Well-posedness and scattering of the Chern-Simons-Schrödinger system

Zhuo Min Lim

Cambridge Centre for Analysis

and St. John's College

University of Cambridge

June 2017

This dissertation is submitted for
the degree of Doctor of Philosophy.

Statement of Originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specically indicated in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other university or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other university or similar institution.

Chapter 1 serves as an introduction to work presented in this thesis, and contains precise statements of the results proved in the thesis, as well as an overview of the history of these and related problems.

In Chapter 2, the contents of Section 2.1 to 2.6 are entirely my own original work. They have previously appeared as a preprint [34], which has since been submitted. Section 2.7 contains the statements and proofs of several results in the literature, which were used in the rest of Chapter 2.

Chapter 3 is entirely my own original work and has not yet been prepared for publication.

Abstract

The subject of the present thesis is the Chern-Simons-Schrödinger system, which is a gauge-covariant Schrödinger system in two spatial dimensions with a long-range electromagnetic field. The present thesis studies two aspects of the system: that of well-posedness and that of the long-time behaviour.

The first main result of the thesis concerns the large-data well-posedness of the initial-value problem for the Chern-Simons-Schrödinger system. We impose the Coulomb gauge to remove the gauge-invariance, in order to obtain a well-defined initial-value problem. We prove that, in the Coulomb gauge, the Chern-Simons-Schrödinger system is locally well-posed in the Sobolev spaces H^s for $s \geqslant 1$, and that the solution map satisfies a weak Lipschitz continuity estimate. The main technical difficulty is the presence of a derivative nonlinearity, which rules out the naive iteration scheme for proving well-posedness. The key idea is to retain the non-perturbative part of the derivative nonlinearity in the principal operator, and to exploit the dispersive properties of the resulting paradifferential-type principal operator, in particular frequency-localised Strichartz estimates, using adaptations of the U^p and V^p spaces introduced by Koch and Tataru in other contexts.

The other main result of the thesis characterises the large-time behaviour in the case where the interaction potential is the defocusing cubic term. We prove that the solution to the Chern-Simons-Schrödinger system in the Coulomb gauge, starting from a localised finite-energy initial datum, will scatter to a free Schrödinger wave at large times. The two crucial ingredients here are the discovery of a new conserved quantity, that of a pseudo-conformal energy, and the cubic null structure discovered by Oh and Pusateri, which reveals a subtle cancellation in the long-range electromagnetic effects. By exploiting pseudo-conformal symmetry, we also prove the existence of wave operators for the Chern-Simons-Schrödinger system in the Coulomb gauge: given a localised finite-energy final state, there exists a unique solution which scatters to that prescribed state.

Acknowledgments

First and foremost I would like to thank two very bright, young mathematicians who have helped me with my research.

- Sung-Jin Oh was the first person to introduce me to research in nonlinear dispersive and wave equations, and in particular gauge theories. Discussing mathematics with him had been extremely pleasant and inspiring, and I have certainly learnt a lot from these experiences.
- I would also like to thank Jason Murphy for teaching me about nonlinear Schrödinger equations and in particular the use of pseudo-conformal methods. In particular, our discussions in Princeton in January 2016 provided the key insights leading to the results in Chapter 3 of this thesis.

I am very grateful for their patience with me and their encouragement, and wish them the very best in their future research careers.

I thank my friends in the Cambridge Centre for Analysis, especially those in my cohort (Adam Kashlak, Ben Jennings, David Driver, Davide Piazzoli, Dominic Dold, Eavan Gleeson, Ellen Powell, Franca Hoffmann, Karen Habermann, Sam Forster), as well as Federico Pasqualotto and Maxime van de Moortel, for their friendship and encouragement, and for all the discussions, mathematical or otherwise, that we had over the years. I certainly learnt a lot from them over the years. I wish them the very best in their future endeavours.

My years studying for a PhD were a difficult period for me, and I would not have made it through without the love from my family. Neither would I have made it without the support of several other groups of friends, including my housemates at 15 Madingley Road, the Chapel community at St John's, and many of my former supervisees. Thanks for all your kind words of understanding and the many pats on the back.

I gratefully acknowledge generous financial support from a graduate research scholarship, awarded by St. John's College over the course of my PhD.

Finally, I thank David Stuart for formally being my advisor and for his help in administrative matters, and I also thank Mihalis Dafermos and Nikolaos Bournaveas for taking the time out to examine this thesis.

Contents

Chapter 1

Introduction

1.1 The Chern-Simons-Schrödinger system

The present thesis is concerned with the Chern-Simons-Schrödinger system [23, 24], which is a gauge-covariant Schrödinger system in $\mathbb{R}_t \times \mathbb{R}^2_x$. Precisely, it has the form

$$\begin{cases} \mathrm{i}\,\mathbf{D}_t\phi + \mathbf{D}_i\mathbf{D}_i\phi = 2V'(|\phi|^2)\phi \ , \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2 \ , \\ \partial_t A_i - \partial_i A_0 = -\epsilon_{ij}\mathrm{Im}\left(\overline{\phi}\,\mathbf{D}_j\phi\right) \end{cases} \tag{1.1}$$

where $\mathbf{D}_t, \mathbf{D}_1, \mathbf{D}_2$ are the covariant derivative operators defined by

$$\mathbf{D}_t := \partial_\alpha + \mathrm{i}A_0 \ ,$$

$$\mathbf{D}_i := \partial_i + \mathrm{i}A_i \ , \quad i = 1, 2 \ ,$$

and V is a (possibly zero) polynomial of the form

$$V(\rho) = c_2\rho^2 + \ldots + c_d\rho^d \tag{1.2}$$

for some $d \geqslant 2$, such that $c_2, \ldots, c_d$ are real numbers. Here and in the rest of this section, repeated indices are always summed over, and ϵ_{ij} denotes the standard anti-symmetric 2-form with $\epsilon_{12} = 1$.

The Chern-Simons-Schrödinger system (1.1) is a non-relativistic Lagrangian field theory whose action is given by

$$\mathcal{L}_{\mathrm{CSS}}(\phi, A) := \iint\limits_{\mathbb{R}_t\times\mathbb{R}^2_x} \left(\frac{1}{2}\mathrm{Im}\left(\overline{\phi}\mathbf{D}_t\phi\right) + \frac{1}{2}\left|\mathbf{D}_x\phi\right|^2 + V(|\phi|^2)\right) \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\iint\limits_{\mathbb{R}_t\times\mathbb{R}^2_x} A \wedge \mathrm{d}A$$

where $A := A_0\mathrm{d}t + A_1\mathrm{d}x_1 + A_2\mathrm{d}x_2$ is the electromagnetic potential 1-form. It describes the effective dynamics of a large system of non-relativistic charged quantum particles, interacting with each other via an interaction potential V, and also with the self-generated electromagnetic field. The Schrödinger field ϕ is commonly called the "condensate wave-function" in the physics literature, and describes the local quantum state of the particle system. The physical interpretation of the last two equations of (1.1) is that the electric field is proportional to a rotation of the matter current, while the magnetic field is proportional to the local charge density. The Chern-Simons-Schrödinger system (1.1) has been proposed as a theoretical model for various condensed matter phenomena such as the quantum Hall effect and high temperature superconductivity.

The Chern-Simons-Schrödinger system (1.1) enjoys the following two conservation laws: that of the total mass,

$$\mathcal{M}(t) := \frac{1}{2}\int_{\mathbb{R}^2} |\phi(t,x)|^2 \,\mathrm{d}x = \mathcal{M}(0) ,$$

and that of the energy

$$\mathcal{E}(t) := \int_{\mathbb{R}^2} \left(\frac{1}{2}\left|\mathbf{D}_x\phi(t,x)\right|^2 + V(|\phi(t,x)|^2)\right) \mathrm{d}x = \mathcal{E}(0) .$$

Another important aspect of the Chern-Simons-Schrödinger system (1.1) is that of gauge-invariance. Indeed, if (ϕ, A) solves (1.1), then so does

$$\left(\mathrm{e}^{\mathrm{i}\chi}\phi, A + \mathrm{d}\chi\right)$$

for any sufficiently well-behaved function $\chi : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{R}$. Such a gauge-invariance introduces an unnecessary degree of freedom in the space of solutions. In order that the evolution of (1.1) be well-defined, this gauge-invariance must be eliminated by imposing an additional constraint equation, that is, by fixing a gauge. There are at least two gauge choices available.

- Throughout this thesis, we will work in the *Coulomb gauge*, which is defined by the condition

  $$\partial_1 A_1 + \partial_2 A_2 = 0 .$$

  With the Coulomb gauge condition, straightforward manipulations reduce (1.1) to the following system,

  $$\begin{cases} (\partial_t - \mathrm{i}\triangle)\,\phi = -2A_x\cdot\nabla\phi - \mathrm{i}A_0\phi - \mathrm{i}|A_x|^2\phi - 2\mathrm{i}V'(|\phi|^2)\phi , \\ \quad -\triangle A_i = -\frac{1}{2}\epsilon_{ij}\partial_j\left(|\phi|^2\right) , \\ \quad -\triangle A_0 = -\mathrm{Im}\left(\nabla\overline{\phi}\wedge\nabla\phi\right) - \mathrm{rot}\left(A_x|\phi|^2\right) . \end{cases} \tag{1.3}$$

  Here, we have denoted the cross product $a \wedge b := a_1 b_2 - a_2 b_1$, and $A_x := (A_1, A_2)$ the spatial components of A, and $\nabla := (\partial_1, \partial_2)$ the spatial derivatives.

  Observe that in the Coulomb gauge, the electromagnetic potentials A_0, A_i are no longer dynamical variables, but are uniquely determined at each time t by solving a Poisson equation. In particular, for the initial value problem, one only prescribes $\phi(0) := \phi^{\mathrm{in}}$ as the initial datum.

- Another possible gauge choice is the *heat gauge* introduced in [13], which is defined by the condition

  $$A_0 = \partial_1 A_1 + \partial_2 A_2 .$$

In the heat gauge, (1.1) reduces to the following system

$$\begin{cases} (\partial_t - \mathrm{i}\triangle)\,\phi = -2A_x \cdot \nabla\phi - (1+\mathrm{i})A_0\phi - \mathrm{i}|A_x|^2\phi - 2\mathrm{i}V'(|\phi|^2)\phi \;, \\ (\partial_t - \triangle)\,A_i = -\epsilon_{ij}\left(\tfrac{1}{2}\partial_j\left(|\phi|^2\right) - \mathrm{Im}\left(\overline{\phi}\partial_j\phi\right) - A_j|\phi|^2\right) \;, \\ (\partial_t - \triangle)\,A_0 = -\epsilon_{ij}\partial_i\left(\mathrm{Im}\left(\overline{\phi}\partial_j\phi\right) + A_j|\phi|^2\right) \;. \end{cases} \tag{1.4}$$

Note that, in the heat gauge, the heat evolution imposed by the gauge choice destroys the time reversibility of the Chern-Simons-Schrödinger system.

After imposing the heat gauge, one still needs to impose initial conditions for A in (1.4), but has the freedom to do so in any way consistent with the last equation in (1.1). Perhaps the most natural way to impose initial conditions for A is to require $A_0(0) = 0$, leading to

$$\begin{cases} \phi(0) =: \phi^{\mathrm{in}} \;, \\ A_i(0) = \tfrac{1}{2}\epsilon_{ij}\triangle^{-1}\partial_j\left(|\phi^{\mathrm{in}}|^2\right) \;, \\ A_0(0) = 0 \;. \end{cases} \tag{1.5}$$

We will not study the Chern-Simons-Schrödinger system in the heat gauge (1.4) in this thesis, but only remark that the choice of the heat gauge is crucial in the small-data low-regularity well-posedness theory of [36].

1.2 Statement of results

The first major result in the thesis, proved in Chapter 2, is that the Chern-Simons-Schrödinger system in the Coulomb gauge, (1.3), is locally well-posed for large initial data in H^s, $s \geqslant 1$. Denoting by $\mathbb{B}_{H^s}(D)$ the closed ball in H^s of radius D, we state this result as follows.

Theorem 1.1 (Well-posedness)**.** *Let $s \geqslant 1$. Recall V is a (possibly zero) polynomial of the form (1.2).*

(i) For any $D > 0$, there exists $T = T(s, D) > 0$ such that, given any initial datum $\phi^{\mathrm{in}} \in \mathbb{B}_{H^s}(D)$, there exists a unique solution $\phi \in C_{\mathrm{b}}((-T,T), H^s)$ to (1.3) with $\phi(0) = \phi^{\mathrm{in}}$, which is the unique uniform limit of smooth solutions.

(ii) With $D > 0$ and $T = T(s, D)$ as above, the solution map

$$\mathbb{B}_{H^s}(D) \ni \phi(0) \mapsto \phi \in C_b((-T,T), H^s)$$

is continuous, and satisfies the local-in-time weak Lipschitz bound

$$\left\|\phi - \phi'\right\|_{L_t^\infty((-T,T),H^{s-1})} \leqslant C\left\|\phi(0) - \phi'(0)\right\|_{H^{s-1}} \;. \tag{1.6}$$

Moreover, persistence of regularity holds: for any $D_1 > 0$, there exists $T_\star = T_\star(s, D_1) > 0$ and $C_\star = C_\star(s, D_1) > 0$ such that any H^s solution ϕ, whose initial datum satisfisfies

$$\|\phi(0)\|_{H^1} \leqslant D_1 \;,$$

can be continued to $(-T_\star, T_\star)$*, with the bound*

$$\|\phi\|_{L_t^\infty((-T_\star,T_\star),H^s)} \leqslant C_\star \|\phi(0)\|_{H^s} \ . \tag{1.7}$$

Therefore we have the following blow-up criterion: A maximal-in-time H^s *solution* ϕ *to (1.3) is global if and only if* $\|\phi(t)\|_{H^1}$ *does not blow up in finite time.*

In particular, if V *is a nonzero polynomial with* $c_d > 0$*, so that the conserved energy is coercive, then (1.3) is globally well-posed in* H^s*,* $s \geqslant 1$*.*

The rest of the thesis concern the long-time behaviour of (1.3) in the particular case where the interaction potential V gives a defocusing cubic nonlinearity. We define, for $s > 0$, the spaces

$$\Sigma^s := \left\{ w \in H^s(\mathbb{R}^2) \;\middle|\; |x|^s w \in L^2(\mathbb{R}^2) \right\} \tag{1.8}$$

with the norm

$$\|w\|_{\Sigma^s} := \|w\|_{H^s} + \||x|^s w\|_{L_x^2} \ .$$

Then we have the following results establishing a form of asymptotic completeness for (1.3).

Theorem 1.2 (Scattering)**.** *Assume* $V(|\phi|^2) = \frac{1}{4}\kappa|\phi|^4$ *with* $\kappa > 0$*. Suppose* $\phi \in C([0,\infty), H^1)$ *is a solution to (1.3) such that* $\phi(0) \in \Sigma^1$*. Then there exists* $\phi_\infty \in \Sigma^1$ *such that*

$$\lim_{t\to\infty} \left\|\phi(t) - \mathrm{e}^{\mathrm{i}t\triangle}\phi_\infty\right\|_{L_x^2} = 0 \ .$$

Theorem 1.3 (Existence of wave operators)**.** *Assume* $V(|\phi|^2) = \frac{1}{4}\kappa|\phi|^4$ *with* $\kappa > 0$*. Given* $\phi_\infty \in \Sigma^1$*, there exists a unique solution* $\phi \in C([0,\infty), H^1)$ *to (1.3) such that* $\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t) \in L_t^\infty([0,\infty), \Sigma^1)$*, and*

$$\lim_{t\to\infty} \left\|\phi(t) - \mathrm{e}^{\mathrm{i}t\triangle}\phi_\infty\right\|_{L_x^2} = 0 \ .$$

Theorems 1.2 and 1.3 are proved in Chapter 3.

1.3 History of the problem and related models

Gauge theories from geometry and physics have been fertile sources of interesting problems in PDE theory. Formally gauge fields are connections on some vector bundle, or more generally on some principal bundle, over a smooth manifold. These arise in physics when the underlying manifold models space or space-time, and where the bundle encodes local states of particles, and the curvature of the connection describes fundamental forces such as electromagnetism or nuclear forces. On physical grounds, one is usually interested in gauge fields which formally extremise some Lagrangian action; often the corresponding Euler-Lagrange equations then give a gauge-invariant PDE system. Once the gauge invariance is eliminated by fixing a gauge choice, one derives a well-defined evolution PDE system.

The present section aims to give an informal overview of certain gauge theories studied by the nonlinear dispersive equations community, and with view towards motivating certain concepts that have been found useful in the study of gauge theories related to the Chern-Simons-Schrödinger system.

We will not develop in any generality the geometric and topological underpinnings of gauge theories, and refer the interested reader to standard textbooks such as [37, 38]. Neither will we be very much concerned with elliptic or parabolic systems (i.e. static solutions extremising energy functionals, or the downward gradient flows of such functionals); our focus will be toward evolution problems of the wave or dispersive type.

1.3.1 Geometric dispersive equations as gauge theories

These are equations governing maps $\phi : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathcal{M}$ for some Riemannian manifold $\mathcal{M}$. We will be primarily concerned with providing a gentle overview of wave maps and Schrödinger maps; the reader may refer to [57] for a more technical account.

In this subsection, for convenience we will use Greek letters $\alpha, \beta, \ldots$ for space-time indices, where $\alpha = 0$ refers to the time coordinate and $\alpha = 1, \ldots, d$ refer to the spatial coordinates. We continue to use Latin letters $i, j, \ldots$ to denote spatial coordinates $1, \ldots, d$.

There are two ways to formulate geometric dispersive equations as PDE.

- In the *extrinsic formulation*, one assumes $\mathcal{M}$ to be isometrically embedded as a hypersurface in some Euclidean space $\mathbb{R}^{m+1}$, where $m := \dim \mathcal{M}$. One then views ϕ as a map $\mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^{m+1}$ that happens to take values in $\mathcal{M}$.
- In the *intrinsic formulation*, one considers the derivatives $\partial_\alpha \phi$ as sections of $\phi^*\mathrm{T}\mathcal{M}$ covering ϕ. Since the domain space $\mathbb{R}_t \times \mathbb{R}^d_x$ is simply-connected, one can choose an orthonormal frame for $\phi^*\mathrm{T}\mathcal{M}$ and express $\partial_\alpha \phi$ in their components $\psi_\alpha : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^m$ with respect to this frame. The pullback of the Levi-Civita connection on $\mathcal{M}$ then gives rise to gauge-covariant derivatives $\mathbf{D}_\alpha$.

The intrinsic formulation is arguably the more natural of the two formulations, and will be the focus of the discussion in this section.

We remark that its main drawback, compared to the extrinsic formulation, is the difficulty arising from the fact that the curvature components $[\mathbf{D}_\alpha, \mathbf{D}_\beta]$ at a point (t, x) will in general depend not just on the derivatives $\partial_\alpha \phi(t, x)$, but also on the target point $\phi(t, x)$ itself. Thus, the intrinsic formulation is mainly useful in the case where the target manifold $\mathcal{M}$ has constant curvature, i.e. $\mathcal{M}$ is either a sphere or a hyperbolic space, so that this dependence on $\phi(t, x)$ does not manifest.

We now explain how to derive a gauge theory from the intrinsic formulation of a geometric dispersive equation. Let us be given a map $\phi : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathcal{M}$, where $\mathcal{M}$ is a Riemannian manifold with constant curvature $\kappa \in \{\pm 1\}$. Suppose we have chosen an orthonormal frame $\mathrm{e} = \{\mathrm{e}_1, \ldots, \mathrm{e}_m\}$ for $\phi^*\mathrm{T}\mathcal{M}$. We can then define the corresponding *differentiated fields* $\psi_\alpha : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^m$ by

$$\partial_\alpha \phi =: \sum_{a=1}^{m} (\psi_\alpha)_a \, \mathrm{e}_a \, .$$

With respect to the frame e we also define the *connection coefficients* $A_\alpha : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^{m\times m}$ by

$$(\phi^*\nabla)_\alpha \, \mathrm{e}_b =: \sum_{a=1}^{m} (A_\alpha)_{ab} \, \mathrm{e}_a$$

where ∇ is the Levi-Civita connection on $\mathcal{M}$. Note that since e is an orthonormal frame, A_α take values in the Lie algebra $\mathfrak{o}(m)$ of $m \times m$ anti-symmetric matrices. With respect to the frame e, the *covariant derivative operators* $\mathbf{D}_\alpha$ are then defined as

$$\mathbf{D}_\alpha := \partial_\alpha + A_\alpha$$

acting on functions $w : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^m$. Equivalently, for any section $\sum_{a=1}^m w_a \mathrm{e}_a$ of $\phi^*\mathrm{T}\mathcal{M}$, we have

$$(\phi^*\nabla)_\alpha \left(\sum_{a=1}^{m} w_a \mathrm{e}_a \right) = \sum_{a=1}^{m} (\mathbf{D}_\alpha w)_a \, \mathrm{e}_a \ .$$

We observe the following two important constraint equations in the above framework.

- As the Levi-Civita connection is torsion-free we have $\nabla_{\partial_\alpha \phi} \partial_\beta \phi = \nabla_{\partial_\beta \phi} \partial_\alpha \phi$, which translates to the constraint
$$\mathbf{D}_\alpha \psi_\beta = \mathbf{D}_\beta \psi_\alpha \ . \tag{1.9}$$
- Since $\mathcal{M}$ has constant curvature κ, it holds that $[\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = \kappa(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$ for any vector fields X, Y, Z on $\mathcal{M}$. On pulling back via ϕ, we find the constraint
$$\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = \kappa \left(\psi_\alpha \psi_\beta^\top - \psi_\beta \psi_\alpha^\top \right) \ . \tag{1.10}$$

Now, the differentiated fields ψ_α and the connection coefficients A_α were defined above only with respect to a frame e. Gauge-invariance is simply the freedom to work with a different frame. A different frame $\widetilde{\mathrm{e}} = \{\widetilde{\mathrm{e}}_1, \ldots, \widetilde{\mathrm{e}}_m\}$ will be related to the original frame e by $\widetilde{\mathrm{e}}_b = \sum_{a=1}^m U_{ab} \mathrm{e}_a$ where $U : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}^{m\times m}$ takes values in the orthogonal matrices. Correspondingly

$$\widetilde{\psi}_\alpha = U\psi_\alpha \ , \quad \widetilde{A}_\alpha = U^{-1} A_\alpha U + U^{-1} \partial_\alpha U \ .$$

We note that the constraint equations (1.9) and (1.10) are automatically satisfied for $\widetilde{\psi}_\alpha$ and $\widetilde{A}_\alpha$.

In the study of geometric dispersive equations, one needs to specify the orthonormal frame in order to obtain a well-defined evolution equation for the differentiated fields ψ_α; this is called *fixing the gauge*. It is natural to choose the orthonormal frame depending on ϕ so as to maximise the advantage to the analysis. Indeed, in recent years, several major advances in obtaining optimal results on well-posedness and large-data behaviour have relied crucially on making an appropriate gauge choice.

Example 1.4. The *Coulomb gauge* is the gauge choice defined by imposing

$$\sum_{i=1}^{d} \partial_i A_i = 0 \ .$$

Substituting into (1.10) yields

$$\triangle A_\alpha = \sum_{i=1}^{d} \left(\kappa \partial_i \left(\psi_i \psi_\alpha^\top - \psi_\alpha \psi_i^\top \right) - \partial_i \left[A_i, A_\alpha \right] \right) \ .$$

Thus, at least when the ψ_α are small in some sense, the preceding equation can be solved uniquely to yield A_α in terms of ψ_α. This gives a frame e, depending on ψ_α, whose connection coefficients are A_α.

If the ψ_α are large and $m \geqslant 3$ so that gauge group is not commutative, then the Coulomb gauge need not exist uniquely; this non-uniqueness is known as the *Gribov ambiguity.* $\square$

We now introduce the wave maps equation. Endow the domain space $\mathbb{R}_t \times \mathbb{R}^d_x$ with the standard $(- + \cdots +)$ Minkowski metric, with respect to which we raise and lower space-time indices and employ the Einstein summation convention of implicitly summing over each pair of repeated upper and lower indices.

Definition 1.5. We say that a map $\phi : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathcal{M}$ is a *wave map* if it is a formal critical point of the Lagrangian functional

$$\mathcal{L}_{\mathrm{WM}}(\phi) := \iint\limits_{\mathbb{R}_t \times \mathbb{R}^d_x} \langle \partial^\alpha \phi(t,x), \partial_\alpha \phi(t,x) \rangle_{\phi(t,x)} \, \mathrm{d}x \, \mathrm{d}t \, . \qquad \square$$

Using $\langle \partial^\alpha \phi, \partial_\alpha \phi \rangle = \langle \psi^\alpha, \psi_\alpha \rangle$ and effecting the variation, we obtain

$$\mathbf{D}^\alpha \psi_\alpha = 0 \, . \tag{1.11}$$

The intrinsic formulation of the wave maps equation therefore consists of (1.9), (1.10) and (1.11).

We also note that, in the positive curvature case when $\mathcal{M} = \mathbb{S}^m \subset \mathbb{R}^{m+1}$ the wave maps equation gives $\Box\phi = (\phi \cdot \Box\phi)\phi$ where $\Box := \partial^\alpha \partial_\alpha$ is the usual d'Alambertian operator. This gives the extrinsic formulation

$$\Box\phi = -\left(\partial^\alpha \phi \cdot \partial_\alpha \phi\right) \phi \, . \tag{1.12}$$

The wave maps equation is arguably the simplest geometric wave equation, and naturally generalises the classical harmonic maps equation to Lorentzian domains. Moreover, quite apart from its mathematical interest, it also arises as a sigma model in physics; for example, [16] proposed the wave maps equation, with $d = 3$ and $\mathcal{M} = \mathbb{S}^3$, as a model for interactions between subatomic particles known as pions.

Tremendous progress was made over the past two decades on understanding various mathematical aspects of the wave maps equation, on issues such as critical well-posedness and large-data behaviour in the energy-critical $d = 2$ setting. Again, we will refrain from providing a detailed overview of all the mathematical work that has arisen out of the study of wave maps, but refer the interested reader to [51], Chapter 6, to [57], and to the excellent recent textbook [15].

We now observe that the wave maps equation is invariant under the scaling symmetry

$$\phi(t,x) \mapsto \phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \, .$$

Under this scaling symmetry the $\dot{H}^{\frac{d}{2}}$ norm of ϕ is invariant. By standard scaling heuristics one would expect ill-posedness of the initial value problem below the $\dot{H}^{\frac{d}{2}}$ regularity, and one could at best hope for local well-posedness above the $\dot{H}^{\frac{d}{2}}$ regularity and small-data global well-posedness at the $\dot{H}^{\frac{d}{2}}$ regularity.

The problem becomes especially interesting in $d = 2$ spatial dimensions, when the critical $\dot{H}^1$ norm coincides with the conserved energy.

Almost optimal local well-posedness of the wave maps equation (1.12) in H^s for $s > \frac{d}{2}$ and $d \geqslant 2$ was obtained by Klainerman-Machedon [28] and by Klainerman-Selberg [29], by exploiting the null structure in (1.12) in the framework of hyperbolic Sobolev spaces. For the critical well-posedness of (1.12), two major breakthroughs occurred at the turn of the century. Firstly, Tataru introduced Besov-space variants of the hyperbolic Sobolev spaces in [54] and null-frame spaces in [55], with which small-data global well-posedness in the scaling-invariant Besov space $\dot{B}^{\frac{d}{2}}_{2,1}$ was shown for $d \geqslant 2$. Secondly, Tao introduced a geometric renormalisation procedure in [48] to prove global regularity of local solutions with small $\dot{H}^{\frac{d}{2}}$ initial data when $d \geqslant 5$, and refined the procedure in [49], using Tataru's null frame spaces, to obtain the analogous result for all $d \geqslant 2$. The concepts provided by these advances were crucial to the small-data global well-posedness of (1.12) in the Sobolev space $\dot{H}^{\frac{d}{2}}$, finally obtained by Tataru in [56].

Having obtained small-data global well-posedness in the energy-critical setting for wave maps in $d = 2$ spatial dimensions, the natural next step is to investigate the large-data scenario. Motivated by a conjecture of Klainerman that large-energy wave maps into negatively curved targets should be globally regular, Tao proposed in [50] a new geometric renormalisation procedure for the wave maps equation in the intrinsic formulation, (1.11), when the target is a hyperbolic space, i.e. $\kappa = -1$. The key new idea is a new gauge choice for (1.11) which we explain now.

Recall from our discussion in Example 1.4 that due to the Gribov ambiguity the Coulomb gauge need not exist uniquely for large energy. Another drawback of the Coulomb gauge is that it exhibits, in low dimensions, very poor high $\times$ high $\to$ low frequency interactions, which are not amenable to analysis at the critical regularity.

To circumvent these limitations, Tao introduced a new gauge choice, the *caloric gauge*, which is based on the harmonic map heat flow. To define the caloric gauge, augment the wave map ϕ by adding in a new "heat time" coordinate $s \in [0, \infty)$, so that $\phi = \phi(t, x, s)$. For any fixed physical time t, demand $\phi(t, \cdot, s = 0)$ to be the original wave map, and that $\phi(t, \cdot, s)$ solves the harmonic map heat flow in the "heat time" s, i.e.

$$\psi_s = \mathbf{D}_1\psi_1 + \mathbf{D}_2\psi_2 \ .$$

Under the assumption of negative curvature of the target $\mathcal{M}$, or under a "small mass" assumption of the initial datum when $\mathcal{M}$ has positive curvature, the harmonic map heat flow with the $\dot{H}^1$ initial datum $\phi(t, \cdot, s = 0)$ will converge to a constant map $\phi(\infty)$ as $s \to \infty$. The caloric gauge is then constructed by choosing an orthonormal basis for $\mathrm{T}_{\phi(\infty)}\mathcal{M}$ and parallel-transporting this basis back to $s = 0$ via the heat flow. More precisely, plugging the parallel-transport condition

$$A_s = 0$$

into (1.10), one gets

$$\partial_s A_\alpha = \kappa \left(\psi_s\psi_\alpha^\top - \psi_\alpha\psi_s^\top\right) \ ,$$

and therefore the caloric gauge condition is given by

$$A_\alpha(t,x) = -\kappa \int_0^\infty \left(\psi_s \psi_\alpha^\top - \psi_\alpha \psi_s^\top\right)(t,x,s)\,\mathrm{d}s \; .$$

Apart from the fact that the caloric gauge exists for large-energy maps into hyperbolic space where the Coulomb gauge does not exist, the caloric gauge also eliminates the bad frequency interactions present in the Coulomb gauge. We refer the reader to [50] and to [51], Chapter 6, for further heuristic discussion.

In [52], Tao outlines an ambitious programme to prove global regularity of large-energy wave maps from $\mathbb{R}_t \times \mathbb{R}_x^2$ into hyperbolic space. The programme was subsequently completed in a series of unpublished articles [53].

Eventually, however, Sterbenz and Tataru, using different arguments, obtained a more general global regularity and scattering result for energy-critical wave maps to more general targets, including the hyperbolic spaces. Their main result in [46] is that an energy-dispersion criterion is sufficient to imply global regularity. Then, in [47], they showed that solutions with energy below that of the first nontrivial harmonic map do indeed satisfy the energy-dispersion criterion. In particular, since there are no nontrivial harmonic maps from $\mathbb{R}_x^2$ to hyperbolic spaces, energy-critical wave maps to hyperbolic spaces are global.

Nevertheless, the caloric gauge proved to be of crucial importance in the study of another important geometric dispersive equation, namely the Schrödinger map equation. In what follows we shall describe only Schrödinger maps with target $\mathcal{M} = \mathbb{S}^2$, although Schrödinger maps can also be defined with any Kähler manifold as a target.

Definition 1.6. A map $\phi : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{S}^2 \subset \mathbb{R}^3$ is a *Schrödinger map* if it satisfies the Schrödinger maps equation,

$$\partial_t \phi = \phi \times \triangle \phi \; . \tag{1.13}$$

□

Apart from its geometric interest, the Schrödinger maps equation (1.13) also arises in physics as a Heisenberg model for ferromagnetic spin systems [40].

Like the wave maps equation, the Schrödinger maps equation exhibits a scaling symmetry: The Schrödinger maps equation is invariant under

$$\phi(t,x) \mapsto \phi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \; .$$

Under this scaling symmetry the $\dot{H}^{\frac{d}{2}}$ norm of ϕ is invariant, and, again, scaling heuristics dictate that a small-data global well-posedness at the $\dot{H}^{\frac{d}{2}}$ regularity would be optimal. In particular, the problem becomes especially interesting in two spatial dimensions, where the critical Sobolev regularity $\dot{H}^1$ coincides with that of the conserved energy.

