

# Profinite rigidity and surface bundles over the circle

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## Abstract

If  $M$  is a compact 3-manifold whose first betti number is 1, and  $N$  is a compact 3-manifold such that  $\pi_1 N$  and  $\pi_1 M$  have the same finite quotients, then  $M$  fibres over the circle if and only if  $N$  does. We prove that groups of the form  $F_2 \rtimes \mathbb{Z}$  are distinguished from one another by their profinite completions. Thus, regardless of betti number, if  $M$  and  $N$  are punctured torus bundles over the circle and  $M$  is not homeomorphic to  $N$ , then there is a finite group  $G$  such that one of  $\pi_1 M$  and  $\pi_1 N$  maps onto  $G$  and the other does not.

## 1 Introduction

When one wants to understand a finitely presented group it is natural to explore its finite quotients, and this is a well-trodden path in many contexts. For instance, one might try to prove that a presentation does not represent the trivial group by exhibiting a map onto a non-trivial finite group, or one might try to prove that two groups are not isomorphic by counting maps to small finite groups. The potential of such techniques depends on the extent to which the groups being studied are determined by the totality of their finite quotients. If the groups  $\Gamma$  at hand are residually finite, i.e. every finite subset injects into some finite quotient, then it is reasonable to expect that one will be able to detect many properties of  $\Gamma$  from the totality of its finite quotients.

Attempts to lend precision to this observation, and to test its limitations, have surfaced repeatedly over the last forty years. There has been a particular resurgence of interest in recent years in the context of low-dimensional topology, where the central problem is that of distinguishing between compact 3-manifolds  $M$  and  $N$  by finding a finite quotient of  $\pi_1 M$  that is not a quotient of  $\pi_1 N$ . In more sophisticated terminology, one wants to develop a complete understanding of the circumstances in which fundamental groups

of non-homeomorphic manifolds  $M$  and  $N$  can have isomorphic *profinite completions*  $\widehat{\pi_1 M}$  and  $\widehat{\pi_1 N}$ . (The profinite completion  $\widehat{G}$  of a discrete group  $G$  is the inverse limit of the inverse system of finite quotients of  $G$ .)

There has been a good deal of progress on this question recently: Boileau and Friedl [7] proved, among other things, that for closed 3-manifolds with  $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$ , being fibred is an invariant of the profinite completion; using very different methods, Bridson and Reid [9] proved that if  $M$  is a compact manifold with non-empty boundary that fibres and has first betti number 1, and if  $N$  is a compact 3-manifold with  $\widehat{\pi_1 N} \cong \widehat{\pi_1 M}$ , then  $N$  has non-empty boundary and fibres; and if  $\pi_1 M$  has the form  $F_r \rtimes \mathbb{Z}$ , with  $F_r$  free of rank  $r$ , then so does  $\pi_1 N$  (but we do not know if the actions of  $\mathbb{Z}$  on  $F_r$  can be different). It follows, for example, that the complement of the figure-8 knot is distinguished from all other 3-manifolds by the profinite completion of its fundamental group [7, 9].

In the negative direction, Funar [13] pointed out that old results of Stebe [22] imply that torus bundles over the circle with Sol geometry cannot, in general, be distinguished from one another by the profinite completions of their fundamental groups  $\mathbb{Z}^2 \rtimes \mathbb{Z}$ . By adapting arguments of Baumslag [5], Hempel [15] exhibited a similar phenomenon among bundles with higher genus fibres and finite monodromy; see also [23].

In this paper we advance the understanding of profinite rigidity for surface bundles over the circle in two ways. First, taking up the theme of [9], we show that in the case of punctured-torus bundles over the circle, the monodromy of the bundle is determined by the profinite completion of the fundamental group and, moreover, profinite rigidity persists if one drops the hypothesis  $b_1(M) = 1$ . Secondly, we extend the fibring theorems of Boileau–Friedl and Bridson–Reid to the case of all bundles  $M$  with compact fibre and  $b_1(M) = 1$  (Theorem C).

To state the first of these results more precisely, we define  $\Sigma_{1,1}$  to be the once-punctured torus, and for any  $\phi$  in the extended mapping class group  $\text{Mod}^\pm(\Sigma_{1,1}) \cong \text{GL}(2, \mathbb{Z}) \cong \text{Out}(F_2)$ , let  $M_\phi$  be the mapping torus of (a homeomorphism representing)  $\phi$ . Let  $F_2$  denote the non-abelian free group of rank 2.

