Anti-de Sitter particles and manifest (super)isometries

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Starting from the classical action for a spin-zero particle in a D-dimensional anti-Sitter (AdS) spacetime, we recover the Breitenlohner-Freedman bound by quantization. For \( D = 4, 5, 7 \), and using an \( SU(2; \mathbb{K}) \) spinor notation for \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \), we find a bi-twistor form of the action for which the AdS isometry group is linearly realised, although only for zero mass when \( D = 4, 7 \), in agreement with previous constructions. For zero mass and \( D = 4 \), the conformal isometry group is linearly realised. We extend these results to the superparticle in the maximally supersymmetric “AdS\( \times S^n \)” string/M-theory vacua, showing that quantization yields a 128+128 component supermultiplet. We also extend them to the null string.

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Actions governing the dynamics of particles, strings or branes are generally invariant under the isometries, and possibly conformal isometries, of the background spacetime, but these symmetries may be realized non-linearly. In some cases it is possible to make manifest the full symmetry group by re-expressing the action in terms of new variables that transform linearly with respect to it.

A well-known example [1] is the twistor formalism for massless particles in 4-dimensional Minkowski spacetime (\( \text{Mink}_4 \)); this makes manifest an invariance under the \( SU(2, 2) \) conformal isometry group of \( \text{Mink}_4 \) because a twistor is essentially a spinor of this group. The supersupertwistor [2] extension of this construction to the \( \mathcal{N} = 4 \) massless superparticle manifests the \( SU(2, 2|4) \) superconformal symmetry of its action [3], allowing a simple demonstration that its quantization yields the \( \mathcal{N} = 4 \) Maxwell supermultiplet. Similar constructions are possible for \( \text{Mink}_{3,6} \) [4]; these rely on the fact that the conformal isometry group of \( \text{Mink}_d \) for \( d = 2 + \dim \mathbb{K} \), where \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \), is isomorphic to \( Sp(4; \mathbb{K}) \), defined as preserving a skew-\( \mathbb{K} \)-hermitian quadratic form on \( \mathbb{K}^4 \) [5].

The conformal isometry group of \( \text{Mink}_d \) is also the isometry group of \( D \)-dimensional anti-de Sitter space (\( \text{AdS}_D \)) for \( D = d + 1 \). Some years ago it was noticed by Claus et al. [6] that the action for a particle in \( \text{AdS}_5 \) could be expressed in terms of bi-twistors of \( \text{Mink}_4 \). A geometric interpretation of this construction was supplied by Cederwall [7], who also showed that a similar bi-twistor construction for \( \text{AdS}_{4,7} \) could work only for zero mass.

Here we present a simple variant of the Claus et al. construction that applies uniformly to \( \text{AdS}_{4,5,7} \). Although the resulting linearly-realized \( Sp(4; \mathbb{K}) \) symmetry group is the AdS isometry group only for zero mass, this mismatch can be eliminated in the \( \mathbb{K} = \mathbb{C} \) case by a re-definition of the twistor variables. We thereby recover the result of Claus et al. for \( \text{AdS}_5 \), and confirm the conclusions of Cederwall for \( \text{AdS}_{4,7} \) by algebraic means.

Although linear realization of the \( \text{AdS}_D \) isometry group limits our bi-twistor construction for \( D = 4, 7 \) to zero mass, a bonus for \( D = 4 \) is that the conformal isometry group of \( \text{AdS}_4 \) is also linearly realized.

Anti-de Sitter vacua arise naturally in supergravity theories. In particular the \( \text{AdS}_{4,5,7} \) cases arise through the maximally supersymmetric “AdS\( \times S^n \)” vacua of string/M-theory in 10/11 dimensions, in which context they can also be interpreted as the near-horizon geometries of, respectively, the M2-brane, D3-brane and M5-brane [8]. The corresponding isometry supergroups are as follows (the \( O(n; \mathbb{K}) \) subgroup of \( \text{OSp}(n|4; \mathbb{K}) \) is defined to preserve a \( \mathbb{K} \)-hermitian quadratic form on \( \mathbb{K}^n \)):

\[
M_2 : \text{AdS}_4 \times S^7 : \text{OSp}(8|4; \mathbb{R}) \supset \text{Spin}(8) \times \text{Spin}(2, 3)
\]
\[
D_3 : \text{AdS}_5 \times S^5 : \text{OSp}(4|4; \mathbb{C}) \supset \text{U}(4) \times \text{Spin}(2, 4)
\]
\[
M_5 : \text{AdS}_7 \times S^4 : \text{OSp}(2|4; \mathbb{H}) \supset \text{USp}(4) \times \text{Spin}(2, 6)
\]

In the D3-brane case, the \( \text{AdS}/\text{CFT} \) correspondence relates a four-dimensional \( N = 4 \) Yang-Mills theory to IIB superstring theory in the \( \text{AdS}_5 \times S^5 \) background [9], and the superstring ground states should be described by a superparticle invariant under the \( \text{OSp}(4|4; \mathbb{C}) \) \( \supset \text{SU}(2, 2|4) \) isometries of this background.