We now formulate the Schrödinger maps equation intrinsically as a gauge theory. As before, assume we have selected an orthonormal frame $\{\mathrm{e}_1, \mathrm{e}_2\}$ for $\phi^* \mathrm{T}\mathbb{S}^2$ that is consistent with the orientation of $\mathbb{S}^2$.

With respect to this frame we may associate differentiated fields ψ_α. However, it is fruitful here to exploit the complex structure of $\mathbb{S}^2$ and consider ψ_α as taking values in $\mathbb{C}$, by writing

$$\psi_\alpha = (\psi_\alpha)_1 + \mathrm{i}\,(\psi_\alpha)_2 \ .$$

The connection coefficients A_α take values in the Lie algebra $\mathfrak{o}(2)$ of skew-symmetric 2×2 matrices, which is a 1-dimensional Lie algebra and in particular is abelian. We may thus, instead, re-define A_α to be real-valued, and re-define the covariant derivative operators by

$$\mathbf{D}_\alpha := \partial_\alpha + \mathrm{i}A_\alpha \ .$$

Recalling also $\kappa = 1$, the equation (1.10) for the curvature components becomes

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \mathrm{Im}\left(\psi_\alpha \overline{\psi_\beta}\right) \ . \tag{1.14}$$

The Schrödinger maps equation (1.13) is then written in terms of the differentiated fields as

$$\psi_t = \mathrm{i} \sum_{i=1}^{d} \mathbf{D}_i \psi_i$$

which, on differentiating and applying (1.9) and (1.14), reduces to

$$\mathbf{D}_t \psi_j = \mathrm{i} \sum_{\ell=1}^{d} \mathbf{D}_\ell \mathbf{D}_\ell \psi_j - \sum_{\ell=1}^{d} \mathrm{Im}\left(\overline{\psi_\ell}\psi_j\right) \psi_\ell \ . \tag{1.15}$$

The Schrödinger maps equation in intrinsic formulation thus consists of (1.15) along with the constraint equations (1.9) and (1.14). Expanding (1.15) gives the equivalent equation

$$(\mathrm{i}\partial_t + \triangle)\,\psi_j = -2\mathrm{i} \sum_{\ell=1}^{d} A_\ell \partial_\ell \psi_j - \mathrm{i}\left(\sum_{\ell=1}^{d} \partial_\ell A_\ell\right)\psi_j + \left(A_t + |A_x|^2\right)\psi_j - \mathrm{i}\sum_{\ell=1}^{d} \mathrm{Im}\left(\overline{\psi_\ell}\psi_j\right)\psi_\ell \ .$$

Note that the critical regularity of (1.15) is $\dot{H}^{\frac{d}{2}-1}$ in the differentiated fields ψ_j. As usual, scaling heuristics dictate that this is the best regularity at which one can hope for well-posedness.

Remark 1.7. The alert reader would have noticed the similarity between the intrinsic formulation of the Schrödinger maps equation, (1.15), and the Chern-Simons-Schrödinger system (1.1) with a cubic nonlinearity, i.e. with $V(|\phi|^2) = \frac{\kappa}{4}|\phi|^4$. In fact, much recent work on the Chern-Simons-Schrödinger system was influenced by progress on understanding the Schrödinger maps equation.

Smith [45] explores in greater detail the relationship between the Schrödinger maps equation and the Chern-Simons-Schrödinger system with a cubic nonlinearity. There, he derived the energy-critical Schrödinger maps equation in $d = 2$ spatial dimensions as the Euler-Lagrange equation associated to the action

$$\begin{aligned}\mathcal{L}_{\mathrm{SM}}(\phi, A) := &\iint_{\mathbb{R}_t \times \mathbb{R}_x^2} \left(\mathrm{Re}\left(\overline{\psi_2}\,\mathbf{D}_t\psi_1\right) - \sum_{j=1}^{2}\mathrm{Im}\left(\overline{\mathbf{D}_j\psi_2}\,\mathbf{D}_j\psi_1\right)\right) \mathrm{d}x\,\mathrm{d}t \\ &+ \frac{1}{2}\iint_{\mathbb{R}_t \times \mathbb{R}_x^2} \left(|\psi_1|^2 + |\psi_2|^2\right) \mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint_{\mathbb{R}_t \times \mathbb{R}_x^2} A \wedge \mathrm{d}A\end{aligned}$$

subject to the torsion-free constraint (1.9). Notice the remarkable appearance of the Chern-Simons term $\iint A \wedge \mathrm{d}A$ in the action. □

The first critical well-posedness results for the Schrödinger maps equation were obtained by Ionescu-Kenig [21, 22] and independently by Bejenaru [2], where small-data global well-posedness of the Schrödinger maps equation in the extrinsic formulation, (1.13), in the critical Besov space $\dot{B}^{\frac{d}{2}}_{2,1}$ for $d \geqslant 3$. Well-posedness in critical Sobolev spaces was later addressed in [3] for $d \geqslant 4$, by working in the intrinsic formulation (1.15) and imposing the Coulomb gauge; there the authors proved the small-data global well-posedness of (1.15) with initial data $\psi_j \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d_x)$.

Unlike the wave maps equation, the Schrödinger maps equation is only first-order in time, and the nonlinear term $-2\mathrm{i}\sum_\ell A_\ell \partial_\ell \psi_j$ involving a derivative term is consequently more difficult to handle in a perturbative manner. A major advance in the above works is the discovery and exploitation of a smoothing estimate for the Schrödinger propagator. For $\mathrm{e} \in \mathbb{S}^{d-1}$, define for $p, q \geqslant 1$ the function space $L^{p,q}_{\mathrm{e}}$ by the norm

$$\|f\|_{L^{p,q}_{\mathrm{e}}} := \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \int_{\mathrm{e}^\perp} \left| f(t, r\mathrm{e} + x') \right|^q \, \mathrm{d}\mathcal{H}^{d-1}(x') \, \mathrm{d}t \right]^{\frac{p}{q}} \mathrm{d}r \right)^{\frac{1}{p}} \tag{1.16}$$

with the obvious modification when either $p = \infty$ or $q = \infty$. Then, when $d \geqslant 3$, for any $\phi \in L^2_x(\mathbb{R}^d)$ whose Fourier transform is supported in $\{\frac{\lambda}{2} \leqslant |\xi| \leqslant 2\lambda, \xi \cdot \mathrm{e} \geqslant \frac{1}{2}|\xi|\}$, one has, for example, the estimate

$$\left\| \mathrm{e}^{\mathrm{i}t\triangle} \phi \right\|_{L^{2,\infty}_{\mathrm{e}}} \leqslant C \lambda^{-\frac{1}{2}} \left\| \phi \right\|_{L^2_x} . \tag{1.17}$$

Notice that crucial gain of half a derivative of the solution from the initial datum. By using the TT^* argument, one then obtains a gain of one full derivative from the nonlinearity, which is the key to closing the iterative argument.

For the interesting energy-critical problem of obtaining small-data global well-posedness of (1.15) with initial data $\psi_j \in L^2(\mathbb{R}^2_x)$, there are at least two additional hurdles. The first is the failure of the smoothing estimate (1.17) in $d = 2$ spatial dimensions. The other is the extremely bad high $\times$ high $\to$ low interactions in the Coulomb gauge, which becomes problematic in low spatial dimensions. In the major work [4], the authors overcame the former hurdle by introducing suitable refinements of the $L^{p,q}_{\mathrm{e}}$ spaces, and the latter by imposing Tao's caloric gauge in place of the Coulomb gauge. In doing so, they were able to finally prove the small-data global well-posedness of the energy-critical Schrödinger maps equation in $d = 2$ spatial dimensions.

Following the small-data theory for energy-critical Schrödinger maps, the next step is to understand the large-energy behaviour. Based on the results obtained for wave maps, it is natural to expect an energy-dispersion continuation criterion for Schrödinger maps, and in particular global regularity of Schrödinger maps with energy below the *threshold energy*, i.e. the energy of the first nontrivial harmonic map. This programme was essentially completed by Smith. In [42], Smith showed that the caloric gauge exists at all energies below the threshold energy, so that large-energy well-posedness can be obtained. Then, in [44], Smith demonstrated energy-dispersion as a continuation criterion under the additional assumption of an a priori $L^4_{t,x}$ bound on ψ_j. This conditional $L^4_{t,x}$ assumption was later removed in [43].

1.3.2 Earlier results on the Chern-Simons-Schrödinger system

To the author's knowledge, most previous works on the Chern-Simons-Schrödinger system (1.1) dealt with a cubic interaction where $V(|\phi|^2) = \frac{\kappa}{4}|\phi|^4$. **All results in this subsection are restricted to this case.** We also note that in this case, the Chern-Simons-Schrödinger system is invariant under the scaling symmetry

$$\phi(t,x) \mapsto \frac{1}{\lambda}\phi\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right) , \quad A_i(t,x) \mapsto \frac{1}{\lambda}A_i\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right) , \quad A_0(t,x) \mapsto \frac{1}{\lambda^2}A_0\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right) ,$$

under which the L^2_x norm of ϕ is also invariant. Hence, the Chern-Simons-Schrödinger system, with a cubic nonlinearity, is mass-critical.

Well-posedness in H^2 of the Chern-Simons-Schrödinger system in the Coulomb gauge, (1.3), was established by Bergé-de Bouard-Saut [6], by invoking an abstract theorem of Kato. By means of a regularisation argument, they also established, in the same paper, global existence of H^1 solutions for H^1 initial data having sufficiently small total mass, but they did not prove that such solutions are unique. In addition, in the focusing case $\kappa < 0$ they proved the existence of the blow-up solutions to (1.3) using a virial argument. Unconditional uniqueness in $L^\infty_t H^1$ of solutions for (1.3) was later demonstrated by Huh in [20] using clever energy estimates, but the continuous dependence of these H^1 solutions on their initial data remains open. We note that neither of these approaches require exploiting the dispersive features of (1.3).

Motivated by the spectacular progress on the Cauchy problem for the Schrödinger maps equation, Liu-Smith-Tataru investigated the low-regularity well-posedness of the Chern-Simons-Schrödinger system [36]. In this very difficult and technical work, they obtained the almost-optimal local well-posedness for small initial data in H^ε for $\varepsilon > 0$ of the Chern-Simons-Schrödinger system in the heat gauge, (1.4). The obvious difficulty is that, unlike the Schrödinger maps equation, the caloric gauge does not exist in the setting of the Chern-Simons-Schrödinger system as there is no natural analogue of the harmonic map heat flow, while the Coulomb gauge still exhibits still exhibits bad frequency interactions. The choice of the heat gauge serves to improve the regularity of the electromagnetic potentials A_j in the frequency region $|\tau| \gg |\xi|$. Several other key ideas in their work include:

- The discovery and exploitation of some null structure in the cubic derivative nonlinearities.
- A second iteration to solve the heat equation for the electromagnetic potentials A_α. This is where the small-data hypothesis is crucial.
- The use of lateral U^p and V^p spaces to take advantage of smoothing estimates in the $L^{p,q}_{\mathrm{e}}$ type spaces, defined in (1.16).
- Meshing together these U^p and V^p spaces with angular spaces, which are necessary to treat the difficult long-range interactions in the Chern-Simons-Schrödinger system.

Unfortunately, in contrast to the successful use of the caloric gauge to obtain optimal well-posedness of the Schrödinger maps equation at the critical regularity, the heat gauge still does not exhibit all the cancellations necessary to obtain critical well-posedness of the Chern-Simons-Schrödinger system. Therefore the Liu-Smith-Tataru result remains the best low-regularity well-posedness result available.

The situation is markedly improved in the symmetry-reduced setting of equivariant solutions, which are called *vortex solutions* in the physics literature. These are solutions of the Chern-Simons-Schrödinger system in the Coulomb guage, (1.3), which have the form

$$\phi(t,x) = \mathrm{e}^{\mathrm{i}m\theta}\phi(t,r) \ , \quad A = A_0(t,r)\,\mathrm{d}t + A_\theta(t,r)\,\mathrm{d}\theta$$

for some $m \in \mathbb{Z}$. Under this ansatz, (1.3) reduces to

$$\begin{cases} (\mathrm{i}\partial_t + \triangle)\,\phi = \dfrac{2m}{r^2}A_\theta\phi + A_0\phi + \dfrac{1}{r^2}A_\theta^2\phi + \kappa|\phi|^2\phi \ , \\ \partial_r A_0 = \dfrac{1}{r}\,(m + A_\theta)\,\phi \ , \\ \partial_t A_\theta = r\mathrm{Im}\left(\overline{\phi}\partial_r\phi\right) \ , \\ \partial_r A_\theta = -\dfrac{1}{2}r|\phi|^2 \ . \end{cases} \tag{1.18}$$

Liu-Smith studied the system (1.18) in [35]. Notice that there are no derivative nonlinearities on the right-hand side of the first equation of (1.18), and already this makes the Cauchy problem considerably simpler. A simple direct iteration argument yields the critical well-posedness of (1.18) in L^2_x; more precisely, this includes small-data global-wellposedness and large-data local existence and uniqueness in L^2_x, with the solution map being Lipschitz continuous on a sufficiently small open ball in L^2_x, and also on compact subsets of L^2_x. To study large-data well-posedness and the behaviour of large-data solutions, they adapted the arguments of Killip-Tao-Visan [27] in using the Kenig-Merle concentration-compactness-rigidity paradigm; one supposes for a contradiction that there existed a large-data solution which does not exist globally, then one can construct a minimal blow-up solution, show that it satisfies a certain phase-space localisation property (namely, *almost-periodicity modulo scaling*) as well as additional regularity properties, and finally rule out the existence of such a solution by virial and Morawetz identities. In this way, Liu-Smith showed, among other results, that in the case $\kappa > -1$ all L^2_x solutions to (1.18) are global and scattering, while in the case $\kappa \leqslant -1$ solutions with L^2_x norm below that of the minimal nontrivial standing-wave solution are also global and scattering.

The only other work, known to the author, that studies the long-time behaviour of the Chern-Simons-Scrhödinger system is that of Oh-Pusateri [39]. They obtained global existence and scattering for solutions in the Coulomb gauge with initial data small in Σ^2, which was defined in (1.8). They built upon the bootstrap argument of Hayashi-Naumkin [19], which established scattering for various nonlinear Schrödinger equations for initial data small in Σ^2. The key novelty of [39] was the discovery of a strongly cubic null structure in the Chern-Simons-Schrödinger system, which reveals a cancellation of the long-range electromagnetic effects. This null structure allowed them to close their bootstrap argument and obtain the optimal decay rate

$$\|\phi(t)\|_{L^\infty_x} \leqslant C(u^{\mathrm{in}})|t|^{-1} \ . \tag{1.19}$$

We remark that in the proof of our scattering result in Chapter 3, the cubic null structure also plays a crucial role, even though the solutions we consider are less regular and in particular do not obey the decay rate (1.19).

Chapter 2

Large data local well-posedness in the energy space of the Chern-Simons-Schrödinger system

The present chapter is devoted to the proof of Theorem 1.1. Observe that (1.3) is time-reversible, therefore we will, in the rest of this chapter, focus exclusively on proving well-posedness forward in time.

2.1 Overview of the proof

The primary difficulty in establishing a well-posedness result for (1.3) at limited regularity, when energy methods alone are insufficient, is the presence of the nonlinear term $2A_x \cdot \nabla\phi$, involving a derivative of ϕ, in the right-hand side of the first equation of (1.3). Indeed, the application of standard dispersive estimates, such as the Strichartz estimates, in the direct iteration scheme incurs a loss of derivatives on the right-hand side, and the estimates will fail to close.

To make matters worse, the electromagnetic interaction is long-range in the sense that A_x does not decay more quickly than $|x|^{-1}$ for large $|x|$. This slow decay can be seen from the representation formula

$$A_i(t,x) = \frac{1}{4\pi}\epsilon_{ij}\int_{\mathbb{R}^2}\frac{x_j - y_j}{|x-y|^2}\,|\phi(t,y)|^2\,\mathrm{d}y$$

given by the Biot-Savart law. The slow decay causes severe difficulty in using local smoothing estimates, such as those in [26], to recover the loss of derivatives by performing estimates in appropriate weighted function spaces.

The above considerations suggest that the difficult nonlinearity $2A_x \cdot \nabla\phi$ is non-perturbative, and motivates the strategy in the present work. Our strategy is primarily inspired by the proof, due to Bejenaru-Tataru, of global well-posedness in the energy space of the Maxwell-Schrödinger system [5].

We perform a paraproduct decomposition on this derivative nonlinearity $2A_x \cdot \nabla\phi$. For a time-dependent spatial 1-form $B = B_1 \mathrm{d}x_1 + B_2 \mathrm{d}x_2 : [0,T) \times \mathbb{R}^2 \to \mathbb{R}^2$, define the operators $\mathfrak{P}_B$ and $\mathfrak{Q}_B$ by

$$\mathfrak{P}_B w := \sum_{\lambda \geqslant 1} \left[\mathrm{P}_{\leqslant 2^{-5}\lambda} B_i \, \mathrm{P}_\lambda \partial_i w + \mathrm{P}_\lambda \left(\mathrm{P}_{\leqslant 2^{-5}\lambda} B_i \, \partial_i w \right) \right] ,$$
$$\mathfrak{Q}_B w := \sum_{\lambda \geqslant 1} \left[\mathrm{P}_\lambda B_i \, \mathrm{P}_{< 2^5\lambda} \partial_i w + \mathrm{P}_{<2^5\lambda} \left(\mathrm{P}_\lambda B_i \, \partial_i w \right) \right] ,$$

where P_λ are inhomogeneous Littlewood-Paley frequency restriction operators, i.e. P_1 restricts to all low frequencies, and the sum above is taken over dyadic frequencies. We refer the reader to the next section for an explanation of the notations. We can then write

$$2A_x \cdot \nabla\phi = \mathfrak{P}_{A_x}\phi + \mathfrak{Q}_{A_x}\phi \, .$$

Heuristically, the term $\mathfrak{Q}_{A_x}\phi$ is well-behaved pertubatively. Indeed, because the derivative acts on a low frequency term in the term $\mathfrak{Q}_{A_x}\phi$, we expect to this term to obey better bounds than $\phi \nabla A_x$. Now, from the second equation in (1.3), we expect ∇A_x to have the regularity of $|\phi|^2$. Therefore, the term $\mathfrak{Q}_{A_x}\phi$ should be better behaved than the standard power nonlinearity $|\phi|^2\phi$, and in particular should be amenable to a perturbative treatment.

The term $\mathfrak{P}_{A_x}\phi$ is the truly non-perturbative part of the derivative nonlinearity $2A_x \cdot \nabla\phi$. Therefore, we retain it in our principal operator and rewrite the first equation of (1.3) as the quasilinear evolution equation,

$$\left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_{A_x} \right) \phi = -\mathfrak{Q}_{A_x}\phi - \mathrm{i}A_0\phi - \mathrm{i}|A_x|^2\phi - 2\mathrm{i}V'(|\phi|^2)\phi \, . \tag{2.1}$$

An essential feature of the present chapter, then, is that of understanding the principal operators of the form $(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)$. At the very least, we require that the homogeneous linear equation

$$\left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B \right) u = 0 \tag{2.2}$$

should be well-posed in Sobolev spaces, and the solutions should moreover satisfy appropriate dispersive estimates. To this end, we need to impose the conditions that $B \in L^\infty_t([0,T), L^\infty_x)$, $\partial_1 B_1 + \partial_2 B_2 = 0$ and $\nabla B \in L^1_t([0,T), L^\infty_x)$, and we shall call such time-dependent spatial 1-forms *admissible forms*. Note that the condition $\partial_1 B_1 + \partial_2 B_2 = 0$ formally guarantees that the evolution of (2.2) conserves the L^2_x norm. We show that, provided B is an admissible form, (2.2) can be uniquely solved in Sobolev spaces on the time interval $[0,T)$, and the solutions satisfy Strichartz estimates with a loss of derivatives.

In order to utilise this functional framework for solving the inhomogeneous equation

$$\left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B \right) u = f \tag{2.3}$$

in an appropriate Sobolev space H, we define the associated U^p and V^p spaces [31, 32, 18], namely $U^p_B H$ and $V^p_B H$, which are adapted to the principal operator $(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)$. This gives us a functional calculus for solving (2.3) in the spaces $U^2_B H$. The construction of our functional framework is accomplished in Section 2.3.

We can now apply our functional calculus to solve (1.3) using the following iteration scheme

$$\begin{cases} \left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_{A_x^{[n-1]}}\right)\phi^{[n]} = -\mathfrak{Q}_{A_x^{[n]}}\phi^{[n]} - \mathrm{i}A_0^{[n]}\phi^{[n]} - \mathrm{i}\left|A_x^{[n]}\right|^2\phi^{[n]} - 2\mathrm{i}V'\left(\left|\phi^{[n]}\right|^2\right)\phi^{[n]}\ , \\ -\triangle A_i^{[n]} = -\frac{1}{2}\epsilon_{ij}\partial_j\left(\left|\phi^{[n]}\right|^2\right)\ , \\ -\triangle A_0^{[n]} = -\mathrm{Im}\left(\nabla\overline{\phi^{[n]}}\wedge\nabla\phi^{[n]}\right) - \mathrm{rot}\left(A_x^{[n]}\left|\phi^{[n]}\right|^2\right)\ , \\ \phi^{[n]}(0) = \phi^{\mathrm{in}}\ , \end{cases} \tag{2.4}$$

which is initialised with $A_x^{[0]} = 0$. Our functional calculus now allows us to solve (2.4) at each iteration n, via a contraction mapping argument, in the function space $U^2_{A_x^{[n-1]}}H$ where H is chosen to be a generalised Sobolev space containing H^s. The key point is that every $A_x^{[n]}$ generated by this iterative scheme will be an admissible form whose size depends only on the size D of the initial datum ϕ^{in}. As a consequence, the existence time of (2.4) is bounded below independently of n, and the $L_t^\infty H$ norm of the iterates $\phi^{[n]}$ are also bounded above independently of n. These are accomplished in Section 2.4.

The convergence of the iteration scheme (2.4) is addressed in Section 2.5. We are able to obtain a weak Lipschitz bound between the iterates,

$$\left\|\phi^{[n+1]} - \phi^{[n]}\right\|_{L_t^\infty H^{s-1}} \leqslant \frac{1}{2}\left\|\phi^{[n]} - \phi^{[n-1]}\right\|_{L_t^\infty H^{s-1}}$$

which shows that the iterates $\{\phi^{[n]}\}_{n=1}^\infty$ converge in $L_t^\infty H^{s-1}$. On the other hand, the iterates $\{\phi^{[n]}\}_{n=1}^\infty$ are bounded uniformly in $L_t^\infty H^{\mathfrak{m}}$ for some generalised Sobolev space $H^{\mathfrak{m}}$, such that the embedding $H^s \hookrightarrow H^{\mathfrak{m}}$ is compact. Thus, by interpolation, the iterates $\{\phi^{[n]}\}_{n=1}^\infty$ converge in $L_t^\infty H^s$ as well, and it is straightforward to check that the limit is the desired solution to the system (1.3). The same arguments also prove the continuity of the solution map, and the weak Lipschitz bound between two solutions.

2.2 Notations and preliminaries

We fix $s \geqslant 1$, and recall that V has the form (1.2); in the event that $V = 0$, we set $d = 2$ for convenience. All constants in this chapter are allowed to depend on coefficients of the polynomial V, but, unless otherwise stated, not on any other parameters. If A and B are nonnegative quantities, we write $A \lesssim B$ if there is a constant C such that $A \leqslant CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

Throughout this chapter we will use the standard Lebesgue spaces $L_x^r := L^r(\mathbb{R}_x^2)$, mixed space-time Lebesgue spaces $L_t^qL_x^r$, and spaces $C_b([0,T), X)$ of continuous bounded functions where X is a Banach space of functions on $\mathbb{R}_x^2$. Very often in this chapter, the time interval is not taken to be all of $\mathbb{R}$, but rather a finite time interval $[0,T)$ for some $T > 0$. For ease of notation, we therefore denote $L_t^qL_x^r[T] := L_t^q([0,T), L_x^r)$ and $C_\mathrm{b}X[T] := C_\mathrm{b}([0,T), X)$.

2.2.1 Fourier analysis

We will occasionally require Fourier transforms over the spatial variables x, but never over the time variable t. Our convention for the Fourier transform will be

$$\widehat{u}(\xi) := \mathcal{F}u(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}x\cdot\xi} u(x)\,\mathrm{d}x \ .$$

We denote the Riesz transform by

$$\mathfrak{R}_i := \frac{\partial_i}{|\nabla|} \ .$$

It is a standard fact in harmonic analysis that the Riesz transforms are bounded linear maps $L^p(\mathbb{R}^2_x) \to L^p(\mathbb{R}^2_x)$ for every $p \in (1,\infty)$; see, for example, [14], Chapter 4.

We will very often make use of the Biot-Savart law,

$$\frac{\partial_i}{(-\triangle)} F(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^2} F(y)\,\mathrm{d}y \ .$$

This formula is amenable to the Hardy-Littlewood-Sobolev inequality for functions supported at low frequencies, when Bernstein's inequality does not directly apply due to the presence of the singular Fourier multiplier $|\mathrm{D}_x|^{-1}$.

We now recall the inhomogeneous Littlewood-Paley decomposition. Denote by

$$\mathfrak{D} := \left\{ 2^k \;\middle|\; k \in \mathbb{Z}_{\geqslant 0} \right\}$$

the set of all dyadic frequencies. Fix, once and for all, a smooth, radial, non-increasing function $\varphi_1 : \mathbb{R}^2_\xi \to \mathbb{R}$ such that $\varphi_1(\xi) \equiv 1$ on $|\xi| \leqslant 1$, and $\varphi_1(\xi) \equiv 0$ on $|\xi| \geqslant 2$. For $\lambda \in \mathfrak{D}, \lambda \geqslant 2$, set

$$\varphi_\lambda(\xi) := \varphi_1\left(\tfrac{1}{\lambda}\xi\right) - \varphi_1\left(\tfrac{2}{\lambda}\xi\right) \ .$$

For all $\lambda \in \mathfrak{D}$, we define $\mathrm{P}_\lambda := \varphi_\lambda(\mathrm{D}_x)$ the standard Littlewood-Paley restriction. Equivalently,

$$\mathrm{P}_\lambda u(x) = \int_{\mathbb{R}^2} \widecheck{\varphi_\lambda}(x-y) u(y)\,\mathrm{d}y \ .$$

Henceforth, we will reserve the letters λ, μ, ν for dyadic frequencies, i.e. elements of $\mathfrak{D}$. For example, when summing over λ, μ, ν, the summation is implicitly taken over all of $\mathfrak{D}$ unless otherwise stated.

Using the Littlewood-Paley decomposition, we define the inhomogeneous Besov spaces in the usual way:

$$\|u\|_{B^s_{p,r}} := \left(\sum_\lambda \lambda^{sr} \|\mathrm{P}_\lambda u\|^r_{L^p_x} \right)^{\frac{1}{r}}$$

with the obvious modification when $r = \infty$.

We define

$$\mathrm{P}_{\leqslant\lambda} := \sum_{\mu \leqslant \lambda} \mathrm{P}_\mu \ , \quad \mathrm{P}_{<\lambda} := \mathrm{P}_{\leqslant \frac{1}{2}\lambda} \ .$$

We will also, for ease of exposition, abuse notation in using the following operators

$$\mathrm{P}_{\ll\lambda} := \mathrm{P}_{\leqslant 2^{-m}\lambda} \ , \quad \mathrm{P}_{\lesssim\lambda} := \mathrm{P}_{\leqslant 2^m\lambda} \ , \quad \mathrm{P}_{\approx\lambda} := \mathrm{P}_{\lesssim\lambda} - \mathrm{P}_{\ll\lambda} \ ,$$

where m denotes fixed universal positive integers, whose values may change from line to line and can be appropriately chosen by the reader if so desired.

In this chapter, we will equip the Sobolev space H^σ with the equivalent Besov space norm,

$$\|w\|_{H^\sigma}^2 := \sum_\lambda \lambda^{2\sigma} \|\mathrm{P}_\lambda w\|_{L_x^2} .$$

These norms will be consistent with those of the following family of function spaces.

Definition 2.1. A *Sobolev weight* is a function $\mathfrak{m} : \mathfrak{D} \to (0,\infty)$ such that $\mathfrak{m}(1) = 1$, and there exist constants $c \leqslant 1$ and $C \geqslant 1$ such that

$$c \leqslant \frac{\mathfrak{m}(2\lambda)}{\mathfrak{m}(\lambda)} \leqslant C \quad \text{for all } \lambda \in \mathfrak{D} .$$

Given a Sobolev weight $\mathfrak{m}$, define the *generalised Sobolev space* $H^{\mathfrak{m}} \subset \mathcal{S}'(\mathbb{R}_x^2)$ to be the Hilbert space whose inner product is given by

$$(v,w)_{H^{\mathfrak{m}}} := \sum_\lambda \mathfrak{m}(\lambda)^2 \int_{\mathbb{R}^2} \mathrm{P}_\lambda v(x)\,\overline{\mathrm{P}_\lambda w(x)}\,\mathrm{d}x .$$

Moreover, for a Sobolev weight $\mathfrak{m}$, define the quantities $[\mathfrak{m}]_\star, [\mathfrak{m}]^\star, [\mathfrak{m}]$ by

$$[\mathfrak{m}]_\star := \inf_\lambda \log_2\left(\frac{\mathfrak{m}(2\lambda)}{\mathfrak{m}(\lambda)}\right) , \quad [\mathfrak{m}]^\star := \sup_\lambda \log_2\left(\frac{\mathfrak{m}(2\lambda)}{\mathfrak{m}(\lambda)}\right) ,$$
$$[\mathfrak{m}] := \max\left(-[\mathfrak{m}]_\star, [\mathfrak{m}]^\star\right) .$$

□

Remark 2.2. It follows directly from the definition that

$$\mathfrak{m}(\lambda) \leqslant 2^{k[\mathfrak{m}]}\mathfrak{m}(\mu) \quad \text{whenever } \left|\log_2\left(\frac{\lambda}{\mu}\right)\right| \leqslant k . \tag{2.5}$$

Lemma 2.3. *Let $\mathfrak{m}$ be a Sobolev weight. Let $(H^{\mathfrak{m}})^*$ be the dual space of $H^{\mathfrak{m}}$ extending the L_x^2 self-duality. Then $(H^{\mathfrak{m}})^*$ is isomorphic to $H^{\mathfrak{m}^{-1}}$ with equivalent norms. More precisely,*

$$C_1([\mathfrak{m}])\,\|v\|_{H^{\mathfrak{m}^{-1}}} \leqslant \|v\|_{(H^{\mathfrak{m}})^*} \leqslant C_2([\mathfrak{m}])\,\|v\|_{H^{\mathfrak{m}^{-1}}} \tag{2.6}$$

for some constants $C_2 \geqslant C_1 > 0$ depending only on $[\mathfrak{m}]$.

Proof. Given $v \in H^{\mathfrak{m}^{-1}}$, we may set

$$w := \sum_\mu \mathfrak{m}(\mu)^{-2}\mathrm{P}_\mu v .$$

Clearly $w \in H^{\mathfrak{m}}$, and from (2.5) we have

$$\|w\|_{H^{\mathfrak{m}}} \leqslant C([\mathfrak{m}])\,\|v\|_{H^{\mathfrak{m}^{-1}}} .$$

Therefore,

$$(w,v)_{L_x^2} = \sum_\mu \mathfrak{m}(\mu)^{-2}\left(\mathrm{P}_\mu v, v\right)_{L_x^2} \geqslant \sum_\mu \mathfrak{m}(\mu)^{-2}\|\mathrm{P}_\mu v\|_{L_x^2}^2 \geqslant C([\mathfrak{m}])\,\|v\|_{H^{\mathfrak{m}^{-1}}}\|w\|_{H^{\mathfrak{m}}} .$$

This verifies the first inequality in (2.6).