**Theorem A.** *Let  $\phi_1, \phi_2 \in \text{Out}(F_2)$  and let  $\Gamma_i = F_2 \rtimes_{\phi_i} \mathbb{Z}$ . If  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ , then  $\phi_1$  is conjugate to  $\phi_2$  in  $\text{Out}(F_2) = \text{GL}(2, \mathbb{Z})$ , hence  $\Gamma_1 \cong \Gamma_2$  and  $M_{\phi_1}$  is homeomorphic to  $M_{\phi_2}$ .*

**Corollary B.** *Let  $M$  be a hyperbolic 3-manifold that fibres over the circle with fibre a one-holed torus, and let  $N$  be a compact connected 3-manifold. If  $\widehat{\pi_1 N} \cong \widehat{\pi_1 M}$ , then  $N$  is homeomorphic to  $M$ .*

To deduce this corollary, first observe that since  $M$  is hyperbolic, its first betti number is 1 and  $\pi_1 M$  has the form  $F_2 \rtimes_{\phi} \mathbb{Z}$  with  $\phi \in \mathrm{GL}(2, \mathbb{Z})$  a hyperbolic matrix. [9, Theorem B] states that  $N$  has non-empty boundary, is a bundle with compact fibre of euler characteristic  $-1$ , and  $\pi_1 N \cong F_2 \rtimes_{\psi} \mathbb{Z}$ . Theorem A tells us that  $\psi$ , which describes the monodromy of  $N$ , is conjugate to  $\phi$  and therefore is hyperbolic. The one-holed torus is the only compact surface of Euler characteristic  $-1$  that supports a hyperbolic automorphism, so  $N$  is a once-holed torus bundle with the same monodromy as  $M$ , and hence  $N \cong M$ .

Although our results are concrete, the key facts that we exploit are abstract properties of mapping class groups. For Theorem A we use the fact that  $\mathrm{Mod}^{\pm}(\Sigma_{1,1})$  is *omnipotent* and enjoys the *congruence subgroup property* (see Section 2). Corresponding results for mapping class groups of surfaces of higher complexity are beyond the reach of current techniques. However, if one assumes those properties of mapping class groups, then one can obtain similar results for bundles with higher-genus fibre (see Theorem 2.4); our proof of Theorem A is presented in a manner that emphasizes this general strategy.

Our other main result completes one step in the strategy by establishing that fibring is a profinite invariant for manifolds with first betti number 1: this is achieved by combining Theorem C with the corresponding result in the case of manifolds with boundary [9].

**Theorem C.** *Let  $M$  be a closed orientable hyperbolic 3-manifold with first betti number  $b_1(M) = 1$  that is a bundle with fibre a closed surface  $\Sigma$  of genus  $g$ . Let  $N$  be a compact 3-manifold with  $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$ . Then  $N$  is also a closed orientable hyperbolic 3-manifold with  $b_1(N) = 1$  that is a bundle with fibre a closed surface of genus  $g$ .*

In [7], this theorem was proved under the assumption that  $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$ , using different methods: we avoid their use of twisted Alexander polynomials, relying instead on topological arguments.

Throughout, we assume that the reader is familiar with elementary facts about profinite groups, as described in [20] for example.

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## 2 Congruence omnipotence

In this section we define the notion of *congruence omnipotence* in  $\text{Out}(G)$ , for  $G$  a finitely generated group. Our main theorem (Theorem 2.4) asserts that, when it holds, congruence omnipotence enables one to deduce profinite rigidity results for mapping tori  $G \rtimes \mathbb{Z}$ .

**Definition 2.1.** Let  $G$  be a finitely generated group and let  $H \subseteq \text{Out}(G)$  be a subgroup. A finite quotient  $H \rightarrow Q$  is a  *$G$ -congruence quotient* if it factors through  $\pi : H \rightarrow P \subset \text{Out}(G/K)$  where  $K$  is a characteristic subgroup of finite index in  $G$  and  $\pi$  is the restriction of the natural map  $\text{Out}(G) \rightarrow \text{Out}(G/K)$ . We say that  $\text{Out}(G)$  *has the congruence subgroup property* if every finite quotient of  $\text{Out}(G)$  is a  $G$ -congruence quotient. More generally, we say that a subgroup  $H \subseteq \text{Out}(G)$  has the  *$G$ -congruence subgroup property* if every finite quotient of  $H$  is a  $G$ -congruence quotient.

*Remark.* Care is needed in the above definition: there may be distinct groups  $G_1$  and  $G_2$  with  $\text{Out}(G_1) \cong \text{Out}(G_2)$  such that every finite quotient is congruence with respect to  $G_1$  but not with respect to  $G_2$ . For instance, this phenomenon occurs with  $\text{Out}(F_2) \cong GL(2, \mathbb{Z}) = \text{Out}(\mathbb{Z}^2)$ , which has the congruence subgroup property with respect to  $F_2$  but not  $\mathbb{Z}^2$ . Thus “ $\text{Out}(G)$  has the congruence subgroup property” is a statement about  $G$  and not the abstract group  $\text{Out}(G)$ .

*Omnipotence* was first defined by Wise in the context of free and hyperbolic groups.

**Definition 2.2.** Let  $\Gamma$  be a group. Elements  $\gamma_1, \gamma_2 \in \Gamma$  of infinite order are said to be *independent* if no non-zero power of  $\gamma_1$  is conjugate to a non-zero power of  $\gamma_2$  in  $\Gamma$ . An  $m$ -tuple  $(\gamma_1, \dots, \gamma_m)$  of elements is *independent* if  $\gamma_i$  and  $\gamma_j$  are independent whenever  $1 \leq i < j \leq m$ . The group  $\Gamma$  is said to be *omnipotent* if, for every independent  $m$ -tuple  $(\gamma_1, \dots, \gamma_m)$  of elements in  $\Gamma$ , there exists a positive integer  $\kappa$  such that, for every  $m$ -tuple of positive integers  $(e_1, \dots, e_m)$  there is a homomorphism to a finite group

$$q : \Gamma \rightarrow Q$$

such that  $o(q(\gamma_i)) = \kappa e_i$  for  $i = 1, \dots, m$ , where  $o(g)$  denotes the order of a group element  $g$ . For a subgroup  $H$  of  $\Gamma$ , if we wish to emphasize that an  $m$ -tuple of elements is independent in  $H$ , we will say that the tuple is *H-independent*.

