Agreeing in networks: unmatched disturbances, algebraic constraints and optimality

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Abstract

This paper considers a problem of output agreement in heterogeneous networks with dynamics on the nodes as well as on the edges. The control and disturbance signals entering the nodal dynamics are "unmatched" meaning that some nodes are only subject to disturbances and not to the actuating signals. To further enrich our model and motivated by synchronization problems in physical networks, we accommodate (solvable) algebraic constraints resulting in a fairly general and heterogeneous network. It is shown that appropriate dynamic feedback controllers achieve output agreement on a desired vector, in the presence of physical coupling and despite the influence of constant as well as time-varying disturbances. Furthermore, we address the case of an optimal steady-state deployment of the control effort over the network by suitable distributed controllers. As a case study, the proposed results are applied to a heterogeneous microgrid.

Key words: Output agreement, algebraic constraints, heterogeneous networks, unmatched disturbances

1 Introduction

A central theme in cooperative control is an agreement among the agents on a certain quantity of interest. The most notable instances are distributed optimization [27], consensus [19], formation control [18], and synchronization, see e.g. [24], [15], [25].

Output synchronization problem has been studied for various models including linear [31] as well as nonlinear agents' dynamics [32]. Compared to the vast amount of literature on output synchronization with various nodal/agent dynamics, relatively few works have considered dynamics on the links, see e.g. [7], [5], [30]. The dynamics on the links could arise from the physical coupling present in the network [30] or as a consequence of distributed controllers located on the links [5].

The study of output agreement/regulation problem in the presence of disturbances has been motivated by numerous applications in balancing demand and supply, power networks, and hydraulic networks. In this framework, the demands/loads are interpreted as external disturbances affecting the network dynamics, see e.g. [8], [6], [10].

In certain applications, [10], [11], control and disturbance signals enter the network dynamics via separated nodes resulting in the case of an *unmatched* disturbance-control scheme. This scheme is ubiquitous in heterogeneous distribution networks where producers/generators are distinguished from consumers/loads. The resulting heterogeneity in the role of the nodes clearly adds to the complexity of the control [20].

A desirable feature in balancing demand and supply in distribution networks is to share the overall demands among the suppliers. This has led to control schemes ensuring an optimal steady-state supply distribution over the network [6], [11]. For a more general and unifying view over the relationship between (passivity-based) cooperative control and network optimization see [7].

In this paper, we consider the nodal dynamics as non-identical nonlinear port-Hamiltonian systems; see [29] for more information on port-Hamiltonian systems. This nodal dynamics is subject to external disturbances. In addition, we consider that a subset of nodal dynamics is governed by algebraic constraints. These constraints could be the result of mismatch in the dynamic order of the agents [33], or an approximation of fast subdynamics in singularly perturbed models [14]. The algebraic constraints we consider here are solvable meaning that they can be expressed in terms of other state variables of the network. However, obviously, the presence of such con-

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straints adds to the heterogeneity of the network, and thus complicates the analysis.

We consider the physical coupling to be “undamped”, and given by a single integrator with a nonlinear output map. We first show that an equilibrium of the network, if exists, is attractive and thus output agreement is locally achieved for the network. Next, we include controller dynamics on some nodes to guarantee output agreement on a prescribed setpoint, in the presence of physical coupling and disturbance signals. We treat an *unmatched* control-disturbance scheme meaning that control signals and disturbances may act on different subsets of nodes. Both constant as well as time-varying disturbances are incorporated in the design via decentralized integral and decentralized internal model based controllers, respectively. Furthermore, by appropriate distributed controllers, we include “steady-state” optimality ensuring a desired deployment of the control effort over the entire network. Time-varying disturbances and the optimal deployment of the control effort are treated in accordance with output regulation theory [5, 21] where disturbance signals are generated by suitable exosystems.

As a case study, we consider a heterogeneous microgrid consisting of synchronous generators, droop-controlled inverters, and frequency dependent loads, where the goal is to guarantee a zero frequency deviation for all the nodes of the grid, and to optimally distribute the active power.

The main contribution of the current manuscript is to consider *simultaneously* i) multivariable nonlinear nodal dynamics, ii) dynamic physical coupling, iii) algebraic constraints, iv) unmatched time-varying disturbances, and v) optimality constraints in the output agreement problem.

Our analysis here is implicitly based on passivity and incremental passivity property inspired by [2], [3], [7], [5], [30].

This paper is organized as follows. The analysis of output agreement problem is carried out in Section 2, whereas the control design is treated in Section 3. Section 4 is devoted to the case study of microgrids. Conclusions are provided in Section 5. The formal proofs of the proposed results are collected in Appendix.

Notation Apart form the standard notation, we use the following conventional notation. We use superscripts for vectors and matrices to indicate their domain of definition. In particular, let x_j with $j \in \mathcal{I}$ be a set of vectors. Then, by x^i we mean $x^i = \mathrm{col}(x_j)$ with $j \in \mathcal{I}_i \subseteq \mathcal{I}$. For a set of matrices, we define $A^i = \mathrm{blockdiag}(A_j)$ with $j \in \mathcal{I}_i \subseteq \mathcal{I}$. We remove the superscript in case $\mathcal{I}_i = \mathcal{I}$.

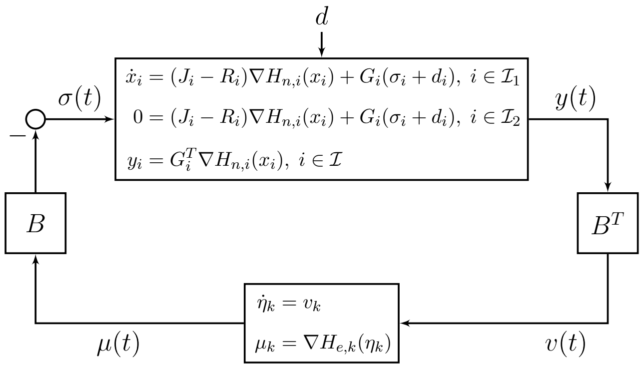

Fig. 1. Block-diagram of the network model

2 Network model and attractivty analysis

We define a dynamical network on a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We partition the vertex set of $\mathcal{G}$ into two distinct subsets, $\mathcal{V} := \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. To each vertex of $\mathcal{G}$, we associate the following port-Hamiltonian types of dynamics:

$$\dot{x}_i = (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + d_i) \quad i \in \mathcal{I}_1 \tag{1a}$$

$$0 = (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + d_i) \quad i \in \mathcal{I}_2 \tag{1b}$$

$$y_i = G_i^T \nabla H_{n,i}(x_i) \quad i \in \mathcal{I} \tag{1c}$$

where $x_i \in \mathbb{R}^n$, J_i is a skew symmetric matrix, R_i is a positive definite matrix, $G_i \in \mathbb{R}^{n \times m}$ is the input matrix, $\sigma_i \in \mathbb{R}^m$ accounts for the physical coupling, $d_i \in \mathbb{R}^m$ is a constant vector, and the Hamiltonian $H_{n,i} : \mathbb{R}^n \to \mathbb{R}$ is strictly convex in an open convex set $\Omega_n \subseteq \mathbb{R}^n$ for each i. Note that the set $\mathcal{I}_1$ indexes the nodes whose dynamics are given by differential equations, whereas $\mathcal{I}_2$ indexes the ones given by algebraic equations.

To each edge of $\mathcal{G}$, we associate the following dynamics:

$$\dot{\eta}_k = v_k \tag{2a}$$

$$\mu_k = \nabla H_{e,k}(\eta_k) \tag{2b}$$

where $\eta_k \in \mathbb{R}^m$, the Hamiltonian $H_{e,k} : \mathbb{R}^m \to \mathbb{R}$ is strictly convex in an open convex set $\Omega_e \subseteq \mathbb{R}^m$, and $k = 1, 2, \ldots, M$. The interconnection law is given by

$$v = (B^T \otimes I_m)y, \quad \sigma = -(B \otimes I_m)\mu \tag{3}$$

where B is the incidence matrix of $\mathcal{G}$, $v = \mathrm{col}(v_k)$, $y = \mathrm{col}(y_i)$, and $\sigma = \mathrm{col}(\sigma_i)$ with $k = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$; see Figure 1.

Then, the edge dynamics (2), the nodal dynamics (1), and the interconnection law (3) can be written compactly as

$$\dot{\eta} = (B^T \otimes I)G^T \nabla H_n(x) \tag{4a}$$

$$\mu = \nabla H_e(\eta) \tag{4b}$$

$$\dot{x}^1 = (J^1 - R^1)\nabla H_n^1(x^1) - G^1(B^1 \otimes I)\nabla H_e(\eta) + G^1 d^1 \tag{4c}$$

$$0 = (J^2 - R^2)\nabla H_n^2(x^2) - G^2(B^2 \otimes I)\nabla H_e(\eta) + G^2 d^2 \tag{4d}$$

$$y = G^T \nabla H_n(x) \tag{4e}$$

where B^1 and B^2 denote the submatrices obtained from B by collecting the rows indexed by $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively.

Let $x = \mathrm{col}(x^1, x^2)$ and $d = \mathrm{col}(d^1, d^2)$. The constant vector d is a *partially controllable* disturbance and will be made more explicit in the next section. Now, suppose that $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ is an equilibrium of system (4) with $\dot{\bar{x}} = 0$ and $\dot{\bar{\eta}} = 0$. Then, we have

$$0 = (B^T \otimes I)G^T \nabla H_n(\bar{x}) \tag{5a}$$

$$0 = (J^1 - R^1)\nabla H_n^1(\bar{x}^1) - G^1(B^1 \otimes I)\nabla H_e(\bar{\eta}) + G^1 d^1 \tag{5b}$$

$$0 = (J^2 - R^2)\nabla H_n^2(\bar{x}^2) - G^2(B^2 \otimes I)\nabla H_e(\bar{\eta}) + G^2 d^2. \tag{5c}$$

Observe that the equation (5a) yields an output agreement condition

$$G_i^T \nabla H_{n,i}(\bar{x}_i) = G_j^T \nabla H_{n,j}(\bar{x}_j), \quad \forall i, j \in \mathcal{I}. \tag{6}$$

Hence, we obtain that $G^T \nabla H_n(\bar{x}) = \mathbb{1}_N \otimes y^*$ for some constant vector $y^* \in \mathbb{R}^n$. The other two equations can be written together as

$$0 = (J - R)\nabla H_n(\bar{x}) - G(B \otimes I)\nabla H_e(\bar{\eta}) + Gd. \tag{7}$$

This implies that

$$\mathbb{1}_N \otimes y^* = G^T (J - R)^{-1} G((B \otimes I)\nabla H_e(\bar{\eta}) - d). \tag{8}$$

Notice that nonsingularity of the matrix $J - R$ follows from the positive definiteness of R. In case the matrix G is equal to the identity matrix, by multiplying both hand sides of (8) from the left by $(\mathbb{1}_N^T \otimes I_n)(J - R)$, we obtain that

$$\sum_{i=1}^{N} (J_i - R_i) y^* = -\sum_{i=1}^{N} d_i. \tag{9}$$

Hence, $y^* = \nabla H_{n,i}(\bar{x}_i)$ is computed as

$$y^* = -(\sum_{i=1}^{N} (J_i - R_i))^{-1} \sum_{i=1}^{N} d_i. \tag{10}$$

Then, noting that $\mathbb{1}_N \otimes y^* = \nabla H_n(\bar{x})$, the constant vector $\bar{x} \in (\Omega_n)^N$ is unique in this case. It is worth mentioning that in the case $n = 1$, we have $J = 0$, and (10) is simplified to $y^* = \frac{\mathbb{1}^T d}{\mathbb{1}^T R \mathbb{1}}$.

