

Five Vortex Equations

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Abstract

The Taubes equation for Abelian Higgs vortices is generalised to five distinct $U(1)$ vortex equations. These include the Popov and Jackiw–Pi vortex equations, and two further equations. The Baptista metric, a conformal rescaling of the background metric by the squared Higgs field, gives insight into these vortices, and shows that vortices can be interpreted as conical singularities superposed on the background geometry. When the background has a constant curvature adapted to the vortex type, then the vortex equation is integrable by a reduction to Liouville’s equation, and the Baptista metric has a constant curvature too, apart from its conical singularities. The conical geometry is fairly easy to visualise in some cases.

Keywords: $U(1)$ vortex, Baptista metric, Conical singularities, Liouville equation

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1 Introduction

There are a number of vortex equations [1, 2, 3], defined on surfaces of suitable curvature, that can be explicitly solved. We say that these vortex equations are integrable. The known examples are (i) the Taubes equation of the Abelian Higgs model, defined on a hyperbolic surface of constant negative curvature [4, 5, 6], (ii) the Popov vortex equation defined on a sphere of constant positive curvature [7, 8], and (iii) the Jackiw–Pi vortex equation defined on a flat plane or torus [9, 10, 11]. In this paper, we investigate more systematically how these integrable vortex equations arise, and discover that there are really five examples. The first new example we call (iv) the Bradlow vortex equation – it is a reinterpretation of the Bradlow limit [12] of the Taubes equation for vortices on a compact hyperbolic surface, where the vortex number saturates its upper bound. The second new example is (v) a vortex equation defined on a hyperbolic surface, generalising an equation found by Ambjørn and Olesen [13, 14].

Our approach builds on the geometric insights of Baptista [15], who interpreted vortices on a smooth surface in terms of a new metric – the Baptista metric – which is a conformal rescaling of the background metric. If the background metric is ds_0^2 and the vortex Higgs field is ϕ , then Baptista’s metric is $ds^2 = |\phi|^2 ds_0^2$. This metric is not smooth. It has singularities at the vortex centres, where $\phi = 0$. For a vortex of unit winding, the metric has a conical singularity with cone angle 4π . The construction of integrable vortices is then closely related to the purely geometrical problem of constructing surface metrics with given curvature and conical singularities. This problem has been studied, in particular, by Troyanov [16].

We do not solve in full generality the five vortex equations, but summarise solutions that are known, and find some new ones. Solutions are obtained using local holomorphic mappings between surfaces. Sometimes these maps are globally defined, and can be given explicitly. This method may generate all solutions, but not always, and it appears that further solutions must be constructed by patching local holomorphic maps together, with a twist.

We describe in some detail the intrinsic Baptista geometry of a number of vortices, in the integrable cases. That is, we describe the geometry of the Baptista metric ds^2 , without splitting it into its factors ds_0^2 and $|\phi|^2$. We also note that this metric is an Einstein metric with conical singularities, in the presence of a cosmological constant [17, 18]. From this perspective, vortices become point particles of negative mass, quite different from the usual insight

that they are smooth solitons on a smooth background surface.

2 Abelian Higgs Vortices

All the vortex equations considered here are variants of the first-order Bogomolny equations [19] of the Abelian Higgs model, which we review first. These equations model critically coupled vortices that neither attract nor repel each other, so the vortices are static, 2-dimensional soliton or multi-soliton solutions, occupying a flat or curved surface.

More precisely, the equations are defined on a Riemann surface M_0 having a metric compatible with its complex structure. M_0 may be compact, or open with a boundary at infinity. Initially we allow M_0 to have an arbitrary curvature, but later we will specialise to surfaces of constant curvature. In terms of a local complex coordinate $z = x_1 + ix_2$ the metric is

$$ds_0^2 = \Omega_0(dx_1^2 + dx_2^2) = \Omega_0 dzd\bar{z}, \quad (1)$$

where Ω_0 is a position-dependent conformal factor.

The fields we need are a U(1) gauge potential a and a complex Higgs field ϕ . Globally, a is a connection on a U(1) line bundle over M_0 , and ϕ is a section of the bundle. Locally, we represent the connection as a real 1-form $a = a_1 dx_1 + a_2 dx_2 = a_z dz + a_{\bar{z}} d\bar{z}$. The connection has 2-form field strength $f = da$, and we suppose that the first Chern number of the bundle,

$$N = \frac{1}{2\pi} \int_{M_0} f = \frac{1}{2\pi} \int_{M_0} f_{12} d^2x, \quad (2)$$

is a positive integer. In component notation $f_{12} = \partial_1 a_2 - \partial_2 a_1$, and the physical magnetic field strength on M_0 is $B = \frac{1}{\Omega_0} f_{12}$. More invariantly it is $*f$, the Hodge dual of f .

The Bogomolny equations are

$$D_1 \phi + i D_2 \phi = 0, \quad (3)$$

$$\frac{1}{\Omega_0} f_{12} = 1 - |\phi|^2, \quad (4)$$

where $D_j = \partial_j - ia_j$ is the gauge covariant derivative. They are also usefully written in terms of the complex coordinate z as

$$D_{\bar{z}} \phi = 0, \quad (5)$$

$$-\frac{2i}{\Omega_0} f_{z\bar{z}} = 1 - |\phi|^2, \quad (6)$$

The pair of Bogomolny equations (5) and (6) can be simplified to a single scalar equation as follows. Equation (5), expanded out, is

$$\partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0, \quad (7)$$

and has the solution $a_{\bar{z}} = -i\partial_{\bar{z}}\log\phi$. Since the gauge group is $U(1)$, a_z is the complex conjugate of $a_{\bar{z}}$, so $a_z = i\partial_z\log\bar{\phi}$, and therefore

$$f_{z\bar{z}} = \partial_z a_{\bar{z}} - \partial_{\bar{z}} a_z = -i\partial_z\partial_{\bar{z}}(\log\phi + \log\bar{\phi}) = -i\partial_z\partial_{\bar{z}}\log|\phi|^2. \quad (8)$$

The second Bogomolny equation (6) therefore reduces to

$$-\frac{2}{\Omega_0}\partial_z\partial_{\bar{z}}\log|\phi|^2 = 1 - |\phi|^2. \quad (9)$$

It is convenient to change notation, by setting $|\phi|^2 = e^{2u}$, and to note that the naive Laplacian is $\nabla^2 = 4\partial_z\partial_{\bar{z}}$. Equation (9) then takes the final form

$$-\frac{1}{\Omega_0}\nabla^2 u = 1 - e^{2u}, \quad (10)$$

known as the Taubes equation [1, 3]. The left-hand side is the negative of the covariant (Beltrami) Laplacian of u , and is still the magnetic field strength.