We focus on a more restrictive form of omnipotence that is adapted to our purposes. Two motivating examples that we have in mind are: (i) where  $G$  is a closed surface group or a free group and  $H = \text{Out}(G)$ ; and (ii) where  $G$  is a free group and  $H \subseteq \text{Out}(G)$  is the mapping class group of a punctured surface. In these contexts, there is usually a favoured class of elements for which one expects omnipotence to hold, e.g. pseudo-Anosovs in the case of mapping class groups, or fully irreducible elements in the case of  $\text{Out}(F_n)$ ; we therefore work with subsets  $\mathcal{S} \subseteq H$ . We also insist that the finite quotients obtained should be  $G$ -congruence quotients.

**Definition 2.3.** Let  $G$  be a finitely generated group, let  $H$  be a subgroup of  $\text{Out}(G)$  and let  $\mathcal{S}$  be a subset of  $H$ . We say<sup>1</sup> that  $\mathcal{S}$  is *(G, H)-congruence omnipotent* if, for every  $m$  and every  $H$ -independent  $m$ -tuple  $(\phi_1, \dots, \phi_m)$  of elements of  $\mathcal{S}$ , there is a constant  $\kappa$  such that, for any  $m$ -tuple of positive integers  $(n_1, \dots, n_m)$ , there is a  $G$ -congruence quotient  $q : H \rightarrow Q$  such that  $o(q(\phi_i)) = \kappa n_i$  for all  $i$ .

*Remark.* If  $\text{Out}(G)$  is omnipotent and has the congruence subgroup property, then the set of infinite-order elements is  $G$ -congruence omnipotent in  $\text{Out}(G)$ .

Our most general theorem shows how congruence omnipotence can be used as a tool for establishing profinite rigidity for the mapping tori associated to automorphisms of a fixed group  $G$ .

**Theorem 2.4.** *Let  $G$  be a finitely generated group, let  $H \subseteq \text{Out}(G)$  be a subgroup and let  $\mathcal{S}$  be a  $(G, H)$ -congruence omnipotent subset. Let  $\phi_1, \phi_2 \in \mathcal{S}$ , let  $\Gamma_i = G \rtimes_{\phi_i} \mathbb{Z}$  and suppose that  $b_1(\Gamma_i) = 1$  for  $i = 1, 2$ . If  $\widehat{\Gamma}_1 = \widehat{\Gamma}_2$  then there is an integer  $n$  such that  $\phi_1^n$  is conjugate in  $H$  to  $\phi_2^{\pm n}$ .*

The key observation in the proof of Theorem 2.4 is contained in the following lemma.

**Lemma 2.5.** *Let  $\Gamma_i = G \rtimes_{\phi_i} \mathbb{Z}$  for  $i = 1, 2$ . If  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$  and  $b_1(\Gamma_i) = 1$  for  $i = 1, 2$ , then the image of  $\phi_1$  and  $\phi_2$  generate the same cyclic subgroup in the outer automorphism group of any characteristic quotient of  $G$ .*

<sup>1</sup>if  $H = \text{Out}(G)$  we abbreviate this to “ $G$ -congruence omnipotent”

*Proof.* We fix an identification  $\widehat{\Gamma}_1 = \widehat{\Gamma}_2$ . The unique epimorphism  $\Gamma_i \rightarrow \mathbb{Z}$  defines a short exact sequence

$$1 \rightarrow \widehat{G} \rightarrow \widehat{\Gamma}_i \rightarrow \widehat{\mathbb{Z}} \rightarrow 1.$$

If  $K < G$  is a characteristic subgroup of finite index, then the canonical map  $G \rightarrow G/K$  defines an epimorphism  $\widehat{G} \rightarrow G/K$ . Since  $\widehat{K}$  is normal in  $\widehat{\Gamma}_i$ , the action of  $\widehat{\mathbb{Z}}$  on  $\widehat{G}$  induced by conjugation in  $\widehat{\Gamma}_i$  descends to an action on  $\widehat{G}/\widehat{K} = G/K$ , defining a cyclic subgroup  $C < \text{Out}(G/K)$ , of order  $m$  say. The righthand factor of  $\Gamma_i = G \rtimes_{\phi_i} \mathbb{Z}$  is dense in  $\widehat{\mathbb{Z}}$ , so the image of  $\phi_i$  generates  $C$  for  $i = 1, 2$ .  $\square$

*Proof of Theorem 2.4.* Suppose first that  $\phi_1$  and  $\phi_2$  are  $H$ -independent. Since  $\mathcal{S}$  is  $(G, H)$ -congruence omnipotent, there exists a  $G$ -congruence quotient  $q : H \rightarrow Q$  such that  $o(q(\phi_1)) \neq o(q(\phi_2))$ . By definition,  $q$  factors as

$$H \rightarrow P \rightarrow Q$$

with  $H \rightarrow P$  the restriction of the natural map  $\text{Out}(G) \rightarrow \text{Out}(G/K)$  for some characteristic subgroup  $K$  of finite index in  $G$ . Since the images of  $\phi_1$  and  $\phi_2$  have distinct orders in  $Q$ , the images of  $\phi_1$  and  $\phi_2$  cannot generate the same cyclic subgroup of  $\text{Out}(G/K)$ , contradicting Lemma 2.5.