By replacing (10) in (8) with $G = I$, the term $(B \otimes I)\nabla H_e(\bar{\eta})$ is explicitly computed. Hence, $\bar{\eta} \in (\Omega_e)^M$ is in general not unique. However, in case the graph $\mathcal{G}$ is a tree, the incidence matrix B has full column rank, and thus $\bar{\eta}$ is unique. Note that an equilibrium $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ does not always exist, and in particular we need to assume the *feasibility conditions* (6) and (7). The following theorem investigates attractivity properties of an invariant set containing this equilibrium, in which (6) holds.

Theorem 1 *Suppose that $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ is an equilibrium of* (4). *Then there exists a region of the state space, which includes $(\bar{x}, \bar{\eta})$, such that any solution (x, η) of* (4) *starting in this region asymptotically converges to an invariant set of* (4) *where* (6) *holds.*

Proof. See Appendix.

Note that Theorem 1 implies that the network (4) reaches an output agreement providing that there exist constant vectors $(\bar{x}, \bar{\eta}) \in (\Omega_n)^N \times (\Omega_e)^M$ satisfying (6), (7), and thus (8). The vector y^* resulting from this agreement could be different than the desired one, due to the dependency on the disturbance d, see (8) and (10). Hence, we investigate next the possibility to influence this vector by an appropriate control scheme.

3 Controlling unmatched disturbances

In this section, we treat certain control problems related to network dynamics (4). These problems involve steering the vector y^* to a desired point. To this end, we actuate some nodes of the network, and further partition the nodal dynamics (1) as

$$\begin{aligned}
\dot{x}_i &= (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + u_i + \delta_i) && i \in \mathcal{I}_{11} \\
\dot{x}_i &= (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + \delta_i) && i \in \mathcal{I}_{12} \\
0 &= (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + u_i + \delta_i) && i \in \mathcal{I}_{21} \\
0 &= (J_i - R_i)\nabla H_{n,i}(x_i) + G_i(\sigma_i + \delta_i) && i \in \mathcal{I}_{22} \\
y_i &= G_i^T \nabla H_{n,i}(x_i) && i \in \mathcal{I}
\end{aligned} \tag{11}$$

where $\mathcal{I}_1 = \mathcal{I}_{11} \cup \mathcal{I}_{12}$, $\mathcal{I}_2 = \mathcal{I}_{21} \cup \mathcal{I}_{22}$, $\mathcal{I}_{11} \neq \varnothing$, and G_i has a full column rank for each i. Here, the $u_i \in \mathbb{R}^m$ components are treated as control signals which are applied to a subset of the nodes, namely $\mathcal{I}_c := \mathcal{I}_{11} \cup \mathcal{I}_{21}$. The vectors $\delta_i \in \mathbb{R}^m, i \in \mathcal{I}$, are unknown disturbance

	Differential Equation	Algebraic Equation
Controlled	$\mathcal{I}_{11}$	$\mathcal{I}_{21}$
Uncontrolled	$\mathcal{I}_{12}$	$\mathcal{I}_{22}$

Table 1. Node partitions in (11)

signals affecting the nodal dynamics. Table 1 clarifies further the four subsets of nodes in (11): the set $\mathcal{I}_{11}$ indexes the nodes whose dynamics are given by differential equations and are directly controlled. The set $\mathcal{I}_{12}$ indexes the nodes whose dynamics are given by differential equations, but are not directly controlled. The set $\mathcal{I}_{21}$ indicates the nodes the dynamics of which are given by algebraic equations and are directly controlled. Finally the set $\mathcal{I}_{22}$ indicates the nodes the dynamics of which are given by algebraic equations, but are not directly controlled.

Note that as the nodes in $\mathcal{I}_{12}$ and $\mathcal{I}_{22}$ are not directly controlled, our treatment here incorporates the case of an *unmatched* control-disturbance scheme.

Remark 2 The model (11) is fairly general and captures a variety of control scenarios as a special case. For instance, in case of *matched* disturbances, we have $\mathcal{I}_C = \mathcal{I}$, and in case the algebraic constraints are absent, $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$ are empty sets.

The overall network dynamics now can be written as

$$\dot{\eta} = (B^T \otimes I)G^T \nabla H_n(x) \tag{12a}$$

$$\begin{aligned}\dot{x}^{11} = (J^{11} - R^{11})\nabla H_n^{11}(x^{11}) - G^{11}(B^{11} \otimes I)\nabla H_e(\eta) \\ + G^{11}u^{11} + G^{11}\delta^{11}\end{aligned} \tag{12b}$$

$$\begin{aligned}\dot{x}^{12} = (J^{12} - R^{12})\nabla H_n^{12}(x^{12}) \\ - G^{12}(B^{12} \otimes I)\nabla H_e(\eta) + G^{12}\delta^{12}\end{aligned} \tag{12c}$$

$$\begin{aligned}0 = (J^{21} - R^{21})\nabla H_n^{21}(x^{21}) - G^{21}(B^{21} \otimes I)\nabla H_e(\eta) \\ + G^{21}u^{21} + G^{21}\delta^{21}\end{aligned} \tag{12d}$$

$$\begin{aligned}0 = (J^{22} - R^{22})\nabla H_n^{22}(x^{22}) \\ - G^{22}(B^{22} \otimes I)\nabla H_e(\eta) + G^{22}\delta^{22}\end{aligned} \tag{12e}$$

$$y = G^T \nabla H_n(x). \tag{12f}$$

Our goal here is to design dynamic feedback controllers u^{11} and u^{21} such that output agreement (6) is guaranteed for the network, for a prescribed vector y^*, in the presence of network coupling and disturbance signals. If such u^{11} and u^{21} exist, we say that the output agreement problem is *solvable*. Obviously, this may not be always plausible, and by (12) we obtain the following condition

$$\mathbb{1} \otimes y^* = G^T \nabla H_n(\bar{x})$$

$$\begin{aligned}
0 &= (J^{11} - R^{11})\nabla H_n^{11}(\bar{x}^{11}) \\
&\quad - G^{11}(B^{11} \otimes I)\nabla H_e(\bar{\eta}) + G^{11}\bar{u}^{11} + G^{11}\delta^{11} \\
0 &= (J^{12} - R^{12})\nabla H_n^{12}(\bar{x}^{12}) \\
&\quad - G^{12}(B^{12} \otimes I)\nabla H_e(\bar{\eta}) + G^{12}\delta^{12} \\
0 &= (J^{21} - R^{21})\nabla H_n^{21}(\bar{x}^{21}) \\
&\quad - G^{21}(B^{21} \otimes I)\nabla H_e(\bar{\eta}) + G^{21}\bar{u}^{21} + G^{21}\delta^{21} \\
0 &= (J^{22} - R^{22})\nabla H_n^{22}(\bar{x}^{22}) \\
&\quad - G^{22}(B^{22} \otimes I)\nabla H_e(\bar{\eta}) + G^{22}\delta^{22}
\end{aligned} \tag{13}$$

The equations above, in fact, depict the steady state solution of the network identified by $\bar{x}$, $\bar{\eta}$, and $\bar{u}$. Let d_i be defined as

$$d_i = \begin{cases} \overline{u}_i + \delta_i & i \in \mathcal{I}_c, \\ \delta_i & i \notin \mathcal{I}_c. \end{cases} \tag{14}$$

Since G_i has a full column rank, and d_i is constant, by (13) we obtain the following feasibility condition.

Feasibility condition: there exist constant vectors $\bar{x} \in (\Omega_n)^N$, $\bar{\eta} \in (\Omega_e)^M$, d^{11}, d^{21} such that

$$\begin{aligned}
\mathbb{1} \otimes y^* &= G^T \nabla H_n(\bar{x}) \\
0 &= (J^{11} - R^{11})\nabla H_n^{11}(\bar{x}^{11}) \\
&\quad - G^{11}(B^{11} \otimes I)\nabla H_e(\bar{\eta}) + G^{11}d^{11} \\
0 &= (J^{12} - R^{12})\nabla H_n^{12}(\bar{x}^{12}) \\
&\quad - G^{12}(B^{12} \otimes I)\nabla H_e(\bar{\eta}) + G^{12}d^{12} \\
0 &= (J^{21} - R^{21})\nabla H_n^{21}(\bar{x}^{21}) \\
&\quad - G^{21}(B^{21} \otimes I)\nabla H_e(\bar{\eta}) + G^{21}d^{21} \\
0 &= (J^{22} - R^{22})\nabla H_n^{22}(\bar{x}^{22}) \\
&\quad - G^{22}(B^{22} \otimes I)\nabla H_e(\bar{\eta}) + G^{22}d^{22}
\end{aligned} \tag{15}$$

Remark 3 Finding a solution that fulfills the feasibility condition is in general a difficult task, in fact, they are tantamount to solving the so-called regulation equations, whose solution very much depends on the structure of the system. Nevertheless, systems for which the condition is fulfilled are known and we refer the interested reader to [5, Proposition 3] and [10], the latter investigating the feasibility conditions for the case study of Section 4.

3.1 Constant disturbances

First, we consider the case of constant disturbances. In particular, assume that δ_i is constant for each $i \in \mathcal{I}$. If in addition the control actions are constant, i.e. $u_i = \bar{u}_i$ for each $i \in \mathcal{I}_c$, then d_i can be defined as in (14), and the network dynamics reduces to (4). This again may result in an undesired vector y^* due to the presence of unknown terms δ_is. To achieve output agreement on a prescribed vector y^* dynamic compensation is needed, and we have the following result.

Theorem 4 *Consider the decentralized controller*

$$\dot{\xi}_i = y^* - G_i^T \nabla H_{n,i}(x_i) \tag{16a}$$

$$u_i = \xi_i \tag{16b}$$

with $i \in \mathcal{I}_{11} \cup \mathcal{I}_{21}$*. Assume that the feasibility condition* (15) *holds. Let* $\xi = \mathrm{col}(\xi_i)$ *and* $\bar{\xi} = \bar{u}$*. Then, there exists a region of the state space, including* $(\bar{x}, \bar{\eta}, \bar{\xi})$*, such that any solution* (x, η, ξ) *of the network originating from this region asymptotically converges to an invariant set of* (12),(16)*, in which* $G_i^* \nabla H_{n,i}(\bar{x}_i) = y^*$ *for each* $i \in \mathcal{V}$*.*

Proof. See Appendix.