The first Bogomolny equation (5) implies that ϕ is gauge-covariantly holomorphic. ϕ can therefore have zeros, but only of positive multiplicity. These are interpreted as the centres of vortices with positive integer winding. It can be shown that the sum of the windings around all the vortex centres is the Chern number N . As ϕ is zero at each vortex centre, u has a logarithmic singularity there, and approaches $-\infty$. The Taubes equation (10) is therefore incomplete, and should be supplemented by delta functions [1]. We will not include these. Instead, we regard the Taubes equation as only valid away from the vortex centres, and require that if there is a vortex centre at Z with winding n , then u has the asymptotic behaviour

$$u \sim n \log|z - Z| + \text{constant} \quad (11)$$

as z approaches Z .

Vortex solutions of the Taubes equation should have no further singularities, so if M_0 is compact then u will have a global maximum value, and it is interesting to consider the maximum principle in this context. The Laplacian of u is non-positive at the location of the global maximum of u , so the

left-hand side of equation (10) and hence the right-hand side is non-negative. Therefore $u \leq 0$ at the maximum, and hence $u \leq 0$ everywhere. In known examples, the maximum value of u is negative. u can have more than one local maximum (typically, between the vortex centres), and at all of these, u is negative.

If M_0 is non-compact, for example the flat plane or the hyperbolic plane, then we impose the condition $|\phi| = 1$ or equivalently $u = 0$ on the boundary. Again, u is assumed to have no singularities apart from those at the vortex centres. In this situation, we also have $u \leq 0$ everywhere. If not, then u would have a maximum positive value at some point of M_0 interior to the boundary. But this would again contradict the Taubes equation. Therefore u has supremum 0, attained on the boundary at infinity.

Note that if u is everywhere negative, then the right-hand side of (10) is everywhere positive, so the magnetic field B is everywhere positive. The magnetic field has its maximum value 1 at the vortex centres, where $|\phi|^2 = 0$. Physically, the Taubes equation describes the Meissner effect in a superconductor, where in the absence of vortices there is no magnetic flux penetration, and $|\phi| = 1$ everywhere. The vortices introduce magnetic flux defects into the superconductor, accompanying the zeros of ϕ .

3 More Vortex Equations

Further vortex equations on M_0 arise by changing the coefficients in equations (4) and (6). The general vortex equations we consider are

$$D_1\phi + iD_2\phi = 0, \tag{12}$$

$$\frac{1}{\Omega_0}f_{12} = -C_0 + C|\phi|^2, \tag{13}$$

or equivalently

$$D_{\bar{z}}\phi = 0, \tag{14}$$

$$-\frac{2i}{\Omega_0}f_{z\bar{z}} = -C_0 + C|\phi|^2, \tag{15}$$

with C_0 and C taking any real, constant values.

As before, we can use the first equation to eliminate the gauge potential, and by setting $|\phi|^2 = e^{2u}$, the second equation becomes

$$-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u}. \tag{16}$$

Now, we can simultaneously rescale C_0 and C by a positive real factor, and absorb this into the metric. We can also rescale C alone by a positive real factor, and absorb this into a constant shift of u (a rescaling of $|\phi|$). Therefore, without loss of generality, we may fix C_0 and C to each take one of the three standard values $-1, 0$ or 1 . There are therefore nine distinct equations of type (16).

Vortex solutions have the property that ϕ has zeros, but no singularities, and equation (14) implies that the Chern number N is positive, because the vortex windings are positive. Of the nine equations, only five can have such vortex solutions. The left-hand side of equation (16) is the magnetic field, and for its integral to be positive, the right-hand side must admit positive values. This excludes the four cases $C_0 = 0$ or $C_0 = 1$, combined with $C = -1$ or $C = 0$.

The remaining cases are the Taubes equation (10), with $C_0 = C = -1$, the Jackiw–Pi vortex equation [9, 10] with $C_0 = 0, C = 1$,

$$-\frac{1}{\Omega_0}\nabla^2 u = e^{2u}, \quad (17)$$

and the Popov vortex equation [7] with $C_0 = C = 1$,

$$-\frac{1}{\Omega_0}\nabla^2 u = -1 + e^{2u}. \quad (18)$$

A further case is the equation with $C_0 = -1, C = 0$,

$$-\frac{1}{\Omega_0}\nabla^2 u = 1, \quad (19)$$

that we shall call the Bradlow vortex equation. Notice that the magnetic field has constant strength 1 here. More usually, one refers to the Bradlow limit of the Taubes equation for vortices [12]. This is where N attains its maximum allowed value on a compact surface, and the Higgs field vanishes everywhere. The second Bogomolny equation then says that the magnetic field is 1, as in equation (19). Our Bradlow vortex equation is different in that it allows a non-vanishing Higgs field ϕ satisfying $D_{\bar{z}}\phi = 0$ in the background of the constant magnetic field. Its solutions are therefore similar to what were considered previously as vortex solutions close to the Bradlow limit [20] or as dissolving vortices [21], where the magnetic field was almost constant

and the Higgs field small. The final case is the vortex equation with $C_0 = -1$, $C = 1$,

$$-\frac{1}{\Omega_0}\nabla^2 u = 1 + e^{2u}. \quad (20)$$

This equation, in its flat space version, appeared in Ambjørn and Olesen's study of the instability of strong magnetic fields in electroweak gauge theory [13, 14]. Notice that the magnetic field has strength 1 at the vortex centres, as for Taubes vortices, but the strength is enhanced away from these centres. This is an anti-Meissner effect.