Therefore,  $\phi_1$  and  $\phi_2$  are not  $H$ -independent, so there are positive integers  $n_1$  and  $n_2$  such that  $\phi_1^{n_1}$  is conjugate to  $\phi_2^{\pm n_2}$  in  $H$ . It remains to prove that  $n_1 = n_2$ . By congruence omnipotence applied to the 1-tuple  $(\phi_1)$ , there is a characteristic subgroup  $K$  of finite index in  $G$  such that  $n_1 n_2$  divides  $o(q(\phi_1))$ , where  $q : \text{Out}(G) \rightarrow \text{Out}(G/K)$  is the natural homomorphism. In particular, we have

$$o(q(\phi_1))/n_1 = o(q(\phi_1^{n_1})) = o(q(\phi_2^{n_2})) = o(q(\phi_2))/n_2$$

and so, since Lemma 2.5 implies that  $o(q(\phi_1)) = o(q(\phi_2))$ , we have  $n_1 = n_2$  as claimed.  $\square$

## 2.1 $\text{Out}(F_2)$ is congruence omnipotent

The congruence subgroup property for  $\text{Out}(F_2)$  was established by Asada [3]; alternative proofs were given by Bux–Ershov–Rapinchuk [11] and Ellenberg–McReynolds [12].

**Theorem 2.6** (Asada [3]). *For any finite quotient  $\text{Out}(F_2) \rightarrow Q$  there is a characteristic finite-index subgroup  $K$  of  $F_2$  such that the quotient map factors as*

$$\text{Out}(F_2) \rightarrow \text{Out}(F_2/K) \rightarrow Q .$$

Wise proved that finitely generated free groups are omnipotent and later extended his proof to virtually special groups [25]. Bridson and Wilton gave a more direct proof that virtually free groups are omnipotent [10]. As  $\text{Out}(F_2)$  is virtually free, in the light of Theorem 2.6 we have:

**Proposition 2.7.** *The set of elements of infinite order in  $\text{Out}(F_2)$  is  $F_2$ -congruence omnipotent.*

### 3 Profinite rigidity for punctured-torus bundles

Our proof of Theorem A relies on a number of elementary calculations in  $\text{GL}(2, \mathbb{Z}) \cong \text{Out}(F_2)$ ; we have relegated these to an appendix, so as not to disturb the flow of our main argument. The reader may wish to read that appendix before proceeding with this section.

#### 3.1 Reducing to the hyperbolic case

**Theorem 3.1.** *Let  $\phi_1, \phi_2 \in \text{Out}(F_2)$ , let  $\Gamma_i = F_2 \rtimes_{\phi_i} \mathbb{Z}$  and suppose that  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ .*

1. *If  $\phi_1$  is hyperbolic then  $\phi_2$  is hyperbolic.*
2. *If  $\phi_1$  is not hyperbolic, then  $\phi_2$  is conjugate<sup>2</sup> to  $\phi_1$ .*

*Proof.* Item (1) was proved in [9, Proposition 3.2] by arguing  $\phi$  is hyperbolic if and only if  $b_1(\Gamma_{\phi^r}) = 1$  for all  $r > 0$ . (The “only if” implication follows easily from Lemma A.3.) We therefore proceed to prove Item (2).

Proposition A.2 tells us that the list of abelianisations of  $\Gamma_{\phi}$  calculated in Lemma A.3 covers all non-hyperbolic  $\phi$ . If all of these groups were non-isomorphic then we would be done, but there remain two ambiguities for which we make special arguments.

First, to distinguish  $\widehat{\Gamma}_{-I}$  from  $\widehat{\Gamma}_{-U(n)}$  with  $n > 0$  even, we can invoke Lemma 2.5: since  $b_1(\Gamma_{-I}) = b_1(\Gamma_{-U(n)}) = 1$ , the automorphisms of  $H_1(F_2, \mathbb{Z}/(n+1))$  induced by  $-I$  and  $-U(n)$  would have the same order if  $\widehat{\Gamma}_{-I} \cong \widehat{\Gamma}_{-U(n)}$ , but the former has order 2 and the latter is

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

which has order  $n+1$ .

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<sup>2</sup> $\phi$  is always conjugate to  $\phi^{-1}$  if  $\phi$  is not hyperbolic

Second, to distinguish  $\widehat{\Gamma}_\epsilon$  from  $\widehat{\Gamma}_{U(2)}$ , we note that  $t^2$  is central in  $\Gamma_\epsilon = F_2 \rtimes_\epsilon \langle t \rangle$  and has infinite order in  $H_1(\Gamma_\epsilon, \mathbb{Z})$ , so if  $Q$  is any finite quotient of  $\Gamma_\epsilon$  then the abelianisation of  $Q/Z(Q)$  is a quotient of  $\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ . On the other hand,  $F_2 \rtimes_{U(2)} \mathbb{Z}$  maps onto the mod-3 Heisenberg group

$$(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \rtimes_{-B} \mathbb{Z}_3,$$

and the quotient of this group by its centre is  $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ .  $\square$

During the course of the proof of Theorem A, we will need to argue that  $\widehat{\Gamma}_\phi \not\cong \widehat{\Gamma}_{-\phi}$ . The required calculation can be found in [9, Lemma 3.5], which we reproduce below for the reader's convenience.