We turn to the second inequality in (2.6). Given $v \in H^{\mathfrak{m}^{-1}}$, $w \in H^{\mathfrak{m}}$, we have from the Cauchy-Schwarz inequality that

$$\left|(w,v)_{L_x^2}\right| \leqslant \sum_{\lambda} \left|(\mathrm{P}_\lambda w, \mathrm{P}_{\approx\lambda} v)_{L_x^2}\right| \leqslant \|w\|_{H^{\mathfrak{m}}} \left(\sum_{\lambda} \mathfrak{m}(\lambda)^{-2} \|\mathrm{P}_{\approx\lambda} v\|_{L_x^2}^2 \right)^{\frac{1}{2}} .$$

Since (2.5) gives us

$$\sum_{\lambda} \mathfrak{m}(\lambda)^{-2} \|\mathrm{P}_{\approx\lambda} v\|_{L_x^2}^2 \leqslant C\left([\mathfrak{m}]\right) \|v\|_{H^{\mathfrak{m}^{-1}}}^2 ,$$

we deduce the second inequality in (2.6). $\square$

2.2.2 Strichartz estimates

We now recall the well-known Strichartz estimates [25, 9] for the Schrödinger equation in two space dimensions.

Definition 2.4. We say $(q,r) \in [2,\infty]^2$ is a *Strichartz pair* if $\frac{2}{q} + \frac{2}{r} = 1$ and $r < \infty$.

Lemma 2.5 (Strichartz estimates)**.** *Suppose (q_1, r_1) and (q_2, r_2) are Strichartz pairs. Assume $u : [0,T) \times \mathbb{R}_x^2 \to \mathbb{C}$ is an L_x^2 solution to the Schrödinger equation,*

$$\left(\partial_t - \mathrm{i}\triangle\right) u = f .$$

Then the estimate

$$\|u\|_{L_t^{q_1} L_x^{r_1}[T]} \lesssim \|u(0)\|_{L_x^2} + \|f\|_{L_t^{q_2'} L_x^{r_2'}[T]}$$

holds with the implicit constant depending on q_1, q_2 but not on T. Here q_2', r_2' denote the Hölder conjugates of q_2, r_2 respectively, i.e. $1 = \frac{1}{q_2} + \frac{1}{q_2'} = \frac{1}{r_2} + \frac{1}{r_2'}$.

Proof. This is a well-known standard result; see, for example, [1], Section 8.2. $\square$

2.2.3 U^p and V^p spaces

As mentioned in Section 2.1, we need a functional framework built on spaces of the U^p and V^p type [31, 32, 18]. In this section, we recall the definitions and basic properties of these spaces, and refer the reader to [30] for a systematic exposition.

Definition 2.6. Let $T > 0$ and let X be a separable Banach space over $\mathbb{C}$. Let $p \in [1,\infty)$. We define a $U^p X[T]$ *atom* to be a function $a : [0,T) \to X$ of the form

$$a(t) = \sum_{k=0}^{K-1} \mathbb{1}_{[t_k, t_{k+1})}(t) a_k$$

where $K \in \mathbb{Z}_{>0}$, $0 = t_0 < t_1 < \ldots < t_K = T$, and $\sum_{k=0}^{K-1} \|a_k\|_X^p = 1$.

The Banach space $U^p X[T]$ is defined to be the atomic space over the $U^p X[T]$ atoms. More precisely, $U^p X[T]$ consists of all functions $a : [0,T) \to X$ admitting a representation

$$u = \sum_{j=1}^{\infty} c_j a_j , \quad a_j \text{ are } U^p X[T] \text{ atoms} , \quad \{c_j\}_{j=1}^{\infty} \in \ell^1 ,$$

equipped with the norm

$$\|u\|_{U^pX[T]} := \inf\left\{ \sum_{j=1}^{\infty} |c_j| \;\middle|\; u = \sum_{j=1}^{\infty} c_j a_j \ ,\ \{c_j\}_{j=1}^{\infty} \in \ell^1 \ ,\ a_j \text{ are } U^pX[T] \text{ atoms} \right\} \ .$$

We define $\mathrm{D}U^pX[T]$ to be the space of distributional derivatives of functions in $U^pX[T]$, equipped with the norm

$$\|f\|_{\mathrm{D}U^pX[T]} := \left\| \int_0^t f(t')\,\mathrm{d}t' \right\|_{U^pX[T]} \ . \qquad \square$$

Observe that, if $0 < T_1 < T_2$ then the restriction map

$$u \mapsto \mathbb{1}_{[0,T_1)}(t)u$$

is continuous linear $U^pX[T_2] \to U^pX[T_1]$ and satisfies

$$\|u\|_{U^pX[T_1]} := \left\|\mathbb{1}_{[0,T_1)}(t)u\right\|_{U^pX[T_1]} \leqslant \|u\|_{U^pX[T_2]} \ .$$

Definition 2.7. Let $T > 0$ and let X be a separable Banach space over $\mathbb{C}$. Let $p \in [1, \infty)$. We define $V^pX[T]$ to be the Banach space of functions $v : [0, T) \to X$ with the norm

$$\|v\|_{V^pX[T]} := \sup_{\mathfrak{t}} \left(\sum_{k=0}^{K-1} \|v(t_{k+1}) - v(t_k)\|_X^p \right)^{\frac{1}{p}}$$

where the supremum is taken over all partitions $\mathfrak{t} = \{t_k\}_{k=0}^{K}$ with $0 = t_0 < t_1 < \ldots < t_K = T$, and we define $v(T) := 0$.

Observe that a $V^pX[T]$ function possesses left and right limits at every $t \in [0, T)$. We define $V^p_{\mathrm{rc}}X[T]$ to be the closed subspace of $V^pX[T]$ of right-continuous functions $[0, T) \to X$. $\square$

We will require the following two crucial properties of the U^p and V^p spaces.

Lemma 2.8 (Embeddings)**.** *Let $T > 0$ and let X be a separable Banach space over $\mathbb{C}$. Let $1 \leqslant p < q < \infty$. Then we have the continuous embeddings*

$$U^pX[T] \hookrightarrow V^p_{\mathrm{rc}}X[T] \hookrightarrow U^qX[T] \hookrightarrow L^\infty_t X[T]$$

whose operator norms depend only on p, q and not on T or X.

Proof. This is proved in [18]. For the reader's convenience, we provide the proof in the appendix to this chapter. $\square$

Lemma 2.9 (Duality)**.** *Let $T > 0$ and let X be a separable Banach space over $\mathbb{C}$ such that X^* is also separable. Let $p \in (1, \infty)$ and let $p' := \frac{p}{p-1}$ be the Hölder conjugate of p. Then*

$$(\mathrm{D}U^pX[T])^* = V^{p'}_{\mathrm{rc}}X^*[T]$$

in the sense that, for $f \in \mathrm{D}U^pX[T]$,

$$\|f\|_{\mathrm{D}U^pX[T]} = \sup\left\{ \left| \int_0^T \langle v(t), f(t) \rangle_{X^*,X}\,\mathrm{d}t \right| \;\middle|\; v \in V^{p'}_{\mathrm{rc}}X^*[T], \|v\|_{V^{p'}_{\mathrm{rc}}X^*[T]} \leqslant 1 \right\} \ .$$

Proof. Again, the proof can be found in [18]. For the reader's convenience, we provide the proof in the appendix to this chapter. $\square$

2.3 The modified principal operator

The goal of this section is to establish the basic properties of solutions to the linear equation

$$(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)\, u = 0 \ , \tag{2.7}$$

and then use these properties to define function spaces for constructing the iterates in the iteration scheme (2.4). The hypotheses we require on B are summarised in the following definition.

Definition 2.10. Let $B = B_1(t,x)\mathrm{d}x_1 + B_2(t,x)\mathrm{d}x_2$ be a time-dependent spatial 1-form defined on $[0,1) \times \mathbb{R}^2_x$. We say that B is an *admissible form*, if $B \in L^\infty_t L^\infty_x[1]$, and $\nabla B \in L^1_t L^\infty_x[1]$, and

$$\partial_1 B_1 + \partial_2 B_2 \equiv 0 \ . \qquad \square$$

The first basic question is that of whether (2.7) gives a well-defined evolution in the generalised Sobolev spaces $H^{\mathfrak{m}}$. The following key Proposition will be proved in the Subsections 2.3.1 and 2.3.2.

Proposition 2.11. *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$ and $t_0 \in [0,T)$. Then, given $u^{\mathrm{in}} \in H^{\mathfrak{m}}$, there exists a unique solution $u \in L^\infty_t H^{\mathfrak{m}}[T]$ to (2.7) with $u(t_0) = u^{\mathrm{in}}$. Moreover, this solution satisfies*

$$\|u\|_{L^\infty_t H^{\mathfrak{m}}[T]} \leqslant C \mathrm{e}^{C_1 \|\nabla B\|_{L^1_t L^\infty_x[1]}} \left\|u^{\mathrm{in}}\right\|_{H^{\mathfrak{m}}} \tag{2.8}$$

where $C, C_1 > 0$ are constants depending only on $[\mathfrak{m}]$.

Remark 2.12. Eventually, when establishing the continuity of the solution map in Theorem 1.1, we will choose $\mathfrak{m}$ depending on the profile of the initial datum. In Proposition 2.11 and other results in this section, the fact that the various constants depend only on $[\mathfrak{m}]_\star$ and $[\mathfrak{m}]^\star$, and not on the finer details of $\mathfrak{m}$, will be crucial for the fact that the existence time in Theorem 1.1 depends only on the size of the initial datum, and not on its profile.

2.3.1 Proof of the uniqueness statement of Proposition 2.11

We first address the issue of uniqueness. Of course, if $H^{\mathfrak{m}}$ is a sufficiently regular space, say if $H^{\mathfrak{m}} = H^s$ for $s \geqslant 2$, then unconditional uniqueness in $L^\infty_t H^{\mathfrak{m}}[T]$ follows from a simple energy argument. For lower regularity $H^{\mathfrak{m}}$ spaces, we have to work harder.

Lemma 2.13. *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Then any solution $u \in L^\infty_t H^{\mathfrak{m}}[T]$ to (2.7) satisfies the differential inequality*

$$\partial_t \|\mathrm{P}_\mu u(t)\|_{L^2_x} \leqslant C \|\nabla B(t)\|_{L^\infty_x} \sum_{\lambda \,:\, \left|\log_2\left(\frac{\lambda}{\mu}\right)\right| \leqslant 5} \|\mathrm{P}_\lambda u(t)\|_{L^2_x} \tag{2.9}$$

where C is a universal constant independent of $\mathfrak{m}$.

Proof. Since u solves (2.7), we have

$$(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)\, \mathrm{P}_\mu u = (\mathfrak{P}_B \mathrm{P}_\mu - \mathrm{P}_\mu \mathfrak{P}_B)\, u \ . \tag{2.10}$$

Now, from the definition we have

$$\begin{aligned}(\mathfrak{P}_B \mathrm{P}_\mu - \mathrm{P}_\mu \mathfrak{P}_B)\, u &= \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \left[\mathrm{P}_{<2^{-5}\lambda} B_i\ \mathrm{P}_\lambda \mathrm{P}_\mu \partial_i u - \mathrm{P}_\mu \left(\mathrm{P}_{<2^{-5}\lambda} B_i\ \mathrm{P}_\lambda \partial_i u\right)\right] \\ &\quad + \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \mathrm{P}_\lambda \left[\mathrm{P}_{<2^{-5}\lambda} B_i\ \mathrm{P}_\mu \partial_i u - \mathrm{P}_\mu \left(\mathrm{P}_{<2^{-5}\lambda} B_i\ \partial_i u\right)\right] \\ &=: \mathrm{I} + \mathrm{II}\,.\end{aligned}$$

We claim the estimate

$$\|\mathrm{I}(t)\|_{L^2_x} \lesssim \|\nabla B(t)\|_{L^\infty_x} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \|\mathrm{P}_\lambda u(t)\|_{L^2_x}\ . \tag{2.11}$$

Indeed, recalling that $\partial_1 B_1 + \partial_2 B_2 = 0$, we have

$$\begin{aligned}&\left|\mathrm{P}_{<2^{-5}\lambda} B_i\ \mathrm{P}_\lambda \mathrm{P}_\mu \partial_i u - \mathrm{P}_\mu\big(\mathrm{P}_{<2^{-5}\lambda} B_i\ \mathrm{P}_\lambda \partial_i u\big)\right|(t,x) \\ &\quad = \left|\int_{\mathbb{R}^2} \widetilde{\varphi_\mu}(x-y)\left(\mathrm{P}_{<2^{-5}\lambda} B_i(t,x) - \mathrm{P}_{<2^{-5}\lambda} B_i(t,y)\right) \partial_i \mathrm{P}_\lambda u(t,y)\ \mathrm{d}y\right| \\ &\quad = \left|\int_{\mathbb{R}^2} \partial_i \widetilde{\varphi_\mu}(x-y)\left(\mathrm{P}_{<2^{-5}\lambda} B_i(t,x) - \mathrm{P}_{<2^{-5}\lambda} B_i(t,y)\right) \mathrm{P}_\lambda u(t,y)\ \mathrm{d}y\right| \\ &\quad \lesssim \|\nabla B(t)\|_{L^\infty_x} \int_{\mathbb{R}^2} |x-y|\, |\nabla \widetilde{\varphi_\mu}(x-y)|\, |\mathrm{P}_\lambda u(t,y)|\ \mathrm{d}y\,.\end{aligned}$$

Applying Young's convolution inequality, and noting that $\|x \nabla \widetilde{\varphi_\mu}\|_{L^1_x}$ is a constant independent of μ, we obtain

$$\left\|\mathrm{P}_{<2^{-5}\lambda} B_i(t)\ \mathrm{P}_\lambda \mathrm{P}_\mu \partial_i u(t) - \mathrm{P}_\mu\big(\mathrm{P}_{<2^{-5}\lambda} B_i(t)\ \mathrm{P}_\lambda \partial_i u(t)\big)\right\|_{L^2_x} \lesssim \|\nabla B(t)\|_{L^\infty_x} \|\mathrm{P}_\lambda u(t)\|_{L^2_x}\ .$$

Summing up over λ gives the desired estimate (2.11).

We can prove a similar estimate for $\mathrm{II}(t)$. Precisely, we have

$$\|\mathrm{II}(t)\|_{L^2_x} \lesssim \|\nabla B(t)\|_{L^\infty_x} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \|\mathrm{P}_\lambda u(t)\|_{L^2_x}\ . \tag{2.12}$$

Indeed, observe that only frequency components of u near μ will make a nonzero contribution to the sum defining $\mathrm{II}(t)$. Therefore, we have

$$\mathrm{II}(t) = \sum_{\nu\,:\,\left|\log_2\left(\frac{\nu}{\mu}\right)\right|\leqslant 5} \mathrm{P}_\nu \left(\sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \left[\mathrm{P}_{<2^{-5}\nu} B_i\ \mathrm{P}_\lambda \mathrm{P}_\mu \partial_i u - \mathrm{P}_\mu \left(\mathrm{P}_{<2^{-5}\nu} B_i\ \mathrm{P}_\lambda \partial_i u\right)\right]\right).$$

For each ν, the expression $\mathrm{P}_\nu(\cdots)$ above can be estimated in the exact same manner as our estimate of $\mathrm{I}(t)$. Then, since we sum only over finitely many ν, we obtain (2.12) as a result.

By combining the estimates (2.11), (2.12), we obtain

$$\|\left((\mathfrak{P}_B \mathrm{P}_\mu - \mathrm{P}_\mu \mathfrak{P}_B)\, u\right)(t)\|_{L^2_x} \lesssim \|\nabla B(t)\|_{L^\infty_x} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \|\mathrm{P}_\lambda u(t)\|_{L^2_x}\ .$$

Hence, multiplying (2.10) by $\overline{\mathrm{P}_\mu u}$ and integrating by parts, which is justified since the terms in (2.10) are smooth, we obtain

$$\partial_t \|\mathrm{P}_\mu u(t)\|^2_{L^2_x} \lesssim \|\nabla B(t)\|_{L^\infty_x} \|\mathrm{P}_\mu u(t)\|_{L^2_x} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \|\mathrm{P}_\lambda u(t)\|_{L^2_x}$$

which gives (2.9). □

Proof of the uniqueness statement of Proposition 2.11. By linearity, we shall only need to prove that any $L_t^\infty H^{\mathfrak{m}}[T]$ solution to (2.7) with initial datum $u(t_0) = 0$ must necessarily be zero.

Let $\varepsilon_0 = \varepsilon_0([\mathfrak{m}]) > 0$ be a small constant to be chosen later. Choose a sufficiently large positive integer K such that, for any interval $I \subset [0,T)$ of length $\leqslant 2TK^{-1}$, we have $\|\nabla B\|_{L_t^1(I,L_x^\infty)} \leqslant \varepsilon_0$. Write $[0,T)$ as the union of the $K-1$ overlapping small intervals $[kTK^{-1},(k+2)TK^{-1})$ with $0 \leqslant k \leqslant K-2$. Therefore it suffices to show, if J is one of these small intervals and there exists $t_J \in J$ such that $u(t_J) = 0$, then u is zero on J.

For $t \in J$, integrating (2.9) from t_J to t gives

$$\mathfrak{m}(\mu)\left\|\mathrm{P}_\mu u(t)\right\|_{L_x^2} \leqslant C([\mathfrak{m}]) \int_J \left\|\nabla B(t')\right\|_{L_x^\infty} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \mathfrak{m}(\lambda)\left\|\mathrm{P}_\lambda u(t')\right\|_{L_x^2} \mathrm{d}t'$$

where the constant $C([\mathfrak{m}])$ comes from (2.5). Squaring both sides and applying Cauchy-Schwarz, we obtain

$$\mathfrak{m}(\mu)^2\left\|\mathrm{P}_\mu u(t)\right\|_{L_x^2}^2 \leqslant C([\mathfrak{m}])\varepsilon_0 \int_J \left\|\nabla B(t')\right\|_{L_x^\infty} \sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5} \mathfrak{m}(\lambda)^2\left\|\mathrm{P}_\lambda u(t')\right\|_{L_x^2}^2 \mathrm{d}t' \ .$$

By summing over μ and taking the supremum over $t \in J$, we deduce

$$\|u\|_{L_t^\infty H^{\mathfrak{m}}[J]}^2 \leqslant C([\mathfrak{m}])\varepsilon_0^2\,\|u\|_{L_t^\infty H^{\mathfrak{m}}[J]}^2 \ .$$

Hence, if ε_0 were chosen small enough so that $C([\mathfrak{m}])\varepsilon_0^2 < 1$, then $\|u\|_{L_t^\infty H^{\mathfrak{m}}[J]} = 0$ as required. □

2.3.2 Proof of the existence statement of Proposition 2.11

We now turn our attention to the existence statement of Proposition 2.11. We first prove existence of solutions in the special case $H^{\mathfrak{m}} = L_x^2$. This is accomplished in Lemma 2.14 by extracting a weak-star limit of solutions to regularised equations, which is possible due to the condition $\partial_1 B_1 + \partial_2 B_2 = 0$.

Lemma 2.14. *Let B be an admissible form. Let $T \in (0,1]$ and $t_0 \in [0,t_0)$. Then, given $u^{\mathrm{in}} \in L_x^2$, there exists $u \in C_t L_x^2[T]$ solving (2.7) such that u is the unique $L_t^\infty L_x^2[T]$ weak-star limit of solutions to the regularised equations*

$$\begin{cases} (\partial_t - \mathrm{i}\triangle)\, u_\mu = \chi_\mu(\mathrm{D}_x)\mathfrak{P}_B u_\mu \ , \\ \qquad u_\mu(t_0) = \chi_\mu(\mathrm{D}_x)u^{\mathrm{in}} \end{cases} \tag{2.13}$$

as $\mathfrak{D} \ni \mu \to \infty$, where χ_μ is the indicator function of the ball of radius μ in $\mathbb{R}^2$. Furthermore,

$$\|u(t)\|_{L_x^2} = \left\|u^{\mathrm{in}}\right\|_{L_x^2} \quad \textit{for all } t \in [0,T) \ . \tag{2.14}$$

Proof. The proof is a standard application of the energy method. The point is, since $\partial_1 B_1 + \partial_2 B_2 = 0$, the operator $\mathfrak{P}_B$ is formally skew-symmetric on L_x^2, and so the evolution of $(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)$ conserves the L_x^2 norm. We provide the details for the sake of completeness.

For ease of exposition, we assume that $t_0 = 0$ and remark that the proof below immediately generalises to any other initial time in $[0,1)$.

For every $\mu \in \mathfrak{D}$, the right-hand side of the evolution equation in (2.13) is continuous linear on L^2_x with norm $\lesssim \mu\|B(t)\|_{L^\infty_x}$. Hence, (2.13) has a unique solution for given initial datum $u^{\mathrm{in}} \in L^2_x$. This solution has compact frequency support and is thus smooth. Therefore, we may multiply by $\overline{u_\mu}$ and integrate by parts to obtain $\partial_t \|u_\mu\|^2_{L^2_x} = 0$. We conclude $\|u_\mu\|_{L^\infty_t L^2_x[T]} \leqslant \|u^{\mathrm{in}}\|_{L^2_x}$.

By weak-star sequential compactness we may extract a subsequence $u_{\mu_k} \xrightarrow{\star} u$ in $L^\infty_t L^2_x[T]$. Then we have

$$\|u\|_{L^\infty_t L^2_x[T]} \leqslant \|u^{\mathrm{in}}\|_{L^2_x} \ .$$

In particular, by linearity, this limit is unique: If $\|u^{\mathrm{in}}\|_{L^2_x} = 0$ then $u = 0$.

Since $\mathfrak{P}_B$ is formally skew-symmetric, for any $v \in C_{\mathrm{b}}([0,T], H^1)$ we have

$$\begin{aligned}
\int_0^T (u(t), v(t))_{L^2_x}\,\mathrm{d}t &= \lim_{k\to\infty}\int_0^T (u_{\mu_k}(t), v(t))_{L^2_x}\,\mathrm{d}t \\
&= \lim_{k\to\infty}\int_0^T \left(\mathrm{e}^{\mathrm{i}t\triangle}u^{\mathrm{in}} + \int_0^t \mathrm{e}^{\mathrm{i}(t-t')\triangle}\mathfrak{P}_{B(t')}u_{\mu_k}(t')\,\mathrm{d}t'\ ,\ \chi_{\mu_k}(\mathrm{D}_x)v(t)\right)_{L^2_x}\mathrm{d}t \\
&= \lim_{k\to\infty}\left(u^{\mathrm{in}}\ ,\ \chi_{\mu_k}(\mathrm{D}_x)\int_0^T \mathrm{e}^{-\mathrm{i}t\triangle}v(t)\,\mathrm{d}t\right)_{L^2_x} \\
&\quad + \lim_{k\to\infty}\int_0^T \left(u_{\mu_k}(t')\ ,\ \mathfrak{P}_{B(t')}\int_{t'}^T \mathrm{e}^{-\mathrm{i}(t-t')\triangle}\chi_{\mu_k}(\mathrm{D}_x)v(t')\,\mathrm{d}t\right)_{L^2_x}\mathrm{d}t' \\
&= \left(u^{\mathrm{in}}, \int_0^T \mathrm{e}^{-\mathrm{i}t\triangle}v(t)\,\mathrm{d}t\right)_{L^2_x} \\
&\quad + \int_0^T \left(u(t'), \mathfrak{P}_{B(t')}\int_{t'}^T \mathrm{e}^{-\mathrm{i}(t-t')\triangle}v(t)\,\mathrm{d}t\right)_{L^2_x}\mathrm{d}t' \\
&= \int_0^T \left\langle \mathrm{e}^{\mathrm{i}t\triangle}u^{\mathrm{in}} + \int_0^t \mathrm{e}^{\mathrm{i}(t-t')\triangle}\mathfrak{P}_{B(t')}u(t')\,\mathrm{d}t'\ ,\ v(t)\right\rangle_{H^{-1},H^1}\mathrm{d}t\ .
\end{aligned}$$

This verifies that

$$u(t) = \mathrm{e}^{\mathrm{i}t\triangle}u^{\mathrm{in}} + \int_0^t \mathrm{e}^{\mathrm{i}(t-t')\triangle}\mathfrak{P}_{B(t')}u(t')\,\mathrm{d}t'$$

as Bochner integrals into H^{-1}. In particular, u solves (2.7) with initial datum u^{in}.

Now, we may also solve (2.13) backwards from any time in $[0,T)$. By applying the same argument above, we have

$$\|u(0)\|_{L^2_x} \leqslant \|u(t)\|_{L^2_x} \ .$$

This verifies (2.14).

Finally, as $\partial_t u \in L^\infty_t H^{-2}[T]$, we have $u \in C_{\mathrm{b}}H^{-2}[T]$. Since $\|u(t)\|_{L^2_x}$ is conserved and L^2_x is a uniformly convex space, we deduce that $u \in C_{\mathrm{b}}L^2_x[T]$. □

We must now upgrade our L^2_x existence result to other $H^{\mathfrak{m}}$ spaces. Given an initial datum u^{in}, it is natural to split u^{in} into its frequency components $\mathrm{P}_\nu u^{\mathrm{in}}$ and solve (2.7) to get an L^2_x solution u_ν with initial datum $u_\nu(t_0) = \mathrm{P}_\nu u^{\mathrm{in}}$ for each ν. Then, by linearity, an obvious candidate for the solution with initial datum u^{in} is $u = \sum_\nu u_\nu$. However, since the evolution of (2.7) does not preserve the frequency support, it is not immediately obvious that the sum $\sum_\nu u_\nu$ converges in $L^\infty_t H^{\mathfrak{m}}[T]$.

The fact that B is an admissible form will be sufficient to guarantee this convergence. The key idea is that initial datum, localised about a frequency scale ν, will launch a solution which, within a fixed time interval, transfers only a very small amount of mass to frequency scales vastly different from ν. The following lemma contains the precise, quantitative formulation of this idea.

Lemma 2.15. *Let B be any admissible form. Let $T \in (0,1]$ and let $t_0 \in [0,T)$. Let $\nu \in \mathfrak{D}$ and let v be a solution on $[0,T)$ to (2.7), whose initial data $v(t_0) \in L^2_x$ is frequency supported in $\{\frac{1}{2}\nu \leqslant |\xi| \leqslant 2\nu\}$. Then for $\ell \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\|\mathrm{P}_\mu v\|_{L^\infty_t L^2_x[T]} \leqslant \frac{\Big(C_0 \|\nabla B\|_{L^1_t L^\infty_x[1]}\Big)^\ell}{\ell!} \|v(t_0)\|_{L^2_x} \quad \textit{whenever } \left|\log_2\left(\frac{\mu}{\nu}\right)\right| \geqslant 5\ell \,. \tag{2.15}$$

Here $C_0 > 0$ is a universal constant independent of T, ν or ℓ.

Proof. For ease of exposition, we shall assume $t_0 = 0$ and remark that the proof for general t_0 is similar. Put $C_0 := 20C$ where C is the constant appearing in (2.9). It suffices to prove the stronger estimate

$$\begin{aligned}\|\mathrm{P}_\mu v(t)\|_{L^2_x} \leqslant C_0^\ell \int_0^t \int_0^{t_\ell} \cdots \int_0^{t_2} \prod_{m=1}^{\ell} \|\nabla B(t_m)\|_{L^\infty_x} \,\mathrm{d}t_1 \cdots \mathrm{d}t_\ell \, \|v(0)\|_{L^2_x} \\ \text{whenever } \left|\log_2\left(\frac{\mu}{\nu}\right)\right| \geqslant 5\ell \,,\end{aligned} \tag{2.16}$$

for $\ell \in \mathbb{Z}_{\geqslant 0}$, where, when $\ell = 0$, the integral is defined to be 1.

We establish (2.16) by induction on ℓ. The conservation of L^2_x norm, from Lemma 2.14, gives the base case $\ell = 0$. For $\ell \geqslant 1$, plugging the induction hypothesis for $\ell - 1$ into every summand on the right-hand side of (2.9), we obtain

$$\partial_t \|\mathrm{P}_\mu v(t)\|_{L^2_x} \leqslant C_0 \|\nabla B(t)\|_{L^\infty_x} C_0^{\ell-1} \int_0^t \cdots \int_0^{t_2} \prod_{m=1}^{\ell-1} \|\nabla B(t_m)\|_{L^\infty_x} \,\mathrm{d}t_1 \cdots \mathrm{d}t_{\ell-1} \|v(0)\|_{L^2_x} \,. \tag{2.17}$$

Since $|\log_2(\mu/\nu)| \geqslant 5\ell \geqslant 5$, we have by definition that $\mathrm{P}_\mu v(0) = 0$. Therefore, a direct integration of (2.17) yields

$$\|\mathrm{P}_\mu v(t)\|_{L^2_x} \leqslant C_0^\ell \int_0^t \int_0^{t_\ell} \cdots \int_0^{t_2} \prod_{m=1}^{\ell} \|\nabla B(t_m)\|_{L^\infty_x} \,\mathrm{d}t_1 \cdots \mathrm{d}t_\ell \, \|v(0)\|_{L^2_x}$$

which completes the induction step. □

Proof of the existence statement of Proposition 2.11 and of (2.8). Let $w \in H^{\mathfrak{m}}$ be given. Let u_ν be the solution of (2.7) with initial data $u_\nu(t_0) = \mathrm{P}_\nu w$. To complete the proof of Proposition 2.11, it suffices to prove

$$\left\| \Big\{ \mathfrak{m}(\mu) \|\mathrm{P}_\mu u_\nu(t)\|_{L^2_x} \Big\}_{\mu,\nu} \right\|_{\ell^2_\mu \ell^1_\nu} \leqslant C \mathrm{e}^{C_1 \|\nabla B\|_{L^1_t L^\infty_x[1]}} \|w\|_{H^{\mathfrak{m}}} \tag{2.18}$$

for C, C_1 as in the statement of Proposition 2.11. Indeed,

$$\sum_\mu \mathfrak{m}(\mu)^2 \left\| \mathrm{P}_\mu \sum_\nu u_\nu(t) \right\|^2_{L^2_x} \leqslant \left\| \Big\{ \mathfrak{m}(\mu) \|\mathrm{P}_\mu u_\nu(t)\|_{L^2_x} \Big\}_{\mu,\nu} \right\|^2_{\ell^2_\mu \ell^1_\nu}$$

which shows that the desired solution $u = \sum_\nu u_\nu$ belongs to $L^\infty_t H^{\mathfrak{m}}[T]$ and satisfies the claimed estimate (2.8).

Now, recall that from the definitions, we have

$$\mathfrak{m}(\mu) \leqslant 2^{5(\ell+1)[\mathfrak{m}]}\mathfrak{m}(\nu) \quad \text{whenever } 5\ell \leqslant \left|\log_2\left(\frac{\mu}{\nu}\right)\right| < 5(\ell+1)\ .$$

Therefore, using Lemma 2.15, we have

$$\mathfrak{m}(\mu)\sum_{\nu}\|\mathrm{P}_\mu u_\nu(t)\|_{L^2_x} \leqslant 2^{5[\mathfrak{m}]}\sum_{\ell=0}^{\infty}\sum_{\nu\,:\,5\ell\leqslant\left|\log_2\left(\frac{\nu}{\mu}\right)\right|<5(\ell+1)}\frac{\left(C_0\,\|\nabla B\|_{L^1_tL^\infty_x[1]}\right)^\ell}{\ell!}2^{5\ell[\mathfrak{m}]}\mathfrak{m}(\nu)\,\|\mathrm{P}_\nu w\|_{L^2_x}\ .$$

We set $C_1 = C_1([\mathfrak{m}]) := C_0 2^{5[\mathfrak{m}]}$ once and for all. Then, by Cauchy-Schwarz,

$$\begin{aligned}&\left(\mathfrak{m}(\mu)\sum_{\nu}\|\mathrm{P}_\mu u_\nu(t)\|_{L^2_x}\right)^2\\ &\qquad\leqslant C([\mathfrak{m}])\mathrm{e}^{C_1\|\nabla B\|_{L^1_tL^\infty_x[1]}}\sum_{\ell=0}^{\infty}\left(\sum_{\nu\,:\,5\ell\leqslant\left|\log_2\left(\frac{\nu}{\mu}\right)\right|<5(\ell+1)}\frac{\left(C_1\,\|\nabla B\|_{L^1_tL^\infty_x[1]}\right)^\ell}{\ell!}\mathfrak{m}(\nu)^2\,\|\mathrm{P}_\nu w\|^2_{L^2_x}\right)\ .\end{aligned}$$

Summing over μ then gives (2.18). $\square$

2.3.3 Local-in-time Strichartz estimates

Having proved Proposition 2.11 in the preceding two sections, we now show that the corresponding solutions enjoy local-in-time Strichartz estimates with loss of derivatives.