This motivates a generalization of the twistor formulation of particle dynamics in AdS to a supertwistor formulation of the superparticle. A direct construction based on AdS supergeometry would involve a complicated expansion in superspace coordinates but a simple Minko\( \text{d} \) supersymmetrization suffices since the other supersymmetries are then implied. This is reminiscent of the “hidden” supersymmetries of the massive superparticle [10]; as in that case, all supersymmetries become manifest in a supertwistor formulation, as anticipated by Cederwall [7]. For the cases corresponding to the above table, we find that the supertwistor form of the superparticle action involves a total of 8 fermi oscillators, so quantization will yield a supermultiplet of \( 2^8 = 128 + 128 \) independent states, as expected for a maximally-supersymmetric graviton supermultiplet in the AdS\( \times S \) background.

Our constructions are based on the fact that \( \text{AdS}_D \) can be foliated by Minkowski spacetimes of dimension \( d = D - 1 \), so it is convenient to choose coordinates
adapted to this foliation. We will begin by showing how the Breitenlohner-Freedman (BF) bound on the mass-squared of scalar fields in AdS [11] follows from a semi-classical quantization of the particle in such a background given that the motion on Minkowski “slices” is non-tachyonic.

We start from the phase-space form of the action, invariant under reparametrizations of the particle’s world-line, which is embedded in a $D$-dimensional spacetime with metric $g_{MN}$ in local coordinates $x^M$:

$$S = \int dt \left\{ \dot{x}^M p_M - \frac{1}{2} \epsilon \left( g^{MN} p_M p_N + m^2 \right) \right\}. \quad (1)$$

We use a “mostly plus” signature convention, and $\epsilon(t)$ is a Lagrange multiplier for the mass-shell constraint. Given an AdS$_D$ background of radius $R$, we may choose the metric to be

$$ds^2 = g_{MN} dx^M dx^N = \frac{R^2}{z^2} \left( \eta_{mn} dx^m dx^n + dz^2 \right), \quad (2)$$

where $\{x^m; m = 0, 1, \ldots, d - 1\}$ are Minkowski coordinates for the Mink$_d$ “slices”, which are the hypersurfaces of constant $z$. AdS infinity is at $z = 0$ and there is a Killing horizon at $z = \infty$.

We can now rewrite the action as

$$S = \int dt \left\{ \dot{z}^m p_m + \dot{z} p_z - \frac{1}{2} \epsilon \left( p^2 + \Delta^2 \right) \right\}, \quad (3)$$

where $R^2 \epsilon = z^2 \epsilon$ and

$$p^2 = \eta^{mn} p_m p_n, \quad \Delta^2 = p_z^2 + (mR/z)^2. \quad (4)$$

Let us remark here that the physical phase space has dimension $2D - 2 = 2d$ because the constraint also generates a gauge invariance, thereby lowering the dimension by 2, and this must be the physical phase-space dimension of any equivalent action in other variables.

A feature of the action (3) is that $\Delta$ is a constant of the motion. Consequently, the motion within the $(x, p)$ subspace of phase space is that of a free particle of mass $\Delta$ in Mink$_d$. The mass $m$ affects directly only the motion in the $(z, p_z)$ phase-plane. For $m = 0$ we have $\dot{p}_z = 0$ and the motion in this phase plane is linear. For $m^2 > 0$ it is convenient to choose $\Delta > 0$ and to write

$$p_z = \Delta \cos \varphi, \quad \frac{mR}{z} = \Delta \sin \varphi, \quad (5)$$

for angular variable $\varphi$; the motion in the $(\Delta, \varphi)$ plane is circular. Notice that $z = \infty$ whenever $\sin \varphi = 0$, which tells us that the particle will pass through two Killing horizons of AdS as $\varphi$ increases by $2\pi$. Because of the periodic identification of the global time coordinate of AdS and the fact that there is only one future and one past Killing horizon in one period, a timelike geodesic will return to the same point in spacetime after crossing both Killing horizons. In this case we should identify $\varphi$ with $\varphi + 2\pi$. However, a particle that crosses a Killing horizon of the simply-connected cover of AdS will never return to the same point in spacetime or even the same point in space, so we should not assume that $\varphi$ is periodically identified in this case.