**Lemma 3.2.**

1.  $b_1(\Gamma_\phi) = 1$  if and only if  $1 + \det \phi \neq \text{tr } \phi$ .
2. If  $b_1(\Gamma_\phi) = 1$  then  $H_1(\Gamma_\phi, \mathbb{Z}) \cong \mathbb{Z} \oplus T$ , where  $|T| = |1 + \det \phi - \text{tr } \phi|$ .

*Proof.* By choosing a representative  $\phi_* \in \text{Aut}(F_2)$ , we get a presentation for  $\Gamma_\phi$ ,

$$\langle a, b, t \mid tat^{-1} = \phi_*(a), tbt^{-1} = \phi_*(b) \rangle.$$

By abelianising, we see that  $H_1(\Gamma_\phi, \mathbb{Z})$  is the direct sum of  $\mathbb{Z}$  (generated by the image of  $t$ ) and  $\mathbb{Z}^2$  modulo the image of  $\phi - I$ . The image of  $\phi - I$  has finite index if and only if  $\det(\phi - I)$  is non-zero, and a trivial calculation shows that this determinant is  $1 - \text{tr } \phi + \det \phi$ . If the index is finite, then the quotient has order  $|\det(\phi - I)|$ .  $\square$

### 3.2 End of the proof: the hyperbolic case

*Proof of Theorem A.* By Theorem 3.1, we may assume that both  $\phi_1$  and  $\phi_2$  are hyperbolic. In particular,  $b_1(\Gamma_1) = b_1(\Gamma_2) = 1$  and so, by Proposition 2.7, we may invoke Theorem 2.4 to deduce that there is an  $n$  such that  $\phi_1^n = \phi_2^{\pm n}$ . It then follows from Lemma A.1 that  $\phi_1$  is conjugate to  $\pm \phi_2^{\pm 1}$ . To remove the possibility that  $\phi_1$  is conjugate to  $-\phi_2^{\pm 1}$  we use Lemma 3.2 to compare the order of the torsion subgroup in  $H_1(\Gamma_\phi, \mathbb{Z})$  with that in  $H_1(\Gamma_{-\phi}, \mathbb{Z})$ , noting that  $\det(-\phi) = \det \phi$  but  $\text{tr}(-\phi) = -\text{tr } \phi$ .  $\square$

## 4 Closed hyperbolic bundles with $b_1(M) = 1$

We now shift our attention to *closed* 3-manifolds. Our purpose in this section is to prove Theorem C. We therefore assume that  $M$  is a closed, orientable,



hyperbolic 3-manifold with  $b_1M = 1$ , fibring over the circle with fibre  $\Sigma$ , and that  $N$  is a closed, orientable 3-manifold with  $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ . We have been informed by Boileau and Friedl that the methods of their paper [7] can be used to give a different proof of Theorem C. Extending the result to bundles with  $b_1(M) > 1$  lies beyond the present scope of both our techniques and theirs.

#### 4.1 The main argument

By arguing as in Theorem 4.1 of [9], we may assume that  $N$  is aspherical, closed and orientable. Since the finite abelian quotients of  $\pi_1 N$  coincide with those of  $\pi_1 M$ , we also see that  $b_1(N) = 1$ . And by [24] we know that  $N$  is hyperbolic.

Set  $\Delta = \pi_1(N)$  and  $\Gamma = \pi_1(M)$ . There is a unique epimorphism  $\Delta \rightarrow \mathbb{Z}$ , and dual to this we can find a closed embedded non-separating incompressible surface  $S \subset N$ . If  $S$  is a fibre we will be done, for exactly as in Lemma 3.1 of [9], we get  $\widehat{\pi_1(S)} \cong \widehat{\pi_1(\Sigma)}$ , from which it easily follows (by noting, for example, that  $H_1(\Sigma, \mathbb{Z})$  is determined by  $\widehat{\pi_1 \Sigma}$ ) that  $S$  is homeomorphic to  $\Sigma$ .

Thus, in order to complete our proof of the theorem, it suffices to derive a contradiction from the assumption that  $S$  is not a fibre. The well-known dichotomy of Bonahon and Thurston [8] implies that if  $S$  is not a fibre then it is quasi-Fuchsian.

Let  $H = \pi_1(S) < \Delta$ , let  $G = \pi_1 \Sigma$ , let  $K < \Delta$  be the kernel of the unique epimorphism  $\Delta \rightarrow \mathbb{Z}$ , let  $\overline{K}$  denote the closure of  $K$  in  $\widehat{\Delta}$ , and note that  $H < K$ . It is elementary to see that  $\Gamma$  induces the full profinite topology on  $G$  (see [9, Lemma 2.2] for example), and it follows from Agol's virtually special theorem [1] that, since  $H$  is quasiconvex and hence a virtual retract, the full profinite topology is induced on  $H$  (see, for example, [4, (L.16), p. 120]).