Proposition 2.16. *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$ and $t_0 \in [0,T)$, and let $w \in H^{\mathfrak{m}}$. Let u be the solution to (2.7) with initial data $u(t_0) = w$. Let (q,r) be a Strichartz pair. Then the estimate*

$$\|\mathrm{P}_\mu u\|_{L^q_tL^r_x[T]} \leqslant C\left(1+\|B\|_{L^\infty_tL^\infty_x[1]}\right)\mathrm{e}^{C_1\|\nabla B\|_{L^1_tL^\infty_x[1]}}\mu^{\frac{1}{q}}\mathfrak{m}(\mu)^{-1}\,\|w\|_{H^{\mathfrak{m}}} \tag{2.19}$$

holds for some constants $C = C([\mathfrak{m}],q) > 0$ and $C_1 = C_1([\mathfrak{m}]) > 0$.

Proof. Following the strategy of [7], we divide $[0,T)$ into disjoint intervals each of length $\leqslant \mu^{-1}$, so that there are $\leqslant \mu$ such intervals. Consider one such interval $J = [t_1,t_2)$. Applying the usual Strichartz estimate to

$$(\partial_t - \mathrm{i}\triangle)\,\mathrm{P}_\mu u = -\mathrm{P}_\mu\mathfrak{P}_B u$$

over the interval J, we obtain

$$\begin{aligned}\|\mathrm{P}_\mu u\|_{L^q_t(J,L^r_x)} &\lesssim \|\mathrm{P}_\mu u(t_1)\|_{L^2_x} + |J|\mu\,\|\mathrm{P}_\mu\mathfrak{P}_B u\|_{L^\infty_t(J,L^2_x)}\\ &\lesssim \|\mathrm{P}_\mu u(t_1)\|_{L^2_x} + \|B\|_{L^\infty_tL^\infty_x[1]}\sum_{\lambda\,:\,\left|\log_2\left(\frac{\lambda}{\mu}\right)\right|\leqslant 5}\|\mathrm{P}_\lambda u\|_{L^\infty_t(J,L^2_x)}\ .\end{aligned}$$

Using (2.8) to bound the right-hand side, we obtain

$$\|\mathrm{P}_\mu u\|_{L^q_t(J,L^r_x)} \leqslant C\left([\mathfrak{m}],q\right)\left(1+\|B\|_{L^\infty_tL^\infty_x[1]}\right)\mathrm{e}^{C_1([\mathfrak{m}])\|\nabla B\|_{L^1_tL^\infty_x[1]}}\mathfrak{m}(\mu)^{-1}\,\|w\|_{H^{\mathfrak{m}}}\ . \tag{2.20}$$

Note that the right-hand side of (2.20) is now independent of J. Hence, raising (2.20) to the q-th power and summing over the intervals J, and recalling that there are $\leqslant \mu$ such intervals, we obtain (2.19). $\square$

2.3.4 Adapted U^p and V^p spaces

Having now established the basic properties of solutions to the linear homogeneous equation (2.7), we define the function spaces which we will use to construct the iteration scheme (2.4).

Notation 2.17. Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. For $t, t_0 \in [0,1)$, denote

$$\mathfrak{S}_B(t,t_0)w := U(t)$$

where U solves (2.7) on $[0,1)$ with initial data $U(t_0) = w \in H^{\mathfrak{m}}$.

Definition 2.18. Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$.

Let $p \in [1,\infty)$. Define $U^p_B H^{\mathfrak{m}}[T]$ to be the Banach space of functions $u : [0,T) \to H^{\mathfrak{m}}$ such that $\mathfrak{S}_B(0,t)u(t)$ belongs to $U^p H^{\mathfrak{m}}[T]$. The $U^p_B H^{\mathfrak{m}}[T]$ norm is given by

$$\|u\|_{U^p_B H^{\mathfrak{m}}[T]} := \|\mathfrak{S}_B(0,t)u(t)\|_{U^p H^{\mathfrak{m}}[T]} \ .$$

Define $\mathrm{D}U^p_B H^{\mathfrak{m}}[T]$ to consist of functions $f : [0,T) \times \mathbb{R}^2_x \to \mathbb{C}$ such that $\mathfrak{S}_B(0,t)f(t) \in \mathrm{D}U^p H^{\mathfrak{m}}[T]$, equipped with the norm

$$\|f\|_{\mathrm{D}U^p_B H^{\mathfrak{m}}[T]} := \|\mathfrak{S}_B(0,t)f(t)\|_{\mathrm{D}U^p H^{\mathfrak{m}}[T]} = \left\| \int_0^t \mathfrak{S}_B(0,t')f(t')\,\mathrm{d}t' \right\|_{U^p H^{\mathfrak{m}}[T]} .$$

Lastly, define $V^p_B H^{\mathfrak{m}}[T]$ to be the Banach space of functions $v : [0,T) \to H^{\mathfrak{m}}$ such that $\mathfrak{S}_B(0,t)v(t)$ belongs to $V^p_{\mathrm{rc}} H^{\mathfrak{m}}[T]$, equipped with the norm

$$\|v\|_{V^p_B H^{\mathfrak{m}}[T]} := \|\mathfrak{S}_B(0,t)v(t)\|_{V^p H^{\mathfrak{m}}[T]} \ . \qquad \square$$

As a first consequence of the definitions, of the uniqueness statement in Proposition 2.11, and of Duhamel's formula, we have the following result.

Lemma 2.19. *Let B be an admissible form and $\mathfrak{m}, \mathfrak{n}$ be Sobolev weights with $\mathfrak{n} \leqslant \mathfrak{m}$, so that $H^{\mathfrak{m}} \hookrightarrow H^{\mathfrak{n}}$. Let $T \in (0,1]$ and $p \in [1,\infty)$.*

Suppose $u \in L^\infty_t H^{\mathfrak{n}}[T]$ and $u(0) = u^{\mathrm{in}} \in H^{\mathfrak{m}}$ and

$$(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B)\, u = f$$

with $f \in \mathrm{D}U^p_B H^{\mathfrak{m}}[T]$. Then, in fact, u must be given by

$$u(t) = \mathfrak{S}_B(t,0)u^{\mathrm{in}} + \int_0^t \mathfrak{S}_B(t,t')f(t')\,\mathrm{d}t' \ , \tag{2.21}$$

and in particular, $u \in U^p_B H^{\mathfrak{m}}[T]$ and

$$\|u\|_{U^p_B H^{\mathfrak{m}}[T]} \lesssim \left\|u^{\mathrm{in}}\right\|_{H^{\mathfrak{m}}} + \|f\|_{\mathrm{D}U^p_B H^{\mathfrak{m}}[T]} \ . \tag{2.22}$$

Proof. Let v be given by the right-hand side of (2.21). Clearly, $v \in U^p_B H^{\mathfrak{m}}[T]$ and satisfies

$$\|v\|_{U^p_B H^{\mathfrak{m}}[T]} \lesssim \left\|u^{\mathrm{in}}\right\|_{H^{\mathfrak{m}}} + \|f\|_{\mathrm{D}U^p_B H^{\mathfrak{m}}[T]} \ .$$

Now, by Proposition 2.11 and the atomic structure of $U^p_B H^{\mathfrak{m}}[T]$, we have $U^p_B H^{\mathfrak{m}}[T] \hookrightarrow L^\infty_t H^{\mathfrak{m}}[T]$. Thus, $u - v \in L^\infty_t H^{\mathfrak{n}}[T]$. But $u - v$ is a solution to (2.7) with $(u-v)(0) = 0$. Hence, by the uniqueness statement in Proposition 2.11, we have $u - v = 0$. $\square$

Observe that Lemma 2.8 generalises immediately to the above function spaces. More precisely, we have the following embedding result.

Lemma 2.20 (Embeddings)**.** *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$ and let $1 \leqslant p < q < \infty$. Then we have the continuous embeddings*

$$U_B^p H^{\mathfrak{m}}[T] \hookrightarrow V_B^p H^{\mathfrak{m}}[T] \hookrightarrow U_B^q H^{\mathfrak{m}}[T]$$

whose operator norms depend on p, q and not on T or B or $\mathfrak{m}$.

To use the Duhamel formula in Lemma 2.19, we will need to estimate the $\mathrm{D}U_B^p H^{\mathfrak{m}}[T]$ norm of the various nonlinearities we encounter. Such estimates can be efficiently obtained using the following duality result, which is the obvious generalisation of Lemma 2.9.

Lemma 2.21 (Duality)**.** *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$ and $p \in (1,\infty)$, and let $p' := \frac{p}{p-1}$. Then*

$$(\mathrm{D}U_B^p H^m[T])^* = V_B^{p'} H^{\mathfrak{m}^{-1}}[T]$$

in the sense that

$$\|f\|_{\mathrm{D}U^p H^{\mathfrak{m}}[T]} \leqslant C\,(p, [\mathfrak{m}]) \sup_v \left| \int_0^T \int_{\mathbb{R}_x^2} \overline{v(t,x)}\, f(t,x)\, \mathrm{d}x \right|$$

where the supremum is taken over all $v \in V_B^{p'} H^{\mathfrak{m}^{-1}}[T]$ with $\|v\|_{V_B^{p'} H^{\mathfrak{m}^{-1}}[T]} \leqslant 1$.

Proof. From Lemma 2.14 we have that $\mathfrak{S}_B(t_1, t_0)$ are unitary maps on L_x^2. Additionally, Lemma 2.3 guarantees that $H^{\mathfrak{m}^{-1}}$ and $(H^{\mathfrak{m}})^*$ are isomorphic with equivalent norms. Hence, Lemma 2.21 follows immediately from Lemma 2.9. $\square$

Our next Lemma shows that generalises the energy and Strichartz estimates, established earlier for free solutions to (2.7), to arbitrary $U_B^p H^{\mathfrak{m}}[T]$ functions.

Lemma 2.22. *Let B be an admissible form and $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$ and $p \in [1,\infty)$, and let (q,r) be a Strichartz pair. Then we have the estimates*

$$\|u\|_{L_t^\infty H^{\mathfrak{m}}[T]} \leqslant C\,([\mathfrak{m}])\, \mathrm{e}^{C_1([\mathfrak{m}])\|\nabla B\|_{L_t^1 L_x^\infty[1]}} \|u\|_{U_B^p H^{\mathfrak{m}}[T]} \tag{2.23}$$

and

$$\|\mathrm{P}_\mu u\|_{L_t^q L_x^r[T]} \leqslant C\,([\mathfrak{m}], q) \left(1 + \|B\|_{L_t^\infty L_x^\infty[1]}\right) \mathrm{e}^{C_1([\mathfrak{m}])\|\nabla B\|_{L_t^1 L_x^\infty[1]}} \mu^{\frac{1}{q}} \mathfrak{m}(\mu)^{-1} \|u\|_{U_B^q H^{\mathfrak{m}}[T]} \tag{2.24}$$

Proof. Due to the atomic structure of the $U_B^p H^{\mathfrak{m}}[T]$ spaces, the asserted estimates are immediate consequences of Propositions 2.11 and 2.16. $\square$

With the above machinery, the following result, which lets us compare U^p spaces associated to different admissible forms, is now straightforward.

Proposition 2.23. *Let $\mathfrak{m}$ be a Sobolev weight and B, Γ be admissible forms. Let $T \in (0,1]$. Let $p \in (1,\infty)$. Then we have the embedding $U^p_B H^{\mathfrak{m}}[T] \hookrightarrow U^p_\Gamma H^{\lambda^{-1}\mathfrak{m}}[T]$ with*

$$\|u\|_{U^p_\Gamma H^{\lambda^{-1}\mathfrak{m}}[T]} \leqslant C([\mathfrak{m}]) \mathrm{e}^{C_1([\mathfrak{m}])\left(\|\nabla B\|_{L^1_t L^\infty_x[1]} + \|\nabla \Gamma\|_{L^1_t L^\infty_x[1]}\right)} T \, \|B - \Gamma\|_{L^\infty_t L^\infty_x[T]} \|u\|_{U^p_B H^{\mathfrak{m}}[T]} \ .$$

Proof. Suppose first that $u = \mathfrak{S}_B(t, t_0) w$ is a free solution on $[0, T)$ to (2.7) with $w \in H^{\mathfrak{m}}$, so that

$$(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_\Gamma)\, u = \mathfrak{P}_{\Gamma - B} u \ . \tag{2.25}$$

Now, observe that $\|\mathfrak{P}_b w\|_{H^{\lambda^{-1}\mathfrak{m}}[T]} \leqslant C([\mathfrak{m}]) \|b\|_{L^\infty_x} \|w\|_{H^{\mathfrak{m}}[T]}$. Therefore, for $v \in V^{p'}_\Gamma H^{\lambda \mathfrak{m}^{-1}}[T]$, we have

$$\left| \int_0^T \int_{\mathbb{R}^2_x} \overline{v}\, \mathfrak{P}_{\Gamma - B} u \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant C([\mathfrak{m}]) T \, \|v\|_{L^\infty_t H^{\lambda \mathfrak{m}^{-1}}[T]} \|B - \Gamma\|_{L^\infty_t L^\infty_x[T]} \|u\|_{L^\infty_t H^{\mathfrak{m}}[T]} \ .$$

Using Lemma 2.22, we obtain

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^2_x} \overline{v}\, \mathfrak{P}_{\Gamma - B} u \, \mathrm{d}x \, \mathrm{d}t \right| &\leqslant C([\mathfrak{m}]) \mathrm{e}^{C_1([\mathfrak{m}])\left(\|\nabla B\|_{L^1_t L^\infty_x[1]} + \|\nabla \Gamma\|_{L^1_t L^\infty_x[1]}\right)} \\ &\quad \cdot T \, \|v\|_{V^{p'}_\Gamma H^{\lambda \mathfrak{m}^{-1}}[T]} \|B - \Gamma\|_{L^\infty_t L^\infty_x[1]} \|w\|_{H^{\mathfrak{m}}} \ . \end{aligned}$$

Thus, by the duality principle of Lemma 2.21,

$$\|\mathfrak{P}_{\Gamma - B} u\|_{\mathrm{D}U^p_B H^{\lambda^{-1}\mathfrak{m}}[T]} \leqslant C([\mathfrak{m}]) \mathrm{e}^{C_1([\mathfrak{m}])\left(\|\nabla B\|_{L^1_t L^\infty_x[1]} + \|\nabla \Gamma\|_{L^1_t L^\infty_x[1]}\right)} T \, \|B - \Gamma\|_{L^\infty_t L^\infty_x[1]} \|w\|_{H^{\mathfrak{m}}} \ .$$

Plugging into the Duhamel formula in Lemma 2.19, we find

$$\|u\|_{U^p_B H^{\lambda^{-1}\mathfrak{m}}[T]} \leqslant C([\mathfrak{m}]) \mathrm{e}^{C_1([\mathfrak{m}])\left(\|\nabla B\|_{L^1_t L^\infty_x[1]} + \|\nabla \Gamma\|_{L^1_t L^\infty_x[1]}\right)} T \, \|B - \Gamma\|_{L^\infty_t L^\infty_x[1]} \|w\|_{H^{\mathfrak{m}}} \ .$$

This proves Proposition 2.23 in the special case when u is a free solution to (2.7).

In particular, if now u is a $U^p_B H^{\mathfrak{m}}[T]$ atom, then

$$\|u\|_{U^p_B H^{\lambda^{-1}\mathfrak{m}}[T]} \leqslant C([\mathfrak{m}]) \mathrm{e}^{C_1([\mathfrak{m}])\left(\|\nabla B\|_{L^1_t L^\infty_x[1]} + \|\nabla \Gamma\|_{L^1_t L^\infty_x[1]}\right)} T \, \|B - \Gamma\|_{L^\infty_t L^\infty_x[1]} \ .$$

The assertion of Proposition 2.23 now follows from the atomic structure of $U^p_B H^{\mathfrak{m}}[T]$. □

2.4 Construction of the iteration scheme

The goal of the present section is to set up the iteration scheme (2.4), and show that the iterates $\phi^{[n]}$ exist on a common time interval $T = T(\|\phi^{\mathrm{in}}\|_{H^s})$. The convergence of the iteration scheme to a solution of the Chern-Simons-Schrödinger system in the Coulomb gauge, (1.3), will be addressed in the next section.

2.4.1 Setting up the iteration scheme

For convenience, we introduce the following notation. Define the bilinear maps $\mathcal{N}_t^2, \mathcal{N}_1^2, \mathcal{N}_2^2$ and the quadrilinear maps $\mathcal{N}_t^4, \mathcal{N}_x^4$ on $\mathcal{S}(\mathbb{R}_x^2, \mathbb{C})$, by

$$\begin{aligned}
\mathcal{N}_i^2[u_1, u_2] &:= \epsilon_{ij} \frac{\partial_j}{(-\triangle)} (u_1 u_2) \ , \\
\mathcal{N}_t^2[u_1, u_2] &:= (-\triangle)^{-1} (\nabla u_1 \wedge \nabla u_2) \ , \\
\mathcal{N}_t^4[u_1, u_2, u_3, u_4] &:= \frac{\mathrm{rot}}{(-\triangle)} \left(\mathcal{N}_x^2[u_1, u_2] u_3 u_4\right) \ , \\
\mathcal{N}_x^4[u_1, u_2, u_3, u_4] &:= -\mathrm{i}\mathcal{N}_x^2[u_1, u_2] \cdot \mathcal{N}_x^2[u_3, u_4] \ ,
\end{aligned}$$

where we have also denoted $\mathcal{N}_x^2 := (\mathcal{N}_1^2, \mathcal{N}_2^2)$. We warn the reader that, despite the similar notation, $\mathcal{N}_x^2$ is $\mathbb{C}^2$-valued while $\mathcal{N}_x^4$ is $\mathbb{C}$-valued. Define also the trilinear map

$$\mathcal{Q}[u_1, u_2, u_3] := \sum_\lambda \left[\mathrm{P}_\lambda \mathcal{N}_x^2[u_1, u_2] \cdot \nabla \mathrm{P}_{<2^5\lambda} u_3 + \mathrm{P}_{<2^5\lambda}\left(\mathrm{P}_\lambda \mathcal{N}_x^2[u_1, u_2] \cdot \nabla u_3\right)\right] \ ,$$

so that, in the notation of the Introduction, $\mathcal{Q}[\overline{\phi}, \phi, \phi] = \mathfrak{Q}_{A_x}\phi$ for a solution ϕ to (1.3).

Then the Chern-Simons-Schrödinger system in the Coulomb gauge, (1.3), can be written as

$$\begin{cases}
(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_{A_x})\,\phi = \mathcal{Q}\left[\overline{\phi}, \phi, \phi\right] + \mathcal{N}_t^2\left[\overline{\phi}, \phi\right]\phi + \mathcal{N}_t^4\left[\overline{\phi}, \phi, \overline{\phi}, \phi\right]\phi \\
\qquad\qquad + \mathcal{N}_x^4\left[\overline{\phi}, \phi, \overline{\phi}, \phi\right]\phi - 2\mathrm{i}V'\left(|\phi|^2\right)\phi \ , \\
A_x = -\frac{1}{2}\mathcal{N}_x^2\left[\overline{\phi}, \phi\right] \ .
\end{cases} \tag{2.26}$$

Similarly, the iteration scheme (2.4) can be written succinctly as

$$\begin{cases}
\left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_{A_x^{[n-1]}}\right)\phi^{[n]} = \mathcal{Q}\left[\overline{\phi^{[n]}}, \phi^{[n]}, \phi^{[n]}\right] + \mathcal{N}_t^2\left[\overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} \\
\qquad\qquad + \mathcal{N}_t^4\left[\overline{\phi^{[n]}}, \phi^{[n]}, \overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} \\
\qquad\qquad + \mathcal{N}_x^4\left[\overline{\phi^{[n]}}, \phi^{[n]}, \overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} - 2\mathrm{i}V'\left(\left|\phi^{[n]}\right|^2\right)\phi^{[n]} \ , \\
A_x^{[n]} = -\frac{1}{2}\mathcal{N}_x^2\left[\overline{\phi^{[n]}}, \phi^{[n]}\right] \ , \\
\phi^{[n]}(0) = \phi^{\mathrm{in}} \ .
\end{cases} \tag{2.27}$$

We record the following easy estimate, which will play a key role in formulating the existence result for the iteration scheme, Theorem 2.25.

Lemma 2.24. *We have the estimate*

$$\left\|\mathcal{N}_x^2[u_1, u_2]\right\|_{L_x^\infty} \leqslant C\,\|u_1\|_{H^1}\,\|u_2\|_{H^1} \ . \tag{2.28}$$

Proof. By Bernstein's inequality, it suffices to prove the stronger estimate

$$\left\|\mathcal{N}_x^2[u_1, u_2]\right\|_{B^1_{4,\infty}} \leqslant C\,\|u_1\|_{H^1}\,\|u_2\|_{H^1} \ . \tag{2.29}$$

By Hardy-Littlewood-Sobolev,

$$\left\|\mathrm{P}_\mu \mathcal{N}_x^2[\mathrm{P}_\lambda u_1, \mathrm{P}_{\leqslant\lambda} u_2]\right\|_{L_x^4} \leqslant C\,\|\mathrm{P}_\lambda u_1\,\mathrm{P}_{\leqslant\lambda} u_2\|_{L_x^{\frac{4}{3}}} \leqslant C\lambda^{-1}\,\|u_1\|_{H^1}\,\|u_2\|_{H^1} \ .$$

Summing up over $\lambda \gtrsim \mu$, and noting that $\mathcal{N}_x^2[u_1, u_2]$ is symmetric in u_1, u_2, we obtain

$$\left\|\mathrm{P}_\mu \mathcal{N}_x^2\left[u_1, u_2\right]\right\|_{L_x^4} \leqslant C\mu^{-1}\left\|u_1\right\|_{H^1}\left\|u_2\right\|_{H^1}$$

which is (2.29). □

2.4.2 Statement of the existence result

We will construct the iterates to (2.27) by solving the more general initial value problem

$$\begin{cases} \left(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_B\right)\psi = \mathcal{Q}\left[\overline{\psi}, \psi, \psi\right] + \mathcal{N}_t^2\left[\overline{\psi}, \psi\right]\psi + \mathcal{N}_t^4\left[\overline{\psi}, \psi, \overline{\psi}, \psi\right]\psi \\ \qquad\qquad + \mathcal{N}_x^4\left[\overline{\psi}, \psi, \overline{\psi}, \psi\right]\psi - 2\mathrm{i}V'\left(|\psi|^2\right)\psi\,, \\ \psi(0) = \psi^{\mathrm{in}} \in H^{\mathfrak{m}}\,. \end{cases} \tag{2.30}$$

Recall that $s \geqslant 1$ is fixed and d is the degree of V. We impose the following hypotheses.

(I) $\mathfrak{m}$ is a Sobolev weight satisfying

$$s \leqslant [\mathfrak{m}]_\star \leqslant [\mathfrak{m}]^\star \leqslant s + \frac{1}{8}\,. \tag{2.31}$$

Note that, in particular, this implies

$$\lambda^s \leqslant \mathfrak{m}(\lambda) \leqslant \lambda^{s+\frac{1}{8}}\,,$$

and more generally

$$\left(\frac{\lambda}{\mu}\right)^s \leqslant \frac{\mathfrak{m}(\lambda)}{\mathfrak{m}(\mu)} \leqslant \left(\frac{\lambda}{\mu}\right)^{s+\frac{1}{8}} \qquad \text{whenever } \lambda \geqslant \mu\,.$$

(II) B is an admissible form which satisfies

$$\|\nabla B\|_{L_t^1 L_x^\infty[1]} \leqslant 1\,. \tag{2.32}$$

Under these hypotheses, Lemma 2.24 and (2.23) guarantee the existence of a constant $K > 0$, which we fix once and for all, such that

$$\left\|\mathcal{N}_x^2\left[\psi_1, \psi_2\right]\right\|_{L_t^\infty L_x^\infty[T]} \leqslant \frac{K}{2}\left\|\psi_1\right\|_{U_B^2 H^1[T]}\left\|\psi_2\right\|_{U_B^2 H^1[T]}\,. \tag{2.33}$$

The main result of this section is that the iterates to the iteration scheme (2.27) can be constructed, and they satisfy certain useful bounds. More precisely, we have the following.

Theorem 2.25. *Assume the hypotheses (I), (II) above. Let $M > 0$ and let $\psi^{\mathrm{in}} \in H^{\mathfrak{m}}$ with $\|\psi^{\mathrm{in}}\|_{H^{\mathfrak{m}}} \leqslant M$. Suppose additionally that*

$$\|B\|_{L_t^\infty L_x^\infty[1]} \leqslant KM^2 \tag{2.34}$$

where K is the constant appearing in (2.33).

Then, for sufficiently small $T = T(s, M) \leqslant 1$, there exists a unique solution $\psi \in U_B^2 H^{\mathfrak{m}}[T]$ to the initial value problem (2.30). This solution satisfies

$$\|\psi\|_{U_B^2 H^{\mathfrak{m}}[T]} \leqslant 2M\,.$$

Moreover, letting Γ be the extension by zero of $-\frac{1}{2}\mathcal{N}_x^2[\overline{\psi}, \psi]$ from $[0, T)$ to $[0, 1)$, we have that Γ is an admissible form which also verifies hypothesis (II) and (2.34).

The basic idea of the proof of Theorem 2.25 is to choose T so that an appropriate contraction map can be set up in the same

$$\mathfrak{E}_{M,T} := \left\{ \psi \in U_B^2 H^{\mathfrak{m}}[T] \,\middle|\, \|\psi\|_{U_B^2 H^{\mathfrak{m}}[T]} \leqslant 2M \right\} .$$

The task of proving Theorem 2.25 thus reduces to establishing multilinear estimates for each nonlinearity on the right-hand side of (2.30).

2.4.3 Preliminary bounds

In proving our multilinear estimates we will heavily rely on the estimates in Lemma 2.22. Due to our hypotheses (I) and (II), and also because of (2.34), the estimates provided by Lemma 4.13 simplify considerably. For ease of exposition we will re-state these estimates here.

Definition 2.26. Let $\mathfrak{m}$ be a Sobolev weight. Let $T \in (0,1]$. We define the seminorm $\|\cdot\|_{F^{\mathfrak{m}}[T]}$ on functions $\psi : [0,T) \times \mathbb{R}_x^2 \to \mathbb{C}$ by

$$\|\psi\|_{F^{\mathfrak{m}}[T]} := \|\psi\|_{L_t^\infty H^{\mathfrak{m}}[T]} + \sup_\mu \mu^{-\frac{1}{4}} \mathfrak{m}(\mu) \|\mathrm{P}_\mu \psi\|_{L_t^4 L_x^4[T]} + \sup_\mu \mu^{-\frac{1}{2d}} \mathfrak{m}(\mu) \|\mathrm{P}_\mu \psi\|_{L_t^{2d} L_x^{\frac{2d}{d-1}}[T]} .$$

For $\sigma \in \mathbb{R}$, we define F^σ to be $F^{\mathfrak{m}}$ corresponding to $\mathfrak{m}(\lambda) = \lambda^\sigma$.

Lemma 2.27. *Let $\mathfrak{m}$ be a Sobolev weight such that $[\mathfrak{m}] \leqslant C(s)$. Assume the hypothesis (II) and assume B satisfies (2.34). Let $T \in (0,1]$. If $\psi \in V_B^2 H^{\mathfrak{m}}[T]$, and $\widetilde{\psi}$ is either ψ or $\overline{\psi}$, then we have*

$$\left\|\widetilde{\psi}\right\|_{F^{\mathfrak{m}}[T]} \leqslant C(s)\,(1+M)^2\, \|\psi\|_{V_B^2 H^{\mathfrak{m}}[T]} . \tag{2.35}$$

Proof. Due to the $V_{\mathrm{rc}}^2 \hookrightarrow U^4 \hookrightarrow U^{2d}$ embedding, (2.35) is simply a restatement of Lemma 2.22. ☐

Lemma 2.28. *Let $\mathfrak{m}$ be a Sobolev weight such that $[\mathfrak{m}] \leqslant C(s)$. Let $T \in (0,1]$. Then*

$$\|\mathrm{P}_\mu \psi\|_{L_t^4 L_x^\infty[T]} \leqslant C(s) \mu^{\frac{3}{4}} \mathfrak{m}(\mu)^{-1} \|\psi\|_{F^{\mathfrak{m}}[T]}$$

and

$$\|\mathrm{P}_\mu \psi\|_{L_t^{2d} L_x^\infty[T]} \leqslant C(s) \mu^{1-\frac{1}{2d}} \mathfrak{m}(\mu)^{-1} \|\psi\|_{F^{\mathfrak{m}}[T]} .$$

In particular, if $\psi \in F^1[T]$ then $\psi \in L_t^4 L_x^\infty[T] \cap L_t^{2d} L_x^\infty[T]$ with

$$\|\psi\|_{L_t^4 L_x^\infty[T]} + \|\psi\|_{L_t^{2d} L_x^\infty[T]} \leqslant C \|\psi\|_{F^1[T]} .$$

Proof. Using Bernstein's inequality, we have

$$\|\mathrm{P}_\mu \psi\|_{L_t^4 L_x^\infty[T]} \leqslant C(s) \mu^{\frac{1}{2}} \|\mathrm{P}_\mu \psi\|_{L_t^4 L_x^4[T]}$$

and also

$$\|\mathrm{P}_\mu \psi\|_{L_t^{2d} L_x^\infty[T]} \leqslant C(s) \mu^{\frac{d-1}{d}} \|\mathrm{P}_\mu \psi\|_{L_t^{2d} L_x^{\frac{2d}{d-1}}[T]} .$$

The lemma follows from recalling the definition of $F^{\mathfrak{m}}[T]$. ☐

2.4.4 Estimates for the gauge fields

In this subsection, we collect various space-time estimates for the gauge fields $\mathcal{N}_t^2, \mathcal{N}_x^2, \mathcal{N}_t^4$, which we will need for our multilinear estimates, and also for our difference estimates in Section 2.5.

Lemma 2.29. *Assume the hypothesis (I). Let $T \in (0,1]$. Then*

$$\left\|\mathrm{P}_\mu \mathcal{N}_x^2\left[\psi_1, \psi_2\right]\right\|_{L_t^2 L_x^\infty[T]} \leqslant C(s) \mu^{-\frac{1}{4}} \mathfrak{m}(\mu)^{-1}\left(\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]}+\left\|\psi_1\right\|_{F^1[T]}\left\|\psi_2\right\|_{F^{\mathfrak{m}}[T]}\right) . \tag{2.36}$$

Proof. For the case $\mu=1$, the Bernstein and Hardy-Littlewood-Sobolev inequalities give

$$\left\|\mathrm{P}_1 \mathcal{N}_x^2\left[\psi_1, \psi_2\right]\right\|_{L_t^2 L_x^\infty[T]} \lesssim T^{\frac{1}{2}}\left\|\psi_1 \psi_2\right\|_{L_t^\infty L_x^1[T]} \leqslant C\left\|\psi_1\right\|_{F^1[T]}\left\|\psi_2\right\|_{F^1[T]} .$$

For $\mu \geqslant 2$, Bernstein's inequality and Lemma 2.28 give

$$\begin{aligned}
\left\|\mathrm{P}_\mu \mathcal{N}_x^2\left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_{\leqslant \lambda} \psi_2\right]\right\|_{L_t^2 L_x^\infty[T]} & \lesssim \mu^{-1}\left\|\mathrm{P}_\lambda \psi_1\right\|_{L_t^4 L_x^\infty[T]}\left\|\psi_2\right\|_{L_t^4 L_x^\infty[T]} \\
& \lesssim \mu^{-1} \lambda^{\frac{3}{4}} \mathfrak{m}(\lambda)^{-1}\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]} \\
& \leqslant C(s) \mu^{-1+s} \lambda^{\frac{3}{4}-s} \mathfrak{m}(\mu)^{-1}\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]}
\end{aligned}$$

where the last inequality is due to hypothesis (I). Summing over $\lambda \gtrsim \mu$ and noting the symmetry of $\mathcal{N}_x^2[\psi_1, \psi_2]$ in ψ_1, ψ_2, we obtain (2.36). □

Lemma 2.30. *Assume the hypothesis (I). Let $T \in (0,1]$. Then, for $\mu \geqslant 2$,*

$$\left\|\mathrm{P}_\mu \mathcal{N}_t^2\left[\psi_1, \psi_2\right]\right\|_{L_t^\infty L_x^1[T]} \leqslant C(s) \mu^{-1} \mathfrak{m}(\mu)^{-1}\left(\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]}+\left\|\psi_1\right\|_{F^1[T]}\left\|\psi_2\right\|_{F^{\mathfrak{m}}[T]}\right) . \tag{2.37}$$

We also have the estimate

$$\left\|\mathcal{N}_t^2\left[\psi_1, \psi_2\right]\right\|_{L_t^4 L_x^\infty[T]} \leqslant C\left\|\psi_1\right\|_{F^1[T]}\left\|\psi_2\right\|_{F^1[T]} . \tag{2.38}$$

Proof. For the proof of (2.37), we have

$$\begin{aligned}
\left\|\mathrm{P}_\mu \mathcal{N}_t^2\left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_{\leqslant \lambda} \psi_2\right]\right\|_{L_t^\infty L_x^1[T]} & \lesssim \mu^{-1}\left\|\mathrm{P}_\lambda \psi_1\right\|_{L_t^\infty L_x^2[T]}\left\|\nabla \psi_2\right\|_{L_t^\infty L_x^2[T]} \\
& \lesssim \mu^{-1} \mathfrak{m}(\lambda)^{-1}\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]} \\
& \leqslant C(s) \mu^{-1} \mathfrak{m}(\mu)^{-1}\left(\frac{\mu}{\lambda}\right)^s\left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]}\left\|\psi_2\right\|_{F^1[T]}
\end{aligned}$$

where the last inequality follows from the hypothesis (I). Summing over $\lambda \gtrsim \mu$, and noting that $\mathcal{N}_t^2[\psi_1, \psi_2]$ is skew-symmetric in ψ_1, ψ_2, we obtain (2.37).

