Aspects of fluid dynamics and the fluid/gravity correspondence

Ashok Thillaisundaram

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

This dissertation is submitted for the degree of
Doctor of Philosophy

Trinity College

July 2017

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. It is not substantially the same as any that I have submitted or is being concurrently submitted for a degree, diploma, or other qualification. The research described in this dissertation was carried out at the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge between October 2010 and September 2014. Except where reference is made to the work of others, all the results are original, mainly based on the following works:

- Ashok Thillaisundaram, "Forced fluid dynamics from gravity in arbitrary dimensions", JHEP 1403 (2014) 138.
- Ashok Thillaisundaram, "Holographic derivation of the entropy current for an anomalous charged fluid with background electromagnetic fields"
- Ashok Thillaisundaram, "Stability of equilibrium in fluid dynamics."

Chapter 1 is an introduction, chapters 2 and 5 are reviews, and chapters 3, 4, and 6 contain original research.

Ashok Thillaisundaram
July 2017

Acknowledgements

I am very grateful to my PhD supervisor Malcolm Perry for his advice and encouragement. It is also a pleasure to thank Rajesh Gopakumar, Veronika Hubeny, Shiraz Minwalla, Mukund Rangamani, Harvey Reall, and David Tong for useful discussions and correspondence.

I owe an infinite debt of gratitude to my friends, my parents, my sister, and my twin for their support throughout my PhD.

My PhD was supported by the Cambridge Commonwealth Trust, the Department of Applied Mathematics and Theoretical Physics, and Trinity College, Cambridge.

Abstract

This thesis considers various extensions to the fluid/gravity correspondence as well as problems fundamental to the study of fluid dynamics. The fluid/gravity correspondence is a map between the solutions of the Navier-Stokes equations of fluid dynamics and the solutions of the Einstein equations in one higher spatial dimension. This map arose within the context of string theory and holography and is a specific realisation of a much wider class of dualities known as the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence.

The first chapter is an introduction; the second chapter reviews the fluid/gravity correspondence. The next two chapters extend existing work on the fluid/gravity map. Our first result concerns the fluid/gravity map for forced fluid dynamics in arbitrary spacetime dimensions. Forced fluid flows are of particular interest as they are known to demonstrate turbulent behaviour. For the case of a fluid with a dilaton-dependent forcing term, we present explicit expressions for the dual bulk metric, the fluid dynamical stress tensor and Lagrangian to second order in boundary spacetime derivatives. Our second result concerns fluid flows with multiple anomalous currents in the presence of external electromagnetic fields. It has recently been shown using thermodynamic arguments that the entropy current for such anomalous fluids contains additional first order terms proportional to the vorticity and magnetic field. Using the fluid/gravity map, we replicate this result using gravitational methods.

The final two chapters consider questions related to the equations of fluid dynamics themselves; these chapters do not involve the fluid/gravity correspondence. The first of these chapters is a review of the various constraints that must be satisfied by the transport coefficients. In the final chapter, we derive the constraints obtained by requiring that the equilibrium fluid configurations are linearly stable to small perturbations. The inequalities that we obtain here are slightly weaker than those found by demanding that the divergence of the entropy current is non-negative.

Table of contents

Chapter 1

Introduction

1.1 AdS/CFT and holographic fluid dynamics

The AdS/CFT correspondence [1, 2] offers a novel perspective on quantum field theories. This duality enables us to translate a problem in field theory into a gravitational/string theoretic language. The hope of course is that, where conventional field theoretic methods have failed, the dual description may prove to be more intuitive and the problem therefore more tractable. This is especially true for strongly coupled field theories and there is significant interest in obtaining holographic descriptions of certain strongly coupled phenomena [3, 4].

Considerable progress has already been made in obtaining the holographic duals of equilibrium field theory configurations. Two canonical examples in the AdS/CFT dictionary are the Schwarzschild-AdS black brane which corresponds to a thermal state in the field theory and the Reissner-Nordstrom-AdS black brane which corresponds to a thermal state at finite density. Now, while an understanding of equilibrium states is useful to describe certain physical phenomena (the study of phase transitions, for example, need not include any explicit time-dependence), many interesting strongly coupled phenomena are dynamical. Hence, there is considerable physical motivation for the holographic study of non-equilibrium behaviour [5].

Unfortunately, while the holographic study of non-equilibrium dynamics is of much greater interest, it is also correspondingly much more difficult. Some headway can be made by focusing on small deviations away from equilibrium. The study of these small amplitude perturbations is known as linear response theory and the holographic methodology involved is part of the standard AdS/CFT toolkit [6–11]. However, the regime of validity of linear response theory does not cover large amplitude, violent perturbations away from equilibrium and, in such cases, often progress can only be made using numerical methods [12–14].

If we are motivated by the desire to obtain *analytically* the holographic dual of a certain class of interesting, non-trivial non-equilibrium phenomena (beyond the reach of linear response theory), a natural starting point would be fluid dynamics. We will now provide some intuition for this statement. For the sake of having a concrete example, we consider the most familiar case of the AdS/CFT correspondence: the duality between *SU(N)* $\mathcal{N} = 4$ Super Yang-Mills theory and Type IIB string theory on $\mathrm{AdS}_5{\times}\mathrm{S}^5$. For generic values of N and coupling λ, both sides of this duality are fairly complicated theories. If we are interested in obtaining analytic, time-dependent solutions with the aim of studying non-equilibrium phenomena, it is well worth considering a limit in which the dynamics will simplify. A natural way forward would be to take $N \to \infty$ in the 't Hooft limit; the bulk theory now becomes classical Type IIB string theory. If we further take the strong coupling limit ($\lambda \to \infty$), the massive string states decouple and the bulk theory simplifies to Type IIB supergravity. Now, while progressing from Type IIB string theory on $\mathrm{AdS}_5{\times}\mathrm{S}^5$ to Type IIB supergravity certainly is a step in the right direction, more can still be done. Type IIB supergravity on $\mathrm{AdS}_5{\times}\mathrm{S}^5$ has several consistent truncations to reduced, decoupled subsectors of dynamics; we focus on the simplest of these which is pure Einstein gravity with negative cosmological constant[1],

$$E_{AB} \equiv R_{AB} - \frac{1}{2}R g_{AB} + \Lambda g_{AB} = 0, \quad \Lambda \equiv -\frac{d(d-1)}{2}. \tag{1.1}$$

It is worth emphasising here that this result applies with much greater universality than implied above. There are an infinite number of field theories possessing gravitational duals; all of which admit large N and strong coupling limits. And in these limits, the bulk theories will generically simplify to two derivative Einstein gravity interacting with other fields. Regardless of the specific nature of these interactions, these bulk theories of gravity will certainly admit $\mathrm{AdS}_{d+1} \times M_I$ as a solution (M_I is some internal manifold). Bulk dynamics with these characteristics all possess consistent truncations to pure Einstein gravity with negative cosmological constant. In this sense, the dynamics described by equation (1.1) is the universal subsector of dynamics for an infinite class of bulk theories. And from the field theory perspective, bearing in mind that the bulk graviton is dual to the boundary field theory stress tensor, we see that pure Einstein

[1] The bulk spacetime has $d+1$ dimensions. Also, we have set the AdS curvature radius to unity.

gravity with negative cosmological constant is the universal dual bulk description of the stress tensor dynamics of an infinite class of strongly coupled field theories.

Let us now pause to summarise our current position. Motivated by our desire to obtain analytic, time-dependent holographic solutions, we were led to the universal subsector of dynamics (1.1) which is the dual dynamics of the boundary stress tensor. Yet even now, attempting to classify all time-dependent solutions to the Einstein equations is far from an easy task. And on the field theory side as well, the full behaviour of the stress tensor is still very nontrivial. To achieve further progress it is again worth limiting our attention to a simpler case. A promising path that we could take would be to focus only on stress tensor dynamics for field theory configurations which are locally equilibrated. Such configurations are governed by fluid dynamics [15]; and the key fluid dynamical equations of motion simply follow from conservation of the stress tensor[2],

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.2}$$

Perhaps constructing bulk time-dependent solutions of (1.1) dual to boundary fluid dynamics is a more realistic aim?

This goal was concretely achieved in [16] where the authors explicitly constructed asymptotically AdS_5 long wavelength solutions to the Einstein equations with negative cosmological constant which are dual to solutions of the four-dimensional conformal relativistic Navier-Stokes equations. It should be stressed here that this work constitutes a derivation of *nonlinear* fluid dynamics from gravity and thus is valid for fluid dynamical solutions with arbitrarily large velocity amplitudes. This is distinct from previous work on holographic linearised hydrodynamics [17–20] which is only valid for small amplitude perturbations about equilibrium configurations. Work on obtaining the holographic dual of nonlinear fluid dynamics was in some sense pioneered by [21–23]; here, the authors considered nonlinear solutions dual to Bjorken flow, a particular boost invariant flow.

This duality between long wavelength solutions of the Einstein equations and solutions of nonlinear boundary fluid dynamics has become known as the *fluid/gravity correspondence* [24, 25]. Subsequent work soon after the seminal paper [16] generalised this map to arbitrary spacetime dimensions [26, 27]. Many new lines of research have also developed to consider further interesting generalisations. In [28], Bhattacharyya

[2]Greek indices label boundary coordinates.

et al extended this result to nonrelativistic fluids and holographically obtained the incompressible nonrelativistic Navier-Stokes equations. Work has also been done on constructing the bulk duals of non-conformal fluids [29], of charged fluids [30, 31], of superfluids [32–34], and of anomalous fluids [35].

1.2 Outline of thesis

Having presented a brief introduction to holographic hydrodynamics we now give a fairly detailed outline of the remaining chapters of this thesis.

Chapter 2 contains a review of the fluid/gravity correspondence. In particular, we walk through the methodology for obtaining the bulk metric highlighting important principles and assumptions. This calculation is shown in full detail.

Chapters 3 and 4 extend existing results on the fluid/gravity correspondence in several interesting directions. In chapter 3, we focus on forced fluid dynamics. We consider long wavelength solutions to the Einstein-dilaton system with negative cosmological constant which are dual, under the AdS/CFT correspondence, to solutions of the conformal relativistic Navier-Stokes equations with a dilaton-dependent forcing term. Certain forced fluid flows are known to exhibit turbulence; holographic duals of forced fluid dynamics are therefore of particular interest as they may aid efforts towards an explicit model of holographic steady state turbulence. In recent work, Bhattacharyya *et al* [36] have constructed long wavelength asymptotically locally AdS_5 bulk spacetimes with a slowly varying boundary dilaton field which are dual to forced fluid flows on the 4-dimensional boundary. In this chapter, we generalise their work to arbitrary spacetime dimensions; we explicitly compute the dual bulk metric, the fluid dynamical stress tensor and Lagrangian to second order in a boundary derivative expansion.

In chapter 4, our focus is on the holographic derivation of the fluid dynamical entropy current for anomalous fluids. We construct long wavelength asymptotically locally AdS_5 spacetimes with slowly varying (background) gauge fields which are solutions to the $U(1)^n$ Einstein-Maxwell-Chern-Simons system. These bulk spacetimes are dual to (3 + 1)-dimensional fluid flows with n anomalous currents in the presence of external electromagnetic fields. We utilise the area form on the outer horizon to holographically compute an entropy current for the dual fluid to first order in boundary derivatives. Our resulting expression contains additional terms proportional to the vorticity and magnetic field and thus provides holographic confirmation of the entropy current calculated by Son and Surowka for hydrodynamics with triangle anomalies [37]. We then restrict our bulk metric to describe the fluid/gravity model of the chiral magnetic effect (CME) [38] and again holographically obtain the entropy current. As expected, our calculation replicates the result produced using standard hydrodynamic/thermodynamic arguments.

Chapters 5 and 6 consider problems fundamental to the study of fluid dynamics. Although research in these areas was inspired by recent developments in string theory, these problems concern fluid dynamics alone and are independent of holography. Chapter 5 contains review material. We give an overview of the constraints imposed on the transport coefficients in fluid dynamics. The main question here is what constraints must the transport coefficients satisfy to ensure that the effective theory of fluid dynamics is completely consistent with an underlying microscopic field theory. Various sets of constraints are currently being explored in the literature and work is being done towards conclusively identifying an exhaustive set of physical constraints. One source of constraints is the requirement that fluid dynamics is consistent with a local form of the second law of thermodynamics; we must demand that the local divergence of the entropy current is always non-negative [39]. Another source of constraints comes from demanding that the solutions to the equations of fluid dynamics are consistent with existence of an equilibrium partition function [40]. We consider the constraints from both these principles for the case of an uncharged fluid at first order.

Chapter 6 develops further the programme of research towards a complete set of physical constraints for the effective theory of fluid dynamics. As stated above, if we demand that the local divergence of the entropy current is non-negative, we obtain various constraints on the transport coefficients. It is well-known that this requirement gives inequality-type constraints on the viscosities; what is perhaps less well-known is that you also obtain equality-type constraints between transport coefficients at each order in the derivative expansion. Now, if we consider the constraints arising from demanding consistency with an equilibrium partition function, it turns out, quite miraculously, that we obtain exactly the same equality-type constraints which result from restricting to non-negative divergence entropy currents. In this chapter, we consider the following question: What further requirement related to equilibrium fluid solutions do we need in order to obtain the inequality-type constraints as well? Here we investigate the constraints arising from demanding that all equilibrium configurations are dynamically stable. We look at the simplest example of a fluid on a flat spacetime background and perform a linear stability analysis. It turns out that the inequalities that we obtain are slightly weaker than those obtained from the local form of the second law of thermodynamics.

Chapter 2

Review of the fluid/gravity correspondence

2.1 Relativistic fluid dynamics

We begin with an overview of relativistic fluid dynamics. For the purpose of having a concrete example, we will first work with the relativistic Navier-Stokes equation which describes a relativistic viscous fluid. The dynamics of such a fluid are governed by the following system of equations:

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= 0, \\ T_{\mu\nu} &= p(g_{\mu\nu} + u_\mu u_\nu) + \rho u_\mu u_\nu + \tau_{\mu\nu}. \end{aligned} \tag{2.1}$$

The overarching physical principle here is simply the conservation of the stress tensor $T_{\mu\nu}$. The first two terms of the stress tensor represent the ideal fluid component (constructed from pressure p, fluid velocity u_μ, energy density ρ, and background spacetime $g_{\mu\nu}$) whereas the final part $\tau_{\mu\nu}$ introduces the viscosity. This term $\tau_{\mu\nu}$ is given by the following expression:

$$\tau_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \zeta\theta(g_{\mu\nu} + u_\mu u_\nu). \tag{2.2}$$

Here $\sigma_{\mu\nu}$ is the fluid shear tensor, a transverse[1] traceless symmetric rank 2 tensor constructed from first order fluid velocity derivatives. The coefficients η and ζ represent

[1]By transverse we mean orthogonal to the fluid velocity vector u_μ.

the shear and bulk viscosities while θ is the divergence of the fluid velocity field $\nabla_\mu u^\mu$.

Now, purely as an instructive tool, let us introduce a dummy variable ϵ and perform the following transformation of variables:

$$x = \epsilon \tilde{x}.$$

With the above transformation, the first order derivative terms in $\tau_{\mu\nu}$ pick up a factor of ϵ:

$$\tilde{\tau}_{\mu\nu}(\tilde{x}) = \epsilon \tau_{\mu\nu}(\epsilon \tilde{x}). \tag{2.3}$$

And in the $\epsilon \to 0$ limit, we see that this coordinate transformation effectively suppresses the first order derivative terms relative to the zeroth order terms. It is useful to now understand physically what this coordinate transformation represents. It essentially magnifies the coordinate axis; a change in $\tilde{x}$ corresponds to a very small change in x for $\epsilon \ll 1$. And regardless of the scale of variation present in $T_{\mu\nu}$, if you take ϵ small enough, $\tilde{\tau}_{\mu\nu}(\tilde{x})$ will be suppressed and $\tilde{T}_{\mu\nu}$ will vary very slowly in $\tilde{x}$. Although there are only first order derivatives present in the stress tensor for the relativistic Navier-Stokes equation, it is straightforward to see that if higher order derivatives were present, each order would be suppressed relative to the previous order by a factor of ϵ. As such, in this limit of *slow variation* it makes sense to work with an effective theory valid up till a specific order of derivatives; a *long wavelength* effective theory. Also, observe that in the $\epsilon \to 0$ limit, we magnify the coordinate axis and essentially zoom into a small local area in the coordinate space. And with all higher derivatives suppressed, the relativistic Navier-Stokes equations will locally admit constant solutions and will thus trivially reduce to equilibrium thermodynamics.

In the above paragraph, we used a coordinate transformation to artificially induce very slow variation. And under such conditions, we concluded that the equations of fluid dynamics are best described by a long wavelength effective theory specified to an arbitrary order in derivatives. We also noted that locally, fluid dynamics will reduce to equilibrium thermodynamics. However, this limit of slow variation is not just the outcome of an artificial coordinate transformation. For fluid dynamics, there is actually a concrete physical reason for why all fluid dynamical quantities must be slowly varying; the equations are only meaningful if this is the case. We will now give a brief intuitive explanation of the reasoning behind this.

For the equations of fluid dynamics, the canonical variables are those of thermodynamics (energy density ρ, pressure p as functions of temperature T) and a fluid velocity field u^μ. These will be functions of the coordinate space x^μ and will have spatial and temporal profiles. For a value to be assigned to the temperature field T at point x^μ say, the fluid must have locally equilibrated at that point. And by this we mean that within a spatial region of sufficient size and over a sufficient period of time (length and timescales are determined by the microscopic physics of the system), local thermal equilibrium must have been reached and thermodynamic quantities can thus be meaningfully assigned. Because of these physical constraints, the fluid dynamical variables must necessarily have slowly-varying spatial and temporal profiles relative to these equilibrium length and timescales. It then follows that fluid dynamics, as argued in the previous paragraph, should naturally be expressed as an effective theory valid up to a specified order in derivatives.

We now move away from our concrete example of the relativistic Navier-Stokes equations and present fluid dynamics as an effective theory expressed as an expansion in derivatives. The governing equations follow from the conservation of the stress tensor

$$\nabla^\mu T_{\mu\nu} = 0$$

but here we allow the stress tensor to be organised as an expansion in derivatives:

$$T_{\mu\nu} = \sum_{l=0}^{\infty} T^{(l)}_{\mu\nu} \tag{2.4}$$

where $T^{(l)}_{\mu\nu}$ contains terms of order l in derivatives. The Navier-Stokes stress tensor only contained terms till first order in derivatives with

$$T^{(1)}_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \zeta\theta(g_{\mu\nu} + u_\mu u_\nu). \tag{2.5}$$

It is possible to constrain the first order terms of a general fluid dynamical stress tensor to take the above form using symmetry arguments. Indeed, this applies to all orders in derivatives; symmetry considerations restrict the allowed terms at each order. It is instructive to work through the first order case as it will illustrate some subtleties. We will now proceed to do this.

First, it is important to note that the stress tensor should only contain independent terms. And here by independence we mean that the equations of motion should not impose relations between any two terms. To illustrate this in more detail, let us consider the zeroth order stress tensor:

$$T^{(0)}_{\mu\nu} = p(g_{\mu\nu} + u_\mu u_\nu) + \rho u_\mu u_\nu.$$

The equations of motion, $\nabla^\mu T_{\mu\nu} = 0$, will contain first order terms derived from the zeroth order stress tensor $T^{(0)}_{\mu\nu}$. These are given by the following:

$$\nabla^\mu T^{(0)}_{\mu\nu} = \nabla^\mu p(g_{\mu\nu} + u_\mu u_\nu) + (p+\rho)(\nabla^\mu u_\mu u_\nu + u_\mu \nabla^\mu u_\nu) + (u_\mu \nabla^\mu \rho) u_\nu = 0 \tag{2.6}$$

For clarity, we decompose this into its spatial and temporal components using the projection tensor, $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

$$\begin{aligned} u^\nu \nabla^\mu T^{(0)}_{\mu\nu} &: u_\mu \nabla^\mu \rho + (p+\rho)\nabla_\mu u^\mu = 0 \\ P^{\nu\alpha} \nabla^\mu T^{(0)}_{\mu\alpha} &: P^\nu_\mu \nabla^\mu p + (p+\rho) u^\mu \nabla_\mu u^\nu = 0 \end{aligned} \tag{2.7}$$

From these two equations we can see that the first order derivatives of the pressure and energy density fields, p and ρ, can be expressed in terms of fluid velocity derivatives (this is further simplified for conformal fluids). Thus when enumerating all possible independent terms for our first order stress tensor we need only consider fluid velocity derivatives.

Now, consider the first order derivative for the fluid velocity field $\nabla_\mu u_\nu$; it is straightforward to use the velocity field u_μ and projection tensor $P_{\mu\nu}$ to decompose this into its scalar, vector, and tensor components. We have two scalars:

$$u^\mu u^\nu \nabla_\mu u_\nu, \quad P^{\mu\alpha} P^\beta_\mu \nabla_\alpha u_\beta. \tag{2.8}$$

The first term vanishes ($\nabla_\mu (u_\nu u^\nu) = 0$) and the second is proportional to the divergence $\nabla_\mu u^\mu$. For the vector components, we have:

$$u^\nu P^{\mu\alpha} \nabla_\nu u_\alpha, \quad u^\nu P^{\mu\alpha} \nabla_\alpha u_\nu \tag{2.9}$$

The first term is equivalent to $u^\nu \nabla_\nu u_\mu$ and the second vanishes. And finally for the tensor components, we have:

$$P^{\mu\alpha} P^{\nu\beta} \nabla_\alpha u_\beta \tag{2.10}$$

This term can be decomposed into traceless symmetric and antisymmetric components and a trace component:

$$P^{\mu\alpha}P^{\nu\beta}\nabla_{(\alpha}u_{\beta)} - \frac{1}{d-1}\nabla_\alpha u^\alpha P^{\mu\nu}, \quad P^{\mu\alpha}P^{\nu\beta}\nabla_{[\alpha}u_{\beta]}, \quad \nabla_\alpha u^\alpha P^{\mu\nu}. \tag{2.11}$$

Here the round and square brackets denote symmetrisation and antisymmetrisation respectively. The first term was defined earlier; it is the shear tensor $\sigma^{\mu\nu}$. The second is the vorticity tensor $\omega^{\mu\nu}$.

Given this decomposition of $\nabla_\mu u_\nu$ into its scalar, vector, and tensor components

$$\nabla_\mu u^\mu, \quad u^\nu\nabla_\nu u_\mu, \quad \sigma_{\mu\nu}, \quad \omega_{\mu\nu}, \quad \nabla_\alpha u^\alpha P_{\mu\nu} \tag{2.12}$$

we can now enumerate all allowed first order terms for the stress tensor. Here we are interested in only symmetric tensors. Thus,

$$T^{(1)}_{\mu\nu} = \alpha\nabla_\alpha u^\alpha P_{\mu\nu} + \beta\sigma_{\mu\nu} + \gamma u^\alpha\nabla_\alpha u_{(\mu}u_{\nu)} \tag{2.13}$$

There is one final subtlety that we need to mention. Not all the terms in the above equation are physically meaningful. The velocity field u^μ only has physical significance in (local) equilibrium (as it is a thermodynamic variable). If you can introduce a transformation of u^μ which reduces to the identity in equilibrium, then all such transformations of u^μ would be physically equivalent. It is possible to use this 'field definition ambiguity' to transform away the third term $u^\alpha\nabla_\alpha u_{(\mu}u_{\nu)}$. The first two terms however are invariant under such transformations and therefore are the only physically relevant terms. This ambiguity is fixed by a choice of frame, usually chosen to be the Landau frame. This is defined in the following manner:

$$T_{\mu\nu}u^\nu = -\rho u_\mu. \tag{2.14}$$

Or equivalently,

$$T^{(l)}_{\mu\nu}u^\nu = 0 \text{ for } l \geq 1. \tag{2.15}$$

The first order stress tensor therefore reduces to:

$$T^{(1)}_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \zeta\theta P_{\mu\nu} \tag{2.16}$$

where the coefficients represent the bulk and shear viscosities as we found for the relativistic Navier-Stokes equations. As mentioned previously, this procedure can be extended to all orders in derivatives. This concludes our overview of relativistic fluid dynamics.

2.2 Gravity dual

Given we now have an understanding of the main features of relativistic fluid dynamics, how should we go about constructing a gravitational dual (via the AdS/CFT correspondence) to a fluid dynamical configuration? There are two parts to this question: First, what system of equations governs the dynamics of the gravitational dual? And second, how do we solve this system of equations to determine the precise gravitational configuration dual to a certain fluid?

We begin with the first part. We presented this argument in the introductory chapter but we repeat it here for completeness. Different field theories have different dual gravitational dynamics; some of these fairly complicated. However the vast majority of them are two derivative gravity systems with various additional fields. Observe that, regardless of how complicated the field theory is, we expect its dynamics to reduce to that of a fluid at sufficiently high temperatures. Thus, for all gravitational systems dual to these various field theories, there should always be a reduced subsector of dynamics which is dual to fluid dynamics. In this sense, there should be a *universal subsector* on the gravitational side since fluid dynamical behaviour is universally exhibited by many field theories.

All of these dual two derivative gravity systems must necessarily contain Einstein gravity with a negative cosmological constant;

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = 0; \Lambda < 0 \tag{2.17}$$

otherwise they would not admit AdS_{d+1} as a solution. Furthermore, the field theory stress tensor, whose conservation determines the equations of fluid dynamics, is dual to spacetime metric. It is therefore reasonable to assume that pure gravity (with a negative cosmological constant) is all that is needed to obtain the dual description of fluid dynamics.

Armed with this assumption, we can now address the second part: How do we solve the Einstein equations with negative cosmological constant to determine the precise spacetimes dual to fluid dynamical configurations? We can draw some intuition from the previous section on relativistic fluid dynamics. There we explained that the fluid dynamical stress tensor is naturally expressed as an expansion in derivatives. Also, locally, fluid dynamics should reduce to equilibrium thermodynamics. And given that the field theory stress tensor is dual to the bulk spacetime metric, it seems reasonable that the bulk spacetime metric should admit the following expansion:

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} + g_{ab}^{(2)} + g_{ab}^{(3)} + \cdots \tag{2.18}$$

where $g_{ab}^{(n)}$ contains terms of order n in *boundary spacetime* derivatives. Furthermore, we should expect $g_{AB}^{(0)}$ to locally approximate a uniform black brane, signifying local thermodynamic equilibrium.

Given this, our zeroth order ansatz $g_{ab}^{(0)}$ could be the boosted Schwarzschild-AdS$_{d+1}$ metric with the thermodynamic variables u_μ and T promoted to spacetime-varying fields, $u_\mu(x^\nu)$ and $T(x^\nu)$. This is given below:

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2 f(br)} + r^2\left(-f(br)u_\mu u_\nu dx^\mu dx^\nu + P_{\mu\nu}dx^\mu dx^\nu\right) \\ f(br) &= 1 - \frac{1}{(br)^d}, \quad b = \frac{d}{4\pi T} \end{aligned} \tag{2.19}$$

where r is the radial coordinate and μ, ν are boundary spacetime coordinates.

However, there is a problem with using this metric as our zeroth order ansatz and it is quite a subtle one. The issue lies with how tubes of constant x^μ extend into the bulk from the boundary. Consider what would happen if a sudden forcing were applied at a point y^μ on the boundary. This forcing would affect all areas causally connected to y^μ; we can define a future light cone $C(y^\mu)$ emanating from the point y^μ confined to the boundary. Now consider the bulk region $B(y^\mu)$ formed by extending $C(y^\mu)$ into the bulk along tubes of constant x^μ. We would expect this area to be causally connected to y^μ; it should be within the future bulk light cone emanating from y^μ. This condition is not met if we use the Schwarzschild-AdS$_{d+1}$ coordinates (see Figure 2.1). Instead, if we use Eddington-Finkelstein coordinates:

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu, \tag{2.20}$$

then tubes of constant x^μ extend along null geodesics into the bulk and this problem is no longer an issue. With this in mind, it seems appropriate to use the Eddington-Finkelstein formulation as our zeroth order ansatz. We are now in a position to

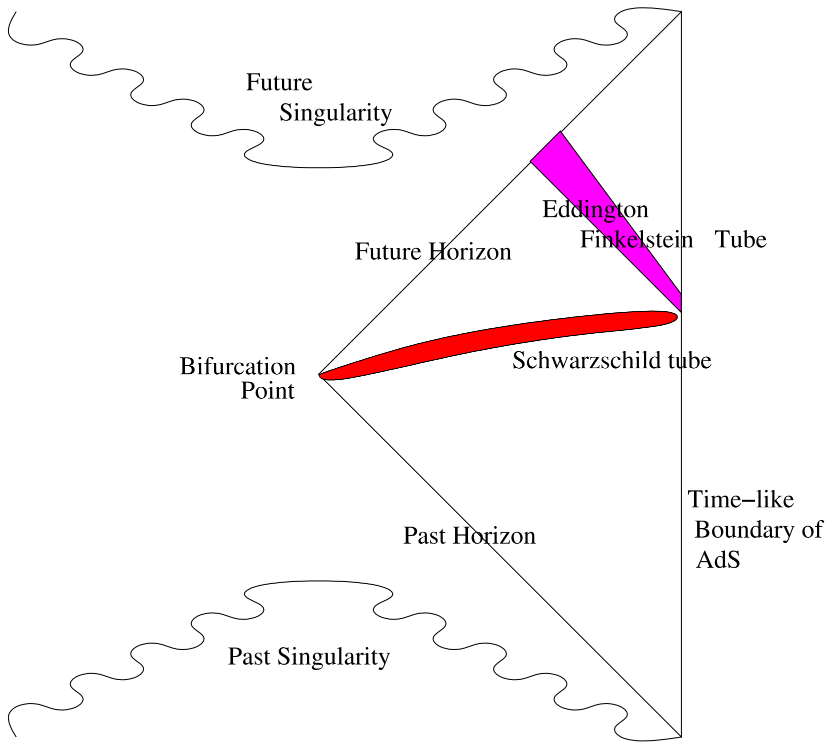

Fig. 2.1 This shows the Penrose diagram of a uniform black brane; tubes of constant x^μ are shown in both the Eddington-Finkelstein and Schwarzschild coordinates. This figure is taken from [26].

perturbatively solve the bulk Einstein equations using an expansion in boundary derivatives. This will be our focus for the remainder of this chapter.

2.3 First order equations

In this section, we will derive and solve the first order equations for the fluid/gravity correspondence. For the purposes of our review, we will work to first order only as that is sufficient to highlight the most important conceptual issues.

2.3.1 Einstein equations and metric ansatz

As always, our starting point is the Einstein equations:

$$E_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R - 6 g_{ab} = 0. \tag{2.21}$$

We are working in 5 spacetime dimensions and we choose our units such that $R_{AdS} = 1$. With these choices, the Ricci scalar can be calculated by taking a contraction of the above equation:

$$g^{ab}E_{ab} = 0 \implies R = -20. \tag{2.22}$$

Our Einstein equations then simplify to just:

$$R_{ab} + 4g_{ab} = 0. \tag{2.23}$$

To proceed further, it is necessary to rewrite the above equation using just the metric, its inverse, and their derivatives. We can then easily substitute our ansatz for the fluid/gravity metric into the equation. The definitions for the Christoffel symbols, Riemann tensor, and Ricci tensor are given below:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(g_{bd,a} + g_{ad,b} - g_{ab,d}\right) \tag{2.24}$$

$$R_{abc}{}^d = \Gamma^d_{ac,b} - \Gamma^d_{bc,a} + \Gamma^e_{ab}\Gamma^d_{eb} - \Gamma^e_{bc}\Gamma^d_{ea} \tag{2.25}$$

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{cb,a} + \Gamma^e_{ab}\Gamma^c_{ec} - \Gamma^e_{cb}\Gamma^c_{ea}. \tag{2.26}$$

Expressing the Ricci tensor in terms of the metric gives us:

$$\begin{aligned}
R_{ab} = &\frac{1}{2}g^{cd}{}_{,c}\left(g_{bd,a} + g_{ad,b} - g_{ab,d}\right) + \frac{1}{2}g^{cd}\left(g_{ad,bc} - g_{ab,dc}\right) \\
&- \frac{1}{2}g^{cd}{}_{,a}\left(g_{bd,c} + g_{cd,b} - g_{cb,d}\right) - \frac{1}{2}g^{cd}\left(g_{cd,ba} - g_{cb,da}\right) \\
&+ \frac{1}{4}g^{ed}g^{cf}g_{cf,e}\left(g_{bd,a} + g_{ad,b} - g_{ab,d}\right) \\
&- \frac{1}{4}g^{ed}g^{cf}g_{ef,a}\left(g_{bd,c} + g_{cd,b} - g_{cb,d}\right) \\
&- \frac{1}{4}g^{ed}g^{cf}g_{af,e}\left(g_{bd,c} + g_{cd,b} - g_{cb,d}\right) \\
&+ \frac{1}{4}g^{ed}g^{cf}g_{ea,f}\left(g_{bd,c} + g_{cd,b} - g_{cb,d}\right).
\end{aligned} \tag{2.27}$$

Using some symmetric-antisymmetric pair cancellations, this simplifies slightly to:

$$\begin{aligned} R_{ab} = &\frac{1}{2}g^{cd}{}_{,c}\left(g_{bd,a}+g_{ad,b}-g_{ab,d}\right)+\frac{1}{2}g^{cd}\left(g_{ad,bc}-g_{ab,dc}\right)\\ &-\frac{1}{2}g^{cd}{}_{,a}g_{cd,b}-\frac{1}{2}g^{cd}\left(g_{cd,ba}-g_{cb,da}\right)\\ &+\frac{1}{4}g^{ed}g^{cf}g_{cf,e}\left(g_{bd,a}+g_{ad,b}-g_{ab,d}\right)\\ &-\frac{1}{4}g^{ed}g^{cf}g_{ef,a}g_{cd,b}\\ &+\frac{1}{2}g^{ed}g^{cf}g_{ea,f}\left(g_{bd,c}-g_{cb,d}\right). \end{aligned} \tag{2.28}$$

Clearly, to evaluate these expressions at first order in boundary derivatives we require the metric till first order. We have previously established that the zeroth order terms should locally resemble boosted black branes

$$g^{(0)}_{ab}dx^a dx^b = -2u_\mu dx^\mu dr - r^2 f(br)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu.$$

We now need to parametrise the first order terms. It is useful to first choose a gauge to avoid complications with coordinate diffeomorphisms. We utilise the gauge chosen in the seminal paper [16]; this is known as the 'background field' gauge

$$g_{rr}=0, \quad g_{r\mu}\propto u_\mu, \quad Tr\left(\left(g^{(0)}\right)^{-1}g^{(n)}\right)=0 \quad \forall n>0. \tag{2.29}$$

And for our parametrisation of $g^{(1)}$, we take advantage of the $SO(3)$ invariance of the Einstein equations

$$g^{(1)}_{ab}dx^a dx^b = -3h_1 dv dr + r^2 h_1 dx_i dx^i + \frac{k_1}{r^2}dv^2 + 2\frac{j_i^{(1)}}{r^2}dv dx^i + r^2\alpha^{(1)}_{ij}dx^i dx^j \tag{2.30}$$

where the unknown functions $h_1, k_1, j_i^{(1)}, \alpha^{(1)}_{ij}$ parametrise the first order metric terms. The resulting equations will decouple into separate scalar, vector, and tensor components.