Remark 5 Note that in case the controller at a node $i \in \mathcal{I}_{11}$ or $i \in \mathcal{I}_{21}$ does not have access to the desired output y^*, one can set u_i to a constant, namely a nominal value, and incorporate the node i in the subdynamics of (11) corresponding to the uncontrolled nodes indexed by $\mathcal{I}_{12}$ or $\mathcal{I}_{22}$, respectively.

3.2 Time-varying disturbances

In this subsection, we discuss the output agreement problem for possibly time-varying disturbances.

As the feasibility condition (15) readily implies that δ^{12} and δ^{22} are constant vectors, we restrict the time-varying disturbances to nodal dynamics defined on $\mathcal{I}_c = \mathcal{I}_{11} \cup \mathcal{I}_{21}$. Following the internal model framework (see e.g. [5, 21]), we consider the case where the disturbance signals are generated by an exosystem, namely

$$\dot{w}_i = s_i(w_i) \tag{17a}$$

$$\delta_i = P_i w_i. \tag{17b}$$

for each $i \in \mathcal{I}_c$. Here, $s_i : \mathbb{R}^r \to \mathbb{R}^r$, $P_i \in \mathbb{R}^{m \times r}$, and $w_i \in \mathbb{R}^r$ is the state of the exosystem. We assume that system (17) is *incrementally passive*, that is for any two solutions w_i and w_i' of (17a) we have [21, Def. 1], [5, Assump. 1]

$$(w_i - w_i')^T (s_i(w_i)) - s_i(w_i')) \leqslant 0. \tag{18}$$

An important subclass of (17a) satisfying (18) is given by $s(w) = Sw$ with S being a skew-symmetric matrix. In this case, the exosystem (17) generates linear combinations of constant and sinusoidal signals. For distribution networks, this is motivated by spectral decomposition of load patterns [1], ocean wave energy [12] and wind energy [17, 28] indicating that the net load can indeed be approximated by a superposition of a constant and a few sinusoidal signals. Now, we have the following result.

Theorem 6 *Assume that the feasibility condition* (15) *holds. Suppose that* δ_i *is given by* (17) *with an incrementally passive map* s_i *satisfying* (18) *for each* $i \in \mathcal{I}_c$*. Consider the decentralized internal model based controller*

$$\dot{\xi}_i = y^* - G_i^T \nabla H_{n,i}(x_i) \tag{19a}$$

$$\dot{\zeta}_i = s_i(\zeta) - P_i^T (y^* - G_i^T \nabla H_{n,i}(x_i)) \tag{19b}$$

$$u_i = \xi_i - P_i \zeta_i \tag{19c}$$

with $i \in \mathcal{I}_c$*. Let* $\xi = \mathrm{col}(\xi_i)$*,* $\bar{\xi} = \mathrm{col}(d^{11}, d^{21})$*,* $\zeta = \mathrm{col}(\zeta_i)$*,* $\bar{\zeta}(0) = \mathrm{col}(w^{11}(0), w^{21}(0))$*. Then, there exists a region of the state space, including* $(\bar{x}, \bar{\eta}, \bar{\xi}, \bar{\zeta}(0))$*, such that any solution* (x, η, ξ, ζ) *of the system* (12), (19)*, originating from this region asymptotically converges to an invariant set of the system in which* $G_i^* \nabla H_{n,i}(\bar{x}_i) = y^*$ *for each* $i \in \mathcal{V}$*.*

Proof. See Appendix.

3.3 An optimal deployment of the control effort

In the previous subsection, the control input u has been designed such that output agreement s achieved on a desired vector y^*. The control signal $\bar{u}$ in both cases (16) and (19) is determined by the initialization of the system and the controller. Next, we aim at adding an *optimality* property into the picture and investigate possible optimal deployment of the steady-state control effort together with the desired network agreement. Motivated by the applications in distribution and in particular power networks, the notion of optimal deployment here refers to suitable cost minimization at the nodes [26], [8], [11]. In particular, let

$$C_i(\bar{u}_i) = \frac{1}{2} \bar{u}_i^T Q_i \bar{u}_i \tag{20}$$

be the cost of the steady-state control effort at node $i \in \mathcal{I}_c$, where $Q_i \in \mathbb{R}^m \times \mathbb{R}^m$ is a positive definite matrix. We aim at minimizing the total generation cost given by

$$\min_{\bar{u}} = \frac{1}{2} \sum_{i \in \mathcal{I}_c} \bar{u}_i^T Q_i \bar{u}_i. \tag{21}$$

under the constraint of output agreement (6) with a desired y^*.

To make the analysis more concise, we restrict our attention in this subsection to the case where $G_i = I$ for each

i. Then, similar to (9), we obtain the following "supply-demand" matching constraint

$$\sum_{i=1}^{N}(J_i - R_i)y^* + \sum_{i\in\mathcal{I}_c} \bar{u}_i + \sum_{i=1}^{N} \delta_i = 0 \tag{22}$$

By standard Lagrange multipliers method, the vector $\bar{u}$ which minimizes (21) subject to (22) is obtained as

$$\bar{u}_i = Q_i^{-1}\lambda \tag{23}$$

where $\lambda \in \mathbb{R}^n$ is given by

$$\lambda = -(\sum_{i\in\mathcal{I}_c} Q_i^{-1})^{-1}(\sum_{i=1}^{N}(J_i - R_i)y^* + \sum_{i=1}^{N} \delta_i) \tag{24}$$

Recall that, by (13), $\bar{u}_i + \delta_i$ is constant for each $i \in \mathcal{I}_c$. This together with the optimality condition (23) substantially restricts the set of time-varying disturbance signals that can be accommodated in the control design. To see this more clearly, let the disturbance signal δ_i be decomposed as $\delta_i(t) = \bar{\delta}_i + \tilde{\delta}_i(t)$ where $\bar{\delta}_i$ is a constant vector, for each $i \in \mathcal{I}_c$. Now, as $\bar{u}_i + \delta_i$ has to be constant, we obtain the following constraint

$$-Q_i^{-1}(\sum_{i\in\mathcal{I}_c} Q_i^{-1})^{-1}(\sum_{i\in\mathcal{I}_c} \tilde{\delta}_i) = -\tilde{\delta}_i. \tag{25}$$

Note that the left hand side of (25) is the time-varying component of $\bar{u}_i$ whereas the right hand side is the time varying component of $-\delta_i$. Clearly, by (25), it is necessary that $\tilde{\delta}_i \in \text{im}(Q_i^{-1})$. Let $\tilde{\delta}_i$ be written as $\tilde{\delta}_i = Q_i^{-1}v_i$ for some (time varying) vector v_i. Then (25) simplifies to

$$(\sum_{i\in\mathcal{I}_c} Q_i^{-1})^{-1}(\sum_{i\in\mathcal{I}_c} Q_i^{-1}v_i) = v_i. \tag{26}$$

Hence, as the left hand side of (26) is independent of i, we obtain that $v_i = v_j$ for every $i, j \in \mathcal{I}_c$. Consequently, $\tilde{\delta}_i$ is equal to $Q_i^{-1}v$ for some vector v. It is easy to observe that this choice satisfies (25). Therefore, we conclude that $\bar{u}_i + \delta_i$ with $\bar{u}_i$ given by (23) is constant if and only if $\tilde{\delta}_i = Q_i^{-1}v$ for some (time varying) vector v. To fulfill this condition, we restrict the class of admissible disturbance signals to

$$\delta_i = \bar{\delta}_i + Q_i^{-1}v \tag{27}$$

where $\bar{\delta}_i$ is a constant vector, and v is generated by an exosystem, namely

$$\dot{w} = s(w) \tag{28a}$$
$$v = Pw \tag{28b}$$

with s defining an incrementally passive map, as before. Note that the time varying disturbance signals now read as

$$\delta_i = \bar{\delta}_i + Q_i^{-1}Pw$$

where w is a solution to (28a). Also note that, by (23) and (25), we have

$$\bar{u}_i = Q_i^{-1}\bar{\lambda} - Q_i^{-1}Pw \tag{29}$$

where $\bar{\lambda}$ denotes the constant component of λ, i.e.

$$\bar{\lambda} = -(\sum_{i\in\mathcal{I}_c} Q_i^{-1})^{-1}(\sum_{i=1}^{N}(J_i - R_i)y^* + \sum_{i=1}^{N} \bar{\delta}_i) \tag{30}$$

and $\bar{\delta}_i = \delta_i$ for each $i \notin \mathcal{I}_c$. Therefore, in this case the feasibility condition (15) is modified as follows.

Feasibility condition with optimality: For a given $y^* \in \Omega_n$, there exists a constant vector $\bar{\eta}$ such that

$$\begin{aligned}
0 =&(J^{11} - R^{11})(\mathbb{1} \otimes y^*) - (B^{11} \otimes I)\nabla H_e(\bar{\eta}) \\
&+ (Q^{11})^{-1}(\mathbb{1} \otimes \bar{\lambda}) + \bar{\delta}^{11} \\
0 =&(J^{12} - R^{12})(\mathbb{1} \otimes y^*) - (B^{12} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{12} \\
0 =&(J^{21} - R^{21})(\mathbb{1} \otimes y^*) - (B^{21} \otimes I)\nabla H_e(\bar{\eta}) \\
&+ (Q^{21})^{-1}(\mathbb{1} \otimes \bar{\lambda}) + \bar{\delta}^{21} \\
0 =&(J^{22} - R^{22})(\mathbb{1} \otimes y^*) - (B^{22} \otimes I)\nabla H_e(\bar{\eta}) + \delta^{22}
\end{aligned} \tag{31}$$

To solve the output agreement problem with an optimal deployment of the control effort, we move away from the decentralized controllers, and propose distributed internal-model based controllers at the nodes. To this end, we consider a *communication* layer, and introduce a communication graph, say $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$, which is undirected and connected. Note that $\mathcal{G}_c$ may be different from the graph $\mathcal{G}$ which describes the physical coupling of the network. The main result of this subsection is now stated in the following theorem.