We may also allow $m^2 < 0$ as long as $\Delta^2 > 0$, which implies that

$$(mR)^2 > -(zp_z)^2. \quad (6)$$

Although $(zp_z)^2$ is non-zero on spacelike geodesics there is otherwise no classical restriction on its value, which could be zero. However, the quantum uncertainty principle implies that its smallest value is $(\Delta z \Delta p_z)^2 = (\hbar/2)^2$. Quantum mechanics therefore implies the inequality

$$(mR/\hbar)^2 > -\frac{1}{4}. \quad (7)$$

This is not yet a bound on the mass parameter $M$ of the Klein-Gordon equation obeyed by the particle’s wavefunction. For $m = 0$ the classical action (3) is invariant under the conformal isometry group of AdS$_D$ and a quantization preserving this symmetry will yield a Klein-Gordon equation with mass parameter $M_c$ satisfying $(M_cR)^2 = -D(D - 2)/4$ [12]. The Klein-Gordon mass-parameter $M$ is therefore given by $M^2 = M_c^2 + (m\hbar)^2$, and the bound it satisfies is

$$(MR)^2 \geq (M_cR)^2 - \frac{1}{4} = -d^2/4. \quad (8)$$

We have allowed for equality here without obvious justification; apart from this detail, we have now recovered the BF bound for a scalar field in an AdS spacetime of arbitrary dimension $D = d + 1$ [13].

This result suggests that we should allow all values of $m^2$ for which $\Delta^2 > 0$. Of particular relevance here is the fact that in all such cases

$$\dot{z} p_z = -zp_z \Delta^{-1} \dot{\Delta} + \frac{z}{\Delta} \left( \cdots \right). \quad (9)$$

Using this result, and ignoring a total derivative, we deduce that the action (3) is equivalent to

$$S = \int dt \left\{ \dot{z}^m p_m - \frac{z p_z}{\Delta} \Delta^{-1} \dot{\Delta} - \frac{1}{2} \epsilon \left( p^2 + \Delta^2 \right) \right\}. \quad (10)$$

For $m = 0$ we have $\Delta = p_z$. For $m^2 > 0$ we have $zp_z = mR \cot \varphi$, which implies that $\varphi$ is the remaining phase space coordinate (and for $m = i|m|$ we have $zp_z = mR \coth \psi$ where $\Delta$ can have either sign and $\psi = -i\varphi$).

For $d = 3, 4, 6$ we may replace the Mink$_d$ coordinates by a $2 \times 2$ hermitian matrix $X$ over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Similarly, we may replace the $d$-momentum by a $2 \times 2$ hermitian matrix $\mathcal{P}$ such that $\det \mathcal{P} = -p^2$ (hermitian quaternionic matrices have an intrinsically defined real determinant [14, 15]). We then have

$$\dot{z}^m p_m = \frac{1}{2} \text{tr} (X \mathcal{P} + \mathcal{P} X) = \frac{1}{2} \text{tr} (X \mathcal{P}), \quad (11)$$
where “tr” indicates the real part of the matrix trace. We now write
\[ P = \mp \mathbb{U} \mathbb{U}^\dagger, \] (12)
where \( \mathbb{U} \) is a new 2 \( \times \) 2 matrix variable and the top/bottom sign is for positive/negative \( p^0 \). The mass-shell constraint is now
\[ \det(\mathbb{U} \mathbb{U}^\dagger) = \Delta^2. \] (13)
Effectively, we have replaced the \( d \)-momentum by a pair of 2-component Minkowski spinors, alias 2-vectors of \( SL(2; \mathbb{K}) \) [16]. This has introduced a new gauge invariance since \( \mathbb{U} \) is acted upon from the left by \( SL(2; \mathbb{K}) \) but from the right by \[ O(2; \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}) = O(2), U(2), Spin(5). \] (14)
This ensures that \( \mathbb{U} \) is determined by the \( d \) real variables \( p_m \) up to an \( O(2; \mathbb{K}) \) gauge transformation. We now find that
\[ \dot{x}^m p_m = \text{tr}_R (\mathbb{U} \mathbb{W}^\dagger) + \frac{d}{dt} (\cdots), \quad \mathbb{W}_0 = \pm \mathbb{X} \mathbb{U}. \] (15)
From the definition of \( \mathbb{W}_0 \), which is also acted upon by \( SL(2; \mathbb{K}) \) from the left and by \( O(2; \mathbb{K}) \) from the right, it follows that
\[ \mathbb{U}^\dagger \mathbb{W}_0 - \mathbb{W}_0^\dagger \mathbb{U} = 0. \] (16)
In the context of a particle in Minkowski 3,4,6 of mass \( \Delta \), we would take the Lagrangian to be \( L = \text{tr}_R (\mathbb{U} \mathbb{W}_0^\dagger) \) and impose the identity (16) as a constraint with a Lagrange multiplier. The component constraints span the Lie algebra of \( O(2; \mathbb{K}) \) with respect to the Poisson brackets implied by (15), and hence generate the required \( O(2; \mathbb{K}) \) gauge invariance of the action; they are the spin-shell constraints of the bi-twistor action for the massive particle in Minkowski 3,4,6 [17–19] (and they also arise in other contexts, e.g. [20]). Of course, in this context we would also need to impose the new \( O(2; \mathbb{K}) \)-invariant but \( Sp(4; \mathbb{K}) \)-violating mass-shell constraint (13).