We turn to the proof of (2.38). By Bernstein (and also Hardy-Littlewood-Sobolev for $\mu=1$) and Lemma 2.28, we have

$$\begin{aligned}
\left\|\mathrm{P}_\mu \mathcal{N}_t^2\left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_{\leqslant \lambda} \psi_2\right]\right\|_{L_t^4 L_x^\infty[T]} & \lesssim \mu^{\frac{1}{2}}\left\|\mathrm{P}_\lambda \psi_1 \nabla \mathrm{P}_{\leqslant \lambda} \psi_2\right\|_{L_t^4 L_x^{\frac{4}{3}}[T]} \\
& \lesssim \mu^{\frac{1}{2}}\left\|\mathrm{P}_\lambda \psi_1\right\|_{L_t^4 L_x^4[T]}\left\|\nabla \mathrm{P}_{\leqslant \lambda} \psi_2\right\|_{L_t^\infty L_x^2[T]} \\
& \lesssim \mu^{\frac{1}{2}} \lambda^{-\frac{3}{4}}\left\|\psi_1\right\|_{F^1[T]}\left\|\psi_2\right\|_{F^1[T]}
\end{aligned}$$

The right-hand side is summable over $\lambda \gtrsim \mu$. Since $\mathcal{N}_t^2$ is skew-symmetric, we have (2.38). □

Lemma 2.31. *Assume the hypothesis (I). Let $T \in (0,1]$. Then*

$$\left\|\mathrm{P}_\mu \mathcal{N}_t^4\left[\psi_1,\psi_2,\psi_3,\psi_4\right]\right\|_{L_t^2 L_x^\infty[T]} \leqslant C(s)\mu^{-\frac{1}{4}}\mathfrak{m}(\mu)^{-1}\sum_{\ell=1}^4 \|\psi_\ell\|_{F^{\mathfrak{m}}[T]}\prod_{\substack{k=1\\k\neq\ell}}^4 \|\psi_k\|_{F^1[T]} . \tag{2.39}$$

Proof. We first deal with the case $\mu = 1$. By Bernstein, Hardy-Littlewood-Sobolev, and Lemma 2.24, we find

$$\left\|\mathrm{P}_1 \mathcal{N}_t^4\left[\psi_1,\psi_2,\psi_3,\psi_4\right]\right\|_{L_t^2 L_x^\infty[T]} \lesssim \left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\psi_3\psi_4\right\|_{L_t^\infty L_x^1[T]} \lesssim \prod_{\ell=1}^4 \|\psi_\ell\|_{F^1[T]}$$

as required.

Suppose now $\mu \geqslant 2$. Then, by Bernstein,

$$\begin{aligned}
\left\|\mathrm{P}_\mu \mathcal{N}_t^4\left[\psi_1,\psi_2,\psi_3,\psi_4\right]\right\|_{L_t^2 L_x^\infty[T]} &\lesssim \left\|\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\psi_3\psi_4\right\|_{L_t^2 L_x^2[T]} \\
&\quad + \mu^{-1}\left\|\mathrm{P}_{\ll\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\mathrm{P}_{\approx\mu}\left(\psi_3\psi_4\right)\right\|_{L_t^2 L_x^\infty[T]} \\
&\lesssim \left\|\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^2 L_x^\infty[T]}\|\psi_3\|_{L_t^\infty L_x^4[T]}\|\psi_4\|_{L_t^\infty L_x^4[T]} \\
&\quad + \mu^{-1}\left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^\infty L_x^\infty[T]}\left\|\mathrm{P}_{\approx\mu}\left(\psi_3\psi_4\right)\right\|_{L_t^2 L_x^\infty[T]} \\
&=: \mathrm{I} + \mathrm{II} .
\end{aligned}$$

Using Lemma 2.29 and the Sobolev embedding $H^1 \hookrightarrow L_x^4$,

$$\mathrm{I} \leqslant C(s)\mu^{-\frac{1}{4}}\mathfrak{m}(\mu)^{-1}\left(\|\psi_1\|_{F^{\mathfrak{m}}[T]}\|\psi_2\|_{F^1[T]} + \|\psi_1\|_{F^1[T]}\|\psi_2\|_{F^{\mathfrak{m}}[T]}\right)\|\psi_3\|_{F^1[T]}\|\psi_4\|_{F^1[T]} .$$

Now, we have

$$\begin{aligned}
\|\mathrm{P}_\lambda\psi_3\,\mathrm{P}_{\leqslant\lambda}\psi_4\|_{L_t^2 L_x^\infty[T]} &\lesssim \|\mathrm{P}_\lambda\psi_3\|_{L_t^4 L_x^\infty[T]}\|\psi_4\|_{L_t^4 L_x^\infty[T]} \\
&\lesssim \lambda^{\frac{3}{4}}\mathfrak{m}(\lambda)^{-1}\|\psi_3\|_{F^{\mathfrak{m}}[T]}\|\psi_4\|_{F^1[T]} \\
&\leqslant C(s)\lambda^{\frac{3}{4}-s}\mathfrak{m}(\mu)^{-1}\mu^s\|\psi_3\|_{F^{\mathfrak{m}}[T]}\|\psi_4\|_{F^1[T]}
\end{aligned}$$

where the last inequality is due to the hypothesis (I). Summing over $\lambda \gtrsim \mu$ we obtain, by symmetry,

$$\|\mathrm{P}_{\approx\mu}(\psi_3\psi_4)\|_{L_t^2 L_x^\infty[T]} \leqslant C(s)\mu^{\frac{3}{4}}\mathfrak{m}(\mu)^{-1}\left(\|\psi_3\|_{F^{\mathfrak{m}}[T]}\|\psi_4\|_{F^1[T]} + \|\psi_3\|_{F^1[T]}\|\psi_4\|_{F^{\mathfrak{m}}[T]}\right) .$$

Hence, by Lemma 2.24,

$$\mathrm{II} \leqslant C(s)\mu^{-\frac{1}{4}}\mathfrak{m}(\mu)^{-1}\|\psi_1\|_{F^1[T]}\|\psi_2\|_{F^1[T]}\left(\|\psi_3\|_{F^{\mathfrak{m}}[T]}\|\psi_4\|_{F^1[T]} + \|\psi_3\|_{F^1[T]}\|\psi_4\|_{F^{\mathfrak{m}}[T]}\right) .$$

The proof is complete. □

2.4.5 Multilinear estimates

We now estimate each of the nonlinearities in (2.30) in $\mathrm{D}U_B^2 H^{\mathfrak{m}}[T]$. This is accomplished with the aid of the duality principle, Lemma 2.21.

Lemma 2.32. *Assume the hypotheses (I), (II). Let $T \in (0,1]$. Then*

$$\|\mathcal{Q}\left[\psi_1,\psi_2,\psi_3\right]\|_{\mathrm{D}U_B^2 H^{\mathfrak{m}}[T]} \leqslant C(s)T^{\frac{1}{2}}\sum_{\ell=1}^3 \|\psi_\ell\|_{F^{\mathfrak{m}}[T]}\prod_{\substack{k=1\\k\neq\ell}}^3 \|\psi_k\|_{F^1[T]} .$$

Proof. By duality, it suffices to prove the estimate

$$\left|\int_0^T \int_{\mathbb{R}^2} \overline{\psi_0}\, \mathcal{Q}\left[\psi_1, \psi_2, \psi_3\right]\, \mathrm{d}x\, \mathrm{d}t\right| \leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]} . \tag{2.40}$$

Using Lemma 2.29, we have

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_\mu \mathcal{N}_x^2\left[\psi_1, \psi_2\right] \cdot \nabla \mathrm{P}_\lambda \psi_3\, \mathrm{d}x\, \mathrm{d}t\right| \\
&\quad\lesssim T^{\frac{1}{2}} \left\|\mathrm{P}_\nu \psi_0\right\|_{L_t^\infty L_x^2[T]} \left\|\mathrm{P}_\mu \mathcal{N}_x^2\left[\psi_1, \psi_2\right]\right\|_{L_t^2 L_x^\infty[T]} \left\|\mathrm{P}_\lambda \psi_3\right\|_{L_t^\infty H^1[T]} \\
&\quad\leqslant C(s) T^{\frac{1}{2}} \mathfrak{m}(\nu) \mu^{-\frac{1}{4}} \mathfrak{m}(\mu)^{-1} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]} \\
&\quad\leqslant C(s) T^{\frac{1}{2}} \nu^{s+\frac{1}{8}} \mu^{-\frac{1}{4}-s} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]}
\end{aligned}$$

where the last two inequalities follow from the hypotheses (I), (II). Now, the right-hand side is summable over $\{\mu \approx \max(\lambda, \nu)\}$ to give (2.40). □

Lemma 2.33. *Assume the hypotheses (I), (II). Assume also that B satisfies (2.34). Let $T \in (0,1]$. Then*

$$\left\|\mathcal{N}_t^2\left[\psi_1, \psi_2\right] \psi_3\right\|_{\mathrm{D}U_B^2 H^{\mathfrak{m}}[T]} \leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{2}} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]} .$$

Proof. By duality, it suffices to prove the estimate

$$\left|\int_0^T \int_{\mathbb{R}^2} \overline{\psi_0} \mathcal{N}_t^2\left[\psi_1, \psi_2\right] \psi_3\, \mathrm{d}x\, \mathrm{d}t\right| \leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]} . \tag{2.41}$$

By (2.38), Hölder's inequality gives

$$\begin{aligned}
&\sum_\nu \left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_{\lesssim\nu} \mathcal{N}_t^2\left[\psi_1, \psi_2\right]\, \mathrm{P}_{\approx\nu} \psi_3\, \mathrm{d}x\, \mathrm{d}t\right| \\
&\quad\lesssim \int_0^T \left\|\mathcal{N}_t^2\left[\psi_1, \psi_2\right]\right\|_{L_x^\infty} \sum_\nu \left\|\mathrm{P}_\nu \psi_0\right\|_{L_x^2} \left\|\mathrm{P}_{\approx\nu} \overline{\psi_3}\right\|_{L_x^2}\, \mathrm{d}t \\
&\quad\lesssim T^{\frac{3}{4}} \left\|\psi_0\right\|_{L_t^\infty H^{\mathfrak{m}-1}[T]} \left\|\mathcal{N}_t^2\left[\psi_1, \psi_2\right]\right\|_{L4_t L_x^\infty[T]} \left\|\psi_3\right\|_{l_t^\infty H^{\mathfrak{m}}[T]} \\
&\quad\leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \left\|\psi_1\right\|_{F^1[T]} \left\|\psi_2\right\|_{F^1[T]} \left\|\psi_3\right\|_{F^{\mathfrak{m}}[T]}
\end{aligned}$$

where the last inequality follows from Lemma 2.30 and the hypotheses (I) and (II).

By the Littlewood-Paley trichotomy, to prove (2.41) it remains to show

$$\begin{aligned}
&\left(\sum_{\mu\approx\lambda\gg\nu} + \sum_{\nu\approx\lambda\gg\mu}\right) \left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_\mu \mathcal{N}_t^2\left[\psi_1, \psi_2\right]\, \mathrm{P}_\lambda \psi_3\, \mathrm{d}x\, \mathrm{d}t\right| \\
&\quad\leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^3 \left\|\psi_k\right\|_{F^1[T]} .
\end{aligned}$$

For this, we note using (2.37) and Lemmas 2.27, 2.28 that

$$
\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_\mu \mathcal{N}_t^2\left[\psi_1, \psi_2\right]\, \mathrm{P}_\lambda \psi_3 \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\lesssim T^{\frac{1}{2}} \left\|\mathrm{P}_\nu \psi_0\right\|_{L_t^4 L_x^\infty[T]} \left\|\mathrm{P}_\mu \mathcal{N}_t^2\left[\psi_1, \psi_2\right]\right\|_{L_t^\infty L_x^1[T]} \left\|\mathrm{P}_\lambda \psi_3\right\|_{L_t^4 L_x^\infty[T]} \\
&\leqslant C(s)(1+M)^2 T^{\frac{1}{2}} \nu^{\frac{3}{4}} \mathfrak{m}(\nu) \mu^{-1} \mathfrak{m}(\mu)^{-1} \lambda^{\frac{3}{4}-1} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^3 \left\|\psi_k\right\|_{F^1[T]} \\
&\leqslant C(s)(1+M)^2 T^{\frac{1}{2}} \nu^{\frac{3}{4}+s} \mu^{-1-s} \lambda^{-\frac{1}{4}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^3 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^3 \left\|\psi_k\right\|_{F^1[T]} .
\end{aligned}
$$

The right-hand side is summable over $\{\mu \approx \lambda \gg \nu\}$ and over $\{\mu \approx \nu \gg \lambda\}$, as desired. □

Lemma 2.34. *Assume the hypotheses (I), (II). Let $T \in (0,1]$. Then*

$$
\left\|\mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right] \psi_5\right\|_{\mathrm{D}U_B^2 H^{\mathfrak{m}}[T]} \leqslant C(s) T^{\frac{1}{2}} \sum_{\ell=1}^5 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^5 \left\|\psi_k\right\|_{F^1[T]} .
$$

Proof. By duality, it suffices to prove the estimate

$$
\left|\int_0^T \int_{\mathbb{R}^2} \overline{\psi_0} \mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right] \psi_5 \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^5 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^5 \left\|\psi_k\right\|_{F^1[T]} . \tag{2.42}
$$

Firstly, using Hölder's inequality and Lemma 2.31, we have

$$
\begin{aligned}
&\sum_\nu \left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_{\lesssim \nu} \mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right]\, \mathrm{P}_{\approx \nu} \psi_5 \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\lesssim \int_0^T \left\|\mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right](t)\right\|_{L_x^\infty} \sum_\nu \left\|\mathrm{P}_\nu \psi_0(t)\right\|_{L_x^2} \left\|\mathrm{P}_{\approx \nu} \psi_5(t)\right\|_{L_x^2} \mathrm{d}t \\
&\quad\leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{L_t^\infty H^{\mathfrak{m}-1}[T]} \left\|\mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right]\right\|_{L_t^2 L_x^\infty[T]} \left\|\psi_5\right\|_{L_t^\infty H^{\mathfrak{m}}[T]} \\
&\quad\leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \left(\prod_{\ell=1}^4 \left\|\psi_\ell\right\|_{F^1[T]}\right) \left\|\psi_5\right\|_{F^{\mathfrak{m}}[T]} .
\end{aligned}
$$

By the Littlewood-Paley trichotomy, to prove (2.42) it remains to show that

$$
\begin{aligned}
&\left(\sum_{\mu \approx \nu \gg \lambda} + \sum_{\mu \approx \lambda \gg \nu}\right) \left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_\mu \mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right]\, \mathrm{P}_\lambda \psi_5 \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\leqslant C(s) T^{\frac{1}{2}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^5 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^5 \left\|\psi_k\right\|_{F^1[T]} .
\end{aligned} \tag{2.43}
$$

For this, using Hölder's inequality and Lemma 2.31 again, we find

$$
\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\psi_0}\, \mathrm{P}_\mu \mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right]\, \mathrm{P}_\lambda \psi_5 \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\lesssim T^{\frac{1}{2}} \left\|\mathrm{P}_\nu \psi_0\right\|_{L_t^\infty L_x^2[T]} \left\|\mathrm{P}_\mu \mathcal{N}_t^4\left[\psi_1, \psi_2, \psi_3, \psi_4\right]\right\|_{L_t^2 L_x^\infty[T]} \left\|\mathrm{P}_\lambda \psi_5\right\|_{L_t^\infty L_x^2[T]} \\
&\leqslant C(s) T^{\frac{1}{2}} \mathfrak{m}(\nu) \mu^{-1} \mathfrak{m}(\mu)^{-1} \lambda^{-1} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \left(\sum_{\ell=1}^4 \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1 \\ k \neq \ell}}^4 \left\|\psi_k\right\|_{F^1[T]}\right) \left\|\psi_5\right\|_{F^1[T]} .
\end{aligned}
$$

Due to hypothesis (I), the right-hand side is summable over $\{\mu \approx \max(\nu, \lambda)\}$. Therefore we obtain (2.43) as required. □

Lemma 2.35. *Assume the hypotheses (I), (II). Let $T \in (0,1]$. Then*

$$\left\|\mathcal{N}_x^4\left[\psi_1,\psi_2,\psi_3,\psi_4\right]\psi_5\right\|_{\mathrm{DU}_B^2 H^{\mathfrak{m}}[T]} \leqslant C(s) T^{\frac{1}{2}} \sum_{\ell=1}^{5}\|\psi_\ell\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\ k\neq\ell}}^{5}\|\psi_k\|_{F^1[T]} \ .$$

Proof. By duality, it suffices to prove the estimate

$$\left|\int_0^T \int_{\mathbb{R}^2} \overline{\psi_0}\mathcal{N}_x^4\left[\psi_1,\psi_2,\psi_3,\psi_4\right]\psi_5 \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s) T^{\frac{1}{2}} \|\psi_0\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{5}\|\psi_\ell\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\ k\neq\ell}}^{5}\|\psi_k\|_{F^1[T]} \ . \tag{2.44}$$

By the Littlewood-Paley trichotomy and symmetry, it suffices to verify the estimates

$$\begin{aligned}\sum_{\mu,\nu\,:\,\mu\gtrsim\nu}&\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu\overline{\psi_0}\,\mathrm{P}_\mu\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\cdot \mathrm{P}_{\lesssim\mu}\mathcal{N}_x^2\left[\psi_3,\psi_4\right]\,\psi_5 \,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leqslant C(s) T^{\frac{1}{2}} \|\psi_0\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{5}\|\psi_\ell\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\ k\neq\ell}}^{5}\|\psi_k\|_{F^1[T]}\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}\sum_{\nu}&\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu\overline{\psi_0}\,\mathrm{P}_{\ll\nu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\cdot \mathrm{P}_{\ll\nu}\mathcal{N}_x^2\left[\psi_3,\psi_4\right]\,\mathrm{P}_{\approx\nu}\psi_5 \,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leqslant C(s) T^{\frac{1}{2}} \|\psi_0\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{5}\|\psi_\ell\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\ k\neq\ell}}^{5}\|\psi_k\|_{F^1[T]} \ .\end{aligned} \tag{2.46}$$

Using Lemmas 2.24 and 2.29, a typical summand on the left-hand side of (2.45) is controlled by

$$\begin{aligned}&T^{\frac{1}{2}}\|\mathrm{P}_\nu\psi_0\|_{L_t^\infty L_x^2[T]}\left\|\mathrm{P}_\mu\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^2 L_x^\infty[T]}\left\|\mathcal{N}_x^2\left[\psi_3,\psi_4\right]\right\|_{L_t^\infty L_x^\infty[T]}\|\psi_5\|_{L_t^\infty L_x^2[T]}\\ &\leqslant C(s) T^{\frac{1}{2}}\mathfrak{m}(\nu)\mu^{-\frac{1}{4}}\mathfrak{m}(\mu)^{-1}\|\psi_0\|_{V_B^2 H^{\mathfrak{m}-1}[T]}\\ &\quad\cdot\left(\|\psi_1\|_{F^{\mathfrak{m}}[T]}\|\psi_2\|_{F^1[T]}+\|\psi_1\|_{F^1[T]}\|\psi_2\|_{F^{\mathfrak{m}}[T]}\right)\|\psi_3\|_{F^1[T]}\|\psi_4\|_{F^1[T]}\|\psi_5\|_{F^1[T]} \ .\end{aligned}$$

Summing over $\{\mu \gtrsim \nu\}$, we obtain (2.45) as required.

As for (2.46), we use Lemma 2.24 to obtain the the left-hand side by

$$\begin{aligned}&\int_0^T \left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right](t)\right\|_{L_x^\infty}\left\|\mathcal{N}_x^2\left[\psi_3,\psi_4\right](t)\right\|_{L_x^\infty}\left(\sum_\nu \|\mathrm{P}_\nu\psi_0(t)\|_{L_x^2}\|\mathrm{P}_{\approx\nu}\psi_5(t)\|_{L_x^2}\right)\mathrm{d}t\\ &\quad\leqslant C(s) T\|\psi_0\|_{L_t^\infty H^{\mathfrak{m}-1}[T]}\left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^\infty L_x^\infty[T]}\left\|\mathcal{N}_x^2\left[\psi_3,\psi_4\right]\right\|_{L_t^\infty L_x^\infty[T]}\|\psi_5\|_{L_t^\infty H^{\mathfrak{m}}[T]}\\ &\quad\leqslant C(s) T^{\frac{1}{2}}\|\psi_0\|_{V_B^2 H^{\mathfrak{m}-1}[T]}\left(\prod_{k=1}^{4}\|\psi_k\|_{F^1[T]}\right)\|\psi_5\|_{F^{\mathfrak{m}}[T]} \ .\end{aligned}$$

This verifies (2.46) and hence completes the proof of (2.44). □

Lemma 2.36. *Assume the hypotheses (I), (II). Assume also that B satisfies (2.34). Let $T \in (0,1]$. Let $b \in \{2,\dots,d\}$. Then we have the estimate*

$$\|\psi_1\cdots\psi_{2b-1}\|_{\mathrm{DU}_B^2 H^{\mathfrak{m}}[T]} \leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{d}} \sum_{\ell=1}^{2b-1}\|\psi_\ell\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\ k\neq\ell}}^{2b-1}\|\psi_k\|_{F^1[T]} \ .$$

Proof. By duality, it suffices to prove the estimate

$$\left|\int_0^T \int_{\mathbb{R}^2} \overline{\psi_0}\psi_1\psi_2\psi_3\cdots\psi_{2b-1}\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{d}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{2b-1} \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^{2b-1} \left\|\psi_k\right\|_{F^1[T]} \ . \tag{2.47}$$

By the Littlewood-Paley trichotomy and symmetry, (2.47) follows from the two estimates

$$\begin{aligned}\sum_\nu &\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu\overline{\psi_0}\,\mathrm{P}_{\approx\nu}\psi_1\,\mathrm{P}_{\lesssim\nu}\psi_2\,\mathrm{P}_{\lesssim\nu}\psi_3\cdots\mathrm{P}_{\lesssim\nu}\psi_{2b-1}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant C(s)T^{\frac{1}{d}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{2b-1} \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^{2b-1} \left\|\psi_k\right\|_{F^1[T]}\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}\sum_{\lambda,\nu\,:\,\lambda\gg\nu} &\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu\overline{\psi_0}\,\mathrm{P}_\lambda\psi_1\,\mathrm{P}_{\approx\lambda}\psi_2\,\mathrm{P}_{\lesssim\lambda}\psi_3\cdots\mathrm{P}_{\lesssim\lambda}\psi_{2b-1}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant C(s)\left(1+M\right)^2 T^{\frac{1}{d}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \sum_{\ell=1}^{2b-1} \left\|\psi_\ell\right\|_{F^{\mathfrak{m}}[T]} \prod_{\substack{k=1\\k\neq\ell}}^{2b-1} \left\|\psi_k\right\|_{F^1[T]} \ .\end{aligned} \tag{2.49}$$

Using Lemma 2.28, we easily obtain (2.48) as follows,

$$\begin{aligned}\sum_\nu &\left|\int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu\overline{\psi_0}\,\mathrm{P}_{\approx\nu}\psi_1\,\mathrm{P}_{\lesssim\nu}\psi_2\,\mathrm{P}_{\lesssim\nu}\psi_3\cdots\mathrm{P}_{\lesssim\nu}\psi_{2b-1}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant \int_0^T \sum_\nu \left\|\mathrm{P}_\nu\psi_0(t)\right\|_{L_x^2} \left\|\mathrm{P}_{\approx\nu}\psi_1(t)\right\|_{L_x^2} \left\|\psi_2(t)\right\|_{L_x^\infty} \left\|\psi_3(t)\right\|_{L_x^\infty}\cdots\left\|\psi_{2b-1}(t)\right\|_{L_x^\infty}\mathrm{d}t \\ &\leqslant C(s)T^{\frac{d-b+1}{d}} \left\|\psi_0\right\|_{L_t^\infty H^{\mathfrak{m}-1}[T]} \left\|\psi_1\right\|_{L_t^\infty H^{\mathfrak{m}}[T]} \left\|\psi_2\right\|_{L_t^{2d}L_x^\infty[T]} \left\|\psi_3\right\|_{L_t^{2d}L_x^\infty[T]}\cdots\left\|\psi_{2b-1}\right\|_{L_t^{2d}L_x^\infty[T]} \\ &\leqslant C(s)T^{\frac{1}{d}} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]} \left\|\psi_2\right\|_{F^1[T]} \left\|\psi_3\right\|_{F^1[T]}\cdots\left\|\psi_{2b-1}\right\|_{F^1[T]} \ .\end{aligned}$$

We now turn to the proof of (2.49). Using Lemma 2.22, a typical summand on the left-hand side of (2.49) is controlled by

$$\begin{aligned}C(s)&T^{\frac{d-b+1}{d}} \left\|\mathrm{P}_\nu\psi_0\right\|_{L_t^{2d}L_x^\infty[T]} \left\|\mathrm{P}_\lambda\psi_1\right\|_{L_t^\infty L_x^2[T]} \left\|\mathrm{P}_{\approx\lambda}\psi_2\right\|_{L_t^\infty L_x^2[T]} \prod_{\ell=3}^{2b-1} \left\|\psi_\ell\right\|_{L_t^{2d}L_x^\infty[T]} \\ &\leqslant C(s)T^{\frac{1}{d}}\left(1+M\right)^2 \nu^{1-\frac{1}{2d}}\mathfrak{m}(\nu)\mathfrak{m}(\lambda)^{-1}\lambda^{-1} \left\|\psi_0\right\|_{V_B^2 H^{\mathfrak{m}-1}[T]} \left\|\psi_1\right\|_{F^{\mathfrak{m}}[T]} \left\|\psi_2\right\|_{F^1[T]} \prod_{\ell=3}^{2b-1} \left\|\psi_\ell\right\|_{F^1[T]} \ .\end{aligned}$$

Summing up over $\{\lambda \gg \nu\}$, we obtain (2.49) as required. □

2.4.6 Proof of Theorem 2.25

Let $T \in (0,1]$ be fixed later. Define $\varSigma : \mathfrak{E}_{M,T} \to U_B^2 H^{\mathfrak{m}}[T]$ by

$$\begin{aligned}\varSigma(\psi)(t) := \mathfrak{S}_B(t,0)\psi^{\mathrm{in}} + \int_0^t \mathfrak{S}_B(t,t')\Big[&\mathcal{Q}\left[\overline{\psi},\psi,\psi\right](t') + \left(\mathcal{N}_t^2\left[\overline{\psi},\psi\right]\psi\right)(t') \\ &+ \left(\mathcal{N}_t^4\left[\overline{\psi},\psi,\overline{\psi},\psi\right]\psi\right)(t') + \left(\mathcal{N}_x^4\left[\overline{\psi},\psi,\overline{\psi},\psi\right]\psi\right)(t') - 2\mathrm{i}\left(V'\left(|\psi|^2\right)\psi\right)(t')\Big]\,\mathrm{d}t' \ .\end{aligned}$$

Our goal is to show that $\varSigma$ defines a contraction map $\mathfrak{E}_{M,T} \to \mathfrak{E}_{M,T}$.

Suppose $\psi' \in \mathfrak{E}_{M,T}$. By Lemma 2.27, we have

$$\left\|\widetilde{\psi'}\right\|_{F^{\mathfrak{m}}[T]} \leqslant C(s)(1+M)^2 M$$

for every $\psi \in \mathfrak{E}_{M,T}$, where $\widetilde{\psi'}$ is either ψ' or $\overline{\psi'}$. Now, applying Lemmas 2.32, 2.33, 2.34, 2.35, 2.36, bounding the $F^1[T]$ norm above by the $F^{\mathfrak{m}}[T]$ norm, we have

$$\begin{aligned}
\left\|\mathcal{Q}\left[\overline{\psi'}, \psi', \psi'\right]\right\|_{\mathrm{D}U^2_B H^{\mathfrak{m}}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^6 M^3 \,, \\
\left\|\mathcal{N}^2_t\left[\overline{\psi'}, \psi'\right]\psi'\right\|_{\mathrm{D}U^2_B H^{\mathfrak{m}}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^8 M^3 \,, \\
\left\|\mathcal{N}^4_t\left[\overline{\psi'}, \psi'\,\overline{\psi'}, \psi'\right]\psi'\right\|_{\mathrm{D}U^2_B H^{\mathfrak{m}}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^{10} M^5 \,, \\
\left\|\mathcal{N}^4_x\left[\overline{\psi'}, \psi'\,\overline{\psi'}, \psi'\right]\psi'\right\|_{\mathrm{D}U^2_B H^{\mathfrak{m}}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^{10} M^5 \,, \\
\left\|V'\left(|\psi'|^2\right)\psi'\right\|_{\mathrm{D}U^2_B H^{\mathfrak{m}}[T]} &\leqslant C(s) T^{\frac{1}{d}} (1+M)^{4d} M^{2d-1} \,.
\end{aligned}$$

Summing these up, we obtain

$$\left\|\varSigma(\psi')\right\|_{U^2_B H^{\mathfrak{m}}[T]} \leqslant M + C(s) T^{\frac{1}{2}} (1+M)^{14} M + C(s) T^{\frac{1}{d}} (1+M)^{6d-2} M \,.$$

If ψ'' is another element of $\mathfrak{E}_{M,T}$, then a similar argument shows

$$\left\|\varSigma(\psi') - \varSigma(\psi'')\right\|_{U^2_B H^{\mathfrak{m}}[T]} \leqslant C(s)\left(T^{\frac{1}{2}}(1+M)^{14} + T^{\frac{1}{d}}(1+M)^{6d-2}\right)\left\|\psi' - \psi''\right\|_{U^2_B H^{\mathfrak{m}}[T]} \,.$$

Hence, we see that by choosing $T = T(s,M) \in (0,1]$ sufficiently small, we could ensure that $\varSigma$ indeed defines a contraction map $\mathfrak{E}_{M,T} \to \mathfrak{E}_{M,T}$.

The unique fixed point $\psi \in \mathfrak{E}_{M,T}$ is then the desired solution to (2.30). Moreover, by (2.33),

$$\left\|\mathcal{N}^2_x\left[\overline{\psi}, \psi\right]\right\|_{L^\infty_t L^\infty_x[1]} \leqslant 2KM^2 \,.$$

Thus, letting $\varGamma$ be the extension by zero of $-\frac{1}{2}\mathcal{N}^2_x[\overline{\psi}, \psi]$ to $[0,1)$, we have that $\varGamma$ satisfies (2.34).

It remains to check that $\varGamma$ verifies the hypothesis (II), provided we choose $T(s,M)$ smaller if necessary. For this, we need the following estimate.