It now seems as though we are in a position to obtain the precise form of the fluid/gravity equations. We have the expression for the Einstein equations in terms of the metric, its inverse, and their derivatives, as well as an ansatz for the metric up to first order; it should be straightforward to input the metric ansatz into the Einstein equations and consider what it evaluates to. However, there are two subtleties that first need to be addressed. The first subtlety is fairly intuitive. Given that our formulation of the fluid/gravity setup requires that the metric admit an expansion in terms of boundary derivatives, it is natural to expect that the fluid velocity field and temperature field should also admit corrections order by order in boundary derivatives. After all, the fluid velocity field and temperature field are parameters of the boundary stress tensor which is dual to the bulk metric. Our velocity and temperature fields, defined via:

$$
\begin{aligned}
u^v &= \frac{1}{\sqrt{1-\beta^2}}, \quad u^i = \frac{\beta_i}{\sqrt{1-\beta^2}}, \\
b &= \frac{1}{\pi T},
\end{aligned} \tag{2.31}
$$

should admit the following expansions:

$$
\begin{aligned}
\beta_i &= \beta_i^{(0)} + \beta_i^{(1)} + O\left(\epsilon^2\right) \\
b &= b^{(0)} + b^{(1)} + O\left(\epsilon^2\right).
\end{aligned} \tag{2.32}
$$

The second subtlety is perhaps slightly more subtle. It turns out it is easier to solve these equations locally utilising a Taylor expansion of the metric ansatz around some arbitrary point and then later covariantising the resulting local solution. Our metric ansatz to first order thus becomes:

$$
\begin{aligned}
ds^2 = {} & 2dvdr - r^2 f(r) dv^2 + r^2 dx_i dx^i \\
& - 2x^\mu \partial_\mu \beta_i^{(0)} dx^i dr - 2x^\mu \partial_\mu \beta_i^{(0)} r^2 \left(1 - f(r)\right) dx^i dv - 4\frac{x^\mu \partial_\mu b^{(0)}}{r^2} dv^2 \\
& - 3h_1 dvdr + r^2 h_1 dx_i dx^i + \frac{k_1}{r^2} dv^2 + 2\frac{j_i^{(1)}}{r^2} dvdx^i + r^2 \alpha_{ij}^{(1)} dx^i dx^j
\end{aligned} \tag{2.33}
$$

And without loss of generality, we take $b = 1$ and $u^\mu = (1, 0, 0, 0)$ at our chosen point; note that the derivatives $\partial_\mu \beta_i^{(0)}$ and $\partial_\mu b^{(0)}$ in the above expression are the values evaluated here also. That is all that we will say at the moment; a more complete explanation of the necessity of this local expansion will be deferred to subsection 2.3.6.

2.3.2 Inverse metric, g^{ab}

In this subsection, we obtain the inverse metric g^{ab} to first order in boundary spacetime derivatives. This will be required to evaluate the terms in the Einstein equation. The inverse metric to first order should obey the following:

$$\begin{aligned} & g_{ab}g^{bc} = \delta_a^c \\ \implies & \left(g_{ab}^{(0)} + g_{ab}^{(1)}\right)\left(g^{(0)bc} + g^{(1)bc}\right) = \delta_a^c \\ \implies & g_{ab}^{(0)}g^{(0)bc} + g_{ab}^{(0)}g^{(1)bc} + g_{ab}^{(1)}g^{(0)bc} + O\left(\epsilon^2\right) = \delta_a^c \\ \implies & g_{ab}^{(0)}g^{(0)bc} = \delta_a^c \quad \text{and} \quad g_{ab}^{(0)}g^{(1)bc} + g_{ab}^{(1)}g^{(0)bc} = 0. \end{aligned} \tag{2.34}$$

Previously we obtained the metric at zeroth and first order. We repeat these below in matrix form (basis in terms of r, v, i):

$$g_{ab}^{(0)} = \begin{bmatrix} 0 & 1 & -x^\mu \partial_\mu \beta_i^{(0)} \\ 1 & -\left(r^2 f + \frac{4x^\mu \partial_\mu b^{(0)}}{r^2}\right) & -x^\mu \partial_\mu \beta_i^{(0)} r^2 \left(1-f\right) \\ -x^\mu \partial_\mu \beta_i^{(0)} & -x^\mu \partial_\mu \beta_i^{(0)} r^2 \left(1-f\right) f^2 & r^2 \delta_{ij} \end{bmatrix},$$

$$g_{ab}^{(1)} = \begin{bmatrix} 0 & -\frac{3}{2}h_1 & 0 \\ -\frac{3}{2}h_1 & \frac{k_1}{r^2} & r^2\left(1-f\right) j_i^{(1)} \\ 0 & r^2\left(1-f\right) j_i^{(1)} & r^2 h_1 \delta_{ij} + r^2 \alpha_{ij} \end{bmatrix}.$$

We can calculate the inverse metric at zeroth order simply using $g_{ab}^{(0)}g^{(0)bc} = \delta_a^c$:

$$g^{(0)ab} = \begin{bmatrix} r^2 f + \frac{4x^\mu \partial_\mu b^{(0)}}{r^2} & 1 & x^\mu \partial_\mu \beta_i^{(0)} \\ 1 & 0 & \frac{x^\mu \partial_\mu \beta_i^{(0)}}{r^2} \\ x^\mu \partial_\mu \beta_i^{(0)} & \frac{x^\mu \partial_\mu \beta_i^{(0)}}{r^2} & \frac{1}{r^2}\delta_{ij} \end{bmatrix}.$$

With this, we now compute the product $g_{ab}^{(1)}g^{(0)bc}$:

$$g_{ab}^{(1)}g^{(0)bc} = \begin{bmatrix} -\frac{3}{2}h_1 & 0 & 0 \\ -\frac{3}{2}r^2 f + \frac{k_1}{r^2} & -\frac{3}{2}h_1 & (1-f)j_i^{(1)} \\ r^2(1-f)j_i^{(1)} & 0 & h_1\delta_{ij} + \alpha_{ij} \end{bmatrix}$$

and using the relation $g_{ab}^{(0)}g^{(1)bc} + g_{ab}^{(1)}g^{(0)bc} = 0$, we obtain the first order component of the inverse metric:

$$g^{(1)ab} = \begin{bmatrix} 3h_1 r^2 f - \frac{k_1}{r^2} & \frac{3}{2}h_1 & -(1-f)j_i^{(1)} \\ \frac{3}{2}h_1 & 0 & 0 \\ -(1-f)j_i^{(1)} & 0 & -\frac{1}{r^2}\left(h_1\delta_{ij} + \alpha_{ij}\right) \end{bmatrix}.$$

2.3.3 First order equations: Scalar components

Here we will compute the Einstein equations at first order. There are two types of terms which will appear; those which contain the unknown functions $h_1, k_1, j_i^{(1)}, \alpha_{ij}$ and those which do not. The terms which contain those functions form the differential operator for our resulting equations and the other terms will be the source terms for those differential equations.

Our focus in this subsection will be on the scalar equations (the following subsections deal with the vector and tensor components).

E_{rr}

We begin with E_{rr}:

$$
\begin{aligned}
E_{rr} &= R_{rr} + 4g_{rr} \\
&= g^{ab}{}_{,a} g_{rb,r} - \frac{1}{2} g^{ab}{}_{,a} g_{rr,b} + g^{ab} g_{rb,ra} - \frac{1}{2} g^{ab} g_{rr,ba} \\
&- \frac{1}{2} g^{ab}{}_{,r} g_{ab,r} - \frac{1}{2} g^{ab} g_{ab,rr} \\
&+ \frac{1}{2} g^{cb} g^{ad} g_{ad,c} g_{rb,r} - \frac{1}{4} g^{cb} g^{ad} g_{ad,c} g_{rr,b} \\
&- \frac{1}{4} g^{cb} g^{ad} g_{cd,r} g_{ab,r} \\
&+ \frac{1}{2} g^{cb} g^{ad} g_{cr,d} g_{rb,a} - \frac{1}{2} g^{cb} g^{ad} g_{cr,d} g_{ar,b} + 4g_{rr}
\end{aligned}
\tag{2.35}
$$

First we calculate the terms which give rise to the differential operator, E_{rr}^{diff}. To do this, we express the metric g_{ab} in terms of zeroth and first order terms $g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)}$. Then, we isolate the terms which contain $g_{ab}^{(1)}$ (or its inverse) and are first order in boundary spacetime derivatives. This can essentially be thought of as a simple problem in combinatorics. For example, consider the product of terms, $\frac{1}{2} g^{cb} g^{ad} g_{cr,d} g_{rb,a}$. The condition that it must contain $g_{ab}^{(1)}$ or its inverse immediately leads to four options. Either the first term could be $g^{(1)cb}$ and all the other terms $g^{(0)}$, or the second term be $g^{(1)ad}$ and again all the others $g^{(0)}$, and so on. When one term is chosen to be $g^{(1)}$ this forces the derivatives of all other terms to be with respect to r since overall it must be of first order. The expression $\frac{1}{2} g^{cb} g^{ad} g_{cr,d} g_{rb,a}$ would thus lead to the terms given below:

$$\frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r}$$
$$+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{rb,r} \tag{2.36}$$

where μ,ν depict boundary spacetime directions v and i. This would of course become more involved at higher orders. Overall, we find that E^{diff}_{rr} comprises of the following terms:

$$\begin{aligned}
E^{diff}_{rr} = &-\frac{1}{2}g^{(1)\mu\nu}{}_{,r}g^{(0)}{}_{\mu\nu,r}-\frac{1}{2}g^{(0)\mu\nu}{}_{,r}g^{(1)}_{\mu\nu,r}-\frac{1}{2}g^{(1)\mu\nu}g^{(0)}_{\mu\nu,rr}-\frac{1}{2}g^{(0)\mu\nu}g^{(1)}_{\mu\nu,rr}\\
&+\frac{1}{2}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r}\\
&+\frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{rb,r}\\
&-\frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r}-\frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r}\\
&-\frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{rr,r}-\frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{rr,r}\\
&-\frac{1}{4}g^{(1)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r}-\frac{1}{4}g^{(0)cb}g^{(1)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r}\\
&-\frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(1)}_{cd,r}g^{(0)}_{ab,r}-\frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(1)}_{ab,r}\\
&+\frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r}\\
&+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{rb,r}+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{rb,r}\\
&-\frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r}-\frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r}\\
&-\frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cr,r}g^{(0)}_{ar,r}-\frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(1)}_{ar,r}\\
&+4g^{(1)}_{rr}
\end{aligned} \tag{2.37}$$

Most of these terms evaluate to zero but below we show the nonzero terms:

$$
\begin{aligned}
E_{rr}^{diff} &= -\frac{1}{2}\cdot 2r\cdot\left(\frac{-3h_1}{r^2}\right)' \quad -\frac{1}{2}\cdot\frac{-2}{r^3}\cdot\left(3r^2h_1\right)' \quad -\frac{1}{2}\cdot 2\cdot\frac{-3h_1}{r^2} \quad -\frac{1}{2}\cdot\frac{1}{r^2}\cdot\left(3h_1r^2\right)'' \\
&+0+0 \\
&+0 \quad +\frac{1}{2}\cdot\frac{6}{r}\cdot\frac{-3h_1'}{2} \\
&-0-0 \\
&-0-0 \\
&-\frac{1}{4}\cdot\frac{-h_1}{r^2}\cdot\frac{1}{r^2}\cdot 2r\cdot 2r\cdot 3 \quad -\frac{1}{4}\cdot\frac{-h_1}{r^2}\cdot\frac{1}{r^2}\cdot 2r\cdot 2r\cdot 3 \\
&-\frac{1}{4}\cdot\frac{1}{r^2}\cdot\frac{1}{r^2}\left(r^2h_1\right)'\cdot 2r\cdot 3 \quad -\frac{1}{4}\cdot\frac{1}{r^2}\cdot\frac{1}{r^2}\left(r^2h_1\right)'\cdot 2r\cdot 3 \\
&+0+0 \\
&+0+0 \\
&-0-0 \\
&-0-0 \\
&+4\cdot 0 \\
&= -\frac{15}{2}\frac{h_1'}{r}-\frac{3}{2}h_1''.
\end{aligned}
\tag{2.38}
$$

The $'$ symbol denotes a derivative with respect to r.

And now for the source terms, E_{rr}^{source}; we are interested in terms which do not contain $g^{(1)}$ and hence do not contain any of the unknown functions $h_1, k_1, j_i^{(1)}, \alpha_{ij}^{(1)}$. There must only be zeroth order metric terms and their derivatives; overall the terms must be of first order. We obtain:

$$
\begin{aligned}
E_{rr}^{source} &= g^{(0)\mu b}{}_{,\mu}g^{(0)}_{rb,r} - \frac{1}{2}g^{(0)\mu r}{}_{,\mu}g^{(0)}_{rr,r} - \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(0)}_{rr,\mu} + g^{(0)\mu b}g^{(0)}_{rb,r\mu} \\
&- g^{(0)r\mu}g^{(0)}_{rr,r\mu} \\
&+ \frac{1}{2}g^{(0)\mu b}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{rb,r} - \frac{1}{4}g^{(0)\mu r}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,\mu} \\
&+ g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,r}g^{(0)}_{rb,\mu} - g^{(0)cr}g^{(0)a\mu}g^{(0)}_{cr,\mu}g^{(0)}_{ar,r}.
\end{aligned}
\tag{2.39}
$$

In fact, all of these terms evaluate to zero. So the final equation for E_{rr} at first order is just:

$$
-\frac{3}{2}h_1'' - \frac{15}{2}\frac{h_1'}{r} = 0. \tag{2.40}
$$

E_{rv}

We now repeat the computation for the E_{rv} component of the Einstein equations. The procedure is essentially the same, so from here on we leave out some of the algebra.

$$\begin{aligned}
E_{rv} &= R_{rv} + 4g_{rv} \\
&= \frac{1}{2}g^{ab}{}_{,a}g_{vb,r} + \frac{1}{2}g^{ab}{}_{,a}g_{rb,v} - \frac{1}{2}g^{ab}{}_{,a}g_{rv,b} + \frac{1}{2}g^{ab}g_{rb,va} - \frac{1}{2}g^{ab}g_{rv,ba} \\
&- \frac{1}{2}g^{ab}{}_{,r}g_{ab,v} - \frac{1}{2}g^{ab}g_{ab,vr} + \frac{1}{2}g^{ab}g_{av,br} \\
&+ \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{vb,r} + \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{rb,v} - \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{rv,b} \\
&- \frac{1}{4}g^{cb}g^{ad}g_{cd,r}g_{ab,v} \\
&+ \frac{1}{2}g^{cb}g^{ad}g_{cr,d}g_{vb,a} - \frac{1}{2}g^{cb}g^{ad}g_{cr,d}g_{av,b} + 4g_{rv}
\end{aligned} \tag{2.41}$$

In terms of zeroth and first order metric terms, the differential part becomes:

$$\begin{aligned}
E_{rv} = &\frac{1}{2}g^{(1)r\mu}{}_{,r}g^{(0)}_{v\mu,r} + \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(1)}_{v\mu,r} - \frac{1}{2}g^{(1)\mu r}g^{(0)}_{\mu v,rr} + \frac{1}{2}g^{(0)\mu r}g^{(1)}_{\mu v,rr} \\
&- \frac{1}{2}g^{(1)rr}g^{(0)}_{rv,rr} - \frac{1}{2}g^{(0)rr}g^{(1)}_{rv,rr} + \frac{1}{2}g^{(1)ar}g^{(0)}_{av,rr} + \frac{1}{2}g^{(0)ar}g^{(1)}_{av,rr} \\
&+ \frac{1}{4}g^{(1)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{v\mu,r} + \frac{1}{4}g^{(0)r\mu}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{v\mu,r} \\
&+ \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{v\mu,r} + \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{v\mu,r} \\
&+ \frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{vb,r} + \frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{vb,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{vb,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{vb,r} \\
&- \frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cr,r}g^{(0)}_{av,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cr,r}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(1)}_{av,r} \\
&+ g^{(1)}_{rv}
\end{aligned} \tag{2.42}$$

which reduces to:

$$E^{diff}_{rv} = -12h_1 + -3rh_1' - \frac{1}{2r^3}k_1' + \frac{1}{2r^2}k_1''. \tag{2.43}$$

And for the source terms:

$$
\begin{aligned}
E_{rv}^{source} = & \frac{1}{2}g^{(0)\mu b}{}_{,\mu}g^{(0)}_{vb,r} + \frac{1}{2}g^{(0)rb}{}_{,r}g^{(0)}_{rb,v} - \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(0)}_{rv,\mu} - \frac{1}{2}g^{(0)r\mu}{}_{,\mu}g^{(0)}_{rv,r} \\
& + \frac{1}{2}g^{(0)rb}g^{(0)}_{rb,vr} - g^{(0)r\mu}g^{(0)}_{rv,r\mu} \\
& - \frac{1}{2}g^{(0)ab}{}_{,r}g^{(0)}_{ab,v} - \frac{1}{2}g^{(0)ab}g^{(0)}_{ab,vr} + \frac{1}{2}g^{(0)a\mu}g^{(0)}_{av,\mu r} \\
& + \frac{1}{4}g^{(0)\mu b}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{vb,r} + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,v} \\
& - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rv,\mu} - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{rv,r} \\
& - \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,v} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,r}g^{(0)}_{vb,\mu} + \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,\mu}g^{(0)}_{vb,r} \\
& - \frac{1}{2}g^{(0)cr}g^{(0)a\mu}g^{(0)}_{cr,\mu}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)c\mu}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{av,\mu}
\end{aligned}
\tag{2.44}
$$

which simplifies to:

$$E_{rv}^{source} = -\frac{1}{r}\partial_i\beta_i^{(0)} \tag{2.45}$$

leading to a complete equation of:

$$-12h_1 - 3rh_1' - \frac{1}{2r^3}k_1' + \frac{1}{2r^2}k_1'' - \frac{1}{r}\partial_i\beta_i^{(0)} = 0. \tag{2.46}$$

E_{vv}

The final scalar equation is the vv-component of the Einstein equations.

$$
\begin{aligned}
E_{vv} &= R_{vv} + 4g_{vv} \\
&= g^{ab}{}_{,a}g_{vb,v} - \frac{1}{2}g^{ab}{}_{,a}g_{vv,b} + g^{ab}g_{vb,va} - \frac{1}{2}g^{ab}g_{vv,ba} \\
&\quad - \frac{1}{2}g^{ab}{}_{,v}g_{ab,v} - \frac{1}{2}g^{ab}g_{ab,vv} \\
&\quad + \frac{1}{2}g^{cb}g^{ad}g_{ad,c}g_{vb,v} - \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{vv,b} \\
&\quad - \frac{1}{4}g^{cb}g^{ad}g_{cd,v}g_{ab,v} \\
&\quad + \frac{1}{2}g^{cb}g^{ad}g_{cv,d}g_{vb,a} - \frac{1}{2}g^{cb}g^{ad}g_{cv,d}g_{av,b} + 4g_{vv}
\end{aligned}
\tag{2.47}
$$

In terms of the zeroth and first order metric terms, the differential operator comprises of the following:

$$
\begin{aligned}
E_{vv}^{diff} = & -\frac{1}{2}g^{(1)rr}{}_{,r}g^{(0)}_{vv,r} - \frac{1}{2}g^{(0)rr}{}_{,r}g^{(1)}_{vv,r} - \frac{1}{2}g^{(1)rr}g^{(0)}_{vv,rr} - \frac{1}{2}g^{(0)rr}g^{(1)}_{vv,rr} \\
& - \frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{vv,r} - \frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{vv,r} \\
& - \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{vv,r} - \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{vv,r} \\
& + \frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cv,r}g^{(0)}_{vb,r} + \frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cv,r}g^{(0)}_{vb,r} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cv,r}g^{(0)}_{vb,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cv,r}g^{(1)}_{vb,r} \\
& - \frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cv,r}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cv,r}g^{(0)}_{av,r} \\
& - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cv,r}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cv,r}g^{(1)}_{av,r} \\
& + 4g^{(1)}_{vv}.
\end{aligned} \tag{2.48}
$$

Computing this explicitly gives us:

$$
E_{vv}^{diff} = \left(12r^2 - \frac{12}{r^2}\right)h_1 + \left(3r^3 - \frac{3}{r}\right)h_1' + \frac{1}{2}\frac{f}{r}k_1' - \frac{1}{2}fk_1''. \tag{2.49}
$$

The source terms are as follows:

$$
\begin{aligned}
E_{vv}^{source} = & \; g^{(0)rb}{}_{,r}g^{(0)}_{vb,v} - \frac{1}{2}g^{(0)r\mu}{}_{,\mu}g^{(0)}_{vv,r} - \frac{1}{2}g^{(0)\mu r}{}_{,r}g^{(0)}_{vv,\mu} + g^{(0)rb}g^{(0)}_{vb,vr} \\
& - g^{(0)r\mu}g^{(0)}_{vv,r\mu} \\
& + \frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{vb,v} - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{vv,\mu} - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{vv,r} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cv,r}g^{(0)}_{vb,\mu} + \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cv,\mu}g^{(0)}_{vb,r} \\
& - \frac{1}{2}g^{(0)cr}g^{(0)a\mu}g^{(0)}_{cv,\mu}g^{(0)}_{av,r} - \frac{1}{2}g^{(0)c\mu}g^{(0)ar}g^{(0)}_{cv,r}g^{(0)}_{av,\mu}
\end{aligned} \tag{2.50}
$$

which reduces to:

$$
E_{vv}^{source} = \partial_i\beta_i^{(0)}\left(r + \frac{1}{r^3}\right) - \frac{6\partial_v b^{(0)}}{r^3}. \tag{2.51}
$$

Thus the complete equation is:

$$
\left(12r^2 - \frac{12}{r^2}\right)h_1 + \left(3r^3 - \frac{3}{r}\right)h_1' + \frac{1}{2}\frac{f}{r}k_1' - \frac{1}{2}fk_1'' + \partial_i\beta_i^{(0)}\left(r + \frac{1}{r^3}\right) - \frac{6\partial_v b^{(0)}}{r^3} = 0. \tag{2.52}
$$

2.3.4 First order equations: Vector components

E_{ri}

Similarly, here we perform the same calculations for the vector components starting with E_{ri}:

$$\begin{aligned}
E_{ri} = & \frac{1}{2}g^{ab}{}_{,a}g_{ib,r} + \frac{1}{2}g^{ab}{}_{,a}g_{rb,i} - \frac{1}{2}g^{ab}{}_{,a}g_{ri,b} + \frac{1}{2}g^{ab}g_{rb,ia} - \frac{1}{2}g^{ab}g_{ri,ba} \\
& - \frac{1}{2}g^{ab}{}_{,r}g_{ab,i} - \frac{1}{2}g^{ab}g_{ab,ir} + \frac{1}{2}g^{ab}g_{ai,br} \\
& + \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{ib,r} + \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{rb,i} - \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{ri,b} \\
& - \frac{1}{4}g^{cb}g^{ad}g_{cd,r}g_{ab,i} \\
& + \frac{1}{2}g^{cb}g^{ad}g_{cr,d}g_{ib,a} - \frac{1}{2}g^{cb}g^{ad}g_{cr,d}g_{ai,b} \\
& + 4g_{ri}.
\end{aligned} \tag{2.53}$$

The differential part in terms of the zeroth and first order metric components is given by:

$$\begin{aligned}
E^{diff}_{ri} = & \frac{1}{2}g^{(1)r\mu}{}_{,r}g^{(0)}_{i\mu,r} + \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(1)}_{i\mu,r} + \frac{1}{2}g^{(1)\mu r}g^{(0)}_{\mu i,rr} + \frac{1}{2}g^{(0)\mu r}g^{(1)}_{\mu i,rr} \\
& + \frac{1}{4}g^{(1)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{i\mu,r} + \frac{1}{4}g^{(0)r\mu}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{i\mu,r} \\
& + \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{i\mu,r} + \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{i\mu,r} \\
& + \frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{ib,r} + \frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{ib,r} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{ib,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{ib,r} \\
& - \frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{ai,r} - \frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cr,r}g^{(0)}_{ai,r} \\
& - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cr,r}g^{(0)}_{ai,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(1)}_{ai,r} \\
& + 4g^{(1)}_{ri},
\end{aligned} \tag{2.54}$$

which evaluates to:

$$E^{diff}_{ri} = \frac{1}{2r^2}j^{(1)\prime\prime}_i - \frac{3}{2r^3}j^{(1)\prime}_i. \tag{2.55}$$

And for the source terms, we find:

$$
\begin{aligned}
E^{source}_{ri} &= \frac{1}{2}g^{(0)\mu b}{}_{,\mu}g^{(0)}_{ib,r} + \frac{1}{2}g^{(0)rb}{}_{,r}g^{(0)}_{rb,i} - \frac{1}{2}g^{(0)\mu r}{}_{,\mu}g^{(0)}_{ri,r} - \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(0)}_{ri,\mu} \\
&+ \frac{1}{2}g^{(0)rb}g^{(0)}_{rb,ir} - g^{(0)r\mu}g^{(0)}_{ri,r\mu} \\
&- \frac{1}{2}g^{(0)ab}{}_{,r}g^{(0)}_{ab,i} - \frac{1}{2}g^{(0)ab}g^{(0)}_{ab,ir} + \frac{1}{2}g^{(0)a\mu}g^{(0)}_{ai,\mu r} \\
&+ \frac{1}{4}g^{(0)\mu b}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{ib,r} + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,i} - \frac{1}{4}g^{(0)\mu r}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{ri,r} \\
&- \frac{1}{4}g^{(0)\mu r}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{ri,\mu} \\
&- \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,i} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,\mu}g^{(0)}_{ib,r} + \frac{1}{2}g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,r}g^{(0)}_{ib,\mu} \\
&- \frac{1}{2}g^{(0)c\mu}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{ai,\mu} - \frac{1}{2}g^{(0)cr}g^{(0)a\mu}g^{(0)}_{cr,\mu}g^{(0)}_{ai,r}
\end{aligned}
\tag{2.56}
$$

which evaluates to just:

$$E^{source}_{ri} = \frac{3}{2r}\partial_v\beta^{(0)}_i. \tag{2.57}$$

Thus the complete equation is:

$$\frac{1}{2r^2}j^{(1)\prime\prime}_i - \frac{3}{2r^3}j^{(1)\prime}_i + \frac{3}{2r}\partial_v\beta^{(0)}_i = 0. \tag{2.58}$$

E_{vi}

The final vector equation is given by the vi-component of the Einstein equations.

$$
\begin{aligned}
E_{vi} &= R_{vi} + 4g_{vi} \\
&= \frac{1}{2}g^{ab}{}_{,a}g_{ib,v} + \frac{1}{2}g^{ab}{}_{,a}g_{vb,i} - \frac{1}{2}g^{ab}{}_{,a}g_{vi,b} + \frac{1}{2}g^{ab}g_{vb,ia} - \frac{1}{2}g^{ab}g_{vi,ba} \\
&- \frac{1}{2}g^{ab}{}_{,v}g_{ab,i} - \frac{1}{2}g^{ab}g_{ab,iv} + \frac{1}{2}g^{ab}g_{ai,bv} \\
&+ \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{ib,v} + \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{vb,i} - \frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{vi,b} \\
&- \frac{1}{4}g^{cb}g^{ad}g_{cd,v}g_{ab,i} \\
&+ \frac{1}{2}g^{cb}g^{ab}g_{cv,d}g_{ib,a} - \frac{1}{2}g^{cb}g^{ad}g_{cv,d}g_{ai,b} + 4g_{vi}
\end{aligned}
\tag{2.59}
$$

The differential operator can be calculated as follows:

$$
\begin{aligned}
E_{vi}^{diff} = & -\frac{1}{2} g^{(1)rr}{}_{,r} g^{(0)}_{vi,r} - \frac{1}{2} g^{(0)rr}{}_{,r} g^{(1)}_{vi,r} - \frac{1}{2} g^{(1)rr} g^{(0)}_{vi,rr} - \frac{1}{2} g^{(0)rr} g^{(1)}_{vi,rr} \\
& - \frac{1}{4} g^{(1)rr} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{vi,r} - \frac{1}{4} g^{(0)rr} g^{(1)ad} g^{(0)}_{ad,r} g^{(0)}_{vi,r} \\
& - \frac{1}{4} g^{(0)rr} g^{(0)ad} g^{(1)}_{ad,r} g^{(0)}_{vi,r} - \frac{1}{4} g^{(0)rr} g^{(0)ad} g^{(0)}_{ad,r} g^{(1)}_{vi,r} \\
& + \frac{1}{2} g^{(1)cb} g^{(0)rr} g^{(0)}_{cv,r} g^{(0)}_{ib,r} + \frac{1}{2} g^{(0)cb} g^{(1)rr} g^{(0)}_{cv,r} g^{(0)}_{ib,r} \\
& + \frac{1}{2} g^{(0)cb} g^{(0)rr} g^{(1)}_{cv,r} g^{(0)}_{ib,r} + \frac{1}{2} g^{(0)cb} g^{(0)rr} g^{(0)}_{cv,r} g^{(1)}_{ib,r} \\
& - \frac{1}{2} g^{(1)cr} g^{(0)ar} g^{(0)}_{cv,r} g^{(0)}_{ai,r} - \frac{1}{2} g^{(0cr} g^{(1)ar} g^{(0)}_{cv,r} g^{(0)}_{ai,r} \\
& - \frac{1}{2} g^{(0)cr} g^{(0)ar} g^{(1)}_{cv,r} g^{(0)}_{ai,r} - \frac{1}{2} g^{(0cr} g^{(0)ar} g^{(0)}_{cv,r} g^{(1)}_{ai,r} \\
& + 4 g^{(1)}_{vi}
\end{aligned}
\tag{2.60}
$$

which, when evaluated, simplifies to just:

$$
E_{vi}^{diff} = -\frac{1}{2} f j_i^{(1)\prime\prime} + \frac{3}{2} \frac{f}{r} j_i^{(1)\prime} \tag{2.61}
$$

For the source terms, we find:

$$
\begin{aligned}
E_{vi}^{source} = & \frac{1}{2} g^{(0)rb}{}_{,r} g^{(0)}_{ib,v} + \frac{1}{2} g^{(0)rb}{}_{vb,i} - \frac{1}{2} g^{(0)r\mu}{}_{,r} g^{(0)}_{vi,\mu} - \frac{1}{2} g^{(0)\mu r}{}_{,\mu} g^{(0)}_{vi,r} \\
& + \frac{1}{2} g^{(0)rb} g^{(0)}_{vb,ir} - g^{(0)r\mu} g^{(0)}_{vi,r\mu} \\
& + \frac{1}{2} g^{(0)ar} g^{(0)}_{ai,rv} \\
& + \frac{1}{4} g^{(0)rb} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{ib,v} + \frac{1}{4} g^{(0)rb} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{vb,i} \\
& - \frac{1}{4} g^{(0)r\mu} g^{(0)ad} g^{(0}_{ad,r} g^{(0)}_{vi,\mu} - \frac{1}{4} g^{(0)\mu r} g^{(0)ad} g^{(0)}_{ad,\mu} g^{(0)}_{vi,r} \\
& + \frac{1}{2} g^{(0)cb} g^{(0)r\mu} g^{(0)}_{cv,\mu} g^{(0)}_{ib,r} + \frac{1}{2} g^{(0)cb} g^{(0)\mu r} g^{(0)}_{cv,r} g^{(0)}_{ib,\mu} \\
& - \frac{1}{2} g^{(0)cr} g^{(0)a\mu} g^{(0)}_{cv,\mu} g^{(0)}_{ai,r} - \frac{1}{2} g^{(0)c\mu} g^{(0)ar} g^{(0)}_{cv,r} g^{(0)}_{ai,\mu}
\end{aligned}
\tag{2.62}
$$

which evaluates to:

$$
E_{vi}^{source} = \frac{2}{r^3} \partial_i b^{(0)} - \frac{1}{2r^3} \partial_v \beta_i^{(0)} - \frac{3}{2} r \partial_v \beta_i^{(0)}. \tag{2.63}
$$

And thus for E_{vi}, we are left with the following complete equation:

$$-\frac{1}{2}fj_i^{(1)\prime\prime}+\frac{3}{2}\frac{f}{r}j_i^{(1)\prime}+\frac{2}{r^3}\partial_i b^{(0)}-\frac{1}{2r^3}\partial_v\beta_i^{(0)}-\frac{3}{2}r\partial_v\beta_i^{(0)}=0. \tag{2.64}$$

2.3.5 First order equations: Tensor components

E_{ij}

The tensor components are slightly more involved. There will be a traceless tensor part and a scalar trace.

$$\begin{aligned}
E_{ij} &= R_{ij}+4g_{ij}\\
&=\frac{1}{2}g^{ab}{}_{,a}g_{jb,i}+\frac{1}{2}g^{ab}{}_{,a}g_{ib,j}-\frac{1}{2}g^{ab}{}_{,a}g_{ij,b}+\frac{1}{2}g^{ab}g_{ib,ja}-\frac{1}{2}g^{ab}g_{ij,ba}\\
&-\frac{1}{2}g^{ab}{}_{,i}g_{ab,j}-\frac{1}{2}g^{ab}g_{ab,ji}+\frac{1}{2}g^{ab}g_{aj,bi}\\
&+\frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{jb,i}+\frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{ib,j}-\frac{1}{4}g^{cb}g^{ad}g_{ad,c}g_{ij,b}\\
&-\frac{1}{4}g^{cb}g^{ad}g_{cd,i}g_{ab,j}\\
&+\frac{1}{2}g^{cb}g^{ad}g_{ci,d}g_{jb,a}-\frac{1}{2}g^{cb}g^{ad}g_{ci,d}g_{aj,b}+4g_{ij}
\end{aligned} \tag{2.65}$$

In terms of the zeroth and first order metric components, the differential part comprises of the following:

$$\begin{aligned}
E_{ij}^{diff} &= -\frac{1}{2}g^{(1)rr}{}_{,r}g^{(0)}_{ij,r}-\frac{1}{2}g^{(0)rr}{}_{,r}g^{(1)}_{ij,r}-\frac{1}{2}g^{(1)rr}g^{(0)}_{ij,rr}-\frac{1}{2}g^{(0)rr}g^{(1)}_{ij,rr}\\
&-\frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{ij,r}-\frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{ij,r}\\
&-\frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{ij,r}-\frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{ij,r}\\
&+\frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{ci,r}g^{(0)}_{jb,r}+\frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{ci,r}g^{(0)}_{jb,r}\\
&+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{ci,r}g^{(0)}_{jb,r}+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{ci,r}g^{(1)}_{jb,r}\\
&-\frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{ci,r}g^{(0)}_{aj,r}-\frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{ci,r}g^{(0)}_{aj,r}\\
&-\frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{ci,r}g^{(0)}_{aj,r}-\frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{ci,r}g^{(1)}_{aj,r}\\
&+4g^{(1)}_{ij}
\end{aligned} \tag{2.66}$$

and this evaluates to:

$$\begin{aligned} E^{diff}_{ij} &= -12r^2 h_1 \delta_{ij} + \left(\frac{-11}{2}r^3 + \frac{7}{2r}\right) h_1' \delta_{ij} - \frac{r^4 f}{2} h_1'' \delta_{ij} + \frac{1}{r} k_1' \delta_{ij} \\ &+ \left(\frac{-5}{2}r^3 + \frac{1}{2r}\right) \alpha^{(1)\prime}_{ij} - \frac{1}{2} r^4 f \alpha^{(1)\prime\prime}_{ij}. \end{aligned} \tag{2.67}$$

And the source terms:

$$\begin{aligned} E^{source}_{ij} &= \frac{1}{2} g^{(0)rb}{}_{,r} g^{(0)}_{jb,i} + \frac{1}{2} g^{(0)rb}{}_{,r} g^{(0)}_{ib,j} - \frac{1}{2} g^{(0)r\mu}{}_{,r} g^{(0)}_{ij,\mu} - \frac{1}{2} g^{(0)r\mu}{}_{,\mu} g^{(0)}_{ij,r} + \frac{1}{2} g^{(0)rb} g^{(0)}_{ib,jr} \\ &- g^{(0)r\mu} g^{(0)}_{ij,r\mu} + \frac{1}{2} g^{(0)ar} g^{(0)}_{aj,ri} \\ &+ \frac{1}{4} g^{(0)rb} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{jb,i} + \frac{1}{4} g^{(0)rb} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{ib,j} \\ &- \frac{1}{4} g^{(0)r\mu} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{ij,\mu} - \frac{1}{4} g^{(0)r\mu} g^{(0)ad} g^{(0)}_{ad,\mu} g^{(0)}_{ij,r} \\ &+ \frac{1}{2} g^{(0)cb} g^{(0)r\mu} g^{(0)}_{ci,r} g^{(0)}_{jb,\mu} + \frac{1}{2} g^{(0)cb} g^{(0)r\mu} g^{(0)}_{ci,\mu} g^{(0)}_{jb,r} \\ &- \frac{1}{2} g^{(0)cr} g^{(0)a\mu} g^{(0)}_{ci,\mu} g^{(0)}_{aj,r} - \frac{1}{2} g^{(0)c\mu} g^{(0)ar} g^{(0)}_{ci,r} g^{(0)}_{aj,\mu} \end{aligned} \tag{2.68}$$

which simplifies to:

$$E^{source}_{ij} = \left(-3r\partial_{(i}\beta^{(0)}_{j)} + r\partial_k \beta^{(0)}_k \delta_{ij}\right) - 2r\partial_k \beta^{(0)}_k \delta_{ij}. \tag{2.69}$$

The terms in brackets form the traceless component and the second part is the trace. The tensor component thus becomes an equation for the traceless tensor:

$$-\frac{1}{2} r^4 f \alpha^{(1)\prime\prime}_{ij} + \left(\frac{-5}{2}r^3 + \frac{1}{2r}\right) \alpha^{(1)\prime}_{ij} - 3r\partial_{(i}\beta^{(0)}_{j)} + r\partial_k \beta^{(0)}_k \delta_{ij} = 0 \tag{2.70}$$

and an equation for the scalar trace:

$$-12r^2 h_1 + \left(\frac{-11}{2}r^3 + \frac{7}{2r}\right) h_1' - \frac{r^4 f}{2} h_1'' + \frac{k_1'}{r} - 2r\partial_k \beta^{(0)}_k = 0. \tag{2.71}$$

2.3.6 Solutions

In this subsection, we aim to solve the differential equations that we have just derived and obtain the first order metric terms. Below we reproduce all the equations that we have derived thus far:

$$
\begin{aligned}
rr &: -\frac{3}{2}h_1'' - \frac{15}{2}\frac{h_1'}{r} = 0 \\
rv &: -12h_1 - 3rh_1' - \frac{1}{2r^3}k_1' + \frac{1}{2r^2}k_1'' - \frac{1}{r}\partial_i\beta_i^{(0)} = 0 \\
vv &: \left(12r^2 - \frac{12}{r^2}\right)h_1 + \left(3r^3 - \frac{3}{r}\right)h_1' + \frac{1}{2}\frac{f}{r}k_1' - \frac{1}{2}fk_1'' + \partial_i\beta_i^{(0)}\left(r + \frac{1}{r^3}\right) - \frac{6\partial_v b^{(0)}}{r^3} = 0 \\
ri &: \frac{1}{2r^2}j_i^{(1)\prime\prime} - \frac{3}{2r^3}j_i^{(1)\prime} + \frac{3}{2r}\partial_v\beta_i^{(0)} = 0 \\
vi &: -\frac{1}{2}fj_i^{(1)\prime\prime} + \frac{3}{2}\frac{f}{r}j_i^{(1)\prime} + \frac{2}{r^3}\partial_i b^{(0)} - \frac{1}{2r^3}\partial_v\beta_i^{(0)} - \frac{3}{2}r\partial_v\beta_i^{(0)} = 0 \\
ij \text{ traceless} &: -\frac{1}{2}r^4 f\alpha_{ij}^{(1)\prime\prime} + \left(\frac{-5}{2}r^3 + \frac{1}{2r}\right)\alpha_{ij}^{(1)\prime} - 3r\partial_{(i}\beta_{j)}^{(0)} + r\partial_k\beta_k^{(0)}\delta_{ij} = 0 \\
ij \text{ trace} &: -12r^2h_1 + \left(\frac{-11}{2}r^3 + \frac{7}{2r}\right)h_1' - \frac{r^4 f}{2}h_1'' + \frac{k_1'}{r} - 2r\partial_k\beta_k^{(0)} = 0.
\end{aligned}
\tag{2.72}
$$

By considering the following linear combinations of the above equations, we also find these constraints:

$$
\begin{aligned}
r^2 fE_{rr} + E_{vr} = 0 &: 12r^3h_1 + \left(3r^4 - 1\right)h_1' - k_1' = -6r^2\frac{\partial_i\beta_i^{(0)}}{3} \\
r^2 fE_{rv} + E_{vv} = 0 &: \partial_v b^{(0)} = \frac{\partial_i\beta_i^{(0)}}{3} \\
r^2 fE_{ri} + E_{vi} = 0 &: \partial_i b^{(0)} = \partial_v\beta_i^{(0)}.
\end{aligned}
\tag{2.73}
$$

For the first equation above we have to substitute for h_1'' and k_1'' using the ij-trace and vv-component equations to get the result shown.