Applying the maximum principle to the Popov equation (18), we see that $u \geq 0$ at its maximum, and as u approaches $-\infty$ at the vortex centres, u takes all negative values. For the remaining equations other than the Taubes equation, the right-hand side is positive for all u , so there is no further constraint on the value of u at its maximum.

4 Energy and Stability

The five vortex equations can all be derived using a Bogomolny rearrangement of a suitable energy functional [19]. The energy is not positive definite in all cases. The equations guarantee that the energy is stationary, though not always minimal. The vortices are therefore not necessarily stable.

The energy expression is of the type familiar in the Abelian Higgs model, but with non-standard coefficients. For general values of C_0 and C , consider the (potential) energy

$$E = \int_{M_0} \left\{ \frac{1}{\Omega_0^2} f_{12}^2 - \frac{2C}{\Omega_0} (\overline{D_1\phi} D_1\phi + \overline{D_2\phi} D_2\phi) + (-C_0 + C|\phi|^2)^2 \right\} \Omega_0 d^2x. \quad (21)$$

This is positive definite if $C \leq 0$, but not otherwise. We rewrite the energy as

$$\begin{aligned} E = \int_{M_0} & \left\{ \left(\frac{1}{\Omega_0} f_{12} + C_0 - C|\phi|^2 \right)^2 \right. \\ & \left. - \frac{2C}{\Omega_0} (\overline{D_1\phi} - i\overline{D_2\phi})(D_1\phi + iD_2\phi) \right\} \Omega_0 d^2x \\ & + \int_{M_0} (-2C_0 f_{12} + 2C f_{12} |\phi|^2 + 2Ci(\overline{D_1\phi} D_2\phi - \overline{D_2\phi} D_1\phi)) d^2x, \end{aligned} \quad (22)$$

where the terms in the second integral (which has no Ω_0 factors) compensate for completing the squares. Next we use the identity

$$\overline{D_1\phi}D_2\phi - \overline{D_2\phi}D_1\phi = \partial_1(\bar{\phi}D_2\phi) - \partial_2(\bar{\phi}D_1\phi) + if_{12}|\phi|^2, \quad (23)$$

which combines the covariant Leibniz rule with the commutator $[D_1, D_2] = -if_{12}$, to obtain

$$\begin{aligned} E &= \int_{M_0} \left\{ \left(\frac{1}{\Omega_0} f_{12} + C_0 - C|\phi|^2 \right)^2 \right. \\ &\quad \left. - \frac{2C}{\Omega_0} (\overline{D_1\phi} - i\overline{D_2\phi}) (D_1\phi + iD_2\phi) \right\} \Omega_0 d^2x \\ &\quad + \int_{M_0} (-2C_0 f_{12} + 2Ci(\partial_1(\bar{\phi}D_2\phi) - \partial_2(\bar{\phi}D_1\phi))) d^2x. \end{aligned} \quad (24)$$

The final two terms are total derivatives and integrate to zero. More invariantly, their integral is that of the globally-defined, exact 2-form $2Ci d(\bar{\phi}D\phi)$. f_{12} integrates to 2π times the Chern number N . Therefore

$$\begin{aligned} E &= \int_{M_0} \left\{ \left(\frac{1}{\Omega_0} f_{12} + C_0 - C|\phi|^2 \right)^2 \right. \\ &\quad \left. - \frac{2C}{\Omega_0} (\overline{D_1\phi} - i\overline{D_2\phi}) (D_1\phi + iD_2\phi) \right\} \Omega_0 d^2x \\ &\quad - 4\pi C_0 N. \end{aligned} \quad (25)$$

The energy E is stationary and has the value $-4\pi C_0 N$, provided the Bogomolny equations (12) and (13) are satisfied. E is minimised if $C \leq 0$. The vortices are then stable, but we are also interested in cases where C is positive. Taubes and Bradlow vortices are stable (although the Higgs field does not contribute to the energy in the Bradlow case, and the equation $D_1\phi + iD_2\phi = 0$ has to be imposed separately). Popov vortices are unstable, as are the vortices satisfying equation (20). The Jackiw–Pi vortices are also unstable by this criterion, but this is not of much significance, as these vortices arise most naturally in Chern–Simons field theory, where the dynamics is different [10, 22].

Euler–Lagrange equations can be derived from the energy E ; these are the second-order, static field equations for vortices. They are satisfied for all types of vortices satisfying the appropriate first-order Bogomolny equations, because E is stationary. This can be checked by differentiation.

5 Vortices as Conical Geometry

The original metric ds_0^2 on M_0 is smooth, but it was suggested some time ago [23] that it is useful to consider for a vortex solution the metric

$$ds^2 = e^{2u} ds_0^2, \quad (26)$$

the original metric conformally rescaled by the squared Higgs field $|\phi|^2 = e^{2u}$. This modified metric has been studied in depth for Abelian Higgs vortices by Baptista [15], so we refer to it as the Baptista metric. ds^2 defines an intrinsic geometry of a vortex solution, and it is sometimes easier to describe and visualise this intrinsic geometry, rather than separate ds^2 into ds_0^2 and e^{2u} . The Baptista metric is useful for all five of our vortex equations, although it has different properties in the various cases. Notice that it tends to reduce lengths and areas near vortex centres, because e^{2u} is close to zero.

The Baptista metric is not a regular Riemannian metric, because it vanishes at the vortex centres. Taubes showed that the asymptotic form of u near a vortex centre is as in equation (11). For a vortex with unit winding ($n = 1$), centred at the origin $Z = 0$ for convenience, $e^{2u} \sim \mu|z|^2$ with μ a positive constant. The background metric is locally $\Omega_0(0)dzd\bar{z}$ with $\Omega_0(0)$ positive, so the Baptista metric is locally $\mu\Omega_0(0)|z|^2dzd\bar{z}$. In polar coordinates, this is a multiple of $r^2(dr^2 + r^2d\theta^2)$. Using the change of variables $\rho = \frac{1}{2}r^2$ and $\chi = 2\theta$, the metric becomes $d\rho^2 + \rho^2d\chi^2$, a flat metric whose polar angle χ runs from 0 to 4π . The metric is therefore conical, with cone angle 4π . The conical excess is 2π . For a vortex centre of multiplicity n , the cone angle would be $2(n+1)\pi$, with excess $2n\pi$. The Baptista metric is not truly a flat cone, because there are higher-order metric corrections, and generally there is a non-zero curvature as the conical singularity is approached.