Theorem 7 *Suppose that, for each $i \in \mathcal{I}_c$, δ_i is given by* (27) *where v is generated by* (28) *and s is an incrementally passive map satisfying* (18). *Assume that the feasibility condition* (31) *holds. Consider the distributed internal model-based controller*

$$\dot{\xi}_i = \sum_{\{i,j\}\in E_c} (\xi_j - \xi_i) + Q_i^{-1}\nu_i \tag{32a}$$

$$\dot{\zeta}_i = \sum_{\{i,j\}\in E_c} (\zeta_j - \zeta_i) + s(\zeta_i) - P^T Q_i^{-1}\nu_i \tag{32b}$$

$$u_i = Q_i^{-1}\xi_i - Q_i^{-1}P\zeta_i \tag{32c}$$

where $\nu_i = y^* - \nabla H_{n,i}(x_i)$ and $i \in \mathcal{I}_c$. Let $\xi = \mathrm{col}(\xi_i)$, $\bar{\xi} = \mathbb{1} \otimes \bar{\lambda}$, $\zeta = \mathrm{col}(\zeta_i)$, $\bar{\zeta}(0) = \mathbb{1} \otimes w(0)$. Then, there exists a region of the state space, including $(\bar{x}, \bar{\eta}, \bar{\xi}, \bar{\zeta}(0))$, such that any solution (x, η, ξ, ζ) of the system (12),(32), originating from this region asymptotically converges to an invariant set of the system in which $\nabla H_{n,i}(\bar{x}_i) = y^*$ for each $i \in \mathcal{V}$. Moreover, the vector u_i asymptotically converges to the optimal $\bar{u}_i$ given by (23).

Proof. See Appendix.

Remark 8 It is easy to observe that in the case δ_i is constant for each i, the controller (32) in Theorem 7 can be replaced by its subdynamics (32a), with $u_i = Q_i^{-1}\xi$.

Remark 9 The convergence region stated in Theorems 1, 4, 6, and 7 is given by the forward invariant compact level set Ω_c which is implicitly characterized in the proofs provided in Appendix. The explicit expression of Ω_c depends on Ω_n, Ω_e and the shape of the level sets of the associated Lyapunov functions. In case $\Omega_n = \mathbb{R}^n$ and $\Omega_e = \mathbb{R}^m$, in view of the strict convexity assumption, the adopted Lyapunov functions are radially unbounded, thus the set Ω_c is equal to the whole state space and the convergence is global.

4 Case study

We consider a (fairly) general heterogeneous microgrid which consists of synchronous generators, droop-controlled inverters, and frequency dependent loads. We partition the buses, i.e. the nodes of $\mathcal{G}$, into three sets, namely $\mathcal{V}_G$, $\mathcal{V}_I$, and $\mathcal{V}_L$, corresponding to the set of synchronous generators, inverters, and loads, respectively.

The dynamics of each synchronous generator is governed by the so-called *swing equation*, and is given by [16]:

$$M_i\ddot{\theta}_i = -A_i\dot{\theta}_i + u_i - P_i + \delta_i, \qquad i \in \mathcal{V}_G, \tag{33}$$

where

$$P_i = \sum_{\{i,j\}\in\mathcal{E}} \mathrm{Im}(Y_{ij})V_iV_j \sin(\theta_i - \theta_j) \tag{34}$$

is the active nodal injection at node i. Here, $M_i > 0$ is the moment of inertia, $A_i > 0$ is the damping constant, u_i is the local controllable power generation, and δ_i is the local load at node $i \in V_G$. The value of $Y_{ij} \in \mathbb{C}$ is equal to the admittance of the branch $\{i,j\} \in \mathcal{E}$, and θ_i is the voltage angle at node i. Also, V_i is the voltage magnitude at node i, and is assumed to be constant.

For the droop-controlled inverters, we consider the following first-order model [22], [11]

$$A_i\dot{\theta}_i = u_i - P_i + \delta_i\ , \qquad i \in \mathcal{V}_I \tag{35}$$

where A_i is known as the droop coefficient, u_i is the injection power at node (inverter) i, δ_i is the local load at inverter i, and $\dot{\theta}_i$ indicates the frequency deviation from the nominal frequency of the network, $i \in \mathcal{V}_I$. The term P_i has the same expression as in (34).

As for nodal dynamics corresponding to the loads, we consider frequency dependent loads given by the first-order system

$$A_i\dot{\theta}_i = \delta_i - P_i\ , \qquad i \in \mathcal{V}_L \tag{36}$$

The load model above appears in the so called *structure preserving* power network model proposed in [4]. A derivation from first principles can also be found in [23, Ch. 7]. Again, here $\dot{\theta}_i$ is the frequency deviation, $A_i > 0$ is the damping coefficient, P_i is given by (34), and δ_i is the constant power consumption at node $i \in \mathcal{V}_L$.

To write the system in a compact form, we need the following nomenclature. For each $k = 1, 2, \ldots, M$, let γ_k be defined as $\gamma_k = (\mathrm{Im}\, Y_{ij})V_iV_j$ with $\{i,j\}$ being the k^{th} edge of the graph, where the edge numbers are in accordance with the incidence matrix B. We define the diagonal matrix Γ as $\Gamma = \mathrm{diag}(\gamma_k)$. Let the matrices B_G, B_I, and B_L be obtained from B by collecting the rows indexed by $\mathcal{V}_G$, V_I, and V_L, respectively. We define the vectors and matrices M_G, A_G, θ_G, and u_G, as $M_G = \mathrm{diag}(M_i)$, $A_G = \mathrm{diag}(A_i)$, $\theta_G = \mathrm{col}(\theta_i)$, $u_G = \mathrm{col}(u_i)$, and $\delta_G = \mathrm{col}(\delta_i)$ where $i \in V_G$. The vectors and matrices A_I, θ_I, and u_I are defined as $A_I = \mathrm{diag}(A_i)$, $\theta_I = \mathrm{col}(\theta_i)$, $u_I = \mathrm{col}(u_i)$, and $\delta_G = \mathrm{col}(\delta_i)$ with $i \in \mathcal{V}_I$. In addition, let $A_L = \mathrm{diag}(A_i)$, $\theta_L = \mathrm{col}(\theta_i)$ and $\delta_L = \mathrm{col}(\delta_i)$ where $i \in \mathcal{V}_L$. Finally, let $P = \mathrm{col}(P_i)$, $\theta = \mathrm{col}(\theta_G, \theta_I, \theta_L)$, and $\underline{\sin}(x) := \mathrm{col}(\sin(x_i))$ for a given vector x. Then, it is easy to observe that the dynamics of the synchronous generators, the inverters, and the loads can be written compactly as:

$$M_G\ddot{\theta}_G + A_G\dot{\theta}_G = -B_G\Gamma\underline{\sin}(B^\top\theta) + u_G + \delta_G \tag{37a}$$

$$A_I\dot{\theta}_I = -B_I\Gamma\underline{\sin}(B^\top\theta) + u_I + \delta_I \tag{37b}$$

$$A_L\dot{\theta}_L = -B_L\Gamma\underline{\sin}(B^\top\theta) + \delta_L \tag{37c}$$

Note that this is the same model as [9], see also [33]. By defining $\eta = B^T\theta$, $\omega_G = \dot{\theta}_G$, $\omega_I = \dot{\theta}_I$, $\omega_L = \dot{\theta}_L$, and $\dot{\theta} = \omega = \mathrm{col}(\omega_G, \omega_I, \omega_L)$, the network dynamics (37), admits the following representation

$$\dot{\eta} = B^T\omega \tag{38a}$$

$$M_G\dot{\omega}_G + A_G\omega_G = -B_G\Gamma\underline{\sin}(\eta) + u_G + \delta_G \tag{38b}$$

$$A_I\omega_I = -B_I\Gamma\underline{\sin}(\eta) + u_I + \delta_I \tag{38c}$$

$$A_L\omega_L = -B_L\Gamma\underline{\sin}(\eta) + \delta_L \tag{38d}$$

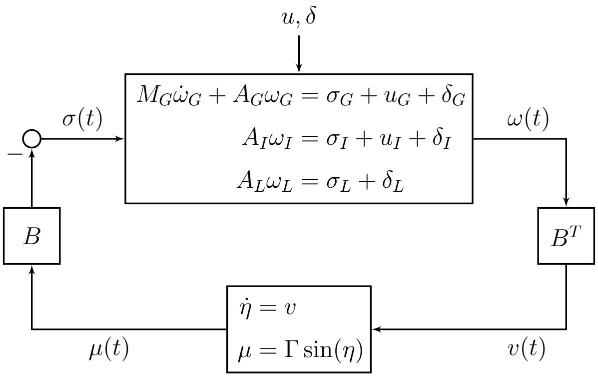

Fig. 2. Block-diagram of the power network model

Figure 2 relates the dynamics above to our problem formulation in Figure 1. Here, the vectors σ_G, σ_I, and σ_L are defined as $\mathrm{col}(\sigma_i)$ for $i = \mathcal{V}_G$, $i = \mathcal{V}_I$, and $i \in \mathcal{V}_L$, respectively. As can be seen from the figure, $\eta \in \mathbb{R}^M$ is the state of the subsystem corresponding to the edge dynamics in (2), and its derivative is driven by the differences of the frequencies (see also [5, Sec. 7]). In addition, the vector μ indicates the active power flow at each edge. The frequency variables constitute the output of the subsystem representing the nodal dynamics. It is worth mentioning that the change of coordinates $\eta = B^T\theta$ is also consistent with the port-Hamiltonian modeling of power networks; see e.g. [29, Sec. 12.3].

Now, let $p_G = M_G\omega_G$, $H_G = \frac{1}{2}p_G^T M_G^{-1} p_G$ and $H_e = -\mathbb{1}^T\Gamma\underline{\cos}(\eta)$. Also let $p_I = M_I\omega_I$, $p_L = M_L\omega_L$, $H_I = \frac{1}{2}p_I^T M_I^{-1} p_I$, and $H_L = \frac{1}{2}p_L^T M_L^{-1} p_L$ for some positive definite diagonal matrices M_I and M_L. In fact, the matrices M_I and M_L can be interpreted as *virtual* masses of the (massless) vertices in $\mathcal{V}_I$ and $\mathcal{V}_L$, respectively. Then, (38) can be written as

$$\dot{\eta} = B^T\nabla H_T(p) \tag{39a}$$
$$\dot{p}_G = -A_G\nabla H_G(p_G) - B_G\nabla H_e(\eta) + u_G + \delta_G \tag{39b}$$
$$0 = -A_I\nabla H_I(p_I) - B_I\nabla H_e(\eta) + u_I + \delta_I \tag{39c}$$
$$0 = -A_L\nabla H_L(p_L) - B_L\nabla H_e(\eta) + \delta_L \tag{39d}$$

where $p = \mathrm{col}(p_G, p_I, p_L)$ and $H_T = H_G + H_I + H_L$. Note that from the port-Hamiltonian modelling viewpoint [30], the equations (38c) and (38d) are interpreted as *damping* relations. Introducing "formal" choices of H_I and H_L allow us to capture a more general class of systems and write the system dynamics in the form of (4).

