Interventionist Routing Algorithm for single-block warehouse: Application and Complexity

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1 Introduction

Nowadays, in many warehouses and distribution centres, the order-picking operation is executed in uncertain environments, mainly due to the stochastic nature customer orders inter-arrival time [1]. Warehouses satisfying customer orders placed via internet or manufacturing warehouses that support Just-In-Time are examples of such environments. Motivated by the needs of increasing the flexibility of order-picking operation in uncertain environments, the authors developed an interventionist routing algorithm which allows the picker to be updated by any newly arrival customer requests (e.g. new orders or order amendment) and re-calculate an optimal order picking route regardless the current location of the order-picker.

This report aims to present the detailed procedures for applying the Interventionist Routing Algorithm (IRA) as well as discuss its complexity. The report is structured as follows. Section 2 briefly discuss the three stages of applying the algorithm. Section 3 presents all the transition tables needed for using the algorithm. Detailed procedures of applying IRA are presented using flow charts in Section 4. Finally, a mathematical proof for the linearity of IRA in terms of its complexity is presented in Section 5 .

For the interested reader, more details about the rationale of the algorithm, order-picking modelling, numerical examples and simulations using the algorithm can be found in [2].

2 Algorithm description

This section discuss the three stages of applying the Interventionist Routing Algorithm (IRA).

Before constructing the order-picking route, we firstly need to examine the possible route (arc) configurations for a picker to access or leave a particular pick-aisle, and to cross between two adjacent pick-aisles.

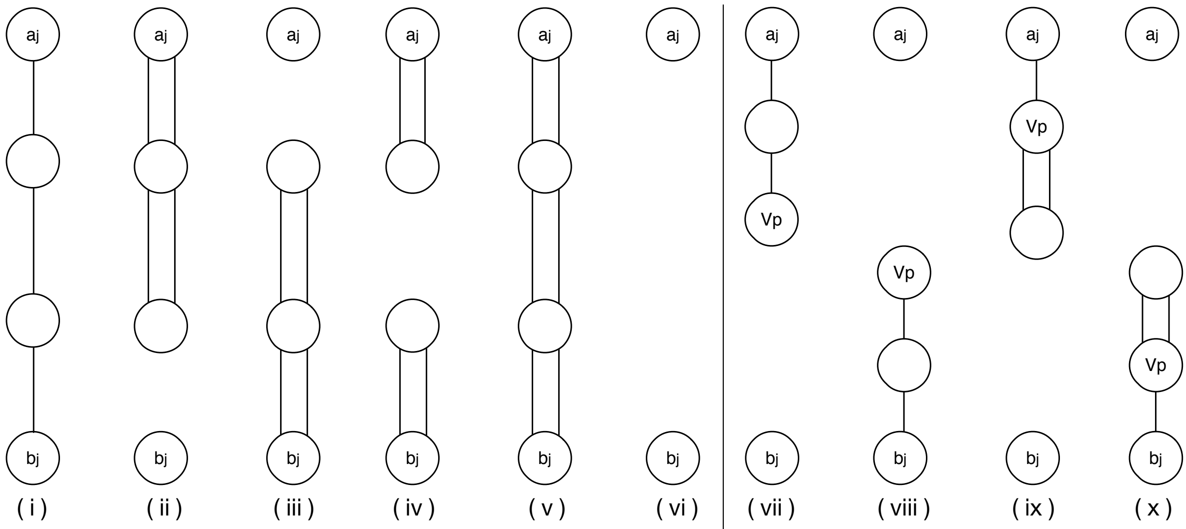
In the configuration for pick-aisle, a minimum length route cannot have more than two arcs between any pair of vertices [3]. Therefore, as shown in Figure 1 beside the 6 possible arcs configurations (defined in [3]) to access an aisle if the picker is at either endpoint of the aisle, we introduced 4 possible configurations to leave an aisle if the picker is in the aisle. It is noted that case (iv) is configured by determining the largest gap between the two item vertices in that aisle because it can only be applied if there are more than two storage locations to be visited in the aisle; and it is the only configuration that may not be unique.

Similarly, in the configuration for cross-aisles, a minimum length route cannot have more than two arcs between any pair of aisle endpoints vertices. Since the total number of arc between two adjacent aisles should always be odd for one-way and even for round-trip, there are only 4 possible arc configurations for cross-overs in one-way and another 4 possible configurations in Round-trip areas, as shown in Figure 2.

Having identified all possible arc configurations, we can now describe the three main stages of IRA.

Stage 1: Initiation

In order to construct the order-picking route, we first need to convert the depot, current location of order-pickers and all requested items on the pick-list into the graph model as



(a) Picker entering aisle j

(b) Picker at point v_p in aisle j

Figure 1: Possible arc configurations for a picker to access or leave any aisle j in a minimum route subgraph

described in [2]. We number the aisles in an increasing matter from left to right unless a) the picker's current location is on the right-hand side of the depot aisle or b) the picker's current location is on the depot aisle and there are no items to be picked on the right-hand side of the depot. In these special cases, we number the aisles in an increasing matter from right to left.

Secondly, we determine the situations by the current location of the picker when a customer order/request is added and a new order-picking route is required. This algorithm assumes that a new route will not be regenerated when the picker is traveling on the cross-aisle in between any two adjacent aisles. This is because a) the distance between two adjacent aisles is relative small compared to the length of an aisle, and b) the cross-aisles in practice are generally too narrow for an order-picker with a travelling device to change the direction. Therefore, a new order-picking route is constructed in two situations:

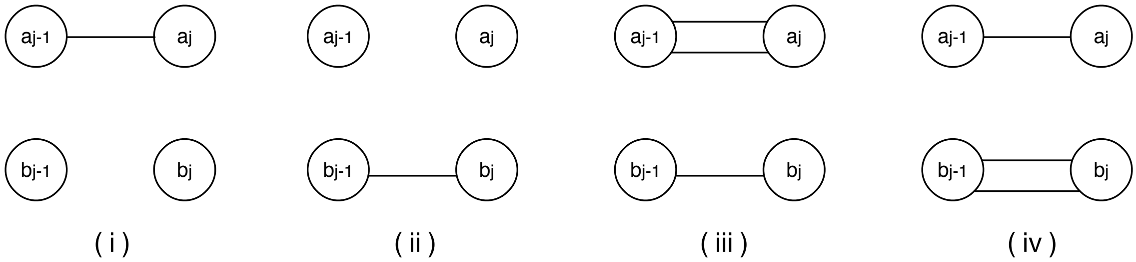
Situation 1: the order-picker is inside a pick-aisle;

Situation 2: the order-picker is at one of the endpoints (a_j or b_j) of an aisle.