To solve for h_1 we utilise the rr-component of the Einstein equations. This can be rewritten as the following first order differential equation which can then easily be solved:

$$
\left(r^5h_1'\right)' = 0. \tag{2.74}
$$

This gives us the solution below:

$$
h_1 = s + \frac{t}{r^4} \tag{2.75}
$$

where s and t are functions of the boundary spacetime coordinates.

And now for k_1, it is simplest to use our first constraint equation which after substituting for h_1 becomes:

$$12r^3 s + \frac{4t}{r^5} - k_1' = -6r^2 \frac{\partial_i \beta_i^{(0)}}{3} \tag{2.76}$$

which integrates to:

$$k_1 = \frac{2}{3} r^3 \partial_i \beta_i^{(0)} + 3r^4 s - \frac{t}{r^4} + u. \tag{2.77}$$

To determine the functions s, t, and u, we use three principles. First, the resulting metric cannot lead to deformations which violate our boundary conditions at $r = \infty$. The term $3r_4 s$ in the expression for k_1 would lead to a field theory metric deformation and hence we must set $s = 0$. Second, any coefficients which can arise from gauge transformations can be chosen to be zero. The coordinate transformation $r' = r\left(1 + \frac{a}{r^4}\right)$ can generate terms proportional to $\frac{1}{r^4}$ so we set $t = 0$. And third, if a metric term leads to terms in $T_{\mu\nu}$ which are essentially redefinitions of the temperature and velocity fields then this ambiguity is fixed via our choice of the Landau frame (as discussed in section 2.1). This is the case for u which vanishes if we demand $u_\mu^{(0)} T^{\mu\nu} = 0$.

For the vector sector, the function $j_i^{(1)}$ can be determined using the ri-component of the Einstein equations. This can be rewritten as:

$$\left(\frac{1}{r^3} j_i^{(1)\prime}\right)' = -\frac{3}{r^2} \partial_v \beta_i^{(0)} \tag{2.78}$$

which has the following solution:

$$j_i^{(1)} = \partial_v \beta_i^{(0)} r^3 + a_i r^4 + c_i. \tag{2.79}$$

The function a_i introduces a deformation to the boundary metric at $r = \infty$ and must be set to zero; c_i vanishes according to the Landau frame condition.

And in the tensor sector, we require the traceless part of the ij-component of the Einstein equation which is equivalent to:

$$\left(r^5 f \alpha_{ij}^{(1)}\right)' = -6r^2 \sigma_{ij} \tag{2.80}$$

where $\sigma_{ij} = \partial_{(i} \beta_{j)}^{(0)} - \frac{1}{3} \partial_k \beta_k^{(0)} \delta_{ij}$ is the boundary fluid tensor $\sigma_{\mu\nu}$ restricted to the boundary spatial coordinates and evaluated locally. This can be integrated to obtain:

$$\alpha_{ij}^{(1)} = 6\sigma_{ij}^{(0)} \int_r^\infty \frac{dx}{f x^5} \int_1^x y^2 dy \tag{2.81}$$

where note that the integral limits do not lead to any terms which violate our boundary conditions. The integral can be further simplified:

$$\int_r^\infty \frac{dx}{fx^5}\int_1^x y^2 dy = \frac{1}{3}\int_r^\infty \frac{dx\,(x^3-1)}{x\,(x^4-1)} = \frac{1}{3}\int_r^\infty \frac{dx\,(x^2+x+1)}{x\,(x^2+1)\,(x+1)} =: \frac{1}{3}F(r). \quad (2.82)$$

Combining our results across all sectors gives us the first order metric shown below:

$$\begin{aligned} ds^2 &= 2dvdr - r^2 f(r)dv^2 + r^2 dx_i dx^i \\ &- 2x^\mu \partial_\mu \beta_i^{(0)} dx^i dr - 2x^\mu \partial_\mu \beta_i^{(0)} r^2 \left(1 - f(r)\right) dx^i dv - 4\frac{x^\mu \partial_\mu b^{(0)}}{r^2} dv^2 \\ &+ \frac{2}{3} r \partial_i \beta_i^{(0)} dv^2 + 2r\partial_v \beta_i^{(0)} dv dx^i + 2r^2 F(r)\sigma_{ij} dx^i dx^j. \end{aligned} \quad (2.83)$$

Clearly this solution is only valid locally. However, we can easily write down a covariant expression which reduces to the above when evaluated close to our chosen point. Such a solution, given below, would then be valid globally.

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ &+ 2r^2 bF(br)\sigma_{\mu\nu} dx^\mu dx^\nu + \frac{2}{3} r u_\mu u_\nu \partial_\lambda u^\lambda dx^\mu dx^\nu - r u^\lambda \partial_\lambda \left(u_\mu u_\nu\right) dx^\mu dx^\nu. \end{aligned} \quad (2.84)$$

Before ending this subsection, there are two important points that we should mention. First, the constraint equations:

$$\begin{aligned} r^2 f E_{rv} + E_{vv} = 0 &: \partial_v b^{(0)} = \frac{\partial_i \beta_i^{(0)}}{3} \\ r^2 f E_{ri} + E_{vi} = 0 &: \partial_i b^{(0)} = \partial_v \beta_i^{(0)}. \end{aligned} \quad (2.85)$$

These have an especially simple dual boundary interpretation. They are equivalent to the conservation of the boundary stress tensor $\nabla_\mu T^{\mu\nu} = 0$ at first order in derivatives. And second, we stated earlier that it would be simpler to solve the Einstein equations locally before covariantising to obtain a global solution but gave no justification for this. We will do so now. The reason for this is the constraint equations. Perhaps at first order these are simple to solve but at higher orders they are far more nontrivial. By expanding locally about a point, finding solutions to the constraint equations reduces to simply matching various derivatives evaluated at our chosen point. This is more straightforward and we can then use these relations between derivatives to reduce our source terms to a subset of independent terms. We saw an analogue of this when we reviewed relativistic fluid dynamics; it was necessary to use the equations of motion to find independent terms for our stress tensor at each order. This ends our discussion of the derivation and solving of the first order equations.

2.4 Abstraction to higher orders

In this section, we comment on the structure of the differential equations and on how they generalise to higher orders. At a given order n, the equations take the schematic form below:

$$\mathbb{H}\left[g^{(0)}\right] g^{(n)}(r, x^{\mu}) = s_n. \tag{2.86}$$

Observe that since $g^{(n)}$ is already of order n in boundary derivatives, the differential operator is necessarily linear. Further, the differential operator solely involves the radial coordinate r and all its coefficients must be zeroth order functions. In this sense, $\mathbb{H}$ is ultralocal in the field theory directions; it cannot have any boundary derivatives. We also note that $\mathbb{H}$ is a second order differential operator which is the same at each order in the perturbation theory. It is second order because it inherits the structure of the Einstein equations, and it is independent of n because the exact same combinations of zeroth order functions and partial derivatives in r which act on $g^{(n)}$ will also act on $g^{(m)}$ for all $m \geq 1$, and hence we must have the same homogeneous operator at all orders in the perturbation theory. The source terms, however, which consist of boundary derivatives acting on lower order functions, will be different at each order.

It is certainly worth pausing here to emphasise what this means. In this long wavelength limit, the Einstein equations have reduced to a system of inhomogeneous second order linear differential equations in the variable r alone. Here we stress that we have not sacrificed the nonlinearity of the Einstein equations; the fact that the differential operator is linear is an advantage that we obtain by working perturbatively order by order in boundary derivatives. This deceptive linearity, coupled with the ultralocality in the boundary directions, is what makes the Einstein equations so much more tractable in this long wavelength limit.

We now comment further on the nature of these equations. In $d+1$ spacetime dimensions, this system of equations will provide us with $\frac{(d+1)(d+2)}{2}$ equations. Only $\frac{d^2+d+2}{2}$ of these equations will explicitly involve the unknown function $g^{(n)}$; and of these, one will prove to be redundant. For the first order case, these are the second order dynamical equations involving h_1, k_1, $j_i^{(1)}$, and $\alpha_{ij}^{(1)}$ that we derived. The remaining d

will only involve boundary derivatives of lower order terms, $g^{(m)}$ for $m \leq n-1$. These are our constraint equations.

The dynamical equations can always be solved by direct integration for an arbitrary source s_n; and subject to imposing regularity at $r > 0$, our boundary conditions at $r = \infty$, and fixing the ambiguity associated with redefinitions of the velocity and temperature fields via our choice of the Landau frame, a unique solution can be obtained. The constraint equations, on the other hand, impose relations between boundary derivatives of $g^{(m)}$ for $m \leq n-1$. And since these $g^{(m)}$ are themselves constructed from appropriate derivatives of the velocity ($u^\mu(x)$) and temperature ($T(x)$) fields, the constraint equations are ultimately relations constraining the allowed forms of $u^\mu(x)$ and $T(x)$. These constraint equations will always be the equations of conservation of the boundary stress tensor at one order lower,

$$\nabla_\mu T^{\mu\nu}_{(n-1)} = 0. \tag{2.87}$$

Thus we see that this methodology generalises in a fairly straightforward way to higher orders. We will see a detailed example at second order in the next chapter.

Chapter 3

Forced fluid dynamics from gravity in arbitrary dimensions

3.1 Introduction

A particularly interesting research direction that will be the focus of this chapter is the construction of bulk duals for forced fluid flows [36]. Solutions of fluid dynamics with particular forcing terms are known to exhibit turbulence, which is a phenomenon that is not well understood. A holographic understanding of turbulence may well provide new insights on this topic. Research along these lines has already begun; some examples in the literature relating to holographic turbulence are [41–44]. Another interesting aspect of turbulence in relativistic fluids is that its behaviour in $(2+1)$-spacetime dimensions is remarkably different to higher dimensions. In $(2+1)$-dimensions, energy dissipates by cascading from short to long wavelengths. The opposite behaviour is displayed in $(3+1)$ and higher-dimensional spacetimes. It is therefore of use to investigate holographic models of forced fluids in multiple spacetime dimensions.

In this chapter, we consider long wavelength solutions to the Einstein-dilaton system *in arbitrary spacetime dimensions*,

$$R_{AB} + dg_{AB} - \frac{1}{2}\partial_A\Phi\partial_B\Phi = 0, \tag{3.1}$$

$$\nabla^2\Phi = 0. \tag{3.2}$$

These bulk metrics are dual to the forced fluid dynamical motions of boundary field theories with actions of the form,

$$S = \int \sqrt{g} e^{-\phi} \mathcal{L}. \tag{3.3}$$

The boundary fluid obeys the following equation of motion,

$$\nabla_\mu T^{\mu\nu} = e^{-\phi} \mathcal{L} \nabla^\nu \phi, \tag{3.4}$$

which effectively is the equations for relativistic fluid dynamics with an explicit dilaton-dependent forcing term. Stating our results more explicitly, we construct long wavelength, asymptotically locally AdS_{d+1} bulk solutions with a slowly-varying boundary dilaton field and a weakly curved boundary metric to second order in a boundary derivative expansion. We also explicitly compute the fluid dynamical stress tensor and Lagrangian to second order in the derivative expansion thus generalising to arbitrary dimensions previous work by Bhattacharyya *et al* [36] which was specific to a five-dimensional bulk spacetime.

This chapter is organised as follows. In section 3.2 we present a review of the Weyl covariant notation for conformal fluid dynamics developed in [45] which we will be using throughout the rest of the chapter. Our main results are contained in section 3.3; we present explicit solutions to the Einstein-dilaton equations valid for arbitrary spacetime dimensions to second order in a boundary derivative expansion, as well as expressions for the boundary stress tensor and Lagrangian accurate to the same order. Our calculations are shown in great detail throughout. Section 3.4 has a discussion of our results and their significance.

3.2 Manifest Weyl covariance

We begin by reviewing the Weyl covariant formalism introduced in [45] for conformal relativistic fluid dynamics. This formalism allows for more compact notation. Also, as we shall see in the final subsection, the components of the bulk metric can be classified according to how they transform under Weyl rescaling; thus, it is convenient to adopt a formalism which makes their Weyl transformation properties manifest.

3.2.1 Regulation and Weyl symmetry

The aim of this subsection is to elaborate on a well-known subtlety in the AdS/CFT correspondence relating to the interpretation of the boundary field theory. This subtlety in interpretation leads to the Weyl covariant nature of the boundary fluid dynamics.

We begin by noting that to obtain the dual field theory interpretation of a bulk solution, one needs to regulate the solution near the boundary on slices of constant r, the radial coordinate. More concretely, the bulk solution will be interpreted as a state of the dual field theory on a background whose metric is related to the induced metric on the regulated boundary. However, there is a well-known ambiguity in the choice of the radial coordinate. To illustrate this further, consider the following parametrisation of AdS:

$$ds^2 = -2u_\mu dx^\mu dr + r^2 g_{\mu\nu} dx^\mu dx^\nu. \tag{3.5}$$

With this choice of coordinates, the dual field theory is considered to live on a background whose metric is given precisely by $g_{\mu\nu}$. If we instead choose a different radial coordinate $\tilde{r}$, given by a constant rescaling of r, and replace $g_{\mu\nu}$ and u_μ as follows:

$$r = \lambda^{-1}\tilde{r}, \quad u_\mu = \lambda \tilde{u}_\mu, \quad g_{\mu\nu} = \lambda^2 \tilde{g}_{\mu\nu}, \tag{3.6}$$

for constant λ, the bulk metric takes the following (invariant) form:

$$ds^2 = -2\tilde{u}_\mu dx^\mu d\tilde{r} + \tilde{r}^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu. \tag{3.7}$$

Regulating on surfaces of constant $\tilde{r}$ gives us a field theory vacuum state on a background $\tilde{g}_{\mu\nu} = \lambda^{-2} g_{\mu\nu}$. This equivalence between boundary metrics related by a constant rescaling arises from the dilatational symmetry of AdS, $SO(1,1)$. The full symmetry group of AdS, however, is the conformal group, $SO(d,2)$, and although this symmetry is not explicitly manifest in the choice of coordinates (3.5), the bulk AdS spacetime must therefore be dual to a field theory state defined on a space with any of the infinite number of metrics Weyl equivalent to $g_{\mu\nu}$; this reflects the Weyl symmetry of the dual field theory.

Now, bulk spacetimes dual to fluid dynamics are asymptotically locally AdS. As such, the boundary fluid dynamics should correspondingly be Weyl invariant. However, in contrast to AdS spacetime (3.5), this boundary Weyl symmetry is explicitly manifest in the bulk metric. By this we mean that if we choose to regulate the fluid dynamical bulk spacetime using a locally rescaled radial coordinate, $r = e^{-\chi(x^\mu)}\tilde{r}$, and perform the following simultaneous replacements:

$$r = e^{-\chi}\tilde{r}, \quad u_\mu = e^{\chi}\tilde{u}_\mu, \quad b = e^{\chi}\tilde{b}, \quad g_{\mu\nu} = e^{2\chi}\tilde{g}_{\mu\nu}, \tag{3.8}$$

the form of the bulk metric will remain invariant. We will now proceed to prove this. As previously established, bulk spacetimes dual to fluid dynamics admit an expansion in boundary derivatives of the form,

$$g_{AB} = g^{(0)}_{AB} + g^{(1)}_{AB} + g^{(2)}_{AB} + g^{(3)}_{AB} + \cdots , \tag{3.9}$$

where the zeroth order contribution, $g^{(0)}_{AB}$, is given by:

$$ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu. \tag{3.10}$$

Now, if we perform the simultaneous replacements (3.8), the bulk metric will take the form:

$$\tilde{g}_{AB} = \tilde{g}^{(0)}_{AB} + \tilde{g}^{(1)}_{AB} + \tilde{g}^{(2)}_{AB} + \tilde{g}^{(3)}_{AB} + \cdots , \tag{3.11}$$

where the terms are functions of the new rescaled variables:

$$\tilde{g}^{(n)}_{AB} \equiv \tilde{g}^{(n)}_{AB}(\tilde{r}, \tilde{u}_\mu, \tilde{b}) \quad \forall n. \tag{3.12}$$

But note that the form of the zeroth order contribution, $\tilde{g}^{(0)}_{AB}$, remains invariant under (3.8), *i.e.*:

$$\begin{aligned} &ds^2 = -2\tilde{u}_\mu dx^\mu d\tilde{r} - \tilde{r}^2 f(\tilde{b}\tilde{r}) \tilde{u}_\mu \tilde{u}_\nu dx^\mu dx^\nu + \tilde{r}^2 \tilde{P}_{\mu\nu} dx^\mu dx^\nu, \\ &\tilde{P}_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{u}_\mu \tilde{u}_\nu. \end{aligned} \tag{3.13}$$

However, we equally could have directly used the expression (3.13) as our zeroth order ansatz to perturbatively construct a bulk spacetime with (3.13) as the fluid dynamical initial data. In doing so, we would have obtained a bulk spacetime identical to (3.9) at each order except with the variables r, u_μ, and b replaced by $\tilde{r}, \tilde{u}_\mu$, and $\tilde{b}$. Now, recall that our perturbative procedure constructs a *unique* bulk spacetime for a specified zeroth order ansatz. Hence we are forced to conclude that the bulk spacetime obtained from (3.9) by performing the simultaneous replacements (3.8) must be the same as the bulk spacetime obtained directly from the zeroth order ansatz (3.13); this would be identical to (3.9) at all orders except with r, u_μ, and b replaced by $\tilde{r}, \tilde{u}_\mu$, and $\tilde{b}$. This concludes our proof; the bulk spacetime dual to fluid dynamics is therefore invariant under the simultaneous replacements (3.8).

Observe that, from the perspective of the boundary, (3.8) is nothing more than a boundary Weyl transformation with u_μ and b transforming as Weyl tensors of weight -1. It is thus convenient to adopt a Weyl covariant formalism; we develop this further in the next subsection.

3.2.2 Weyl covariant derivative

A Weyl covariant tensor is a quantity that transforms homogeneously under a Weyl transformation. More specifically, a tensor of weight w transforms as follows:

$$\mathcal{Q}^{\mu\cdots}_{\nu\cdots} = e^{-w\chi(x)}\tilde{\mathcal{Q}}^{\mu\cdots}_{\nu\cdots} \tag{3.14}$$

under a Weyl rescaling, $g_{\mu\nu} = e^{2\chi(x)}\tilde{g}_{\mu\nu}$. The main obstruction to maintaining explicit Weyl covariance is that ordinary covariant derivatives of Weyl covariant tensors are not themselves Weyl covariant. This problem can be circumvented by introducing a 'Weyl covariant derivative'; this was the main technical innovation of [45]. The action of the Weyl covariant derivative on an arbitrary tensor $\mathcal{Q}^{\mu\cdots}_{\nu\cdots}$ of weight w is defined by:

$$\begin{aligned}\mathcal{D}_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} \equiv \nabla_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} &+ w\mathcal{A}_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots} \\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right]\mathcal{Q}^{\alpha\cdots}_{\nu\cdots} + \cdots \\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right]\mathcal{Q}^{\mu\cdots}_{\alpha\cdots} - \cdots .\end{aligned} \tag{3.15}$$

The Weyl connection, $\mathcal{A}_\mu$, is constructed from the fluid velocity field, u_μ, as follows:

$$\mathcal{A}_\mu \equiv u^\lambda\nabla_\lambda u_\mu - \frac{\nabla_\lambda u^\lambda}{d-1}u_\mu = \tilde{\mathcal{A}}_\mu + \partial_\mu\chi. \tag{3.16}$$

As we can see from the last equality, this expression for $\mathcal{A}_\mu$ transforms in a similar manner to a metric connection under a Weyl transformation. This is what enables us to construct a derivative that is Weyl covariant, as done in (3.15); the parts of $\nabla_\lambda\mathcal{Q}^{\mu\cdots}_{\nu\cdots}$ which do not transform homogeneously are cancelled by the terms involving $\mathcal{A}_\mu$. It can further be shown that the Weyl covariant derivative of a tensor of weight w is itself a tensor of weight w.

We will now introduce several Weyl covariant tensors that will be used throughout the rest of the chapter. The following tensors are naturally constructed from the Weyl covariant derivative:

$$
\begin{aligned}
&[\mathcal{D}_\mu, \mathcal{D}_\nu]\, V_\lambda \equiv w\mathcal{F}_{\mu\nu}V_\lambda + \mathcal{R}_{\mu\nu\lambda}{}^{\alpha}V_\alpha \quad \text{with}\\
&\qquad \mathcal{F}_{\mu\nu} \equiv \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu \quad \text{and}\\
&\qquad \mathcal{R}_{\mu\nu\lambda\sigma} \equiv R_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu}g_{\lambda\sigma} - \delta^{\alpha}_{[\mu}g_{\nu][\lambda}\delta^{\beta}_{\sigma]}\left(\nabla_\alpha \mathcal{A}_\beta + \mathcal{A}_\alpha\mathcal{A}_\beta - \frac{\mathcal{A}^2}{2}g_{\alpha\beta}\right).
\end{aligned}
\tag{3.17}
$$

We will also make use of the following two contractions obtained from the Weyl covariant Riemann tensor, $\mathcal{R}_{\mu\nu\lambda\sigma}$:

$$
\begin{aligned}
\mathcal{R}_{\mu\nu} &\equiv \mathcal{R}_{\mu\lambda\nu}{}^{\lambda} = R_{\mu\nu} + (d-2)(\nabla_\mu\mathcal{A}_\nu + \mathcal{A}_\mu\mathcal{A}_\nu - \mathcal{A}^2 g_{\mu\nu}) + g_{\mu\nu}\nabla_\lambda\mathcal{A}^\lambda + \mathcal{F}_{\mu\nu};\\
\mathcal{R} &\equiv \mathcal{R}_\lambda{}^{\lambda} = R + 2(d-1)\nabla_\lambda\mathcal{A}^\lambda - (d-2)(d-1)\mathcal{A}^2.
\end{aligned}
\tag{3.18}
$$

And finally we define the shear strain rate, $\sigma_{\mu\nu}$, and vorticity, $\omega_{\mu\nu}$, of the boundary fluid:

$$
\begin{aligned}
\sigma_{\mu\nu} &\equiv \mathcal{D}_{(\mu}u_{\nu)};\\
\omega_{\mu\nu} &\equiv \mathcal{D}_{[\mu}u_{\nu]}.
\end{aligned}
\tag{3.19}
$$

3.2.3 Independent Weyl invariant tensors

Here, we classify all Weyl invariant scalars, transverse[1] vectors, and symmetric traceless transverse tensors up till second order in derivatives; this will be of importance in the following subsection. There are two subtleties involved in this classification that we should first highlight. Note that the equations of motion, $\nabla_\mu T^{\mu\nu} = e^{-\phi}\mathcal{L}\nabla^\nu\phi$, impose relations between various Weyl covariant quantities; thus, in our counting, we only list Weyl tensors which are independent on-shell. And also, since the dilaton is Weyl invariant, any Weyl invariant tensor can be multiplied by a function of ϕ to get another independent Weyl invariant quantity; we will neglect this complication in our classification as well. The methodology behind listing independent tensors was explained in detail in the previous review chapter in subsection 2.1 on relativistic fluid dynamics so we leave out some of the detail here.

We begin with the zeroth order Weyl invariant tensors. We aim to construct Weyl invariants using the boundary dilaton field, ϕ, the boundary metric, $g_{\mu\nu}$, and the fluid dynamical quantities, b and u_μ. The boundary dilaton ϕ is a Weyl invariant scalar while b and u_μ transform homogeneously under Weyl rescalings with weight -1. It

[1]By transverse we mean orthogonal to u^μ.

thus follows that there are no nontrivial Weyl invariant scalars, transverse vectors, or symmetric traceless transverse tensors at zeroth order in derivatives.

To obtain the Weyl invariants at first order, we must consider the first order relations imposed by the equations of motion, $\nabla_\mu T^{\mu\nu} = e^{-\phi}\mathcal{L}\nabla^\nu\phi$. It is easy to see that these relations arise from the zeroth order contributions to the stress tensor, $T^{\mu\nu}$, and Lagrangian, $\mathcal{L}$. For the stress tensor, the zeroth order contribution is simply that of a perfect fluid, $b^{-d}\left(g^{\mu\nu} + du^\mu u^\nu\right)$. And for the Lagrangian, there can be no zeroth order contribution. The reason for this is as follows: if we set ϕ to be a constant, the Einstein-dilaton system must consistently truncate to the Einstein equations with negative cosmological constant. Correspondingly, the boundary fluid dynamics must reduce to that of the unforced case. Thus, the lowest order contribution to the Lagrangian must be proportional to a derivative of ϕ; there can be no zeroth order terms. Analysing the resulting first order relations, it can be shown that first order partial derivatives of b can be expressed as derivatives of u_μ. This result was derived explicitly in the previous review chapter when we calculated the first order constraint equations. It follows that there is only one independent Weyl invariant scalar at first order (which can be taken to be $bu^\mu\mathcal{D}_\mu\phi$), one Weyl invariant transverse vector $\left(P_\mu^\nu\mathcal{D}_\nu\phi\right)$, and one Weyl invariant symmetric traceless transverse tensor $(b^{-1}\sigma_{\mu\nu})$.

For the second order Weyl invariant tensors, we must similarly consider the relations imposed at second order by the equations of motion; these originate from the first order contributions to $T^{\mu\nu}$ and $\mathcal{L}$. The stress tensor, $T^{\mu\nu}$, transforms with weight $d+2$; and thus, using our previous classification of first order Weyl invariants, we can deduce that the first order contribution to $T^{\mu\nu}$ must be proportional to $b^{1-d}\sigma^{\mu\nu}$. And for the Lagrangian, which transforms with weight d, we can similarly conclude that the first order term must be of the form $b^{1-d}u^\mu\mathcal{D}_\mu\phi$. The two derivative relations which result can be used to express the partial derivatives of b to second order in terms of derivatives of u_μ and ϕ. There are thus seven independent Weyl invariant scalars:

$$\begin{gathered} b^2\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad b^2\omega_{\mu\nu}\omega^{\mu\nu}, \quad b^2\mathcal{R}, \\ b^2P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu\phi, \quad b^2u^\mu u^\nu\mathcal{D}_\mu\mathcal{D}_\nu\phi, \quad b^2P^{\mu\nu}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi, \quad \text{and} \quad b^2u^\mu u^\nu\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi, \end{gathered} \tag{3.20}$$

six Weyl invariant transverse vectors:

$$\begin{gathered} bP_\mu^\nu\mathcal{D}_\lambda{\sigma_\nu}^\lambda, \quad bP_\mu^\nu\mathcal{D}_\lambda{\omega_\nu}^\lambda, \quad bP_\mu^\nu u^\lambda\mathcal{D}_\nu\mathcal{D}_\lambda\phi, \quad bP_\mu^\nu u^\lambda\mathcal{D}_\nu\phi\mathcal{D}_\lambda\phi, \\ b{\sigma_\mu}^\lambda\mathcal{D}_\lambda\phi, \quad \text{and} \quad b{\omega_\mu}^\lambda\mathcal{D}_\lambda\phi, \end{gathered} \tag{3.21}$$

and eight Weyl invariant symmetric traceless transverse tensors:

$$\begin{gathered} u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu}, \quad \sigma_{\mu\nu} u^\lambda \mathcal{D}_\lambda \phi, \quad C_{\mu\alpha\nu\beta} u^\alpha u^\beta, \quad \omega_\mu{}^\lambda \sigma_{\lambda\nu} + \omega_\nu{}^\lambda \sigma_{\lambda\mu}, \\ \frac{1}{2}\left[P_\mu^\alpha P_\nu^\beta + P_\nu^\alpha P_\mu^\beta - \frac{2}{d-1} P^{\alpha\beta} P_{\mu\nu}\right] \mathcal{D}_\alpha \mathcal{D}_\beta \phi, \quad \left[P_\mu^\alpha P_\nu^\beta - \frac{1}{d-1} P^{\alpha\beta} P_{\mu\nu}\right] \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi, \\ \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{1}{d-1} P_{\mu\nu} \sigma_{\alpha\beta} \sigma^{\alpha\beta}, \quad \text{and} \quad \omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-1} P_{\mu\nu} \omega_{\alpha\beta} \omega^{\alpha\beta}. \end{gathered} \tag{3.22}$$

3.2.4 Weyl covariant form of the fluid dynamical metric

In this final subsection, we demonstrate that it is possible to use boundary Weyl invariance to constrain the form of the bulk metric. In more detail, we show that because the bulk metric is invariant under the simultaneous replacements (3.8), the components of the bulk metric can be classified according to how they transform under boundary Weyl rescalings.

Before we proceed further, we must first choose a gauge for the bulk metric. We use the same gauge[2] as [26], which is specified by:

$$g_{rr} = 0, \quad g_{r\mu} = -u_\mu. \tag{3.24}$$

This gauge has the nice geometric interpretation that lines of constant x^μ are ingoing null geodesics with r being an affine parameter along them. Also, note that this gauge choice is invariant under the transformation (3.8).

Now, observe that, consistent with our gauge choice (3.24), we can parametrise our bulk metric as follows:

$$ds^2 = -2u_\mu dx^\mu \left(dr + \mathcal{V}_\nu \left(r, u_\alpha, b\right) dx^\nu\right) + \mathcal{G}_{\mu\nu}\left(r, u_\alpha, b\right) dx^\mu dx^\nu \quad \text{with } \mathcal{G}_{\mu\nu} \text{ transverse.} \tag{3.25}$$

We aim to determine how the functions $\mathcal{V}_\nu$ and $\mathcal{G}_{\mu\nu}$ transform under (3.8) which effectively is a boundary Weyl transformation. Recall that the fluid dynamical bulk metric

[2] Some early work on the fluid/gravity correspondence [16, 36] used a different gauge, given by:

$$g_{rr} = 0, \quad g_{r\mu} \propto u_\mu, \quad Tr\left(\left(g^{(0)}\right)^{-1} g^{(m)}\right) = 0 \quad (m > 0). \tag{3.23}$$

All of our results can be recast in this gauge by making an appropriate change of variables. In fact, we used this gauge in the previous review chapter.

is invariant under the simultaneous replacements (3.8), thus, under this transformation, the bulk metric (3.25) becomes:

$$\begin{aligned} ds^2 &= -2\tilde{u}_\mu dx^\mu \left(d\tilde{r} + \mathcal{V}_\nu\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right) dx^\nu\right) + \mathcal{G}_{\mu\nu}\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right) dx^\mu dx^\nu \\ &= -2u_\mu dx^\mu \left(dr + e^{-\chi}\mathcal{V}_\nu\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right) dx^\nu + r\partial_\nu\chi dx^\nu\right) + \mathcal{G}_{\mu\nu}\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right) dx^\mu dx^\nu. \end{aligned} \tag{3.26}$$

By comparing the two equivalent metrics (3.25) and (3.26), we can deduce the transformation properties of $\mathcal{V}_\nu$ and $\mathcal{G}_{\mu\nu}$:

$$\mathcal{V}_\nu\left(r, u_\alpha, b\right) = e^{-\chi}\left[\mathcal{V}_\nu\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right) + \tilde{r}\partial_\nu\chi\right] \quad \text{and} \quad \mathcal{G}_{\mu\nu}\left(r, u_\alpha, b\right) = \mathcal{G}_{\mu\nu}\left(\tilde{r}, \tilde{u}_\alpha, \tilde{b}\right). \tag{3.27}$$

It follows that $\mathcal{V}_\nu - r\mathcal{A}_\nu$ must be a linear sum of Weyl covariant vectors (both transverse and non-transverse) of weight $+1$ with coefficients that are arbitrary functions of br. Similarly, $\mathcal{G}_{\mu\nu}$ must be a linear sum of Weyl invariant tensors. These Weyl covariant vectors of weight $+1$ and the Weyl invariant tensors can easily be obtained from our classification in the previous subsection. The functions of br, however, must be determined by direct calculation.

In keeping with explicit Weyl covariance, we now choose a slightly different starting ansatz, $g^{(0)}_{AB}$:

$$ds^2 = -2u_\mu dx^\mu \left(dr + \left(r\mathcal{A}_\nu + \frac{r^2 f(br)}{2}u_\nu\right) dx^\nu\right) + r^2 P_{\mu\nu} dx^\mu dx^\nu. \tag{3.28}$$

Using this ansatz, we can perturbatively solve the Einstein equations and determine the functions $\mathcal{V}_\nu$ and $\mathcal{G}_{\mu\nu}$ to any order in boundary derivatives. We present the results of such a calculation to second order in the next section.