Now, it is easy to observe that (39) has a similar structure/properties as (12), with $\Omega_n = \mathbb{R}$, $\Omega_e = (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\mathcal{I}_{12} = \varnothing$. The primary control goal here is to achieve a zero frequency deviation for the power network. As $\nabla H_T = w$, this is in accordance with our definition of output agreement with $y^* = 0$. Moreover, we would like to achieve an optimal steady-state distribution of the power in the sense of (23). In this case, (23) reads as

$$\bar{u}_i = q_i^{-1}\lambda \tag{40}$$

where

$$\lambda = -(\sum_i q_i)^{-1}(\mathbb{1}^T\delta_G + \mathbb{1}^T\delta_I + \mathbb{1}^T\delta_L).$$

First, we consider the constant demand case, i.e. δ_G, δ_I, and δ_L are constant vectors. The feasibility condition (31) in this case amounts for the existence of a constant vector $\bar{\eta} \in (-\frac{\pi}{2}, \frac{\pi}{2})^M$ such that

$$0 = -B_G\nabla H_e(\bar{\eta}) + \bar{u}_G + \delta_G \tag{41a}$$
$$0 = -B_I\nabla H_e(\bar{\eta}) + \bar{u}_I + \delta_I \tag{41b}$$
$$0 = -B_L\nabla H_e(\bar{\eta}) + \delta_L \tag{41c}$$

where $\bar{u}_i$ is given by (40) for each $i \in \mathcal{V}_G \cup \mathcal{V}_I$. Now, assume that the feasibility condition (41) holds. Then, by Theorem 7, the controller

$$\dot{\xi}_i = \sum_{\{i,j\}\in\mathcal{E}_c} (\xi_j - \xi_i) - q_i^{-1}\omega_i \tag{42a}$$
$$u_i = q_i^{-1}\xi_i, \qquad i \in \mathcal{V}_G \cup \mathcal{V}_I \tag{42b}$$

achieves zero frequency deviation , and moreover u_i asymptotically converges to the optimal $\bar{u}_i$ given by (40).

Now, consider the case where a proper subset of generators, say $\mathcal{V}_F \subset \mathcal{V}_G$, encounter some failures. In particular, assume that u_i is not appropriately actuated, and is equal to some unknown constant vector for each $i \in \mathcal{V}_F$. Then, for the nodes in the *fail mode*, subdynamics (39b) reads as

$$\dot{p}_F = -A_F\nabla H_F(p_F) - B_F\nabla H_e(\eta) + \delta_F \tag{43}$$

where we have used the subscript "F" to distinguish the subdynamics above from the nominal subdynamics (39b). Assume that there exists $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})^M$ such that (41) and

$$0 = -B_F\nabla H_e(\bar{\eta}) + \delta_F$$

are satisfied. Note that (41a) has to be modified accordingly to exclude the faulty generators, and that (43) has the same structure as (12c). Then, by Theorem 7, we conclude that the controller (42) achieves a zero frequency deviation, and we have optimal steady state distribution of the power, given by (40), despite the failures in the nodal dynamics $\mathcal{V}_G$.

Similarly, absence or failure of actuation in inverters can be incorporated in our design, as this results in dynamics analogous to that of the loads.

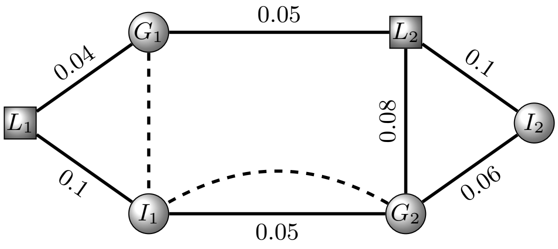

Fig. 3. The solid lines denote the transmission lines, and the dashed lines depict the communication links.

Next, we consider the case where the local loads δ_G and δ_I are not constant anymore and are subject to variations. Then, the control goal is to achieve zero frequency deviation and an optimal distribution of the power despite these variations. Note that δ_L has to be constant due to (41c). Similar to Subsection 3.3, for each $i \in \mathcal{V}_{GI} = \mathcal{V}_G \cup \mathcal{V}_I$, let δ_i be decomposed as $\delta_i = \bar{\delta}_i + q_i^{-1}v$, where $\bar{\delta}_i$ is a constant vector, and v is generated by the exosystem (17). For instance, the loads δ_G and δ_I may be given as the sum of constant and sinusoidal signals. Then the feasibility condition (31) takes the following form

$$0 = -(B_G \otimes I)\nabla H_e(\bar{\eta}) + Q_G^{-1}(\mathbb{1} \otimes \bar{\lambda}) + \bar{\delta}_G \tag{44a}$$
$$0 = -(B_I \otimes I)\nabla H_e(\bar{\eta}) + Q_I^{-1}(\mathbb{1} \otimes \bar{\lambda}) + \bar{\delta}_I \tag{44b}$$
$$0 = -(B_L \otimes I)\nabla H_e(\bar{\eta}) + \delta_L \tag{44c}$$

where

$$\bar{\lambda} = (\sum_{i \in \mathcal{V}_{GI}} q_i^{-1})^{-1}(\sum_{i \in \mathcal{V}_{GI}} \bar{\delta}_i)$$

in this case. Now, assuming that there exists a constant vector $\bar{\eta}$ satisfying (44), the controller

$$\dot{\xi}_i = \sum_{\{i,j\} \in E_c} (\xi_j - \xi_i) - q_i^{-1}\omega_i \tag{45a}$$
$$\dot{\zeta}_i = \sum_{\{i,j\} \in E_c} (\zeta_j - \zeta_i) + s(\zeta_i) - q_i^{-1}P^T\omega_i \tag{45b}$$
$$u_i = q_i^{-1}\xi_i - q_i^{-1}P\zeta_i \tag{45c}$$

achieves a zero frequency deviation by Theorem 7. Moreover, u_i asymptotically converges to the optimal $\bar{u}_i$ given by (40).

Next, we conclude this section by a numerical example of a microgrid consisting of two generators, two inverters, and two loads. The interconnection topology is depicted in Figure 3. The synchronous generators and inverters are assumed to have additional shunt loads. The microgrid parameters are chosen as: $M_{G_1} = 4.49$, $M_{G_2} = 4.22$, $A_{G_1} = 1.38$, $A_{G_2} = 1.42$, $A_{I_1} = 1.60$, $A_{I_2} = 1.22$,

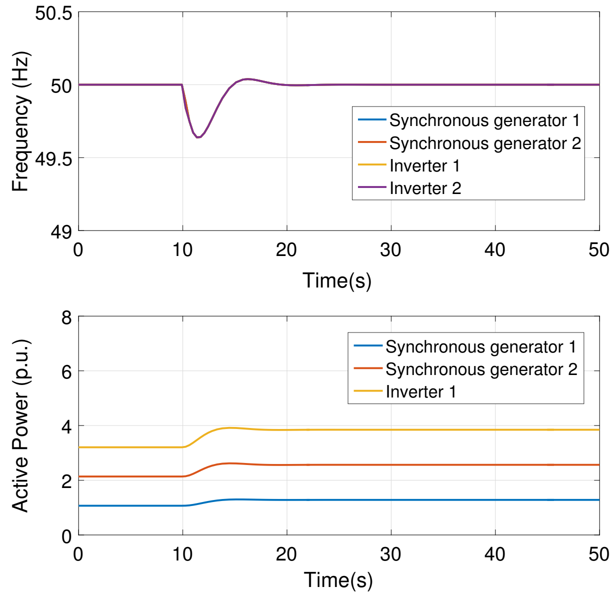

Fig. 4. Frequency regulation and power generation with controllers (42): constant load.

$A_{L_1} = 1.00$, $A_{L_2} = 1.00$. The line inductances are chosen as shown in Figure 3.

As the first scenario, we employ the controllers (42) in the synchronous generators and the inverter I_1, and the control input u_{I_2} is set to a constant. The system is initially at steady state with a constant load. At time $t = 10$, loads L_1 and L_2 are increased by 10 percent of their original values. The frequency evolution and the active power injections are depicted in Figure 4. It is observed that the system is regulating the frequency at 50 Hz (the frequencies at the various nodes are so similar to each other that no difference can be noticed in the plot), and the generation costs are minimized meaning that power is proportionally shared (namely with a ratio given by $q_{G_1}^{-1} = 1$, $q_{I_2}^{-1} = 2$, and $q_{G_2}^{-1} = 3$).

In the second scenario, at time $t = 50$, we modulate the shunt loads by sinusoidal signals with a period of 30 seconds, and apply the controllers (45). Again, as shown in Figure 5, the controllers achieve frequency regulation together with the optimal power sharing.

5 Conclusions

We have investigated the problem of output agreement in heterogeneous networks with port-Hamiltonian nodal dynamics, dynamic physical coupling, and algebraic constraints. We have considered the case where control and disturbance signals may act on different subsets of nodes and the disturbances are generated by exosystems. As observed, the output variables of the network asymptotically converge to the same vector. We have discussed how

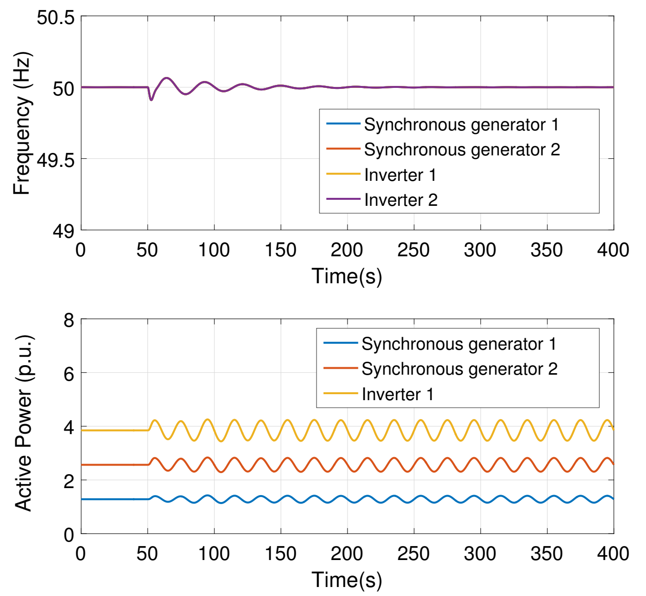

Fig. 5. Frequency regulation and power generation with controllers (45): time-varying load.

this vector can be steered to a desired one by applying decentralized controllers for the case of constant and time-varying disturbances. Moreover, we have shown that appropriate distributed controllers achieve output agreement together with an optimal steady-state distribution of the control effort over the network. The proposed results are applied to a heterogeneous microgrid consisting of synchronous generators, droop control inverters, and frequency-dependent loads. As observed in the case study, the control scheme considered in this paper allows us to cope with failures in nodal dynamics. Incorporating failures on the links is a subject of future research.