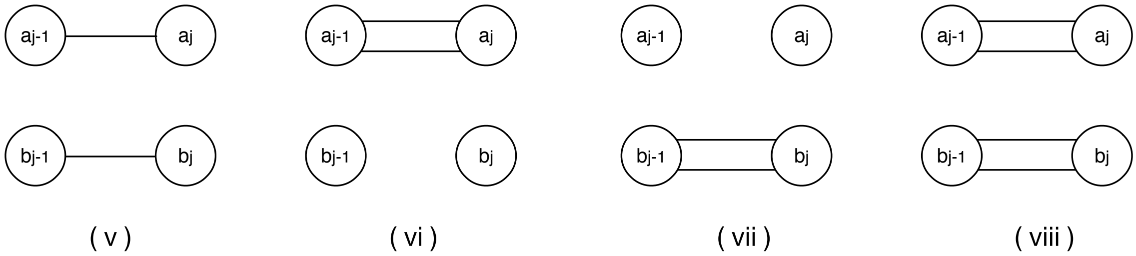
In Situation 1, in order to decide which endpoint the order-picker should exist the pick-aisle, the algorithm will:

1. assume no item is requested from the current pick-aisle in question;
2. calculate the potential travel distances of leaving from either exit;
3. determine the overall travel distance in addition to the distance of travelling to the associated exit;
4. choose the exit (endpoint) with the shortest overall travel distance and generate the associated routing.

In order to generate the route for Situation 1, the algorithm will calculate that for Situation 2 twice by assuming the order-picker is at the endpoint a and then b respectively. (Therefore, for Situation 2, we can go to the next step directly.)



(a) Picker Cross-Aisle Options For One-Way area



(b) Picker Cross-Aisle Options For Round-Trip area

Figure 2: Possible arc configurations for cross-aisle between aisle $j-1$ and aisle j in minimum route subgraph

We denote aisles from left to right, the first pick-aisle as aisle f and the last one is aisle r, the algorithm will then construct all equivalence classes of the L_j PRS for each aisle $j = f, f+1, \ldots, r-1, r$ in sequence. Each equivalence classes of ${L_f}^+$ PRSs (i.e. $(U, U, 1C), (E, 0, 1C), (0, E, 1C), (E, E, 2C), (E, E, 1C)$) can be determined by one of the 5 arcs configurations in Figure 1a correspondingly.

If the order-picker is to the right of the depot or the in the depot aisle, and requested items are all located his left, the algorithm will 'flip' the warehouse by renumbering the aisles starting from the right-most pick aisle to the left-most one.

Stage 2: Transitions

The transition from ${L_{j-1}}^+$ to L_j^-

This transition determines which of the 8 possible ways in Figure 2 can be added to the equivalence classes of the ${L_{j-1}}^+$ PRS in order to obtain the L_j^- PRS in each of its equivalence classes. An example is provided using Table 3a. The L_j^- PRS in *(U,E,2C)* class can be obtained by applying the arc configuration (iv) in Figure 2 to two of the equivalence classes of ${L_{j-1}}^+$ PRS, *(0,E,1C)* class and *(U,E,2C)* class respectively. The minimum length PRS is obtain by taking the smallest value of the PRSs in these equivalence classes, which is the combination that derives the shortest travel distance for this class.

Such transition is enabled by applying the appropriate tables based on the travelling area that aisle $j-1$ and j are located in. Specifically:

- If aisle $j-1$ and j are both in Round-Trip(RT) area, and the One-Way(OW) area does not exist before aisle $j-1$, apply Table 2a.
- If aisle $j-1$ and j are both in RT area, and the OW area exists before aisle $j-1$, apply Table 2b.

- If aisle $j-1$ is in RT area or $j-1=f$, and aisle j is in OW area, apply Table 3a or 3b if the order-picker starts from endpoint a or b respectively.
- If aisle $j-1$ and j are both in OW area, apply Table 5.
- if aisle $j-1$ is in OW area, and aisle j is in RT area, apply Table 6.
- if aisle $j-1$ is the depot aisle, and the picking route will start from the endpoint a (i.e. the order-picker will start from the head of the depot aisle a), apply Table 2c.

The transition from L_j^- to L_j^+

Similarly, this transition determines which of the 6 possible ways in Figure 1a can be added to each equivalence classes of the L_j^- PRS in order to obtain the L_j^+ PRS in each of its equivalence classes.

Such transition is enabled by applying the appropriate tables based on the travelling area that aisle $j-1$ and j are located in. Specifically:

- If aisle $j-1$ and j are both in RT area, and the OW area does not exist before aisle $j-1$, apply Table 1a.
- If aisle $j-1$ and j are both in RT area, and the OW area exists before aisle $j-1$, apply Table 1b.
- If aisle $j-1$ is in RT area or $j-1=f$, and aisle j is in OW area, apply Table 4a or 4b if the order-picker starts from endpoint a or b respectively.
- If aisle $j-1$ and j are both in OW area, apply Table 4c.
- if aisle $j-1$ is in OW area, and aisle j is in RT area, apply Table 1b.
- if aisle $j-1$ is the depot aisle, and the picking route will start from the endpoint a (i.e. the order-picker will start from the head of the depot aisle a), apply Table 1c.

Stage 3: Route construction

After determining the ${L_r}^+$ PRS for each equivalence class, the minimum length route subgraph (i.e. the minimum order-picking route) is the minimum length ${L_r}^+$ PRS in (E, 0, 1C), (0, E, 1C) and (E, E, 1C) if aisle r is in a RT area [3], or in (0, U, 1C) and (E, U, 1C) if aisle r is in OW area, given that $f \neq d$ in both cases.