The isomorphism  $\widehat{\Gamma} \cong \widehat{\Delta}$  identifies  $\widehat{G}$  with  $\overline{K}$  (each being the kernel of the unique epimorphism  $\widehat{\Gamma} \rightarrow \widehat{\mathbb{Z}}$ ). Thus  $\widehat{H} = \overline{H}$  is a subgroup of  $\widehat{G}$ . To complete the proof of the theorem, we need two lemmas. The first of these lemmas relies on Agol's theorem [1], while the second is based on a standard exercise about duality groups at a prime  $p$  that was drawn to our attention by P. Zalesskii.

**Lemma 4.1.** *There is a finite index subgroup  $\Delta_0 < \Delta$  such that:*

1.  $H < \Delta_0$  ;

2. there exists an epimorphism  $f : \Delta_0 \rightarrow F_2$  (a free group of rank 2) such that  $H < \ker f$ ).

**Lemma 4.2.**  $[\widehat{G} : \widehat{H}] < \infty$ .

We defer the proofs of these lemmas for a moment while we complete the proof of the theorem.

Define  $\Gamma_0 = \Gamma \cap \widehat{\Delta}_0$  and  $G_0 = G \cap \Gamma_0$ . The surjection  $\Delta_0 \rightarrow F_2$  of Lemma 4.1 induces an epimorphism  $\widehat{\Gamma}_0 = \widehat{\Delta}_0 \rightarrow \widehat{F}_2$ , the kernel of which contains  $\widehat{H}$ . It follows from Lemma 4.2 that the image of  $\widehat{G}_0$  in  $\widehat{F}_2$  is finite. But  $\widehat{F}_2$  is torsion-free, so in fact the image of  $\widehat{G}_0$  must be trivial, which means that  $\widehat{\Gamma}_0 = \widehat{\Delta}_0 \rightarrow \widehat{F}_2$  factors through the abelian group  $\widehat{\Gamma}_0/\widehat{G}_0 \cong \widehat{\mathbb{Z}}$ , which is impossible. This contradiction completes the proof of Theorem C.  $\square$

## 4.2 Proofs of lemmas

*Proof of Lemma 4.1.* We first argue that there are infinitely many double cosets  $H \backslash \Delta / H$ . The group  $\Delta$  acts on the Bass–Serre tree  $T$  corresponding to the splitting of  $\Delta$  obtained by cutting  $N$  along  $S$ . Let  $e$  be the edge of  $T$  stabilized by  $H$ . The set of double cosets  $H \backslash \Delta / H$  is in bijection with the orbits of the edges of  $T$  under the action of  $H$  on  $T$ . Since  $H$  acts on  $T$  by isometries and there are edges of  $T$  at arbitrarily large distance from  $e$ , it follows that  $H \backslash \Delta / H$  is infinite.

By [1],  $\Delta$  is virtually special. Combining the results of [14] and [17], the double cosets in  $H \backslash \Delta / H$  are separable. Hence there exists a subgroup  $\Delta_0$  of finite index in  $\Delta$ , containing  $H$ , so that  $|\Delta_0 \backslash \Delta / H| \geq 4$ . Let  $N_0$  be the covering space of  $N$  corresponding to  $\Delta_0$ . Then the complete preimage  $S_0 \subseteq N_0$  of the surface  $S$  is embedded, and the components of  $S_1, \dots, S_k$  of  $S_0$  naturally correspond to the double cosets  $\Delta_0 \backslash \Delta / H$ . If  $S_1$  is the component corresponding to the trivial double coset  $\Delta_0 H = \Delta_0$ , then  $S_1$  is homeomorphic to  $S$ , since  $\Delta_0$  contains  $H$ . Choose three components  $S_2, S_3, S_4$  of  $S_0$ , each distinct from  $S_1$ . Let  $X$  be the dual graph to the decomposition of  $N_0$  obtained by cutting along  $S_2 \cup S_3 \cup S_4$ . Then  $X$  has three non-separating edges, and hence the fundamental group  $F$  of  $X$  is free and non-abelian. Finally,  $H$  is in the kernel of the natural epimorphism  $q : \Delta_0 \rightarrow F$ , since  $q$  is induced by a continuous map  $N_0 \rightarrow X$  that crushes  $S_1$  to a vertex.  $\square$

We now turn to Lemma 4.2. We are grateful to Pavel Zalesskii for drawing our attention to [21, p. 44, Exercise 5(b)], which guides the proof.

*Proof of Lemma 4.2.* Suppose for a contradiction that  $[\widehat{G}_0 : \widehat{H}] = \infty$ . Since  $\widehat{H}$  is closed, we may choose nested finite-index subgroups  $U_i$  in  $G_0$  so that the intersection  $\bigcap_i \widehat{U}_i = \widehat{H}$ .