Appendix

Poof of Theorem 1: From (4a), we have

$$\dot{\eta} = (B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1) + (B^2 \otimes I)^T (G^2)^T \nabla H_n^2(x^2) \tag{46}$$

By (4d), we obtain that

$$\begin{aligned}\dot{\eta} = &(B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1)\\ &+ (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2\\ &\cdot ((B^2 \otimes I) \nabla H_e(\eta) - d^2)\end{aligned}$$

Next, we study the asymptotic behavior of the following subdynamics of (4)

$$\begin{aligned}\dot{\eta} = &(B^1 \otimes I)^T (G^1)^T \nabla H_n^1(x^1)\\ &+ (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2 (B^2 \otimes I) \nabla H_e(\eta)\\ &- (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} G^2 d^2) \end{aligned} \tag{47a}$$

$$\begin{aligned}\dot{x}^1 = &(J^1 - R^1) \nabla H_n^1(x^1)\\ &- G^1 (B^1 \otimes I) \nabla H_e(\eta) + G^1 d^1\end{aligned} \tag{47b}$$

Let W_n and W_e be defined as

$$W_n(x^1, \bar{x}^1) = H_n^1(x^1) - H_n^1(\bar{x}^1) - (\nabla H_n^1(\bar{x}^1))^T (x^1 - \bar{x}^1) \tag{48}$$

and

$$W_e(\eta, \bar{\eta}) = H_e(\eta) - H_e(\bar{\eta}) - (\nabla H_e(\bar{\eta}))^T (\eta - \bar{\eta}) \tag{49}$$

where $(\bar{x}^1, \bar{\eta})$ is an equilibrium of (47). Following [13], W_n identifies a positive definite map with a strict local minimum at $x^1 = \bar{x}^1$. Also W_e defines a positive definite map with a strict local minimum at $\eta = \bar{\eta}$. Noting that $\dot{\bar{x}}^1 = 0$, we have

$$\begin{aligned}\dot{W}_n = &(\nabla H_n^1(x^1))^T \dot{x}^1 - (\nabla H_n^1(\bar{x}^1))^T (\dot{x}^1 - \dot{\bar{x}}^1)\\ = &(\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T (\dot{x}^1 - \dot{\bar{x}}^1)\\ = &(\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T\\ &\cdot (J^1 - R^1)(\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))\\ &- (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T\\ &\cdot G^1 (B^1 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned}$$

In addition, noting that $\dot{\bar{\eta}} = 0$ we have

$$\begin{aligned}\dot{W}_e = &(\nabla H_e(\eta))^T \dot{\eta} - (\nabla H_e(\bar{\eta}))^T (\dot{\eta} - \dot{\bar{\eta}})\\ = &(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (\dot{\eta} - \dot{\bar{\eta}})\\ = &(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^1 \otimes I)^T\\ &\cdot (G^1)^T (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))\\ &+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^2 \otimes I)^T (G^2)^T\\ &\cdot (J^2 - R^2)^{-1} G^2 (B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned} \tag{50}$$

Let $W_T := W_n + W_e$. Then, we have

$$\begin{aligned}\dot{W}_T = &(\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))^T (J^1 - R^1)\\ &\cdot (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1))\\ &+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^2 \otimes I)^T (G^2)^T\\ &\cdot (J^2 - R^2)^{-1} G^2 (B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned}$$

where we have used the fact that d^1 and d^2 are constant.

Now, note that for any skew-symmetric matrix J and a positive definite matrix R, we have $-2R = (J - R) + (J - R)^T < 0$, and thus $(J - R)^{-1} + (J - R)^{-T} < 0$. Hence, $z^T (J - R) z < 0$ and $z^T (J - R)^{-1} z < 0$ for any nonzero vector z. Therefore, we conclude that $\dot{W}_T \leqslant 0$.

Observe that W_T has a strict local minimum at $x = \bar{x}^1$ and $\eta = \bar{\eta}$, and hence one can construct a compact level set $\Omega_c \subseteq (\Omega_n)^{|\mathcal{I}_1|} \times (\Omega_e)^M$ around $(\bar{x}^1, \bar{\eta})$ which is forward invariant. This implies that on the interval of definition of a solution to system (4), the variables x^1 and η are bounded. Hence, by (4d), the variables $\nabla H_n^2(x^2)$ are also bounded, and a solution to (4) exists for all t.

Then by invoking LaSalle invariance principle, on the invariant set $\dot{W}_T = 0$, we have

$$\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1) = 0 \tag{51a}$$

$$G^2(B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) = 0. \tag{51b}$$

Due to the strict convexity of H_n^1, (51a) yields $x^1 = \bar{x}^1$. Besides, (47a) admits the following incremental model

$$\begin{aligned}\dot{\eta} = &(B^1 \otimes I)^T (G^1)^T (\nabla H_n^1(x^1) - \nabla H_n^1(\bar{x}^1)) \\ &+ (B^2 \otimes I)^T (G^2)^T (J^2 - R^2)^{-1} \\ &\cdot G^2 (B^2 \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned}$$

Therefore, by (51), we obtain that $\dot{\eta} = 0$ on the invariant set, and thus output agreement (6) holds. ■

Proof of Theorem 6: [2] By the algebraic equation (12d), the controller (19) can be written as

$$\dot{\xi}^{11} = (\mathbb{1} \otimes y^*) - (G^{11})^T \nabla H_n^{11}(x^{11}) \tag{52a}$$

$$\begin{aligned}\dot{\zeta}^{11} = &s^{11}(\zeta^{11}) - (P^{11})^T (\mathbb{1} \otimes y^*) \\ &+ (G^{11} P^{11})^T \nabla H_n^{11}(x^{11})\end{aligned} \tag{52b}$$

$$\begin{aligned}\dot{\xi}^{21} = &(\mathbb{1} \otimes y^*) - (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot ((B^{21} \otimes I) \nabla H_e(\eta) - \xi^{21} + P^{21} \zeta^{21} - P^{21} w^{21})\end{aligned} \tag{52c}$$

$$\begin{aligned}\dot{\zeta}^{21} = &s^{21}(\zeta^{21}) - (P^{21})^T (\mathbb{1} \otimes y^*) \\ &+ (G^{21} P^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot ((B^{21} \otimes I) \nabla H_e(\eta) - \xi^{21} + P^{21} \zeta^{21} - P^{21} w^{21})\end{aligned} \tag{52d}$$

$$u^{11} = \xi^{11} - P^{11} \zeta^{11} \tag{52e}$$

$$u^{21} = \xi^{21} - P^{21} \zeta^{21}. \tag{52f}$$

Moreover, we have

$$\begin{aligned}\dot{\eta} = &(B^{11} \otimes I)^T (G^{11})^T \nabla H_n^{11}(x^{11}) \\ &+ (B^{12} \otimes I)^T (G^{12})^T \nabla H_n^{12}(x^{12}) \\ &+ (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot ((B^{21} \otimes I) \nabla H_e(\eta) - \xi^{21} + P^{21} \zeta^{21} - P^{21} w^{21}) \\ &+ (B^{22} \otimes I)^T (G^{22})^T (J^{22} - R^{22})^{-1} G^{22} \\ &\cdot ((B^{22} \otimes I) \nabla H_e(\eta) - \delta^{22})\end{aligned} \tag{53}$$

[2] The proof of Theorem 4 is provided afterwards.

Hence, (52) together with (12b), (12c), and (53) defines a dynamical system with ordinary differential equations, the solution of which exists and is unique. By comparing the two solutions (x, η, ξ, ζ, w) and $(\bar{x}, \bar{\eta}, \bar{\xi}, \bar{\zeta}, w)$, we obtain the incremental system dynamics

$$\begin{aligned}\dot{\eta} - \dot{\bar{\eta}} = &(B^{11} \otimes I)^T (G^{11})^T (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &+ (B^{12} \otimes I)^T (G^{12})^T (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ &+ (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &- (B^{21} \otimes I)^T (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (\xi^{21} - \bar{\xi}^{21} - P^{21}(\zeta^{21} - \bar{\zeta}^{21})) \\ &+ (B^{22} \otimes I)^T (G^{22})^T (J^{22} - R^{22})^{-1} G^{22} \\ &\cdot (B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned} \tag{54a}$$

$$\begin{aligned}\dot{x}^{11} - \dot{\bar{x}}^{11} = &(J^{11} - R^{11})(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &- G^{11}(B^{11} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &+ G^{11}(\xi^{11} - \bar{\xi}^{11} - P^{11}(\zeta^{11} - \bar{\zeta}^{11}))\end{aligned} \tag{54b}$$

$$\begin{aligned}\dot{x}^{12} - \dot{\bar{x}}^{12} = &(J^{12} - R^{12})(\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ &- G^{12}(B^{12} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned} \tag{54c}$$

with incremental controller dynamics

$$\dot{\xi}^{11} - \dot{\bar{\xi}}^{11} = -(G^{11})^T (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \tag{55a}$$

$$\begin{aligned}\dot{\zeta}^{11} - \dot{\bar{\zeta}}^{11} = &s^{11}(\zeta^{11}) - s^{11}(\bar{\zeta}^{11}) \\ &+ (G^{11} P^{11})^T (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))\end{aligned} \tag{55b}$$

$$\begin{aligned}\dot{\xi}^{21} - \dot{\bar{\xi}}^{21} = &-(G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &+ (G^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (\xi^{21} - \bar{\xi}^{21} - P^{21}(\zeta^{21} - \bar{\zeta}^{21}))\end{aligned} \tag{55c}$$

$$\begin{aligned}\dot{\zeta}^{21} - \dot{\bar{\zeta}}^{21} = &s^{21}(\zeta^{11}) - s^{21}(\bar{\zeta}^{11}) \qquad \text{(55d)} \\ &+ (G^{21} P^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &- (G^{21} P^{21})^T (J^{21} - R^{21})^{-1} G^{21} \\ &\cdot (\xi^{21} - \bar{\xi}^{21} - P^{21}(\zeta^{21} - \bar{\zeta}^{21}))\end{aligned} \tag{55e}$$

Now, let $\bar{\xi}^{11} = d^{11}$, $\bar{\xi}^{21} = d^{21}$, $\dot{\bar{\zeta}}^{11} = s^{11}(\bar{\zeta}^{11})$, and $\dot{\bar{\zeta}}^{21} = s^{21}(\bar{\zeta}^{21})$. Also let

$$\bar{u}^{11} = \bar{\xi}^{11} - P^{11} \bar{\zeta}^{11}, \qquad \bar{u}^{21} = \bar{\xi}^{21} - P^{21} \bar{\zeta}^{21}.$$

Then, by the feasibility condition (15), it is easy to see that $(\bar{x}, \bar{\eta}, \bar{\xi}, \bar{\zeta}, w)$ is a valid solution to (12) by initializing

$\bar{\zeta}^{11}$ as $\bar{\zeta}^{11}(0) = w^{11}(0)$, and $\bar{\zeta}^{21}$ as $\bar{\zeta}^{21}(0) = w^{21}(0)$. Note that $\bar{x}$ and $\bar{\eta}$ are constant vectors satisfying (15).