Eventually, each equivalence class of L_j^- and L_j^+ (for $j = f, f+1, \ldots, r-1, r$) PRS can be determined by following the trace of obtaining the chosen equivalence class of ${L_r}^+$ PRS, and therefore the associated order-picking route can be constructed from aisle f to aisle r accordingly.

3 Tables for constructing the order-picking routes

This section presents the tables for the calculations of the transition from first pick aisle to the last one.

Table 1: Transition from L_j^- to L_j^+ in Round-Trip equivalent classes from adding each of the arc configuration in Figure 1a

#	L_j^- Equivalence Classes	Arc Configuration from Figure 1a (i)	(ii)	(iii)	(iv)	(v)	(vi)[a]
1	(U, U, 1C)	(E, E, 1C)	(U,U,1C)	(U, U, 1C)	(U, U, 1C)	(U, U, 1C)	(U, U, 1C)
2	(E, 0, 1C)	(U, U, 1C)	(E, 0, 1C)	(E, E, 2C)	(E, E, 2C)	(E, E, 1C)	(E, 0, 1C)
3	(0, E, 1C)	(U, U, 1C)	(E, E, 2C)	(0, E, 1C)	(E, E, 2C)	(E, E, 1C)	(0, E, 1C)
4	(E, E, 1C)	(U, U, 1C)	(E, E, 1C)	(E, E, 1C)	(E, E, 1C)	(E, E, 1C)	(E, E, 1C)
5	(E, E, 2C)	(U, U, 1C)	(E, E, 2C)	(E, E, 2C)	(E, E, 2C)	(E, E, 1C)	(E, E, 2C)
6	(0, 0, 0C)[b]	(U, U, 1C)	(E, 0, 1C)	(0, E, 1C)	(E, E, 2C)	(E, E, 1C)	(0, 0, 0C)
7	(U, U, 2C)	(E, E, 1C)	(U, U, 2C)	(U, U, 2C)	(U, U, 2C)	(U, U, 1C)	(U, U, 2C)

Table 1a contains rows 1–6
Table 1b contains rows 1–7
Table 1c contains rows 1,2,3,4,6,7

[a] This is not a feasible configuration if there is any item to be picked in aisle j.
[b] This class can occur only if there are no items to be picked to the left of aisle j.

Table 2: Transition from L_{j-1}^+ to L_j^- in Round-Trip equivalent classes from adding each of the arc configuration in Figure 2b

#	L_{j-1}^+ Equivalence Classes	Arc Configuration from Figure 2 (v)	(vi)	(vii)	(viii)
1	(U, U, 1C)	(U, U, 1C)	—[a]	—[a]	—[a]
2	(E, 0, 1C)	—[a]	(E, 0, 1C)	—[a]	(E, E, 2C)
3	(0, E, 1C)	—[a]	—[a]	(0, E, 1C)	(E, E, 2C)
4	(E, E, 1C)	—[a]	(E, 0, 1C)	(0, E, 1C)	(E, E, 1C)
5	(E, E, 2C)	—[a]	—[a]	—[a]	(E, E, 2C)
6	(U, U, 2C)	(U, U, 2C)	—[a]	—[a]	—[a]

Table 2a contains rows 1–5
Table 2b contains rows 1–6

L_{j-1}^+ Equivalence Classes	Arc configuration from Figure 2 (v)	(vi)	(vii)	(viii)
(U, U, 1C)	—[a]	(E, 0, 1C)	(0, E, 1C)	(E, E, 1C)
(E, 0, 1C)	(U, U, 2C)	—[a]	—[a]	—[a]
(0, E, 1C)	(U, U, 2C)	—[a]	—[a]	—[a]
(E, E, 1C)	—[a]	—[a]	—[a]	—[a]
(E, E, 2C)	(U, U, 2C)	—[a]	—[a]	—[a]

Table 2c: if the picker starts the tour on the head of the depot aisle.

[a]No completion can connect the graph

Table 3: Transition from L^+_{j-1} in Round-Trip equivalent classes to L^-_j in One-Way equivalent classes from adding each of the arc configuration in Figure 2a

L^+_{j-1} Equivalence	Arc configuration from Figure 2			
Classes	(i)	(ii)	(iii)	(iv)
(U, U, 1C)	—[a]	(0, U, 1C)	(E, U, 1C)	—[a]
(E, 0, 1C)	(U, 0, 1C)	—[a]	—[a]	—[a]
(0, E, 1C)	—[a]	—[a]	—[a]	(U, E, 2C)
(E, E, 1C)	(U, 0, 1C)	—[a]	—[a]	(U, E, 1C)
(E, E, 2C)	—[a]	—[a]	—[a]	(U, E, 2C)

Table 3a: for picker start at the head (a-point) of the aisle

L^+_{j-1} Equivalence	Arc configuration from Figure 2			
Classes	(i)	(ii)	(iii)	(iv)
(U, U, 1C)	(U, 0, 1C)	—[a]	—[a]	(U, E, 1C)
(E, 0, 1C)	—[a]	—[a]	(E, U, 2C)	—[a]
(0, E, 1C)	—[a]	(0, U, 1C)	—[a]	—[a]
(E, E, 1C)	—[a]	(0, U, 1C)	(E, U, 1C)	—[a]
(E, E, 2C)	—[a]	—[a]	(E, U, 2C)	—[a]

Table 3b: for picker start at the head (a-point) of the aisle

[a]No completion can connect the graph

Table 4: Transition from L^-_j to L^+_j in One-Way equivalent classes from adding each of the arc configuration in Figure 1a

	L^-_j Equivalence	Arc configuration of Figure 1					
#	Classes	(i)	(ii)	(iii)	(iv)	(v)	(vi)[a]
1	(U, 0, 1C)	(E, U, 1C)	(U, 0, 1C)	(U, E, 2C)	(U, E, 2C)	(U, E, 1C)	(U, 0, 1C)
2	(0, U, 1C)	(U, E, 1C)	(E, U, 2C)	(0, U, 1C)	(E, U, 2C)	(E, U, 1C)	(0, U, 1C)
3	(E, U, 1C)	(U, E, 1C)	(E, U, 1C)	(E, U, 1C)	(E, U, 1C)	(E, U, 1C)	(E, U, 1C)
4	(E, U, 2C)	(U, E, 1C)	(E, U, 2C)	(E, U, 2C)	(E, U, 2C)	(E, U, 1C)	(E, U, 2C)
5	(U, E, 1C)	(E, U, 1C)	(U, E, 1C)	(U, E, 1C)	(U, E, 1C)	(U, E, 1C)	(U, E, 1C)
6	(U, E, 2C)	(E, U, 1C)	(U, E, 2C)	(U, E, 2C)	(U, E, 2C)	(U, E, 1C)	(U, E, 2C)

Table 4a contains rows 1,2,3,5,6
Table 4b contains rows 1,2,3,4,5
Table 4c contains rows 1–6

[a]This is not a feasible configuration if there is any item to be picked in aisle j.