We fix a prime  $p$ , consider the map  $f : \Delta_0 \rightarrow F_2$  provided by Lemma 4.1, and let  $\hat{f} : \widehat{\Delta}_0 \rightarrow \widehat{F}_2^{(p)}$  be the composition of the induced map on profinite completions and the projection from  $\widehat{F}_2$  to the pro- $p$  completion of  $F_2$ . Since  $\widehat{F}_2^{(p)}$  is non-abelian,  $\hat{f}$  certainly does not factor through the quotient map  $\widehat{\Delta}_0 \rightarrow \widehat{\Delta}_0/\widehat{G}_0 \cong \widehat{\mathbb{Z}}$ , so the closed subgroup  $\widehat{L} := \hat{f}(\widehat{G}_0)$  is non-trivial. Choose an infinite nested sequence of open subgroups  $V_i \subseteq \widehat{L}$  with trivial intersection and let  $W_i := \widehat{U}_i \cap \hat{f}^{-1}(V_i)$ . Then  $\bigcap_i W_i = \widehat{H}$  and  $p$  divides the index  $[W_i : W_{i+1}]$  for infinitely many  $i$ , so, passing to a subsequence, we may assume that  $p$  divides  $[W_i : W_{i+1}]$  for all  $i$ .

The end of the argument is a standard exercise about duality groups at the prime  $p$  (cf. [21, p. 44, Exercise 5(b)]). We consider continuous cohomology with coefficients in the finite field  $\mathbb{F}_p$ . As a finite-index subgroup of a surface group, each  $W_i \cap G_0$  is a surface group, hence it is good in the sense of Serre, which means that each of the restriction maps  $H^2(W_i, \mathbb{F}_p) \rightarrow H^2(W_j, \mathbb{F}_p) \cong \mathbb{F}_p$  is multiplication by  $[W_i : W_j]$ . Since

$$H^2(\widehat{H}, \mathbb{F}_p) = H^2\left(\bigcap_i W_i, \mathbb{F}_p\right) = \varinjlim H^2(W_i, \mathbb{F}_p)$$

and  $p$  divides  $[W_i : W_{i+1}]$ , we conclude that  $H^2(\widehat{H}, \mathbb{F}_p) = 0$ , which is a contradiction, since  $H$  is also a surface group.  $\square$

## 5 Surfaces of higher complexity and free groups of higher rank

As far as we know, the hypotheses of Theorem 2.4 may hold in very great generality. In the general context of mapping class groups, the congruence subgroup property is open, as is omnipotence for pseudo-Anosov elements. Likewise, in the context of outer automorphism groups of free groups, the congruence subgroup property is open, as is omnipotence for hyperbolic automorphisms.

**Question 5.1.** Let  $\Sigma$  be a surface of finite type. Might the set of pseudo-Anosovs in the mapping class group  $\text{Mod}(\Sigma)$  be  $\pi_1\Sigma$ -congruence omnipotent?

A positive answer to Question 5.1 would have significant ramifications. For example, it would immediately imply that if  $M$  is a closed hyperbolic 3-manifold with  $b_1(M) = 1$  and  $N$  is a compact 3-manifold with  $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$  then  $M$  and  $N$  share a common finite cyclic cover (of the same degree over  $N$  and  $M$ ); in particular they are cyclically commensurable.

The closedness hypothesis assures that the manifolds have homeomorphic fibres. Less obviously, if  $M$  and  $N$  are hyperbolic knot complements in  $S^3$  (or in an integral homology sphere) with  $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ , then [9, Theorem 7.2] implies that the fibres are homeomorphic, so again a positive answer to Question 5.1 would imply that then  $M$  and  $N$  share a common finite cyclic cover. These observations gain further interest in the context of the following conjecture of the second author and G. Walsh [19].

**Conjecture 5.2.** *Let  $K \subset S^3$  be a hyperbolic knot. There are at most 3 distinct knot complements in the commensurability class of  $S^3 \setminus K$ .*

Conjecture 5.2 was proved in [6] in the “generic case”, namely when  $K$  has no hidden symmetries (see [19] or [6] for the definition of hidden symmetry). At present the only knots that are known to have hidden symmetries are the figure-eight knot and the two dodecahedral knots of Aitchison and Rubinstein [2]. The dodecahedral knots are known to be the only knots in their commensurability class [16], and their fundamental groups are distinguished by their profinite completions using [9], since one is fibred and the other is not. Since the figure-eight knot group is distinguished from all 3-manifold groups by its profinite completion, the proviso concerning hidden symmetries in the following result may be unnecessary.

**Proposition 5.3.** *Let  $K \subset S^3$  be a fibred hyperbolic knot. If Question 5.1 has a positive answer and  $K$  has no hidden symmetries, then there is no other hyperbolic knot  $K'$  such that  $\pi_1(S^3 \setminus K)$  and  $\pi_1(S^3 \setminus K')$  have the same profinite completion.*

*Proof.* If there were such a  $K'$ , then by [9, Theorem 7.2]  $K'$  would be fibred with fibre of the same genus. A positive answer to Question 5.1 would imply that the complements of  $K$  and  $K'$  had a common finite cyclic cover of the same degree (in the light of Theorem 2.4). In particular the knot groups would be commensurable and the complements would have the same volume. But Theorem 1.7 of [6] shows that the complements in the commensurability class of a hyperbolic knot that has no hidden symmetries each have a different volume.  $\square$

**Corollary 5.4.** *Let  $K \subset S^3$  be a hyperbolic knot that admits a Lens Space surgery. If Question 5.1 has a positive answer and  $K$  has no hidden symmetries, then there is no other hyperbolic knot  $K'$  such that  $\pi_1(S^3 \setminus K)$  and  $\pi_1(S^3 \setminus K')$  have the same profinite completion.*

*Proof.* The result follows from Proposition 5.3 on noting that Y. Ni [18] proved that a (hyperbolic) knot that admits a Lens Space surgery is fibred.  $\square$

*Remark.* (i) Theorem 1.4 of [6] establishes that if the complements of knots without hidden symmetries are commensurable, then they are actually cyclically commensurable (in line with our results).