Now, consider again the Lyapunov function

$$V = W_n + W_e + W_c$$

where W_n is given by (48), W_e is given by (49), and

$$\begin{aligned} W_c = &\frac{1}{2}(\xi^{11} - \bar{\xi}^{11})^T(\xi^{11} - \bar{\xi}^{11}) + \frac{1}{2}(\zeta^{11} - \bar{\zeta}^{11})^T(\zeta^{11} - \bar{\zeta}^{11}) \\ &+ \frac{1}{2}(\xi^{21} - \bar{\xi}^{21})^T(\xi^{21} - \bar{\xi}^{21}) + \frac{1}{2}(\zeta^{21} - \bar{\zeta}^{21})^T(\zeta^{21} - \bar{\zeta}^{21}) \end{aligned} \tag{56}$$

By the use of incremental model (54)-(55), $\dot{W}_n$ and $\dot{W}_e$ are computed as

$$\begin{aligned} \dot{W}_n = &(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T(\dot{x}^{11} - \dot{\bar{x}}^{11}) \\ &+ (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T(\dot{x}^{12} - \dot{\bar{x}}^{12}) \\ =&(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T(J^{11} - R^{11}) \\ &\cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &- (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T(G^{11}) \\ &\cdot (B^{11} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &+ (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T G^{11}(\xi^{11} - \bar{\xi}^{11}) \\ &- (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T G^{11} P^{11}(\zeta^{11} - \bar{\zeta}^{11}) \\ &+ (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T(J^{12} - R^{12}) \\ &\cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ &- (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T G^{12} \\ &\cdot (B^{12} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \tag{57}$$

and

$$\begin{aligned} \dot{W}_e = &(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(\dot{\eta} - \dot{\bar{\eta}}) \\ =& (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{11} \otimes I)^T \\ &\cdot (G^{11})^T(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{12} \otimes I)^T \\ &\cdot (G^{12})^T(\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ &+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{21} \otimes I)^T(G^{21})^T \\ &\cdot (J^{21} - R^{21})^{-1}G^{21}(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &- (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{21} \otimes I)^T(G^{21})^T \\ &\cdot (J^{21} - R^{21})^{-1}G^{21}(\xi^{21} - \bar{\xi}^{21}) \\ &+(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{21} \otimes I)^T \\ &\cdot (G^{21})^T(J^{21} - R^{21})^{-1}G^{21}P^{21}(\zeta^{21} - \bar{\zeta}^{21}) \qquad (58) \\ &+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{22} \otimes I)^T(G^{22})^T \\ &\cdot (J^{22} - R^{22})^{-1}G^{22}(B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \end{aligned} \tag{59}$$

Moreover, we have

$$\begin{aligned} \dot{W}_c = &- (\xi^{11} - \bar{\xi}^{11})^T(G^{11})^T(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &+ (\zeta^{11} - \bar{\zeta}^{11})^T(s^{11}(\zeta^{11}) - s^{11}(\bar{\zeta}^{11})) \\ &+ (\zeta^{11} - \bar{\zeta}^{11})^T(G^{11}P^{11})^T \\ &\cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &- (\xi^{21} - \bar{\xi}^{21})^T(G^{21})^T(J^{21} - R^{21})^{-1}G^{21} \\ &\cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &+ (\xi^{21} - \bar{\xi}^{21})^T(G^{21})^T(J^{21} - R^{21})^{-1}G^{21} \\ &\cdot (\xi^{21} - \bar{\xi}^{21} - P^{21}(\zeta^{21} - \bar{\zeta}^{21})) \\ &+ (\zeta^{21} - \bar{\zeta}^{21})^T(s^{21}(\zeta^{21}) - s^{21}(\bar{\zeta}^{21})) \\ &+ (\zeta^{21} - \bar{\zeta}^{21})^T(G^{21}P^{21})^T(J^{21} - R^{21})^{-1}G^{21} \\ &\cdot (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &- (\zeta^{21} - \bar{\zeta}^{21})^T(G^{21}P^{21})^T(J^{21} - R^{21})^{-1}G^{21} \\ &\cdot (\xi^{21} - \bar{\xi}^{21} - P^{21}(\zeta^{21} - \bar{\zeta}^{21})) \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \dot{V} &= \dot{W}_n + \dot{W}_e + \dot{W}_c \\ &= (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T(J^{11} - R^{11}) \\ &\quad \cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\ &\quad + (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T(J^{12} - R^{12}) \\ &\quad \cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\ &\quad + (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T(B^{22} \otimes I)^T(G^{22})^T \\ &\quad \cdot (J^{22} - R^{22})^{-1}G^{22}(B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &\quad + (\zeta^{11} - \bar{\zeta}^{11})^T(s^{11}(\zeta^{11}) - s^{11}(\bar{\zeta}^{11})) \\ &\quad + (\zeta^{21} - \bar{\zeta}^{21})^T(s^{21}(\zeta^{21}) - s^{11}(\bar{\zeta}^{11})) \\ &\quad + z^T(J^{21} - R^{21})^{-1}z \end{aligned}$$

where

$$\begin{aligned} z =&G^{21}(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\ &- G^{21}(\xi^{21} - \bar{\xi}^{21}) + G^{21}P^{21}(\zeta^{21} - \bar{\zeta}^{21}) \end{aligned}$$

By incremental passivity assumption (18), it is easy to observe that $\dot{V} \leqslant 0$. Observe that the solution

$$\begin{aligned} &(x^{11}, x^{12}, \eta, \xi^{11}, \xi^{21}, \zeta^{11}, \zeta^{21}) \\ &\qquad = (\bar{x}^{11}, \bar{x}^{12}, \bar{\eta}, \bar{\xi}^{11}, \bar{\xi}^{21}, \bar{\zeta}^{11}, \bar{\zeta}^{21}) \end{aligned}$$

is a strict local minimum of V for all time, and thus one can find a compact level set Ω_c around this solution within which every other solution $(x^{11}, x^{12}, \eta, \xi^{11}, \xi^{21}, \zeta^{11}, \zeta^{21})$ evolves. Then, noting that the w_i variables are bounded, so are also the $(\bar{x}^{11}, \bar{x}^{12}, \bar{\eta}, \bar{\xi}^{11}, \bar{\xi}^{21}, \bar{\zeta}^{11}, \bar{\zeta}^{21})$ variables, and one can conclude boundedness of state components and the existence of a solution to (12) for all t. Now by invoking the LaSalle invariance principle, on the invariant set $\dot{V} = 0$, we have

$$\begin{aligned}\nabla H_n^{11}(x^{11}) &= \nabla H_n^{11}(\bar{x}^{11})\\ \nabla H_n^{12}(x^{12}) &= \nabla H_n^{12}(\bar{x}^{12})\\ (B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) &= 0,\end{aligned}$$

and

$$\begin{aligned}&G^{21}(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\\ &\quad - G^{21}(\xi^{21} - \bar{\xi}^{21}) + G^{21}P^{21}(\zeta^{21} - \bar{\zeta}^{21}) = 0.\end{aligned}$$

Hence, by (54a), we obtain that $\dot{\eta} = \dot{\bar{\eta}} = 0$. Besides, by (55a), we obtain that $\dot{\xi}^{11} = \dot{\bar{\xi}}^{11} = 0$ on the invariant set. Then, by (52a), we obtain that

$$(G^{11})^T \nabla H_n^{11}(x^{11}) = (G^{11})^T \nabla H_n^{11}(\bar{x}^{11}) = \mathbb{1} \otimes y^*.$$

This together with $\dot{\eta} = 0$ implies that $y_i = y^*$ for each i. ∎

Proof of Theorem 4: We construct the proof from that of Theorem 6. To this end, we set s_i to zero, and define the vector $\chi_i = \mathrm{col}(\xi_i, \zeta_i)$ for each $i \in \mathcal{I}_c$. Then the dynamics (19) reduces to

$$\begin{aligned}\dot{\chi}_i &= F_i(y^* - G_i^T \nabla H_{n,i}(x_i))\\ u_i &= F_i^T \chi_i\end{aligned}$$

where $F_i = \mathrm{col}(I_n, -P_i)$. This is essentially the same controller as in (16) modulo the presence of the matrices F_i and F_i^T. It is easy to observe that these matrices do not harm the analysis, and the proof follows from the result of Theorem 6.

Proof of Theorem 7: The controller (32) can be written in compact as

$$\begin{bmatrix}\dot{\xi}^{11}\\ \dot{\xi}^{21}\end{bmatrix} = -(L_c \otimes I)\begin{bmatrix}\xi^{11}\\ \xi^{21}\end{bmatrix} + Q^{-1}\begin{bmatrix}\mathbb{1} \otimes y^* - \nabla H_n^{11}(x^{11})\\ \mathbb{1} \otimes y^* - \nabla H_n^{21}(x^{21})\end{bmatrix} \tag{60a}$$

$$\begin{aligned}\begin{bmatrix}\dot{\zeta}^{11}\\ \dot{\zeta}^{21}\end{bmatrix} &= -(L_c \otimes I)\begin{bmatrix}\zeta^{11}\\ \zeta^{21}\end{bmatrix} + \begin{bmatrix}\underline{s}(\zeta^{11})\\ \underline{s}(\zeta^{21})\end{bmatrix}\\ &\quad - (I \otimes P^T)Q^{-1}\begin{bmatrix}\mathbb{1} \otimes y^* - \nabla H_n^{11}(x^{11})\\ \mathbb{1} \otimes y^* - \nabla H_n^{21}(x^{21})\end{bmatrix}\end{aligned} \tag{60b}$$

$$\begin{bmatrix}u^{11}\\ u^{21}\end{bmatrix} = Q^{-1}\begin{bmatrix}\xi^{11}\\ \xi^{21}\end{bmatrix} - Q^{-1}(I \otimes P)\begin{bmatrix}\zeta^{11}\\ \zeta^{21}\end{bmatrix} \tag{60c}$$

where L_c denotes the Laplacian matrix of G_c, $Q = \mathrm{blockdiag}(Q_i)$ with $i \in \mathcal{I}_c$, $\underline{s}(\zeta^{11}) = \mathrm{col}(s(\zeta_j))$ with $j \in \mathcal{I}_{11}$, $\underline{s}(\zeta^{21}) = \mathrm{col}(s(\zeta_j))$ with $j \in \mathcal{I}_{21}$, and in this case $H_n^{21}(x^{21})$ is equal to

$$\begin{aligned}&(J^{21} - R^{21})^{-1}\\ &\cdot ((B^{21} \otimes I)\nabla H_e(\eta) - (Q^{21})^{-1}(\xi^{21} - (I \otimes P)\zeta^{21}) - \delta^{21})\end{aligned}$$