However, we are dealing with a particle in AdS\(_5\) and an action (10) for which \( \Delta \) is a phase-space coordinate. In this context we may interpret the new mass-shell condition as providing an expression for \( \Delta \) in terms of \( \mathbb{U} \), which is such that
\[ \Delta^{-1} \bar{\Delta} = \text{tr}_R (\bar{\mathbb{U}} \mathbb{V}), \quad \mathbb{V} \equiv \mathbb{U}^{-1}. \] (17)
We remark that the left and right inverses of \( \mathbb{U} \) are equal even for \( \mathbb{K} = \mathbb{H} \) [21]. Taking into account (15), we now have
\[ \dot{x}^m p_m - \frac{z p_z}{\Delta} \bar{\Delta} = \text{tr}_R (\bar{\mathbb{U}} \mathbb{W}^\dagger) + \frac{d}{dt} (\cdots), \] (18)
where
\[ \mathbb{W} = \pm \mathbb{X} \mathbb{U} - z p_z \mathbb{V}^\dagger. \] (19)
This expression for \( \mathbb{W} \) implies the identity
\[ \mathbb{G} := \mathbb{U}^\dagger \mathbb{W} - \mathbb{W}^\dagger \mathbb{U} \equiv 0, \] (20)
which again becomes a constraint to be imposed by an anti-\( \mathbb{K} \)-hermitian Lagrange multiplier \( \mathbb{L} \) in the action. There is no longer any mass-shell constraint, so the action is
\[ S = \int dt \text{tr}_R \{ \mathbb{U} \mathbb{W}^\dagger - \mathbb{L} \mathbb{G} \}. \] (21)
There are \( (3 \dim \mathbb{K} - 2) \) first-class constraints on \( 8 \dim \mathbb{K} \) variables, yielding a physical phase space of dimension \( 2(\dim \mathbb{K} + 2) = 2d \), as required.

The \( 4 \times 2 \) matrix with \( \mathbb{K} \)-hermitian conjugate \( (\mathbb{U}^\dagger, \mathbb{W}^\dagger) \) is pair of Minkowski 3,4,6 twistors; i.e. a bi-twistor, acted upon from the left by \( Sp(4; \mathbb{K}) \) and from the right by \( O(2; \mathbb{K}) \). The Noether charges for the \( Sp(4; \mathbb{K}) \) invariance of the action (21) are the gauge-invariant bi-twistor bilinears
\[ \mp \mathbb{U} \mathbb{W}^\dagger = -\mathbb{X} \mathbb{P} \mp z p_z \mathbb{V}^\dagger, \] (22)
except that the imaginary part of \( \text{tr}(\mathbb{U} \mathbb{W}^\dagger) \) should be omitted for \( d = 4 \) since this is the trace of \( \mathbb{G} \). The last line uses the mass-shell constraint (13) and the relation
\[ \pm \Delta^2 \mathbb{V}^\dagger \mathbb{V} = \mathbb{P}^\dagger \equiv \mathbb{P} - \text{tr}_R \mathbb{F}. \] (23)
The matrix \( \mathbb{P} \) represents the \( d \)-vector \( \eta^m p_m \), and is such that \( \det \mathbb{P} = -p^2 \) and \( \text{tr}_R (\mathbb{P}^\dagger \mathbb{P}) = 2p^2 \).