Table 5: Transition from L^+_{j-1} to L^-_j in One-Way equivalent classes from adding each of the arc configuration in Figure 2b

#	L^+_{j-1} Equivalence Classes	Arc configuration from Figure 2 (vi)	(vii)	(viii)	(ix)
1	(U, 0, 1C)	(U, 0, 1C)	—[a]	—[a]	(U, E, 2C)
2	(0, U, 1C)	—[a]	(0, U, 1C)	(E, U, 2C)	—[a]
3	(E, U, 1C)	—[a]	(0, U, 1C)	(E, U, 1C)	—[a]
4	(E, U, 2C)	—[a]	—[a]	(E, U, 2C)	—[a]
5	(U, E, 1C)	(U, 0, 1C)	—[a]	—[a]	(U, E, 1C)
6	(U, E, 2C)	—[a]	—[a]	—[a]	(U, E, 2C)

Table 5a contains rows 1,2,3,5,6
Table 5b contains rows 1–5
Table 5c contains rows 1–6

[a]No completion can connect the graph

Table 6: Transition from L^+_{j-1} in One-Way equivalent classes to L^-_j in Round-Trip equivalent classes from adding each of the arc configuration in Figure 2a

L^+_{j-1} Equivalence Classes	Arc configuration from Figure 2 (i)	(ii)	(iii)	(iv)
(U, 0, 1C)	(U, U, 2C)	—[a]	—[a]	—[a]
(0, U, 1C)	—[a]	—[a]	(0, E, 1C)	—[a]
(E, U, 1C)	—[a]	(E, 0, 1C)	(0, E, 1C)	(E, E, 1C)
(E, U, 2C)	—[a]	—[a]	—[a]	(E, E, 2C)
(U, E, 1C)	(U, U, 1C)	—[a]	—[a]	—[a]
(U, E, 2C)	(U, U, 2C)	—[a]	—[a]	—[a]

[a]No completion can connect the graph

4 Flow charts for route construction procedures

This section presents the detail procedures of applying the proposed Interventionist Routing Algorithm (IRA). Figure 3 and Figure 4 shows the initiation of applying the IRA. The former one describes the procedure when the current location of the picker is within an aisle, and the latter shows the set-up procedure when the current location of the picker is on either endpoint of the aisle. Finally, the procedures of route construction (i.e. the procedures of applying the tables in 3) for Round-Trip area and One-Way area are depicted in Figure 5 and Figure 6 respectively.