Baptista derived a simple relation between the curvature of the background metric and the curvature of the new (Baptista) metric. We present this for the general vortex equation $-\frac{1}{\Omega_0}\nabla^2u = -C_0 + Ce^{2u}$. We start with the formula for the Gaussian curvature of the background,

$$K_0 = -\frac{1}{2\Omega_0}\nabla^2\log\Omega_0. \quad (27)$$

The Baptista metric, with conformal factor $\Omega = e^{2u}\Omega_0$, has Gaussian curvature

$$K = -\frac{1}{2e^{2u}\Omega_0}\nabla^2(2u + \log\Omega_0), \quad (28)$$

so the curvatures are related by

$$-\frac{1}{\Omega_0}\nabla^2 u = -K_0 + Ke^{2u}. \quad (29)$$

This is a well known, purely geometrical identity [24, 25], discussed in the context of metrics with conical singularities by Troyanov [16]. In addition, u satisfies the vortex equation, so

$$-C_0 + Ce^{2u} = -\frac{1}{\Omega_0}\nabla^2 u = -K_0 + Ke^{2u}. \quad (30)$$

Baptista's version of this equation, obtained by multiplying by Ω_0 , is

$$(K_0 - C_0)\Omega_0 = (K - C)\Omega. \quad (31)$$

Intrinsically, this relates linear combinations of the curvature 2-form and Kähler 2-form of the background metric and Baptista metric. From it, Baptista derived a superposition principle for Taubes vortices [15]. Equation (31) is not algebraic, despite its appearance, because the curvature formulae involve the Laplacian.

A basic property of the Baptista metric is the relation between its area

$$A = \int_{M_0} \Omega d^2x \quad (32)$$

and the background area of M_0 ,

$$A_0 = \int_{M_0} \Omega_0 d^2x. \quad (33)$$

If M_0 is smooth and compact, and of genus \mathfrak{g}_0 , then by the Gauss–Bonnet theorem,

$$\frac{1}{2\pi} \int_{M_0} K_0 \Omega_0 d^2x = 2 - 2\mathfrak{g}_0. \quad (34)$$

For an N -vortex solution, the curvature of the Baptista metric on M_0 satisfies

$$\frac{1}{2\pi} \int_{M_0} K \Omega d^2x - N = 2 - 2\mathfrak{g}_0, \quad (35)$$

because each conical singularity with conical excess 2π contributes -1 to the Gauss–Bonnet integral, and the topology of M_0 is unchanged. Integrating the equation (31), and using these formulae, we find

$$CA = C_0A_0 + 2\pi N. \quad (36)$$

This has the following consequences for the five vortex equations in standard form. For the Taubes equation with $C_0 = C = -1$, $A = A_0 - 2\pi N$. The Baptista area A is smaller than the original area A_0 , implying Bradlow's upper bound on the vortex number $2\pi N \leq A_0$, because A has to be non-negative. It is not possible to have more vortices than this satisfying the Bogomolny equations. If $A_0 = 2\pi N_0$ for some integer N_0 , and $N = N_0$, then we are at the Bradlow limit of the Taubes equation. Here $A = 0$, so the Higgs field and the Baptista metric are both zero. Genuine Taubes vortices require $N < N_0$.

Our Bradlow vortex equation (19), with $C_0 = -1$ and $C = 0$, allows for vortices in this limit. Solutions only exist if $A_0 = 2\pi N$ (as the magnetic field strength is 1, so its integral is A_0). The Higgs field satisfies $D_{\bar{z}}\phi = 0$ in the background of the constant magnetic field, which is the equation for Lowest Landau Level states. ϕ can be non-zero, and its magnitude can be rescaled by an arbitrary constant, so the area A is arbitrary.

For the Popov vortices with $C_0 = C = 1$, A is larger than A_0 and there is no constraint on N (although we shall see later that N must be even). This implies that the average of e^{2u} over M_0 is greater than 1, so u must be strictly positive at its maximum, and take all values between this maximum and $-\infty$, a stronger result than what we obtained using the maximum principle. For the Jackiw–Pi vortices with $C_0 = 0$ and $C = 1$, the Baptista metric has area $A = 2\pi N$.

For the vortices satisfying equation (20), with $C_0 = -1$ and $C = 1$, the vortex number has to satisfy $2\pi N > A_0$ for A to be positive. This is a novel lower bound on the vortex number. If $2\pi N = A_0$ we again have a Bradlow limit, with degenerating vortices.

6 Integrable Vortices

When the curvature K_0 of the background surface M_0 is constant, with a special value adapted to the vortex equation, we call the vortex equation integrable. The special curvature values are those that make each side of the Baptista equation (31) vanish. The vortex equation is therefore integrable if $K_0 = C_0$. So for the Taubes, Bradlow and final type of vortices, with standard coefficients, M_0 needs to be hyperbolic, with curvature $K_0 = -1$. For the Jackiw–Pi vortices the background needs to be flat, with $K_0 = 0$, and for the Popov vortices the background needs to be spherical, with $K_0 = 1$.

The Baptista metric then has constant curvature $K = C$, except at the conical singularities, so in these five cases it is respectively hyperbolic, flat, spherical, spherical and spherical.

Finding the geometry of these integrable vortices is closely related to the problem of starting with a smooth, constant curvature Riemann surface M_0 and constructing on it another constant curvature metric (possibly with different curvature) that additionally has a number of conical singularities, each with conical excess 2π . This requires solving Liouville's equation. Solutions of Liouville's equation can locally be expressed in terms of a holomorphic function f , and the conical singularities correspond to ramification points of f , where the derivative of f vanishes.