For \( m = 0 \), these Noether charges are those associated with invariance under the AdS\(_5\) isometry group. In the \( D = 4 \) case there is a larger linearly-realized symmetry because there is an antisymmetric second-order invariant tensor of the \( SO(2) \) gauge group. Using the corresponding matrix \( \mathbb{E} \), and noting that \( \mathbb{U}^\dagger \mathbb{W} \) is \( O(2) \) invariant, we can write down an additional \( 4 + 1 = 5 \) quadratic Noether charges: \( \mathbb{E} \mathbb{W}^\dagger \) and \( \mathbb{U}^\dagger \mathbb{W} + \mathbb{W}^\dagger \mathbb{U} \). The full set of quadratic charges (omitting \( \mathbb{G} \) itself) spans the Lie algebra (with respect to Poisson brackets) of the AdS\(_5\) conformal isometry group \( SO(2, 4) \).

When \( m \neq 0 \) the expression for \( \mathbb{W} \mathbb{W}^\dagger \) in (22) contains an additional term that is not linear in momenta. This shows that the linearly realized \( Sp(4; \mathbb{K}) \) symmetry group is no longer the \( Sp(4; \mathbb{K}) \) isometry group and it explains how the action (21) manages to be independent of the mass \( m \). In the \( \mathbb{K} = \mathbb{C} \) case, and \( m^2 > 0 \), this conclusion can be changed by setting
\[ \mathbb{W} = \mathbb{W}^\dagger + i(mR) \mathbb{V}^\dagger. \] (24)
Replacing \( \mathbb{W} \mathbb{W}^\dagger \) by \( \mathbb{W}^\dagger \mathbb{W}^\dagger \) eliminates the unwanted \( m \)-dependent term in this Noether charge. At the same time, the action in terms of \( \mathbb{W} \) is unchanged from (21) except that the \( 2 \times 2 \) anti-hermitian matrix constraint function now takes the form
\[ \mathbb{G} = \mathbb{U}^\dagger \mathbb{W} - \mathbb{W}^\dagger \mathbb{U} + 2imR. \] (25)
In other words, the $U(1)$ constraint function $\frac{1}{2} \text{tr} \mathcal{G}$ has been shifted by $2i m R$, as found directly in the AdS$_5$ construction of [6]. This possibility is available only for $\mathbb{K} = \mathbb{C}$ because there is no imaginary unit for $\mathbb{K} = \mathbb{R}$ and a choice of one for $\mathbb{K} = \mathbb{H}$ breaks the Spin(5) gauge invariance. This difficulty can be circumvented by using a quartet of twistors, instead of a bi-twistor, but only at the cost of introducing second-class constraints [7].

We now return to the action (10) and extend its manifest Poincaré invariance on Mink$_4$ slices to an $N$-extended super-Poincaré invariance. In the $SI(2; \mathbb{K})$ notation this is achieved by the replacement [22]

$$\hat{X} \rightarrow \hat{X} + \sum_{i=1}^N \left( \Theta_i^i \dot{\Theta}^i - \dot{\Theta}_i^i \Theta^i \right),$$

(26)

where the $N$ anticommuting 2-component spinors $\Theta^i$ are acted upon from the left by $O(N; \mathbb{K})$ and from the right by $SI(2; \mathbb{K})$. We have adopted the convention that $\mathbb{K}$-conjugation (in contrast to $\mathbb{K}$-hermitian conjugation) does not change the order of anticommuting factors, so the addition to $\dot{X}$ is hermitian. This construction ensures the existence of $N SI(2; \mathbb{K})$ spinor supercharges $Q^i$.

Next, we proceed as before to the twistor form of the action, introducing the new anticommuting Lorentz scalar variables

$$\Xi^i = \Theta^i \mathbf{U},$$

(27)

which are acted upon from the left by $O(N; \mathbb{K})$ and from the right by the $O(2; \mathbb{K})$ gauge group. One finds, omitting a total derivative, that the action is

$$S = \int dt \text{tr}_\mathbb{R} \left\{ \hat{W}^i \dot{\Xi}^i - \Xi^i \dot{\Xi}^i - \mathbb{L} \mathcal{G} \right\},$$

(28)

where now

$$\mathbb{W} = \pm \left( \mathbf{X} \mathbf{U} - \Theta^i \Xi^i \right) - z P \mathbf{U}^l,$$

