Most of the contribution to the integrals comes from low energies; for the Anderson phase shifts:

$$\int_1^\infty \frac{dz}{\xi\eta^3} \sin 2\delta_{33} = 0.65, \text{ of which 50\% of the contribution}$$

occurs below 30 Mev and 10% arises above the resonance,

$$\int_1^\infty \frac{dz}{\xi\eta^3} \sin 2\delta_{31} = -0.13, \quad 50\% \text{ below 30 Mev,}$$

$$\int_1^\infty \frac{dz}{\xi\eta^3} \sin 2\delta_{13} = 0.06, \quad \text{mostly from values near the resonance,}$$

and $\int_1^\infty \frac{dz}{\xi\eta^3} \sin 2\delta_{11} = -0.08$, 50% below 30 Mev.

Using only the 33-phase shift, we would have gotten $f^2 = 0.069$ so that the contribution of the small p-phases is about 25%. Since so much of the contribution to the integral arises from energies near threshold, we shall modify the integral in order to separate out this contribution in terms of the scattering lengths. The quantity under the integral is $\frac{\sin 2\delta}{\eta^3}$ which is constant near threshold when the phases are small. We shall subtract a term from the integrand which just cancels the threshold value of the integrand. From (5.13), $\frac{2}{\pi}\int_1^\infty \frac{dz}{z\xi} = 1$; equation (5.10) may be modified to read:

$$\bar{\sigma}/2 = G^2/K\mu + \frac{2}{\pi}\int_1^\infty \frac{dz}{\xi}\left[\mathrm{Re}\, g^{e}(z) - \frac{\mathrm{Re}\, g^{e}(1)}{z}\right] + \mathrm{Re}\, g^{e}(1)$$

As a rough first approximation we shall drop the small integral term. Writing out $\mathrm{Re}\, g^{e}(1)$ and dropping the small terms arising from $D^{e}(1)$, we have

$$3 f^2 \sim \delta_{33}^{\circ} - \delta_{31}^{\circ} + \frac{1}{2}(\delta_{13}^{\circ} - \delta_{11}^{\circ}) - \frac{3\mu^3 \sigma}{32\pi K}$$

If we use Anderson's values for the scattering lengths:

$$\delta_{33}^{\circ} = 0.248, \quad \delta_{31}^{\circ} = -0.042, \quad \delta_{11}^{\circ} = -0.0175, \quad \delta_{13}^{\circ} = +0.0048$$

Then, including the σ correction.

$$f^2 \sim 0.096$$

The scattering lengths obtained in section VII along with Orear's value for the 33-phase shift yield a similar value:

$$\delta_{33}^{\circ} = 0.235, \quad \delta_{31}^{\circ} = -0.038, \quad \delta_{11}^{\circ} = -0.12, \quad \delta_{13}^{\circ} = -0.03$$

and

$$f^2 \sim 0.098$$

A more accurate sum rule may be obtained from (5.9). We shall specialize this relation to threshold; however the integral is quite singular in this limit and we again perform a subtraction under the integral sign. From (5.13) we can deduce that

$$\lim_{\bar{z} \to 1+} \frac{2}{\pi} P \int_1^{\infty} \frac{dz}{z^2 - \bar{z}^2} \frac{1}{z \xi} = -1$$

Then we reduce the singularity of the integral:

$$\frac{G^2}{\kappa\mu} = \lim_{\bar{z}\to 1} \frac{\mathrm{Im}\, g^e(\bar{z})}{\xi} - \mathrm{Re}\, g^e(1)$$

$$+ \frac{2}{\pi}\int_1^\infty \frac{dz}{z^2-1}\,\frac{1}{\xi}\left[\mathrm{Re}\, g^e(z) - \frac{1}{z}\,\mathrm{Re}\, g^e(1)\right] \tag{5.15}$$

There is no trouble with the limit of the principal part function, since the integrand after subtraction has an integrable singularity at $z = 1$. The threshold term is related to the squares of the s-wave scattering lengths. It is

$$\lim_{\bar{z}\to 1} \frac{\mathrm{Im}\, g^e(\bar{z})}{\bar{\xi}} = \frac{2\pi\mu}{3(\kappa+\mu)}\left[(\delta_1^\circ)^2 + 2(\delta_3^\circ)^2\right]$$

which is less than 1% of the coupling constant term. The contribution to $g^e(z)$ from the direct amplitude is also about this magnitude. The integral is rapidly convergent and thus easy to evaluate with the experimental data. Dropping all the higher waves in $g^e(z)$ we have

$$3f^2 = \delta_{33}^\circ - \delta_{31}^\circ + \frac{1}{2}(\delta_{13}^\circ - \delta_{11}^\circ)$$

$$- \left[\lambda_{33} - \lambda_{31} + \frac{1}{2}(\lambda_{13} - \lambda_{11})\right] \tag{5.16}$$

in which

$$\lambda = \frac{1}{\pi}\int_1^\infty \frac{dz}{\xi^3}\left[\frac{\sin 2\delta}{\eta^3} - \frac{2\delta^\circ}{z}\right] \tag{5.17}$$

$$G^2/\kappa\mu = \lim_{\bar{z}\to 1} \frac{\mathrm{Im}\, g^e(\bar{z})}{\xi} - \mathrm{Re}\, g^e(1)$$

$$+ \frac{2}{\pi}\int_1^\infty \frac{dz}{z^2-1}\frac{1}{\xi}\left[\mathrm{Re}\, g^e(z) - \frac{1}{z}\mathrm{Re}\, g^e(1)\right] \tag{5.15}$$

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$$- \left[\lambda_{33} - \lambda_{31} + \frac{1}{2}(\lambda_{13} - \lambda_{11})\right] \tag{5.16}$$

in which

$$\lambda = \frac{1}{\pi}\int_1^\infty \frac{dz}{\xi^3}\left[\frac{\sin 2\delta}{\eta^3} - \frac{2\delta^\circ}{z}\right] \tag{5.17}$$

The relation (5.15) differs from the previously obtained relation only by the term in $\bar{\sigma}$, the threshold term, and the integral. Combining these two relations, we find

$$\bar{\sigma}/2 - \frac{2\pi\mu}{3(\kappa+\mu)}\left[(\delta_1^0)^2 + 2(\delta_3^0)^2\right] = \frac{2}{\pi}\int_1^\infty \frac{z^2\,dz}{\xi^3} \times \left(\mathrm{Re}\, g^c(z) - \mathrm{Re}\, g^c(1)/z\right) \tag{5.18}$$

This equation connects the high energy cross section with the low energy phase shifts.

We evaluate (5.16) using the Anderson phase shifts. The integrals are

λ_{33}	$= 0.0404$	λ_{13}	$= 0.0029$
λ_{31}	$= -0.0060$	λ_{11}	$= -0.0033$

The contribution from the scattering lengths is

$$3f^2 \sim 0.301$$

and the correction from the integral is 20% of this, being -0.050 leaving a coupling constant

$$f^2 = 0.084$$

This is in excellent agreement with the value $f^2 = 0.086$ obtained from the same data by relation (5.10). An alternative way of looking at these values is to use them to compute the limit of the total scattering cross section in the limit of infinite energy in terms of the Anderson phases. We had previously

$$f^2 = 0.090 - \bar{\sigma}\mu^3/32\pi K$$

or

$$\bar{\sigma} = (\sigma^+ + \sigma^-)_\infty = 4/\mu^2 \text{ or } 80 \text{ mb.}$$

as opposed to the value of 60 mb that corresponds to the way we have interpreted the experimental indications. This is a dubious way of calculating the asymptotic cross section. It depends on the assumptions we have made about the high energy behavior of the theory, and the agreement we have obtained, which indicates that the high energy contributions to the integrals cancel, is probably fortuitous. This coupling constant evaluation has given a lower value for the coupling constant than was obtained by Davidon and Goldberger[21] who found $f^2 = 0.10$, using the same data. The reason for this is probably that the relation (5.15) is more suitable for the calculation of the coupling constant than is the relation they used, which would correspond to (4.6). The integral in our relation is more rapidly convergent by a factor of z^2, and, furthermore, in the relation they used, f^2 was determined as the difference of two numbers 3/2 and 1/2 of its size. The contributions to the sum rule that we have used are mostly additive. The value that we have found is in excellent agreement with that given by Haberschaim: $f^2 = 0.082$.[20]

Another approach to this computation of the coupling constant is to modify equation (5.9) so that it could be used to represent the experimental data as a straight line whose intercept would be the coupling constant. This would provide, in addition to a value for the coupling constant, a stringent check on the experimental data. Performing subtractions to remove the singular behavior of the integrals, (5.9) can easily be brought to the form

$$G^2/K_\mu + \bar{z}^2 \left[\operatorname{Re} g^e(1) - \frac{2}{\pi} \int_1^\infty \frac{dz}{\xi^3} \chi(z) \right]$$

$$= \frac{\bar{z}}{\bar{\xi}} \operatorname{Im} g^e(\bar{z}) + \frac{2}{\pi} \bar{z}^2 \bar{\xi}^2 \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \frac{1}{\xi^3} \left(\chi(z) - \frac{\bar{z}}{z} \chi(\bar{z}) \right) \tag{5.19}$$

where

$$\chi(z) = \operatorname{Re} g^e(z) - \frac{1}{z} \operatorname{Re} g^e(1)$$

The integrals over the experimental phases are rapidly convergent. If the right hand side of (5.19) were graphed against $\bar{z}^2$, the data could be extrapolated to zero to determine the coupling constant. An alternative approach is to use the fact that the threshold value of $\frac{\operatorname{Im} g^e(z)}{\xi}$ is insignificant and graph

$$G^2/K_\mu = - \frac{\bar{z}}{\bar{\xi}^3} \operatorname{Im} g^e(\bar{z}) - \frac{2}{\pi} \bar{z}^2 \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \, 1/\xi^3 \left(\chi(z) - \bar{z}/z \, \chi(\bar{z}) \right) \tag{5.20}$$

which should be a straight horizontal line.