3.3 Derivation of fluid/gravity equations and their solutions

The Einstein-dilaton system is governed by the following equations:

$$\begin{aligned} &E_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} - \frac{1}{2}\partial_a\Phi\partial_b\Phi + \frac{1}{4}\left(\partial\Phi\right)^2 g_{ab} = 0, \\ &g^{ab}\nabla_a\nabla_b\Phi = 0. \end{aligned} \tag{3.29}$$

And upon resubstituting for the Ricci scalar R and the cosmological constant Λ, this then simplifies to:

$$\begin{aligned} E_{ab} \equiv R_{ab} + d g_{ab} - \frac{1}{2}\partial_a\Phi\partial_b\Phi = 0, \\ g^{ab}\nabla_a\nabla_b\Phi = 0. \end{aligned} \tag{3.30}$$

We now proceed in an analogous manner to what was done in the previous chapter. Field theory intuition again tells us that the metric should be a slowly-varying function in the boundary coordinates and that we should thus be aiming to solve the equations (3.30) perturbatively to a certain accuracy in boundary derivatives. However, for the Einstein-dilaton system, the metric couples to the dilaton, and so we must further require that the dilaton be slowly-varying in the boundary directions as well. Hence, it must also admit an expansion in boundary derivatives:

$$\Phi = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \cdots . \tag{3.31}$$

We must again address the issue of choosing the zeroth order ansatz, but this is just a straightforward generalisation of the unforced case. The Einstein-dilaton system admits uniform black brane solutions of the following form,

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \\ \Phi &= \phi_0, \end{aligned} \tag{3.32}$$

where ϕ_0 is a constant. Patching together tubes of uniform black brane solutions with different parameter values gives us in principle our zeroth order ansatz. However, as discussed in the previous section, we choose a Weyl covariant form of this:

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu \left(dr + \left(r\mathcal{A}_\nu + \frac{r^2 f(br)}{2} u_\nu \right) dx^\nu \right) + r^2 P_{\mu\nu} dx^\mu dx^\nu \\ \Phi &= \phi(x). \end{aligned} \tag{3.33}$$

Note that the velocity and temperature have become functions of boundary coordinates $u_\mu(x)$ and $b(x)$.

With all of this in hand, we can now substitute the expansions for the metric and the dilaton into the Einstein and dilaton equations and examine the structure of the resulting equations. The equations at order n in the derivative expansion can be schematically represented as:

$$\mathbb{H}\left[g^{(0)}\right] g^{(n)}(r, x^{\mu}) = s_n, \tag{3.34}$$

$$\mathbb{H}^{\Phi}\left[g^{(0)}\right] \Phi^{(n)}(r, x^{\mu}) = s_n^{\Phi}, \tag{3.35}$$

where $\mathbb{H}^{\Phi}$ and s_n^{Φ} are the differential operator and source terms for Φ respectively. The dynamical equations of (3.34) together with the equation for the dilaton (3.35) are sufficient to determine $g^{(n)}$ and $\Phi^{(n)}$. The remaining d constraint equations reduce to the equations of forced fluid dynamics (3.4).

3.3.1 Einstein equations: First order

We begin with the Einstein equations at first order in boundary derivatives.

Inverse metric to first order

We require the zeroth and first order components or the metric and its inverse. Our zeroth order metric ansatz was introduced in the previous subsection on Weyl covariance. It is shown below in matrix form (basis in terms of r,μ) Taylor expanded to first order about $x^{\mu} = 0$:

$$g_{ab}^{(0)} = \begin{bmatrix} 0 & \left(-u_{\nu}^{(0)} - x^{\lambda}\partial_{\lambda}u_{\nu}^{(0)}\right) \\ \left(-u_{\mu}^{(0)} - x^{\lambda}\partial_{\lambda}u_{\mu}^{(0)}\right) & \left(r^2\eta_{\mu\nu} + r^{2-d}u_{\mu}^{(0)}u_{\nu}^{(0)} - dr^{2-d}x^{\lambda}\partial_{\lambda}b^{(0)}u_{\mu}^{(0)}u_{\nu}^{(0)} + 2r^{2-d}u_{(\mu}^{(0)}x^{\lambda}\partial_{\lambda}u_{\nu)}^{(0)}\right) \end{bmatrix}.$$

Our first order metric terms are parametrised according to the Weyl covariant form (and gauge choice) that we discussed in the last section on Weyl covariance. It too is given below in matrix form:

$$g_{ab}^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & \left(-2ru_{(\mu}^{(0)}\mathcal{A}_{\nu)}^{(1)} - 2u_{(\mu}^{(0)}\mathcal{V}_{\nu)}^{(1)} + \mathcal{G}_{\mu\nu}^{(1)}\right) \end{bmatrix}.$$

The inverse metric to first order is calculated according to the following formula:

$$\begin{aligned} g_{ab}g^{bc} &= \delta_a^c \\ \left(g_{ab}^{(0)} + g_{ab}^{(1)}\right)\left(g^{(0)bc} + g^{(1)bc}\right) &= \delta_a^c \\ \implies g_{ab}^{(0)}g^{(0)bc} &= \delta_a^c \text{ and} \\ g_{ab}^{(0)}g^{(1)bc} + g_{ab}^{(1)}g^{(0)bc} &= 0. \end{aligned} \tag{3.36}$$

The inverse metric is then found to be (in matrix form):

$$g^{(0)ab} = \begin{bmatrix} r^2 f + dx^\lambda \partial_\lambda b^{(0)} r^{2-d} & u^{(0)\nu} + x^\lambda \partial_\lambda u^{(0)\nu} \\ u^{(0)\mu} + x^\lambda \partial_\lambda u^{(0)\mu} & \frac{1}{r^2} P^{(0)\mu\nu} + \frac{2}{r^2} u^{(0)(\mu} x^\lambda \partial_\lambda u^{(0)\nu)} \end{bmatrix}$$

$$g^{(1)ab} = \begin{bmatrix} -2u^{(0)\lambda}\mathcal{A}_\lambda^{(1)} r - 2u^{(0)\lambda}\mathcal{V}_\lambda^{(1)} & -\frac{1}{r}P^{(0)\mu\lambda}\mathcal{A}_\lambda^{(1)} - \frac{1}{r^2}P^{(0)\mu\lambda}\mathcal{V}^{(1)} \\ -\frac{1}{r}p^{(0)\mu\lambda}\mathcal{A}_\lambda^{(1)} - \frac{1}{r^2}P^{(0)\mu\lambda}\mathcal{V}_\lambda^{(1)} & -\frac{1}{r^4}\mathcal{G}^{(1)\mu\nu} \end{bmatrix}.$$

The remaining calculation is analogous to that which we considered in our review in the previous chapter. However, with our Weyl covariant formulation of the metric, we only need compute the rr, $r\mu$, and $\mu\nu$-components of E_{ab}. For each of these, we calculate the differential operator and the source terms.

E_{rr}

The differential operator comprises of the following terms:

$$
\begin{aligned}
E_{rr} = &-\frac{1}{2}g^{(1)\mu\nu}{}_{,r}g^{(0)}_{\mu\nu,r} - \frac{1}{2}g^{(0)\mu\nu}{}_{,r}g^{(1)}_{\mu\nu,r} - \frac{1}{2}g^{(1)\mu\nu}g^{(0)}_{\mu\nu,rr} - \frac{1}{2}g^{(0)\mu\nu}g^{(1)}_{\mu\nu,rr} \\
&+ \frac{1}{2}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r} \\
&+ \frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{rb,r} \\
&- \frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r} \\
&- \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{rr,r} \\
&- \frac{1}{4}g^{(1)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r} - \frac{1}{4}g^{(0)cb}g^{(1)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r} \\
&- \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(1)}_{cd,r}g^{(0)}_{ab,r} - \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(1)}_{ab,r} \\
&+ \frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{rb,r} \\
&- \frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r} - \frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cr,r}g^{(0)}_{ar,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(1)}_{ar,r} \\
&+ dg^{(1)}_{rr}.
\end{aligned}
\tag{3.37}
$$

These terms evaluate explicitly to:

$$
\begin{aligned}
E_{rr}^{diff} &= \frac{1}{2}\left(\frac{4}{r^5}\mathcal{G}^{(1)\mu\nu} - \frac{1}{r^4}\mathcal{G}^{(1)\prime\mu\nu}\right)\left(2r\eta_{\mu\nu} + (2-d)r^{1-d}u^{(0)}_\mu u^{(0)}_\nu\right)\\
&- \frac{1}{2}\left(-\frac{2}{r^3}P^{(0)\mu\nu}\right)\left(-2u^{(0)}_{(\mu}\mathcal{A}^{(1)}_{\nu)} - 2u^{(0)}_{(\mu}\mathcal{V}^{(1)\prime}_{\nu)} + \mathcal{G}^{(1)\prime}_{\mu\nu}\right)\\
&- \frac{1}{2}\left(-\frac{1}{r^4}\mathcal{G}^{(1)\mu\nu}\right)\left(\eta_{\mu\nu} + (2-d)(1-d)r^{-d}u^{(0)}_\mu u^{(0)}_\nu\right)\\
&- \frac{1}{2}\left(\frac{1}{r^2}P^{(0)\mu\nu}\right)\left(-2u^{(0)}_{(\mu}\mathcal{V}^{(1)\prime\prime}_{\nu)} + \mathcal{G}^{(1)\prime\prime}_{\mu\nu}\right)\\
&+ 0 + 0\\
&+ 0 + 0\\
&- 0 - 0\\
&- 0 - 0\\
&- \frac{1}{2}\left(-\frac{1}{r^4}\mathcal{G}^{(1)\rho\nu}\right)\left(\frac{1}{r^2}P^{(0)\mu\delta}\right)\left(2r\eta_{\rho\delta} + (2-d)r^{1-d}u^{(0)}_\rho u^{(0)}_\delta\right)\left(2r\eta_{\mu\nu} + (2-d)r^{1-d}u^{(0)}_\mu u^{(0)}_\nu\right)\\
&- \frac{1}{2}\left(\frac{1}{r^2}P^{(0)\rho\nu}\right)\left(\frac{1}{r^2}P^{(0)\mu\delta}\right)\left(-2u^{(0)}_{(\rho}\mathcal{A}^{(1)}_{\delta)} - 2u^{(0)}_{(\rho}\mathcal{V}^{(1)\prime}_{\delta)} + \mathcal{G}^{(1)\prime}_{\rho\delta}\right)\left(2r\eta_{\mu\nu} + (2-d)r^{1-d}u^{(0)}_\mu u^{(0)}_\nu\right)\\
&+ 0
\end{aligned}
\tag{3.38}
$$

which simplifies to:

$$
E_{rr}^{diff} = -\frac{\mathcal{G}^{(1)\prime\prime\mu}{}_\mu}{2r^2} + \frac{\mathcal{G}^{(1)\prime\mu}{}_\mu}{r^3} - \frac{\mathcal{G}^{(1)\mu}{}_\mu}{r^4}. \tag{3.39}
$$

As for the source terms:

$$
\begin{aligned}
E_{rr}^{source} &= g^{(0)\mu b}{}_{,\mu}g^{(0)}_{rb,r} - \frac{1}{2}g^{(0)\mu r}_{,\mu}g^{(0)}_{rr,r} - \frac{1}{2}g^{(0)r\mu}{}_{,r}g^{(0)}_{rr,\mu} + g^{(0)\mu b}g^{(0)}_{rb,r\mu}\\
&- g^{(0)r\mu}g^{(0)}_{rr,r\mu}\\
&+ \frac{1}{2}g^{(0)\mu b}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{rb,r} - \frac{1}{4}g^{(0)\mu r}g^{(0)ad}g^{(0)}_{ad,\mu}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)r\mu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,\mu}\\
&+ g^{(0)cb}g^{(0)r\mu}g^{(0)}_{cr,r}g^{(0)}_{rb,\mu} - g^{(0)cr}g^{(0)a\mu}g^{(0)}_{cr,\mu}g^{(0)}_{ar,r}
\end{aligned}
\tag{3.40}
$$

which actually all evaluate to zero:

$$
E_{rr}^{source} = 0. \tag{3.41}
$$

$E_{r\mu}$

The differential operator is given by:

$$\begin{aligned}
E^{diff}_{r\mu} &= \frac{1}{2}g^{r\mu}{}_{,r}g^{(0)}_{\mu\nu,r} + \frac{1}{2}g^{(0)r\nu}{}_{,r}g^{(1)}_{\mu\nu,r} + \frac{1}{2}g^{(1)\nu r}g^{(0)}_{\nu\mu,rr} + \frac{1}{2}g^{(0)\nu r}g^{(1)}_{\nu\mu,rr} \\
&+ \frac{1}{4}g^{(1)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r} + \frac{1}{4}g^{(0)r\nu}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r} \\
&+ \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu\nu,r} + \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu\nu,r} \\
&+ \frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{cr,r}g^{(0)}_{\mu b,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{cr,r}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(1)}_{\mu b,r} \\
&- \frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{cr,r}g^{(0)}_{a\mu,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{cr,r}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(1)}_{a\mu,r}
\end{aligned} \tag{3.42}$$

which computes to:

$$\begin{aligned}
E^{diff}_{r\mu} &= \frac{1}{2}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(1)\prime\prime}_{\lambda} + \frac{d-3}{2r}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(1)\prime}_{\lambda} - \frac{d-2}{r^2}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(1)}_{\lambda} \\
&- u^{(0)\nu}\mathcal{V}^{(1)\prime\prime}_{\nu}u^{(0)}_{\mu} - \frac{d-1}{r}u^{(0)\nu}\mathcal{V}^{(1)\prime}_{\nu}u^{(0)}_{\mu} \\
&- \frac{1}{2r^2}\left(2-(2-d)r^{-d}\right)\mathcal{G}^{(1)\rho}{}_{\rho}u^{(0)}_{\mu} + \frac{1}{4r}\left(2-(2-dd)r^{-d}\right)\mathcal{G}^{(1)\prime\rho}{}_{\rho}u^{(0)}_{\mu} \\
&- \frac{\partial_\rho u^{(0)\rho}}{r}u^{(0)}_{\mu} - \frac{d-1}{2r}u^{(0)\rho}\partial_\rho u^{(0)}_{\mu}.
\end{aligned} \tag{3.43}$$

The first line involves operator terms transverse to $u^{(0)}_{\mu}$, the second and third lines are differential operator terms in the direction of $u^{(0)}_{\mu}$. The final line involves terms derived from $\mathcal{A}_\mu$ which are technically source terms but we include them here since we included $\mathcal{A}_\mu$ in our parametrisation of the first order metric for the purpose of Weyl covariance.

And the source terms can be calculated from:

$$
\begin{aligned}
E^{source}_{r\mu} = & \frac{1}{2} g^{(0)\nu b}{}_{,\nu} g^{(0)}_{\mu b,r} + \frac{1}{2} g^{(0)rb}{}_{,r} g^{(0)}_{rb,\mu} \frac{1}{2} g^{(0)\nu r}{}_{,\nu} g^{(0)}_{r\mu,r} - \frac{1}{2} g^{(0)r\nu}{}_{,r} g^{(0)}_{r\mu,\nu} \\
& + \frac{1}{2} g^{(0)rb} g^{(0)}_{rb,\mu r} - g^{(0)r\nu} g^{(0)}_{r\mu,r\nu} \\
& - \frac{1}{2} g^{(0)ab}{}_{,r} g^{(0)}_{ab,\mu} - \frac{1}{2} g^{(0)ab} g^{(0)}_{ab,\mu r} + \frac{1}{2} g^{(0)a\nu} g^{(0)}_{a\mu,\nu r} \\
& + \frac{1}{4} g^{(0)\nu b} g^{(0)ad} g^{(0)}_{ad,\nu} g^{(0)}_{\mu b,r} + \frac{1}{4} g^{(0)rb} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{rb,\mu} - \frac{1}{4} g^{(0)\nu r} g^{(0)ad} g^{(0)}_{ad,\nu} g^{(0)}_{r\mu,r} \\
& - \frac{1}{4} g^{(0)\nu r} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{r\mu,\nu} \\
& - \frac{1}{4} g^{(0)cb} g^{(0)ad} g^{(0)}_{cd,r} g^{(0)}_{ab,\mu} \\
& + \frac{1}{2} g^{(0)cb} g^{(0)r\nu} g^{(0)}_{cr,\nu} g^{(0)}_{\mu b,r} + \frac{1}{2} g^{(0)cb} g^{(0)r\nu} g^{(0)}_{cr,r} g^{(0)}_{\mu b,\nu} \\
& - \frac{1}{2} g^{(0)c\nu} g^{(0)ar} g^{(0)}_{cr,r} g^{(0)}_{a\mu,\nu} - \frac{1}{2} g^{(0)cr} g^{(0)a\nu} g^{(0)}_{cr,\nu} g^{(0)}_{a\mu,r}
\end{aligned}
\tag{3.44}
$$

which gives:

$$
E^{source}_{r\mu} = \frac{1}{r} u^{(0)}_{\mu} \partial_\nu u^{(0)\nu} + \frac{1}{2} \frac{d-1}{r} u^{(0)\nu} \partial_\nu u^{(0)}_{\mu}. \tag{3.45}
$$

$E_{\mu\nu}$

The differential operator is given by:

$$
\begin{aligned}
E^{diff}_{\mu\nu} = & \frac{1}{2} g^{(1)rr}{}_{,r} g^{(0)}_{\mu\nu,r} - \frac{1}{2} g^{(0)rr}{}_{,r} g^{(1)}_{\mu\nu,r} - \frac{1}{2} g^{(1)rr} g^{(0)}_{\mu\nu,rr} - \frac{1}{2} g^{(0rr} g^{(1)}_{\mu\nu,rr} \\
& - \frac{1}{4} g^{(1)rr} g^{(0)ad} g^{(0)}_{ad,r} g^{(0)}_{\mu\nu,r} - \frac{1}{4} g^{(0)rr} g^{(1)ad} g^{(0)}_{ad,r} g^{(0)}_{\mu\nu,r} \\
& - \frac{1}{4} g^{(0)rr} g^{(0)ad} g^{(1)}_{ad,r} g^{(0)}_{\mu\nu,r} - \frac{1}{4} g^{(0)rr} g^{(0)ad} g^{(0)}_{ad,r} g^{(1)}_{\mu\nu,r} \\
& + \frac{1}{2} g^{(1)cb} g^{(0)rr} g^{(0)}_{c\mu,r} g^{(0)}_{\nu b,r} + \frac{1}{2} g^{(0)cb} g^{(1)rr} g^{(0)}_{c\mu,r} g^{(0)}_{\nu b,r} \\
& + \frac{1}{2} g^{(0)cb} g^{(0)rr} g^{(1)}_{c\mu,r} g^{(0)}_{\nu b,r} + \frac{1}{2} g^{(0)cb} g^{(0)rr} g^{(0)}_{c\mu,r} g^{(1)}_{\nu b,r} \\
& - \frac{1}{2} g^{(1)cr} g^{(0)ar} g^{(0)}_{c\mu,r} g^{(0)}_{a\nu,r} - \frac{1}{2} g^{(0)cr} g^{(1)ar} g^{(0)}_{c\mu,r} g^{(0)}_{a\nu,r} \\
& - \frac{1}{2} g^{(0)cr} g^{(0)ar} g^{(1)}_{c\mu,r} g^{(0)}_{a\nu,r} - \frac{1}{2} g^{(0)cr} g^{(0)ar} g^{(0)}_{c\mu,r} g^{(1)}_{a\nu,r} \\
& + d g^{(1)}_{\mu\nu}
\end{aligned}
\tag{3.46}
$$

which evaluates to:

$$
\begin{aligned}
E^{diff}_{\mu\nu} &= 2(2-d)fu^{(0)\lambda}\mathcal{V}^{(1)}_{\lambda}u^{(0)}_{\mu}u^{(0)}_{\nu} + 2(2-d)fu^{(0)}_{(\mu}\mathcal{V}^{(1)}_{\nu)} \\
&- 2rfu^{(0)\lambda}\mathcal{V}^{(1)\prime}_{\lambda}u^{(0)}_{\mu}u^{(0)}_{\nu} + (d-3)rfu^{(0)}_{(\mu}\mathcal{V}^{(1)\prime}_{\nu)} \\
&+ r^2 fu^{(0)}_{(\mu}\mathcal{V}^{(1)\prime\prime}_{\nu)} \\
&+ 2(d-2)u^{(0)\lambda}\mathcal{V}^{(1)}_{\lambda}P^{(0)}_{\mu\nu} + 2ru^{(0)\lambda}\mathcal{V}^{(1)\prime}_{\lambda}P^{(0)}_{\mu\nu} \\
&- \frac{r^2 f}{2}\mathcal{G}^{(1)\prime\prime}_{\mu\nu} - \frac{(d-3)r}{2}\mathcal{G}^{(1)\prime}_{\mu\nu} - \frac{3}{2}r^{1-d}\mathcal{G}^{(1)\prime}_{\mu\nu} + (d-2)\mathcal{G}^{(1)}_{\mu\nu} + 2r^{-d}\mathcal{G}^{(1)}_{\mu\nu} \\
&+ 2r\partial_\rho u^{(0)\rho}\eta_{\mu\nu} + \partial_\rho u^{(0)\rho}u^{(0)}_{\mu}u^{(0)}_{\nu}r^{1-d} + r\partial_\rho u^{(0)\rho}u^{(0)}_{\mu}u^{(0)}_{\nu} \\
&+ (d-1)r^{1-d}u^{(0)\rho}u^{(0)}_{(\mu}\partial_\rho u^{(0)}_{\nu)} - (d-1)ru^{(0)\rho}u^{(0)}_{(\mu}\partial_\rho u^{(0)}_{\nu)}.
\end{aligned}
\tag{3.47}
$$

The source terms are:

$$
\begin{aligned}
E^{source}_{\mu\nu} &= \frac{1}{2}g^{(0)rb}{}_{,r}g^{(0)}_{\nu b,\mu} + \frac{1}{2}g^{(0)rb}{}_{,r}g^{(0)}_{\mu b,\nu} - \frac{1}{2}g^{(0)r\rho}{}_{,r}g^{(0)}_{\mu\nu,\rho} - \frac{1}{2}g^{(0)r\rho}{}_{,\rho}g^{(0)}_{\mu\nu,r} + \frac{1}{2}g^{(0)rb}g^{(0)}_{\mu b,\nu r} \\
&- g^{(0)r\rho}g^{(0)}_{\mu\nu,r\rho} + \frac{1}{2}g^{(0)ar}g^{(0)}_{a\nu,r\mu} \\
&+ \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\nu b,\mu} + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu b,\nu} \\
&- \frac{1}{4}g^{(0)r\rho}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,\rho} - \frac{1}{4}g^{(0)r\rho}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\mu\nu,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)r\rho}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,\rho} + \frac{1}{2}g^{(0)cb}g^{(0)r\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{\nu b,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)a\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{a\nu,r} - \frac{1}{2}g^{(0)c\rho}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,\rho}
\end{aligned}
\tag{3.48}
$$

which upon substituting for the metric and its derivatives become:

$$
\begin{aligned}
E^{source}_{\mu\nu} &= r^{1-d}u^{(0)\rho}u^{(0)}_{(\mu}\partial_\rho u^{(0)}_{\nu)} + (1-d)r\partial_{(\mu}u^{(0)}_{\nu)} - dr^{1-d}\partial_{(\mu}b^{(0)}u^{(0)}_{\nu)} \\
&- \frac{d}{2}(d-1)r^{1-d}u^{(0)\rho}\partial_\rho b^{(0)}u^{(0)}_{\mu}u^{(0)}_{\nu} - r\eta_{\mu\nu}\partial_\rho u^{(0)\rho} - \frac{2-d}{2}r^{1-d}u^{(0)}_{\mu}u^{(0)}_{\nu}\partial_\rho u^{(0)\rho}.
\end{aligned}
\tag{3.49}
$$

Constraint equations at first order

The constraint equations are obtained by contracting the Einstein equations with n^A, the normal to the constant r hypersurface: $E^{\Phi}_{AB}n^B = 0$. The boundary components of this equation give us the following relation:

$$\partial_\mu b^{(0)} = \left(u^{(0)\lambda} \partial_\lambda u^{(0)}_\mu - \frac{\partial_\lambda u^{(0)\lambda}}{d-1} u^{(0)}_\mu \right) \tag{3.50}$$

which can be re-expressed as (the b has been added on dimensional grounds):

$$\partial_\mu b = \mathcal{A}_\mu b. \tag{3.51}$$

This relation is equivalent to the equations of forced fluid dynamics $\nabla_\mu T^{\mu\nu} = e^{-\phi}\mathcal{L}\nabla^\nu \phi$ at first order in derivatives. This explicitly confirms our expectation that our perturbative procedure constructs bulk spacetimes dual to solutions of forced fluid dynamics. At first order, the relevant terms in this equation stem from the zeroth order contributions to the stress tensor $T^{\mu\nu}$ and Lagrangian $\mathcal{L}$; and since the Lagrangian only consists of first and higher orders terms, the equations at this order are the same as the unforced case.

Solution at first order

The rr-component of the Einstein equations, as found previously, gives us the following equation:

$$-\frac{\mathcal{G}^{(1)\prime\prime\mu}{}_\mu}{2r^2} + \frac{\mathcal{G}^{(1)\prime\mu}{}_\mu}{r^3} - \frac{\mathcal{G}^{(1)\mu}{}_\mu}{r^4} = 0. \tag{3.52}$$

Subject to our boundary conditions at $r = \infty$ and our Landau frame requirement this integrates to give $\mathcal{G}^{(1)\mu}{}_\mu = 0$.

The $r\mu$-component of the Einstein equations simplifies to become:

$$\begin{aligned} \frac{1}{2} P^{(0)}{}_\mu{}^\lambda \mathcal{V}^{(1)\prime\prime}_\lambda + \frac{d-3}{2r} P^{(0)}{}_\mu{}^\lambda \mathcal{V}^{(1)\prime}_\lambda - \frac{d-2}{r^2} P^{(0)}{}_\mu{}^\lambda \mathcal{V}^{(1)}_\lambda &= 0, \\ -u^{(0)\nu}\mathcal{V}^{(1)\prime\prime}_\nu - \frac{d-1}{r} u^{(0)\nu}\mathcal{V}^{(1)\prime}_\nu &= 0. \end{aligned} \tag{3.53}$$

We have separated this into its transverse and non-transverse components. Note that the non-operator terms arising from $E^{diff}_{r\mu}$ exactly cancel the terms in $E^{source}_{r\mu}$ leaving effectively no source terms. Also $\mathcal{G}^{(1)\mu}{}_\mu$ which appears in $E^{diff}_{r\mu}$ was set to zero as we just determined. Solving this subject to all conditions again gives zero, $\mathcal{V}^{(1)}_\mu = 0$.

And finally for the $\mu\nu$-component of the Einstein equation, we only require the transverse traceless part to determine the traceless part of $\mathcal{G}^{(1)}_{\mu\nu}$, which is all that remains to be found. This is given by:

$$-\frac{(br)^2-(br)^{2-d}}{2}\tilde{\mathcal{G}}^{(1)\prime\prime}_{\mu\nu}-\frac{(br)}{2}(d-3)\tilde{\mathcal{G}}^{(1)\prime}_{\mu\nu}-\frac{3(br)^{1-d}}{2}\tilde{\mathcal{G}}^{(1)\prime}_{\mu\nu}+(d-2+2(br)^{-d})\tilde{\mathcal{G}}^{(1)}_{\mu\nu}$$
$$=(d-1)(br)\sigma_{\mu\nu}. \tag{3.54}$$

with:

$$\sigma_{\mu\nu}:=\frac{1}{2}P_\mu{}^\alpha P_\nu{}^\beta\nabla_{(\mu}u_{\nu)}-\frac{1}{d-1}\nabla_\lambda u^\lambda=\mathcal{D}_{(\mu}u_{\nu)}. \tag{3.55}$$

Here, we have used $\tilde{\mathcal{G}}_{\mu\nu}\equiv\mathcal{G}_{\mu\nu}-\frac{1}{d-1}\mathcal{G}^\alpha_\alpha P_{\mu\nu}$ though essentially there is no difference since we found the trace of $\mathcal{G}^{(1)}_{\mu\nu}$ to be zero. Further, we have reintroduced the factors of b for Weyl covariance. This integrates (using the same methodology described in the previous review chapter) to give:

$$\mathcal{G}^{(1)}_{\mu\nu}=2(br)^2F(br)\frac{1}{b}\sigma_{\mu\nu} \tag{3.56}$$

where the function $F(br)$ is:

$$F(br)\equiv\int_{br}^{\infty}\frac{y^{d-1}-1}{y(y^d-1)}dy.$$

Our appropriately covariantised metric at first order is thus:

$$ds^2=-2u_\mu dx^\mu\left[dr+\left(rA_\nu+\frac{r^2f(br)}{2}u_\nu\right)dx^\nu\right]+\left[r^2P_{\mu\nu}+2(br)^2F(br)\frac{1}{b}\sigma_{\mu\nu}\right]dx^\mu dx^\nu. \tag{3.57}$$

Since the dilaton terms appearing in the Einstein equations E_{ab} are at least second order in derivatives, we do not see any contribution from the dilaton at first order here.

3.3.2 Einstein equations: Second order

In this subsection, we consider the Einstein equations at second order in boundary derivatives.

Subtleties at second order

At second order there are several subtleties that we must account for. First, note that our previous calculation at first order gave us constraint equations that imposed relations on $u^{(0)}$ and $b^{(0)}$. This generalises to all orders in boundary derivatives. At each order, the constraint equations impose relations at one order lower. At second order, we must therefore expand the velocity and temperature fields to first order as follows:

$$\begin{aligned} u_\mu &= u_\mu^{(0)} + u_\mu^{(1)} + O(\epsilon^2) \\ b &= b^{(0)} + b^{(1)} + O(\epsilon^2). \end{aligned} \tag{3.58}$$

A second subtlety is that there exists an ambiguity in definition between $b^{(0)}$ and $b^{(1)}$ and similarly for u_μ. If parts of $u_\mu^{(1)}$ solve the first order fluid equations of motion then they can be absorbed into the definition of $u_\mu^{(0)}$. We use this ambiguity to set $u_\mu^{(1)}$ to zero. Given that we are working locally around $x^\mu = 0$, this means that we can set all derivatives of $u_\mu^{(1)}$ at $x^\mu = 0$ to be zero. The actual values for $b^{(1)}$ and $u_\mu^{(1)}$ at $x^\mu = 0$ can be set to be zero by the same coordinate transformations that fixed $u_\mu^{(0)} = (-1, 0, 0, 0)$ and $b^{(0)} = 1$, as explained in the previous review chapter. Thus, we only need consider the derivatives of $b^{(1)}$ at second order.

The final subtlety that we need to consider concerns how we ensure that the first order fluid equations of motion continue to hold at second order. In the last section, our constraint equations were equivalent to:

$$\nabla_\mu T^{(0)\mu\nu} = 0 \tag{3.59}$$

where $T^{(0)\mu\nu} \propto b^{-d} \left(g^{\mu\nu} + d u^\mu u^\mu \right)$ and there are no forcing terms since the dilaton terms only appear at second order. But here at second order in boundary derivatives, we are now working to second order in a Taylor expansion about $x^\mu = 0$. Expanding the above relation to second order in derivatives about $x^\mu = 0$ imposes further conditions between derivative terms given by:

$$\nabla_\lambda \nabla_\mu T^{(0)\mu\nu}|_{x^\rho=0} = 0. \tag{3.60}$$

This evaluates explicitly to the following:

$$
\begin{aligned}
&+d(d+1)\partial_\lambda b^{(0)}\partial_\mu b^{(0)}\left(\eta^{\mu\nu}+du^{(0)\mu}u^{(0)\nu}\right)-d^2\partial_\mu b^{(0)}\partial_\lambda u^{(0)\mu}u^{(0)\nu}-d^2\partial_\mu b^{(0)}\partial_\lambda u^{(0)\nu}u^{(0)\mu}\\
&-d^2\partial_\lambda b^{(0)}\partial_\mu u^{(0)\mu}u^{(0)\nu}-d^2\partial_\lambda b^{(0)}u^{(0)\mu}\partial_\mu u^{(0)\nu}-d\partial_\lambda\partial_\mu b^{(0)}\eta^{\mu\nu}-d^2\partial_\lambda\partial_\mu b^{(0)}u^{(0)\mu}u^{(0)\nu}\\
&+d\partial_\lambda\partial_\mu u^{(0)\mu}u^{(0)\nu}+du^{(0)\mu}\partial_\lambda\partial_\mu u^{(0)\nu}+d\partial_\mu u^{(0)\mu}\partial_\lambda u^{(0)\nu}+d\partial_\lambda u^{(0)\mu}\partial_\mu u^{(0)\nu}=0.
\end{aligned}
\tag{3.61}
$$

This is used to reduce our source terms to a subset of independent terms.