For some of the equations we are discussing, this construction of vortex solutions is well known. We briefly review these cases. Then we discuss cases that have not been considered before, and find a few novel solutions.

The reduction to Liouville's equation is simple. Let us write the Baptista conformal factor as $\Omega = e^{2v}$. Then the curvature formula $K = -\frac{1}{2\Omega}\nabla^2 \log \Omega$ becomes $\nabla^2 v = -Ke^{2v}$, and therefore

$$\nabla^2 v = -Ce^{2v} \tag{37}$$

when $K = C$. This is Liouville's equation when C is non-zero, but the Bradlow case $C = 0$ can be included too.

The general solution of equation (37) in a simply connected region of the z -plane is

$$\Omega = e^{2v} = \frac{4}{(1 + C|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2, \tag{38}$$

where f is a holomorphic function. This formula is also valid if $C = 0$, as v is then the sum of the holomorphic function $\frac{1}{2} \log \left(2 \frac{df}{dz} \right)$ and its complex conjugate, and therefore satisfies Laplace's equation. Locally we also have an explicit expression for the background conformal factor Ω_0 . In suitably chosen local coordinates,

$$\Omega_0 = \frac{4}{(1 + C_0|z|^2)^2}. \tag{39}$$

The solution of all the integrable vortex equations is therefore locally

$$|\phi|^2 = e^{2u} = \frac{\Omega}{\Omega_0} = \frac{(1 + C_0|z|^2)^2}{(1 + C|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2, \tag{40}$$

with the values of C_0 and C as in the vortex equation.

One may fix the gauge by choosing the Higgs field itself to be

$$\phi = \frac{1 + C_0|z|^2}{1 + C|f(z)|^2} \frac{df}{dz}. \quad (41)$$

The vortex centres are the ramification points of f , where its derivative vanishes. Here, ϕ is zero. If f near Z_0 has the expansion

$$f(z) \sim f_0 + \nu(z - Z_0)^{n+1} \quad (42)$$

then the ramification number is n , and there are n coincident vortices at Z_0 .

The original application of these formulae was Witten's construction of Taubes vortices on the hyperbolic plane [4]. Here, $C_0 = C = -1$ and f is a holomorphic map from the hyperbolic plane to itself, mapping boundary to boundary. In the Poincaré disc model, f needs to be a Blaschke rational function

$$f(z) = \prod_{m=1}^{N+1} \frac{z - a_m}{1 - \overline{a_m}z} \quad (43)$$

with $|a_m| < 1$. Inside the disc, $\frac{df}{dz}$ has N zeros, so there are N vortices, and the Higgs field satisfies the boundary condition $|\phi| = 1$. The simplest example is where $f(z) = z^{N+1}$, and the expression (41) for the Higgs field is

$$\phi = \frac{1 - |z|^2}{1 - |z|^{2(N+1)}} (N+1)z^N = \frac{(N+1)z^N}{1 + |z|^2 + \dots + |z|^{2N}}. \quad (44)$$

Here, there are N coincident vortices at the origin.

Similar formulae have been used to construct solutions of the Popov vortex equation [8], with $C_0 = C = 1$. These are vortices on a unit sphere, and are constructed using a meromorphic function f , a map from the Riemann sphere to itself. To obtain a finite vortex number, the map must again be rational, of the form

$$f(z) = \frac{p(z)}{q(z)}, \quad (45)$$

where p and q are any polynomials with no common root. If p and q (and hence f) have degree n , then $\frac{df}{dz}$ has $2n - 2$ zeros. The vortex number for Popov vortices is therefore an even number, $N = 2n - 2$. It has been shown by Chen et al. [26] that this construction of integrable Popov vortices, using global rational functions, gives all solutions.

The Riemann–Hurwitz formula states that if f is a globally-defined holomorphic map of degree n from a compact surface of genus \mathfrak{g}_0 to a compact surface of genus \mathfrak{g} , then the ramification number (vortex number) N is given by

$$2 - 2\mathfrak{g}_0 + N = n(2 - 2\mathfrak{g}). \quad (46)$$

If the map is from a sphere to a sphere then $\mathfrak{g}_0 = \mathfrak{g} = 0$, so $N = 2n - 2$, confirming the vortex number given above.

Also known are solutions of the Jackiw–Pi vortex equation, both on the flat plane, and on a torus. Let us focus on solutions on a torus [11], which have the form (40) with $C_0 = 0$ and $C = 1$,

$$|\phi|^2 = \frac{1}{(1 + |f(z)|^2)^2} \left| \frac{df}{dz} \right|^2. \quad (47)$$

The simplest solutions are where f is a globally-defined holomorphic map from the torus to a sphere, i.e. a meromorphic function on the torus. f is then an elliptic function, with the double periodicity of the torus. For an elliptic function, $\mathfrak{g}_0 = 1$ and $\mathfrak{g} = 0$, and the degree n is the number of poles (counted with multiplicity). The vortex number is then $N = 2n$ according to the Riemann–Hurwitz formula. The simplest elliptic functions have degree 2 and give solutions with four vortices.

f may not be globally defined, and there is a Jackiw–Pi vortex solution where f is elliptic but the periods of f are twice the periods of the torus itself. This solution is due to Olesen [27], and has vortex number $N = 1$. It has been presented in slightly simpler form, and given a braneworld interpretation, in [28]. When the torus is glued together, f transforms, but in such a way that $|\phi|^2$ is smooth. It would be useful to more systematically investigate vortices constructed from local holomorphic maps f , suitably glued together.