Section VI:

The Derivative Relations and the 33-Phase Shift

We shall now develop the relativistic generalization of the equations used by Chew and Low[14] to derive the effective range formulae. We have, in the two dispersion relations for $g^e(z)$ and $g^o(z)$, essentially two relations for the four p-wave phases, if we assume that the contribution of the d-waves and higher waves is negligible in the low energy region of interest. We can obtain two further relations of the p-phases by considering the derivatives of the amplitudes with respect to the cosine of the angle of scattering in the center-of-mass system. This derivative is simply related to the derivative with respect to x, the dimensionless measure of the momentum transfer. Since

$$x = \frac{pp' - \kappa^2}{\kappa\mu} = \mu/\kappa \; \eta^2 (1 - \cos\theta)$$

then

$$\frac{\partial}{\partial x} = - \frac{\kappa}{\mu} \; 1/\eta^2 \; \frac{\partial}{\partial \cos\theta}$$

because η is a function of z alone. Upon differentiating the equations (3.23) and writing the result in the forward direction, we have:

$$\mathrm{Re}\, \frac{\partial g^o(\bar{z})}{\partial x} = \frac{2}{\pi} P \int_1^\infty \frac{z\,dz}{z^2 - \bar{z}^2} \,\mathrm{Im}\, \frac{\partial g^o}{\partial x} + \frac{1}{\pi} \int_1^\infty \frac{dz}{(z+\bar{z})^2} \,\mathrm{Im}\, g^o(z)$$

$$+ \frac{1}{2} \frac{G^2/\kappa\mu}{(\bar{z} - \nu)^2}$$

$$\frac{\partial}{\partial \nu} \operatorname{Re} f^o(\bar{z}) = \frac{2}{\pi} \bar{z} P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \frac{\partial \operatorname{Im} f^o(z)}{\partial \nu} - \frac{1}{\pi} \int_1^\infty \frac{dz}{(z+\bar{z})^2} \operatorname{Im} f^o(z)$$

$$\frac{\partial}{\partial \nu} \operatorname{Re} g^e(z) = \frac{2}{\pi} \bar{z} P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} \frac{\partial \operatorname{Im} g^e(z)}{\partial \nu} - \frac{1}{\pi} \int_1^\infty \frac{dz}{(z+\bar{z})^2} \operatorname{Im} g^e \qquad (6.1)$$

$$- \frac{1}{2} \frac{G^2/\kappa\mu}{(\bar{z} - \nu)^2}$$

$$\frac{\partial}{\partial \nu} \operatorname{Re} f^e(\bar{z}) - \frac{\partial}{\partial \nu} \operatorname{Re} f^e(\bar{z}') = \frac{2}{\pi} (\bar{z}^2 - \bar{z}'^2) P \int_1^\infty \frac{z\,dz}{(z^2 - \bar{z}^2)(z^2 - \bar{z}'^2)} \frac{\partial \operatorname{Im} f^e}{\partial \nu}$$

$$- \frac{1}{\pi}(\bar{z}^2 - \bar{z}'^2) \int_1^\infty \frac{dz}{(z+\bar{z})^2 (z+\bar{z}')^2} \operatorname{Im} f^e - \frac{2}{\pi}(\bar{z} - \bar{z}') \int_1^\infty \frac{z\,dz \operatorname{Im} f^e}{(z+\bar{z})^2 (z+\bar{z}')^2}$$

Differentiating (3.28) yields the connection to the direct and spin-flip amplitudes, in the forward direction:

$$\frac{\partial g}{\partial \nu} = \frac{1}{2\beta E}\left[\frac{\partial D}{\partial \nu} + \frac{\kappa\mu}{\beta} S\right] + \frac{\kappa}{\beta E}\frac{\partial S}{\partial \nu}$$

$$\frac{\partial f}{\partial \nu} = \frac{\alpha}{2\beta E}\left[\frac{\partial D}{\partial \nu} + \frac{\kappa\mu}{\beta} S\right] - \frac{z\kappa\mu}{\beta E}\frac{\partial S}{\partial \nu} \qquad (6.2)$$

Using the phase shift expansions (3.32) and keeping only the s- and the p-phases, the quantities in the brackets are, for the odd isotopic index term:

$$\frac{\partial D^o}{\partial \nu} + \frac{\kappa\mu}{\beta} S^o = \frac{8\pi E\kappa}{\mu^2 \eta^3} [a_{33} - a_{13}] \qquad (6.3)$$

and for the even isotopic index term:

$$\frac{\partial D^e}{\partial \nu} + \frac{\kappa\mu}{\beta} S^e = - \frac{8\pi E\kappa}{\mu^2 \eta^3} [2a_{33} + a_{13}] \qquad (6.4)$$

The d-wave terms enter into the term $\frac{\partial S}{\partial x}$ in a significant manner, since they become weighted with an additional factor of κ^2/μ^2. From the phase shift expansion, these terms are, using d_{35} and d_{33} for the 3/2 isotopic spin $J = 5/2$ and $J = 3/2$ states respectively, d_{15} and d_{13} for the 1/2 isotopic spin state.

$$\frac{\kappa}{\beta E}\frac{\partial S^e}{\partial x} = -\frac{\kappa^2}{\mu^4}\,{}^1\!/_{\eta^5}\,8\pi\left(d_{13} - d_{15} + 2d_{33} - 2d_{35}\right)$$

$$\frac{\kappa}{\beta E}\frac{\partial S^o}{\partial x} = -\frac{\kappa^2}{\mu^4}\,{}^1\!/_{\eta^5}\,8\pi\left(d_{13} - d_{15} - d_{33} + d_{35}\right) \tag{6.5}$$

The derivative relations must be combined in such a way as to eliminate these d-waves on the left hand side. The combination of the $f(z)$ and $g(z)$ amplitudes required is

$$\frac{\partial f}{\partial x} + \mu z\frac{\partial g}{\partial x} = \frac{1}{2\kappa}\left[\frac{\partial D}{\partial x} + \frac{\kappa\mu}{\beta}S\right]$$

since $\kappa(\alpha + \mu z) = \beta E$. For the odd isotopic index combination this is

$$\frac{1}{2\kappa}\mathrm{Re}\left[\frac{\partial D^o}{\partial x} + \frac{\kappa\mu}{\beta}S^o\right]_{\bar{z}} = \frac{G^2}{2\kappa}\frac{\bar{z}}{(\bar{z}-\lambda)^2}$$

$$+\frac{\bar{z}}{\pi\kappa}P\int_1^\infty \frac{dz}{z^2-\bar{z}^2}\,\mathrm{Im}\left[\frac{\partial D^o}{\partial x} + \frac{\kappa\mu}{\beta}S^o\right] + \frac{1}{\pi}\int_1^\infty \frac{dz}{(z+\bar{z})^2}\,\mathrm{Im}\left(\bar{z}\mu g^o - f^o\right) \tag{6.6}$$

We shall simplify the even isotopic index equations by assuming that $\frac{\partial f^e}{\partial x}$ goes to zero for high energies. That is, we will not use a difference relation for $\frac{\partial f^e}{\partial \nu}$. This corresponds to a strict p-wave approximation: we cut off the phase shift expansion so that the relation for f^e involving the Λ - term converges, and we assume that the Λ -term is independent of pp'. We do this primarily as a convenience in handling the equations; when we finally derive an effective range formula for the 33-phase shift, we shall use the difference relations. The even isotopic index relation is then:

$$\frac{1}{2K} \mathrm{Re}\left[\frac{\partial D^e}{\partial \nu} + \frac{K\mu}{\beta} S^e\right]_{\bar{z}} = -\frac{1}{2} \frac{G^2}{K} \frac{\bar{z}}{(\bar{z}-\nu)^2}$$

$$+ \frac{1}{K\pi} P \int_1^\infty \frac{z\,dz}{z^2 - \bar{z}^2} \,\mathrm{Im}\left[\frac{\partial D^e}{\partial \nu} + \frac{K\mu}{\beta} S^e\right]_{\bar{z}} + \frac{1}{\pi}\int_1^\infty \frac{dz}{(z+\bar{z})^2}\,\mathrm{Im}\,(f^e - \mu\bar{z}g^e) \tag{6.7}$$

Combining these to obtain relations for the 3/2 and 1/2 isotopic spin states, each of which, by (6.3) and (6.4), contains only one phase shift on the left hand side, yields:

$$-\frac{12\pi\bar{E}}{\mu^2\bar{\eta}^3}\sin\delta_{33}\cos\delta_{33} = -\frac{G^2}{K}\,\bar{z}/(\bar{z}-\nu)^2$$

$$-\frac{2}{3\pi} P\int_1^\infty dz\,\frac{12\pi E}{\mu^2\eta^3}\left[\frac{2z+\bar{z}}{z^2-\bar{z}^2}\right]\sin^2\delta_{33} \tag{6.8}$$

$$-\frac{2}{3\pi}\int_1^\infty dz\,\frac{12\pi E}{\mu^2\eta^3}\,\frac{1}{z+\bar{z}}\,\sin^2\delta_{13}$$

$$+\frac{1}{3\pi}\int_1^\infty dz\,\mathrm{Im}\left[\frac{2(f^1-\mu\bar{z}g^1)+(f^3-\mu\bar{z}g^3)}{(z+\bar{z})^2}\right]$$

$$-\frac{12\pi\bar{E}}{\mu^2\bar{\eta}^3}\sin\delta_{13}\cos\delta_{13} = \frac{1}{2}\,\frac{G^2}{K}\,\frac{\bar{z}}{(\bar{z}-\nu)^2}$$

$$-\frac{4}{3\pi}\int_1^\infty dz\,\frac{12\pi E}{\mu^2\eta^3}\,\frac{1}{(z+\bar{z})}\sin^2\delta_{33}$$

$$-\frac{2}{3\pi}P\int_1^\infty dz\,\frac{12\pi E}{\mu^2\eta^3}\left(\frac{z+2\bar{z}}{z^2-\bar{z}^2}\right)\sin^2\delta_{13} \qquad (6.9)$$

$$+\frac{1}{3\pi}\int_1^\infty dz\,\frac{-(f^1-\mu\bar{z}g^1)+4(f^3-\mu\bar{z}g^3)}{(z+\bar{z})^2}$$

The singular integrals in these relations only involve terms of the same isotopic and ordinary spin as occur on the left hand sides. Since we know from experiment and from the analysis of the static theory that the 33-state is dominant, we shall drop all but the 33-phase shift under the integrals. Then the only contribution from the undifferentiated amplitudes is

$$f^3-\mu\bar{z}g^3 \approx \frac{8\pi K}{\mu^2\eta^3}(z+\bar{z})\,a_{33} + \frac{8\pi\alpha}{\beta\mu}\,\frac{1}{\eta}\,a_{33}$$

Rewriting the relation for the 33-phase shift and splitting up the term $\frac{2z+\bar{z}}{z^2-\bar{z}^2}$ into partial fractions, we have

$$\frac{\bar{E}}{\bar{\eta}^3}\sin\delta_{33}\cos\delta_{33} = \frac{\mu^2}{3K}g^2\frac{\bar{z}}{(\bar{z}-\nu)^2} + \frac{1}{\pi}P\int_1^\infty dz\,\frac{E}{\eta^3}\,\frac{\sin^2\delta_{33}}{z-\bar{z}}$$

$$+\frac{1}{3\pi}\int_1^\infty \frac{E}{\eta^3}\sin^2\delta_{33}\,\frac{dz}{z+\bar{z}} \qquad (6.10)$$

$$-\frac{2}{9\pi}\int_1^\infty \frac{K}{\eta^3}\sin^2\delta_{33}\,\frac{dz}{z+\bar{z}}$$

$$-\frac{2\mu}{9\pi}\int_1^\infty dz\,\frac{\alpha}{\beta\eta}\sin^2\delta_{33}\,\frac{1}{(z+\bar{z})^2}$$

We shall drop the last three terms on the right hand side. The third and fourth terms cancel each other to leave a slowly varying contribution of the order of 10% of the second term. The last term is of the order of $\mu/5K$ with respect to the second term, and has the opposite sign to the difference between the third and fourth terms. This leaves

$$\bar{E}/\bar{\eta}^3 \sin\delta_{33}\cos\delta_{33} = \frac{\mu^2}{3K}g^2\frac{\bar{z}}{(\bar{z}-\lambda)^2} + \frac{1}{\pi}P\int_1^\infty dz\, \frac{E}{\eta^3}\,\frac{\sin^2\delta_{33}}{z-\bar{z}} \tag{6.11}$$

The δ_{33} sum rule developed in section V can be used to evaluate the scattering length. For equation (6.11) we have

$$\delta_{33}^0 \sim \frac{\mu^2 g^2}{K^2}\left(\frac{1}{3}\,\frac{1}{(1-\lambda)^2}\,\frac{K}{K+\mu} + \frac{3}{8}\,\frac{E_R}{K+\mu}\,\frac{1}{z_R-1}\right)$$

which is, taking $z_R = 2.4$, $g^2 = 15$, $\delta_{33}^0 = 0.22$. The same result is obtained from (6.10) since the additional terms are small. The numerical values for the different terms are:

$$\delta_{33}^0 = \frac{4}{3}f^2\left[1.02 + 0.92 + 0.06 - 0.01\right]$$

from (6.10). If z_R is taken to be 2.3, δ_{33}^0 would be 0.23. This value is not too bad, considering the fact that we are working with an approximate equation ignoring the large high energy contributions to this equation in a true theory.