Inverse metric to second order

As we established previously, the fluid/gravity metric to first order is given by:

$$
\begin{aligned}
ds^2 = &-2u_\mu dx^\mu\left(dr+r\mathcal{A}_\nu dx^\nu\right)+r^2 g_{\mu\nu}dx^\mu dx^\nu+\frac{1}{(br)^d}r^2u_\mu u_\nu dx^\mu dx^\nu\\
&+2(br)^2F(br)\frac{1}{b}\sigma_{\mu\nu}dx^\mu dx^\nu.
\end{aligned}
\tag{3.62}
$$

Adding on our parametrisation of the second order metric terms gives us:

$$
\begin{aligned}
ds^2 = &-2u_\mu dx^\mu\left(dr+r\mathcal{A}_\nu dx^\nu\right)+r^2 g_{\mu\nu}dx^\mu dx^\nu+\frac{1}{(br)^d}r^2u_\mu u_\nu dx^\mu dx^\nu\\
&+2(br)^2F(br)\frac{1}{b}\sigma_{\mu\nu}dx^\mu dx^\nu\\
&+-2u_\mu dx^\mu\mathcal{V}^{(2)}_\nu dx^\nu+\mathcal{G}^{(2)}_{\mu\nu}.
\end{aligned}
\tag{3.63}
$$

After Taylor expanding about $x^\mu=0$ to second order, we obtain:

$$
g^{(0)}_{ab}=\begin{bmatrix}
0 & \begin{array}{c}-u^{(0)}_\mu\\ -x^\lambda\partial_\lambda u^{(0)}_\mu\\ -\frac{1}{2}x^\lambda x^\rho\partial_\lambda\partial_\rho u^{(0)}_\mu\end{array}\\
\begin{array}{c}-u^{(0)}_\nu\\ -x^\lambda\partial_\lambda u^{(0)}_\nu\\ -\frac{1}{2}x^\lambda x^\rho\partial_\lambda\partial_\rho u^{(0)}_\nu\end{array} & \begin{array}{c}r^2\eta_{\mu\nu}+r^{2-d}u^{(0)}_\mu u^{(0)}_\nu\\ -d.r^{2-d}x^\lambda\partial_\lambda b^{(0)}u^{(0)}_\mu u^{(0)}_\nu+2r^{2-d}x^\lambda\partial_\lambda u^{(0)}_{(\mu}u^{(0)}_{\nu)}\\ +\frac{r^2}{2}x^\lambda x^\rho\partial_\lambda\partial_\rho g_{\mu\nu}+r^{2-d}x^\lambda x^\rho N_{\lambda\rho\mu\nu}\end{array}
\end{bmatrix}.
$$

$$g^{(1)}_{ab} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{matrix} 2r^2F(r)\sigma^{(1)}_{\mu\nu} - 2ru^{(0)}_{(\mu}\mathcal{A}^{(1)}_{\nu)} - r^{2-d}db^{(1)}u^{(0)}_\mu u^{(0)}_\nu \\ + r^2x^\lambda F(r)O_{\lambda\mu\nu} + 2r^3x^\lambda\partial_\lambda b^{(0)}F'(r)\sigma^{(1)}_{\mu\nu} \\ + rx^\lambda M_{\lambda\mu\nu} - r^{2-d}x^\lambda d\partial_\lambda b^{(1)}u^{(0)}_\mu u^{(0)}_\nu \\ + r^{2-d}d(d+1)b^{(1)}x^\lambda\partial_\lambda b^{(0)}u^{(0)}_\mu u^{(0)}_\nu \\ - 2r^{2-d}db^{(1)}x^\lambda\partial_\lambda u^{(0)}_{(\mu}u^{(0)}_{\nu)} \end{matrix} \end{bmatrix}$$

$$g^{(2)}_{ab} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{matrix} -2u^{(0)}_{(\mu}\mathcal{V}^{(2)}_{\nu)} + \mathcal{G}^{(2)}_{\mu\nu} \\ + \frac{d(d+1)}{2}{b^{(1)}}^2r^{2-d}u^{(0)}_\mu u^{(0)}_\nu \\ + 2r^2b^{(1)}F(r)\sigma^{(1)}_{\mu\nu} + 2r^3F'(r)b^{(1)}\sigma^{(1)}_{\mu\nu} \end{matrix} \end{bmatrix}$$

where the terms $M_{\lambda\mu\nu}$, $N_{\lambda\rho\mu\nu}$, $O_{\lambda\mu\nu}$ are given by:

$$\begin{aligned} M_{\lambda\mu\nu} = & -2\partial_\lambda u^{(0)}_{(\mu}\mathcal{A}^{(1)}_{\nu)} - 2\partial_\lambda u^{(0)\rho}u^{(0)}_{(\mu}\partial_\rho u^{(0)}_{\nu)} - 2u^{(0)\rho}u^{(0)}_{(\mu}\partial_\lambda\partial_\rho u^{(0)}_{\nu)} + 2u^{(0)\delta}u^{(0)}_{(\mu}\partial_\lambda\Gamma^{\delta(2)}_{\delta\nu)}u^{(0)}_\rho \\ & + \frac{2}{d-1}\partial_\rho u^{(0)\rho}u^{(0)}_{(\mu}\partial_\lambda u^{(0)}_{\nu)} + \frac{2}{d-1}u^{(0)}_\mu u^{(0)}_\nu\partial_\lambda\partial_\rho u^{(0)\rho} + \frac{2}{d-1}u^{(0)}_\mu u^{(0)}_\nu\partial_\lambda\Gamma^{\delta(2)}_{\delta\rho}u^{(0)\rho} \end{aligned} \tag{3.64}$$

$$\begin{aligned} N_{\lambda\rho\mu\nu} = & -\frac{d}{2}\partial_\lambda\partial_\rho b^{(0)}u^{(0)}_\mu u^{(0)}_\nu + \partial_\lambda\partial_\rho u^{(0)}_\mu u^{(0)}_\nu - 2d\partial_\lambda b^{(0)}\partial_\rho u^{(0)}_{(\mu}u^{(0)}_{\nu)} + \partial_\lambda u^{(0)}_{(\mu}\partial_\rho u^{(0)}_{\nu)} \\ & + \frac{d(d+1)}{2}\partial_\lambda b^{(0)}\partial_\rho b^{(0)}u^{(0)}_\mu u^{(0)}_\nu \end{aligned} \tag{3.65}$$

$$\begin{aligned} O_{\lambda\mu\nu} = & \, 2\partial_\lambda b^{(0)}\sigma^{(1)}_{\mu\nu} + 4u^{(0)\alpha}\partial_\lambda u^{(0)}_{(\mu}P^{(0)\beta}{}_{\nu)}\partial_{(\alpha}u^{(0)}_{\beta)} + 4\partial_\lambda u^{(0)\alpha}u^{(0)}_{(\mu}P^{(0)\beta}{}_{\nu)}\partial_{(\alpha}u^{(0)}_{\beta)} \\ & + 2P^{(0)\alpha}{}_\mu P^{(0)\beta}{}_\nu\partial_\lambda\partial_{(\alpha}u^{(0)}_{\beta)} - 2\partial_\lambda\Gamma^{(\rho(2)}_{\alpha\beta)}u^{(0)}_\rho P^{(0)\alpha}{}_\mu P^{(0)\beta}{}_\nu - \frac{4}{d-1}\partial_\lambda u^{(0)}_{(\mu}u^{(0)}_{\nu)}\partial_\alpha u^{(0)\alpha} \\ & - \frac{2}{d-1}\partial_\lambda\partial_\alpha u^{(0)\alpha}P^{(0)}_{\mu\nu} - \frac{2}{d-1}\partial_\lambda\Gamma^{(\alpha(2)}_{\alpha\rho)}u^{(0)\rho}P^{(0)}_{\mu\nu}. \end{aligned} \tag{3.66}$$

Given the complexity of these expressions, we obtain the inverse in a slightly different manner to our first order calculation. For a matrix of the form:

$$g_{ab} = \begin{bmatrix} 0 & -u_\mu \\ -u_\nu & -2u_{(\mu}\mathcal{V}_{\nu)} + \mathcal{G}_{\mu\nu} \end{bmatrix}$$

where $(\mathcal{V}_\mu - r\mathcal{A}_\mu)$ is a linear sum of Weyl coveriant vectors with weight 1 and $\mathcal{G}_{\mu\nu}$ is a linear sum of Weyl invariant forms all of which are transverse ($u^\mu \mathcal{G}_{\mu\nu} = 0$), the inverse matrix will take the following general form:

$$g^{ab} = \begin{bmatrix} -2u^\lambda \mathcal{V}_\lambda + \left(\mathcal{G}^{-1}\right)^{\lambda\rho} \mathcal{V}_\lambda \mathcal{V}_\rho & u^\mu - \left(\mathcal{G}^{-1}\right)^{\mu\lambda} \mathcal{V}_\lambda \\ u^\nu - \left(\mathcal{G}^{-1}\right)^{\nu\lambda} \mathcal{V}_\lambda & \left(\mathcal{G}^{-1}\right)^{\mu\nu} \end{bmatrix}$$

where $\left(\mathcal{G}^{-1}\right)^{\mu\nu}$ satisfies $u_\mu \left(\mathcal{G}^{-1}\right)^{\mu\nu} = 0$ and $\left(\mathcal{G}^{-1}\right)^{\mu\lambda} \mathcal{G}_{\lambda\nu} = P^\mu{}_\nu$.

For our metric till second order,

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu \left(dr + r\mathcal{A}_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu + \frac{1}{(br)^d} r^2 u_\mu u_\nu dx^\mu dx^\nu \\ &+ 2(br)^2 F(br) \frac{1}{b} \sigma_{\mu\nu} dx^\mu dx^\nu \\ &+ -2u_\mu dx^\mu \mathcal{V}^{(2)}_\nu dx^\nu + \mathcal{G}^{(2)}_{\mu\nu} \end{aligned} \tag{3.67}$$

the functions $\mathcal{V}_\mu$ and $\mathcal{G}_{\mu\nu}$ are given by:

$$\mathcal{V}_\mu = r\mathcal{A}_\mu + \frac{r^2 f(br)}{2} u_\mu + \mathcal{V}^{(2)}_\mu \tag{3.68}$$

$$\mathcal{G}_{\mu\nu} = r^2 P_{\mu\nu} + 2(br)^2 F(br) \frac{1}{b} \sigma_{\mu\nu} + \mathcal{G}^{(2)}_{\mu\nu}. \tag{3.69}$$

Using all this, we can calculate $\left(\mathcal{G}^{-1}\right)^{\mu\nu}$. We find the following expression to second order:

$$\left(\mathcal{G}^{-1}\right)^{\mu\nu} = \frac{1}{r^2} P^{\mu\nu} - \frac{2}{r^2} F(br) b \sigma^{\mu\nu} + \frac{4}{r^2} F^2(br) b^2 \sigma^\mu{}_\lambda \sigma^{\lambda\nu} - \frac{1}{r^4} \mathcal{G}^{(2)\mu\nu}. \tag{3.70}$$

This gives us a complete inverse metric as follows:

$$g^{(0)ab} = \begin{bmatrix} r^2 f(br) & u^\mu \\ u^\nu & \frac{1}{r^2} P^{\mu\nu} \end{bmatrix}$$

$$g^{(1)ab} = \begin{bmatrix} -2ru^{\lambda}\mathcal{A}_{\lambda} & -\frac{1}{r}P^{\mu\lambda}\mathcal{A}_{\lambda} \\ -\frac{1}{r}P^{\nu\lambda}\mathcal{A}_{\lambda} & -\frac{2}{r^2}F(br)b\sigma_{\mu\nu} \end{bmatrix}$$

$$g^{(2)ab} = \begin{bmatrix} -2u^{\lambda}\mathcal{V}^{(2)}_{\lambda} + P^{\lambda\rho}\mathcal{A}_{\lambda}\mathcal{A}_{\rho} & \frac{2}{r}F(br)b\sigma^{\mu\lambda}\mathcal{A}_{\lambda} - \frac{1}{r^2}P^{\mu\lambda}\mathcal{V}^{(2)}_{\lambda} \\ \frac{2}{r}F(br)b\sigma^{\nu\lambda}\mathcal{A}_{\lambda} - \frac{1}{r^2}P^{\nu\lambda}\mathcal{V}^{(2)}_{\lambda} & \frac{4}{r^2}F^2(br)b^2\sigma^{\mu}{}_{\lambda}\sigma^{\lambda\nu} - \frac{1}{r^4}\mathcal{G}^{(2)\mu\nu} \end{bmatrix}.$$

Taylor expanding this about $x^{\mu} = 0$ to second order gives:

$$g^{(0)ab} = \begin{bmatrix} \begin{matrix} r^2 - r^{2-d} \\ +dr^{2-d}x^{\lambda}\partial_{\lambda}b^{(0)} \\ +r^{2-d}x^{\lambda}x^{\rho}\left(\frac{d}{2}\partial_{\lambda}\partial_{\rho}b^{(0)} - \frac{d(d+1)}{2}\partial_{\lambda}b^{(0)}\partial_{\rho}b^{(0)}\right) \end{matrix} & \begin{matrix} u^{(0)\mu} \\ +x^{\lambda}\partial_{\lambda}u^{(0)\mu} \\ +\frac{1}{2}x^{\lambda}x^{\rho}\partial_{\lambda}\partial_{\rho}u^{(0)\mu} \end{matrix} \\ \begin{matrix} u^{(0)\nu} \\ +x^{\lambda}\partial_{\lambda}u^{(0)\nu} \\ +\frac{1}{2}x^{\lambda}x^{\rho}\partial_{\lambda}\partial_{\rho}u^{(0)\nu} \end{matrix} & \begin{matrix} \frac{1}{r^2}P^{(0)\mu\nu} \\ +\frac{2}{r^2}x^{\lambda}\partial_{\lambda}u^{(0)(\mu}u^{(0)\nu)} \\ +\frac{1}{r^2}x^{\lambda}x^{\rho}\left(\frac{1}{2}\partial_{\lambda}\partial_{\rho}g^{\mu\nu} + \partial_{\lambda}\partial_{\rho}u^{(0)(\mu}u^{(0)\nu)} + \partial_{\lambda}u^{(0)\mu}\partial_{\rho}u^{(0)\nu}\right) \end{matrix} \end{bmatrix}$$

$$g^{(1)ab} = \begin{bmatrix} \begin{matrix} -2ru^{(0)\lambda}\mathcal{A}^{(1)}_{\lambda} + dr^{2-d}b^{(1)} \\ +rx^{\rho}M_{\rho}{}^{\lambda}{}_{\lambda} \\ +dr^{2-d}x^{\lambda}\partial_{\lambda}b^{(1)} - r^{2-d}d(d+1)b^{(1)}x^{\lambda}\partial_{\lambda}b^{(0)} \end{matrix} & \begin{matrix} -\frac{1}{r}P^{(0)\mu\lambda}\mathcal{A}^{(1)}_{\lambda} \\ +\frac{1}{r}x^{\rho}W_{\rho}{}^{\mu} \end{matrix} \\ \begin{matrix} -\frac{1}{r}P^{(0)\nu\lambda}\mathcal{A}^{(1)}_{\lambda} \\ +\frac{1}{r}x^{\rho}W_{\rho}{}^{\nu} \end{matrix} & \begin{matrix} -\frac{2}{r^2}F(r)\sigma^{(1)\mu\nu} \\ -\frac{1}{r^2}F(r)x^{\lambda}O_{\lambda}{}^{\mu\nu} - \frac{2}{r}x^{\lambda}\partial_{\lambda}b^{(0)}F'(r)\sigma^{(1)\mu\nu} \end{matrix} \end{bmatrix}$$

$$g^{(2)ab} = \begin{bmatrix} \begin{matrix} -2u^{(0)\lambda}\mathcal{V}^{(2)}_{\lambda} + P^{(0)\lambda\rho}\mathcal{A}^{(1)}_{\lambda}\mathcal{A}^{(1)}_{\rho} \\ -r^{2-d}\frac{d(d+1)}{2}{b^{(1)}}^2 \end{matrix} & \frac{2}{r}F(r)\sigma^{(1)\mu\lambda}\mathcal{A}^{(1)}_{\lambda} - \frac{1}{r^2}P^{(0)\mu\lambda}\mathcal{V}^{(2)}_{\lambda} \\ \frac{2}{r}F(r)\sigma^{(1)\nu\lambda}\mathcal{A}^{(1)}_{\lambda} - \frac{1}{r^2}P^{(0)\nu\lambda}\mathcal{V}^{(2)}_{\lambda} & \begin{matrix} \frac{4}{r^2}F^2(r)\sigma^{(1)\mu}{}_{\lambda}\sigma^{(1)\lambda\nu} - \frac{1}{r^4}\mathcal{G}^{(2)\mu\nu} \\ -\frac{2}{r^2}b^{(1)}F(r)\sigma^{(1)\mu\nu} - \frac{2}{r}b^{(1)}F'(r)\sigma^{(1)\mu\nu} \end{matrix} \end{bmatrix}$$

where

$$\begin{aligned} W_{\rho}{}^{\mu} = &-2\partial_{\rho}u^{(0)(\mu}u^{(0)\lambda)}u^{(0)\delta}\partial_{\delta}u^{(0)}_{\lambda} - P^{(0)\mu\lambda}\partial_{\rho}u^{(0)\delta}\partial_{\delta}u^{(0)}_{\lambda} - P^{(0)\mu\lambda}u^{(0)\delta}\partial_{\rho}\partial_{\delta}u^{(0)}_{\lambda} \\ &+ P^{(0)\mu\lambda}u^{(0)\gamma}\partial_{\rho}\Gamma^{\delta(2)}_{\gamma\lambda}u^{(0)}_{\delta}. \end{aligned} \tag{3.71}$$

We are now ready to calculate the differential operator terms and the source terms from the Einstein equations.

E_{rr}

The differential operator terms are given by:

$$
\begin{aligned}
E^{diff}_{rr} = &-\frac{1}{2}g^{(2)\mu\nu}{}_{,r}g^{(0)}_{\mu\nu,r} - \frac{1}{2}g^{(0)\mu\nu}{}_{,r}g^{(2)}_{\mu\nu,r} - \frac{1}{2}g^{(2)\mu\nu}g^{(0)}_{\mu\nu,rr} - \frac{1}{2}g^{(0)\mu\nu}g^{(2)}_{\mu\nu,rr} \\
&+ \frac{1}{2}g^{(2)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)rb}g^{(2)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,r} \\
&+ \frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(2)}_{ad,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(2)}_{rb,r} \\
&- \frac{1}{4}g^{(2)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)rr}g^{(2)ad}g^{(0)}_{ad,r}g^{(0)}_{rr,r} \\
&- \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(2)}_{ad,r}g^{(0)}_{rr,r} - \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(2)}_{rr,r} \\
&- \frac{1}{2}g^{(2)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r} - \frac{1}{2}g^{(0)cb}g^{(2)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r} \\
&- \frac{1}{2}g^{(0)cb}g^{(0)ad}g^{(2)}_{cd,r}g^{(0)}_{ab,r} - \frac{1}{2}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(2)}_{ab,r} \\
&+ \frac{1}{2}g^{(2)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)cb}g^{(2)rr}g^{(0)}_{cr,r}g^{(0)}_{rb,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(2)}_{cr,r}g^{(0)}_{rb,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(2)}_{rb,r} \\
&- \frac{1}{2}g^{(2)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r} - \frac{1}{2}g^{(0)cr}g^{(2)ar}g^{(0)}_{cr,r}g^{(0)}_{ar,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(2)}_{cr,r}g^{(0)}_{ar,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(2)}_{ar,r} \\
&+ dg^{(2)}_{rr}
\end{aligned}
\tag{3.72}
$$

which evaluates to:

$$
E^{diff}_{rr} = -\frac{\mathcal{G}^{(2)\prime\prime\mu}{}_{\mu}}{2r^2} + \frac{\mathcal{G}^{(2)\prime\mu}{}_{\mu}}{r^3} - \frac{\mathcal{G}^{(2)\mu}{}_{\mu}}{r^4} + \left(-\frac{4}{r^2}F^2(r) - \frac{8}{r}F(r)F'(r)\right)\sigma^{(1)\rho\delta}\sigma^{(1)}_{\rho\delta}. \tag{3.73}
$$

And the source terms:

$$\begin{aligned}E_{rr}^{source} = &-\frac{1}{2}g^{(0)\mu\nu}{}_{,\mu}g^{(0)}_{rr,\nu} - \frac{1}{2}g^{(1)r\mu}{}_{,r}g^{(0)}_{rr,\mu} - \frac{1}{2}g^{(0)\mu\nu}g^{(0)}_{rr,\mu\nu} - \frac{1}{2}g^{(1)ab}{}_{,r}g^{(1)}_{ab,r} - \frac{1}{2}g^{(1)ab}g^{(1)}_{ab,rr}\\ &- \frac{1}{4}g^{(1)cb}g^{(1)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,r} - \frac{1}{4}g^{(1)cb}g^{(0)ad}g^{(1)}_{cd,r}g^{(0)}_{ab,r}\\ &- \frac{1}{4}g^{(1)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(1)}_{ab,r} - \frac{1}{4}g^{(0)cb}g^{(1)ad}g^{(1)}_{cd,r}g^{(0)}_{ab,r}\\ &- \frac{1}{4}g^{(0)cb}g^{(1)ad}g^{(0)}_{cd,r}g^{(1)}_{ab,r} - \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(1)}_{cd,r}g^{(1)}_{ab,r}\\ &+ \frac{1}{2}g^{(0)cb}g^{(0)\mu\nu}g^{(0)}_{cr,\nu}g^{(0)}_{rb,\mu} - \frac{1}{2}g^{(0)c\mu}g^{(0)a\nu}g^{(0)}_{cr,\nu}g^{(0)}_{ar,\mu}\\ &- \frac{1}{2}\Phi^{(1)}_{,r}\Phi^{(1)}_{,r}\end{aligned} \tag{3.74}$$

which simplifies to:

$$\begin{aligned}E_{rr}^{source} = &\left(\frac{4}{r^2}F^2(r) + \frac{12}{r}F(r)F'(r) + F'^2(r) + 2F''(r)F(r)\right)\sigma^{(1)\mu\nu}\sigma^{(1)}_{\mu\nu}\\ &+ \frac{1}{r^4}\omega^{(1)}_{\mu\nu}\omega^{(1)\mu\nu}\\ &- \frac{1}{2}\Phi^{(1)}_{,r}\Phi^{(1)}_{,r}\end{aligned} \tag{3.75}$$

where $\omega^{(1)}_{\mu\nu} = \frac{1}{2}\left(\mathcal{D}_\mu u_\nu - \mathcal{D}_\nu u_\mu\right)$.

$E_{r\mu}$

The differential operator terms are:

$$\begin{aligned}E_{r\mu}^{diff} = &\frac{1}{2}g^{(2)r\nu}{}_{,r}g^{(0)}_{\mu\nu,r} + \frac{1}{2}g^{(0)r\nu}{}_{,r}g^{(2)}_{\mu\nu,r} + \frac{1}{2}g^{(2)\nu r}g^{(0)}_{\nu\mu,rr} + \frac{1}{2}g^{(0)\nu r}g^{(2)}_{\nu\mu,rr}\\ &+ \frac{1}{4}g^{(2)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r} + \frac{1}{4}g^{(0)r\nu}g^{(2)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r}\\ &+ \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(2)}_{ad,r}g^{(0)}_{\mu\nu,r} + \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(2)}_{\mu\nu,r}\\ &+ \frac{1}{2}g^{(2)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)cb}g^{(2)rr}g^{(0)}_{cr,r}g^{(0)}_{\mu b,r}\\ &+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(2)}_{cr,r}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{cr,r}g^{(2)}_{\mu b,r}\\ &- \frac{1}{2}g^{(2)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(0)cr}g^{(2)ar}g^{(0)}_{cr,r}g^{(0)}_{a\mu,r}\\ &- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(2)}_{cr,r}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{cr,r}g^{(2)}_{a\mu,r}\end{aligned} \tag{3.76}$$

which computes to give:

$$
\begin{aligned}
E^{diff}_{r\mu} = & \left(\frac{2(d-1)}{r}F(r) + 2F'(r)\right)\sigma^{(1)}{}_{\mu}{}^{\lambda}\mathcal{A}^{(1)}_{\lambda} \\
& + \frac{1}{2}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(2)\prime\prime}_{\lambda} + \frac{1}{2}\frac{d-3}{r}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(2)\prime}_{\lambda} - \frac{d-2}{r^2}P^{(0)}{}_{\mu}{}^{\lambda}\mathcal{V}^{(2)}_{\lambda} \\
& - u^{(0)\nu}\mathcal{V}^{(2)\prime\prime}_{\nu}u^{(0)}_{\mu} - \frac{d-1}{r}u^{(0)\nu}\mathcal{V}^{(2)\prime}_{\nu}u^{(0)}_{\mu} \\
& + 2F^2(r)\left(2-(2-d)r^{-d}\right)\sigma^{(1)}_{\rho\delta}\sigma^{(1)\rho\delta}u^{(0)}_{\mu} - \frac{1}{2r^2}\left(2-(2-d)r^{-d}\right)\mathcal{G}^{(2)\rho}{}_{\rho}u^{(0)}_{\mu} \\
& + \frac{1}{4r}\left(2-(2-d)r^{-d}\right)\mathcal{G}^{(2)\rho\prime}{}_{\rho}u^{(0)}_{\mu}.
\end{aligned}
\tag{3.77}
$$

And for the source terms:

$$
\begin{aligned}
E_{r\mu}^{source} = & \frac{1}{2}g^{(1)\nu b}{}_{,\nu}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)\nu b}{}_{,\nu}g^{(1)}_{\mu b,r} + \frac{1}{2}g^{(1)rb}{}_{,r}g^{(1)}_{\mu b,r} \\
& + \frac{1}{2}g^{(0)rb}{}_{,r}g^{(1)}_{rb,\mu} + \frac{1}{2}g^{(0)\nu b}{}_{,\nu}g^{(0)}_{rb,\mu} + \frac{1}{2}g^{(1)rb}{}_{,r}g^{(0)}_{rb,\mu} \\
& - \frac{1}{2}g^{(1)\nu r}{}_{,\nu}g^{(0)}_{r\mu,r} - \frac{1}{2}g^{(0)r\nu}{}_{,r}g^{(1)}_{r\mu,\nu} - \frac{1}{2}g^{(0)\nu\rho}{}_{,\nu}g^{(0)}_{r\mu,\rho} - \frac{1}{2}g^{(1)r\nu}{}_{,r}g^{(0)}_{r\mu,\nu} \\
& - \frac{1}{2}g^{(0)\nu r}{}_{,\nu}g^{(1)}_{r\mu,r} - g^{(1)rr}{}_{,r}g^{(1)}_{r\mu,r} \\
& + \frac{1}{2}g^{(0)\nu b}g^{(0)}_{rb,\mu\nu} + \frac{1}{2}g^{(0)rb}g^{(1)}_{rb,\mu r} + \frac{1}{2}g^{(1)rb}g^{(0)}_{rb,\mu r} \\
& - \frac{1}{2}g^{(0)\nu\rho}g^{(0)}_{r\mu,\nu\rho} - g^{(0)r\nu}g^{(1)}_{r\mu,r\nu} - g^{(1)r\nu}g^{(0)}_{r\mu,r\nu} - \frac{1}{2}g^{(1)rr}g^{(1)}_{r\mu,rr} \\
& - \frac{1}{2}g^{(0)ab}{}_{,r}g^{(1)}_{ab,\mu} - \frac{1}{2}g^{(1)ab}{}_{,r}g^{(0)}_{ab,\mu} \\
& - \frac{1}{2}g^{(0)ab}g^{(1)}_{ab,\mu r} - \frac{1}{2}g^{(1)ab}g^{(0)}_{ab,\mu r} \\
& + \frac{1}{2}g^{(0)a\nu}g^{(1)}_{a\mu,\nu r} + \frac{1}{2}g^{(1)a\nu}g^{(0)}_{a\mu,\nu r} + \frac{1}{2}g^{(1)ar}g^{(1)}_{a\mu,rr} \\
& + \frac{1}{4}g^{(0)\nu b}g^{(0)ad}g^{(1)}_{ad,\nu}g^{(0)}_{\mu b,r} + \frac{1}{4}g^{(1)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu b,r} + \frac{1}{4}g^{(1)\nu b}g^{(0)ad}g^{(0)}_{ad,\nu}g^{(0)}_{\mu b,r} \\
& + \frac{1}{4}g^{(1)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu b,r} + \frac{1}{4}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu b,r} + \frac{1}{4}g^{(0)\nu b}g^{(1)ad}g^{(0)}_{ad,\nu}g^{(0)}_{\mu b,r} \\
& + \frac{1}{4}g^{(0)rb}g^{(1)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu b,r} + \frac{1}{4}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu b,r} + \frac{1}{4}g^{(0)\nu b}g^{(0)ad}g^{(0)}_{ad,\nu}g^{(1)}_{\mu b,r} \\
& + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(1)}_{\mu b,r} \\
& + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{rb,\mu} + \frac{1}{4}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,\mu} + \frac{1}{4}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,\mu} \\
& + \frac{1}{4}g^{(0)\nu b}g^{(0)ad}g^{(0)}_{ad,\nu}g^{(0)}_{rb,\mu} + \frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{rb,\mu} \\
& - \frac{1}{4}g^{(0)\nu r}g^{(0)ad}g^{(1)}_{ad,\nu}g^{(0)}_{r\mu,r} - \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{r\mu,\nu} - \frac{1}{4}g^{(1)r\nu}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{r\mu,\nu} \\
& - \frac{1}{4}g^{(0)r\nu}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{r\mu,\nu} - \frac{1}{4}g^{(0)\nu\rho}g^{(0)ad}g^{(0)}_{ad,\nu}g^{(0)}_{r\mu,\rho} - \frac{1}{4}g^{(0)r\nu}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{r\mu,\nu} \\
& - \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(1)}_{ab,\mu} - \frac{1}{4}g^{(1)cb}g^{(0)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,\mu} - \frac{1}{4}g^{(0)cb}g^{(1)ad}g^{(0)}_{cd,r}g^{(0)}_{ab,\mu} \\
& - \frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(1)}_{cd,r}g^{(0)}_{ab,\mu} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)r\nu}g^{(1)}_{cr,\nu}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(1)cb}g^{(0)r\nu}g^{(0)}_{cr,\nu}g^{(0)}_{\mu b,r} + \frac{1}{2}g^{(0)cb}g^{(1)r\nu}g^{(0)}_{cr,\nu}g^{(0)}_{\mu b,r} \\
& + \frac{1}{2}g^{(0)cb}g^{(0)\rho\nu}g^{(0)}_{cr,\nu}g^{(0)}_{\mu b,\rho} + \frac{1}{2}g^{(0)cb}g^{(0)r\nu}g^{(0)}_{cr,\nu}g^{(1)}_{\mu b,r} \\
& - \frac{1}{2}g^{(0)cr}g^{(0)a\nu}g^{(1)}_{cr,\nu}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(1)cr}g^{(0)a\nu}g^{(0)}_{cr,\nu}g^{(0)}_{a\mu,r} - \frac{1}{2}g^{(0)cr}g^{(1)a\nu}g^{(0)}_{cr,\nu}g^{(0)}_{a\mu,r} \\
& - \frac{1}{2}g^{(0)c\rho}g^{(0)a\nu}g^{(0)}_{cr,\nu}g^{(0)}_{a\mu,\rho} - \frac{1}{2}g^{(0)cr}g^{(0)a\nu}g^{(1)}_{cr,\nu}g^{(1)}_{a\mu,r} - \frac{1}{2}\Phi^{(1)}_{,r}\Phi^{(0)}_{,\mu}
\end{aligned} \tag{3.78}
$$

which evaluates to:

$$
\begin{aligned}
E^{source}_{r\mu} &= -F'(r)u^{(0)\alpha}u^{(0)\beta}\partial_\mu\partial_\alpha u^{(0)}_\beta + \frac{1}{2}F'(r)P^{(0)\nu\beta}\partial_\mu\partial_\nu u^{(0)}_\beta + \frac{1}{2}F'(r)P^{(0)\nu\beta}\partial_\nu\partial_\beta u^{(0)}_\mu \\
&- \frac{1}{d-1}F'(r)\partial_\mu\partial_\alpha u^{(0)\alpha} - \frac{1}{2r^2}P^{(0)\nu\rho}\partial_\mu\partial_\nu u^{(0)}_\rho + \frac{1}{2r^2}P^{(0)\nu\rho}\partial_\nu\partial_\rho u^{(0)}_\mu \\
&+ \left(2F'(r) + rF''(r)\right)\sigma^{(1)}{}_\mu{}^\nu\partial_\nu b^{(0)} \\
&+ \frac{1}{2}F'(r)\partial_\nu u^{(0)\nu}u^{(0)\alpha}\partial_\alpha u^{(0)}_\mu + \frac{2}{d-1}\frac{F(r)}{r}\partial_\alpha u^{(0)\alpha}u^{(0)\nu}\partial_\nu u^{(0)}_\mu + \frac{1}{2r^2}\partial_\nu u^{(0)\nu}u^{(0)\rho}\partial_\rho u^{(0)}_\mu \\
&+ \frac{1}{2}F'(r)u^{(0)\alpha}\partial_\alpha u^{(0)\mu}\partial_\nu u^{(0)}_\mu - \frac{F(r)}{r}u^{(0)\nu}\partial_\nu u^{(0)\alpha}\partial_\alpha u^{(0)}_\mu - \frac{d-3}{2r^2}u^{(0)\nu}\partial_\nu u^{(0)\rho}\partial_\rho u^{(0)}_\rho \\
&- F'(r)u^{(0)\alpha}\partial_\mu u^{(0)\beta}\partial_\alpha u^{(0)}_\beta - \frac{F(r)}{r}u^{(0)\alpha}\partial_\mu u^{(0)\beta}\partial_\alpha u^{(0)}_\beta + \frac{d-4}{2r^2}u^{(0)\lambda}\partial_\lambda u^{(0)\rho}\partial_\mu u^{(0)}_\rho \\
&+ \left(-2(d-2)\frac{F(r)}{r} - (d+2)F'(r) - rF''(r)\right)\sigma^{(1)}_{\mu\rho}u^{(0)\gamma}\partial_\gamma u^{(0)\rho} - \frac{1}{2}\Phi^{(1)}_{,r}\Phi^{(0)}_{,\mu}.
\end{aligned}
\tag{3.79}
$$

$E_{\mu\nu}$

The differential operator is given by:

$$
\begin{aligned}
E^{diff}_{\mu\nu} &= -\frac{1}{2}g^{(2)rr}{}_{,r}g^{(0)}_{\mu\nu,r} - \frac{1}{2}g^{(0)rr}{}_{,r}g^{(2)}_{\mu\nu,r} - \frac{1}{2}g^{(2)rr}g^{(0)}_{\mu\nu,rr} - \frac{1}{2}g^{(0)rr}g^{(2)}_{\mu\nu,rr} \\
&- \frac{1}{4}g^{(2)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r} - \frac{1}{4}g^{(0)rr}g^{(2)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r} \\
&- \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(2)}_{ad,r}g^{(0)}_{\mu\nu,r} - \frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(2)}_{\mu\nu,r} \\
&+ \frac{1}{2}g^{(2)cb}g^{(0)rr}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,r} + \frac{1}{2}g^{(0)cb}g^{(2)rr}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,r} \\
&+ \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(2)}_{c\mu,r}g^{(0)}_{\nu b,r} + \frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(0)}_{c\mu,r}g^{(2)}_{\nu b,r} \\
&- \frac{1}{2}g^{(2)cr}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,r} - \frac{1}{2}g^{(0)cr}g^{(2)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,r} \\
&- \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(2)}_{c\mu,r}g^{(0)}_{a\nu,r} - \frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(2)}_{a\nu,r} \\
&+ dg^{(2)}_{\mu\nu}.
\end{aligned}
\tag{3.80}
$$

We only require the transverse part of this equation (along with $E_{r\mu}$ and E_{rr}) to determine the second order fluid/gravity metric. The transverse traceless (TT) part computes to:

$$
\begin{aligned}
E^{diff,TT}_{\mu\nu} &= \tilde{\mathcal{G}}^{(2)}_{\mu\nu}\left((d-2)+2r^{2-d}\right)+\tilde{\mathcal{G}}^{(2)\prime}_{\mu\nu}\left(\frac{3-d}{2}r-\frac{3}{2}r^{1-d}\right)+\tilde{\mathcal{G}}^{(2)\prime\prime}_{\mu\nu}\left(-\frac{1}{2}r^2+\frac{1}{2}r^{2-d}\right)\\
&\quad+\left(r^4-r^{4-d}\right)\left(\frac{8}{r^2}F^2(r)\sigma^{(1)}_{\mu\lambda}\sigma^{(1)\lambda}{}_{\nu}-\frac{1}{d-1}\frac{8}{r^2}F^2(r)\sigma^{(1)\alpha\beta}\sigma^{(1)}_{\alpha\beta}P^{(0)}_{\mu\nu}\right)
\end{aligned}
\tag{3.81}
$$

where $\tilde{\mathcal{G}}^{(2)}_{\mu\nu}=\tilde{\mathcal{G}}^{(2)}_{\mu\nu}-\frac{1}{d-1}\tilde{\mathcal{G}}^{(2)\gamma}_{\gamma}P^{(0)}_{\mu\nu}$, is the traceless part of $\tilde{\mathcal{G}}^{(2)}_{\mu\nu}$. The operator part is just the first line:

$$
E^{diff,TT,operator}_{\mu\nu}=\tilde{\mathcal{G}}^{(2)}_{\mu\nu}\left((d-2)+2r^{2-d}\right)+\tilde{\mathcal{G}}^{(2)\prime}_{\mu\nu}\left(\frac{3-d}{2}r-\frac{3}{2}r^{1-d}\right)+\tilde{\mathcal{G}}^{(2)\prime\prime}_{\mu\nu}\left(-\frac{1}{2}r^2+\frac{1}{2}r^{2-d}\right).
\tag{3.82}
$$

For the transverse trace, the operator terms are:

$$
\begin{aligned}
E^{diff,Trace,operator}_{\mu\nu} &= 2ru^{(0)\lambda}\mathcal{V}^{(2)\prime}_{\lambda}+(d-2)2u^{(0)\lambda}\mathcal{V}^{(2)}_{\lambda}-\frac{r^2-r^{2-d}}{2(d-1)}\mathcal{G}^{(2)\prime\prime\alpha}{}_{\alpha}\\
&\quad+\left(\frac{2-d}{d-1}r+\frac{d-4}{2(d-1)}r^{1-d}\right)\mathcal{G}^{(2)\prime\alpha}{}_{\alpha}+\left(\frac{2d-3}{d-1}-\frac{d-3}{d-1}r^{-d}\right)\mathcal{G}^{(2)\alpha}{}_{\alpha}.
\end{aligned}
\tag{3.83}
$$