The controller above admits the following incremental model

$$\begin{aligned}\begin{bmatrix}\dot{\xi}^{11} - \dot{\bar{\xi}}^{11}\\ \dot{\xi}^{21} - \dot{\bar{\xi}}^{21}\end{bmatrix} &= -(L_c \otimes I)\begin{bmatrix}\xi^{11} - \bar{\xi}^{11}\\ \xi^{21} - \bar{\xi}^{21}\end{bmatrix}\\ &\quad - Q^{-1}\begin{bmatrix}\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})\\ \nabla H_n^{21}(x^{21}) - \nabla H_n^{11}(\bar{x}^{21})\end{bmatrix}\end{aligned} \tag{61a}$$

$$\begin{aligned}\begin{bmatrix}\dot{\zeta}^{11} - \dot{\bar{\zeta}}^{11}\\ \dot{\zeta}^{21} - \dot{\bar{\zeta}}^{21}\end{bmatrix} &= -(L_c \otimes I)\begin{bmatrix}\zeta^{11} - \bar{\zeta}^{11}\\ \zeta^{21} - \bar{\zeta}^{21}\end{bmatrix} + \begin{bmatrix}\underline{s}(\zeta^{11}) - \underline{s}(\bar{\zeta}^{11})\\ \underline{s}(\zeta^{21}) - \underline{s}(\bar{\zeta}^{21})\end{bmatrix}\\ &\quad + (I \otimes P^T)Q^{-1}\begin{bmatrix}\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})\\ \nabla H_n^{21}(x^{21}) - \nabla H_n^{11}(\bar{x}^{21})\end{bmatrix}\end{aligned} \tag{61b}$$

$$\begin{bmatrix}u^{11} - \bar{u}^{11}\\ u^{21} - \bar{u}^{21}\end{bmatrix} = Q^{-1}\begin{bmatrix}\xi^{11} - \bar{\xi}^{11}\\ \xi^{21} - \bar{\xi}^{21}\end{bmatrix} - Q^{-1}(I \otimes P)\begin{bmatrix}\zeta^{11} - \bar{\zeta}^{11}\\ \zeta^{21} - \bar{\zeta}^{21}\end{bmatrix} \tag{61c}$$

where

$$\begin{aligned}&H_n^{21}(x^{21}) - H_n^{21}(\bar{x}^{21})\\ &\quad = (J^{21} - R^{21})^{-1}(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})\\ &\qquad - (J^{21} - R^{21})^{-1}(Q^{21})^{-1}\\ &\qquad\quad \cdot (\xi^{21} - \bar{\xi}^{21} - (I \otimes P)(\zeta^{21} - \bar{\zeta}^{21}))\end{aligned}$$

The incremental system dynamics in this case is given by

$$\begin{aligned}\dot{\eta} - \dot{\bar{\eta}} =& (B^{11} \otimes I)^T(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))\\ &+ (B^{12} \otimes I)^T(\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))\\ &+ (B^{21} \otimes I)^T(J^{21} - R^{21})^{-1}(B^{21} \otimes I)\\ &\quad \cdot (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\\ &- (B^{21} \otimes I)^T(J^{21} - R^{21})^{-1}(Q^{21})^{-1}\\ &\quad \cdot (\xi^{21} - \xi^{21} - (I \otimes P)(\zeta^{21} - \zeta^{21}))\\ &+ (B^{22} \otimes I)^T(J^{22} - R^{22})^{-1}(B^{22} \otimes I)\\ &\quad \cdot (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\end{aligned} \tag{62a}$$

$$\begin{aligned}\dot{x}^{11} - \dot{\bar{x}}^{11} =& (J^{11} - R^{11})(\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))\\ &- (B^{11} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))\\ &+ (Q^{11})^{-1}(\xi^{11} - \bar{\xi}^{11} - (I \otimes P)(\zeta^{11} - \bar{\zeta}^{11}))\end{aligned} \tag{62b}$$

$$\dot{x}^{12} - \dot{\bar{x}}^{12} = (J^{12} - R^{12})(\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) - (B^{12} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \tag{62c}$$

Now, let

$$\begin{aligned}
\bar{\xi}^{11} &= \mathbb{1} \otimes \bar{\lambda}, \qquad \bar{\xi}^{21} = \mathbb{1} \otimes \bar{\lambda} \\
\dot{\bar{\zeta}}^{11} &= \underline{s}(\bar{\zeta}^{11}), \quad \zeta^{11}(0) = \mathbb{1} \otimes w(0) \\
\dot{\bar{\zeta}}^{21} &= \underline{s}(\bar{\zeta}^{21}), \quad \zeta^{21}(0) = \mathbb{1} \otimes w(0)
\end{aligned}$$

where $\bar{\lambda}$ is given by (30). Note that

$$\begin{aligned}
\bar{u}^{11} &= (Q^{11})^{-1}\bar{\xi}^{11} - (Q^{11})^{-1}(I \otimes P)\bar{\zeta}^{11} \\
&= -(Q^{11})^{-1}(\mathbb{1} \otimes \bar{\lambda}) - (Q^{11})^{-1}(\mathbb{1} \otimes Pw)
\end{aligned}$$

and similarly

$$\bar{u}^{21} = -(Q^{21})^{-1}(\mathbb{1} \otimes \bar{\lambda}) - (Q^{21})^{-1}(\mathbb{1} \otimes Pw).$$

This coincides with $\bar{u}_i$ given by (29). Hence, by (31), it is easy to observe that $(\bar{x}, \bar{\eta}, \bar{\xi}, \bar{\zeta}, w)$ defines a valid solution to (12) where $y^* = \nabla H_n(\bar{x})$, and $\bar{\eta}$ is a constant vector satisfying (31).

Now consider again the Lyapunov function $V = W_n + W_e + W_c$ where W_n, W_e, and W_c are given by (48), (49), and (56), receptively. Then it is straightforward to investigate that

$$\begin{aligned}
\dot{V} =& (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}))^T (J^{11} - R^{11}) \\
&\cdot (\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11})) \\
&+ (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}))^T (J^{12} - R^{12}) \\
&\cdot (\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12})) \\
&+ (\nabla H_e(\eta) - \nabla H_e(\bar{\eta}))^T (B^{22} \otimes I)^T (J^{22} - R^{22})^{-1} \\
&\cdot (B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) \\
&+ (\zeta^{11} - \bar{\zeta}^{11})^T (\underline{s}(\zeta^{11}) - \underline{s}(\bar{\zeta}^{11})) \\
&+ (\zeta^{21} - \bar{\zeta}^{21})^T (\underline{s}(\zeta^{21}) - \underline{s}(\bar{\zeta}^{11})) \\
&- \tilde{\xi}^T (L_c \otimes I)\tilde{\xi} - \tilde{\zeta}^T (L_c \otimes I)\tilde{\zeta} + z^T (J^{21} - R^{21})^{-1} z
\end{aligned}$$

where

$$\tilde{\xi} = \begin{bmatrix} \xi^{11} - \bar{\xi}^{11} \\ \xi^{21} - \bar{\xi}^{21} \end{bmatrix}, \qquad \tilde{\zeta} = \begin{bmatrix} \zeta^{11} - \bar{\zeta}^{11} \\ \zeta^{21} - \bar{\zeta}^{21} \end{bmatrix},$$

and

$$\begin{aligned}
z = & (B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - (Q^{21})^{-1}(\xi^{21} - \bar{\xi}^{21}) \\
&+ (Q^{21})^{-1}(I \otimes P)(\zeta^{21} - \bar{\zeta}^{21}).
\end{aligned}$$

Hence, we obtain that $\dot{V} \leqslant 0$. Note that boundedness, existence, and uniqueness of solution is guaranteed as before. Now by constructing a forward invariant compact level set Ω_c around $(\bar{x}^{11}, \bar{x}^{12}, \bar{\eta}, \bar{\zeta}^{11}, \bar{\zeta}^{21})$, and invoking the LaSalle invariance principle, on the invariant set we have

$$\nabla H_n^{11}(x^{11}) - \nabla H_n^{11}(\bar{x}^{11}) = 0 \tag{63a}$$

$$\nabla H_n^{12}(x^{12}) - \nabla H_n^{12}(\bar{x}^{12}) = 0 \tag{63b}$$

$$(B^{22} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) = 0 \tag{63c}$$

$$(L_c \otimes I)\tilde{\xi} = 0 \tag{63d}$$

$$(L_c \otimes I)\tilde{\zeta} = 0 \tag{63e}$$

$$(B^{21} \otimes I)(\nabla H_e(\eta) - \nabla H_e(\bar{\eta})) - (Q^{21})^{-1}(\xi^{21} - \bar{\xi}^{21}) + (Q^{21})^{-1}(I \otimes P)(\zeta^{21} - \bar{\zeta}^{21}) = 0 \tag{63f}$$

Therefore, by (62a), we have $\dot{\eta} = \dot{\bar{\eta}} = 0$. Moreover, by (61a) and (63d), we obtain that $\dot{\xi}^{11} = \dot{\bar{\xi}}^{11} = 0$ on the invariant set. In addition, (63d) implies that $\tilde{\xi} = \mathbb{1} \otimes \alpha$ for some vector α. Replacing this into (60a) yields

$$\nabla H_n^{11}(x^{11}) = \nabla H_n^{11}(\bar{x}^{11}) = \mathbb{1} \otimes y^*.$$

This together with $\dot{\eta} = 0$ results in $y_i = y^*$ for each i.

Note that $\xi^{11} = \bar{\xi}^{11} + \mathbb{1} \otimes \alpha = \mathbb{1} \otimes (\alpha + \bar{\lambda})$, and similarly $\xi^{21} = \mathbb{1} \otimes (\alpha + \bar{\lambda})$. In addition, by (63e), we have $\zeta^{11} = \bar{\zeta}^{11} + \mathbb{1} \otimes \beta = \mathbb{1} \otimes (\beta + w)$ and $\zeta^{21} = \mathbb{1} \otimes (\beta + w)$ for some vector β. Now, the system dynamics on the invariant set takes the form

$$\begin{aligned}
0 =& (J^{11} - R^{11})(\mathbb{1} \otimes y^*) - (B^{11} \otimes I)\nabla H_e(\eta) \\
&+ (Q^{11})^{-1}\mathbb{1} \otimes (\bar{\lambda} + \alpha - P\beta) + \bar{\delta}^{11} \\
0 =& (J^{12} - R^{12})(\mathbb{1} \otimes y^*) - (B^{12} \otimes I)\nabla H_e(\eta) + \delta^{12} \\
0 =& (J^{21} - R^{21})(\mathbb{1} \otimes y^*) - (B^{21} \otimes I)\nabla H_e(\eta) \\
&+ (Q^{21})^{-1}\mathbb{1} \otimes (\bar{\lambda} + \alpha - P\beta) + \bar{\delta}^{21} \\
0 =& (J^{22} - R^{22})(\mathbb{1} \otimes y^*) - (B^{22} \otimes I)\nabla H_e(\eta) + \delta^{22}.
\end{aligned}$$

By multiplying the equations above from the left by $\mathbb{1}^T \otimes I$ and taking the sum, we obtain that

$$\bar{\lambda} + \alpha - PB = -(\sum_{i \in \mathcal{I}_c} Q_i^{-1})^{-1}(\sum_{i=1}^{N}(J_i - R_i)y^* + \sum_{i=1}^{N}\bar{\delta}_i)$$

By comparing this to (30), we conclude that $\alpha - P\beta = 0$. Consequently, on the invariant set u_i is equal to the optimal $\bar{u}_i$ given by (29). ■

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