Lemma 2.37. *Let $T \in (0,1]$. Then*

$$\left\|\nabla\mathcal{N}^2_x\left[\psi_1, \psi_2\right]\right\|_{L^1_t L^\infty_x[T]} \lesssim T^{\frac{1}{2}} \left\|\psi_1\right\|_{F^1[T]} \left\|\psi_2\right\|_{F^1[T]} \,.$$

Proof. Recalling that $\mathfrak{R}$ denotes the Riesz transform, observe that $\nabla\mathcal{N}^2_x[v_1, v_2]$ and $\mathfrak{R}^2(v_1 v_2)$ have the same components. By Bernstein's inequality, the boundedness of the Riesz transform on L^4_x, and Lemma 2.28, we have

$$\begin{aligned}
\left\|\nabla\mathcal{N}^2_x\left[\mathrm{P}_\lambda\psi_1, \mathrm{P}_{\leqslant\lambda}\psi_2\right]\right\|_{L^1_t L^\infty_x[T]} &\lesssim T^{\frac{1}{2}}\mu^{\frac{1}{2}} \left\|\mathrm{P}_\lambda\psi_1\,\mathrm{P}_{\leqslant\lambda}\psi_2\right\|_{L^2_t L^4_x[T]} \\
&\lesssim T^{\frac{1}{2}}\mu^{\frac{1}{2}} \left\|\mathrm{P}_\lambda\psi_1\right\|_{L^4_t L^4_x[T]} \left\|\psi_2\right\|_{L^4_t L^\infty_x[T]} \\
&\lesssim T^{\frac{1}{2}}\mu^{\frac{1}{2}}\lambda^{-\frac{3}{4}} \left\|\psi_1\right\|_{F^1[T]} \left\|\psi_2\right\|_{F^1[T]} \,.
\end{aligned}$$

Summing up over $\{\lambda \gtrsim \mu\}$ and noting the symmetry of $\mathcal{N}^2_x[v_1, v_2]$ in v_1 and v_2, we obtain the desired estimate. □

By Lemma 2.37, noting that $\|\psi\|_{F^1[T]} \leqslant C(s)(1+M)^2 M$, we have

$$\|\Gamma\|_{L^1_t L^\infty_x[T]} \leqslant C(s) T^{\frac{1}{2}} (1+M)^4 M^2 .$$

Hence, by choosing $T = T(s, M)$ smaller if necessary, we can ensure that the right-hand side is $\leqslant 1$ and, consequently, (II) holds for Γ.

The proof of Theorem 2.25 is complete. □

2.5 Convergence of the iteration scheme

Using Theorem 2.25, we may inductively construct the iterates $\phi^{[n]}$ of the iteration scheme (2.27) which is initialised with $A_x^{[0]} = 0$. For the proof of Theorem 1.1, it remains to show that the iterates $\phi^{[n]}$ converge to a solution ϕ of the Chern-Simons-Schrödinger system in the Coulomb gauge, (2.26), and to verify the H^s continuity of the solution map and the weak Lipschitz estimate (1.6).

The technical core of both tasks is that of estimating $\|\psi - \psi'\|_{L^\infty_t H^{s-1}[T]}$ where both ψ, ψ' solve (2.30) with possibly different admissible forms B, B' respectively, and possibly different initial data. This is provided for by the following result.

Theorem 2.38. *Let $M > 0$, let $\varepsilon \in (0, 1]$ and suppose $B, B', B^\dagger$ are admissible forms satisfying the hypothesis (II) and (2.34). Assume that $\mathfrak{m}(\lambda) = \lambda^s$. Then, for sufficiently small $T = T(s, M, \varepsilon) \leqslant 1$, the following is true.*

Let $\psi \in U^2_B H^s[T]$ and $\psi' \in U^2_{B'} H^s[T]$ are solutions given by Theorem 2.25 to (2.30), with admissible forms B, B' respectively, such that $\|\psi(0)\|_{H^s} \leqslant M$, $\|\psi'(0)\|_{H^s} \leqslant M$. Then

$$\|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \leqslant \varepsilon \left(\left\|B - B^\dagger\right\|_{L^2_t L^\infty_x[T]} + \left\|B' - B^\dagger\right\|_{L^2_t L^\infty_x[T]} \right) + C(s) \left\|\psi(0) - \psi'(0)\right\|_{H^{s-1}} . \tag{2.50}$$

Note that Proposition 2.23 already guarantees that $\psi, \psi' \in U^2_{B^\dagger} H^{s-1}[T]$. Thus, the left-hand side of (2.50) is finite.

The proof of Theorem 2.38 is straightforward but rather labourious. The rest of this section will be devoted to this proof.

Explicitly, the difference equation for $\psi - \psi'$ can be written

$$\begin{aligned}
(\partial_t - \mathrm{i}\triangle + \mathfrak{P}_{B^\dagger}) \left(\psi - \psi'\right) = {} & \mathfrak{P}_{B^\dagger - B}\psi + \mathfrak{P}_{B' - B^\dagger}\psi' \\
& + \left(\mathcal{Q}\left[\overline{\psi}, \psi, \psi\right] - \mathcal{Q}\left[\overline{\psi'}, \psi', \psi'\right]\right) \\
& + \left(\mathcal{N}^2_t\left[\overline{\psi}, \psi\right]\psi - \mathcal{N}^2_t\left[\overline{\psi'}, \psi'\right]\psi'\right) \\
& + \left(\mathcal{N}^4_t\left[\overline{\psi}, \psi, \overline{\psi}, \psi\right]\psi - \mathcal{N}^4_t\left[\overline{\psi'}, \psi', \overline{\psi'}, \psi'\right]\psi'\right) \\
& + \left(\mathcal{N}^4_x\left[\overline{\psi}, \psi, \overline{\psi}, \psi\right]\psi - \mathcal{N}^4_x\left[\overline{\psi'}, \psi', \overline{\psi'}, \psi'\right]\psi'\right) \\
& - 2\mathrm{i}\left(V'\left(|\psi|^2\right)\psi - V'\left(|\psi'|^2\right)\psi'\right) .
\end{aligned} \tag{2.51}$$

The proof of Theorem 2.38 proceeds in the exact same manner as that of Theorem 2.25. We estimate the $\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]$ norm of each term of the right-hand side of (2.51), by testing against a $V^2_{B^\dagger} H^{-(s-1)}[T]$ function and invoking the duality principle of Lemma 2.21.

2.5.1 Difference estimates for the gauge fields

First we need the following preliminary estimates, which are analogous to those of Lemmas 2.29, 2.30, 2.31. Eventually, the indeterminate function ω will be substituted with $\psi - \psi'$ or its complex conjugate.

Lemma 2.39. *Let $T \in (0, 1]$. Then*

$$\left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\psi_1, \omega \right] \right\|_{L_t^2 L_x^\infty [T]} \leqslant C(s) \mu^{\frac{3}{4} - s} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \tag{2.52}$$

Proof. For $\mu = 1$, Bernstein and Hardy-Littlewood-Sobolev give us

$$\begin{aligned} \left\| \mathrm{P}_1 \mathcal{N}_x^2 \left[\psi_1, \omega \right] \right\|_{L_t^2 L_x^\infty [T]} &\lesssim T^{\frac{1}{4}} \left\| \psi_1 \omega \right\|_{L_t^4 L^{\frac{4}{3}} [T]} \\ &\lesssim \left\| \psi_1 \right\|_{L_t^4 L_x^4 [T]} \left\| \omega \right\|_{L_t^\infty L_x^2 [T]} \\ &\leqslant C(s) \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \end{aligned}$$

Now, suppose instead that $2 \leqslant \mu \in \mathfrak{D}$. By Bernstein,

$$\begin{aligned} \left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_\nu \omega \right] \right\|_{L_t^2 L_x^\infty [T]} &\lesssim \mu^{-\frac{1}{2}} \left\| \mathrm{P}_\lambda \psi_1 \, \mathrm{P}_\nu \omega \right\|_{L_t^2 L_x^4 [T]} \\ &\lesssim \mu^{-\frac{1}{2}} \left\| \mathrm{P}_\lambda \psi_1 \right\|_{L_t^4 L_x^\infty [T]} \left\| \mathrm{P}_\nu \omega \right\|_{L_t^4 L_x^4 [T]} \\ &\leqslant C(s) \mu^{-\frac{1}{2}} \lambda^{\frac{1}{4} - s} \nu^{\frac{1}{4} - (s-1)} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \end{aligned}$$

Performing the relevant summations, we obtain

$$\begin{aligned} \left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\mathrm{P}_{\lesssim \mu} \psi_1, \mathrm{P}_{\approx \mu} \omega \right] \right\|_{L_t^2 L_x^\infty [T]} &+ \left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\mathrm{P}_{\approx \mu} \psi_1, \mathrm{P}_{\lesssim \mu} \omega \right] \right\|_{L_t^2 L_x^\infty [T]} \\ &\leqslant C(s) \mu^{\frac{3}{4} - s} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \end{aligned}$$

By the Littlewood-Paley trichotomy, to prove (2.52) it remains to show

$$\sum_{\lambda \,:\, \lambda \gg \mu} \left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_{\approx \lambda} \omega \right] \right\|_{L_t^2 L_x^\infty [T]} \leqslant C(s) \mu^{\frac{3}{4} - s} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \tag{2.53}$$

For this, we use Bernstein to estimate

$$\begin{aligned} \left\| \mathrm{P}_\mu \mathcal{N}_x^2 \left[\mathrm{P}_\lambda \psi_1, \mathrm{P}_{\approx \lambda} \omega \right] \right\|_{L_t^2 L_x^\infty [T]} &\lesssim \left\| \mathrm{P}_\lambda \psi_1 \, \mathrm{P}_{\approx \lambda} \omega \right\|_{L_t^2 L_x^2 [T]} \\ &\lesssim \left\| \mathrm{P}_\lambda \psi_1 \right\|_{L_t^4 L_x^4 [T]} \left\| \mathrm{P}_{\approx \lambda} \omega \right\|_{L_t^4 L_x^4 [T]} \\ &\leqslant C(s) \lambda^{\frac{1}{4} - s} \lambda^{\frac{1}{4} - (s-1)} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ , \end{aligned}$$

and (2.53) follows immediately. □

Lemma 2.40. *Let $T \in (0, 1]$. Then*

$$\left\| \mathrm{P}_\mu \mathcal{N}_t^2 \left[\psi_1, \omega \right] \right\|_{L_t^\infty L_x^1 [T]} \leqslant C(s) \mu^{-s} \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \quad \text{for } \mu \geqslant 2 \ . \tag{2.54}$$

We also have

$$\left\| \mathrm{P}_{\lesssim \mu} \mathcal{N}_t^2 \left[\psi_1, \omega \right] \right\|_{L_t^\infty L_x^\infty [T]} \leqslant C(s) \mu \left\| \psi_1 \right\|_{F^s[T]} \left\| \omega \right\|_{F^{s-1}[T]} \ . \tag{2.55}$$

Proof. We first observe the preliminary estimate

$$\left\| \mathrm{P}_\mu \left(\nabla \mathrm{P}_{\gg \mu} \psi_1 \, \mathrm{P}_{\gg \mu} \omega \right) \right\|_{L_x^1} \leqslant C(s) \mu^{-2(s-1)} \left\| \psi_1 \right\|_{H^s} \left\| \omega \right\|_{H^{s-1}} \ . \tag{2.56}$$

Indeed, since $s \geqslant 1$, for $\lambda \gtrsim \mu$ we have

$$\|\nabla \mathrm{P}_\lambda \psi_1 \, \mathrm{P}_{\approx\lambda}\omega\|_{L^1_x} \leqslant C(s)\mu^{-(2s-2)}\lambda^{2s-1} \|\mathrm{P}_\lambda\psi_1\|_{L^2_x} \|\mathrm{P}_{\approx\lambda}\omega\|_{L^2_x} \ .$$

Summing over $\lambda \gg \mu$, (2.56) follows using Cauchy-Schwarz.

We turn to the proof of (2.54). Let $\mu \geqslant 2$ be fixed. We have

$$\begin{aligned}\left\|\mathrm{P}_\mu \mathcal{N}^2_t\left[\mathrm{P}_{\approx\mu}\psi_1, \mathrm{P}_{\lesssim\mu}\omega\right]\right\|_{L^\infty_t L^1_x[T]} &\lesssim \mu^{-1}\|\nabla \mathrm{P}_{\approx\mu}\psi_1 \, \mathrm{P}_{\lesssim\mu}\omega\|_{L^\infty_t L^1_x[T]} \\ &\leqslant C(s)\mu^{-s}\|\psi_1\|_{L^\infty_t H^s[T]}\|\omega\|_{L^\infty_t H^{s-1}[T]}\end{aligned}$$

and similarly

$$\left\|\mathrm{P}_\mu \mathcal{N}^2_t\left[\mathrm{P}_{\lesssim\mu}\psi_1, \mathrm{P}_{\approx\mu}\omega\right]\right\|_{L^\infty_t L^1_x[T]} \lesssim C(s)\mu^{-s}\|\psi_1\|_{L^\infty_t H^s[T]}\|\omega\|_{L^\infty_t H^{s-1}[T]} \ .$$

On the other hand, by (2.56),

$$\left\|\mathrm{P}_\mu \mathcal{N}^2_t\left[\mathrm{P}_{\gg\mu}\psi_1, \mathrm{P}_{\gg\mu}\omega\right]\right\|_{L^\infty_t L^1_x[T]} \leqslant C(s)\mu^{1-2s}\|\psi_1\|_{L^\infty_t H^s[T]}\|\omega\|_{L^\infty_t H^{s-1}[T]} \ .$$

Hence, due to the Littlewood-Paley trichotomy, we obtain (2.54).

We now prove (2.55). By Bernstein, Hardy-Littlewood-Sobolev and Hölder, we have

$$\left\|\mathrm{P}_1 \mathcal{N}^2_t\left[\psi_1, \omega\right]\right\|_{L^\infty_t L^\infty_x[T]} \lesssim \|\nabla\psi_1 \, \omega\|_{L^\infty_t L^1_x[T]} \leqslant C(s)\|\psi_1\|_{L^\infty_t H^s[T]}\|\omega\|_{L^\infty_t H^{s-1}[T]} \ .$$

For $\nu \geqslant 2$, Bernstein's inequality and (2.54) give

$$\left\|\mathrm{P}_\nu \mathcal{N}^2_t\left[\psi_1, \omega\right]\right\|_{L^\infty_t L^\infty_x[T]} \leqslant C(s)\nu\|\psi_1\|_{F^s[T]}\|\omega\|_{F^{s-1}[T]}$$

since $2 - s \leqslant 1$. Hence, (2.55) follows by summing the preceding estimates. □

Lemma 2.41. *Let $T \in (0,1]$. Then*

$$\left\|\mathrm{P}_\mu \mathcal{N}^4_t\left[\psi_1, \omega, \psi_2, \psi_3\right]\right\|_{L^2_t L^\infty_x[T]} \leqslant C(s)\mu^{\frac{3}{4}-s}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^3\|\psi_\ell\|_{F^s[T]} \tag{2.57}$$

and

$$\left\|\mathrm{P}_\mu \mathcal{N}^4_t\left[\psi_1, \psi_2, \psi_3, \omega\right]\right\|_{L^2_t L^\infty_x[T]} \leqslant C(s)\mu^{\frac{3}{4}-s}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^3\|\psi_\ell\|_{F^s[T]} \tag{2.58}$$

Proof. We first prove (2.57) in the case $\mu = 1$. By the Bernstein, Hardy-Littlewood-Sobolev and Hölder inequalities, we have

$$\begin{aligned}\left\|\mathrm{P}_1 \mathcal{N}^4_t\left[\psi_1, \omega, \psi_2, \psi_3\right]\right\|_{L^2_t L^\infty_x[T]} &\lesssim \left\|\mathcal{N}^2_x\left[\psi_1, \omega\right]\right\|_{L^2_t L^\infty_x[T]}\|\psi_2\|_{L^\infty_t L^2_x[T]}\|\psi_3\|_{L^\infty_t L^2_x[T]} \\ &\leqslant C(s)\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^3\|\psi_\ell\|_{F^s[T]}\end{aligned}$$

where we have used (2.52) to estimate $\|\mathcal{N}^2_x[\psi_1, \omega]\|_{L^2_t L^\infty_x[T]}$.

Suppose now $2 \leqslant \mu \in \mathfrak{D}$. Then, by Bernstein,

$$\left\|\mathrm{P}_\mu \mathcal{N}^4_t\left[\psi_1, \omega, \psi_2, \psi_3\right]\right\|_{L^2_t L^\infty_x[T]} \lesssim \mu^{-1}\left\|\mathrm{P}_\mu\left(\mathcal{N}^2_x\left[\psi_1, \omega\right]\psi_2\psi_3\right)\right\|_{L^2_t L^\infty_x[T]} \ .$$

By Bernstein, Hölder and (2.52),

$$\begin{aligned}\mu^{-1}\left\|\mathrm{P}_\mu\left(\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\omega\right]\ \psi_2\psi_3\right)\right\|_{L_t^2L_x^\infty[T]} &\lesssim \left\|\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\omega\right]\right\|_{L_t^2L_x^\infty[T]}\left\|\psi_2\right\|_{L_t^\infty L_x^4[T]}\left\|\psi_3\right\|_{L_t^\infty L_x^4[T]}\\ &\leqslant C(s)\mu^{\frac{3}{4}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

On the other hand, for $\lambda \gtrsim \mu$, we have

$$\begin{aligned}&\mu^{-1}\Big\|\mathrm{P}_\mu\left(\mathrm{P}_{\ll\mu}\mathcal{N}_x^2\left[\psi_1,\omega\right]\ \mathrm{P}_\lambda\psi_2\,\mathrm{P}_{\leqslant\lambda}\psi_3\right)\Big\|_{L_t^2L_x^\infty[T]}\\ &\quad\lesssim \left\|\mathcal{N}_x^2\left[\psi_1,\omega\right]\right\|_{L_t^2L_x^\infty[T]}\left\|\mathrm{P}_\lambda\psi_2\right\|_{L_t^\infty L_x^4[T]}\left\|\psi_3\right\|_{L_t^\infty L_x^4[T]}\\ &\quad\leqslant C(s)\lambda^{\frac{1}{2}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

By summing over $\lambda \gtrsim \mu$ and noting the symmetry in ψ_2, ψ_3, we have

$$\mu^{-1}\left\|\mathrm{P}_\mu\left(\mathrm{P}_{\ll\mu}\mathcal{N}_x^2\left[\psi_1,\omega\right]\ \psi_2\psi_3\right)\right\|_{L_t^2L_x^\infty[T]} \leqslant C(s)\mu^{\frac{1}{2}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\ .$$

This completes the proof of (2.57).

We turn to the proof of (2.58). The case $\mu = 1$ is handled in exactly the same fashion as above. Suppose now $2 \leqslant \mu \in \mathfrak{D}$. Then, by Bernstein,

$$\left\|\mathrm{P}_\mu\mathcal{N}_t^4\left[\psi_1,\psi_2,\psi_3,\omega\right]\right\|_{L_t^2L_x^\infty[T]} \lesssim \mu^{-1}\left\|\mathrm{P}_\mu\left(\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\psi_3\omega\right)\right\|_{L_t^2L_x^\infty[T]}\ .$$

By Bernstein, Hölder and (2.36),

$$\begin{aligned}\mu^{-1}\left\|\mathrm{P}_\mu\left(\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\ \psi_3\omega\right)\right\|_{L_t^2L_x^\infty[T]} &\lesssim \mu^{\frac{1}{2}}\left\|\mathrm{P}_{\gtrsim\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^2L_x^\infty[T]}\left\|\psi_3\right\|_{L_t^\infty L_x^4[T]}\left\|\omega\right\|_{L_t^\infty L_x^2[T]}\\ &\leqslant C(s)\mu^{\frac{1}{4}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

On the other hand, by Bernstein, Hölder and (2.28) we have

$$\begin{aligned}&\mu^{-1}\Big\|\mathrm{P}_\mu\left(\mathrm{P}_{\ll\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\ \mathrm{P}_\lambda\psi_3\,\mathrm{P}_{\lesssim\lambda}\omega\right)\Big\|_{L_t^2L_x^\infty[T]}\\ &\quad\lesssim \left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^\infty L_x^\infty[T]}\left\|\mathrm{P}_\lambda\psi_3\right\|_{L_t^4L_x^\infty[T]}\left\|\omega\right\|_{L_t^\infty L_x^2[T]}\\ &\quad\leqslant C(s)\lambda^{\frac{3}{4}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\end{aligned}$$

which is summable over $\lambda \gtrsim \mu$; and also we have

$$\begin{aligned}&\mu^{-1}\Big\|\mathrm{P}_\mu\left(\mathrm{P}_{\ll\mu}\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\ \mathrm{P}_{\ll\mu}\psi_3\,\mathrm{P}_{\approx\mu}\omega\right)\Big\|_{L_t^2L_x^\infty[T]}\\ &\quad\lesssim \mu^{-1}\left\|\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^\infty L_x^\infty[T]}\left\|\psi_3\right\|_{L_t^4L_x^\infty[T]}\left\|\omega\right\|_{L_t^4L_x^\infty[T]}\\ &\quad\leqslant C(s)\mu^{\frac{1}{4}-s}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^3\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

By the Littlewood-Paley trichotomy, (2.58) is proved. □

2.5.2 Difference estimates for nonlinearities

We are now ready to estimate the $\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]$ norm of each term of the right-hand side of (2.51).

Lemma 2.42. *Assume the hypothesis (II). Let $T \in (0,1]$. Let Θ be an admissible form. Then*

$$\|\mathfrak{P}_\Theta \psi_1\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]} \leqslant C(s)T^{\frac{1}{2}}\,\|\Theta\|_{L^2_tL^\infty_x[T]}\,\|\psi_1\|_{F^s[T]}\ .$$

Proof. By duality, it suffices to prove

$$\left|\int_0^T\int_{\mathbb{R}^2_x}\overline{\omega_0}\,\mathfrak{P}_\Theta\psi_1\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s)T^{\frac{1}{2}}\,\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\,\|\Theta\|_{L^2_tL^\infty_x[T]}\,\|\psi_1\|_{F^s[T]}\ . \tag{2.59}$$

By Hölder and Bernstein, we have

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^2_x}\overline{\omega_0}\,\mathfrak{P}_\Theta\psi_1\,\mathrm{d}x\,\mathrm{d}t\right| &\lesssim \int_0^T\sum_\nu\|\mathrm{P}_\nu\omega_0(t)\|_{L^2_x}\,\|\Theta(t)\|_{L^\infty_x}\,\nu\,\|\mathrm{P}_{\approx\nu}\psi_1(t)\|_{L^2_x}\ \mathrm{d}t\\ &\leqslant C(s)T^{\frac{1}{2}}\,\|\omega_0(t)\|_{L^\infty_tH^{-(s-1)}[T]}\,\|\Theta\|_{L^2_tL^\infty_x[T]}\,\|\psi_1\|_{L^\infty_tH^s[T]}\ .\end{aligned}$$

Now use Lemma 2.22 to replace the $L^\infty_tH^{-(s-1)}[T]$ norm of ω_0 by the $V^2_{B^\dagger}H^{-(s-1)}[T]$. We thus obtain (2.59). □

Lemma 2.43. *Assume the hypothesis (II). Let $T \in (0,1]$. Then we have*

$$\|\mathcal{Q}[\psi_1,\omega,\psi_2]\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]} \leqslant C(s)T^{\frac{1}{2}}\,\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]} \tag{2.60}$$

and

$$\|\mathcal{Q}[\psi_1,\psi_2,\omega]\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]} \leqslant C(s)T^{\frac{1}{2}}\,\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]} \tag{2.61}$$

Proof. By duality, the proof of (2.60) reduces to verifying the estimate

$$\left|\int_0^T\int_{\mathbb{R}^2_x}\overline{\omega_0}\,\mathcal{Q}[\psi_1,\omega,\psi_2]\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s)T^{\frac{1}{2}}\,\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\,\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ . \tag{2.62}$$

By Lemma 2.39, we have

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2_x}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}^2_x[\psi_1,\omega]\cdot\nabla\mathrm{P}_\lambda\psi_2\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\quad\lesssim T^{\frac{1}{2}}\,\|\mathrm{P}_\nu\omega_0\|_{L^\infty_tL^2_x[T]}\,\|\mathrm{P}_\mu\mathcal{N}^2_x[\psi_1,\omega]\|_{L^2_tL^\infty_x[T]}\,\lambda\,\|\mathrm{P}_\lambda\psi_2\|_{L^\infty_tL^2_x[T]}\\ &\quad\leqslant C(s)T^{\frac{1}{2}}\nu^{s-1}\mu^{\frac{3}{4}-s}\lambda^{1-s}\,\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\,\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ .\end{aligned}$$

The right-hand side is summable over $\{\mu \gtrsim \max(\lambda,\nu)\}$. This gives (2.60).

We now turn to proving (2.61). By duality, it suffices to prove

$$\left|\int_0^T\int_{\mathbb{R}^2_x}\overline{\omega_0}\,\mathcal{Q}[\psi_1,\psi_2,\omega]\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C(s)T^{\frac{1}{2}}\,\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\,\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ . \tag{2.63}$$

By Lemma 2.29,

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2_x}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\cdot\nabla\mathrm{P}_\lambda\omega\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad\lesssim T^{\frac12}\left\|\mathrm{P}_\nu\omega_0\right\|_{L_t^\infty L_x^2[T]}\left\|\mathrm{P}_\mu\mathcal{N}_x^2\left[\psi_1,\psi_2\right]\right\|_{L_t^2L_x^\infty[T]}\lambda\left\|\mathrm{P}_\lambda\omega\right\|_{L_t^\infty L_x^2[T]}\\&\qquad\leqslant C(s)T^{\frac12}\nu^{s-1}\mu^{-\frac34-s}\lambda^{2-s}\left\|\omega_0\right\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

The right-hand side is summable over $\{\mu\gtrsim\max(\lambda,\nu)\}$ to give (2.63). □

Lemma 2.44. *Assume the hypothesis (II). Assume also that the admissible form B satisfies (2.34). Let $T\in(0,1]$. Then we have*

$$\left\|\mathcal{N}_t^2\left[\psi_1,\omega\right]\psi_2\right\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]}\leqslant C(s)\left(1+M\right)^2T^{\frac12}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\tag{2.64}$$

and

$$\left\|\mathcal{N}_t^2\left[\psi_1,\psi_2\right]\omega\right\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]}\leqslant C(s)\left(1+M\right)^2T^{\frac12}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\tag{2.65}$$

Proof. By duality, the proof of (2.64) reduces to verifying the estimate

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2}\overline{\omega_0}\,\mathcal{N}_t^2\left[\psi_1,\omega\right]\psi_2\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad\leqslant C(s)\left(1+M\right)^2T^{\frac12}\left\|\omega_0\right\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}\tag{2.66}$$

Using (2.55) we have

$$\begin{aligned}&\sum_\nu\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_{\lesssim\nu}\mathcal{N}_t^2\left[\psi_1,\omega\right]\,\mathrm{P}_{\approx\nu}\psi_2\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad\lesssim\int_0^T\left\|\mathcal{N}_t^2\left[\psi_1,\omega\right](t)\right\|_{L_x^\infty}\sum_\nu\left\|\mathrm{P}_\nu\omega_0(t)\right\|_{L_x^2}\left\|\mathrm{P}_{\approx\nu}\psi_2(t)\right\|_{L_x^2}\mathrm{d}t\\&\qquad\leqslant C(s)T\left\|\omega_0\right\|_{L_t^\infty H^{-(s-1)}[T]}\left\|\psi_1\right\|_{F^s[T]}\left\|\omega\right\|_{F^{s-1}[T]}\left\|\psi_2\right\|_{L_t^\infty H^s[T]}\\&\qquad\leqslant C(s)T^{\frac12}\left\|\omega_0\right\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

By the Littlewood-Paley trichotomy, to prove (2.66) it remains to prove

$$\begin{aligned}&\left(\sum_{\mu\approx\lambda\gg\nu}+\sum_{\mu\approx\nu\gg\lambda}\right)\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_t^2\left[\psi_1,\omega\right]\,\mathrm{P}_\lambda\psi_2\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad\leqslant C(s)\left(1+M\right)^2T^{\frac12}\left\|\omega_0\right\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}\tag{2.67}$$

For this, we have from (2.54) and the hypotheses that, for $\mu\geqslant2$,

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_t^2\left[\psi_1,\omega\right]\,\mathrm{P}_\lambda\psi_2\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad\leqslant C(s)T^{\frac12}\left\|\mathrm{P}_\nu\omega_0\right\|_{L_t^4L_x^\infty[T]}\left\|\mathrm{P}_\mu\mathcal{N}_t^2\left[\psi_1,\omega\right]\right\|_{L_t^\infty L_x^1[T]}\left\|\mathrm{P}_\lambda\psi_2\right\|_{L_t^4L_x^\infty[T]}\\&\qquad\leqslant C(s)T^{\frac12}\left(1+M\right)^2\nu^{\frac34+(s-1)}\mu^{-s}\lambda^{\frac34-s}\left\|\omega_0\right\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\left\|\omega\right\|_{F^{s-1}[T]}\prod_{\ell=1}^2\left\|\psi_\ell\right\|_{F^s[T]}\ .\end{aligned}$$

Therefore we have (2.67). Hence we have proved (2.64).

The proof of (2.65) is similar. By duality, it suffices to verify the estimate

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^2}\overline{\omega_0}\,\mathcal{N}_t^2[\psi_1,\psi_2]\,\omega\,\mathrm{d}x\,\mathrm{d}t\right| \\ \leqslant C(s)(1+M)^2T^{\frac12}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ .\end{aligned}\tag{2.68}$$

Indeed, using (2.38) and arguing as above, we obtain

$$\begin{aligned}\sum_\nu\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_{\lesssim\nu}\mathcal{N}_t^2[\psi_1,\psi_2]\ \mathrm{P}_{\approx\nu}\omega\,\mathrm{d}x\,\mathrm{d}t\right| \\ \leqslant C(s)T^{\frac34}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ .\end{aligned}$$

By the Littlewood-Paley trichotomy, to prove (2.68) it remains to prove

$$\begin{aligned}\left(\sum_{\mu\approx\lambda\gg\nu}+\sum_{\mu\approx\nu\gg\lambda}\right)\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_t^2[\psi_1,\psi_2]\ \mathrm{P}_\lambda\omega\,\mathrm{d}x\,\mathrm{d}t\right| \\ \leqslant C(s)(1+M)^2T^{\frac12}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{F^s[T]}\ .\end{aligned}\tag{2.69}$$

Arguing as before, we have from (2.37) and the hypotheses that, for $\mu\geqslant 2$,

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_t^2[\psi_1,\psi_2]\ \mathrm{P}_\lambda\omega\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\lesssim T^{\frac12}\|\mathrm{P}_\nu\omega_0\|_{L_t^4L_x^\infty[T]}\left\|\mathrm{P}_\mu\mathcal{N}_t^2[\psi_1,\psi_2]\right\|_{L_t^\infty L_x^1[T]}\|\mathrm{P}_\lambda\omega\|_{L_t^4L_x^\infty[T]} \\ &\leqslant C(s)T^{\frac12}(1+M)^2\nu^{-\frac14+s}\mu^{-s-1}\lambda^{\frac74-s}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{U^2_BH^{s-1}[T]}\prod_{\ell=1}^2\|\psi_\ell\|_{U^2_BH^s[T]}\ .\end{aligned}$$

Thus (2.69) is immediate, and we have completed the proof of (2.65). □

Lemma 2.45. *Assume the hypothesis (II). Let $T\in(0,1]$. Then we have the estimates*

$$\left\|\mathcal{N}_t^4[\psi_1,\omega,\psi_2,\psi_3]\,\psi_4\right\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]}\leqslant C(s)T^{\frac12}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^4\|\psi_\ell\|_{F^s[T]}\ ,\tag{2.70}$$

$$\left\|\mathcal{N}_t^4[\psi_1,\psi_2,\psi_3,\omega]\,\psi_4\right\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]}\leqslant C(s)T^{\frac12}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^4\|\psi_\ell\|_{F^s[T]}\ ,\tag{2.71}$$

$$\left\|\mathcal{N}_t^4[\psi_1,\psi_2,\psi_3,\psi_4]\,\omega\right\|_{\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]}\leqslant C(s)T^{\frac12}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^4\|\psi_\ell\|_{F^s[T]}\ .\tag{2.72}$$

Proof. We first prove (2.70). By duality, it suffices to prove the estimate

$$\left|\int_0^T\int_{\mathbb{R}^2}\overline{\omega_0}\,\mathcal{N}_t^4[\psi_1,\omega,\psi_2,\psi_3]\,\psi_4\,\mathrm{d}x\,\mathrm{d}t\right|\leqslant C(s)T^{\frac12}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^4\|\psi_\ell\|_{F^s[T]}\ .$$

For this, using (2.57) from Lemma 2.41 gives

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^2}\mathrm{P}_\nu\overline{\omega_0}\,\mathrm{P}_\mu\mathcal{N}_t^4[\psi_1,\omega,\psi_2,\psi_3]\ \mathrm{P}_\lambda\psi_4\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant \|\mathrm{P}_\nu\omega_0\|_{L_t^\infty L_x^2[T]}\left\|\mathrm{P}_\mu\mathcal{N}_t^4[\psi_1,\omega,\psi_2,\psi_3]\right\|_{L_t^2L_x^\infty[T]}\|\mathrm{P}_\lambda\psi_4\|_{L_t^\infty L_x^2[T]} \\ &\leqslant C(s)T^{\frac12}\nu^{s-1}\mu^{\frac34-s}\lambda^{-s}\|\omega_0\|_{V^2_{B^\dagger}H^{-(s-1)}[T]}\|\omega\|_{F^{s-1}[T]}\prod_{\ell=1}^4\|\psi_\ell\|_{F^s[T]}\end{aligned}$$

which is certainly summable over the regime where the larger two of $\{\nu, \mu, \lambda\}$ are comparable. Therefore we have proved (2.70).