(ii) We regard Proposition 5.3 as giving further credence to the belief that hyperbolic knot groups are profinitely rigid. This belief is in keeping with a theme that has recently emerged in low-dimensional topology and Kleinian groups exploring the extent to which the fundamental group of a finite volume hyperbolic 3-manifold is determined by the geometry and topology of its finite covers. An aspect of this is the way that a “normalized” version of  $|\text{Tor}(\mathbb{H}_1(M, \mathbb{Z}))|$  behaves in finite covers; it is conjectured that this should determine the volume of the manifold. Since  $\text{Tor}(\mathbb{H}_1(M, \mathbb{Z}))$  is detected at the level of the profinite completion, the volume is thus conjectured to be a profinite invariant.

Turning to the case of  $\text{Out}(F_n)$  we can ask:

**Question 5.5.** Let  $F_n$  be the non-abelian free group of rank  $n$ . Might the set of fully irreducible automorphisms in  $\text{Out}(F_n)$  be  $F_n$ -congruence omnipotent? What about the set of hyperbolic automorphisms?

As above, a positive answer to Question 5.5 would imply that hyperbolic mapping tori  $F_n \rtimes \mathbb{Z}$  with  $b_1 = 1$  and the same profinite genus are commensurable.

## A Appendix: Computations in $GL(2, \mathbb{Z})$

The action of  $\text{Aut}(F_2)$  on  $H_1(F_2, \mathbb{Z})$  gives an epimorphism  $\text{Aut}(F_2) \rightarrow GL(2, \mathbb{Z})$  whose kernel is the group of inner automorphisms. The isomorphism type of  $\Gamma_\phi$  depends only on the conjugacy class of the image of  $\phi$  in  $\text{Out}(F_2) = GL(2, \mathbb{Z})$ , so we may regard  $\phi$  as an element of  $GL(2, \mathbb{Z})$ . We remind the reader that finite-order elements of  $GL(2, \mathbb{Z})$  are termed *elliptic*, infinite-order elements with an eigenvalue of absolute value bigger than 1 are *hyperbolic*, and the other infinite-order elements are *parabolic*.

In this appendix, we collect various standard facts about the algebra of  $GL(2, \mathbb{Z})$ . Each can be checked using elementary algebra (or more elegantly, in some cases, using the action of  $PSL(2, \mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$  on the dual tree to the Farey tessellation of the hyperbolic plane). The first such fact concerns the uniqueness of roots.

**Lemma A.1.** *If  $\phi, \psi \in GL(2, \mathbb{Z})$  are elements of infinite order and  $\phi^n = \psi^n$  for some  $n \neq 0$ , then  $\phi = \pm\psi$ .*

We next recall the classification of non-hyperbolic elements of  $GL(2, \mathbb{Z})$ , up to conjugacy.

**Proposition A.2.** *Every non-hyperbolic element of  $GL(2, \mathbb{Z})$  is conjugate to exactly one of the following elements:*

1.  $\pm I$ ;
2.  $\theta = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ , which has order 3;
3.  $-\theta$ , which has order 6;
4.  $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , which have order 2 and are not conjugate to each other,
5.  $\epsilon\tau$ , which has order 4;
6.  $U(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  with  $n > 0$ ;
7.  $-U(n)$  with  $n > 0$ .

From the obvious presentation  $\Gamma_\phi = \langle a, b, t \mid tat^{-1} = \phi(a), tat^{-1} = \phi(b) \rangle$  we get the presentation

$$H_1(\Gamma_\phi, \mathbb{Z}) = \langle a, b, t \mid [a, b] = [a, t] = [b, t] = 1 = a^{-1}\phi(a) = b^{-1}\phi(b) \rangle$$

from which it is easy to calculate the following.

**Lemma A.3.** *With the notation of Proposition A.2:*

1.  $H_1(\Gamma_I, \mathbb{Z}) \cong \mathbb{Z}^3$  and  $H_1(\Gamma_{-I}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ ;
2.  $H_1(\Gamma_\theta, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/3$ ;

3.  $H_1(\Gamma_{-\theta}, \mathbb{Z}) \cong \mathbb{Z}$ ;
4.  $H_1(\Gamma_\epsilon, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$  and  $H_1(\Gamma_\tau, \mathbb{Z}) \cong \mathbb{Z}^2$ ;
5.  $H_1(\Gamma_{\epsilon\tau}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ ;
6.  $H_1(\Gamma_{U(n)}, \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/n$  if  $n > 0$ ;
7.  $H_1(\Gamma_{-U(n)}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/4$  if  $n$  odd, and  $H_1(\Gamma_{-U(n)}) = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$  if  $n$  even.

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