Equation (6.11) is analogous to the equation for the 33-state derived in the static theory by Chew and Low(14) if the crossing terms have been dropped. We shall write several effective range solutions to this equation, based on the Chew and Low methods. Dropping the b in the denominator of the coupling constant term, we consider the function $r(\bar{z})$ of the complex variable $\bar{z}$ defined by

$$r(\bar{z}) = \frac{g^2\mu^2}{3K}\frac{1}{\bar{z}} + \frac{1}{\pi}\int_1^\infty \frac{dz}{z-\bar{z}} \, E/\eta^3 \sin^2\delta_{33} \tag{6.12}$$

Then, if $\bar{z}$ approaches the real axis from above, the limit of $r(z)$ is

$$r(z) = E/\eta^3 \, e^{i\delta_{33}} \sin\delta_{33} \qquad 1 < z < \infty \tag{6.13}$$

$r(z)$ is an analytic function in the entire plane with a cut line along the positive real axis from one to infinity and a pole with residue $\frac{g^2\mu^2}{3K}$ at the origin. $r(z)$ goes to zero at least as $1/z$ at infinity. If we assume that $r(z)$ has no zeros, we can derive a solution of (6.11). Of course there is a good deal of ambiguity in the solutions that we find; they are defined only to within a meromorphic function which we have assumed to be zero, see Dyson, Dalitz, and Castillejo[28] for a discussion of this point. We shall first examine a solution obtained by introducing a new function

$$h(z) = \frac{1}{r(z)\, z^2}$$

Then $h(z)$ is an analytic function in the entire plane with a cut line from one to infinity along the real axis, a pole with residue $3K/g^2\mu^2$ at the origin, and a $1/z$ behavior at infinity.

A representation for $h(z)$ in terms of its imaginary part along the cut line is

$$h(\bar{z}) = \frac{3\kappa}{g^2\mu^2}\frac{1}{\bar{z}} + \frac{1}{\pi}\int_1^\infty \frac{dz}{z-\bar{z}}\, \mathrm{Im}\, h(z)$$

But from the defining equation for $h(z)$, we see that

$$\mathrm{Re}\, h(z) = \frac{\eta^3}{E z^2}\cot\delta_{33} \qquad \text{for} \qquad 1 < z < \infty$$

and

$$\mathrm{Im}\, h(z) = -\,{\eta^3}/{E z^2}$$

approaching the real axis from above. Taking the real part of the representation yields an effective range expression:

$$\frac{\eta^3 \cot\delta_{33}}{E\, z} = \frac{3\kappa}{g^2\mu^2} - \frac{\bar{z}}{\pi}\, P\int_1^\infty \frac{dz}{z-\bar{z}}\, {\eta^3}/{E z^2} \tag{6.14}$$

The integral on the right hand side is convergent and may be performed to yield

$$P\int_1^\infty \frac{dz}{z-\bar{z}}\,{\eta^3}/{E z^2} = -\,\frac{\bar{\eta}^3}{\bar{E}}\,{1}/{\bar{z}^2}\, \ln\left[\frac{\bar{z}-1+\bar{\xi}}{\bar{\xi}-\bar{z}+1}\right] + \frac{\pi\mu}{\kappa^2}\,\frac{2\bar{z}-a}{\bar{z}^2}$$

$$+ \frac{1}{\kappa}\left[\frac{a^2-1}{a+\bar{z}} - \frac{1}{\bar{z}}\right] + \frac{2\mu}{\kappa}\ln\left[\frac{a+1+\sqrt{a^2-1}}{a+1-\sqrt{a^2-1}}\right]$$

$$\times\, \frac{\sqrt{a^2-1}}{(a+\bar{z})^2}\left(\bar{z}\,(2+a^2) + 3a\right)$$

where $a = {\kappa}/{2\mu}$. This is a slowly varying function of $\bar{z}$ whose value in the range $1 < \bar{z} < 3$ is $\sim \frac{1}{2\mu}$. Then

$$\frac{\eta^3 \cot \delta_{33}}{E\,z} \sim \frac{3\kappa}{g^2\mu^2}\left[1 - z/z_R\right]$$

and the effective range is

$$z_R \approx \frac{3\kappa m}{g^2} \;\; \pi/_{1/2\mu} \sim 8.5$$

for $q^2 = 15$. The scattering length is 0.11 and there is no resonance since for $\bar{z} = 8$, the integral is $0.2/\mu$ corresponding to an effective range of the order of 20. For $g^2 = 50$ one would get an effective range of 2.5 but the scattering length would be 0.5. This is quite different from the static theory result. The static limit of equation (6.14) is just the effective range relationship of the static theory(14) dropping the crossing terms. If ω and k are the meson energy and momentum, then the relation is

$$\frac{k^3 \cot \delta_{33}}{\omega} \approx 3\kappa^2/g^2 - \frac{\omega}{\pi} P \int_\mu^\infty \frac{d\bar{\omega}}{\bar{\omega} - \omega} \; \bar{k}^3/\bar{\omega}^2$$

The integral that yields the effective range is now divergent, and a cut off must be supplied. For a cut off of the order of the nucleon mass, the integral is $\sim 7\mu$ and the effective range is about 4. This is in much better agreement; the divergence has enhanced the result. We have, however, made a specific mistake in finding the solution (6.14). We have imposed the boundary condition on $h(z)$ that it behaves as $1/z$ at infinity which corresponds to, in principle, $\cot \delta_{33}$ approaching a constant at infinity. We were saved from the consequences of this by the fact that the integral in (6.14) behaves logarithmically at infinity and the phase that we have found will approach zero logarithmically at infinity. If we impose a more rapid approach to either zero or 180° at infinity, we would expect $r(z)$ to approach zero considerably more rapidly than $1/z$. If this is to be the case, the analytic function which is related to the reciprocal of $r(z)$ must be given another inverse power of z. If we consider

$$\chi(z) = \frac{1}{r(z)\; z^2(z-1)}$$

this function will have the correct behavior at infinity. The use of this new function is equivalent to using a difference equation originally for $\frac{\partial}{\partial r} f^e$. All the approximations are better, since the integrals converge more rapidly, and the contributions from high energies become less significant. This new function is analytic in the entire plane, has a cut line from one to infinity along the real axis, has a pole at the origin with residue $-\frac{3K}{g^2\mu^2}$, and has a pole at $z = 1$ with residue $\frac{1}{(K+M)\delta^0_{33}}$. Then we can derive the equation

$$\frac{\eta^3 \cot \delta_{33}}{E\, z} = -\frac{3K}{g^2\mu^2}(z-1) + \frac{1}{(K+M)\,\delta^0_{33}}\, z - \frac{z(z-1)}{\pi}\, P\int_1^\infty \frac{d\bar{z}}{\bar{z}-z}\, \frac{\bar{\eta}^3}{\bar{E}\,\bar{z}^2(\bar{z}-1)} \tag{6.15}$$

Ignoring the integral, which is small, we have

$$\frac{\eta^3 \cot \delta_{33}}{E\, z} = \frac{3K}{g^2\mu^2}\left[1 - z/z_r\right] \tag{6.16}$$

and the effective range is

$$z_n = \frac{\delta^0_{33}\,\frac{K+M}{K}}{\delta^0_{33}\left(\frac{K+M}{K}\right) - \frac{4}{3}f^2}$$

for $f^2 = 0.084$ and $\delta^0_{33} = 0.235$ the resonance would be at $z_n = 1.7$. This relation shows a great tendency to go through resonance. For a fixed value of the scattering length, as the coupling constant increases the resonance moves to higher energies, and agreement with experiment is obtainable with $f^2 \sim 0.11$ yielding $z_n \sim 2.4$. We still have a two parameter relation, since the threshold scattering length has been introduced in order to yield the correct high energy behavior, a rôle that is left to the cut-off function in the static theory. A distinct advantage over the static theory is that both of the parameters are observable quantities. The integral in (6.15) yields a positive contribution, the term increasing as $\bar{z}$ for large $\bar{z}$, but the term in $-\bar{z}$ always dominates and the phase goes to 180° at high energies. The functional form of this effective range relationship is different from that proposed by Chew and Low. They proposed that $\frac{\eta^3}{\omega^*}\cot\delta_{33}$ graphed against $\omega^* = E - K$ would show a straight line behavior. We are going to graph $\frac{\eta^3}{Ez}\cot\delta_{33}$ against z, the laboratory energy. The experimental data fit a straight line on this new plot better than they did on a Chew-Low plot, since the high energy points at 220 Mev are lifted onto a straight line with the low energy points. Instead of using (6.16), however, we shall derive a more correct relation by not neglecting the λ in the position of the coupling constant term in (6.11). This correction produces a 10% effect and improves the agreement with experiment. The analytic function related to the reciprocal of $e^{i\delta}\sin\delta$ is then

$$\varphi(z) = \frac{z}{\eta(z)\,(z-\lambda)^3\,(z-1)}$$

which has the usual cut line from one to infinity along the real axis, has a pole at $z=\lambda$ with residue $-\frac{3\kappa}{g^2\mu^2}\,\frac{1}{1-\lambda}$ and one at $z=1$ with residue $\frac{1}{(1-\lambda)^3(\kappa+\mu)\,\delta_{33}^{\circ}}$. Note that $(1-\lambda)^2(\kappa+\mu)=\kappa$ dropping terms in λ^2. Then we have for $\cot\delta_{33}$.

$$\frac{z}{(z-\lambda)^2}\,\frac{\eta^3\cot\delta_{33}}{E} = -\frac{3\kappa\,(z-1)}{g^2\mu^2(1-\lambda)} + \frac{z-\lambda}{(1-\lambda)\,\delta_{33}^{\circ}\,\kappa}$$
$$- \frac{(z-\lambda)(z-1)}{\pi}\,P\int_1^\infty \frac{d\bar{z}}{\bar{z}-z}\,\frac{\bar{\eta}^3}{\bar{E}}\,\frac{\bar{z}}{(\bar{z}-\lambda)^3(\bar{z}-1)} \qquad (6.17)$$