And for the source terms:

$$
\begin{aligned}
E^{source}_{\mu\nu} &= \frac{1}{2}g^{(0)rb}{}_{,r}g^{(1)}_{\nu b,\mu}+\frac{1}{2}g^{(1)rb}{}_{,r}g^{(0)}_{\nu b,\mu}+\frac{1}{2}g^{(0)\rho b}{}_{,\rho}g^{(0)}_{\nu b,\mu}\\
&+\frac{1}{2}g^{(0)rb}{}_{,r}g^{(1)}_{\mu b,\nu}+\frac{1}{2}g^{(1)rb}{}_{,r}g^{(0)}_{\mu b,\nu}+\frac{1}{2}g^{(0)\rho b}{}_{,\rho}g^{(0)}_{\mu b,\nu}\\
&-\frac{1}{2}g^{(1)\rho r}{}_{,\rho}g^{(0)}_{\mu\nu,r}-\frac{1}{2}g^{(0)r\rho}{}_{,r}g^{(0)}_{\mu\nu,\rho}+\frac{1}{2}g^{(1)rr}{}_{,r}g^{(1)}_{\mu\nu,r}-\frac{1}{2}g^{(0)\rho r}{}_{,\rho}g^{(1)}_{\mu\nu,r}\\
&-\frac{1}{2}g^{(1)r\rho}{}_{,r}g^{(0)}_{\mu\nu,\rho}-\frac{1}{2}g^{(0)\rho\gamma}{}_{,\rho}g^{(0)}_{\mu\nu,\gamma}\\
&+\frac{1}{2}g^{(0)rb}g^{(1)}_{\mu b,\nu r}+\frac{1}{2}g^{(0)\rho b}g^{(0)\mu b,\nu\rho}+\frac{1}{2}g^{(1)rb}g^{(0)}_{\mu b,\nu r}\\
&-\frac{1}{2}g^{(0)\rho\gamma}g^{(0)}_{\mu\nu,\rho\gamma}-g^{(0)r\rho}g^{(1)}_{\mu\nu,r\rho}-\frac{1}{2}g^{(1)rr}g^{(1)}_{\mu\nu,rr}-g^{(1)r\rho}g^{(0)}_{\mu\nu,r\rho}\\
&-\frac{1}{2}g^{(0)ab}{}_{,\mu}g^{(0)}_{ab,\nu}-\frac{1}{2}g^{(0)ab}g^{(0)}_{ab,\nu\mu}+\frac{1}{2}g^{(0)ar}g^{(1)}_{a\nu,r\mu}+\frac{1}{2}g^{(0)a\rho}g^{(0)}_{a\nu,\rho\mu}\\
&+\frac{1}{2}g^{(1)ar}g^{(0)}_{a\nu,r\mu}\\
&+\frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\nu b,\mu}+\frac{1}{4}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\nu b,\mu}+\frac{1}{4}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\nu b,\mu}\\
&+\frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\nu b,\mu}+\frac{1}{4}g^{(0)\rho b}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\nu b,\mu}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu b,\nu}+\frac{1}{4}g^{(1)rb}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu b,\nu}+\frac{1}{4}g^{(0)rb}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu b,\nu}\\
&+\frac{1}{4}g^{(0)rb}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu b,\nu}+\frac{1}{4}g^{(0)\rho b}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\mu b,\nu}\\
&-\frac{1}{4}g^{(0)\rho r}g^{(0)ad}g^{(1)}_{ad,\rho}g^{(0)}_{\mu\nu,r}-\frac{1}{4}g^{(0)r\rho}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu\nu,\rho}-\frac{1}{4}g^{(1)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,r}\\
&-\frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu\nu,r}-\frac{1}{4}g^{(1)\rho r}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\mu\nu,r}-\frac{1}{4}g^{(1)rr}g^{(0)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu\nu,r}\\
&-\frac{1}{4}g^{(1)r\rho}g^{(0)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,\rho}-\frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu\nu,r}-\frac{1}{4}g^{(0)\rho r}g^{(1)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\mu\nu,r}\\
&-\frac{1}{4}g^{(0)rr}g^{(1)ad}g^{(0)}_{ad,r}g^{(1)}_{\mu\nu,r}-\frac{1}{4}g^{(0)r\rho}g^{(1)ad}g^{(0)}_{ad,r}g^{(0)}_{\mu\nu,\rho}-\frac{1}{4}g^{(0)rr}g^{(0)ad}g^{(1)}_{ad,r}g^{(1)}_{\mu\nu,r}\\
&-\frac{1}{4}g^{(0)\rho r}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(1)}_{\mu\nu,r}-\frac{1}{4}g^{(0)r\rho}g^{(0)ad}g^{(1)}_{ad,r}g^{(0)}_{\mu\nu,\rho}-\frac{1}{4}g^{(0)\rho\gamma}g^{(0)ad}g^{(0)}_{ad,\rho}g^{(0)}_{\mu\nu,\gamma}\\
&-\frac{1}{4}g^{(0)cb}g^{(0)ad}g^{(0)}_{cd,\mu}g^{(0)}_{ab,\nu}\\
&+\frac{1}{2}g^{(0)cb}g^{(0)r\rho}g^{(1)}_{c\mu,\rho}g^{(0)}_{\nu b,r}+\frac{1}{2}g^{(0)cb}g^{(0)\rho r}g^{(0)}_{c\mu,r}g^{(1)}_{\nu b,\rho}+\frac{1}{2}g^{(1)cb}g^{(1)rr}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,r}\\
&+\frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(1)}_{c\mu,r}g^{(0)}_{\nu b,r}+\frac{1}{2}g^{(1)cb}g^{(0)r\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{\nu b,r}+\frac{1}{2}g^{(1)cb}g^{(0)rr}g^{(0)}_{c\mu,r}g^{(1)}_{\nu b,r}\\
&+\frac{1}{2}g^{(1)cb}g^{(0)\rho r}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,\rho}+\frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(1)}_{c\mu,r}g^{(0)}_{\nu b,r}+\frac{1}{2}g^{(0)cb}g^{(1)r\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{\nu b,r}\\
&+\frac{1}{2}g^{(0)cb}g^{(1)rr}g^{(0)}_{c\mu,r}g^{(1)}_{\nu b,r}+\frac{1}{2}g^{(0)cb}g^{(1)\rho r}g^{(0)}_{c\mu,r}g^{(0)}_{\nu b,\rho}+\frac{1}{2}g^{(0)cb}g^{(0)rr}g^{(1)}_{c\mu,r}g^{(1)}_{\nu b,r}\\
&+\frac{1}{2}g^{(0)cb}g^{(0)r\rho}g^{(0)}_{c\mu,\rho}g^{(1)}_{\nu b,r}+\frac{1}{2}g^{(0)cb}g^{(0)\rho r}g^{(1)}_{c\mu,r}g^{(0)}_{\nu b,\rho}+\frac{1}{2}g^{(0)cb}g^{(0)\rho\gamma}g^{(0)}_{c\mu,\gamma}g^{(0)}_{\nu b,\rho}\\
&-\frac{1}{2}g^{(0)cr}g^{(0)a\rho}g^{(1)}_{c\mu,\rho}g^{(0)}_{a\nu,r}-\frac{1}{2}g^{(0)c\rho}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(1)}_{a\nu,\rho}-\frac{1}{2}g^{(1)cr}g^{(1)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,r}\\
&-\frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(1)}_{c\mu,r}g^{(0)}_{a\nu,r}-\frac{1}{2}g^{(1)cr}g^{(0)a\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{a\nu,r}-\frac{1}{2}g^{(1)cr}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(1)}_{a\nu,r}\\
&-\frac{1}{2}g^{(1)c\rho}g^{(0)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,\rho}-\frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(1)}_{c\mu,r}g^{(0)}_{a\nu,r}-\frac{1}{2}g^{(0)cr}g^{(1)a\rho}g^{(0)}_{c\mu,\rho}g^{(0)}_{a\nu,r}\\
&-\frac{1}{2}g^{(0)cr}g^{(1)ar}g^{(0)}_{c\mu,r}g^{(1)}_{a\nu,r}-\frac{1}{2}g^{(0)c\rho}g^{(1)ar}g^{(0)}_{c\mu,r}g^{(0)}_{a\nu,\rho}-\frac{1}{2}g^{(0)cr}g^{(0)ar}g^{(1)}_{c\mu,r}g^{(1)}_{a\nu,r}\\
&-\frac{1}{2}g^{(0)cr}g^{(0)a\rho}g^{(0)}_{c\mu,\rho}g^{(1)}_{a\nu,r}-\frac{1}{2}g^{(0)c\rho}g^{(0)ar}g^{(1)}_{c\mu,r}g^{(0)}_{a\nu,\rho}-\frac{1}{2}g^{(0)c\rho}g^{(0)a\gamma}g^{(0)}_{c\mu,\gamma}g^{(0)}_{a\nu,\rho}\\
&-\frac{1}{2}\Phi^{(0)}_{,\mu}\Phi^{(0)}_{,\nu}
\end{aligned}
$$

We are interested in only the transverse part. For the transverse traceless part (if we add on the non-operator terms from $E^{diff,TT}_{\mu\nu}$) we obtain the following expression:

$$
\begin{aligned}
E^{source,TT}_{\mu\nu}&=\partial_\rho u^{(0)\rho}\sigma^{(1)}_{\mu\nu}\left(-1+3rF(r)\right)\\
&\quad+u^{(0)\rho}\partial_\rho u^{(0)}_\mu u^{(0)\delta}\partial_\delta u^{(0)}_\nu\left(-2r^2F'(r)-(d-1)rF(r)+(d-3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ u^{(0)\rho}\partial_\rho\partial_{(\mu}u^{(0)}_{\nu)}\left(+(d-3)-2r^2F'(r)-(d-1)rF(r)\right)\\
&-\frac{d-1}{2}rF(r)\left(2\partial_\mu u^{(0)\alpha}\partial_\nu u^{(0)}_\alpha+\partial_\nu u^{(0)\alpha}\partial_\alpha u^{(0)}_\mu+\partial_\mu u^{(0)\alpha}\partial_\alpha u^{(0)}_\lambda+\partial_\mu u^{(0)\alpha}\partial_\alpha u^{(0)}_\nu\right)\\
&+\sigma^{(1)\lambda}_{\mu}\sigma^{(1)}_{\lambda\nu}\cdot 2r^2F'^2(r)\left(r^2-r^{2-d}\right)\\
&+r^2F'(r)\left(\partial^\lambda u^{(0)}_\mu\partial_\lambda u^{(0)}_\nu-\partial_\mu u^{(0)\lambda}\partial_\nu u^{(0)}_\lambda+u^{(0)\lambda}\partial_\lambda u^{(0)}_\nu u^{(0)\rho}\partial_\rho u^{(0)}_\mu-\frac{2}{d-1}\sigma^{(1)}_{\mu\nu}\partial_\rho u^{(0)\rho}\right)\\
&+\partial_\mu u^{(0)\rho}\partial_\nu u^{(0)}_\rho\left(-\frac{1}{2}-\frac{1}{2}r^{-d}\right)\\
&+P^{(0)\rho\gamma}\partial_\gamma u^{(0)}_\mu\partial_\rho u^{(0)}_\nu\left(\frac{1}{2}-\frac{1}{2}r^{-d}\right)\\
&+\frac{1}{2}r^{-d}\partial_\nu u^{(0)}_\gamma\partial_\gamma u^{(0)}_\mu+\frac{1}{2}r^{-d}\partial_\mu u^{(0)}_\gamma\partial^\gamma u^{(0)}_\nu\\
&+\left(-1+\frac{d}{2}\right)\left(\partial_\nu u^{(0)\gamma}\partial_\gamma u^{(0)}_\mu+\partial_\mu u^{(0)\gamma}\partial_\gamma u^{(0)}_\nu\right)+(d-2)C_{\mu\alpha\nu\beta}u^{(0)\alpha}u^{(0)\beta}\\
&-\frac{1}{2}\left(P^{(0)\alpha}_\mu P^{(0)\beta}_\nu\partial_\alpha\phi\partial_\beta\phi-\frac{1}{d-1}P^{(0)}_{\mu\nu}P^{(0)\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right).
\end{aligned}
$$

where $C_{\mu\alpha\nu\beta}$ is the Weyl curvature tensor. In Weyl covariant form, this becomes:

$$
\begin{aligned}
E^{source,TT}_{\mu\nu} &= \left[-(d-4)+2(d-1)(br)F(br)+4(br)^2F'(br)+\frac{2(1-(br)^{2-d})}{(br)^d-1}\right]\left(\sigma_\mu{}^\lambda\sigma_{\lambda\nu}-\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}\right)\\
&+\left[-(d-3)+(d-1)(br)F(br)+2(br)^2F'(br)\right]u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}\\
&+\left[1+(d-1)(br)F(br)+2(br)^2F'(br)\right]\left(\omega_\mu{}^\lambda\sigma_{\lambda\nu}+\omega_\nu{}^\lambda\sigma_{\mu\lambda}\right)\\
&+\left[-(d-2)-2(br)^{-d}\right]\omega_\mu{}^\lambda\omega_{\lambda\nu}+(d-2)C_{\mu\alpha\nu\beta}u^\alpha u^\beta\\
&-\frac{1}{2}\left(P^\alpha_\mu P^\beta_\nu\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi-\frac{1}{d-1}P_{\mu\nu}P^{\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi\right).
\end{aligned}
\tag{3.84}
$$

For the trace, the scalar terms are given by:

$$\begin{aligned}
E^{source}_{\mu\nu,Trace} &= -u^\alpha \mathcal{D}_\lambda \sigma^\lambda_\alpha + u^\alpha \mathcal{D}_\lambda \omega^\lambda{}_\alpha - \frac{\mathcal{R}}{d-1} + \frac{1}{2(d-1)} P^{\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi \\
&+ \left[\frac{2(d-2)}{d-1}(br)^{-d} - \frac{2d-3}{d-1}\right]\omega_{\alpha\beta}\omega^{\alpha\beta} \\
&+ F(br)\left[\frac{-4d}{d-1}(br)^3 F'(br) + \frac{2d}{d-1}(br)^{3-d}F'(br) - \frac{2}{d-1}\left((br)^4 - (br)^{4-d}\right)F''(br)\right]\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\
&+ \left[\frac{-1}{d-1}F'^2(br)\left((br)^4 - (br)^{4-d}\right) + \frac{1}{d-1}(br)^{1-d}\right]\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\
&+ \left[-\frac{1}{d-1}(br)^{2-d}(br-1)F'(br) - 2\right]\sigma_{\alpha\beta}\sigma^{\alpha\beta} \\
&+ \left[\frac{-1}{2(d-1)} - \frac{1}{d-1}(br)^{3-d}F'(br) - \frac{1}{2(d-1)}(br)^{2-d}\frac{(br)^{d-2}-1}{(br)^d-1}\right]u^\alpha u^\beta \mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi \\
&+ \left[\frac{1}{2(d-1)}\left((br)^4 - (br)^{4-d}\right)F'^2(br)\right]u^\alpha u^\beta\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi.
\end{aligned} \tag{3.85}$$

Constraint equations at second order

If we now consider the boundary components of $E^\Phi_{AB}n^B = 0$ at second order, we find that the relation (3.51) acquires a second order correction:

$$\partial_\mu b = \mathcal{A}_\mu b + 2b^2\left[\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}u_\mu - \frac{\mathcal{D}_\lambda\sigma^\lambda{}_\mu}{d}\right] + \frac{b^2}{d}\left[\frac{u^\alpha\mathcal{D}_\alpha\phi u^\beta\mathcal{D}_\beta\phi}{d-1}u_\mu + u^\alpha\mathcal{D}_\alpha\phi P_{\mu\lambda}\mathcal{D}^\lambda\phi\right]. \tag{3.86}$$

This can again be shown to be equivalent to $\nabla_\mu T^{\mu\nu} = e^{-\phi}\mathcal{L}\nabla^\nu\phi$; the additional second order terms arise from the first order contributions to $T^{\mu\nu}$ and $\mathcal{L}$.

Solution at second order

For the rr-component of the Einstein equations, we derived the following equation:

$$\begin{aligned}
&-\frac{\mathcal{G}^{(2)\prime\prime\mu}{}_\mu}{2r^2} + \frac{\mathcal{G}^{(2)\prime\mu}{}_\mu}{r^3} - \frac{\mathcal{G}^{(2)\mu}{}_\mu}{r^4} \\
&= -\left(\frac{4}{r}F(r)F'(r) + F'^2(r) + 2F''(r)F(r)\right)\sigma^{(1)\mu\nu}\sigma^{(1)}_{\mu\nu} - \frac{1}{r^4}\omega^{(1)}_{\mu\nu}\omega^{(1)\mu\nu} + \frac{1}{2}\Phi^{(1)}_{,r}\Phi^{(1)}_{,r}.
\end{aligned} \tag{3.87}$$

We require $\Phi^{(1)}$ which we will determine in the following subsection, so we just quote the answer here for now.

$$\Phi^{(1)} = bu^{\mu}\mathcal{D}_{\mu}\phi F(br). \tag{3.88}$$

Now using the same methodology introduced in our previous review chapter, our equation for $\mathcal{G}^{(2)\mu}{}_{\mu}$ above integrates to give (factors of b included):

$$\mathcal{G}^{(2)\mu}{}_{\mu} = +2(br)^2\left[F^2(br) - K_1(br)\right]\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta} - (br)^2 K_1(br)u^{\alpha}u^{\beta}\mathcal{D}_{\alpha}\phi\mathcal{D}_{\beta}\phi. \tag{3.89}$$

where

$$K_1(br) \equiv \int_{br}^{\infty}\frac{d\xi}{\xi^2}\int_{\xi}^{\infty} dy\ y^2 F'(y)^2.$$

For the $r\mu$-component of the Einstein equations, we only require the transverse part; this simplifies to:

$$\begin{aligned}&\frac{1}{2}P^{(0)\lambda}_{\mu}\mathcal{V}^{(2)\prime\prime}_{\lambda} + \frac{1}{2}\frac{d-3}{r}P^{(0)\lambda}_{\mu}\mathcal{V}^{(2)\prime}_{\lambda} - \frac{d-2}{r^2}P^{(0)\lambda}_{\mu}\mathcal{V}^{(2)}_{\lambda}\\ &= \frac{1}{2}F'(r)u^{(0)\alpha}\mathcal{D}_{\alpha}\phi P^{(0)\lambda}_{\mu}\mathcal{D}_{\lambda}\phi - \frac{1}{2}F'(r)P^{(0)\lambda}_{\mu}\mathcal{D}_{\alpha}\sigma^{(1)\alpha}_{\lambda} - \frac{1}{2r^3}P^{(0)\lambda}_{\mu}\mathcal{D}_{\alpha}\omega^{(1)\alpha}{}_{\lambda}.\end{aligned} \tag{3.90}$$

This is sufficient to determine $P^{\lambda}_{\mu}\mathcal{V}^{(2)}_{\lambda}$.

$$\begin{aligned}P^{\lambda}_{\mu}\mathcal{V}^{(2)}_{\lambda} &= -\frac{1}{2(d-2)}\mathcal{D}_{\lambda}\sigma^{\lambda}{}_{\mu} + \frac{L(br)}{(br)^{d-2}}P^{\lambda}_{\mu}\mathcal{D}_{\alpha}\sigma^{\alpha}_{\lambda} - \frac{1}{2(d-2)}\mathcal{D}_{\lambda}\omega^{\lambda}{}_{\mu}\\ &\quad + \frac{1}{(br)^{d-2}}\left[\frac{(br)^{d-2}}{2(d-2)} + L(br)\right]u^{\alpha}\mathcal{D}_{\alpha}\phi P^{\lambda}_{\mu}\mathcal{D}_{\lambda}\phi\end{aligned} \tag{3.91}$$

where:

$$L(br) \equiv \int_{br}^{\infty}\xi^{d-1}d\xi\int_{\xi}^{\infty}dy\ \frac{y-1}{y^3(y^d-1)}.$$

To solve for $u^\lambda \mathcal{V}^{(2)}_\lambda$ we use the trace of the transverse part of the $\mu\nu$-component of the Einstein equations. This is:

$$
\begin{aligned}
&2ru^{(0)\lambda}\mathcal{V}^{(2)\prime}_\lambda + (d-2)2u^{(0)\lambda}\mathcal{V}^{(2)}_\lambda - \frac{r^2-r^{2-d}}{2(d-1)}\mathcal{G}^{(2)\prime\prime\alpha}{}_\alpha \\
&+\left(\frac{2-d}{d-1}r + \frac{d-4}{2(d-1)}r^{1-d}\right)\mathcal{G}^{(2)\prime\alpha}{}_\alpha + \left(\frac{2d-3}{d-1} - \frac{d-3}{d-1}r^{-d}\right)\mathcal{G}^{(2)\alpha}{}_\alpha \\
&= -u^{(0)\alpha}\mathcal{D}_\lambda\sigma^{(1)\lambda}{}_\alpha + u^{(0)\alpha}\mathcal{D}_\lambda\omega^{(1)\lambda}{}_\alpha - \frac{\mathcal{R}}{d-1} + \frac{1}{2(d-1)}P^{(0)\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi \\
&+\left[\frac{2(d-2)}{d-1}r^{-d} - \frac{2d-3}{d-1}\right]\omega^{(1)}_{\alpha\beta}\omega^{(1)\alpha\beta} \\
&+F(r)\left[\frac{-4d}{d-1}r^3F'(r) + \frac{2d}{d-1}r^{3-d}F'(r) - \frac{2}{d-1}\left(r^4 - r^{4-d}\right)F''(r)\right]\sigma^{(1)}_{\alpha\beta}\sigma^{(1)\alpha\beta} \\
&+\left[\frac{-1}{d-1}F'^2(r)\left(r^4 - r^{4-d}\right) + \frac{1}{d-1}r^{1-d} - \frac{1}{d-1}r^{2-d}(r-1)F'(r) - 2\right]\sigma^{(1)}_{\alpha\beta}\sigma^{(1)\alpha\beta} \\
&+\left[\frac{-1}{2(d-1)} - \frac{1}{d-1}r^{3-d}F'(r) - \frac{1}{2(d-1)}r^{2-d}\frac{r^{d-2}-1}{(br)^d-1}\right]u^{(0)\alpha}u^{(0)\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi \\
&+\left[\frac{1}{2(d-1)}\left(r^4 - r^{4-d}\right)F'^2(r)\right]u^{(0)\alpha}u^{(0)\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi.
\end{aligned}
\tag{3.92}
$$

This integrates to give:

$$
\begin{aligned}
u^\lambda\mathcal{V}^{(2)}_\lambda &= \frac{1}{4(br)^d}\omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{K_2(br)}{2(br)^{d-2}}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{(d-1)} + \frac{\mathcal{R}}{2(d-1)(d-2)} \\
&- \frac{1}{4(d-2)(d-1)}P^{\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi \\
&- \frac{1}{2(br)^{d-2}}\left[\frac{(2d-3)(br)^{d-2}}{2(d-1)(d-2)} + K_3(br)\right]u^\alpha u^\beta\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi
\end{aligned}
\tag{3.93}
$$

where:

$$
\begin{aligned}
K_2(br) \equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\Big[1 - \xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1} \\
+\Big(2(d-1)\xi^d - (d-2)\Big)\int_\xi^\infty dy\ y^2F'(y)^2\Big],
\end{aligned}
$$

$$
\begin{aligned}
K_3(br) \equiv \frac{d-2}{2(d-1)}K_1(br) - \frac{1}{d-1}F(br) + \frac{1}{2(d-1)}H_1(br) \\
+\int_{br}^\infty d\xi\left(\xi^{d-3} - \xi^{d-2}\int_\xi^\infty dy\ y^2F'(y)^2\right).
\end{aligned}
$$

And finally for the traceless part of $\mathcal{G}^{(2)}_{\mu\nu}$, we need the transverse traceless part of the $\mu\nu$-component of the Einstein equations:

$$
\begin{aligned}
&\left(-\frac{1}{2}r^2+\frac{1}{2}r^{2-d}\right)\tilde{\mathcal{G}}^{(2)\prime\prime}_{\mu\nu}+\left(\frac{(3-d)}{2}r-\frac{3}{2}r^{1-d}\right)\tilde{\mathcal{G}}^{(2)\prime}_{\mu\nu}+\left((d-2)+2r^{-d}\right)\tilde{\mathcal{G}}^{(2)}_{\mu\nu}\\
&=\left[-(d-4)+2(d-1)rF(r)+4(br)^2F'(r)+\frac{2(1-r^{2-d})}{r^d-1}\right]\left(\sigma^{(1)\lambda}_{\mu}\sigma^{(1)}_{\lambda\nu}-\frac{\sigma^{(1)}_{\alpha\beta}\sigma^{(1)\alpha\beta}}{d-1}P^{(0)}_{\mu\nu}\right)\\
&+\left[-(d-3)+(d-1)rF(r)+2(r)^2F'(r)\right]u^{(0)\lambda}\mathcal{D}_\lambda\sigma^{(1)}_{\mu\nu}\\
&+\left[1+(d-1)rF(r)+2(r)^2F'(r)\right]\left(\omega^{(1)\lambda}_{\mu}\sigma^{(1)}_{\lambda\nu}+\omega^{(1)\lambda}_{\nu}\sigma^{(1)}_{\mu\lambda}\right)\\
&+\left[-(d-2)-2(r)^{-d}\right]\omega^{(1)\lambda}_{\mu}\omega^{(1)}_{\lambda\nu}+(d-2)C_{\mu\alpha\nu\beta}u^{(0)\alpha}u^{(0)\beta}\\
&-\frac{1}{2}\left(P^{(0)\alpha}{}_{\mu}P^{(0)\beta}{}_{\nu}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi-\frac{1}{d-1}P^{(0)}_{\mu\nu}P^{(0)\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi\right)
\end{aligned}
\tag{3.94}
$$

where $\tilde{\mathcal{G}}^{(2)}_{\mu\nu}\equiv\mathcal{G}^{(2)}_{\mu\nu}-\frac{1}{d-1}\mathcal{G}^{(2)\alpha}{}_{\alpha}P_{\mu\nu}$. We integrate this to obtain:

$$
\begin{aligned}
\tilde{\mathcal{G}}^{(2)}_{\mu\nu}&=2(br)^2\left[\left(F^2(br)-H_1(br)\right)\left(\sigma_\mu{}^\lambda\sigma_{\lambda\nu}-\frac{1}{d-1}\sigma_{\alpha\beta}\sigma^{\alpha\beta}P_{\mu\nu}\right)+\left(H_2(br)-H_1(br)\right)u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}\right]\\
&+2(br)^2\left[H_2(br)\left(\omega_\mu{}^\lambda\sigma_{\lambda\nu}+\omega_\nu{}^\lambda\sigma_{\lambda\mu}\right)-H_1(br)C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right]-\omega_\mu{}^\lambda\omega_{\lambda\nu}\\
&+\frac{(br)^2}{(d-2)}H_1(br)\left[P^\alpha_\mu P^\beta_\nu\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi-\frac{1}{d-1}P_{\mu\nu}P^{\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi\right]
\end{aligned}
\tag{3.95}
$$

with:

$$
H_1(br)\equiv\int_{br}^{\infty}\frac{y^{d-2}-1}{y(y^d-1)}dy
$$

$$
\begin{aligned}
H_2(br)&\equiv\int_{br}^{\infty}\frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi y^{d-3}dy\left[1+(d-1)yF(y)+2y^2F'(y)\right]\\
&=\frac{1}{2}F(br)^2-\int_{br}^{\infty}\frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi\frac{y^{d-2}-1}{y(y^d-1)}dy
\end{aligned}
$$

Thus, the full metric to second order is:

$$
\begin{aligned}
ds^2 = &-2u_\mu dx^\mu \left[dr + \left(rA_\nu + \frac{r^2 f(br)}{2}u_\nu\right)dx^\nu\right] + \left[r^2 P_{\mu\nu} + 2(br)^2 F(br)\frac{1}{b}\sigma_{\mu\nu}\right]dx^\mu dx^\nu \\
&+ \left[\frac{1}{d-2}\mathcal{D}_\lambda \sigma^\lambda{}_{(\mu}u_{\nu)} + \frac{2L(br)}{(br)^{d-2}}u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\alpha \sigma^\alpha_\lambda - \frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_{(\mu}u_{\nu)}\right]dx^\mu dx^\nu \\
&- \frac{2}{(br)^{d-2}}\left[\frac{(br)^{d-2}}{2(d-2)} + L(br)\right]u^\alpha \mathcal{D}_\alpha \phi u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\lambda \phi dx^\mu dx^\nu \\
&- \left[\frac{1}{2(br)^d}\omega_{\alpha\beta}\omega^{\alpha\beta} + \frac{K_2(br)}{(br)^{d-2}}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{(d-1)} + \frac{\mathcal{R}}{(d-1)(d-2)}\right]u_\mu u_\nu dx^\mu dx^\nu \\
&+ \frac{1}{2(d-2)(d-1)}P^{\alpha\beta}\mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi u_\mu u_\nu dx^\mu dx^\nu \\
&+ \frac{1}{(br)^{d-2}}\left[\frac{(2d-3)(br)^{d-2}}{2(d-1)(d-2)} + K_3(br)\right]u^\alpha u^\beta \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi u_\mu u_\nu dx^\mu dx^\nu \\
&+ 2(br)^2\left[\left(F^2(br) - H_1(br)\right)\sigma_\mu{}^\lambda \sigma_{\lambda\nu} + \left(H_2(br) - H_1(br)\right)u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu}\right]dx^\mu dx^\nu \\
&+ 2(br)^2\left[H_2(br)\left(\omega_\mu{}^\lambda \sigma_{\lambda\nu} + \omega_\nu{}^\lambda \sigma_{\lambda\mu}\right) - H_1(br)C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right]dx^\mu dx^\nu - \omega_\mu{}^\lambda \omega_{\lambda\nu}dx^\mu dx^\nu \\
&+ \frac{(br)^2}{(d-2)}H_1(br)\left[P^\alpha_\mu P^\beta_\nu \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi - \frac{1}{d-1}P_{\mu\nu}P^{\alpha\beta}\mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi\right]dx^\mu dx^\nu \\
&+ 2(br)^2\left[H_1(br) - K_1(br)\right]\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu}dx^\mu dx^\nu - \frac{(br)^2}{d-1}K_1(br)u^\alpha u^\beta \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi P_{\mu\nu}dx^\mu dx^\nu .
\end{aligned}
\tag{3.96}
$$

3.3.3 Dilaton equations: First order

Here, we solve the dilaton equation to determine the first and second order dilaton corrections, $\Phi^{(1)}$ and $\Phi^{(2)}$. For clarity, we rewrite the field equation for the dilaton in terms of explicit partial derivatives of the metric and Φ.

$$
E^\Phi := g^{ab}\nabla_a \nabla_b \Phi = g^{ab}\Phi_{,ab} - g^{ab}g^{cd}g_{ad,b}\Phi_{,c} + \frac{1}{2}g^{ab}g^{cd}g_{ab,d}\Phi_{,c}. \tag{3.97}
$$

We begin with the first order equation. The differential operator terms at first order are:

$$
\begin{aligned}
E^{\Phi(1)}_{diff} &= g^{(0)ab}\Phi^{(1)}_{,ab} - g^{(0)ab}g^{(0)cd}g^{(0)}_{ad,b}\Phi^{(1)}_{,c} + \frac{1}{2}g^{(0)ab}g^{(0)cd}g^{(0)}_{ab,d}\Phi^{(1)}_{,c} \\
&= g^{(0)rr}\Phi^{(1)}_{,rr} - g^{(0)ar}g^{(0)rd}g^{(0)}_{ad,r}\Phi^{(1)}_{,r} + \frac{1}{2}g^{(0)ab}g^{(0)rr}g^{(0)}_{ab,r}\Phi^{(1)}_{,r}
\end{aligned}
\tag{3.98}
$$

which evaluates to:

$$E^{\Phi(1)}_{diff} = \left(r^2 - r^{2-d}\right)\Phi^{(1)}_{,rr} + \left((d+1)r - r^{1-d}\right)\Phi^{(1)}_{,r}. \tag{3.99}$$

And for the source terms

$$E^{\Phi(1)}_{source} = 2g^{(0)r\mu}\Phi^{(0)}_{,r\mu} - g^{(0)ar}g^{(0)\mu d}g^{(0)}_{ad,r}\Phi^{(0)}_{,\mu} + \frac{1}{2}g^{(0)ab}g^{(0)\mu r}g^{(0)}_{ab,r}\Phi^{(0)}_{,\mu} \tag{3.100}$$

which equates to:

$$E^{\Phi(1)}_{source} = \frac{d-1}{r}u^{(0)\mu}\Phi^{(0)}_{,\mu}. \tag{3.101}$$

The entire equation at first order is then:

$$\left(r^2 - r^{2-d}\right)\Phi^{(1)}_{,rr} + \left((d+1)r - r^{1-d}\right)\Phi^{(1)}_{,r} + \frac{d-1}{r}u^{(0)\mu}\Phi^{(0)}_{,\mu} = 0 \tag{3.102}$$

which can be rewritten as:

$$\frac{1}{r^{d-1}}\partial_r\left(r^{d+1}\left(1 - \frac{1}{r^d}\partial_r\right)\Phi^{(1)}\right) + \frac{1}{r^{d-1}}\partial_r\left(r^{d-1}u^{(0)\mu}\Phi^{(0)}_{,\mu}\right) = 0. \tag{3.103}$$

This integrates to give:

$$\Phi^{(1)} = u^{(0)\mu}\partial_\mu\phi\int_r^\infty \frac{y^{d-1} - 1}{y^{d+1}f(r)}dr \tag{3.104}$$

3.3.4 Dilaton equations: Second order

Differential operator is:

$$E^{\Phi(2)}_{diff} = g^{(0)rr}\Phi^{(2)}_{,rr} - g^{(0)ar}g^{(0)rd}g^{(0)}_{ad,r}\Phi^{(2}_{,r} + \frac{1}{2}g^{(0)ab}g^{(0)rr}g^{(0)}_{ab,r}\Phi^{(2)}_{,r} \tag{3.105}$$

which again evaluates to:

$$E^{\Phi(2)}_{diff} = \left(r^2 - r^{2-d}\right)\Phi^{(2)}_{,rr} + \left((d+1)r - r^{1-d}\right)\Phi^{(2)}_{,r}. \tag{3.106}$$

And the source terms are given by:

$$\begin{aligned}
E^{\Phi(2)}_{source} &= g^{(0)\mu\nu}\Phi^{(0)}_{\mu\nu} + 2g^{(0)r\mu}\Phi^{(1)}_{,r\mu} + g^{(1)rr}\Phi^{(1)}_{,rr} \\
&- g^{(0)ar}g^{(0)\mu d}g^{(0)}_{ad,r}\Phi^{(1)}_{,\mu} - g^{(1)ar}g^{(0)rd}g^{(0)}_{ad,r}\Phi^{(1)}_{,r} - g^{(0)ar}g^{(1)rd}g^{(0)}_{ad,r}\Phi^{(1)}_{,r} \\
&- g^{(0)ar}g^{(0)rd}g^{(1)}_{ad,r}\Phi^{(1)}_{,r} - g^{(0)a\mu}g^{(0)rd}g^{(0)}_{ad,\mu}\Phi^{(1)}_{,r} - g^{(1)ar}g^{(0)\mu d}g^{(0)}_{ad,r}\Phi^{(0)}_{,\mu} \\
&- g^{(0)ar}g^{(1)\mu d}g^{(0)}_{ad,r}\Phi^{(0)}_{,\mu} - g^{(0)ar}g^{(0)\mu d}g^{(1)}_{ad,r}\Phi^{(0)}_{,\mu} - g^{(0)a\mu}g^{(0)\nu d}g^{(0)}_{ad,\mu}\Phi^{(1)}_{,\nu} \\
&+ \frac{1}{2}g^{(0)ab}g^{(0)\mu r}g^{(0)}_{ab,r}\Phi^{(1)}_{,\mu} + \frac{1}{2}g^{(1)ab}g^{(0)rr}g^{(0)}_{ab,r}\Phi^{(1)}_{,r} + \frac{1}{2}g^{(0)ab}g^{(1)rr}g^{(0)}_{ab,r}\Phi^{(1)}_{,r} \\
&+ \frac{1}{2}g^{(0)ab}g^{(0)rr}g^{(1)}_{ab,r}\Phi^{(1)}_{,r} + \frac{1}{2}g^{(0)ab}g^{(0)r\mu}g^{(0)}_{ab,\mu}\Phi^{(1)}_{,r} + \frac{1}{2}g^{(1)ab}g^{(0)\mu r}g^{(0)}_{ab,r}\Phi^{(0)}_{,\mu} \\
&+ \frac{1}{2}g^{(0)ab}g^{(1)\mu r}g^{(0)}_{ab,r}\Phi^{(0)}_{,\mu} + \frac{1}{2}g^{(0)ab}g^{(0)\mu r}g^{(1)}_{ab,r}\Phi^{(0)}_{,\mu} + \frac{1}{2}g^{(0)ab}g^{(0)\mu\nu}g^{(0)}_{ab,\nu}\Phi^{(0)}_{,\mu}.
\end{aligned} \tag{3.107}$$

This simplifies to:

$$\begin{aligned}
E^{\Phi(2)}_{source} &= \frac{1}{r^2}\nabla^2\phi \\
&+ u^{(0)\mu}u^{(0)\nu}\partial_\mu\partial_\nu\phi\left(\frac{1}{r^2} + 2F'(r) + \frac{1}{r}F(r)(d-1)\right) \\
&+ u^{(0)\mu}\partial_\mu u^{(0)\nu}\partial_\nu\phi\left(2F'(r) + \frac{1}{r^2}(3-d) + \frac{1}{r}F(r)(d-1)\right) \\
&+ \partial_\lambda u^{(0)\lambda}u^{(0)\mu}\partial_\mu\phi\left(\frac{1}{r^2} + \frac{2}{d-1}F'(r) + \frac{1}{r}F(r)\right)
\end{aligned} \tag{3.108}$$

which can be rewritten in the following Weyl covariant form:

$$E^{\Phi(2)}_{source} = \frac{1}{r^2}\mathcal{D}^2\phi + u^\mu u^\nu\mathcal{D}_\mu\mathcal{D}_\nu\phi\left(\frac{1}{r^2} + 2F'(r) + \frac{1}{r}F(r)(d-1)\right). \tag{3.109}$$

It is straightforward to integrate this to obtain:

$$\Phi^{(2)} = b^2\int_{br}^{\infty}\frac{1}{\xi(\xi^d-1)}\int_1^\xi y^{d-3}\left(\mathcal{D}^2\phi + \left(1 + y^2F'(y) + yF(y)(d-1)\right)u^\mu u^\nu\mathcal{D}_\mu\mathcal{D}_\nu\phi\right). \tag{3.110}$$

Thus the dilaton to second order in Weyl covariant form is given by:

$$\Phi = \phi + bu^\mu\mathcal{D}_\mu\phi F(br) + \frac{b^2}{d-2}H_1(br)\mathcal{D}^2\phi + b^2H_2(br)u^\mu u^\nu\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi. \tag{3.111}$$

3.3.5 Stress tensor and Lagrangrian of the dual fluid

We now calculate the boundary stress tensor and Lagrangian using the following formulae [26]:

$$
\begin{aligned}
16\pi G_{d+1} T^{\mu}_{\nu} &= \lim_{r\to\infty} r^d \Big(2(K_{\alpha\beta} h^{\alpha\beta} \delta^{\mu}_{\nu} - K^{\mu}_{\nu}) \\
&\qquad + \bar{\mathfrak{G}}^{\mu}_{\nu} - \frac{d(d-1)}{2}\delta^{\mu}_{\nu} - \frac{1}{d-2}\left(\bar{\nabla}^{\mu}\Phi \bar{\nabla}_{\nu}\Phi - \frac{\delta^{\mu}_{\nu}}{2}(\bar{\nabla}\Phi)^2 \right) \Big) , \\
16\pi G_{d+1} e^{-\phi}\, \mathcal{L} &= -\lim_{r\to\infty} r^d \left(\partial_n \Phi + \frac{1}{d-2}\bar{\nabla}^2\Phi \right) .
\end{aligned}
\tag{3.112}
$$

Here, $h_{\mu\nu}$ is the induced metric on the constant r hypersurface; from this, we obtain the covariant derivative $\bar{\nabla}$ and the corresponding Einstein tensor $\bar{\mathfrak{G}}^{\mu}_{\nu}$. We define n^A to be the outward pointing unit normal of the constant r hypersurface; the extrinsic curvature of the constant r hypersurface is then defined by the Lie derivative of the induced metric, $K_{\mu\nu} \equiv \frac{1}{2}\mathfrak{L}_n h_{\mu\nu}$, and ∂_n is the partial derivative along n^A. In the formulae above, all the indices are raised using the induced metric.