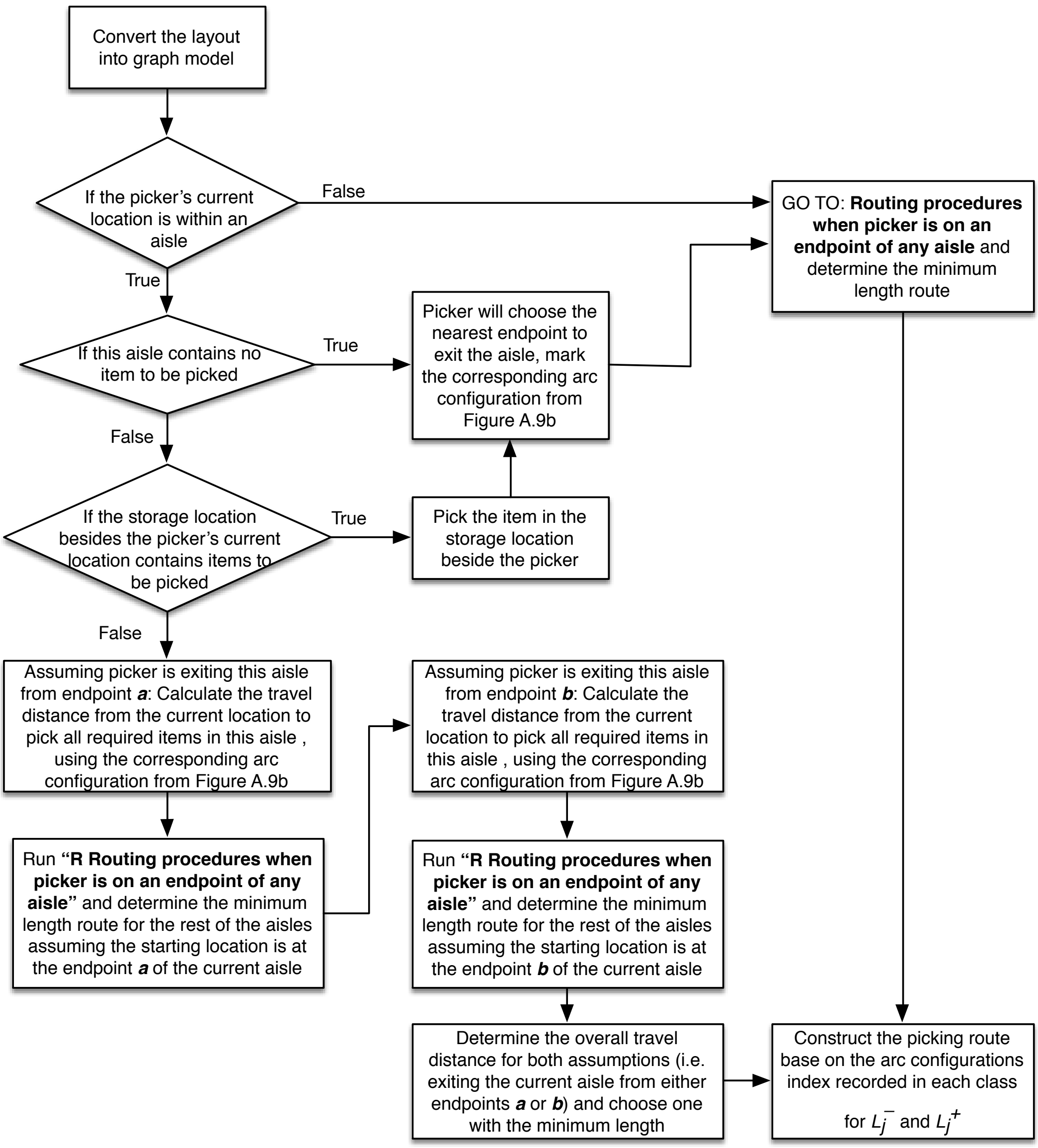


Figure 3: Top-level procedure

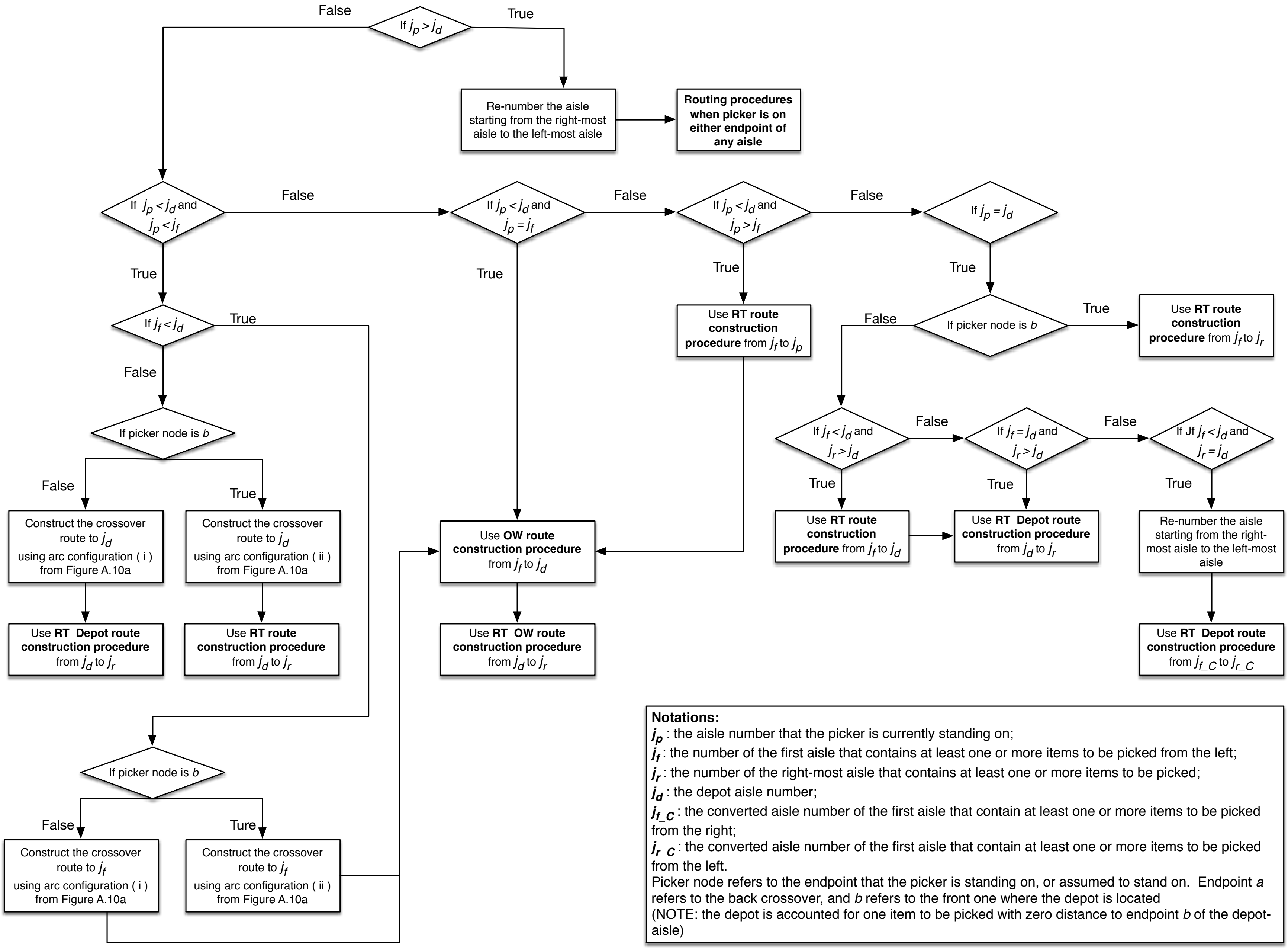


Figure 4: Routing procedures when picker is on an endpoint of any aisle

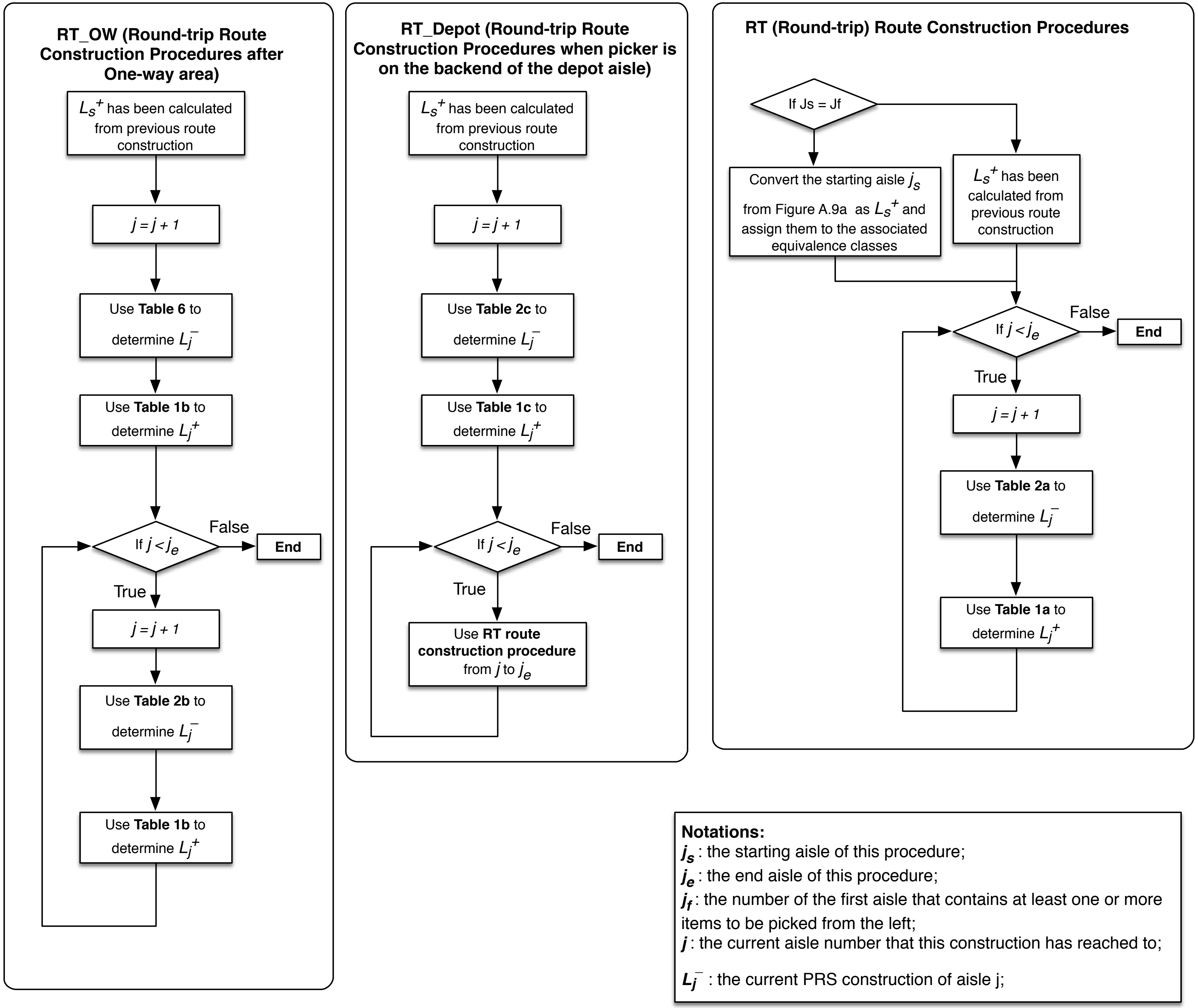


Figure 5: RT (Round-Trip) Route Construction Procedures

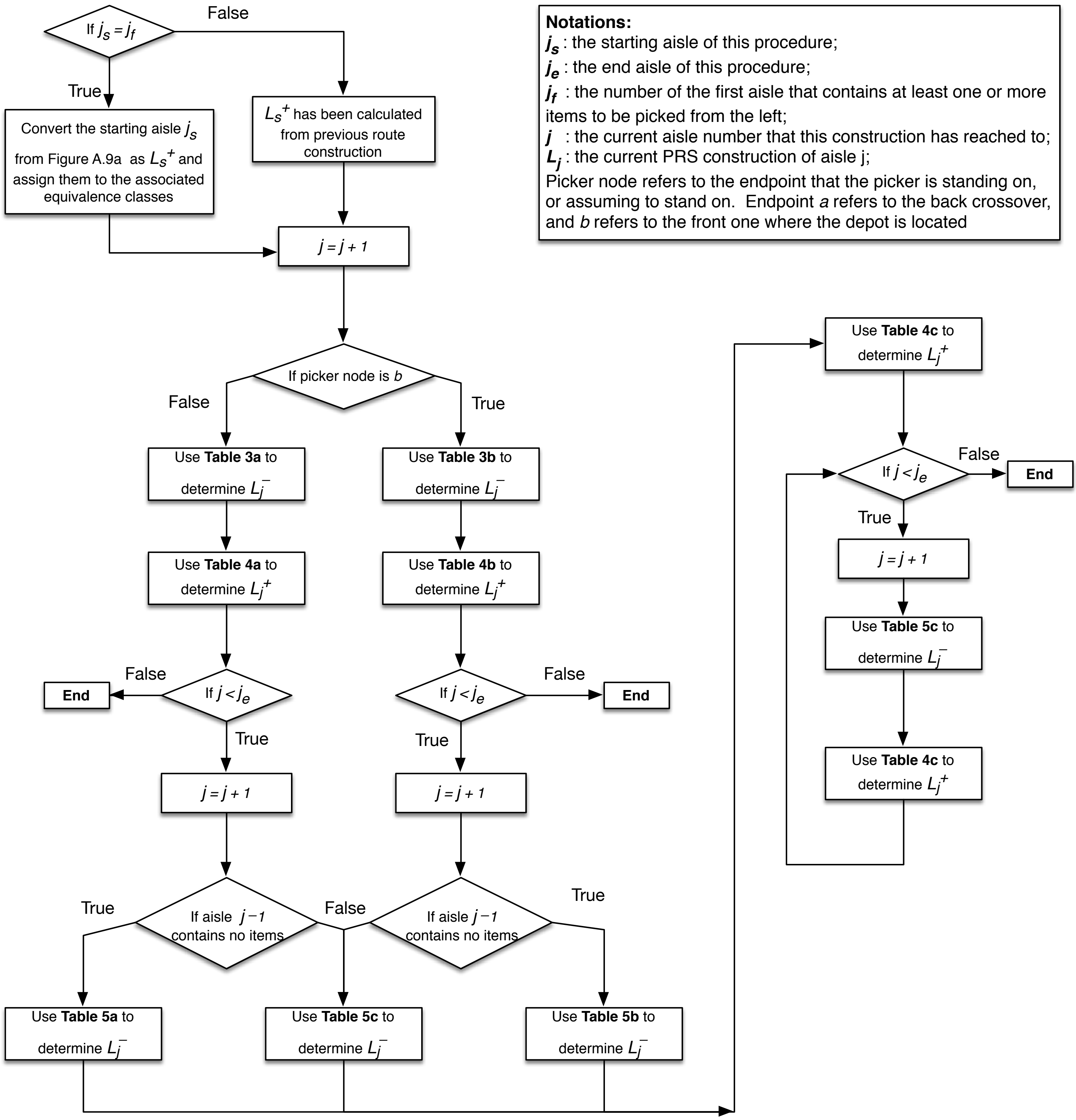


Figure 6: OW (One-way) Route Construction Procedures

5 Mathematical expression of the complexity

In order to examine the complexity of IRA, we therefore need to compare the number of all possible route configurations to the number of route configurations examined by IRA (after reducing the search space). Let us introduce the following notation for the mathematical expression:

Cal: the total number of calculations for determining the optimal route

K: the total number of equivalence classes for any PRS. According to Observations 1 and 2, $K = 12$.

z^-, z^+ : the index number of each equivalence class for L_j^- and L_j^+ respectively

$E_{jz^-}^-$: the total number of possible PRSs for equivalence class z^- of L_j^-

$E_{jz^+}^+$: the total number of possible PRSs for equivalence class z^+ of L_j^+

$\alpha_{jz^-z^+}$: for L_j^-, the total number of possible combinations to form equivalence class z^- using the equivalence class of z^+ from L_{j-1}^+ (determined by the tables for transition from L_{j-1}^+ to L_j^- in **??**)

$\beta_{jz^+z^-}$: for L_j^+, the total number of possible combinations to form equivalence class z^+ using the equivalence class of z^- from L_j^- (determined by the tables for transition from L_j^- to L_j^+ in **??**)

For a picking area with $(r - f)$ aisles, the total number of calculations for determining the optimal route among all possible routes (which is here equal to the number of all possible routes) can be calculated by the following equation[1] when $j = r$:

$$E_j^+ = \sum_{z^+=1}^{K} \sum_{z^-=1}^{K} \beta_{jz^+z^-} \times (\sum_{z^+=1}^{K} \alpha_{jz^-z^+} \times E_{(j-1)z^+}^+) \tag{1}$$

Notice, that this is a recursive equation that runs from $j = f + 1$ (using $E_{fz^+}^+ = 1$ for all z^+) till $j = r$ when the calculation terminates.

Equation 1 demonstrates that the number of calculations is not linear[2] to the number of aisles within the picking range (aisle $j \in [f, r]$)

However, as mentioned above, since the proposed algorithm uses the concept of equivalent PRS and choose the route configuration with the minimum length as the representative PRS, the algorithm do not consider other possible PRS as the calculation proceeds forwardly. i.e. $E_{jz^-}^- = E_{jz^+}^+ = 1$ for any $j \in (f, r)$ as long as there is a possible PRS for equivalence class z^+ of L_j^+ (otherwise $E_{jz^-}^- = E_{jz^+}^+ = 0$). Consequently the number of calculation for the proposed IRA can be determined by:

$$Cal(j) = \sum_{j=f+1}^{r} (E_j^- + E_j^+) = \sum_{j=f+1}^{r} (\sum_{z^-=1}^{K} \alpha_{jz^-} + \sum_{z^+=1}^{K} \beta_{jz^+}) \tag{2}$$

Equation 2 shows that the number of calculation solely depends on the density of the associated tables applied to aisle j, and it can be rearranged as

$$Cal(j) = Cal(j-1) + (\sum_{z^-=1}^{K} \alpha_{jz^-} + \sum_{z^+=1}^{K} \beta_{jz^+})$$

Therefore the number of calculation is polynomial and linear to the number of pick-aisle.

[1] The two main equations of this subsection are mathematically proved in 5.1

[2] Non-linearity is mathematically proved in 5.2

5.1 Proofs of Equations for the number of calculations

This section aims to prove the Equations 1 and 2. Using the notation in that section, for any aisle $f < j < r$, the total number of the equivalence class z^- for L_j^- PRSs is:

$$E_{jz^-}^- = \sum_{z^+=1}^{K} \alpha_{jz^-z^+} \times E_{(j-1)z^+}^+ \tag{3}$$

Therefore, the total number of L_j^- PRSs can be determined by calculating all the equivalence classes:

$$E_j^- = \sum_{z^-=1}^{K} E_{jz^-}^- = \sum_{z^-=1}^{K} \sum_{z^+=1}^{K} \alpha_{jz^-z^+} \times E_{(j-1)z^+}^+ \tag{4}$$

Similarly, the total number of the equivalence class z^+ for L_j^+ PRSs is:

$$E_{jz^+}^+ = \sum_{z^-=1}^{K} \beta_{jz^+z^-} \times E_{jz^-}^- \tag{5}$$

Therefore, the total number of L_j^+ PRSs can be determined by calculating all the equivalence classes:

$$E_j^+ = \sum_{z^+=1}^{K} E_{jz^+}^+ = \sum_{z^+=1}^{K} \sum_{z^-=1}^{K} \beta_{jz^+z^-} \times E_{jz^-}^- \tag{6}$$

Substitute Equation 3 into 6:

$$E_j^+ = \sum_{z^+=1}^{K} \sum_{z^-=1}^{K} \beta_{jz^+z^-} \times (\sum_{z^+=1}^{K} \alpha_{jz^-z^+} \times E_{(j-1)z^+}^+) \tag{7}$$

Since the number of calculation for obtaining the optimal route among all possible route is equal to the number of all possible route subgraphs for executing the picking operation, the number of calculation (i.e. the complexity) for exhaustive searching the optimal route can be determined using Equation 7 when $j = r$.

Since the proposed algorithm uses the concept of equivalent PRS and choose the route configuration with the minimum length as the representative PRS, the algorithm do not consider other possible PRS as the calculation proceeds forwardly. i.e. $E_{jz^-}^- = E_{jz^+}^+ = 1$ for any $j \in (f, r)$ as long as there is a possible PRS for equivalence class z^+ of L_j^+ (otherwise $E_{jz^-}^- = E_{jz^+}^+ = 0$). Consequently, the total number of L_j^- PRSs becomes:

$$E_j^- = \sum_{z^-=1}^{K} E_{jz^-}^- = \sum_{z^-=1}^{K} \alpha_{jz^-}$$

Similarly, the total number of L_j^+ PRSs becomes:

$$E_j^+ = \sum_{z^+=1}^{K} E_{jz^+}^+ = \sum_{z^+=1}^{K} \beta_{jz^+}$$

However, the proposed IRA will calculate all possible PRS as the route is constructed from $j = f$ to $j = r$ in order to determine the minimum length PRS for each j. Therefore the total number of calculations for $(r-f)$ aisles among the picking area using the proposed IRA is:

$$Cal = \sum_{j=f+1}^{r} (E_j^- + E_j^+) = \sum_{j=f+1}^{r} (\sum_{z^-=1}^{K} \alpha_{jz^-} + \sum_{z^+=1}^{K} \beta_{jz^+}) \tag{8}$$