The integral is quite small, less than 5% around 150 Mev and very small at 300 Mev; we shall drop it. The relation that we shall compare with experiment is then

$$\frac{z}{(z-\lambda)^2}\,\frac{\mu}{E}\,\eta^3\cot\delta_{33} = a\left[1 - z/z_r\right] \qquad (6.18)$$

in which

$$a = \frac{\mu}{\kappa(1-\mu)} \left[\frac{3}{4f^2} - \mu/\delta^{\circ}_{33} \right] \tag{6.19}$$

and

$$z_r = \left[\frac{\delta^{\circ}_{33} - \mu \, 4/3 \, f^2}{\delta^{\circ}_{33} - 4/3 \, f^2} \right] \tag{6.20}$$

Such a comparison is embodied in figure I. The experimental data are those points collected by Ashkin, Blaser, Feiner, and Stern.[29] The points lie on a straight line characterized by the equation

$$1.05 - 0.45\, z$$

which corresponds to $f^2 = 0.11$ and $\delta^{\circ}_{33} = 0.25$. The extrapolation for the coupling constant and the scattering length is not very accurate, the dashed line also fits the points and corresponds to $f^2 = 0.10$ and $\delta^{\circ}_{33} = 0.24$. Both these lines resonate near $z = 2.35$ or 188 Mev. The determination of the resonance is better than the Chew-Low plot, since the high energy points are used to fix the straight line, rather than having to extrapolate the low energy data alone. The spread of experimental points corresponds roughly to a 3% spread in the resonance, but the experimental data themselves are only 5 - 10% accurate. In Figure II five more points are incorporated. These are the points reported by Mukhin, Ozerov, Pontecorvo, Grigoriev, and Mitin[30] for 200, 240, 270, 307, and 310 Mev. The dotted line would be the Chew-Low effective range curve at these energies. We should emphasize again that the straight line prediction is based on the difference equation for $f^{e}(z,x)$, and the approximations are that the small phase shifts can be dropped as negligible, that the crossing terms in the 33-phase shift are dropped since they are small, and that the scattering is assumed to be elastic and the d-waves, small. The integral that has been ignored can be computed; it is very small and passes through zero near $z = 3$.

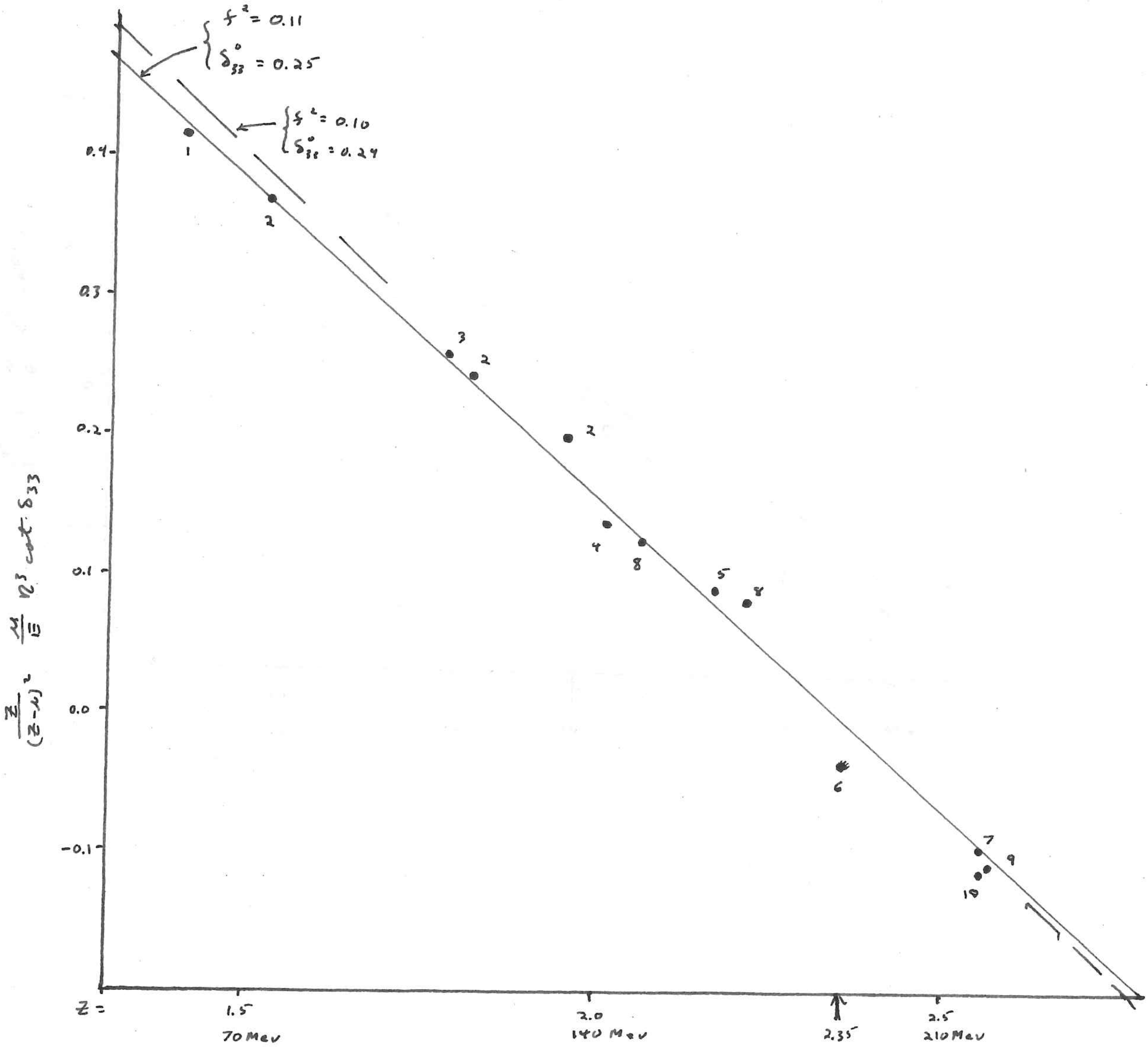

Figure I: The Effective Range Relation

$\frac{Z}{(Z-\mu)^2}\,\frac{\mu}{E}\,\eta^3 \cot \delta_{33}$ graphed against z the dimensionless laboratory energy of the meson. The experimental points are numbered to correspond to the references in Ashkin et al.[29]

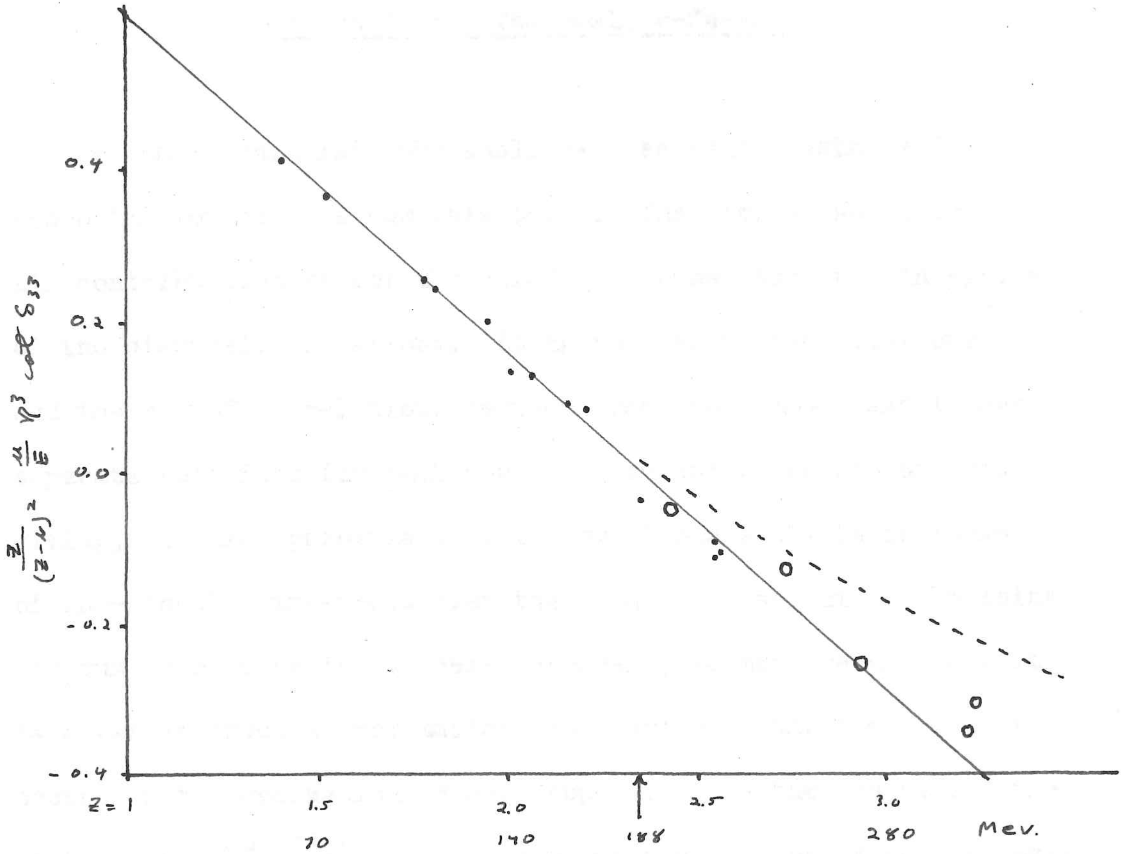

Figure II: The Effective Range Relation for High Energies

$\frac{z}{(z-\mu)^2}\frac{\mu}{E}\eta^3 \cot \delta_{33}$ graphed against z. The circles are the Russian data.[30] The dashed line is the Chew-Low extrapolation as given by Orear.[21]

Section VII: The Small P-Phases

We shall calculate the small p-phase shifts using a 33-approximation and the sum rule (5.8). That is, we shall drop all contributions except the $\sin^2 \delta_{33}$ terms under the integrals of the dispersion relations. Using the derivative relations and the spin-flip relations derived from the $g(z)$ amplitudes, separate relations for each p-wave phase shift can be obtained, giving, in this approximation, the small phase shifts in terms of non-singular integrals over the δ_{33} phase shift. In using the sum rule to evaluate these integrals, we must recall that it is a rather crude approximation and, further, that the z_r that occurs in the evaluation is only approximately the resonance, the maximum in $\frac{\sin^2 \delta_{33}}{\eta^3}$ occurring before the resonance. We make the further assumption that the high energy contributions are negligible, so that we do not have to use difference relations but can use relations (6.8) and (6.9) directly, both to compute δ_{13} and to eliminate δ_{13} and δ_{33} from the spin-flip relations derived from $g^c(z)$ and $g^o(z)$.