The proof of (2.71) is exactly the same, except that (2.58) is used in place of (2.57).

We turn to the proof of (2.72). By duality, it suffices to prove the estimate

$$\left| \int_0^T \int_{\mathbb{R}^2} \overline{\omega_0}\, \mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4]\, \omega \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant C(s) T^{\frac{1}{2}} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ . \tag{2.73}$$

Firstly, using Lemma 2.31, we have

$$\begin{aligned} \sum_\nu & \left| \int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_{\lesssim \nu} \mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4] \ \mathrm{P}_{\approx \nu} \omega \, \mathrm{d}x \, \mathrm{d}t \right| \\ & \lesssim \int_0^T \sum_\nu \|\mathrm{P}_\nu \omega_0(t)\|_{L^2_x} \left\|\mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4] (t)\right\|_{L^\infty_x} \|\mathrm{P}_{\approx\nu} \omega(t)\|_{L^2_x} \, \mathrm{d}t \\ & \leqslant C(s) T^{\frac{1}{2}} \|\omega_0\|_{L^\infty_t H^{-(s-1)}[T]} \left\|\mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4]\right\|_{L^2_t L^\infty_x[T]} \|\omega\|_{L^\infty_t H^{s-1}[T]} \\ & \leqslant C(s) T^{\frac{1}{2}} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ . \end{aligned}$$

By the Littlewood-Paley trichotomy, it remains to prove

$$\begin{aligned} \left(\sum_{\nu \approx \mu \gg \lambda} + \sum_{\lambda \approx \mu \gg \nu} \right) & \left| \int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_\mu \mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4]\, \mathrm{P}_\lambda \omega \, \mathrm{d}x \, \mathrm{d}t \right| \\ & \leqslant C(s) T^{\frac{1}{2}} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ . \end{aligned} \tag{2.74}$$

For this, using Lemma 2.31 we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^2} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_\mu \mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4]\, \mathrm{P}_\lambda \omega \, \mathrm{d}x \, \mathrm{d}t \right| \\ & \leqslant C(s) T^{\frac{1}{2}} \|\mathrm{P}_\nu \omega_0\|_{L^\infty_t L^2_x[T]} \left\|\mathrm{P}_\mu \mathcal{N}_t^4 [\psi_1, \psi_2, \psi_3, \psi_4]\right\|_{L^2_t L^\infty_x[T]} \|\mathrm{P}_\lambda \omega\|_{L^\infty_t L^2_x[T]} \\ & \leqslant C(s) T^{\frac{1}{2}} \nu^{s-1} \mu^{-\frac{1}{4}-s} \lambda^{-(s-1)} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{L^\infty_t H^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ . \end{aligned}$$

Therefore (2.74) follows immediately. □

Lemma 2.46. *Assume the hypothesis (II). Let $T \in (0, 1]$. Then we have the estimates*

$$\left\|\mathcal{N}_x^4 [\psi_1, \psi_2, \psi_3, \omega]\, \psi_4\right\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} \leqslant C(s) T^{\frac{1}{2}} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \tag{2.75}$$

and

$$\left\|\mathcal{N}_x^4 [\psi_1, \psi_2, \psi_3, \psi_4]\, \omega\right\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} \leqslant C(s) T^{\frac{1}{2}} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ . \tag{2.76}$$

Proof. By duality, the proof of (2.75) reduces to proving the estimate

$$\left| \int_0^T \int_{\mathbb{R}^2_x} \overline{\omega_0}\, \mathcal{N}_x^4 [\psi_1, \psi_2, \psi_3, \omega]\, \psi_4 \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant C(s) T^{\frac{1}{2}} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \|\psi_\ell\|_{F^s[T]} \ .$$

This, in turn, follows from using Lemmas 2.24 and 2.39 to obtain

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^2_x} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_{\mu_1} \mathcal{N}^2_x[\psi_1, \psi_2] \cdot \mathrm{P}_{\mu_2} \mathcal{N}^2_x[\psi_3, \omega]\ \mathrm{P}_\lambda \psi_4 \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leqslant T^{\frac{1}{2}} \left\| \mathrm{P}_\nu \omega_0 \right\|_{L^\infty_t L^2_x[T]} \left\| \mathrm{P}_{\mu_1} \mathcal{N}^2_x[\psi_1, \psi_2] \right\|_{L^\infty_t L^\infty_x[T]} \left\| \mathrm{P}_{\mu_2} \mathcal{N}^2_x[\psi_3, \omega] \right\|_{L^2_t L^\infty_x[T]} \left\| \mathrm{P}_\lambda \psi_4 \right\|_{L^\infty_t L^2_x[T]} \\
&\quad \leqslant C(s) T^{\frac{1}{2}} \nu^{s-1} \mu_1^{\frac{1}{4}-s} \mu_2^{\frac{3}{4}-s} \lambda^{-s} \left\| \omega_0 \right\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \left\| \psi_\ell \right\|_{F^s[T]} ,
\end{aligned}$$

and observing that the right-hand side is summable over the regime where the two largest of $\{\nu, \mu_1, \mu_2, \lambda\}$ are comparable.

We now turn to the proof of (2.76). By duality, this reduces to proving

$$\left| \int_0^T \int_{\mathbb{R}^2_x} \overline{\omega_0}\, \mathcal{N}^4_x[\psi_1, \psi_2, \psi_3, \psi_4]\, \omega \,\mathrm{d}x\,\mathrm{d}t \right| \leqslant C(s) T^{\frac{1}{2}} \left\| \omega_0 \right\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \left\| \psi_\ell \right\|_{F^s[T]} .$$

Firstly, we have, using Lemma 2.24, that

$$\begin{aligned}
&\sum_\nu \left| \int_0^T \int_{\mathbb{R}^2_x} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_{\ll\nu} \mathcal{N}^2_x[\psi_1, \psi_2] \cdot \mathrm{P}_{\ll\nu} \mathcal{N}^2_x[\psi_3, \psi_4]\ \mathrm{P}_{\approx\nu} \omega \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leqslant \int_0^T \sum_\nu \left\| \mathrm{P}_\nu \omega_0(t) \right\|_{L^2_x} \left\| \mathcal{N}^2_x[\psi_1, \psi_2](t) \right\|_{L^\infty_x} \left\| \mathcal{N}^2_x[\psi_3, \psi_4](t) \right\|_{L^\infty_x} \left\| \mathrm{P}_{\approx\nu} \omega(t) \right\|_{L^2_x} \,\mathrm{d}t \\
&\quad \leqslant C(s) T \left\| \omega_0 \right\|_{L^\infty_t H^{-(s-1)}[T]} \left\| \mathcal{N}^2_x[\psi_1, \psi_2] \right\|_{L^\infty_t L^\infty_x[T]} \left\| \mathcal{N}^2_x[\psi_3, \psi_4] \right\|_{L^\infty_t L^\infty_x[T]} \left\| \omega \right\|_{L^\infty_t H^{s-1}[T]} \\
&\quad \leqslant C(s) T^{\frac{1}{2}} \left\| \omega_0 \right\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \left\| \psi_\ell \right\|_{F^s[T]} .
\end{aligned}$$

By the Littlewood-Paley trichotomy and symmetry, it remains to show that

$$\begin{aligned}
\sum_{\mu,\nu\,:\,\mu\gtrsim\nu} &\left| \int_0^T \int_{\mathbb{R}^2_x} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_\mu \mathcal{N}^2_x[\psi_1, \psi_2] \cdot \mathrm{P}_{\leqslant\mu} \mathcal{N}^2_x[\psi_3, \psi_4]\ \omega \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \leqslant C(s) T^{\frac{1}{2}} \left\| \omega_0 \right\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \left\| \psi_\ell \right\|_{F^s[T]} .
\end{aligned} \tag{2.77}$$

Indeed, by Lemmas 2.24 and 2.29, we have

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^2_x} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_\mu \mathcal{N}^2_x[\psi_1, \psi_2] \cdot \mathrm{P}_{\leqslant\mu} \mathcal{N}^2_x[\psi_3, \psi_4]\ \omega \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leqslant T^{\frac{1}{2}} \left\| \mathrm{P}_\nu \omega_0 \right\|_{L^\infty_t L^2_x[T]} \left\| \mathrm{P}_\mu \mathcal{N}^2_x[\psi_1, \psi_2] \right\|_{L^2_t L^\infty_x[T]} \left\| \mathcal{N}^2_x[\psi_3, \psi_4] \right\|_{L^\infty_t L^\infty_x[T]} \left\| \omega \right\|_{L^\infty_t L^2_x[T]} \\
&\quad \leqslant C(s) T^{\frac{1}{2}} \nu^{s-1} \mu^{-\frac{1}{4}-s} \left\| \omega_0 \right\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^4 \left\| \psi_\ell \right\|_{F^s[T]} ,
\end{aligned}$$

and then (2.77) follows immediately. □

Lemma 2.47. *Assume the hypothesis (II). Let $T \in (0, 1]$. Let $b \in \{2, \dots, d\}$. Then we have the estimate*

$$\left\| \psi_1 \psi_2 \cdots \psi_{2b-2} \omega \right\|_{\mathrm{DU}^2_{B^\dagger} H^{s-1}[T]} \leqslant C(s) T^{\frac{1}{d}} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^{2b-2} \left\| \psi_\ell \right\|_{F^s[T]} . \tag{2.78}$$

Proof. By duality, it suffices to prove the estimate

$$\left| \int_0^T \int_{\mathbb{R}^2_x} \overline{\omega_0}\, \psi_1 \psi_2 \cdots \psi_{2b-2} \omega \,\mathrm{d}x\,\mathrm{d}t \right| \leqslant C(s) T^{\frac{1}{d}} \left\| \omega_0 \right\|_{\mathrm{V}^2_B H^{-(s-1)}[T]} \left\| \omega \right\|_{F^{s-1}[T]} \prod_{\ell=1}^{2b-2} \left\| \psi_\ell \right\|_{F^s[T]} .$$

For this, using Lemma 2.28 we have

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^2_x} \mathrm{P}_\nu \overline{\omega_0}\, \mathrm{P}_{\mu_1}\psi_1\, \mathrm{P}_{\mu_2}\psi_2 \,\cdots\, \mathrm{P}_{\mu_{2b-2}}\psi_{2b-2} \mathrm{P}_\lambda \omega \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\qquad \leqslant T^{\frac{d-b+1}{d}} \|\mathrm{P}_\nu \omega_0\|_{L^\infty_t L^2_x[T]} \|\mathrm{P}_\lambda \omega\|_{L^\infty_t L^2_x[T]} \prod_{\ell=1}^{2b-2} \|\mathrm{P}_{\mu_\ell}\psi_\ell\|_{L^{2d}_t L^\infty_x[T]} \\
&\qquad \leqslant C(s) T^{\frac{1}{d}} \nu^{s-1} \lambda^{-(s-1)} \mu_1^{1-\frac{1}{2d}-s} \mu_2^{1-\frac{1}{2d}-s} \cdots \mu_{2b-2}^{1-\frac{1}{2d}-s} \|\omega_0\|_{V^2_{B^\dagger} H^{-(s-1)}[T]} \|\omega\|_{F^{s-1}[T]} \prod_{\ell=1}^{2b-2} \|\psi_\ell\|_{F^s[T]} \ .
\end{aligned}$$

Now simply observe that the right-hand side is summable over the regime where the two largest frequencies among $\{\nu, \mu_1, \mu_2, \ldots, \mu_{2b-2}, \lambda\}$ are comparable. Therefore we obtain (2.78). □

2.5.3 Proof of Theorem 2.38

By Theorem 2.25, with $T = T(s, M) \leqslant 1$ sufficiently small, we have the bounds $\|\psi\|_{U^2_B H^s[T]} \leqslant 2M$ and $\|\psi'\|_{U^2_{B'} H^s[T]} \leqslant 2M$. Thus, by Lemma 2.27 and the $U^2 \hookrightarrow V^2_{\mathrm{rc}}$ embedding, we have

$$\|\psi\|_{F^s[T]} + \|\psi'\|_{F^s[T]} \leqslant C(s)\,(1+M)^3 \ .$$

Similarly,

$$\|\psi - \psi'\|_{F^{s-1}[T]} \leqslant C(s)\,(1+M)^2 \|\psi - \psi'\|_{U^2_B H^{s-1}[T]} \ .$$

Now, apply Lemmas 2.42, 2.43, 2.44, 2.45, 2.46, 2.47, with each ψ_ℓ being ψ or ψ' or their complex conjugates, and ω being $\psi - \psi'$ or its complex conjugate. We find, respectively,

$$\begin{aligned}
\|\mathfrak{P}_{B^\dagger - B}\psi\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^3 \|B - B^\dagger\|_{L^2_t L^\infty_x[T]} \ , \\
\|\mathfrak{P}_{B' - B^\dagger}\psi'\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^3 \|B' - B^\dagger\|_{L^2_t L^\infty_x[T]} \ , \\
\|\mathcal{Q}[\overline{\psi}, \psi, \psi] - \mathcal{Q}[\overline{\psi'}, \psi', \psi']\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^8 \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ , \\
\|\mathcal{N}^2_0[\overline{\psi}, \psi]\psi - \mathcal{N}^2_0[\overline{\psi'}, \psi']\psi'\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^{10} \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ , \\
\|\mathcal{N}^4_t[\overline{\psi}, \psi, \overline{\psi}, \psi]\psi - \mathcal{N}^4_t[\overline{\psi'}, \psi', \overline{\psi'}, \psi']\psi'\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^{14} \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ , \\
\|\mathcal{N}^4_x[\overline{\psi}, \psi, \overline{\psi}, \psi]\psi - \mathcal{N}^4_x[\overline{\psi'}, \psi', \overline{\psi'}, \psi']\psi'\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{2}} (1+M)^{14} \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ , \\
\|V'(|\psi|^2)\psi - V'(|\psi'|^2)\psi'\|_{\mathrm{D}U^2_{B^\dagger} H^{s-1}[T]} &\leqslant C(s) T^{\frac{1}{d}} (1+M)^{6d-4} \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ .
\end{aligned}$$

Hence, applying Duhamel's formula to (2.51) and using the above estimates, we obtain

$$\begin{aligned}
\|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \leqslant\ & \|\psi(0) - \psi'(0)\|_{H^{s-1}} \\
&+ C(s) T^{\frac{1}{2}} (1+M)^3 \left(\|B - B^\dagger\|_{L^2_t L^\infty_x[T]} + \|B' - B^\dagger\|_{L^2_t L^\infty_x[T]} \right) \\
&+ C(s) \left(T^{\frac{1}{2}} (1+M)^{14} + T^{\frac{1}{d}} (1+M)^{6d-4} \right) \|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \ .
\end{aligned}$$

Therefore, by choosing $T = T(s, M, \varepsilon)$ even smaller if necessary, we can subtract the last term on the right-hand side from the left, and conclude

$$\|\psi - \psi'\|_{U^2_{B^\dagger} H^{s-1}[T]} \leqslant \varepsilon \left(\|B - B^\dagger\|_{L^2_t L^\infty_x[T]} + \|B' - B^\dagger\|_{L^2_t L^\infty_x[T]} \right) + C \|\psi(0) - \psi'(0)\|_{H^{s-1}} \ .$$

The proof of Theorem 2.38 is complete.

2.6 Completion of the proof of the Theorem 1.1

We are now ready to complete the proof of Theorem 1.1. We first prove the following auxiliary lemma, which will be necessary to upgrade $L_t^\infty H^{s-1}[T]$ convergence to $L_t^\infty H^s[T]$ convergence in the following proofs.

Lemma 2.48. *Let $s \geqslant 1$ and let $\mathcal{K}$ be a compact subset of H^s. Then there exists a Sobolev weight $\mathfrak{m}$ satisfying the hypothesis (I), which additionally satisfies*

$$\lim_{\lambda \to \infty} \frac{\mathfrak{m}(\lambda)}{\lambda} = \infty \tag{2.79}$$

and

$$\sup_{w \in \mathcal{K}} \|w\|_{H^{\mathfrak{m}}} \leqslant 2 \sup_{w \in \mathcal{K}} \|w\|_{H^s} \ . \tag{2.80}$$

Proof. By rescaling, we may assume without loss of generality that

$$\sup_{w \in \mathcal{K}} \|w\|_{H^s} = 1 \ .$$

We claim that, for every $\varepsilon \in (0,1]$, there exists $\Lambda = \Lambda(\varepsilon) \in \mathfrak{D}$ such that

$$\sup_{w \in \mathcal{K}} \left(\sum_{\lambda \geqslant \Lambda} \lambda^{2s} \|\mathrm{P}_\lambda w\|_{L_x^2}^2 \right)^{\frac{1}{2}} \leqslant \varepsilon \ .$$

Indeed, suppose for a contradiction that our claim was false. Then, for every $\mu \in \mathfrak{D}$ there exists $w_\mu \in \mathcal{K}$ such that

$$\left(\sum_{\lambda \,:\, \lambda \geqslant \mu} \lambda^{2s} \|\mathrm{P}_\lambda w_\mu\|_{L_x^2}^2 \right)^{\frac{1}{2}} > \varepsilon \ .$$

Since $\mathcal{K}$ is compact, there exists a subsequence $\mu_m \to \infty$ such that w_{μ_m} converges to some $w_\infty \in \mathcal{K}$ in H^s. As $w_\infty \in H^s$, there exists $\nu \in \mathfrak{D}$ such that

$$\left(\sum_{\lambda \,:\, \lambda \geqslant \nu} \lambda^{2s} \|\mathrm{P}_\lambda w_\infty\|_{L_x^2}^2 \right)^{\frac{1}{2}} \leqslant \frac{\varepsilon}{2} \ .$$

However, the triangle inequality gives, for $\mu_m \geqslant \nu$,

$$\begin{aligned} \left(\sum_{\lambda \,:\, \lambda \geqslant \nu} \lambda^{2s} \|\mathrm{P}_\lambda (w_\infty - w_{\mu_m})\|_{L_x^2}^2 \right)^{\frac{1}{2}} &\geqslant \left(\sum_{\lambda \,:\, \lambda \geqslant \nu} \lambda^{2s} \|\mathrm{P}_\lambda w_{\mu_m}\|_{L_x^2}^2 \right)^{\frac{1}{2}} - \left(\sum_{\lambda \,:\, \lambda \geqslant \nu} \lambda^{2s} \|\mathrm{P}_\lambda w_\infty\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &> \frac{\varepsilon}{2} \end{aligned}$$

which contradicts the aforementioned convergence $w_{\mu_m} \to w_\infty$.

Set $\nu_0 := 1$ and for $m \in \{1, 2, 3, \ldots\}$ let ν_m be the smallest element of $\mathfrak{D}$ strictly greater than ν_{m-1} such that

$$\sup_{w \in \mathcal{K}} \left(\sum_{\lambda \geqslant \nu_m} \lambda^{2s} \|\mathrm{P}_\lambda w\|_{L_x^2}^2 \right)^{\frac{1}{2}} \leqslant 2^{-m} \ .$$

Set $\mathfrak{m}(\lambda) := 2^{\frac{1}{8}m} \lambda^s$ for $\nu_m \leqslant \lambda < \nu_{m+1}$. Clearly $\mathfrak{m}$ satisfies the hypothesis (I) and (2.79), and it is straightforward to verify that (2.80) also holds. ☐

2.6.1 Existence of solutions

Given $D > 0$ as in the statement of Theorem 1.1, let $\varepsilon = \varepsilon(s, D) \in (0, 1]$ be a small constant which we will choose later. Let $M := 2D$, and let $T = T(s, M, \varepsilon)$ be that given in Theorem 2.38.

Now, let the initial data $\phi^{\text{in}} \in H^s$ be given with $\|\phi^{\text{in}}\|_{H^s} \leqslant D$. By Lemma 2.48, we may choose a Sobolev weight $\mathfrak{m}$ satisfying the hypothesis (I) and (2.79), such that $\|\phi^{\text{in}}\|_{H^{\mathfrak{m}}} \leqslant M$. Using Theorem 2.25, starting from $A_x^{[0]} = 0$, we construct the iterates $\phi^{[n]} \in U^2_{A_x^{[n-1]}} H^{\mathfrak{m}}[T]$ solving the iteration scheme (2.27).

We claim that, provided $\varepsilon = \varepsilon(s, D)$ was chosen small enough, $\{\phi^{[n]}\}_{n=1}^{\infty}$ will be a Cauchy sequence $L_t^{\infty} H^s[T]$. Indeed, applying Theorem 2.38 with $B^{\dagger} = A_x^{[n-1]}$ and $(\psi, B) = (\phi^{[n]}, A_x^{[n-1]})$ and $(\psi', B') = (\phi^{[n+1]}, A_x^{[n]})$, we find

$$\left\|\phi^{[n]} - \phi^{[n+1]}\right\|_{U^2_{A_x^{[n-1]}} H^{s-1}[T]} \leqslant \varepsilon \left\|A_x^{[n-1]} - A_x^{[n]}\right\|_{L_t^2 L_x^{\infty}[T]} .$$

By Lemma 2.39 and Lemma 2.27, we may replace the right-hand side as follows,

$$\begin{aligned} \left\|\phi^{[n]} - \phi^{[n+1]}\right\|_{U^2_{A_x^{[n-1]}} H^{s-1}[T]} &\leqslant C(s)\varepsilon \left(\left\|\phi^{[n]}\right\|_{F^s[T]} + \left\|\phi^{[n-1]}\right\|_{F^s[T]}\right) \left\|\phi^{[n-1]} - \phi^{[n]}\right\|_{F^{s-1}[T]} \\ &\leqslant C(s)\varepsilon \left(1 + M\right)^5 \left\|\phi^{[n-1]} - \phi^{[n]}\right\|_{U^2_{A_x^{[n-2]}} H^{s-1}[T]} . \end{aligned}$$

Choose $\varepsilon = \varepsilon(s, D)$ sufficiently small so that $C(s)\varepsilon(1 + M)^5 < \frac{1}{2}$ on the right-hand side. Then

$$\left\|\phi^{[n]} - \phi^{[n+1]}\right\|_{U^2_{A_x^{[n-1]}} H^{s-1}[T]} \leqslant \frac{1}{2} \left\|\phi^{[n-1]} - \phi^{[n]}\right\|_{U^2_{A_x^{[n-2]}} H^{s-1}[T]} .$$

Now, Lemma 2.27 gives the estimate

$$\left\|\phi^{[n]} - \phi^{[n+1]}\right\|_{F^{s-1}[T]} \leqslant C(s) \left(1 + M\right)^2 \left\|\phi^{[n]} - \phi^{[n+1]}\right\|_{U^2_{A_x^{[n-1]}} H^{s-1}[T]} .$$

Thus, $\{\phi^{[n]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $F^{s-1}[T]$ and hence in $L_t^{\infty} H^{s-1}[T]$. On the other hand, Lemma 2.22 also guarantees that $\{\phi^{[n]}\}_{n=1}^{\infty}$ is a bounded sequence in $L_t^{\infty} H^{\mathfrak{m}}[T]$. Due to (2.79), we deduce that $\{\phi^{[n]}\}_{n=1}^{\infty}$ is also a Cauchy sequence in $L_t^{\infty} H^s[T]$, as claimed.

Let ϕ be the limit of $\{\phi^{[n]}\}_{n=1}^{\infty}$ in $L_t^{\infty} H^s[T]$. By Lemma 2.24,

$$A_x^{[n]} \to A_x := -\tfrac{1}{2}\mathcal{N}_x^2\left[\overline{\phi}, \phi\right] \quad \text{in } L_t^{\infty} L_x^{\infty}[T] .$$

Moreover, since H^s controls the L_x^4 norm, it is easy to see that

$$\|\phi\|_{F^s[T]} \leqslant \liminf_{n \to \infty} \left\|\phi^{[n]}\right\|_{F^s[T]} .$$

In particular, $\|\phi\|_{F^s[T]} \leqslant C(s)(1 + M)^3$. By our choice of $T = T(s, M, \varepsilon)$, Lemma 2.37 guarantees that $\|\nabla A_x\|_{L_t^1 L_x^{\infty}[T]} \leqslant 1$. Hence A_x is an admissible form satisfying the hypothesis (II) and also (2.34).

Since $A_x^{[n]} \to A_x$ in $L_t^{\infty} L_x^{\infty}[T]$ and $\phi^{[n]} \to \phi$ in $L_t^{\infty} H^s[T]$, we have

$$\mathfrak{P}_{A_x^{[n-1]}} \phi^{[n]} \to \mathfrak{P}_{A_x} \phi \quad \text{in } L_t^{\infty} H^{s-1}[T] .$$

Since $\phi^{[n]} \to \phi$ in $F^{s-1}[T]$, Lemmas 2.43, 2.44, 2.45, 2.46, 2.47 guarantee that

$$\begin{aligned}
\mathcal{Q}\left[\overline{\phi^{[n]}}, \phi^{[n]}, \phi^{[n]}\right] &\to \mathcal{Q}\left[\overline{\phi}, \phi, \phi\right] \ , \\
\mathcal{N}_0^2\left[\overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} &\to \mathcal{N}_0^2\left[\overline{\phi}, \phi\right]\phi \ , \\
\mathcal{N}_t^4\left[\overline{\phi^{[n]}}, \phi^{[n]}, \overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} &\to \mathcal{N}_t^4\left[\overline{\phi}, \phi, \overline{\phi}, \phi\right]\phi \ , \\
\mathcal{N}_x^4\left[\overline{\phi^{[n]}}, \phi^{[n]}, \overline{\phi^{[n]}}, \phi^{[n]}\right]\phi^{[n]} &\to \mathcal{N}_x^4\left[\overline{\phi}, \phi, \overline{\phi}, \phi\right]\phi \ , \\
V'\left(\left|\phi^{[n]}\right|^2\right)\phi^{[n]} &\to V'\left(|\phi|^2\right)\phi
\end{aligned}$$

in $\mathrm{D}U^2_{B^\dagger}H^{s-1}[T]$ for any admissible form $B^\dagger$, and in particular for $B^\dagger = A_x$.

Hence, $\phi \in U^2_{A_x}H^{s-1}[T]$ is indeed a solution to the Chern-Simons-Schrödinger system in the Coulomb guage, (2.26). Furthermore, since $\phi \in F^s[T]$, Lemmas 2.32, 2.33, 2.34, 2.35, 2.36 guarantee that the right-hand side of (2.26) belongs to $\mathrm{D}U^2_{A_x}H^s[T]$. In particular, $\phi \in U^2_{A_x}H^s[T]$ and $\|\phi\|_{U^2_{A_x}H^s[T]} \leqslant 2M$.

This concludes the proof of the existence of solutions.

2.6.2 Uniqueness of solutions, continuity of the solution map, regularity

The uniqueness of a solution, given initial data, is a consequence of the weak Lipschitz bound (1.6). In turn, the weak Lipschitz bound (1.6) is a straightforward consequence of Theorem 2.38. Indeed, let $\varepsilon = \varepsilon(s, D) \in (0, 1]$ a small constant (possibly smaller than the one chosen before) which we will choose later, and let $T = T(s, M, \varepsilon)$ be given by Theorem 2.38. Then, for two solutions (ϕ, A_x) and (ϕ', A'_x) to the Chern-Simons-Schrödinger system (2.26) with $\phi(0), \phi'(0) \in \mathbb{B}_{H^s}(D)$, we have the estimate

$$\|\phi - \phi'\|_{U^2_{A_x}H^{s-1}[T]} \leqslant \varepsilon \left\|A_x - A'_x\right\|_{L^2_t L^\infty_x[T]} + \left\|\phi(0) - \phi'(0)\right\|_{H^{s-1}} \ .$$

Arguing as before using Lemma 2.39, we have

$$\|\phi - \phi'\|_{U^2_{A_x}H^{s-1}[T]} \leqslant C(s)\varepsilon(1 + D)^5 \|\phi - \phi'\|_{U^2_{A_x}H^{s-1}[T]} + \left\|\phi(0) - \phi'(0)\right\|_{H^{s-1}} \ ,$$

so that, with $\varepsilon = \varepsilon(s, D)$ chosen sufficiently small, we obtain by Lemma 2.22,

$$\left\|\phi - \phi'\right\|_{L^\infty_t H^{s-1}[T]} \leqslant C(s) \left\|\phi - \phi'\right\|_{U^2_{A_x}H^{s-1}[T]} \leqslant C(s) \left\|\phi(0) - \phi'(0)\right\|_{H^{s-1}} \ .$$

This completes the proof of (1.6), which also implies the uniqueness statement for solutions.

Next, we address the issue of the continuity of the solution map into $L^\infty_t H^s[T]$. Let $\phi^{\mathrm{in},[n]}$ be a sequence of initial data converging to ϕ^{in} in H^s. By Lemma 2.48, we may pick a Sobolev weight $\mathfrak{m}$ for $\mathcal{K} := \{\phi^{\mathrm{in},[n]}\}_{n=1}^\infty \cup \{\phi^{\mathrm{in}}\}$. By the weak Lipschitz bound (1.6), the solutions $\phi^{[n]}$ converge to ϕ in $L^\infty_t H^{s-1}[T]$. On the other hand, $\{\phi^{[n]}\}_{n=1}^\infty$ is bounded in $L^\infty_t H^{\mathfrak{m}}[T]$. Hence, (2.80) guarantees that $\{\phi^{[n]}\}_{n=1}^\infty$ is a Cauchy sequence in $L^\infty_t H^s[T]$. Thus, the solution map is continuous.

We now turn to proving the norm growth estimate (1.7). By Theorem 2.25, there exists $T_1 = T_1(D_1) > 0$ such that any H^1 solution ϕ to (1.3) with $\|\phi(0)\|_{H^1} \leqslant D_1$ exists up to $[0, T_1)$, and satisfies $\|\phi\|_{U^2_{A_x}H^1[T_1]} \leqslant 2D_1$ and thus, by Lemma 2.27,

$$\|\phi\|_{F^1[T_1]} \lesssim (1 + D_1)^2 D_1 \ .$$

Moreover A_x satisfies (2.34) with $M = D_1$. If additionally $\phi \in C_{\rm b}([0,T_s),H^s)$ with $T_s \leqslant T_1$, then using Lemmas 2.32, 2.33, 2.34, 2.35, 2.36 respectively, we obtain

$$\begin{aligned}
\left\|\mathcal{Q}\left[\overline{\phi},\phi,\phi\right]\right\|_{\mathrm{D}U^2_{A_x}H^s[T_s]} &\leqslant C(s)T_s^{\frac{1}{2}}\left(1+D_1\right)^6 D_1^2 \left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ ,\\
\left\|\mathcal{N}_0^2\left[\overline{\phi},\phi\right]\phi\right\|_{\mathrm{D}U^2_{A_x}H^s[T_s]} &\leqslant C(s)T_s^{\frac{1}{2}}\left(1+D_1\right)^8 D_1^2 \left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ ,\\
\left\|\mathcal{N}_t^4\left[\overline{\phi},\phi,\overline{\phi},\phi\right]\phi\right\|_{\mathrm{D}U^2_{A_x}H^s[T_s]} &\leqslant C(s)T_s^{\frac{1}{2}}\left(1+D_1\right)^{10} D_1^4 \left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ ,\\
\left\|\mathcal{N}_x^4\left[\overline{\phi},\phi,\overline{\phi},\phi\right]\phi\right\|_{\mathrm{D}U^2_{A_x}H^s[T_s]} &\leqslant C(s)T_s^{\frac{1}{2}}\left(1+D_1\right)^{10} D_1^4 \left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ ,\\
\left\|V'\left(|\phi|^2\right)\phi\right\|_{\mathrm{D}U^2_{A_x}H^s[T_s]} &\leqslant C(s)T_s^{\frac{1}{d}}\left(1+D_1\right)^{4d} D_1^{2d-2} \left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ .
\end{aligned}$$

Summing the above estimates, we conclude from Duhamel's formula that there exists a constant $C_0 = C_0(s) > 0$ such that

$$\left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \leqslant \left\|\phi(0)\right\|_{H^s} + C_0(s)\left(T_s^{\frac{1}{2}}\left(1+D_1\right)^{14} + T_s^{\frac{1}{d}}\left(1+D_1\right)^{6d-2}\right)\left\|\phi\right\|_{U^2_{A_x}H^s[T_s]} \ .$$

Choose $T_\star = T_\star(s,D_1)$ such that $T_\star \leqslant T_1$ and

$$C_0(s)\left(T_s^{\frac{1}{2}}\left(1+D_1\right)^{14} + T_s^{\frac{1}{d}}\left(1+D_1\right)^{6d-2}\right) \leqslant \frac{1}{2} \ .$$

Therefore, if $T_s \leqslant T_\star$ we have $\|\phi\|_{U^2_{A_x}H^s[T_s]} \leqslant 2\|\phi(0)\|_{H^s}$ and hence, by Lemma 2.27,

$$\left\|\phi\right\|_{L^\infty_t H^s[T_s]} \leqslant C(s)\left(1+D_1\right)^2 \left\|\phi(0)\right\|_{H^s} =: C_\star(s,D_1)\left\|\phi(0)\right\|_{H^s} \ .$$

2.6.3 Global regularity

It remains to prove the final statement that, if V is a nonzero polynomial with positive leading coefficient $c_d > 0$, then global well-posedness holds for (1.3) in H^s for every $s \geqslant 1$. For this, it suffices to show that the H^1 norm of a solution ϕ to (1.3) is controlled by the conserved mass $\mathcal{M}(0)$ and energy $\mathcal{E}(0)$.