(29)

which leads to the new $O(2; \mathbb{K})$ generators

$$\mathcal{G} = \mathbf{U}^l \mathbb{W} - \mathbb{W}^l \mathbf{U} \pm 2 \Xi^i \Xi^i.$$  

(30)

The $(4 + N) \times 2$ matrix with $\mathbb{K}$-hermitian conjugate $(\mathbf{U}^l, \mathbb{W}^l, \Xi^i)$ is a bi-supertwistor, acted upon from the right by the $O(2; \mathbb{K})$ gauge group and from the left by $\text{osp}(N|4; \mathbb{K})$. The supersymmetry charges are $Q^i = \Xi^i \mathbf{U}^l$ and $S^i = \mathbf{W} \Xi^i$, which is double the number guaranteed by the construction. In the $\mathbb{K} = \mathbb{C}$ case we can again allow for $m^2 > 0$ by making the substitution (24) in the action, but now we must replace not only the Noether charge $\mathbb{W} \mathbf{U}^l$ by $\tilde{W} \mathbf{U}^l$ but also $S^i$ by

$$\tilde{S}^i = \Xi^i \left[ \tilde{W}^i - \frac{1}{4} \mathbb{W} \text{tr} \mathcal{G} \right],$$

(31)

which is physically equivalent to $\Xi^i \tilde{W}^i$ but the $m$-dependence of $\mathbb{W}$ is cancelled by that of $\text{tr} \mathcal{G}$.

Choosing $N = 8/\dim \mathbb{K}$ we get, for $m = 0$, the invariance supergroups of the String/M-theory “AdS$\times S^7$” vacua tabulated earlier. In each case there are 8 fermi oscillators so we get a supermultiplet of $2^8 = 128 + 128$ states, which is the degeneracy of the expected graviton supermultiplet. In light of the connection between the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and supersymmetric gauge theories in dimensions $d = 3, 4, 6, 10$ [23], our results suggest that there should be some corresponding connection to the maximal gauged supergravity theories in dimensions $D = 4, 5, 7$, and perhaps $D = 11$ with “OSp(1|4; \mathbb{O})” as the AdS$_{11}$ supergroup [24]. Also, the fact that a pair of supertwistors is needed to describe a graviton supermultiplet, whereas a single supertwistor suffices for a 4D Maxwell supermultiplet (to take the $\mathbb{K} = \mathbb{C}$ case) could be viewed as support for the proposal, recently reviewed in [25], that gravity is the “square” of Yang-Mills theory.

Finally, we consider strings in AdS$_D$. A bi-twistor action for the Nambu-Goto string in Mink$_4$ was found in [26] but the constraints are not all quadratic and its extension to an AdS$_D$ background is far from obvious. Here we consider the closed null string in AdS$_{4,5,7}$. As the twistor formulation makes manifest invariance under AdS isometries, and conformal isometries for AdS$_4$, this may be useful for investigations into the proposed link to higher-spin theories [27–29]. A string-inspired twistor model, but without spin-shell constraints, has been used previously for this purpose [30], and higher-spins emerge from the twistor form of the AdS (super)particle when its spin-shell constraints are relaxed [7], but the relation of higher spin theory to the null string remains conjectural.

Following the massless particle example, the standard phase-space action for the closed null string in AdS$_D$ can be put in the form

$$S = \int dt \oint d\sigma \left\{ \dot{X}^m P_m + \dot{Z} P_Z - \frac{1}{2} \ell \left( P^2 + P^2_Z \right) \right. \right.$$  

$$\left. - \ell \left( X^m P_m + Z^l P_Z \right) \right\},$$

(32)

where all variables are now functions of the worldsheet coordinates $(t, \sigma)$ and $\ell$ is the Lagrange multiplier for the string reparametrization constraint. The twistor form of the action is found as before, with the result that

$$S = \int dt \oint d\sigma \left\{ \text{tr}_\mathbb{R} \left( \hat{W}^l \dot{\Xi}^i - \Xi^i \dot{\Xi}^i - \mathbb{L} \mathcal{G} \right) - \ell \mathcal{O} \right\},$$

(33)

where $\mathcal{O}$ is the twistor version of the string reparametrization constraint:

$$\mathcal{O} = \text{tr}_\mathbb{R} \left( \mathbb{W} \mathbf{U}^l - \mathbf{W}^l \mathbf{U}^l \right).$$

(34)

This result has an obvious extension to the null $p$-brane, and supersymmetry may be incorporated as for the particle. The zero-mode contribution is the bi-twistor action for the massless (super)particle.
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