We begin with the 13-phase shift. From (6.9) we have, restricting the undifferentiated amplitudes to the $\sin^2 \delta_{33}$ terms:

$$\frac{\bar{E}}{\bar{\eta}^3} \sin \delta_{13} \cos \delta_{13} = -\frac{1}{6} \frac{\mu^2 g^2}{\kappa} \frac{\bar{z}}{(\bar{z}-\lambda)^2} + \frac{4}{3\pi} \int_1^\infty \frac{dz}{z+\bar{z}} \frac{\sin^2 \delta_{33}}{\eta^3} \left(E - \frac{2}{3}\kappa\right) \tag{7.1}$$

$$- \frac{8}{9\pi} \mu \int_1^\infty dz \frac{\alpha}{\beta \eta} \sin^2 \delta_{33} \frac{1}{(z+\bar{z})^2}$$

The sum rule yields

$$\sin 2\delta_{13} \approx 2\eta^3 \frac{g^2\mu^2}{\bar{E}K}\left[-\frac{1}{6}\frac{\bar{z}}{(\bar{z}-\mu)^2} + \frac{1}{2}\frac{E_n - \frac{2}{3}K}{K(z+z_n)}\right. \tag{7.2}$$

$$\left. - \frac{1}{3}\frac{\alpha_n}{\beta_n K}\eta_n^2 \frac{1}{(z+z_n)^2}\right]$$

where the subscript ' n ' means that the quantity in question is evaluated at $z = z_n$. Then taking z_n = 2.4 and g^2 = 15, we have, upon inserting numerical values,

$$\sin 2\delta_{13} \approx 2\eta^3 \frac{g^2\mu^2}{\bar{E}K}\left\{-0.167\frac{z}{(z-\mu)^2} + 0.325\frac{1}{z+2.4} - \frac{0.156}{(z+2.4)^2}\right\}$$

which yields the following values for δ_{13} :

z	=	1.5	2.0	3.0	4.0
δ_{13}	=	-0.6°	$-1.^\circ$	-0.3°	$+1^\circ$

(7.3)

and at threshold the scattering length is δ_{13}° = -0.03. The qualitative behavior of the phase is, of course, the most trustworthy: that the phase shift is extremely small. A 10% change in z_n will change this phase shift by 50% . For example, if we take z_n to be 2.7, then the phase shift is

z	=	1.5	2.0	3.0	4.0
δ_{13}	=	-0.7°	-1.2°	-0.7°	0.4°

but the qualitative behavior remains unchanged.

For the other p-waves, we combine the relations for g^e and g^o as was done in section IV to get relations for the isotopic spin 3/2 and 1/2 spin-flip amplitudes. Thus

$$\mathrm{Re}\, g^3(z) = \mathrm{Re}\, g^e - \mathrm{Re}\, g^o = - \frac{G^2}{\kappa\mu} \frac{1}{z - \lambda}$$

$$+ \frac{2}{3\pi} P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} (z + 2\bar{z}) \,\mathrm{Im}\, g^3(z) - \frac{2}{3\pi} \int_1^\infty \frac{dz}{z + \bar{z}} \,\mathrm{Im}\, g^1(z)$$

and

$$\mathrm{Re}\, g^1(z) = \mathrm{Re}\, g^e + 2\,\mathrm{Re}\, g^o = - \frac{G^2}{\kappa\mu} \left[\frac{z - 2\lambda}{z^2} \right]$$

$$- \frac{4}{3\pi} \int_1^\infty \frac{dz}{z + \bar{z}} \,\mathrm{Im}\, g^3(z) + \frac{2}{3\pi} P \int_1^\infty \frac{dz}{z^2 - \bar{z}^2} (2z + \bar{z}) \,\mathrm{Im}\, g^1(z)$$

Dropping the small direct amplitude terms and the higher waves, the amplitudes are

$$g^3 = \frac{8\pi\kappa}{\mu^3 \eta^3} (a_{31} - a_{33})$$

and

$$g^1 = \frac{8\pi\kappa}{\mu^3 \eta^3} (a_{11} - a_{13})$$

If these relations are combined with (6.8) and (6.9), the resulting relations for the δ_{11} and δ_{31} phase shifts are

$$\sin 2\delta_{11} = 2\bar{\eta}^3 \Bigg\{ - \frac{1}{2} g^2 \mu^2/\kappa^2 \left[\frac{\bar{z} - 2\lambda}{\bar{z}^2} + \frac{\kappa}{3\bar{E}} \frac{\bar{z}}{(\bar{z} - \lambda)^2} \right]$$

$$+ \frac{4}{3\pi} \int_1^\infty \frac{dz}{z + \bar{z}} \frac{\sin^2 \delta_{33}}{\eta^3} \left[1 + E/\bar{E} - \frac{2}{3} \kappa/\bar{E} \right] \tag{7.4}$$

$$- \frac{8}{9\pi} \, \mu/\bar{E} \int_1^\infty d\bar{z} \, \frac{\alpha}{\beta\eta} \, \sin^2 \delta_{33} \, \frac{1}{(z+\bar{z})^2} \Bigg\}$$

and

$$\sin 2\delta_{31} \approx 2\eta^3 \Bigg\{ - \frac{1}{2} \frac{g^2\mu^2}{\kappa^2} \left[\frac{1}{z-\nu} - \frac{2}{3} \frac{\kappa}{E} \frac{\bar{z}}{\bar{z}-\nu} \right]$$

$$- \frac{1}{\pi} \int_1^\infty d\bar{z} \, \frac{\sin^2 \delta_{33}}{\eta^3} \left(1 - E/\bar{E} \right) \frac{1}{z - \bar{z}}$$

$$+ \frac{1}{3\pi} \int_1^\infty d\bar{z} \, \frac{\sin^2 \delta_{33}}{\eta^3} \, \frac{1}{z+\bar{z}} \left[1 + E/\bar{E} - \frac{2}{3} \frac{\kappa}{\bar{E}} \right] \tag{7.5}$$

$$- \frac{2}{9\pi} \, \mu/\bar{E} \int_1^\infty d\bar{z} \, \frac{\sin^2 \delta_{33}}{(z+\bar{z})^2} \, \frac{\alpha}{\beta\eta} \Bigg\}$$

and the sum rule yields

$$\sin 2\delta_{11} \approx 2\eta^3 \frac{g^2\mu^2}{\kappa^2} \Bigg[- \frac{z - 2\nu}{z^2} - \frac{\kappa}{6E} \frac{z}{(z-\nu)^2}$$

$$+ \frac{1}{2} \frac{1}{z+z_r} \left(1 + \frac{E_r - \frac{2}{3}\kappa}{E_r} \right) - \frac{1}{3} \frac{1}{E(z+z_r)^2} \frac{\alpha_r}{\beta_r} \eta_r^2 \Bigg]$$

and

$$\sin 2\delta_{31} \approx 2\eta^3 \, g^2\mu^2/\kappa^2 \Bigg[- \frac{1}{2(z-\nu)} + \frac{\kappa}{3E} \frac{z}{(z-\nu)^2}$$

$$+ \frac{3}{8} \frac{E_r - E}{z_r - z} \frac{1}{E} + \frac{1}{8} \frac{1}{z+z_r} \left(1 + \frac{E_r - \frac{2}{3}\kappa}{E_r} \right) - \frac{1}{12} \frac{1}{(z+z_r)^2} \frac{\alpha_r \eta_r^2}{E \beta_r} \Bigg]$$

Using the same parameters as for (7.3), we have:

z	=	1.5	2.0	3.0	4.0
δ_{11}	=	-3.2°	-6.6°	-11.5°	-13.6°

(7.6)

δ_{31} =	-1°	-2.6°	-4.0°	-5.4°	
at	70	140	280	420	Mev

and the scattering lengths are

$$\delta_{11}^{\circ} = -0.12 \quad \text{and} \quad \delta_{31}^{\circ} = -0.038$$

Changing z_n from 2.4 to 2.7 changes δ_{31} from -2.6° to -2.8° at $z = 2$ and changes δ_{11} from -6.6° to -7.2° at $z = 2$. These larger phases are more stable with respect to changes in z_n than is the small 13-phase, a 10% change in producing only a 10% change in the phases. The second figures in these phases are not significant.

The δ_{31} that we have calculated is in good agreement with the values obtained by Puppi[31] which are $\delta_{31} = -1.4^\circ$ and -2.6° at 80 and 120 Mev. The other phases are in rough agreement with the Anderson machine-fit phase shifts[19] in that although δ_{13} is not positive, it is several degrees more positive than δ_{31} in the resonance region. The δ_{11} that we have calculated is about twice as large as the Anderson value at 140 Mev, about four degrees more negative at higher energies.

The equations that we have been using break down at high energies, in that this 33-approximation and the sum rule eventually violate unitarity. As written, assuming the p-wave approximation to be valid and the neglect of the higher phase shifts on the left hand side to be justified, the left hand side of the equations involves

$\sin 2\delta$ which must always be less than unity. If we push these equations to energies higher than we have used, the η^3 will dominate and increase indefinitely. For low energies, the estimates of the phases that we have made should be adequate. The contribution of the small phases to the integrals is negligible at low energies, although it must become significant at high energies in a p-wave approximation in order to satisfy unitarity. This is just what occurs in the static theory in which a strict p-wave approximation satisfying unitarity is required; the p-waves show rapidly varying behavior at energies near the cut off.

The static limit of the expressions that we have derived for the phase shifts is easily obtained by setting E equal to κ and dropping all terms of the form M/κ . Then

$$\sin 2\delta_{11} \approx -\frac{16}{3}\eta^3 f^2 \frac{z_r}{z(z+z_r)}$$

and (7.7)

$$\sin 2\delta_{13} = \sin 2\delta_{31} \approx -\frac{4}{3}\eta^3 f^2 \frac{z_r}{z(z+z_r)}$$

This last identity is the usual static theory assertion that $\delta_{13} = \delta_{31}$. We see that in this limit the phases are all negative and $\sin 2\delta_{11}$ is four times as large as $\sin 2\delta_{13}$ or $\sin 2\delta_{31}$. The more general results that we have previously obtained do not have δ_{13} and δ_{31} identical and are in qualitative agreement with experiment in that $\delta_{13} > \delta_{31}$.

Section VIII: The S-Waves

We shall only give a brief discussion of the s-waves, because the quantities of greatest interest, the s-wave scattering lengths, cannot be determined in a p-wave theory even with the use of the dispersion relations. The magnitude of the s-waves is intimately connected with the high energy behavior of the theory, as is known even from the Born approximation, in which the large s-waves arise in a way related to pair production, a high energy phenomenon. We can see by examining the dispersion relations some of the ways in which the s-waves depend upon the high energy behavior of the amplitudes.