5.2 Proof of non-linearity for Equation 1

If we expand Equation 3 for all $z^+ = 1, 2, \ldots, K$:

$$E_{jz^-}^- = \alpha_{jz^-1} \times E_{(j-1)1}^+ + \alpha_{jz^-2} \times E_{(j-1)2}^+ + \cdots + \alpha_{jz^-K} \times E_{(j-1)K}^+$$

Then expand Equation 5 for all $z^- = 1, 2, \ldots, K$

$$\begin{aligned} E_{jz^+}^+ &= \beta_{jz^+1} \times E_{j1}^- + \beta_{jz^+2} \times E_{j2}^- + \cdots + \beta_{jz^+K} \times E_{jK}^- \\ &= \beta_{jz^+1} \times (\alpha_{j11} \times E_{(j-1)1}^+ + \alpha_{j21} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK1} \times E_{(j-1)K}^+) \\ &+ \beta_{jz^+2} \times (\alpha_{j12} \times E_{(j-1)1}^+ + \alpha_{j22} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK2} \times E_{(j-1)K}^+) \\ &+ \ldots \\ &+ \beta_{jz^+K} \times (\alpha_{j1K} \times E_{(j-1)1}^+ + \alpha_{j2K} \times E_{(j-1)2}^+ + \cdots + \alpha_{jKK} \times E_{(j-1)K}^+) \end{aligned}$$

Lastly, we expand and rearrange the Equation 6 for all $z^+ = 1, 2, \ldots, K$:

$$\begin{aligned} E_j^+ &= E_{j1}^+ + E_{j2}^+ + \cdots + E_{jK}^+ \\ &= [\beta_{j11} \times (\alpha_{j11} \times E_{(j-1)1}^+ + \alpha_{j21} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK1} \times E_{(j-1)K}^+) \\ &+ \beta_{j12} \times (\alpha_{j12} \times E_{(j-1)1}^+ + \alpha_{j22} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK2} \times E_{(j-1)K}^+) + \ldots \\ &+ \beta_{j1K} \times (\alpha_{j1K} \times E_{(j-1)1}^+ + \alpha_{j2K} \times E_{(j-1)2}^+ + \cdots + \alpha_{jKK} \times E_{(j-1)K}^+)] \\ &+ [\beta_{j21} \times (\alpha_{j11} \times E_{(j-1)1}^+ + \alpha_{j21} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK1} \times E_{(j-1)K}^+) \\ &+ \beta_{j22} \times (\alpha_{j12} \times E_{(j-1)1}^+ + \alpha_{j22} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK2} \times E_{(j-1)K}^+) + \ldots \\ &+ \beta_{j2K} \times (\alpha_{j1K} \times E_{(j-1)1}^+ + \alpha_{j2K} \times E_{(j-1)2}^+ + \cdots + \alpha_{jKK} \times E_{(j-1)K}^+)] \\ &+ \ldots \\ &+ [\beta_{jK1} \times (\alpha_{j11} \times E_{(j-1)1}^+ + \alpha_{j21} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK1} \times E_{(j-1)K}^+) \\ &+ \beta_{jK2} \times (\alpha_{j12} \times E_{(j-1)1}^+ + \alpha_{j22} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK2} \times E_{(j-1)K}^+) + \ldots \\ &+ \beta_{jKK} \times (\alpha_{j1K} \times E_{(j-1)1}^+ + \alpha_{j2K} \times E_{(j-1)2}^+ + \cdots + \alpha_{jKK} \times E_{(j-1)K}^+)] \\ &= (\alpha_{j11} \times E_{(j-1)1}^+ + \alpha_{j21} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK1} \times E_{(j-1)K}^+) \times (\beta_{j11} + \beta_{j21} + \cdots + \beta_{jK1}) \\ &+ (\alpha_{j12} \times E_{(j-1)1}^+ + \alpha_{j22} \times E_{(j-1)2}^+ + \cdots + \alpha_{jK2} \times E_{(j-1)K}^+) \times (\beta_{j12} + \beta_{j22} + \cdots + \beta_{jK2}) \\ &+ \ldots \\ &+ (\alpha_{j1K} \times E_{(j-1)1}^+ + \alpha_{j2K} \times E_{(j-1)2}^+ + \cdots + \alpha_{jKK} \times E_{(j-1)K}^+) \times (\beta_{j1K} + \beta_{j2K} + \cdots + \beta_{jKK}) \\ &= [\alpha_{j11} \times (\beta_{j11} + \beta_{j21} + \cdots + \beta_{jK1}) + \cdots + \alpha_{j1K} \times (\beta_{j1K} + \beta_{j2K} + \cdots + \beta_{jKK})] \times E_{(j-1)1}^+ \\ &+ [\alpha_{j21} \times (\beta_{j11} + \beta_{j21} + \cdots + \beta_{jK1}) + \cdots + \alpha_{j2K} \times (\beta_{j1K} + \beta_{j2K} + \cdots + \beta_{jKK})] \times E_{(j-1)2}^+ \\ &+ \ldots \\ &+ [\alpha_{jK1} \times (\beta_{j11} + \beta_{j21} + \cdots + \beta_{jK1}) + \cdots + \alpha_{jKK} \times (\beta_{j1K} + \beta_{j2K} + \cdots + \beta_{jKK})] \times E_{(j-1)K}^+ \end{aligned}$$

Therefore, the equation above can be written as:

$$E_{j1}^{+} + E_{j2}^{+} + \cdots + E_{jK}^{+} = A_1 \times E_{(j-1)1}^{+} + A_2 \times E_{(j-1)2}^{+} + \cdots + A_K \times E_{(j-1)K}^{+}$$

where $A_1 \neq A_2 \neq \cdots \neq A_K$, and hence the equation is non-linear with the increase of aisle j.

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