A few Taubes vortex solutions are known on hyperbolic surfaces other than the hyperbolic plane [5], and in particular on the Bolza surface [6], the most symmetric genus 2 surface. The Bolza surface is a double covering of the Riemann sphere, branched over the six vertices of a regular octahedron. Algebraically, it is defined as the complex curve

$$y^2 = (x^4 - 1)x \quad (x, y \in \mathbb{C}), \quad (48)$$

with branch points at $0, 1, i, -1, -i$ supplemented by a branch point at infinity. The Bolza surface has a hyperbolic metric in which the octants of the

Riemann sphere are covered by 16 equilateral, hyperbolic triangles with vertex angles $\frac{\pi}{4}$. At each branch point, eight of these triangles meet smoothly. The Bolza surface can also be modelled as a regular hyperbolic octagon with vertex angles $\frac{\pi}{4}$ and opposite sides identified, and all eight vertices identified to one point. This octagon is one cell of the $\{8, 8\}$ tessellation of the Poincaré disc by octagons.

A compact hyperbolic surface of curvature -1 has area $A_0 = 2\pi(2\mathfrak{g}_0 - 2)$, by the Gauss–Bonnet theorem. Therefore, for the (integrable) Taubes vortex equation, $N < 2\mathfrak{g}_0 - 2$, and there can be no more than one vortex on the $\mathfrak{g}_0 = 2$ Bolza surface. An explicit solution has been found for a vortex located at a branch point (by symmetry, all branch points are equivalent) [6]. In the octagon model, two of the branch points correspond to the centre of the octagon and the vertex of the octagon, and the solution with a vortex at either point is available. The formulae depend on a compound Schwarz triangle map f , which is locally a map from the hyperbolic plane to itself, mapping a triangle to a triangle. One triangle angle is doubled by the map, and this produces the ramification of f required at the vortex centre.

The solutions considered so far were all previously known. Let us briefly mention a class of solutions of the vortex equation (20), with $C_0 = -1$, $C = 1$. This equation is integrable on a hyperbolic surface M_0 with $K_0 = -1$, and requires f to be locally a map from M_0 to a sphere. If f is globally defined, then it is a meromorphic function on M_0 . Meromorphic functions on compact surfaces are plentiful, although not generally easy to write down explicitly. For the Bolza surface defined by equation (48), the simplest meromorphic function is x . The map f is then the canonical covering map from the Bolza surface to the Riemann sphere, of degree 2. f has six ramification points, according to the Riemann–Hurwitz formula (46), and they are the lifts of the six branch points on the sphere. For example, in the neighbourhood of $x = 0$, y is a good local coordinate and $x = y^2 + \dots$. So $f(y) = y^2 + \dots$ and $\frac{df}{dy} = 0$ at $y = 0$. The function x therefore gives a 6-vortex solution of equation (20) on the Bolza surface, a vortex number consistent with the inequality $N > 2\mathfrak{g}_0 - 2$. The Baptista metric is simply the sphere metric $\frac{4}{(1+|x|^2)^2} dx d\bar{x}$ lifted to the double cover. One would need to express x in terms of y to make this lift explicit, which requires solving a quintic.

Integrable Bradlow vortices on a hyperbolic surface M_0 locally involve maps from M_0 to a flat surface. The Baptista metric is then flat, apart from its conical singularities. Such metrics arise from Abelian differentials of the first kind (holomorphic 1-forms), as follows. Given any such differential form

ω on M_0 , there is the beautifully simple metric

$$ds^2 = |\omega|^2. \quad (49)$$

Because ω is closed it can be expressed locally as $\omega = \frac{df}{dz} dz$, so the metric is

$$ds^2 = \left| \frac{df}{dz} \right|^2 dz d\bar{z}, \quad (50)$$

which is flat, being the pullback of the flat metric $dwd\bar{w}$ using the map $w = f(z)$. The metric also has conical singularities where $\omega = 0$, or equivalently at the ramification points of f .

Globally, an Abelian differential of the first kind is a section of the canonical bundle, and has $2\mathfrak{g}_0 - 2$ zeros, where \mathfrak{g}_0 is the genus of M_0 . The vortex number is therefore $N = 2\mathfrak{g}_0 - 2$, as expected for Bradlow vortices. We will describe more explicitly a Bradlow vortex solution on the $\mathfrak{g}_0 = 2$ Bolza surface in the next section.

7 Geometric Interpretation of Vortices

Associated to a vortex solution, there is the conformal modification of the background metric, which we are calling the Baptista metric. Its curvature satisfies equation (31), but note that this is not easy to solve on a general surface, and is not equivalent to the problem of constructing a metric with given curvature. The exceptions are the cases where the vortex equation is integrable. Here, finding a vortex solution is equivalent to finding a metric of constant curvature, with conical singularities of cone angle 4π at specified vortex locations. For some purposes, one may regard the Baptista metric as an intrinsic geometry of a vortex, and in this section, we shall explore this intrinsic geometry further.

Recall that the definition of the Baptista metric (26) implies that the squared Higgs field of a vortex solution on M_0 is the ratio of two conformally equivalent metrics,

$$|\phi|^2 = e^{2u} = \frac{ds^2}{ds_0^2} = \frac{\Omega}{\Omega_0}. \quad (51)$$

For integrable vortices, this is the ratio of a constant curvature metric on M_0 with conical singularities at the vortex locations (the Baptista metric

with curvature C) to the smooth, constant curvature metric on M_0 (the background metric with curvature C_0).

We can verify directly that $|\phi|^2$ satisfies the vortex equation (16). Let us write $\Omega_0 = e^{2t}$ and $\Omega = e^{2v}$, so that

$$e^{2u} = e^{2(v-t)} \quad (52)$$

and therefore $u = v - t$. The general formula (27) for the curvature implies that $\nabla^2 t = -C_0 e^{2t}$, and similarly $\nabla^2 v = -C e^{2v}$. Therefore,

$$\nabla^2 u = -C e^{2v} + C_0 e^{2t}, \quad (53)$$

and dividing by e^{2t} we obtain $e^{-2t} \nabla^2 u = -C e^{2u} + C_0$, which is equivalent to (16).