We find that the boundary stress tensor is given by:

$$
\begin{aligned}
16\pi G_{d+1} T_{\mu\nu} &= b^{-d}\left(g_{\mu\nu} + d u_\mu u_\nu\right) - 2b^{1-d}\sigma_{\mu\nu} \\
&- 2b^{2-d}\tau_\omega \left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \omega_\mu{}^\lambda \sigma_{\lambda\nu} + \omega_\nu{}^\lambda \sigma_{\mu\lambda} \right] \\
&+ 2b^{2-d}\left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu} + C_{\mu\alpha\nu\beta} u^\alpha u^\beta \right] \\
&- \frac{1}{d-2} b^{2-d} \left[P^\alpha_\mu P^\beta_\nu \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi - \frac{1}{d-1} P_{\mu\nu} P^{\alpha\beta} \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi \right]
\end{aligned}
\tag{3.113}
$$

with

$$
b = \frac{d}{4\pi T} \qquad \text{and} \qquad \tau_\omega = \int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)} dy. \tag{3.114}
$$

We further obtain the following expression for the Lagrangian:

$$
16\pi G_{d+1} e^{-\phi}\mathcal{L} = -b^{1-d} u^\mu \mathcal{D}_\mu \phi - \frac{1}{d-2} b^{2-d}\mathcal{D}^2\phi - b^{2-d}\tau_\omega u^\mu u^\nu \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi. \tag{3.115}
$$

3.4 Conclusions

In this chapter, we have constructed asymptotically locally AdS_{d+1} bulk spacetimes with a slowly varying dilaton field which are dual, under the AdS/CFT correspondence, to forced fluid flows on the weakly curved boundary metric. These forced fluid flows satisfy the conformal relativistic Navier-Stokes equations with a dilaton-dependent forcing term. We have also obtained the form of the dual stress tensor and Lagrangian, all to second order in the boundary derivative expansion. Our results further generalise previous work on the fluid/gravity correspondence [16, 36, 26, 27]; in particular, we have generalised the results of [26] to arbitrary spacetime dimensions.

There are several interesting applications of our work which merit further consideration. It would be useful to study in detail holographic models of novel forced fluid flows. One avenue which is worth exploring would be to consider holographic duals of forced fluid flows which exhibit turbulence. A holographic model of turbulence would certainly offer a new perspective on this poorly understood phenomenon, and hopefully new and fruitful insights could then be derived from this. By carefully choosing the form of the forcing term (which is fixed by our choice of $\phi(x)$), it could be possible to stir the boundary fluid into turbulent configurations[3]. A noteworthy point that should be raised here is that unforced fluid flows can exhibit turbulence as well; however, these turbulent phases will be transient. And, in fact, a holographic model of transient turbulence has already been constructed [44]. The key advantage of considering holographic models of forced fluid flows, on the other hand, is the possibility of realising holographic models of *steady state* turbulence; such configurations can only exist with a forcing term as the fluid would otherwise eventually settle down into a non-dissipative configuration.

Also, observe that the expressions that we have obtained are valid for arbitrary spacetime dimensions; this is particularly relevant for the study of turbulence. It is a well-known fact that turbulent phases for relativistic fluids in $2+1$ dimensions display remarkably different behaviour to relativistic fluids in higher dimensions. In $2+1$ dimensions, relativistic fluids display an 'inverse energy cascade'; energy cascades from short to long wavelengths [41]. This is in sharp contrast to the standard cascade observed in higher dimensions which is from long to short wavelengths. The results of this chapter could in principle be used to construct holographic models of turbulence in different dimensions which would then shed light on the source of the discrepancy between the nature of the energy cascades in two spatial dimensions and greater. Such holographic models would also be of interest purely from a gravitational perspective. The construction of such models would suggest that AdS_4 displays qualitatively different

[3]Another possibility of realising turbulence which is potentially more straightforward would be to consider a boundary spacetime which consists of small time-dependent fluctuations away from flat space. These linearised time-dependent fluctuations can effectively act as a forcing term for a fluid on a flat background metric; this is discussed in more detail in the introduction of [36].

instabilities to AdS_{d+1} for $d > 3$. Further, these models may have interesting connections to the weakly turbulent instability of AdS discovered in [46].

Chapter 4

Holographic derivation of the entropy current for an anomalous charged fluid with background electromagnetic fields

4.1 Introduction

In this chapter we consider another extension of the fluid/gravity correspondence; in particular, we study the fluid/gravity model for multiple anomalous currents in the presence of background electromagnetic fields. As a concrete example of this, we will further specify to the fluid/gravity model of the chiral magnetic effect (CME) [38]. The chiral magnetic effect [47] is a phenomenon where a background magnetic field induces a current in the direction of the field in the presence of imbalanced chirality; such imbalances in chirality can arise in topologically nontrivial gluonic configurations in QCD due to the axial anomaly, for example. Early work on the CME derived this current for equilibrium systems [47]. More recently, a hydrodynamic model of the CME was constructed [48, 49]; these results are based on the arguments presented in [37] where the authors considered (holographic) hydrodynamics with triangle anomalies. The results of [37] represent one of the notable triumphs of AdS/CFT; and, as their work is relevant to our discussion, we now pause to elaborate on their paper and to provide some background.

For generic hydrodynamic models, the currents can be expressed as a sum of terms allowed by symmetries, each with an attached transport coefficient. However, in canonical textbook examples [15], some of these terms were disallowed on the grounds that they were inconsistent with the second law of thermodynamics. However, most surprisingly, in recent holographic calculations involving the bulk dual of a charged fluid [30, 31], one of these coefficients was found to be nonzero, implying an incompatibility with the second law. In [37], the authors resolved this issue. By considering fluid dynamics with triangle anomalies, they showed that that these problematic transport coefficients can be included if we allow the entropy current to be modified by the inclusion of additional terms; further, such additional terms are in fact *required* by the presence of triangle anomalies. Their findings have challenged standard lore and have led to a modification of the canonical equations of fluid dynamics.

In the simplest case of a charged fluid with one $U(1)$ current (and a $U(1)^3$ anomaly), the additional terms to the charge current represent contributions to the current in the direction of the vorticity and external magnetic field (if one is present). And it is the term proportional to the magnetic field that has been connected to the chiral magnetic effect in heavy ion collisions. Given that heavy ion collisions probe physics at strong coupling, it is useful to establish holographic models describing the CME; here, calculations at strong coupling can be done using fairly straightforward gravitational methods. The fluid/gravity model of the CME mentioned above [38] is an example of a successful holographic model (see [50–53] for earlier attempts). In [38], the authors holographically modelled a hydrodynamic system involving two currents, denoted by $U(1)_A$ and $U(1)_V$ respectively, in the presence of a background magnetic field. These currents represent the axial and vector currents in QCD, $j_5^\nu = \bar{q}\gamma^\nu\gamma^5 q$ and $j^\nu = \bar{q}\gamma^\nu q$, and we label their associated chemical potentials by μ and μ_5. The chemical potential μ_5 for the $U(1)_A$ axial current, which the authors modelled as anomalous, can be thought of as representing the imbalance in chirality due to the axial anomaly. The presence of the anomalous axial current requires the addition of the extra transport coefficient signifying the existence of a current in the direction of the magnetic field. Their fluid/gravity model correctly reproduced the results for the transport coefficients given in [48, 49] and thus provides holographic confirmation of their hydrodynamic extension of the CME.

In this chapter, we extend the results of [38] by holographically computing the entropy current for the hydrodynamic model of the CME. We first demonstrate that the bulk solutions presented in [38] possess a regular event horizon and we determine its location; then, utilising the area form on the horizon, we construct a gravitational entropy current. Mapping this to the boundary along ingoing null geodesics allows us to determine an entropy current for the corresponding hydrodynamic model of the CME defined on the boundary. The area increase theorem of general relativity guarantees the positive divergence of this entropy current.

More concretely, we first obtain long wavelength bulk solutions dual to a fluid with *multiple anomalous currents* in the presence of *external electromagnetic fields* - this generalises previous work which considered more specific configurations [30, 31, 38]. The dynamics of such bulk spacetimes are governed by the $U(1)^n$ Einstein-Maxwell action with a Chern-Simons term:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - 12 - F^a_{MN}F^{aMN} + \frac{S^{abc}}{6\sqrt{-g}}\epsilon^{PKLMN}A^a_P F^b_{KL}F^c_{MN}\right] \quad (4.1)$$

We calculate the location of the regular event horizon for such fluid bulk duals and determine the expression for the entropy current. These results are in themselves already novel generalisations of existing work - the location of the event horizon and the form of the entropy current thus far have only been calculated for bulk duals of fluids with a single charge without any background electromagnetic fields [54]. Our expression for the entropy current contains additional terms proportional to the vorticity and magnetic field, and is completely consistent with the results of [37]. We then specify to the fluid/gravity model of the CME and holographically calculate its entropy current.

This chapter is organised as follows. In sections 4.2, we compute the bulk dual for a conformal fluid with multiple anomalous currents in the presence of background electromagnetic fields; we also determine the location of the event horizon and calculate the (holographic) entropy current. In section 4.3 we specify to the fluid/gravity model of the CME and holographically calculate its entropy current as well. We end with a discussion of our results in section 4.4.

4.2 Holographic fluid with n anomalous currents with background fields

Here we present the explicit results of our calculations for the fluid/gravity model with n anomalous currents with background electromagnetic fields. In the following subsections, we give the expressions for the bulk metric, (outer) event horizon, and entropy current for this fluid/gravity model. These are already novel results which further generalise existing literature. In the next section, we then specify to the fluid/gravity model of the chiral magnetic effect - these results are of particular interest from a condensed matter perspective.

4.2.1 Bulk metric

We aim to construct bulk gravitational solutions dual to a fluid with n anomalous currents in the presence of background electromagnetic fields. This can be done by obtaining long wavelength solutions to the bulk dynamics described by the $U(1)^n$ Einstein-Maxwell-Chern-Simons action[1]

$$S=\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R-12-F^a_{MN}F^{aMN}+\frac{S^{abc}}{6\sqrt{-g}}\epsilon^{PKLMN}A^a_P F^b_{KL}F^c_{MN}\right]. \quad (4.3)$$

The triangles anomalies for the currents are encoded in the Chern-Simons parameter via the relation:

$$C^{abc}=-\frac{S^{abc}}{4\pi G_5} \quad (4.4)$$

as found in [37].

The equations of motion which result from this action are given by:

$$\begin{aligned} G_{MN}-6g_{MN}+2\left(F^a_{MR}F^{aR}{}_N-\frac{1}{4}g_{MN}F^a_{SR}F^{aSR}\right)&=0,\\ \nabla_M F^{aMP}&=-\frac{S^{abc}}{8\sqrt{-g}}\varepsilon^{PMNKL}F^b_{MN}F^c_{KL}. \end{aligned} \quad (4.5)$$

The bulk fluid duals can then be computed using the method pioneered in [16] (see also [24, 25, 36]). This methodology was explained in detail in the last two chapters so we just make a few comments here on how this naturally extends to the charged case. Note that the equations (4.5) admit the following solution describing a uniform charged black brane corresponding to a field theory state at constant temperature and

[1] Repeated lower case Latin indices imply summation, for example:

$$q^a q^a \equiv \Sigma^n_{a=1}q^a q^a. \quad (4.2)$$

.

chemical potentials:

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu dr - r^2 f(r) u_\mu u_\nu dx^\mu dx^\mu + r^2 P_{\mu\nu} dx^\mu dx^\nu , \\
A^a &= \frac{\sqrt{3} q^a}{2r^2} u_\mu dx^\mu ,
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
f(r) &= 1 - \frac{m}{r^4} + \frac{q^a q^a}{r^6} , \\
P_{\mu\nu} &= \eta_{\mu\nu} + u_\mu u_\nu .
\end{aligned}
\tag{4.7}
$$

The function $f(r)$ has two positive, real solutions which describe the outer and inner horizon of the black brane. The position of the (outer) event horizon is at $r = r_+$ with

$$
r_+ = \frac{\pi T}{2} \left(1 + \sqrt{1 + \frac{8}{3\pi^2} \frac{\mu^a \mu^a}{T^2}} \right) . \tag{4.8}
$$

There is also an inner horizon at $r = r_-$:

$$
r_-^2 = \frac{1}{2} r_+^2 \left(-1 + \sqrt{9 - \frac{8}{\frac{1}{2}\left(1 + \sqrt{1 + \frac{8}{3\pi^2}\frac{\mu^a\mu^a}{T^2}}\right)}} \right) . \tag{4.9}
$$

Here, the chemical potentials μ^a and temperature T can be expressed in terms of bulk quantities as:

$$
\begin{aligned}
T &= \frac{r_+}{2\pi} \left(2 - \frac{q^a q^a}{r_+^6} \right) , \\
\mu^a &= \frac{\sqrt{3} q^a}{2 r_+^2} .
\end{aligned}
\tag{4.10}
$$

To obtain a bulk hydrodynamic dual representing a locally equilibrated boundary configuration, we thus proceed by promoting the constant mass m, charges q^a, and velocity field u^μ to slowly-varying fluid dynamical parameters; we further introduce background gauge fields $\mathfrak{A}^a_\mu$ to model external electromagnetic fields $B^{a\mu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} u_\nu F^a_{\mathfrak{A}\alpha\beta}$ and $E^{a\mu} = F^{a\mu\nu}_{\mathfrak{A}} u_\nu$ (where $F^{a\mu\nu}_{\mathfrak{A}} = \partial^\mu \mathfrak{A}^{a\nu} - \partial^\nu \mathfrak{A}^{a\mu}$). We therefore take the following expressions as our zeroth order ansatz:

$$
\begin{aligned}
ds^2 &= -2u_\mu(x) dx^\mu dr - r^2 f(r, m(x), q^a(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\mu + r^2 P_{\mu\nu}(x) dx^\mu dx^\nu , \\
A^a &= \frac{\sqrt{3} q^a(x)}{2r^2} u_\mu(x) dx^\mu + \mathfrak{A}^a_\mu(x) dx^\mu ,
\end{aligned}
\tag{4.11}
$$

and perturbatively solve the equations (4.5) to a specified order in boundary derivatives.

To first order, we obtain the following expression for the bulk metric:

$$
\begin{aligned}
ds^2 = & -2u_\mu dx^\mu \left(dr + r\mathcal{A}_\nu dx^\nu\right) - r^2 f(r) u_\mu u_\nu dx^\mu dx^\mu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \\
& - \frac{2\sqrt{3} S^{abc} q^a q^b q^c}{8mr^4} u_{(\mu}\omega_{\nu)} dx^\mu dx^\nu - \frac{12r^2}{r_+^7} u_{(\mu} P_{\nu)}^\lambda q^a \mathcal{D}_\lambda q^a F_q(r) dx^\mu dx^\nu \\
& - 2F_B(r) S^{abc} q^a q^b u_{(\mu} B_{\nu)}^c dx^\mu dx^\nu \\
& - 2F_E(r) q^a u_{(\mu} E_{\nu)}^a dx^\mu dx^\nu + \frac{2r^2}{r_+} F_\sigma(r) \sigma_{\mu\nu} dx^\mu dx^\nu
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
F_q(r) &\equiv \frac{1}{3} f(r) \int_r^\infty dr' \frac{r_+^8}{f(r')^2} \left(\frac{r_+}{r'^8} - \frac{3}{4r'^7} \left(1 + \frac{r_+^4}{m}\right)\right), \\
F_\sigma(r) &\equiv \int_{\frac{r}{r_+}}^\infty dr' \frac{r'\left(r'^2 + r' + 1\right)}{(r'+1)\left(r'^4 + r'^2 - \frac{m}{r_+^4} + 1\right)}, \\
F_B(r) &= 3r^2 f(r) \int_\infty^r dr' \frac{1}{r'^5 f(r')^2} \left(\frac{1}{2r'^4} - \frac{r_+^6 + \frac{q^a q^a}{4}}{r'^2 r_+^4 m} \right), \\
F_E(r) &= -r^2 f(r) \int_r^\infty dr' \left(\frac{3\sqrt{3} q^a q^a}{r_+ m r'^3} - \frac{4\sqrt{3}(r' - r_+)}{r'^3} \right) \int_{r'}^\infty dr'' \frac{1}{r''^5 f(r'')^2} \\
&\quad + r^2 f(r) \left(\int_r^\infty dr' \frac{1}{r'^5 f(r')^2} \right) \left(\int_r^\infty dr' \left(\frac{3\sqrt{3} q^a q^a}{r_+ m r'^3} - \frac{4\sqrt{3}(r' - r_+)}{r'^3} \right)\right)
\end{aligned}
\tag{4.13}
$$

where $\omega^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \omega_{\alpha\beta}$. We now pause to elaborate on the significance of these results within the context of existing literature. The results above describe the bulk dual of a fluid with *multiple anomalous currents* in the presence of *background electromagnetic fields*. While similar situations have been studied in the literature [38, 30, 31], their calculations were for more specific configurations. Our results, which are also explicitly Weyl covariant, extend existing work on the fluid/gravity correspondence to greater generality.

4.2.2 Event horizon

With this bulk metric in hand, we can determine the location of the outer horizon following the procedure laid out in [55]. This procedure uses a clever argument which we outline now before working through the calculation in detail.

It is reasonable to assume that the location of the outer horizon should be given by an expansion in boundary derivatives; to first order this would be:

$$r_H = r_0 + r_1 + \cdots \, . \tag{4.14}$$

It is further also reasonable to assume that the bulk metric will settle down to the dual of uniform flow given enough time; this would be a uniform black brane. Note that at this asymptotic time, all boundary derivatives should thus vanish and our expression for r_H should reduce to just r_0 which must then be equal to r_+, the outer horizon for a uniform charged black brane (since we are working with charged fluids). To determine r_H to all orders in boundary derivatives we use the fact that the outer horizon is a null hypersurface. If the outer horizon is specified by the following equation:

$$S_{\mathcal{H}}(r, x) = r - r_H(x) = 0 \tag{4.15}$$

then the condition that it is null is simply:

$$g^{AB}\partial_A S_{\mathcal{H}} \partial_B S_{\mathcal{H}} = 0. \tag{4.16}$$

Solving this to first order should give us the first order correction to the outer horizon location r_1. We will now walk through this calculation.

To evaluate our null condition above we need the inverse metric g^{AB}. If we rewrite our bulk metric in the following form:

$$ds^2 = -2u_\mu dr dx^\mu + \chi_{\mu\nu} dx^\mu dx^\nu \tag{4.17}$$

with $\chi^{\mu\nu}$ defined by $\chi_{\mu\nu}\chi^{\nu\rho} = \delta_\mu{}^\rho$, the inverse metric is given by the expressions below:

$$\begin{aligned} g^{rr} &= -\frac{1}{u_\mu u_\nu \chi^{\mu\nu}}, \\ g^{r\alpha} &= \frac{\chi^{\alpha\beta} u_\beta}{-u_\mu u_\nu \chi^{\mu\nu}}, \\ g^{\alpha\beta} &= \frac{u_\gamma u_\delta \left(\chi^{\alpha\beta}\chi^{\gamma\delta} - \chi^{\alpha\gamma}\chi^{\beta\delta}\right)}{-u_\mu u_\nu \chi^{\mu\nu}}. \end{aligned} \tag{4.18}$$

We would now like to determine the zeroth and first order components of $\chi^{\mu\nu} = \chi^{(0)\mu\nu} + \chi^{(1)\mu\nu} + \cdots$. From $\chi_{\mu\nu}\chi^{\nu\rho} = \delta_\mu{}^\rho$, we obtain the following relations:

$$\begin{aligned} \chi^{(0)}_{\mu\nu}\chi^{(0)\nu\rho} &= \delta_\mu{}^\rho \\ \chi^{(0)\mu\rho}\chi^{(1)}_{\rho\nu} + \chi^{(1)\mu\rho}\chi^{(0)}_{\rho\nu} &= 0. \end{aligned} \tag{4.19}$$

This gives us $\chi^{(0)\mu\nu}$ as below:

$$\chi^{(0)\mu\nu} = -\frac{1}{r^2 f(r)} u^\mu u^\nu + \frac{1}{r^2} P^{\mu\nu}. \tag{4.20}$$

To compute $\chi^{(1)\mu\nu}$ let us first express $\chi^{(1)}_{\mu\nu}$ as:

$$\begin{aligned} \chi^{(1)}_{\mu\nu} &= -2r u_{(\mu}\mathcal{A}_{\nu)} - 2\tilde{F}_\omega(r) u_{(\mu}\omega_{\nu)} - 2u_{(\mu}P^\lambda_{\nu)}\mathcal{D}_\lambda q^a \tilde{F}_q(r) \\ &\quad - 2\tilde{F}^a_B(r) u_{(\mu}B^a_{\nu)} - 2\tilde{F}^a_E(r) u_{(\mu}E^a_{\nu)} + 2\tilde{F}_\sigma(r)\sigma_{\mu\nu} \end{aligned} \tag{4.21}$$

where the functions $\tilde{F}$ are defined implicitly. These have been introduced for notational convenience. The terms $\chi^{(0)\mu\rho}\chi^{(1)}_{\rho\nu}$ and $\chi^{(1)\mu\rho}\chi^{(0)}_{\rho\nu}$ now evaluate to:

$$\begin{aligned} \chi^{(0)\mu\rho}\chi^{(1)}_{\rho\nu} &= -\frac{1}{rf(r)} u^\mu \mathcal{A}_\nu + \frac{1}{rf(r)} u^\rho \mathcal{A}_\rho u^\mu u_\nu - \frac{1}{r} P^{\mu\rho}\mathcal{A}_\rho u_\nu \\ &\quad - \frac{1}{r^2 f(r)} \tilde{F}_\omega(r) u^\mu \omega_\nu - \frac{1}{r^2 f(r)} u^\mu P^\lambda_\nu \mathcal{D}_\lambda q^a \tilde{F}_q(r) - \frac{1}{r^2 f(r)} u^\mu B^a_\nu \tilde{F}^a_B(r) \\ &\quad - \frac{1}{r^2 f(r)} u^\mu E^a_\nu \tilde{F}^a_E(r) \\ &\quad - \frac{1}{r^2}\tilde{F}_\omega(r) u^\mu \omega_\nu - \frac{1}{r^2} u^\mu P^\lambda_\nu \mathcal{D}_\lambda q^a \tilde{F}_q(r) - \frac{1}{r^2} u^\mu B^a_\nu \tilde{F}^a_B(r) - \frac{1}{r^2} u^\mu E^a_\nu \tilde{F}^a_E(r) \\ &\quad + \frac{2}{r^2}\tilde{F}_\sigma(r)\sigma^\mu{}_\nu \\ \chi^{(1)\mu\rho}\chi^{(0)}_{\rho\nu} &= -r^2 f(r)\chi^{(1)\mu\rho} u_\rho u_\nu + r^2 P_{\rho\nu}\chi^{(1)\rho\mu}. \end{aligned} \tag{4.22}$$

We can thus calculate $\chi^{(1)\mu\nu}$ to obtain:

$$\begin{aligned}\chi^{(1)\mu\nu} &= \frac{2}{r^3 f(r)} u^\rho \mathcal{A}_\rho u^\mu u^\nu + \frac{2}{r^3} u^{(\mu} P^{\nu)\rho} \mathcal{A}_\rho + \frac{2}{r^4 f(r)} \omega^{(\mu} u^{\nu)} \tilde{F}_\omega(r) \\ &+ \frac{2}{r^4 f(r)} u^{(\mu} P^{\nu)\lambda} \mathcal{D}_\lambda q^a \tilde{F}_q(r) + \frac{2}{r^4 f(r)} E^{a(\mu} u^{\nu)} \tilde{F}^a_E(r) \\ &+ \frac{2}{r^4 f(r)} B^{a(\mu} u^{\nu)} \tilde{F}^a_B(r) - \frac{2}{r^4} \tilde{F}_\sigma(r) \sigma^{\mu\nu}.\end{aligned} \tag{4.23}$$

Our null condition in terms of our inverse metric is:

$$-\frac{1}{u_\mu u_\nu \chi^{\mu\nu}} \left(1 - 2\chi^{\alpha\beta} u_\beta \partial_\alpha r_H - \left(\chi^{\alpha\beta}\chi^{\gamma\delta} - \chi^{\alpha\gamma}\chi^{\beta\delta}\right) u_\gamma u_\delta \partial_\alpha r_H \partial_\beta r_H\right) = 0. \tag{4.24}$$

Evaluating this to first order using the explicit form of the inverse metric gives:

$$\begin{aligned} r_H^2 f(r_H) \left(1 - \frac{2}{r_H^2 f(r_H)} u^\alpha \partial_\alpha r_H\right) &= 0, \\ \implies r_+^2 f'(r_+) r_1 - 2u^\alpha \partial_\alpha r_+ &= 0 \end{aligned} \tag{4.25}$$

which actually just evaluates to $r_1 = 0$. Thus the first order correction to the location of the outer horizon vanishes. In hindsight, this is unsurprising and follows from Weyl covariance [55]. Bulk fluid dynamical spacetimes are invariant under boundary Weyl transformations of the hydrodynamic variables - this is a reflection of the conformal nature of the boundary fluid dynamics. This boundary Weyl symmetry constrains the form of the terms allowed in the bulk metric. Consequently, the location of the horizon must be given as a sum of Weyl covariant scalars; and since there are no Weyl covariant scalars at first order in boundary derivatives, the outer horizon receives no first order correction.

4.2.3 Entropy current

In this subsection, applying the method discussed in [55], we compute the entropy current to first order in boundary derivatives. The formula they derived for the entropy current is given below:

$$J_S^\mu = \frac{\sqrt{h}}{4G_5} \frac{n^\mu}{n^v} \tag{4.26}$$

where h is the determinant of the 3-metric on the spatial slice of the outer horizon, n^μ is the normal to the outer horizon, and n^v is its v-component which we can take to be

$u_\mu n^\mu$. This formula was obtained by considering the area form on the event horizon then mapping it to the boundary. We do not reproduce their proof here but we will show all the calculations required to evaluate the expression above.

First, we need to determine h. Taking the metric and restricting to a constant v-slice of the event horizon $r = r_H = r_+$ at first order, we obtain:

$$ds^2 = r_+^2 \delta_{ij} dx^i dx^j + 2r_+ F_\sigma(r_+)\sigma_{ij} dx^i dx^j. \tag{4.27}$$

We can evaluate the determinant of this metric perturbatively in boundary derivatives using the expansion:

$$det(I + \epsilon A) = 1 + \epsilon Tr(A) + O(\epsilon^2). \tag{4.28}$$

Fortunately, the trace of σ_{ij} is zero by construction which gives $h = r_+^6$ or $\sqrt{h} = r_+^3$. And for the normal to the horizon $n^A = g^{AB}\partial_B S_{\mathcal{H}}$ ($S_{\mathcal{H}} = r - r_H = 0$ defines the horizon), we are only interested in the boundary directions n^μ:

$$\begin{aligned}
n^\mu &= g^{\mu r}\partial_r S_{\mathcal{H}} + g^{\mu\nu}\partial_\nu S_{\mathcal{H}} \\
&= \frac{r_+^2 f(r_+)}{1 - \frac{2u^\rho \mathcal{A}_\rho}{r_+}} \left(\frac{1}{r_+^2 f(r_+)} u^\mu - \frac{2}{r_+^3 f(r_+)} u^\rho \mathcal{A}_\rho u^\mu - \frac{1}{r_+^3} P^{\mu\rho}\mathcal{A}_\rho - \frac{1}{r_+^4 f(r_+)} \omega^\mu \tilde{F}_\omega(r_+) \right. \\
&\left. - \frac{1}{r_+^4 f(r_+)} P^{\mu\lambda}\mathcal{D}_\lambda q^a \tilde{F}_q(r_+) - \frac{1}{r_+^4 f(r_+)} E^{a\mu}\tilde{F}_E^a(r_+) - \frac{1}{r_+^4 f(r_+)} B^{a\mu}\tilde{F}_B^a(r_+) \right) \\
&+ \frac{r_+^2 f(r_+)}{1 - \frac{2u^\rho \mathcal{A}_\rho}{r_+}} \left(\frac{1}{r_+^2} P^{\mu\nu} \cdot - \frac{1}{r_+^2 f(r_+)} \cdot -\partial_\nu r_+ \right) + O(\epsilon^2).
\end{aligned} \tag{4.29}$$

To first order, this simplifies to just:

$$\begin{aligned}
n^\mu = u^\mu &- \frac{f(r_+)}{r_+} P^{\mu\rho}\mathcal{A}_\rho - \frac{1}{r_+^2}\omega^\mu \tilde{F}_\omega(r_+) - \frac{1}{r_+^2} P^{\mu\lambda}\mathcal{D}_\lambda q^a \tilde{F}_q(r_+) \\
&- \frac{1}{r_+^2} E^{a\mu}\tilde{F}_E^a(r_+) - \frac{1}{r_+^2} B^{a\mu}\tilde{F}_B^a(r_+).
\end{aligned} \tag{4.30}$$

Putting this altogether gives us the following expression for the entropy current:

$$\begin{aligned}
J_S^\mu = \frac{1}{4G_5} \cdot r_+^3 &\left(u^\mu - \frac{f(r_+)}{r_+} P^{\mu\rho}\mathcal{A}_\rho - \frac{1}{r_+^2}\omega^\mu \tilde{F}_\omega(r_+) - \frac{1}{r_+^2} P^{\mu\lambda}\mathcal{D}_\lambda q^a \tilde{F}_q(r_+) \right. \\
&\left. - \frac{1}{r_+^2} E^{a\mu}\tilde{F}_E^a(r_+) - \frac{1}{r_+^2} B^{a\mu}\tilde{F}_B^a(r_+) \right).
\end{aligned} \tag{4.31}$$

If we now evaluate all the integral expressions explicitly (using Mathematica), we obtain:

$$
\begin{aligned}
J_S^\mu = \frac{1}{4G_5}\Bigg(& r_+^3 u^\mu + \frac{3q^a(q^bq^b+2r_+^6)}{4m(2r_+^6-q^bq^b)}P^{\mu\nu}\mathcal{D}_\nu q^a - S^{abc}q^aq^bq^c\frac{\sqrt{3}}{8mr_+^3}\omega^\mu \\
& - S^{abc}q^aq^b\frac{3}{8mr_+}B^{c\mu} + \frac{\sqrt{3}\pi(2r_+^6-q^bq^b)}{2m^2}q^aE^{a\mu}\Bigg).
\end{aligned}
\tag{4.32}
$$

To make contact with existing literature, we rewrite our expression for the entropy current in the following form:

$$
J_S^\mu = su^\mu - \frac{\mu^a}{T}\nu^{a\mu} + D\omega^\mu + D_B^a B^{a\mu}, \tag{4.33}
$$

where

$$
\nu^{a\mu} = -\sigma^{ab}TP^{\mu\nu}\partial_\nu\left(\frac{\mu^b}{T}\right) + \sigma^{ab}E^{b\mu} + \xi^a\omega^\mu + \xi_B^{ab}B^{b\mu}. \tag{4.34}
$$

The entropy density s, electrical conductivity σ^{ab}, and anomalous transport coefficients ξ^a and ξ_B^{ab} are given by:

$$
\begin{aligned}
s &= \frac{r_+^3}{4G_5} \\
\sigma^{ab} &= \frac{\pi T^2 r_+^7}{4G_5m^2}\delta^{ab} \\
\xi^a &= -\frac{3S^{abc}q^bq^c}{16\pi G_5} \\
\xi_B^{ab} &= -\frac{\sqrt{3}\left(3r_+^4+m\right)S^{abc}q^c}{32\pi G_5 m r_+^2}.
\end{aligned}
\tag{4.35}
$$

The expressions for s and σ_{ab} can be obtained by comparison with (4.32)[2]; the anomalous transport coefficients ξ^a and ξ_B^{ab} were holographically computed in [37]. We assume that these parameters are given and use our holographic computation of the entropy current (4.32) to determine the coefficients D and D_B^a. We find that:

$$
\begin{aligned}
D &= \frac{1}{3T}C^{abc}\mu^a\mu^b\mu^c, \\
D_B^a &= \frac{1}{2T}C^{abc}\mu^b\mu^c.
\end{aligned}
\tag{4.36}
$$

[2]We make use of the relations (4.10).

This is in perfect agreement with the results of [37]; here, the authors computed the coefficients D_B^a and D using thermodynamic arguments. Our results thus provide holographic confirmation of their calculation.

4.3 Holographic entropy current for the chiral magnetic effect

In this subsection, we specify to the fluid/gravity model of the chiral magnetic effect. As explained in the introduction, we require just two currents, an axial current and a vector current, with associated gauge fields A^A and A^V, and chemical potentials μ_5 and μ. An external magnetic field is needed as well. To model this, it is sufficient to have just an external vector gauge field $\mathfrak{A}^{V\mu}$ with a nonzero magnetic component B^μ; we set the corresponding electric field E^μ to zero. The anomaly coefficient C^{abc} is determined by $C-$parity [56]:

$$\begin{aligned}
&C^{121} = C^{211} = C^{112} = 0\,,\\
&C^{222} = 0\,,\\
&C^{111} \neq 0\,,\\
&C^{122} = C^{221} = C^{212} \neq 0\,.
\end{aligned} \tag{4.37}$$

We choose $C^{111} = C^{122} \equiv C/3$; this then fixes the Chern-Simons parameter S^{abc} via relation (4.4).

Observe that, with the restrictions imposed thus far, the two currents are both anomalous. This is not consistent with the hydrodynamic model of the CME which requires just the axial current to be anomalous, accounting for the imbalance in chirality caused by the axial anomaly. The vector current should remain conserved. Thish issue can easily be addressed by the addition of Bardeen currents [56] which can be incorporated by including the Bardeen counterterm in the action; the Bardeen currents will restore conservation of the vector current. However, here, we are primarily interested in the form of the coefficients which appear in the expression for the entropy current. These coefficients depend only on the temperature T and chemical potentials μ and μ_5, and are unchanged by the inclusion of Bardeen currents. We therefore neglect this subtlety in our calculation.

The bulk metric is similar to (4.12) with just the inclusion of the modifications described above, so we do not write it here explicitly. And, similarly, there is no first order correction to the location of the outer horizon. For the entropy current, we find the following expressions for the coefficients D and D_B^a:

$$\begin{aligned} D &= \frac{C}{T}\mu^2\mu_5, \\ D_B &= \frac{C}{T}\mu\mu_5. \end{aligned} \tag{4.38}$$

And this is completely consistent with the results of the purely hydrodynamic analysis of [48].