If we consider the relation (3.31) for the odd isotopic index direct amplitude in the forward direction, written in terms of the cross sections and specialized to threshold we have

$$\delta_1^0 - \delta_3^0 = 3b\, g^2\, \mu/_{\kappa+\mu} + \frac{3}{4\pi^2} \frac{\kappa\mu^2}{\kappa+\mu} \int_1^\infty \frac{dz}{\xi} (\sigma - - \sigma +) \tag{8.1}$$

This relation was used by Goldberger, Miyazawa, and Oehme[5] to evaluate the difference in scattering lengths $\delta_1^0 - \delta_3^0$. Our purpose is to repeat their evaluation to show that a large part of the contribution to this difference is given by the second maximum in the pion-nucleon cross section. Using the experimental values collected by Anderson, Davidon and Kruse[6] to evaluate the integral over the total cross sections we find the following:

1) the bound state term, which happens to be the Born approximation, contributes 0.436 for $g^2 = 15$,

2) the contribution of the integral over the first maximum, integrating up to $z = 4$ or 420 Mev is -0.220 leaving $\delta_1^0 - \delta_3^0 = 0.216$, and

3) the contribution from $z = 4$ to $z = 10$, including the second maximum, is 0.057 producing as a final value $\delta_1^0 - \delta_3^0 = 0.27$ in agreement with Orear's[21] value. We have not considered the third maximum.[17] We see that the second maximum contributes 20% of the value. The sum rule approximation that we are using is not in very good agreement, yielding $\delta_1^0 - \delta_3^0 = 0.10$ in a p-wave approximation rather than the 0.22 obtained by considering only the first maximum in the integral over cross sections. Part of the reason for the experimental integral's being smaller is that it contains corrections for the 1/2 isotopic spin interactions which enter, with a negative sign, this integral over a difference in cross sections. Such terms are absent in the sum rule approximation.

The weighted average of the scattering lengths also cannot be determined in a p-wave approximation. In an approximation in which the cross section vanishes at infinity, a p-wave or a cut-off approximation, the even isotopic index weighted average of the s-wave scattering lengths is given by

$$\frac{\delta_1^0 + 2\delta_3^0}{3} = -\lambda^2 G^2 \frac{\mu}{K+\mu} + \frac{1}{4\pi^2} \frac{K+\mu^2}{K+\mu} \int_1^\infty \frac{z\,dz}{\xi} (\sigma^+ + \sigma^-) + \Lambda \qquad (8.2)$$

If the Λ -term is ignored, the weighted average would have to be positive if there is any appreciable amount of scattering. The coupling constant term is -0.011 which is comparable to Orear's[21] value for the weighted s-wave lengths

$$\frac{\delta_1^\circ + 2\delta_3^\circ}{3} = -0.02$$

Or Anderson's value[23]

$$\frac{\delta_1^\circ + 2\delta_3^\circ}{3} = 0.007$$

But the contribution from the integral must be positive. The sum rule estimate for the integral is

$$\frac{\mu^2}{\kappa + \mu} \quad z_n/E_n \quad g^2 \quad \sim \quad 0.5$$

The small observed value of the weighted average implies either the existence of a Λ -term, i.e. a φ^4 meson-meson interaction or a meson-pair interaction, or some more non-linear interaction, or else that the integral over the cross section fails to converge. This behavior of a p-wave approximation, predicting a positive weighted average, as opposed to the Born approximation's large negative value $\frac{\delta_1^\circ + 2\delta_3^\circ}{3} = -g^2 \frac{\mu}{\kappa + \mu}$ can be seen in the Low equation. The only negative contributions to the real part of the threshold scattering amplitude arise either from the Λ -term, from intermediate states containing pairs, or from intermediate states containing three or more mesons. All such states are dropped in the one-meson approximation.

In the light of all this, we shall use the dispersion relations to perform some rough computations of the behavior of the s-waves, assuming that the scattering lengths are known. We shall use the dispersion relations for the direct forward scattering amplitude, derived by Goldberger, to compute the real part of the amplitude in a δ_{33} -sum-rule approximation. Then we shall subtract out the p-wave contributions using the values that we have computed in section VII. An alternative approach would be to eliminate the p-wave contributions directly by using the derivative relations of section VI. We would then have relations that involved the p-wave scattering lengths as additional parameters, but there should only be computational differences between these two approaches. We use the two relations (3.31) both modified to refer to the scattering lengths:

$$\mathrm{Re}\, D^{o}(\bar{z}) - \bar{z}\, \mathrm{Re}\, D^{o}(1) = \frac{2}{\pi}(\bar{z}^2-1)\,\bar{z}\; P\int_1^\infty \frac{dz}{(z^2-\bar{z}^2)(z^2-1)}\, \mathrm{Im}\, D^{o}(z)$$

$$- (\bar{z}^2-1)\, 2f^2/\bar{z}$$

$$\mathrm{Re}\, D^{e}(\bar{z}) - \mathrm{Re}\, D^{e}(1) = \frac{2}{\pi}(\bar{z}^2-1)\; P\int_1^\infty \frac{z\,dz}{(z^2-\bar{z}^2)(z^2-1)}\, \mathrm{Im}\, D^{e}(z)$$

$$+ \frac{\bar{z}^2-1}{\bar{z}^2}\, 2f^2G^2$$

And combine them to form the isotopic spin 3/2 and 1/2 amplitudes. Using the phase shift expansion, dropping the d-waves and the higher phases, and restricting the integrals to $\sin^2\delta_{33}$ terms we have

$$\frac{\bar{E}}{\bar{\eta}} \operatorname{Re}\left[a_1 + a_{11} + 2a_{13}\right] = \frac{\kappa+\mu}{3}\left\{(\delta_1^\circ + 2\delta_3^\circ) + 2\bar{z}(\delta_1^\circ - \delta_3^\circ)\right\}$$

$$- (\bar{z}^2-1)\, r\, g^2 \mu \left[\frac{2\bar{z}-r}{\bar{z}^2}\right] \tag{8.3}$$

$$+ \frac{8}{3\pi}(\bar{z}^2-1)\int_1^\infty \frac{dz}{z+\bar{z}}\ \frac{\bar{E}}{\bar{\eta}}\ \frac{\sin^2\delta_{33}}{\bar{z}^2-1}$$

and

$$\frac{\bar{E}}{\bar{\eta}} \operatorname{Re}\left[a_3 + a_{31} + 2a_{33}\right] = \frac{\kappa+\mu}{3}\left\{(\delta_1^\circ + 2\delta_3^\circ) - \bar{z}(\delta_1^\circ - \delta_3^\circ)\right\}$$

$$+ (\bar{z}^2-1)\,\mu\, r\, g^2\left[\frac{1}{\bar{z}-r}\right] \tag{8.4}$$

$$+ \frac{4}{3\pi}(\bar{z}^2-1)\, P\int_1^\infty \frac{dz\,(2z+\bar{z})\, E \sin^2\delta_{33}}{(z^2-\bar{z}^2)(z^2-1)\,\eta}$$

We use the relation for $g^3(z)$ (4.2) to eliminate the large a_{33} term in (8.3). Since $z^2-1 = E^2\eta^2/\kappa^2$, we get

$$\sin 2\delta_3 \approx -3\sin 2\delta_{31} + 2\cdot\frac{\mu\bar{\eta}}{\bar{E}}\ \frac{\kappa+\mu}{3}\left\{\delta_1^\circ + 2\delta_3^\circ - \bar{z}(\delta_1^\circ - \delta_3^\circ)\right\}$$

$$+ (\bar{z}^2-1)\, 2\,\frac{\bar{\eta}}{\bar{E}}\left\{\frac{g^2}{\bar{z}-r}\left(r - \frac{\mu}{\bar{E}}\right) - \frac{2}{\pi}\int_1^\infty \frac{E-\bar{E}}{z-\bar{z}}\ \frac{\kappa^2}{\mu E\bar{E}}\ \frac{\sin^2\delta_{33}}{\eta^3}\, dz \right. \tag{8.5}$$

$$\left. + \frac{2}{3\pi}\int_1^\infty \frac{\kappa^2}{\mu E\bar{E}}\ \frac{E+\bar{E}}{z+\bar{z}}\ \frac{\sin^2\delta_{33}}{\eta^3}\, dz\right\}$$

and for the 1/2 state

$$\sin 2\delta_1 \approx -\sin 2\delta_{11} - 2\sin 2\delta_{13} +$$

$$+ 2\,\frac{\bar{\eta}}{\bar{E}}\ \frac{\kappa+\mu}{3}\left\{\delta_1^\circ + 2\delta_3^\circ + 2\bar{z}(\delta_1^\circ - \delta_3^\circ)\right\} \tag{8.6}$$

$$+ 2\,\frac{\bar{\eta}\mu}{\bar{E}}(\bar{z}^2-1)\left\{-r g^2\,\frac{2\bar{z}-r}{\bar{z}^2} + \frac{8}{3\pi}\int_1^\infty \frac{dz}{z+\bar{z}}\ \frac{\kappa^2}{E}\ \frac{\sin^2\delta_{33}}{\eta^3}\right\}$$

We shall ignore the self-interaction effect: that the square of the phase shift will enter into a singular integral influencing the sine of twice the phase shift that occurs on the left hand side of our equations. We presume this effect to be small at low energies. It could be taken into account numerically, or by attempting to derive effective range formulas for the s-phases as Goldberger has done.[32] The sum rule applied to (8.5) and (8.6) yields

$$
\begin{aligned}
\sin 2\delta_3 \simeq & -3 \sin 2\delta_{31} + 2\,\frac{\eta}{E}\,\frac{\kappa+\mu}{3}\left(\delta_1^0 + 2\delta_3^0 - z\left(\delta_1^0 - \delta_3^0\right)\right) \\
& + (z^2-1)\, 2\,\frac{\eta\mu}{E}\, g^2 \left[\frac{\nu - \mu/E}{z-\nu} \right. \\
& \left. - \frac{3}{4}\,\frac{E-E_n}{z-z_n}\,\frac{\mu}{E E_n} + \frac{1}{4}\,\frac{E+E_n}{z+z_n}\,\mu/E E_n \right]
\end{aligned}
\qquad (8.7)
$$

and

$$
\begin{aligned}
\sin 2\delta_1 \simeq & - \sin 2\delta_{11} - 2 \sin 2\delta_{13} + 2\,\frac{\eta}{E}\,\frac{\kappa+\mu}{3}\left(\delta_1^0 + 2\delta_3^0 + 2z\left(\delta_1^0 - \delta_3^0\right)\right) \\
& + (z^2-1)\, 2\,\frac{\eta\mu}{E}\, g^2 \left[-\nu\,\frac{2z-\nu}{z^2} + \frac{\mu}{E_n}\,\frac{1}{z+z_n} \right]
\end{aligned}
\qquad (8.8)
$$

In the static limit of these equations, taking $E = \kappa$ and dropping terms in μ/κ, the small p-wave terms given by (7.7) are cancelled by the terms involving the coupling constant on the right hand sides of (8.7) and (8.8). The s-waves are thus not connected with the coupling constant in this limit, but remain interrelated:

$$\sin 2\delta_3 \approx \frac{2}{3}\eta\left[\delta_1^{\circ} + 2\delta_3^{\circ} - z(\delta_1^{\circ} - \delta_3^{\circ})\right] \tag{8.9}$$

$$\sin 2\delta_1 \approx \frac{2}{3}\eta\left[\delta_1^{\circ} + 2\delta_3^{\circ} + 2z(\delta_1 - \delta_3)\right]$$