This geometric description of a vortex solution matches the formula (40) as follows. The background metric on M_0 has constant curvature C_0 , so is locally

$$ds_0^2 = \frac{4}{(1 + C_0 |z|^2)^2} dz d\bar{z}. \quad (54)$$

f is a holomorphic map (at least locally) from M_0 to a second constant curvature Riemann surface M with curvature C . Let this surface have local complex coordinate w and metric

$$\widetilde{ds}^2 = \frac{4}{(1 + C |w|^2)^2} dw d\bar{w}. \quad (55)$$

The map has the local expression $w = f(z)$, so $dw = \frac{df}{dz} dz$. The metric \widetilde{ds}^2 , pulled back to M_0 using the map f , is therefore

$$ds^2 = \frac{4}{(1 + C |f(z)|^2)^2} \left| \frac{df}{dz} \right|^2 dz d\bar{z}, \quad (56)$$

and this is the Baptista metric. It still has curvature C , but also has conical singularities at the ramification points of f , the locations of the vortices. The ratio of the metrics (56) and (54) then gives the formula (40).

The Baptista metric, being the pullback of a constant curvature metric, can sometimes be described explicitly. For example, it was shown in [6] that an $N = 1$ Taubes vortex on the Bolza surface, located at the centre of the hyperbolic Bolza octagon, is obtained using a map f from the octagon

to a hyperbolic square with the same vertex angles, $\frac{\pi}{4}$. The map double covers the square, with a branch point at the centre of the square. The full image winds round the square twice and is therefore itself an octagon with a conical singularity at the centre. Its hyperbolic metric, including the singularity, is the Baptista metric. The pullback by f places this metric on the original octagon. Opposite sides are identified in the same way for both octagons. The octagon with the Baptista metric has half the area of the original Bolza octagon. (Note that f is not a globally-defined map between compact surfaces, because the hyperbolic square by itself does not have opposite sides identified. The appropriate modification of equation (46) is discussed in [6].)

It is rather easier to visualise this geometry if we consider the vortex to be at the vertex of the Bolza octagon, which is equivalent by symmetry to being at the centre. In this case the appropriate map f is from the Bolza octagon with vertex angles $\frac{\pi}{4}$ to a smaller hyperbolic octagon with vertex angles $\frac{\pi}{2}$. Gluing opposite sides of the smaller octagon together creates a cone of angle 4π at the vertex, as required. This glued-together octagon is the intrinsic geometry of the vortex with its Baptista metric, and some of its geometric properties are easy to calculate. This is despite the fact that the map f involves Schwarz triangle functions (and hence hypergeometric functions), so is not very explicit.

These two octagons are shown in Figure 1. The intrinsic, Baptista octagon has half the area of the Bolza octagon. Using hyperbolic trigonometry, one can compare the lengths of their closed geodesics, for example, the geodesics along the boundary connecting vertex to vertex. Assuming the curvature is $K_0 = -1$, the boundary geodesic of the Bolza octagon has length a , where $\cosh a = 5 + 4\sqrt{2}$. The analogous geodesic on the octagon with its Baptista metric and conical singularity has the shorter length \tilde{a} , where $\cosh \tilde{a} = 1 + \sqrt{2}$.

An $N = 2$ Taubes vortex on the Bolza octagon would saturate the Bradlow limit, and the Baptista metric would degenerate and have zero area. We have evaded this degeneracy by introducing the Bradlow vortex equation (19). For this equation on the Bolza surface, the vortex number N must be 2. The solution for the squared Higgs field is given by equation (40) with $C = 0$. The Baptista metric is therefore the pullback of a flat metric to the Bolza surface

$$ds^2 = \left| \frac{df}{dz} \right|^2 dzd\bar{z}. \quad (57)$$

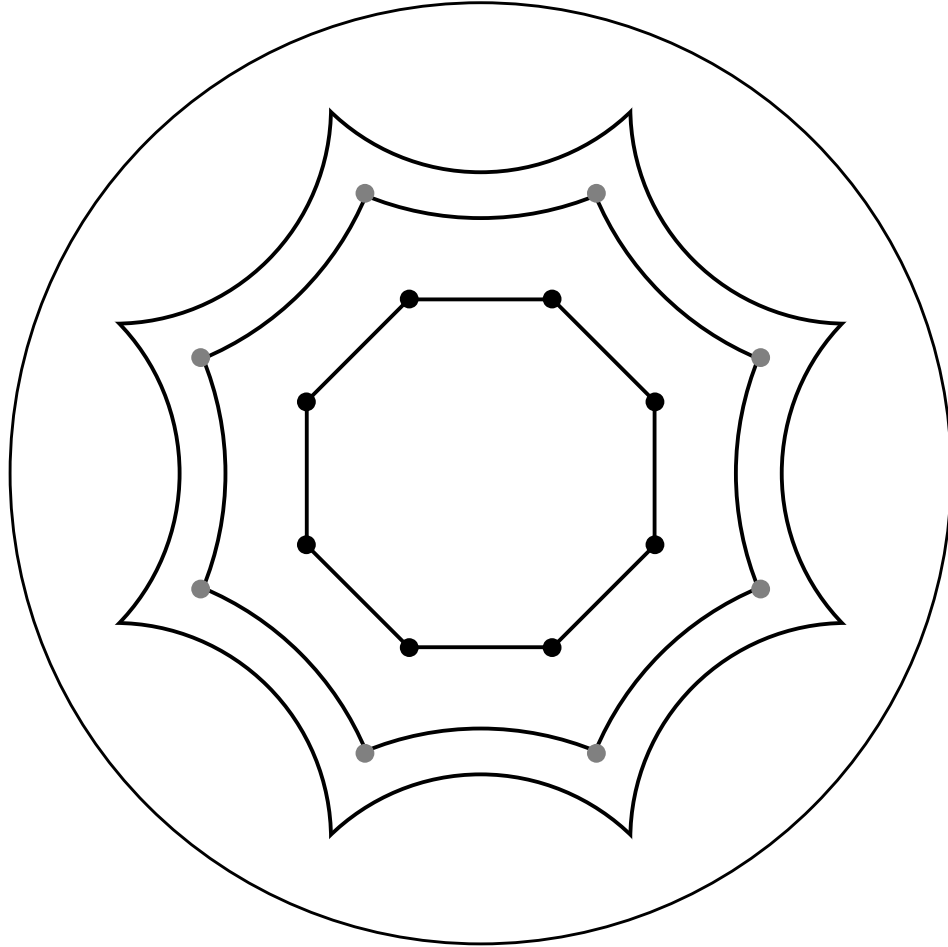


Figure 1: Three octagons: The outer is the Bolza octagon with vertex angle $\frac{\pi}{4}$ superimposed on the Poincaré disc; the middle is the Baptista octagon with vertex angle $\frac{\pi}{2}$ – the intrinsic (hyperbolic) geometry of an $N = 1$ Taubes vortex located at the Bolza octagon vertex; the inner is a flat octagon with vertex angle $\frac{3\pi}{4}$ – the intrinsic geometry of an $N = 2$ Bradlow vortex located at the Bolza octagon vertex. In all cases, opposite edges are identified.