4.4 Discussion

In this chapter, we have constructed long wavelength asymptotically locally AdS_5 bulk spacetimes with slowly varying gauge fields which are solutions of the $U(1)^n$ Einstein-Maxwell-Chern-Simons system. These bulk spacetimes are dual to $(3+1)-$dimensional fluid flows with n anomalous currents in the presence of background electromagnetic fields. Using the methodology introduced in [55], we then computed the entropy current for the boundary fluid flow to first order in boundary spacetime derivatives. These results further generalise existing work on the fluid/gravity correspondence. In particular, holographic computations of the entropy current have previously only been done for a fluid with a single $U(1)$ current (possessing a $U(1)^3$ anomaly) without any background electromagnetic fields present. Finally, we restricted our results to the fluid/gravity model of the chiral magnetic effect, and calculated the corresponding entropy current. Our calculations are all consistent with those done using conventional thermodynamic/hydrodynamic methods.

A very natural generalisation of our work would be to extend our bulk metric (4.12) to second order in boundary derivatives. Then, using standard holographic techniques, the corresponding transport coefficients can easily be obtained. It would certainly be of interest to understand the physical significance of these second order coefficients.

The logical next step would be to extend the expression for the entropy current to second order as well; however, this is a much more intricate problem. From a purely hydrodynamic perspective, the requirement that the divergence of the entropy current be non-negative fixes the first order contribution to the current. This is not true in general at second order and an ambiguity remains. This has been shown explicitly in [39]. Here, the author found that the entropy current (with non-negative divergence) at second order for an uncharged conformal fluid has a four-parameter ambiguity. We should also mention here that these results can also be reproduced by demanding consistency with a partition function [40] instead of using the constraints arising from non-negativity of the divergence of the entropy current. And in fact, this alternative method may be more computationally tractable. It would be very useful to extend the analyses of [39, 40] to a charged fluid with anomalous currents, as studied in this chapter.

If we now consider this issue holographically, the ambiguity in the entropy current at second order should be reflected in the bulk construction as well. Two sources of ambiguity have already been determined in previous literature. First, there exists freedom in how we choose to map the gravitational entropy current from the horizon to the boundary [55]. And second, it is possible to construct entropy currents (with non-negative divergence) on horizons other than the event horizon - this leads to a further source of ambiguity. In [54, 57], the authors constructed an entropy current on a Weyl-invariant apparent horizon and demonstrated that it also satisfies all relevant hydrodynamic constraints. A possible extension of our work would be to utilise their method to construct this alternative entropy current. It would be interesting to see if the ambiguity in the entropy current computed holographically precisely matches the ambiguity one would find using the hydrodynamic analyses of [39, 40].

Chapter 5

Constraints on transport coefficients

5.1 Introduction

Previous chapters have considered the fluid/gravity correspondence. This chapter and the next one, however, do not relate to this correspondence and are completely independent of string theory. Here, we consider problems fundamental to fluid dynamics, specifically the nature of the various constraints that must be imposed on the fluid dynamical transport coefficients. This chapter will just be a review; original results are presented in the next chapter.

Fluid dynamics is by construction a long wavelength effective theory; it provides a *macroscopic* description of field theory configurations which are locally equilibrated. The emphasis here is on the word macroscopic. To ensure that the equations of fluid dynamics are consistent with an underlying microscopic field theory, we need to impose further constraints on the transport coefficients. Otherwise, the fluid may not be physical. It is currently not known what constitutes an exhaustive set of constraints. In this chapter, we review two different physical principles which lead to various sets of constraints.

The first physical principle that we consider is a local form of the second law of thermodynamics. More precisely, we demand that the divergence of the fluid dynamical entropy current is non-negative for all fluid flows. This constraint has been well-established within the fluid dynamics literature for a long time, and is contained within the works of Landau and Lifshitz [15]. Requiring that the divergence of the entropy be positive semi-definite results in several inequality-type constraints; for example, the viscosities must be non-negative. It is perhaps less well-known that equality-type constraints result as well. These reduce the number of independent transport coefficients.

This physical principle is certainly necessary; entropy must increase. However, it is computationally quite involved to obtain these constraints, especially at higher orders in the derivative expansion. Also, the microscopic origins of the second law of thermodynamics are fairly unclear. It would be informative to derive these same constraints from another physical principle which has more transparent microscopic origins.

The second physical principle we consider has both these properties; it is far less computationally intensive and it relates more directly to the microscopic field theory. This principle is the requirement that the solutions of the relativistic Navier-Stokes equations admit equilibrium solutions which are derivable from an equilibrium partition function; this was introduced in [40]. More precisely, they considered stationary background spacetimes and demanded that the fluid dynamical equations admit equilibrium solutions expressed in terms of the background metric fields and their derivatives. Next, they derived the most general equilibrium partition function on such a background and demanded that the stress tensor was consistent with this. As this physical principle deals directly with the partition function, it is more transparent microscopically.

What is surprising is that the equality-type constraints which result from the second law completely match the constraints obtained by requiring consistency with the equilibrium partition function (no inequality-type constraints are obtained from the equilibrium partition function though). This matching is conjectured to hold to all orders in the derivative expansion.

This review chapter is organised as follows. In section 5.2 we derive the constraints which follow from requiring local entropy production for the simplest case of an uncharged fluid at first order. Next in section 5.3 we briefly outline the methodology used to derive the constraints from demanding consistency with an equilibrium partition function. These results will offer a useful point of comparison for our original work in the following chapter.

Before ending this introduction, it is worth making one last point. We stated earlier that the material contained in this chapter and the next is completely independent of string theory and is solely concerned with fluid dynamics. However, the fluid/gravity duality offers us the possibility of translating all of these results into a gravitational language. We could, for example, investigate how these physical principles are interpreted on the bulk gravity side. It is already known that the second law of thermodynamics for the boundary fluid corresponds to the area increase theorem on the bulk side. Yet, the bulk interpretation of the constraints imposed by the equilibrium partition function is still unknown. If this alternative requirement is indeed equivalent to the area increase theorem then it could potentially provide a method of deriving an area increase theorem for higher curvature gravity. This has been discussed in [55].

5.2 Constraints from entropy increase

Our starting point is simply the entropy current. For an uncharged fluid at first order we have:

$$J_S^\mu = s u^\mu. \tag{5.1}$$

Taking its divergence gives us:

$$\nabla_\mu J_S^\mu = u^\mu \partial_\mu s + s \nabla_\mu u^\mu. \tag{5.2}$$

If we work with the pressure p and energy density ρ via the following relations:

$$\begin{aligned} s &= \frac{\rho + p}{T} \\ d\rho &= T ds \end{aligned} \tag{5.3}$$

we can rewrite $\nabla_\mu J_S^\mu$ as:

$$\nabla_\mu J_S^\mu = \frac{1}{T} \left(u^\mu \partial_\mu \rho + (\rho + p) \nabla_\mu u^\mu \right). \tag{5.4}$$

Recall the equation for the conservation of the stress tensor at first order:

$$\begin{aligned} 0 = \nabla_\mu T^{\mu\nu} &= \partial_\mu p \cdot P^{\mu\nu} + \partial_\mu \rho \cdot u^\mu u^\nu + (\rho + p)(\nabla_\mu u^\mu u^\nu + u^\mu \nabla_\mu u^\nu) \\ &+ 2\partial_\mu \eta \sigma^{\mu\nu} + 2\eta \nabla_\mu \sigma^{\mu\nu} \\ &+ \partial_\mu \zeta (\nabla_\mu u^\mu) P^{\mu\nu} + \zeta (\nabla_\mu u^\mu)(\nabla_\mu u^\mu u^\nu + u^\mu \nabla_\mu u^\nu). \end{aligned} \tag{5.5}$$

Taking the component proportional to u^μ, *i.e.* $u_\nu \nabla_\mu T^{\mu\nu}$ we obtain:

$$0 = u_\nu \nabla_\mu T^{\mu\nu} = u^\mu \partial_\mu \rho + (\rho + p)\nabla_\mu u^\mu - 2\eta u_\nu \nabla_\mu \sigma^{\mu\nu} + \zeta(\nabla_\mu u^\mu)^2. \tag{5.6}$$

Thus, the equation for the divergence of the entropy current can be expressed as:

$$\nabla_\mu J_S^\mu = -2\eta u_\nu \nabla_\mu \sigma^{\mu\nu} + \zeta(\nabla_\mu u^\mu)^2. \tag{5.7}$$

Using the chain rule and dropping terms on the boundary, we have:

$$- u_\nu \nabla_\mu \sigma^{\mu\nu} = \nabla_\mu u_\nu \cdot \sigma^{\mu\nu} = \sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{5.8}$$

giving us the following result:

$$\nabla_\mu J_S^\mu = 2\eta \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\mu u^\mu)^2. \tag{5.9}$$

Therefore, in order to have local entropy production, we require $\eta, \zeta \geq 0$. Notice here that there are only inequality relations. This is the case for an uncharged fluid at first order but generally there will be both inequality and equality constraints. However, for our work in the next chapter, the inequality constraints derived here are the only ones that are relevant.

5.3 Constraints from equilibrium partition functions

In this section we consider the constraints from demanding consistency with an equilibrium partition function. As this calculation is not that relevant for our work in the following chapter, we just provide a schematic outline for the sake of completeness.

The fluid stress tensor comprises of dissipative and non-dissipative terms:

$$T_{\mu\nu} = T_{\mu\nu}^{disp} + T_{\mu\nu}^{non-disp}. \tag{5.10}$$

Each term from first order onwards will have an attached transport coefficient. The dissipative terms will vanish when evaluated on equilibrium configurations and thus cannot be constrained by equilibrium considerations. Demanding consistency with an equilibrium partition function will only impose constraints on the transport coefficients arising from $T_{\mu\nu}^{non-disp}$.

Our constraints from this principle are derived as follows. We first parametrise all possible equilibrium solutions. For a fluid on given stationary spacetime background $g_{AB}(x_i)$ (where x_i represent spatial coordinates only) the fluid variables will necessarily be functions of this metric (there are no other characteristic scales in the system). For equilibrium configurations, the velocity and temperature field, u_μ^{eq} and T^{eq} must take the following functional form:

$$\begin{aligned} u_\mu^{eq} &= u_\mu(g_{AB}(x_i)) \\ T^{eq} &= T(g_{AB}(x_i)). \end{aligned} \tag{5.11}$$

As a consequence, the stress tensor will evaluate to:

$$T_{\mu\nu}^{eq} = T_{\mu\nu}^{non-disp}(g_{AB}(x_i)). \tag{5.12}$$

The result above gives us a set of expressions for the stress tensor terms at each order in the derivative expansion.

Another approach to evaluating the stress tensor on equilibrium configurations would be to construct the partition function; again, this must be a function of the background metric fields:

$$\log Z = \log Z(g_{AB}(x_i)). \tag{5.13}$$

The equilibrium stress tensor would then be proportional to this derivative of the partition function:

$$T_{\mu\nu}^{partition} \propto \frac{\delta \log Z}{\delta g_{AB}}. \tag{5.14}$$

Equating these two expressions $T_{\mu\nu}^{partition}$ and $T_{\mu\nu}^{eq}$ will give us equality constraints for the non-dissipative transport coefficients at each order in the derivative expansion.

Note that for an uncharged fluid at first order, the stress tensor only has two transport coefficients η and ζ, the shear and bulk viscosities, which are both dissipative terms. Thus there can be no equality constraints at first order if we restrict to the uncharged case. This matches what we found in the previous section when we derived the constraints using the principle of entropy increase.

Chapter 6

Stability of equilibrium in fluid dynamics

6.1 Introduction

In the previous chapter, we considered two physical principles which led to sets of constraints on the transport coefficients. The first was that the divergence of the fluid dynamical entropy current must be non-negative for all admissible fluid flows. This is a local form of the second law of thermodynamics. The second physical principle that we examined was compatibility with an equilibrium partition function. In general, the equality-type relations which result from the second law exactly match the constraints imposed by compatibility with an equilibrium partition function. The requirement of entropy increase also yielded a set of inequalities; these were not obtained by the equilibrium partition function method.

It is very interesting to consider how exactly these two physical principles are related. For example, if we could recast the second law of thermodynamics in the language of partition functions, it would make its microscopic origins much clearer. As things stand, these two physical principles are not completely equivalent given the additional inequalities that result from the second law. This naturally motivates the question: What further ingredient do we need to add for our equilibrium considerations to yield exactly the same set of constraints as the second law? It was conjectured in [39] that compatibility with an equilibrium partition function together with the requirement that the equilibrium solutions be dynamically stable will be equivalent to positivity of the divergence of the entropy current.

We now pause to consider this conjecture further. It is well-known that the second law of thermodynamics implies stability of equilibrium. This is intuitively clear. The equilibrium configuration would be at a local maximum of the entropy; small perturbations away from this would then evolve back to the same equilibrium given that entropy is constrained to always increase. This conjecture, however, concerns the reverse implication. Can the existence of equilibrium together with the requirement of stability imply the second law for fluid dynamics? This would be an immensely significant statement. It would imply that statements concerning *small perturbations around equilibrium* could result in the same constraints as the second law which holds for *all non-equilibrium fluid flow, however far from equilibrium.*

In this chapter, we consider this conjecture in the simplest possible case. We study uncharged fluid dynamics on a flat spacetime background to first order in the derivative expansion. A linear stability analysis is done and we determine the constraints imposed by stability about equilibrium. It turns out that the inequalities that we obtain are slightly weaker than the inequalities imposed by the second law.

We now outline the structure of this chapter. In section 6.2, we consider small amplitude perturbations about equilibrium for linearised hydrodynamics in flat spacetime; the requirement of linear stability imposes a set of inequalities which we list here. Reasons for the inequivalence between the two sets of inequalities derived from stability of equilibrium and the second law of thermodynamics are then discussed in section 6.3. We end with our conclusions in the final section 6.4.

6.2 Linear stability analysis

We are interested in the equations of relativistic fluid dynamics to first order in the derivative expansion. Our stress tensor is of the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} + \Pi^{\mu\nu}$$

where we define $\Pi^{\mu\nu}$ as follows

$$\Pi^{\mu\nu} = \zeta\theta P^{\mu\nu} + \eta\sigma^{\mu\nu}.$$

The equations for fluid dynamics then follow from $\nabla_\mu T^{\mu\nu} = 0$; we take the projections of this equation in the transverse and longitudinal directions relative to the velocity

$$\begin{aligned}
0 = u_\nu \nabla_\mu T^{\mu\nu} &= -u^\mu \nabla_\mu \rho - \rho \nabla_\mu u^\mu - p \nabla_\mu u^\mu \\
&- \eta \nabla_{(\alpha} u_{\beta)} P^{\mu\alpha} \nabla_\mu u^\beta - (\zeta - \frac{1}{3}\eta) \nabla_\rho u^\rho \cdot \nabla_\mu u^\mu, \\
0 = P^\alpha{}_\nu \nabla_\mu T^{\mu\nu} &= \rho u^\mu \nabla_\mu u^\alpha + \nabla_\mu p \cdot P^{\mu\alpha} + \nabla_\mu \eta \nabla_{(\delta} u_{\beta)} P^{\mu\delta} P^{\alpha\beta} \\
&+ p u^\mu \nabla_\mu u^\alpha + \eta \cdot \frac{1}{2} \left(\nabla_\mu \nabla_\delta u_\beta + \nabla_\mu \nabla_\beta u_\delta \right) P^{\mu\delta} P^{\alpha\beta} \\
&+ \eta \nabla_{(\delta} u_{\beta)} \left(\nabla_\mu u^\mu \cdot u^\delta + u^\mu \nabla_\mu u^\delta \right) P^{\alpha\beta} \\
&+ \eta \nabla_{(\delta} u_{\beta)} P^{\mu\delta} \left(\nabla_\mu u^\alpha \cdot u^\beta \right) \\
&+ \left(\nabla_\mu \zeta - \frac{1}{3} \nabla_\mu \eta \right) \nabla_\rho u^\rho \cdot P^{\mu\alpha} + \left(\zeta - \frac{1}{3}\eta \right) \nabla_\mu \nabla_\rho u^\rho \cdot P^{\mu\alpha} \\
&+ \left(\zeta - \frac{1}{3}\eta \right) \nabla_\rho u^\rho \cdot u^\mu \nabla_\mu u^\alpha.
\end{aligned} \tag{6.1}$$

We consider small amplitude perturbations about equilibrium. Without loss of generality, we can take our equilibrium fluid to be at rest. The fluid velocity and temperature are taken to be our variables and we consider perturbations of the following form:

$$T = T_0 + \delta T(t, \vec{x}), \qquad u^\mu = (1, \vec{0}) + \delta u^\mu(t, \vec{x}). \tag{6.2}$$

Substituting this into (6.1), linearising, and keeping terms to second order in derivatives gives:

$$\begin{aligned}
\delta \left(u_\nu \nabla_\mu T^{\mu\nu} \right) &= -\rho'' \cdot \delta T \cdot u^\mu \nabla_\mu T - \rho' \delta u^\mu \nabla_\mu T - \rho' u^\mu \nabla_\mu \delta T \\
&- \rho' \delta T \nabla_\mu u^\mu - \rho \nabla_\mu \delta u^\mu - p' \delta T \nabla_\mu u^\mu - p \nabla_\mu \delta u^\mu \\
&- \eta' \delta T \nabla_{(\alpha} u_{\beta)} P^{\mu\alpha} \nabla_\mu u^\beta - \eta \nabla_{(\alpha} \delta u_{\beta)} P^{\mu\alpha} \nabla_\mu u^\beta \\
&- \eta \nabla_{(\alpha} u_{\beta)} \left(\delta u^\mu \cdot u^\alpha + u^\mu \cdot \delta u^\alpha \right) \nabla_\mu u^\beta - \eta \nabla_{(\alpha} u_{\beta)} P^{\mu\alpha} \nabla_\mu \delta u^\beta \\
&- \left(\zeta' \cdot \delta T - \frac{1}{3} \eta' \cdot \delta T \right) \nabla_\rho u^\rho \cdot \nabla_\mu u^\mu - 2 \left(\zeta - \frac{1}{3}\eta \right) \nabla_\rho \delta u^\rho \nabla_\mu u^\mu,
\end{aligned} \tag{6.3}$$

$$
\begin{aligned}
\delta\left(P^{\alpha}{}_{\nu}\nabla_{\mu}T^{\mu\nu}\right) = {} & \rho'\delta T u^{\mu}\nabla_{\mu}u^{\alpha} + \rho\delta u^{\mu}\nabla_{\mu}u^{\alpha} + \rho u^{\mu}\nabla_{\mu}\delta u^{\alpha} + p'\nabla_{\mu}\delta T\cdot P^{\mu\alpha} \\
& + p'\nabla_{\mu}T\cdot\delta u^{\mu}\cdot u^{\alpha} + \eta'\nabla_{\mu}T\nabla_{(\delta}\delta u_{\beta)}P^{\mu\delta}P^{\alpha\beta} \\
& + \eta'\nabla_{\mu}T\cdot\nabla_{(\delta}u_{\beta)}\cdot\delta u^{\mu}\cdot u^{\delta}P^{\alpha\beta} + \eta'\nabla_{\mu}T\nabla_{(\delta}u_{\beta)}P^{\mu\delta}\left(\delta u^{\alpha}\cdot u^{\beta} + u^{\alpha}\delta u^{\beta}\right) \\
& + p'\delta T u^{\mu}\nabla_{\mu}u^{\alpha} + p\delta u^{\mu}\nabla_{\mu}u^{\alpha} + p u^{\mu}\nabla_{\mu}\delta u^{\alpha} \\
& + \frac{1}{2}\eta\left(\nabla_{\mu}\nabla_{\delta}\delta u^{\beta} + \nabla_{\mu}\nabla_{\beta}\delta u_{\delta}\right)P^{\mu\delta}P^{\alpha\beta} \\
& + \frac{1}{2}\eta\left(\nabla_{\mu}\nabla_{\delta}u_{\beta} + \nabla_{\mu}\nabla_{\beta}u_{\delta}\right)\left(\delta u^{\mu}u^{\delta} + u^{\mu}\delta u^{\delta}\right)P^{\alpha\beta} \\
& + \frac{1}{2}\eta\left(\nabla_{\mu}\nabla_{\delta}u_{\beta} + \nabla_{\mu}\nabla_{\beta}u_{\delta}\right)P^{\mu\delta}\left(\delta u^{\alpha}\cdot u^{\beta} + u^{\alpha}\delta u^{\beta}\right) \\
& + \eta\nabla_{(\delta}\delta u_{\beta)}\cdot u^{\mu}\nabla_{\mu}u^{\delta}P^{\alpha\beta} + \eta\nabla_{(\delta}u_{\beta)}\left(\nabla_{\mu}\delta u^{\mu}\cdot u^{\delta} + u^{\mu}\nabla_{\mu}\delta u^{\delta}\right)P^{\alpha\beta} \\
& + \eta\nabla_{(\delta}u_{\beta)}u^{\mu}\nabla_{\mu}u^{\delta}\left(\delta u^{\alpha}\cdot u^{\beta} + u^{\alpha}\delta u^{\beta}\right) + \eta'\delta T\nabla_{(\delta}u_{\beta)}P^{\mu\delta}\left(\nabla_{\mu}u^{\alpha}\cdot u^{\beta}\right) \\
& + \eta\nabla_{(\delta}\delta u_{\beta)}P^{\mu\delta}\nabla_{\mu}u^{\alpha}\cdot u^{\beta} + \eta\nabla_{(\delta}u_{\beta)}u^{\mu}\delta u^{\delta}\nabla_{\mu}u^{\alpha}\cdot u^{\beta} \\
& + \eta\nabla_{(\delta}u_{\beta)}P^{\mu\delta}\left(\nabla_{\mu}\delta u^{\alpha}\cdot u^{\beta} + \nabla_{\mu}u^{\alpha}\delta u^{\beta}\right) + \left(\zeta'\nabla_{\mu}T - \frac{1}{3}\eta'\nabla_{\mu}T\right)\nabla_{\rho}\delta u^{\rho}P^{\mu\alpha} \\
& + \left(\zeta - \frac{1}{3}\eta\right)\nabla_{\mu}\nabla_{\rho}\delta u^{\rho}P^{\mu\alpha} + \left(\zeta - \frac{1}{3}\eta\right)\nabla_{\rho}\delta u^{\rho}\cdot u^{\mu}\nabla_{\mu}u^{\alpha}.
\end{aligned}
\tag{6.4}
$$

Removing all derivatives of equilibrium fields, this simplifies to just:

$$
\begin{aligned}
\delta\left(u_{\nu}\nabla_{\mu}T^{\mu\nu}\right) &= -\rho'\partial_t\delta T - \rho\partial_i\delta u^i - p\partial_i\delta u^i \\
\delta\left(P^{\alpha}{}_{\nu}\nabla_{\mu}T^{\mu\nu}\right) &= \rho\partial_t\delta u^i + p'\partial_i\delta T + p\partial_t\delta u^i + \frac{1}{2}\eta\left(\partial^2\delta u^i + \partial_t\partial_j\delta u^j\right) + \left(\zeta - \frac{1}{3}\eta\right)\partial_i\partial_j\delta u^j.
\end{aligned}
\tag{6.5}
$$

Utilising translational invariance, we decompose the perturbation into plane waves; we consider wavenumbers with just an x-component:

$$
\delta T = e^{i\omega t - ikx}\delta T_{\omega,k}, \quad \delta u^i = e^{i\omega t - ikx}\delta u^i_{\omega,k}.
$$

Using this ansatz, we obtain:

$$
i\omega\ \rho'\delta T_{\omega,k} - ik(\rho+p)\ \delta u^x_{\omega,k} = 0, \tag{6.6}
$$

$$
i\omega(\rho+p)\ \delta u^x_{\omega,k} - ikp'\delta T_{\omega,k} - ik\left(\frac{4}{3}ik\eta + ik\zeta\right)\delta u^x_{\omega,k} = 0, \tag{6.7}
$$

$$
i\omega(\rho+p)\ \delta u^y_{\omega,k} - ik\left(ik\eta\right)\delta u^y_{\omega,k} = 0\,. \tag{6.8}
$$

For the transverse velocity perturbation $\delta u^y(t,x)$ we have the following dispersion relation

$$\omega = i\frac{\eta}{\rho+p}k^2 \ . \tag{6.9}$$

And for the longitudinal density and velocity perturbations (sound waves) $\delta T_{\omega,k}$ and $\delta u^x_{\omega,k}$, we find:

$$i\omega - i\frac{k^2}{\omega}p' + k^2\left(\frac{4}{3}\frac{\eta}{\rho+p} + \frac{\zeta}{\rho+p}\right). \tag{6.10}$$

In the hydrodynamic limit $\omega, k \ll 1$, this becomes

$$\omega \ = \pm k c_s + ik^2\left(\tfrac{2}{3}\tfrac{\eta}{\rho+p} + \tfrac{1}{2}\tfrac{\zeta}{\rho+p}\right) \mp \tfrac{k^3}{2c_s}\left(\tfrac{2}{3}\tfrac{\eta}{\rho+p} + \tfrac{1}{2}\tfrac{\zeta}{\rho+p}\right)^2 , \tag{6.11}$$

where we have introduced c_s for the sound speed

$$c_s \equiv \sqrt{\frac{dp}{d\rho}}. \tag{6.12}$$

Having found the plane wave solutions to the linearised equations we now need to determine the condition for stability. The leading order imaginary part of ω must be positive for the perturbations to decay. For the transverse velocity perturbation we require

$$\eta \geq 0,$$

and for the longitudinal perturbations we require

$$\left(\frac{2}{3}\eta + \frac{1}{2}\zeta\right) \geq 0.$$

.

6.3 Reasons for inequivalence between constraints from stability of equilibrium and from entropy increase

We should first state very precisely what problem we have addressed. In this chapter, we have performed a *linear* stability analysis about equilibrium fluid configurations on *flat* spacetime backgrounds; our calculation was further restricted to first order uncharged fluid dynamics. This linear stability analysis enabled us to derive constraints on the transport coefficients; in particular, the shear and bulk viscosities η and ζ. We found the following constraints

$$\eta \geq 0, \quad \left(\frac{2}{3}\eta + \frac{1}{2}\zeta\right) \geq 0.$$

And in the last chapter, we considered the constraints imposed by the second law of thermodynamics for our case of first order uncharged fluid dynamics; the constraints were slightly stronger

$$\eta \geq 0, \quad \zeta \geq 0.$$

Why do these two physical requirements not result in the same constraints on the transport coefficients? It is instructive to closely examine our linear stability analysis to determine what causes this divergence. The fact that our calculation was *linear* and done on a *flat* spacetime background allowed us to utilise translational invariance and consider plane wave solutions. At this crucial step, we effectively reduced our system to a *one-dimensional* system; there was just one direction of note which was given by the wavenumber. We took this to just be the x-direction, without loss of generality. This reduction to a unidirectional system caused the two viscosities to combine. We saw this in equation (6.11). This is a common feature of one-dimensional fluids; the shear and bulk viscosities are no longer independent and combine to form the factor $\left(\frac{2}{3}\eta + \frac{1}{2}\zeta\right)$. This then led to the slightly weaker condition

$$\left(\frac{2}{3}\eta + \frac{1}{2}\zeta\right) \geq 0,$$

instead of

$$\zeta \geq 0.$$

This can perhaps be understood more physically by the fact that dissipation is a *quadratic* effect and therefore a *linear* plane wave analysis is unlikely to capture the full constraints obtained from demanding that entropy increases on all admissible fluid flows.

6.4 Discussion

The conclusion of this chapter is that our linear stability analysis about equilibrium configurations for uncharged fluid dynamics on a flat spacetime background does not yield the same constraints as imposing positivity of the divergence of the entropy current. There are several useful extensions to this calculation that are well worth considering.

First, it would be interesting to extend this calculation to encompass nonlinear stability of equilibrium uncharged fluid configurations on curved backgrounds. A plane wave analysis would not be applicable here and a more sophisticated dynamical systems approach would have to be utilised.

And second, this linear stability analysis could be extended to the charged case; this would be fairly straightforward. It would be worthwhile to check to see if the inequality-type constraints on the conductivities which follow from a linear stability analysis match those derived from the second law.

References

[1] J. M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv Theor Math Phys 2 231-252*, 1998.

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz. Large n field theories, string theory, and gravity. *Phys Rept 323 183-386*, 2000.

[3] S. S. Gubser and A. Karch. From gauge-string duality to strong interactions: A pedestrian's guide. *Ann Rev Nucl Part Sci 59 145-168*, 2009.

[4] S. A. Hartnoll. Lectures on holographic methods for condensed matter physics. *Class Quant Grav 26 224002*, 2009.

[5] V. E. Hubeny and M. Rangamani. A holographic view on physics out of equilibrium. *Adv High Energy Phys 2010 297916*, 2010.

[6] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi. Holographic probes of anti-de sitter spacetimes. *Phys Rev D 59 104021*, 1999.

[7] I. Klebanov and E. Witten. Ads/cft correspondence and symmetry breaking. *Nucl Phys B 556 89*, 1999.

[8] I. Papadimitriou and K. Skenderis. Ads/cft correspondence and geometry. *arXiv:040417 [hep-th]*, 2004.

[9] I. Papadimitriou and K. Skenderis. Correlation functions in holographic rg flows. *JHEP 0410 075*, 2004.

[10] N. Iqbal and H. Liu. Universality of the hydrodynamic limit in ads/cft and the membrane paradigm. *Phys Rev D 79 025023*, 2009.

[11] N. Iqbal and H. Liu. Real-time response in ads/cft with application to spinors. *Fortsch Phys 57 367*, 2009.

[12] S. Caron-Huot, P. M. Chesler, and D. Teaney. Fluctuation, dissipation, and thermalization in non-equilibrium ads black hole geometries. *Phys Rev D 84 026012*, 2011.

[13] P. M. Chesler and L. G. Yaffe. Holography and colliding gravitational shock waves in asymptotically ads spacetime. *Phys Rev Lett 106 021601*, 2011.

[14] P. M. Chesler and L. G. Yaffe. Horizon formation and far-from-equilibrium isotropization in supersymmetric yang-mills plasma. *Phys Rev Lett 102 211601*, 2009.

[15] L. D. Landau and E. M. Lifshitz. *Fluid mechanics*. Oxford: Pergamon Press, 1959.

[16] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani. Nonlinear fluid dynamics from gravity. *JHEP 0802*, 2008.

[17] G. Policastro, D. T. Son, and A. O. Starinets. The shear viscosity of strongly coupled n=4 supersymmetric yang-mills plasma. *Phys Rev Lett 87 081601*, 2001.

[18] G. Policastro, D. T. Son, and A. O. Starinets. From ads/cft correspondence to hydrodynamics. *JHEP 09 043*, 2002.

[19] G. Policastro, D. T. Son, and A. O. Starinets. From ads/cft correspondence to hydrodynamics ii: Sound waves. *JHEP 12 054*, 2002.

[20] D. T. Son and A. O. Starinets. Viscosity, black holes, and quantum field theory. *arXiv:0704.0240 [hep-th]*, 2007.

[21] R. A. Janik and R. Peschanski. Asymptotic perfect fluid dynamics as a consequence of ads/cft. *Phys Rev D 73 045013*, 2006.

[22] R. A. Janik. Viscous plasma evolution from gravity using ads/cft. *Phys Rev Lett 98 022302*, arXiv:0610144 [hep-th].

[23] M. P. Heller and R. A. Janik. Viscous hydrodynamics relaxation time from ads/cft. *Phys Rev D 76 025027*, 2007.

[24] M. Rangamani. Gravity and hydrodynamics: Lectures on the fluid-gravity correspondence. *Class Quant Grav 26 224003*, 2009.

[25] V. E. Hubeny, S. Minwalla, and M. Rangamani. The fluid/gravity correspondence. *arXiv:1107.5780*, 2011.

[26] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma. Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions. *JHEP 0812*, 2008.

[27] M. Haack and A. Yarom. Nonlinear viscous hydrodynamics in various dimensions using ads/cft. *arXiv:0806.4602*, 2008.

[28] S. Bhattacharyya, S. Minwalla, and S. R. Wadia. The incompressible non-relativistic navier-stokes equation from gravity. *JHEP 0908 059*, 2009.

[29] I. Kanitscheider and K. Skenderis. Universal hydrodynamics of non-conformal branes. *JHEP 0904 062*, 2009.

[30] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom. Fluid dynamics of r-charged black holes. *JHEP 0901 055*, 2009.

[31] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, and R. Loganayagam. Hydrodynamics from charged black branes. *JHEP 1101 094*, 2011.

[32] J. Sonner and B. Withers. A gravity derivation of the tisza-landau model in ads/cft. *Phys Rev D 82 026001*, 2010.

[33] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom. A theory of first order dissipative superfluid dynamics. *arXiv:1105.3733*, 2011.

[34] C. P. Herzog, N. Lisker, P. Surowka, and A. Yarom. Transport in holographic superfluids. *JHEP 1108 052*, 2011.

[35] R. Banerjee, P. Chakraborty, S. Dey, B. R. Majhi, and A. K. Mitra. Two dimensional hydrodynamcis with gauge and gravitational anomalies. *arXiv:1307.1313 [hep-th]*, 2013.

[36] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, and S. P. et al Trivedi. Forced fluid dynamics from gravity. *JHEP 0902*, 2009.

[37] D. T. Son and P. Surowka. Hydrodynamics with triangle anomalies. *Phys Rev Lett 103 191601*, 2009.

[38] T. Kalaydzhyan and I. Kirsch. Fluid-gravity model for the chiral magnetic effect. *arXiv:1102.4334 [hep-th]*, 2011.

[39] S. Bhattacharyya. Constraints on the second order transport coefficients of an uncharged fluid. *arXiv:1201.4654*, 2012.

[40] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma. Constraints on fluid dynamics from equilibrium partition functions. *arXiv:1203.3544*, 2012.

[41] F. Carrasco, L. Lehner, R. C. Myers, O. Reula, and A. Singh. Turbulent flows for relativistic conformal fluids in 2+1 dimensions. *Phys Rev D 86*, 2012.

[42] C. Eling, I. Fouxon, and Y. Oz. Gravity and geometrization of turbulence: An intriguing correspondence. *arXiv:1004.2632 [hep-th]*, 2010.

[43] C. Eling and Y. Oz. Holographic vorticity in the fluid/gravity correspondence. *arXiv:1308.1651 [hep-th]*.

[44] A. Adams, P. M. Chesler, and H. Liu. Holographic turbulence. *arXiv:1307.7267 [hep-th]*, 2013.

[45] R. Loganayagam. Entropy current in conformal hydrodynamics. *JHEP 0805*, 2008.

[46] P. Bizon and A. Rostworowski. On weakly turbulent instability of anti-de sitter space. *Phs Rev Lett 107*, 2011.

[47] D. Kharzeev. Parity violation in hot qcd: why it can happen, and how to look for it. *Phys Lett B 633 260-264*, 2006.

[48] A. V. Sadofyev and M. V. Isachenkov. The chiral magnetic effect in the hydrodynamical approach. *Phys Lett B 697 404*, 2011.

[49] S. Pu, J. H. Gao, and Q. Wang. Consistent description of kinetic equation with triangle anomaly. *Phys Rev D 83 094017*, 2011.

[50] G. Lifschytz and M. Lippert. Anomalous conductivity in holographic qcd. *Phys Rev D 80 066005*, 2009.

[51] H. U. Yee. Holographic chiral magnetic conductivity. *JHEP 0911 085*, 2009.

[52] A. Gorsky, P. N. Kopnin, and A. V. Zayakin. Chiral magnetic effect in the soft-wall ads/qcd model. *Phys Rev D 83 014023*, 2011.

[53] V. A. Rubakov. On chiral magnetic effect and holography. *arXiv:1005.1888*, 2010.

[54] G. Plewa and M. Spalinski. Entropy currents from holography in hydrodynamics with charge. *arXiv:1304.6705*, 2013.

[55] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani, and H. S. Reall. Local fluid dynamical entropy from gravity. *JHEP 0806 055*, 2008.

[56] I. Gahramanov, T. Kalaydzhyan, and I. Kirsch. Anistropic hydrodynamics, holography and the chiral magnetic effect. *arXiv:1203.4259*, 2012.

[57] I. Booth, M. P. Heller, G. Plewa, and M. Spalinski. On the apparent horizon in the fluid-gravity duality. *arXiv:1102.2885*, 2011.