This interrelatedness is just an expression of the fact that in a dynamical theory, the s-waves could be represented by two quadratic terms in the field, one even in the meson isotopic indices and the other partially odd, such as the Hamiltonian used by Drell, Friedman, and Zachariasen.[33] These expressions for the s-waves deviate rapidly from straight-line behavior above 30 Mev. If we use Orear's scattering lengths[21] $\delta_3^{\circ} = -0.11$ and $\delta_1^{\circ} = 0.16$, the phases would be

for	z =	1.2	1.5	2.0	
or an energy of		28	70	140	Mev
	δ_1 =	5.6°	13.7°	34°	
	δ_3 =	-4.2°	-8.3°	-16.5°	

while if the phases were proportional to η they would be, at the same energies,

z =	1.2	1.5	2.0
δ_1 =	5.2°	8.4°	12.5°
δ_3 =	-3.6°	-5.8°	-8.6°

These crude expressions violate unitarity at energies slightly higher than these. If we do not take the static limit of these equations, we can compute the phases using the same parameters as in section VII and using these values for the small p-phases. The behavior of δ_1 is considerably improved;

for	z	=	1.5	2.0	3.0	
or			70	140	280	Mev
	δ_1	=	8.8°	15°	23°	
	δ_3	=	-8.1°	-13°	-34°	
at	η	=	.92	1.4	2.0	

These values are for higher energies than were obtainable for the static limit. The δ_1 phase is now quite close to linear in η . The δ_3 phase above 50 Mev falls below the straight line behavior proposed by Orear, and passes near the experimental points at 113 , 165 , 169 , and 217 Mev cited by Orear. Above these energies the δ_3 phase derived this way becomes too large to satisfy unitarity. The δ_1 phase shows no signs of dropping off to become negative by 200 Mev.

Section IX: The D-Waves

We can make an estimate of the size of the d-waves by again using the approximation of retaining only the 33-phase shift under the integrals. That is, we use our knowledge from experiment that the 33-phase shift is the largest at the energies of interest, and that the d-waves are small. We begin by combining the derivative relations of section VI in order to have only the term in $\frac{\partial S}{\partial \nu}$ on the left hand side. Keeping only the largest term in the 33-phase shift from the integrals we have from (6.1) and (6.2):

$$\frac{\partial}{\partial \nu} \mathrm{Re}\, S^{e} = \alpha \frac{\partial}{\partial \nu} g - \frac{\partial f}{\partial \nu} = -\frac{\bar{\alpha} G^2}{2\kappa\mu} \frac{1}{(\bar{z}-\nu)^2} + \frac{16}{\mu^2} P \int_1^\infty \frac{dz}{z^2-\bar{z}^2} \left[\alpha z - \bar{\alpha} \bar{z} \right] \kappa/\beta \, \frac{\sin^2 \delta_{33}}{\eta^3} \tag{9.1}$$

$$\frac{\partial}{\partial \nu} \mathrm{Re}\, S^{o} = \frac{\bar{\alpha} G^2}{2\kappa\mu} \frac{1}{(\bar{z}-\nu)^2} + \frac{8}{\mu^2} \int_1^\infty \frac{dz}{z^2-\bar{z}^2} \left[\bar{\alpha} z - \alpha \bar{z} \right] \kappa/\beta \, \frac{\sin^2 \delta_{33}}{\eta^3}$$

Even this largest contribution from the p-wave integrals is of order μ/κ of the coupling constant term at threshold. The integral falls off more slowly with energy than the coupling constant term, however, so we shall retain it. We shall make a static approximation and take $\alpha \sim \beta \sim 2\kappa$ and $E \sim \kappa$. Then using the sum rule:

$$\frac{\partial}{\partial \nu} \mathrm{Re}\, S^{e} \approx -G^2/\mu z^2 + \frac{6\pi}{\kappa}\, g^2/z+z_r$$

and

$$\frac{\partial}{\partial \nu} \mathrm{Re}\, S^{\circ} = \frac{4\pi g^2}{\mu z^2} + \frac{3\pi}{\kappa} \, g^2/_{z+z_n}$$

Combining these yields the separate isotopic spin amplitudes; introducing d for $\sin \delta^D \cos \delta^D$ we have, from (6.5),

$$\frac{d_{33} - d_{35}}{\eta^5} = - \mu^4/_{\kappa^3} \, \frac{1}{48\pi} \, \frac{\partial \mathrm{Re}\, S^3}{\partial \nu}$$

$$= \tfrac{4}{3} \nu f^2/_{z^2} - \nu^2 f^2/_{z+z_n} \qquad (9.2)$$

$$\frac{d_{13} - d_{35}}{\eta^5} = - \tfrac{2}{3} \nu f^2/_{z^2} - 4 \nu^2 f^2/_{z+z_n}$$

In order to get any information about the separate phases, we must use relations based on the second derivatives of (3.23). We shall assume that there is no contribution from the lower limit of integration to these derivatives and also that there is no need to use the difference relation for f^e ; this corresponds to assumptions about the high energy dependence of the second derivative justified in a cut-off theory. Since the differentiation of the amplitudes with respect to ν brings in a factor of $\kappa/_\mu$ for each derivative, we can pick out the important terms in the second derivative and ignore the rest. The integrals containing the first derivatives of the amplitudes will contain the largest p-wave contributions, and of the two functions $\frac{\partial f}{\partial \nu}$ and $\frac{\partial g}{\partial \nu}$, $\frac{\partial f}{\partial \nu}$ is larger by a factor of 2κ . Thus the only significant parts of the second derivative are, for

$\nu = 0$,

$$\frac{\partial^2}{\partial \nu^2} \text{Re}\, g^o \approx G^2/\kappa\mu \quad 1/(z-\nu)^3$$

$$\frac{\partial^2}{\partial \nu^2} \text{Re}\, f^o \approx -\frac{2}{\pi}\int_1^\infty \frac{dz}{(z+\bar{z})^2}\, \text{Im}\, \frac{\partial f^o}{\partial \nu} \tag{9.3}$$

for the odd isotopic index and

$$\frac{\partial^2}{\partial \nu^2} \text{Re}\, g^e \approx -G^2/\kappa\mu \quad 1/(z-\nu)^3$$

$$\frac{\partial^2}{\partial \nu^2} \text{Re}\, f^e \approx \frac{2}{\pi}\int_1^\infty \frac{dz}{(z+\bar{z})^2}\, \text{Im}\, \frac{\partial f^e}{\partial \nu} \tag{9.4}$$

for the even isotopic index terms. The second derivatives are related to the direct and spin-flip amplitudes by

$$\frac{\partial^2}{\partial \nu^2} f = \frac{\alpha}{2\beta E}\left[\frac{\partial^2}{\partial \nu^2} D + 2\frac{\kappa\mu}{\beta}\frac{\partial S}{\partial \nu}\right] - \frac{Z\kappa\mu}{\beta E}\frac{\partial^2 S}{\partial \nu^2}$$

$$\frac{\partial^2}{\partial \nu^2} g = \frac{1}{2\beta E}\left[\frac{\partial^2}{\partial \nu^2} D + 2\frac{\kappa\mu}{\beta}\frac{\partial S}{\partial \nu}\right] + \frac{\kappa}{\beta E}\frac{\partial^2 S}{\partial \nu^2} \tag{9.5}$$

from (3.28). These expressions must be combined as in section VI to eliminate the f-waves in the second derivative of the spin-flip amplitude. The terms in the brackets are, using the phase shift expansion and taking $E = \kappa$, $\alpha = \beta = 2\kappa$,

$$\text{Re}\left[\frac{\partial^2}{\partial \nu^2} D^e + \frac{2\kappa\mu}{\beta}\frac{\partial S^e}{\partial \nu}\right] = \frac{\pi}{\nu^3}\,\frac{5}{\eta^5}\left[d_{15} + 2 d_{35}\right] \tag{9.6}$$

$$\mathrm{Re}\left[\frac{\partial^2}{\partial r^2} D^0 + \frac{2 i k M}{\beta} \frac{\partial S^0}{\partial r} \right] = \frac{\pi}{k^3} \, {}^5/_{\eta^5} \left[d_{15} - d_{35} \right]$$

Combining these equations we have, using the sum rule and (6.2), (6.3), and (6.4)

$$\frac{d_{15} + 2 d_{35}}{\eta^5} = - \frac{8}{5} k f^2/z^2 - \frac{12}{5} k f^2 \frac{1}{(z+z_n)^2}$$

$$\frac{d_{15} - d_{35}}{\eta^5} = \frac{8}{5} k f^2/z^2 - {}^6/_5 \, k f^2 \frac{1}{(z+z_n)^2} \tag{9.7}$$

Or, since the phase shifts are small enough to write δ for $\sin \delta$

$$\delta^D_{35} \approx - \frac{2}{15} k f^2 \eta^5 \left[{}^8/_{z^2} + {}^3/_{(z+z_n)^2} \right]$$

$$\delta^D_{15} \approx \cdot \frac{2}{15} k f^2 \eta^5 \left[{}^4/_{z^2} - {}^{12}/_{(z+z_n)^2} \right] \tag{9.8}$$

and using (9.2) yields for the other two phase shifts:

$$\delta^D_{33} \approx \frac{2}{15} k f^2 \eta^5 \left[\frac{2}{z^2} - \frac{15}{2} k \frac{1}{z+z_n} - \frac{3}{(z+z_n)^2} \right]$$

$$\delta^D_{13} \approx - \frac{2}{15} k f^2 \eta^5 \left[{}^1/_{z^2} + \frac{12}{(z+z_n)^2} + 30 \, k/_{z+z_n} \right] \tag{9.9}$$

The d-waves predicted by these expressions are very small. The corrections from the integrals are the same size as the coupling constant term for the 15- and 33-states and dominate for the 13-state. If we use the same parameters as were used in section VII, the phase shifts are

at		140 Mev	280 Mev
δ^D_{35}	=	-0.5°	-1.6°
δ^D_{15}	=	0.08°	0.05°
δ^D_{33}	=	0.05°	0.02°
δ^D_{13}	=	-0.3°	-1.6°

The two smaller d-waves are decreasing with energy, as given in this table, and they would change sign at somewhat higher energies. We have assumed that there is no other source of d-waves in the theory other than the direct Yukawa coupling described in principle by a cut off to obtain convergence. If this is not so, and the high energy assumption is certainly dubious, the derivative relation based on $f^e(\nu, z)$ would contain arbitrary d-wave scattering lengths and the values of the d-phases would be modified. Such an additional term could arise even in the convergent theory, for example there could be a strongly angle-dependent $\Lambda(pp')$ -term in the dispersion relation for f^e. Since we expect the dispersion relation for $f^e(z)$ to have to be a difference relation in order to converge, the d-wave scattering lengths will certainly enter into the theory, and thus make it impossible to predict their size from a purely dispersion-theoretic approach. The d-waves that we have computed are far smaller than any found in fitting the experimental data.[30]

Results similar to ours for the small phase shifts have recently been obtained by Chew, Goldberger, Low, and Nambu using the static limit of the dispersion theory equations and a 33-phase shift approximation closely analogous to the one we have used.[34]

Appendix I

Notation and Normalization

We shall use the timelike metric, writing the invariant product of two four-vectors p_μ and q_μ as

$$pq = p_0 q_0 - \vec{p}\cdot\vec{q} \tag{A.1}$$

We take $\hbar = c = 1$. Since furthermore we are interested in the invariant T-matrix, and not specifically interested in non-relativistic limits, we shall use invariant normalizations for the particles. We shall normalize both bosons and fermions the same way: to $2p_0$ particles per unit volume. Then the boson wave function is just e^{-iqx} for a boson of four momentum q, $q^2=\mu^2$, μ is the meson's mass. The fermion wave functions have the normalization

$$\bar{u}(p)\, u(p) = 2\kappa \tag{A.2}$$

for a fermion of four-momentum p, where $p^2 = \kappa^2$.