For a vortex of multiplicity 2 at the Bolza octagon vertex, the intrinsic geometry is now a flat, regular octagon of arbitrary scale size, with its straight, opposite sides identified. This octagon is also shown in Figure 1. The vertex angle is $\frac{3\pi}{4}$, so after gluing, there is a single conical singularity of cone angle 6π . The conical excess is 4π , as expected for a vortex of multiplicity 2. The map f is from the hyperbolic Bolza octagon to the flat octagon, which again involves nontrivial Schwarz triangle maps. The intrinsic geometry of the vortex is fairly easy to visualise, even though the squared Higgs field is not an elementary function.

The intrinsic geometry of Popov vortices is a spherical metric on a unit-sphere background, incorporating conical singularities. For example, the 2-vortex intrinsic geometry is a double covering of the unit sphere branched over a pair of points, with the unit-sphere metric lifted to both sheets. The total area is therefore 8π , twice the original area. On the background sphere, the vortex locations can be at any pair of distinct points. Coincident vortices are not allowed, because a rational map of degree 2 cannot have a single ramification point. It is also known that in the intrinsic spherical geometry with two conical singularities, the cone angles must be equal, and the singularities are at antipodal points [29]. This is intuitively fairly clear. A conical singularity (with an angular deficit or angular excess) at the north pole opens up a wedge bounded by meridians (geodesics), and these meet at the south pole at the same angle. Then these meridians are glued together. The Baptista metric for two Popov vortices must therefore have this geometry with cone angles 4π , for any rational map of degree 2.

It is also easy to describe examples of the intrinsic geometry of the Jackiw–Pi vortices, and of the vortex solutions of equation (20). The background surface should be a flat torus of genus $\mathfrak{g}_0 = 1$ in the Jackiw–Pi case, and a hyperbolic surface of higher genus \mathfrak{g}_0 in the second case. Suppose the background surface is hyperelliptic, a double cover of a sphere with $2\mathfrak{g}_0 + 2$ branch points. (This always holds for surfaces of genus 1 or 2, but only for selected surfaces of higher genus.) A special vortex solution is then obtained by choosing f to be the covering map. The vortex number is $N = 2\mathfrak{g}_0 + 2$ and the vortices are located at the ramification points (the points covering the branch points). The Baptista metric is the underlying spherical metric pulled back to the double cover, and it has conical singularities with cone angle 4π over each branch point. $|\phi|^2$ is the ratio of this lifted spherical metric to the smooth, background flat or hyperbolic metric, but is not an elementary function.

We can go beyond double covers. Suppose $f : M_0 \rightarrow \tilde{M}$ is any globally-defined, branched covering map between compact Riemann surfaces. Both M_0 and \tilde{M} have unique constant curvature metrics. The pullback by f of the metric on \tilde{M} to M_0 is a Baptista metric ds^2 of vortices on M_0 , with vortices located at the ramification points of f , and its ratio to the smooth metric ds_0^2 on M_0 is the squared Higgs field. The relevant vortex equation that is satisfied depends on the two curvatures.

8 Conclusions

We have considered the generalised Taubes equation for vortices on a curved background surface, $-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u}$. By rescaling, both C_0 and C take standard values $-1, 0$ or 1 , but only five combinations of these values allow vortex solutions without singularities. After reviewing Baptista's metric $ds^2 = |\phi|^2 ds_0^2$, where ds_0^2 is the background metric and ϕ the Higgs field, we have seen that the vortex equation is integrable provided the background curvature is constant and equals C_0 . Baptista's metric is then of constant curvature C , but additionally, for an N -vortex solution, it has N conical singularities with cone angles 4π . Solutions of Liouville's equation, locally involving a holomorphic function f , give constant curvature metrics, and the conical singularities arise from ramification points of f . This allows a unified treatment of known solutions for Taubes, Jackiw–Pi and Popov vortices, and also for the two further types of vortex presented here, in all the integrable cases. The squared Higgs field on a compact Riemann surface M_0 is simply the ratio of a constant curvature metric with conical singularities to the unique, smooth constant curvature metric.

For some vortex solutions, including 1- and 2-vortex solutions on the genus-2 Bolza surface, we have described the intrinsic Baptista geometry, bypassing the need for finding the Higgs field explicitly. It would be desirable to extend this intrinsic geometrical picture of vortices to further examples.

For integrable vortices, the Baptista metric with its constant curvature and conical singularities is the spatial part of an Einstein metric in 2+1 dimensions with cosmological constant [17, 18]. The singularities have a 2π conical excess, and therefore correspond to point-particle sources of negative mass. It is a surprise that vortices, which are normally regarded as smooth field configurations on a smooth surface, have such a point-particle interpretation, and it would be interesting if the dynamics of vortices could be

related to the dynamics of gravitating point particles.

The integrable cases of the Taubes and Popov vortex equations, where the background geometry is respectively hyperbolic and spherical, are known to arise from a dimensional reduction of the self-dual Yang–Mills equations in \mathbb{R}^4 [4, 7]. Vortex solutions can therefore be interpreted as instantons with symmetry. A more systematic treatment of the dimensional reduction, allowing for a wider range of symmetry groups and gauge groups, can probably account for all the vortex equations considered here. This is under investigation by Contatto and Dunajski [30].

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