The spinors $u(p)$ obey

$$(\gamma p - \kappa)\, u(p) = 0 \tag{A.3}$$

in which κ is the renormalized mass of the nucleon. We take γ_0 and γ_5 to be Hermitian; the γ_i to be anti-Hermitian. The normalization (A.2) corresponds to a density $\bar{u}\gamma_0 u = 2p_0$.

The density of state factors are then the same for both bosons and fermions and are invariant:

$$\frac{1}{(2\pi)^3}\frac{d^3k}{2k_0} = \frac{d^4k}{(2\pi)^3}\,\delta(k^2-M^2) \tag{A.4}$$

The summation over nucleon spins becomes

$$\sum_s u_s(p)\,\bar{u}_s(p) = \gamma p + K \tag{A.5}$$

differing by the factor $2K$ from the usual projection operator. The T-matrix that we shall define will differ both by the $\sqrt{2q_0}$ factors and by a factor of $2K$ from the usual T-matrix. The identity, the scalar product of one-particle states, has the form

$$\langle q | q' \rangle = (2\pi)^3\, 2q_0\, \delta^3(q-q') \tag{A.6}$$

for bosons and

$$\begin{aligned}\langle p | p' \rangle &= \frac{\bar{u}(p)\,u(p')}{2K}(2\pi)^3\, 2p_0\, \delta^3(p-p') \\ &= \delta_{ss'}\,(2\pi)^3\, 2p_0\, \delta^3(p-p')\end{aligned} \tag{A.7}$$

for fermions where s, s' are the spin indices. The T-matrix is related to the S-matrix by

$$S = 1 + iT \tag{A.8}$$

or for the scattering of a nucleon p', meson q' into a state p, q,

$$S(p,q;p',q') = \langle q|q'\rangle\langle p|p'\rangle + i(2\pi)^4\delta(p+q-p'-q')\,T(p,q,p',q') \tag{A.9}$$

The probability that the scattering takes place is of course proportional to $|T|^2$. For an elastic scattering, the differential cross section is (see Moeller)(35)

$$d\sigma = \frac{1}{4\sqrt{(pq)^2-\kappa^2\mu^2}}\,|T|^2(2\pi)^4\,\delta^4(p+q-p'-q') \times \frac{1}{(2\pi)^3}\frac{d^3p'}{2p_0'}\,\frac{1}{(2\pi)^3}\frac{d^3q'}{2q_0'} \tag{A.10}$$

in which the first factor is the invariant normalization for an incident flux arising from $2q_0$ particles per unit volume striking $2p_0$ particles per unit volume. This is

$$4\sqrt{(pq)^2-\kappa^2\mu^2} = 4k_L\kappa$$

in the laboratory system where the mesons have momentum k_L and is

$$= 4\mu\eta E$$

in the center-of-mass system of total energy E and momentum $\mu\eta$. The Optical theorem follows from (A.8). In the forward direction

$$\mathrm{Im}\,T = \tfrac{1}{2}TT^+ \tag{A.11}$$

in which the sum implicit in the matrix notation is restricted to energy and momentum conserving intermediate states by the delta-function coefficient of the T-matrix in (A.9). Then the total cross section is

$$\sigma = \frac{1}{2\mu\eta E}\,\mathrm{Im}\,T = \frac{1}{2\sqrt{(pq)^2-(\kappa\mu)^2}}\,\mathrm{Im}\,T(pq) \qquad \text{(A.12)}$$

for mesons of laboratory energy pq/κ .

In writing the dispersion relations we use the dimensionless variables

$z = pq/\kappa\mu$ the laboratory energy

and

$x = \frac{pp'-\kappa^2}{\kappa\mu} = \frac{\mu}{\kappa}\,\eta^2(1-\cos\theta)$ in center-of-mass

and

$\xi = \sqrt{z^2-1}$ the laboratory momentum of the incident meson. We also use

E = total center-of-mass energy

$\mu\eta$ = center-of-mass momentum

α = $E+\kappa$

and a factor arising from the normalization of the Dirac equation

$\beta = p_0+\kappa$ in the center-of-mass

These obey the algebraic relations:

$$E = \sqrt{\kappa^2 + \mu^2 + 2\kappa\mu z}$$

and

$$\beta E = \kappa(\alpha + \mu z)$$

When we insert numerical values, we use $\kappa = 6.7\,\mu$.

Appendix II

The T-Matrix

The formulas that we use for the renormalized T-matrix element in terms of Heisenberg field operators were derived from perturbation theory by Low[36] and from an asymptotic formulation of field theory by Lehmann, Symanzik, and Zimmermann.[37] Goldberger[38] applied these expressions to dispersion theory by observing that the T-matrix element defined in terms of the T-product of Heisenberg field operators (or Green's functions) could be re-expressed in terms of the commutator of the operators as far as physical processes were concerned. We shall briefly recapitulate part of the Lehmann, Symanzik, and Zimmermann argument in order to write out the formulas we want for the T-matrix element. We consider eigenstates of the total energy-momentum four-vector P_μ :

$$P_\mu |n\rangle = n_\mu |n\rangle \tag{B.1}$$

where we shall in general surpress all the other quantum numbers necessary to specify the state. Since P_μ is a displacement operator:

$$[P_\mu, O(x)] = -i\partial_\mu O(x) \tag{B.2}$$

for any Heisenberg operator $O(x)$. Thus the coordinate dependence of any operator may be explicitly exhibited and removed in the momentum representation:

$$O(x) = e^{iPx} \; O(0) \; e^{-iPx} \tag{B.3}$$

We shall normally drop the reference to the origin and write O for $O(0)$. The interacting meson field obeys the equation

$$(-\Box^2 - \mu^2)\,\phi(x) = j(x) \tag{B.4}$$

in which μ is the renormalized meson mass and $j(x)$ is the renormalized current containing counter-terms and any interaction. The expressions for the T-matrix are simply derived by observing that the quantities

$$\phi_{q\pm} = \lim_{\sigma \to \pm\infty} i \int d\sigma_\mu \; e^{iqx} \overleftrightarrow{\partial_\mu} \phi(x) \tag{B.5}$$

where

$$q^2 = \mu^2 \qquad f \overleftrightarrow{\partial_\mu} g = -(\partial_\mu f) g + f(\partial_\mu g)$$

behave as annihilation operators for dressed mesons while the Hermitian conjugate quantities behave as creation operators. We assume the commutation relations

$$[\phi_{q+}, \phi^\dagger_{q'+}] = [\phi_{q-}, \phi^\dagger_{q'-}] = \langle q | q' \rangle = 2q_0 (2\pi)^3 \delta^3(q-q') \tag{B.6}$$

Given a one-nucleon state $|p'\rangle$, which may be obtained by operating with an operator constructed from the Fermi fields analogously to (B.5) on the vacuum, and which is assumed to be steady, the scattering state of two particles defined for large negative times is

$$|p', q', -\rangle = \phi^{+}_{q'-}|p'\rangle = \lim_{\sigma \to -\infty} i \int d\sigma_\mu \, \phi(x) \overleftrightarrow{\partial}_\mu \, e^{-iqx} |p'\rangle \qquad \text{(B.7)}$$

and the final state at large positive times is

$$|p, q, +\rangle = \phi^{+}_{q+}|p'\rangle = \lim_{\sigma \to +\infty} i \int d\sigma_\mu \, \phi(x) \overleftrightarrow{\partial}_\mu \, e^{-iqx} |p\rangle \qquad \text{(B.8)}$$

Then the S-matrix element is

$$\langle p, q, + | p', q', - \rangle = \lim_{\sigma \to -\infty} \langle p q + | i \int d\sigma_\mu \, \phi(x) \overleftrightarrow{\partial}_\mu \, e^{-iq'x} |p'\rangle \qquad \text{(B.9)}$$

The surface integral is transformed using Gauss's theorem; the commutation relations (B.6) and the fact that $\phi_{q\pm}|p\rangle = 0$ by heavy particle conservation, since there is no state containing one nucleon with mass less than a nucleon.

$$= \langle p | p' \rangle \langle q | q' \rangle - i \int dx \, \langle p q + | \partial_\mu \left(\phi(x) \overleftrightarrow{\partial}_\mu \, e^{-iq'x} \right) | p' \rangle$$

and using the definition of the current and the translational invariance (B.3)

$$= \langle p | p' \rangle \langle q | q' \rangle - i (2\pi)^4 \delta^4 (p+q-p'-q') \langle p, q+ | j | p' \rangle$$

The T-matrix element for this scattering is then

$$T = - \langle p, q+ | j | p' \rangle \qquad \text{(B.10)}$$

which depends on only three of the possible four four-vectors entering into the interaction of four particles. If the decomposition for the final two particle state is used, this becomes

$$T = -i \int_{+\infty} d\sigma_\mu \, e^{iqx} \overleftrightarrow{\partial_\mu} \langle p | \phi(x) \, j(0) | p' \rangle \qquad \text{(B.11)}$$

and using again the fact that

$$\langle p | j(0) \, \phi_{q+} | p' \rangle = 0$$

since ϕ_{q+} is a meson annihilation operator;

$$= -i \int_{+\infty} d\sigma_\mu \, e^{iqx} \overleftrightarrow{\partial_\mu} \langle p | [\phi(x), j(0)] | p' \rangle$$

If the surface integral is expressed in terms of a volume integral over the half space $x_0 > 0$:

$$= -i \int_0^\infty dx \, \partial_\mu [e^{iqx} \overleftrightarrow{\partial_\mu} \langle p | [\phi(x), j(0)] | p' \rangle]$$

$$- i \int_{\sigma=0} e^{iqx} \overleftrightarrow{\partial_\mu} \langle p | [\phi(x), j(0)] | p' \rangle \, d\sigma_\mu$$

or, using the fact that e^{iqx} obeys the Klein-Gordon equation and the causality condition $[j(x), j(0)] = 0$, $x^2 < 0$

$$T = i \int_{+} e^{iqx} dx \langle p | [j(x), j(0)] | p' \rangle$$
$$- i \int_{\sigma=0} d\sigma_\mu \, e^{iqx} \overleftrightarrow{\partial_\mu} \langle p | [\phi(x), j(0)] | p' \rangle \qquad \text{(B.12}$$

where the + under the integral means that the integral is restricted to the forward light cone.

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