An easy application of Hölder's inequality gives, for $b \in \{2,\ldots,d-1\}$,

$$\left\|\phi(t)\right\|_{L^{2b}_x} \leqslant \left\|\phi(t)\right\|_{L^2_x}^{\frac{d-b}{d-1}} \left\|\phi(t)\right\|_{L^{2d}_x}^{\frac{b-1}{d-1}} \ .$$

Therefore, using Young's inequality, we have for any $\varepsilon > 0$ the estimate

$$\left\|\phi(t)\right\|_{L^{2b}_x}^{2b} \leqslant \varepsilon\left\|\phi(t)\right\|_{L^{2d}_x}^{2d} + C\left(\left\|\phi(t)\right\|_{L^2_x},\varepsilon\right) \ .$$

By choosing $\varepsilon > 0$ sufficiently small depending on $c_d > 0$, and we deduce

$$\frac{1}{2}\left\|\mathbf{D}_x\phi(t)\right\|_{L^2_x}^2 + \frac{c_d}{2}\left\|\phi(t)\right\|_{L^{2d}_x}^{2d} \leqslant \mathcal{E}(t) + C\left(\left\|\phi(t)\right\|_{L^2_x}\right) \leqslant C\left(\mathcal{M}(0),\mathcal{E}(0)\right) \ .$$

Thus, $\|\phi(t)\|_{L^{2d}_x} \leqslant C(\mathcal{M}(0),\mathcal{E}(0))$, and therefore, by an easy interpolation,

$$\left\|\phi(t)\right\|_{L^4_x} \leqslant C\left(\mathcal{M}(0),\mathcal{E}(0)\right) \ .$$

Now, by the Hardy-Littlewood-Sobolev inequality, we have

$$\left\|A_x(t)\right\|_{L^4_x} \lesssim \left\|\left|\phi(t)\right|^2\right\|_{L^{\frac{4}{3}}_x} \lesssim \left\|\phi(t)\right\|_{L^2_x}\left\|\phi(t)\right\|_{L^4_x} \ .$$

Hence, we find

$$\begin{aligned}\|\phi(t)\|_{H^1}^2 &\lesssim \|\phi(t)\|_{L_x^2}^2 + \|\mathbf{D}_x\phi(t)\|_{L_x^2}^2 + \|A_x(t)\|_{L_x^4}^2 \|\phi(t)\|_{L_x^4}^2 \\ &\lesssim \mathcal{M}(0) + \|\mathbf{D}_x\phi(t)\|_{L_x^2}^2 + \mathcal{M}(0)\,\|\phi(t)\|_{L_x^4}^4 \\ &\leqslant C\left(\mathcal{M}(0), \mathcal{E}(0)\right)\end{aligned}$$

from which we conclude that $\|\phi(t)\|_{H^1}$ remains uniformly bounded, controlled by the conserved quantities.

The proof of Theorem 1.1 is complete.

2.7 Appendix: Proofs of basic properties of the U^p and V^p spaces

In this appendix, we provide the proofs of Lemmas 2.8 and 2.9, which describe the properties of the U^p and V^p spaces that we use in this chapter. These proofs are mainly taken from [18], Section 2, while the proof of Lemma 2.9 is taken from [33], Appendix B.

Proof of Lemma 2.8. We first remark that the embeddings $V^p_{\mathrm{rc}}X[T] \hookrightarrow L^\infty_t X[T]$ and $U^q X[T] \hookrightarrow L^\infty_t X[T]$ are trivial, with

$$\|v\|_{L_t^\infty X[T]} \leqslant \|v\|_{V^p X[T]}$$

and

$$\|u\|_{L_t^\infty X[T]} \leqslant \|u\|_{U^q X[T]} \ .$$

Since $U^p X[T]$ atoms are right-continuous $[0,T) \to X$, and a general $U^p X[T]$ function is a uniform limit of $U^p X[T]$ atoms, we see that $U^p X[T]$ functions are right-continuous. To verify the $U^p X[T] \hookrightarrow V^p X[T]$ embedding, it suffices by the atomic structure of $U^p X[T]$ to check that $U^p X[T]$ atoms belong to $V^p X[T]$. Let

$$a(t) = \sum_{k=0}^{K-1} \mathbb{1}_{[t_k, t_{k+1})}(t) a_k$$

be a $U^p X[T]$ atom. Consider a partition $\mathfrak{t} = \{s_j\}_{j=0}^J$ with $0 = s_0 < s_1 < \ldots < s_J = T$. For $0 \leqslant j \leqslant J$ let $k(j) \in \{0, 1, \ldots, K\}$ be maximal such that $t_{k(j)} \leqslant s_j$. Observe that $a(s_{j+1}) - a(s_j) = a_{k(j+1)} - a_{k(j)}$, and in particular vanishes when $k(j+1) = k(j)$. Therefore, we see that

$$\sum_{j=0}^{J-1} \|a(s_{j+1}) - a(s_j)\|_X^p \leqslant 2^p \sum_{k=0}^{K-1} \|a_k\|_X^p \leqslant 2^p \ .$$

Since $\mathfrak{t}$ was arbitrary, we deduce $\|a\|_{V^p X[T]} \leqslant 2$. This establishes the embedding $U^p X[T] \hookrightarrow V^p_{\mathrm{rc}} X[T]$.

It remains to demonstrate the embedding $V^p_{\mathrm{rc}} X[T] \hookrightarrow U^q X[T]$ with $q > p$. Let $v \in V^p_{\mathrm{rc}} X[T]$ be such that $\|v\|_{V^p X[T]} = 1$. We construct, for $n \in \mathbb{Z}_{\geqslant 0}$, partitions $\mathfrak{t}^{[n]}$ of $[0,T)$ and functions $u^{[n]} : [0,T) \to X$ such that

(i) $\mathfrak{t}^{[0]} \subseteq \mathfrak{t}^{[1]} \subseteq \mathfrak{t}^{[2]} \subseteq \ldots$ with

$$\left|\mathfrak{t}^{[n]}\right| \leqslant 2^{(n+1)p} \ .$$

(ii) $u^{[0]} = 0$, and for each n, $u^{[n]}$ is a right-continuous step-function with jumps only at $\mathfrak{t}^{[n]}$, and

$$\left\| u^{[n]} \right\|_{L_t^\infty X[T]} \leqslant 2^{1-n} \ .$$

(iii) Defining $v^{[0]} := v$ and inductively $v^{[n+1]} := v^{[n]} - u^{[n+1]}$, we have $v^{[n]} \in V_{\mathrm{rc}}^p X[T]$ and

$$\left\| v^{[n]} \right\|_{L_t^\infty X[T]} \leqslant 2^{-n} \ .$$

We proceed by induction on n. Defining $\mathfrak{t}^{[0]} := \{0, T\}$, we see that the relevant properties above are satisfied for $n = 0$. Now, having constructed $\mathfrak{t}^{[n]}, u^{[n]}, v^{[n]}$, we construct $\mathfrak{t}^{[n+1]}, u^{[n+1]}, v^{[n+1]}$ as follows. Denote $\mathfrak{t}^{[n]} := \{0 = t_0^{[n]} < \ldots < t_{K_n}^{[n]} = T\}$. For $k = 0, 1, \ldots, K_n$, define $t_{k,0}^{[n+1]} := t_k^{[n]}$ and if $k < K_n$ construct the sequence $t_{k,1}^{[n+1]}, t_{k,2}^{[n+1]}, \ldots$ by

$$t_{k,j}^{[n+1]} := \inf \left\{ t \in \left(t_{k,j-1}^{[n+1]} \, , \, t_{k+1}^{[n]} \right] \; \middle| \; \left\| v(t) - v(t_{k,j-1}^{[n+1]}) \right\|_X > 2^{-n-1} \right\}$$

provided the latter set is nonempty; otherwise $t_{k,j}^{[n+1]} := t_{k+1}^{[n]}$. Since v is right-continuous we see that $\|v(t_{k,j}^{[n+1]}) - v(t_{k,j-1}^{[n+1]})\|_X \geqslant 2^{-n-1}$ and since $v \in V^p X[T]$ the sequence $t_{k,1}^{[n+1]}, t_{k,2}^{[n+1]}, \ldots$ is finite. We may thus define the partition $\mathfrak{t}^{[n+1]} := \{0 = t_0^{[n+1]} < \ldots < t_{K_{n+1}}^{[n+1]} = T\}$ to consist of all the elements $\{t_{k,j}^{[n+1]}\}$. Now, define

$$u^{[n+1]}(t) := \sum_{k=0}^{K_{n+1}} \mathbb{1}_{\left[t_k^{[n+1]} \, , \, t_{k+1}^{[n+1]}\right)}(t) \, v^{[n]}(t_k^{[n+1]})$$

and $v^{[n+1]} := v^{[n]} - u^{[n+1]}$. Then (ii) is trivially satisfied for $u^{[n+1]}$. Next, note that, for each $t \in [0, T)$ there is a unique k such that $t \in [t_k^{[n+1]} \, , \, t_{k+1}^{[n+1]})$, then because $v^{[n+1]} - v$ is a step function with jumps in $\mathfrak{t}^{[n+1]}$, we have $\|v^{[n+1]}(t)\|_X = \|v^{[n]}(t) - v^{[n]}(t_k^{[n+1]})\|_X = \|v(t) - v(t_k^{[n+1]})\|_X \leqslant 2^{-n-1}$. Thus (iii) is verified for $v^{[n+1]}$. Finally, $1 = \|v\|_{V^p}^p \geqslant |\mathfrak{t}^{[n+1]}| 2^{-(n+1)p}$, which verifies (i) for $\mathfrak{t}^{[n+1]}$. This completes the induction step.

We thus have $v = \sum_{n=0}^\infty u^{[n]}$ where the sum strongly converges in $L_t^\infty X[T]$. Now, every $u^{[n]}$, being a right-continuous step function, clearly belongs to $U^q X[T]$, with $\|u^{[n]}\|_{U^q X[T]}^q \leqslant |\mathfrak{t}^{[n]}| \|u^{[n]}\|_{L_t^\infty X[T]}^q$. Plugging in the above bounds, we obtain

$$\left\| u^{[n]} \right\|_{U^q X[T]} \leqslant 2^{1+\frac{p}{q}} 2^{-n\left(1-\frac{p}{q}\right)} \ .$$

Since $q > p$ we conclude that $v \in U^q X[T]$ with $\|v\|_{U^q X[T]} \leqslant C(p, q)$. $\square$

The proof of Lemma 2.9 requires us to first characterise the dual space of $U^p X[T]$. In the rest of this section, X is a separable Banach space over $\mathbb{C}$, such that its dual space X^* is also separable.

Lemma 2.49. *Let $1 < p < \infty$. For a partition $\mathfrak{t} = \{0 = t_0 < \ldots < t_K = T\}$ of $[0, T)$, define the bilinear form $B_{\mathfrak{t}} : U^p X[T] \times V^{p'} X^*[T] \to \mathbb{C}$ by*

$$B_{\mathfrak{t}}(u, v) := \sum_{k=0}^{K-1} \langle u(t_k), v(t_{k+1}) - v(t_k) \rangle_{X, X^*} \ .$$

Then, viewing the family of all partitions of $[0, T)$ as a directed set under inclusion, the limit

$$B(u, v) := \lim_{\mathfrak{t}} B_{\mathfrak{t}}(u, v) \tag{2.81}$$

exists. More precisely, given $u \in U^p X[T]$ *and* $v \in V^{p'} X^*[T]$*, there exists* $B(u,v) \in \mathbb{C}$ *such that for every* $\varepsilon > 0$ *there exists* $\mathfrak{t}$ *such that whenever* $\mathfrak{t}' \supseteq \mathfrak{t}$ *it holds that* $|B_{\mathfrak{t}'}(u,v) - B(u,v)| < \varepsilon$*.*

Moreover, B *defines a continuous bilinear form* $B : U^p X[T] \times V^{p'} X^*[T] \to \mathbb{C}$ *with*

$$|B(u,v)| \leqslant \|u\|_{U^p X[T]} \|v\|_{V^{p'} X^*[T]} \ . \tag{2.82}$$

Proof. Let $\mathfrak{t} = \{0 = t_0 < \ldots < t_K = T\}$ be a partition of $[0,T)$ and $v \in V^{p'} X^*[T]$. Let a be a $U^p X[T]$ atom, so that $a(t) = \sum_{j=0}^{J-1} \mathbb{1}_{[s_j, s_{j+1})}(t) a_j$ for a partition $\mathfrak{s} = \{0 = s_0 < \ldots < s_J = T\}$ of $[0,T)$, with $\sum_{j=0}^{J-1} \|a_j\|_X^p = 1$. Then, for $t_k \in \mathfrak{t}$, there exists a unique $j(k) \in \{0, 1, \ldots, J-1\}$ such that $s_{j(k)} \leqslant t_k < s_{j(k+1)}$. By definition,

$$B_{\mathfrak{t}}(a,v) := \sum_{k=0}^{K-1} \big\langle a_{j(k)} \, , \, v(t_{k+1}) - v(t_k) \big\rangle_{X,X^*} \ .$$

Let $\mathfrak{t}^\star$ be the partition of $[0,T)$ formed by removing all t_{k+1} from $\mathfrak{t}$ whenever $j(k) = j(k+1)$. Writing $\mathfrak{t}^\star := \{0 = t_0^\star < \ldots < t_L^\star = T\}$ and defining $j^\star(\ell) \in \{0, 1, \ldots, J-1\}$ similarly as above, we have that $j^\star(0) < \ldots < j^*(L-1)$, and

$$B_{\mathfrak{t}}(a,v) := \sum_{\ell=0}^{L-1} \big\langle a_{j^\star(\ell)} \, , \, v(t_{\ell+1}^\star) - v(t_\ell^\star) \big\rangle_{X,X^*} \ .$$

Thus, a simple application of Hölder's inequality gives

$$|B_{\mathfrak{t}}(a,v)| \leqslant \|v\|_{V^{p'} X^*[T]} \ .$$

By the atomic structure of $U^p X[T]$, we deduce that

$$|B_{\mathfrak{t}}(u,v)| \leqslant \|u\|_{U^p X[T]} \|v\|_{V^{p'} X^*[T]} \tag{2.83}$$

for any partition $\mathfrak{t}$ of $[0,T)$, and any $u \in U^p X[T]$ and $v \in V^{p'} X^*[T]$.

Now, let $u \in U^p X[T]$ and $v \in V^{p'} X^*[T]$ be given. Write $u = \sum_{j=1}^\infty c_j a_j$ with $\{c_j\}_{j=1}^\infty \in \ell^1$ and a_j being $U^p X[T]$ atoms. Given $\varepsilon > 0$, pick $J \in \mathbb{Z}_{>0}$ so that $\sum_{j=J+1}^\infty |c_j| \leqslant \frac{\varepsilon}{2}(1 + \|v\|_{V^{p'} X^*[T]})^{-1}$. Set $u_J := \sum_{j=1}^J c_j a_j$. Notice that u_J is a right-continuous step function and $\|u - u_J\|_{U^p X[T]} \leqslant \frac{\varepsilon}{2}(1 + \|v\|_{V^{p'} X^*[T]})^{-1}$. Let $\mathfrak{t}$ be the partition of $[0,T)$ subordinate to u_J. Then, for any $\mathfrak{t}' \supseteq \mathfrak{t}$, we have $B_{\mathfrak{t}'}(u_J, v) = B_{\mathfrak{t}}(u_J, v)$, so that

$$\begin{aligned} |B_{\mathfrak{t}'}(u,v) - B_{\mathfrak{t}}(u,v)| &\leqslant |B_{\mathfrak{t}'}(u,v) - B_{\mathfrak{t}'}(u_J,v)| + |B_{\mathfrak{t}}(u,v) - B_{\mathfrak{t}}(u_J,v)| \\ &\leqslant 2 \|u - u_J\|_{U^p X[T]} \|v\|_{V^{p'} X^*[T]} \\ &< \varepsilon \ . \end{aligned}$$

This implies the existence of the limit (2.81). The bound (2.82) is immediate from (2.83). ☐

Lemma 2.50. *Let* $1 < p < \infty$*. Then*

$$(U^p X[T])^* = V^{p'} X^*[T] \ .$$

More precisely, with $B : U^p X[T] \times V^{p'} X^*[T] \to \mathbb{C}$ *defined in the previous Lemma, the map*

$$V^{p'} X^*[T] \ni v \mapsto B(\cdot, v) \in (U^p X[T])^*$$

is an isometric isomorphism.

Proof. Let $L \in (U^p X[T])^*$ be given. For any $t \in [0,T]$ define $v(t) \in X^*$ by

$$\langle w, v(t)\rangle_{X,X^*} := -L\left(\mathbb{1}_{[t,T)}(\cdot)\, w\right) \ .$$

This gives us a function $v : [0,T] \to X^*$ with $v(T) = 0$.

We claim that $v \in V^{p'} X^*[T]$ with

$$\|v\|_{V^{p'} X^*[T]} \leqslant \|L\|_{(U^p X[T])^*} \ . \tag{2.84}$$

Indeed, let $\mathfrak{t} = \{0 = t_0 < \ldots < t_K = T\}$ be any partition and let $\varepsilon \in (0,1)$. Choose $a_0, a_1, \ldots, a_{K-1} \in X$ such that

$$\|a_k\|_X = \frac{\|v(t_{k+1}) - v(t_k)\|_{X^*}^{p'-1}}{\left(\sum_{\ell=1}^{K-1} \|v(t_{\ell+1}) - v(t_\ell)\|_{X^*}^{p'}\right)^{\frac{1}{p}}}$$

and

$$\langle a_k, v(t_{k+1}) - v(t_k)\rangle_{X,X^*} \geqslant (1-\varepsilon)\, \|a_k\|_X\, \|v(t_{k+1}) - v(t_k)\|_{X^*} \ .$$

Then $a := \sum_{k=0}^{K-1} \mathbb{1}_{[t_k,t_{k+1})}(\cdot) a_k$ is a $U^p X[T]$ atom, so $\|a\|_{U^p X[T]} \leqslant 1$. We thus obtain

$$\begin{aligned}
\|L\|_{(U^p X[T])^*} &\geqslant \left| \sum_{k=0}^{K-1} L\left(\mathbb{1}_{[t_k,t_{k+1})}(\cdot)\, a_k\right) \right| \\
&= \left| \sum_{k=0}^{K-1} \langle a_k, v(t_{k+1}) - v(t_k)\rangle_{X,X^*} \right| \\
&\geqslant (1-\varepsilon) \left(\sum_{k=0}^{K-1} \|v(t_{k+1}) - v(t_k)\|_{X^*}^{p'} \right)^{\frac{1}{p'}} \ .
\end{aligned}$$

Since ε and $\mathfrak{t}$ were arbitrary, we deduce (2.84).

To complete the proof of the Lemma, it suffices to show that $B(\cdot, v) = L$. Let $a = \sum_{k=0}^{K-1} \mathbb{1}_{[t_k,t_{k+1})}(\cdot)\, a_k$ be any $U^p X[T]$ atom. Then, if $\mathfrak{t}'$ is any partition containing $\{0 = t_0 < \ldots < t_K = T\}$, we have

$$B_{\mathfrak{t}'}(a, v) = \sum_{k=0}^{K-1} \langle a_k, v(t_{k+1}) - v(t_k)\rangle_{X,X^*} = \sum_{k=0}^{K-1} L\left(\mathbb{1}_{[t_k,t_{k+1})}(\cdot)\, a_k\right) = L(a) \ .$$

Therefore, in the limit,

$$B(a,v) = L(a) \ .$$

Since a was an arbitrary $U^p X[T]$ atom, we see from the atomic structure of $U^p X[T]$ that

$$B(u,v) = L(u)$$

for any $u \in U^p X[T]$, as required. □

Proof of Lemma 2.9. Let $f \in \mathrm{D}U^p X[T]$, so that $u(t) := \int_0^t f(t')\mathrm{d}t'$ defines a function $u \in U^p X[T]$. Notice that it suffices to show, for $v \in V^{p'} X^*[T]$, that

$$B(u,v) = -\int_0^T \langle f(t), v(t)\rangle_{X,X^*} \mathrm{d}t \ . \tag{2.85}$$

This follows from the characterisation of $(U^pX[T])^*$ in Lemma 2.50, and the fact that the integral on the right-hand side remains unchanged if we modify $v(t)$ for countably many $t \in [0,T)$ so that, we could have taken $v \in V^{p'}_{\mathrm{rc}}X^*[T]$.

We may assume that $\|v\|_{V^{p'}X^*[T]} = 1$. Let the functions $v_+, w : [0,T) \to X^*$ be given by

$$v_+(t) := v(t+) \ , \quad w(t) := v(t) - v_+(t) \ .$$

Since v is continuous except at countably many points, we see that w is nonzero only at countably many points. Moreover, it is easy to see that $v_+ \in V^{p'}_{\mathrm{rc}}X^*[T]$.

We claim that $B(u,w) = 0$. Indeed, if $\mathfrak{t} = \{0 = t_0 < \ldots < t_K = T\}$ is a partition, then recalling $u(0) = 0$ we find

$$\begin{aligned} B_{\mathfrak{t}}(u,w) &= -\sum_{k=1}^{K-1} \langle u(t_k) - u(t_{k-1}), w(t_k)\rangle_{X,X^*} \mathrm{d}t \\ &= -\sum_{k=1}^{K-1} \int_{t_{k-1}}^{t_k} \langle f(t), w(t_k)\rangle_{X,X^*} \mathrm{d}t \ . \end{aligned} \tag{2.86}$$

For $m \in \mathbb{Z}_{>0}$, let

$$P_m := \left\{ t \in (0,T) \,\middle|\, \|w(t)\|_{X^*} \geqslant 2^{-m} \right\} \ ,$$

and listing the points of P_m as $s_1 < \ldots < s_N$, denote $s'_j := \max\{\frac{1}{2}(s_{j-1} + s_j), s_j - 4^{-m}T\}$ with $s_0 := 0$, and set $P'_m := \{s'_1, \ldots, s'_j\}$. Let $I^{[m]} := \bigcup_{j=1}^N [s'_j, s_j)$. Since P_m has at most 2^m points, we see that $|I^{[m]}| \leqslant 2^{-m}T$. Now, if $\mathfrak{t} = \{0 = t_0 < \ldots < t_K = T\}$ is any partition containing all points of $P_m \cup P'_m$, then using (2.86) we easily obtain

$$|B_{\mathfrak{t}}(u,w)| \leqslant 2^{-m} \int_0^T \|f(t)\|_X \,\mathrm{d}t + \int_{I^{[m]}} \|f(t)\|_X \,\mathrm{d}t \ .$$

Since u is a well-defined element of $L^\infty_t X[T]$ we have $f \in L^1_t X^*[T]$, and hence the right-hand side vanishes in the limit $m \to \infty$.

In particular, to show (2.85) we may assume without loss of generality that $v \in V^{p'}_{\mathrm{rc}}X^*[T]$.

Fix $q \in (p', \infty)$ and note that, due to the $V^{p'}_{\mathrm{rc}}X^*[T] \hookrightarrow U^qX^*[T]$ embedding given by Lemma 2.8, we may assume that v was a $U^qX^*[T]$ atom. Write $v(t) = \sum_{k=0}^{K-1} \mathbb{1}_{[t_k,t_{k+1})}(t)\, a_k$ and the partition $\mathfrak{t} := \{0 = t_0 < \ldots < t_K = T\}$. Recalling that $u(0) = 0$, we see that for any partition $\mathfrak{t}'$ containing $\mathfrak{t}$,

$$\begin{aligned} B_{\mathfrak{t}'}(u,v) &= B_{\mathfrak{t}}(u,v) \\ &= -\sum_{k=1}^{K-1} \langle u(t_k) - u(t_{k-1}), v(t_k)\rangle_{X,X^*} \\ &= -\sum_{k=1}^{K-1} \int_{t_{k-1}}^{t_k} \langle f(t), v(t_k)\rangle_{X,X^*} \,\mathrm{d}t \\ &= -\int_0^T \langle f(t), v(t)\rangle_{X,X^*} \mathrm{d}t \ . \end{aligned}$$

Thus, (2.85) holds by taking the limit. $\square$

Chapter 3

Asymptotic completeness of the Chern-Simons-Schrödinger system with a defocusing cubic nonlinearity

In this chapter we will prove Theorems 1.2 and 1.3 establishing a kind of asymptotic completeness for the Chern-Simons-Schrödinger system in the Coulomb gauge, for a defocusing cubic nonlinearity. Physically, this describes a repulsive binary interaction between the quantum particles in the system. Henceforth in this chapter, we specialise to the case $V(|\phi|^2) = \frac{1}{4}\kappa|\phi|^4$ with $\kappa > 0$, whereby (1.3) becomes

$$\begin{cases} (\partial_t - \mathrm{i}\triangle)\,\phi = -2A_x \cdot \nabla\phi - \mathrm{i}A_0\phi - \mathrm{i}|A_x|^2\phi - \mathrm{i}\kappa|\phi|^2\phi\,, \\ \quad -\triangle A_i = -\frac{1}{2}\epsilon_{ij}\partial_j\left(|\phi|^2\right)\,, \\ \quad -\triangle A_0 = -\operatorname{Im}\left(\nabla\overline{\phi} \wedge \nabla\phi\right) - \operatorname{rot}\left(A_x|\phi|^2\right)\,. \end{cases} \tag{3.1}$$

3.1 Heuristic ideas

We first remark that the proof of linear scattering, Theorem 1.2, reduces to showing that

$$\int_1^\infty \left\|\left(-2A_x \cdot \nabla\phi - \mathrm{i}A_0\phi - \mathrm{i}|A_x|^2\phi - \mathrm{i}\kappa|\phi|^2\phi\right)(t)\right\|_{L_x^2} \mathrm{d}t < \infty\,. \tag{3.2}$$

Unfortunately, the only global-in-time a priori bounds we currently have do not seem sufficient to deduce (3.2). Indeed, the conserved mass and energy do not give any decay in time, and we do not have estimates on global-in-time Strichartz norms of ϕ. The only other global-in-time a priori bound for H^1 solutions, known to the author, is

$$\|\phi\|_{L_t^4 L_x^8} \lesssim \left\||\nabla|^{\frac{1}{2}}\,\phi\right\|_{L_t^4 L_x^4} \lesssim C\left(\|\phi(0)\|_{H^1}\right)\,,$$

which can be derived using interaction Morawetz estimates (see [11, 12, 41, 10]). However, this bound, too, seems to be inadequate to deduce (3.2).

We also recall that the electromagnetic interaction is long-range in that, from the representation formula

$$A_i(t,x) = \frac{1}{4\pi}\epsilon_{ij}\int_{\mathbb{R}^2}\frac{x_j-y_j}{|x-y|^2}\,|\phi(t,y)|^2\,\mathrm{d}y$$

given by the Biot-Savart law, the electromagnetic potentials A_x do not decay more quickly than $|x|^{-1}$ for large $|x|$. In principle, such a long range interaction could lead to complicated long-time dynamics such as modified scattering or even finite-time blow up. Therefore, in order to prove that solutions to (3.1) scatter to a free Schrödinger wave, one must identify and exploit some kind of cancellation in the long-range effects.

In fact, such a cancellation is indeed present in the long-range electromagnetic interaction, more precisely in the cubic terms $-2A_x\cdot\nabla\phi$ and $-\mathrm{i}A_0'\phi$ where $A_0' := \triangle^{-1}\mathrm{Im}(\nabla\overline{\phi}\wedge\nabla\phi)$ is the quadratic part of A_0. This cancellation was first observed by Oh and Pusateri [39], who named it the *cubic null structure* and exploited it in Fourier space to close their bootstrap argument. We can, however, already heuristically see the cancellation in physical space as follows. Write

$$\begin{aligned}
-2A_x\cdot\nabla\phi-\mathrm{i}A_0'\phi &= \epsilon_{ij}\frac{\partial_j}{\triangle}\left(|\phi|^2\right)\partial_j\phi-\epsilon_{ij}\phi\frac{\partial_i}{\triangle}\left(\overline{\phi}\partial_j\phi\right)\\
&= \frac{1}{2\pi}\epsilon_{ij}\int_{\mathbb{R}^2}\overline{\phi(y)}\,\frac{x_i-y_i}{|x-y|^2}\left[\phi(x)\partial_j\phi(y)-\phi(y)\partial_j\phi(x)\right]\,\mathrm{d}y\ .
\end{aligned}\tag{3.3}$$

Now, let us plug in the approximation $\phi(t)\approx \mathrm{e}^{\mathrm{i}t\triangle}w$ and motivate the idea that this approximation is consistent. We formally have

$$\begin{aligned}
\phi(t,x) &\approx \frac{1}{4\pi\mathrm{i}t}\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}}w(y)\,\mathrm{d}y\\
&\approx \frac{1}{4\pi\mathrm{i}t}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}\frac{x\cdot y}{2t}}w(y)\,\mathrm{d}y = \frac{1}{4\pi\mathrm{i}t}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\,\widehat{w}\left(\frac{x}{2t}\right)\ ,
\end{aligned}\tag{3.4}$$

an expression known as the *Fraunhofer approximation* of the solution ϕ. On differentiating, we expect

$$\partial_j\phi(t,x)\approx\frac{1}{8\pi\mathrm{i}t^2}\left(\partial_{\xi_j}\widehat{w}\left(\frac{x}{2t}\right)+\mathrm{i}x_j\widehat{w}\left(\frac{x}{2t}\right)\right)\ .\tag{3.5}$$

Intuitively, of the two terms on the right-hand side of (3.5), the first term $t^{-2}\nabla_\xi\widehat{w}(x/2t)$ should exhibit better time decay than the second term $t^{-2}x\widehat{w}(x/2t)$; this is true, for example, of the L^p_x norms. However, the contribution of the slowly decaying part $t^{-2}x\widehat{w}(x/2t)$ in (3.3) cancels out exactly. This suggests that our initial approximation $\phi(t)\approx\mathrm{e}^{\mathrm{i}t\triangle}w$ could be consistent.

Working with solutions with initial data in the space Σ^1, defined in (1.8), simultaneously provides new a priori bounds on the solution ϕ, and renders the approximation (3.4) effective. The crux is the discovery of a new conserved quantity for solutions to (3.1), namely that of the pseudo-conformal energy,

$$\mathcal{H}(t) := \int_{\mathbb{R}^2}\left(\frac{1}{2}\left|x\phi(t,x)+2\mathrm{i}t\mathbf{D}_x\phi(t)(x)\right|^2+\kappa t^2\left|\phi(t,x)\right|^4\right)\mathrm{d}x = \mathcal{H}(0)\ .\tag{3.6}$$

This is the direct analogue of the pseudo-conformal energy conservation law for mass-critical nonlinear Schrödinger equations [17]; note that, both the cubic nonlinear Schrödinger equation on $\mathbb{R}^2_x$ and the Chern-Simons-Schrödinger system (3.1) are mass-critical. However, the proof of the pseudo-conformal energy conservation law (3.6) is slightly more involved due to the fact that covariant derivatives do not commute. In fact, (3.6) crucially relies on the very special form of the curvature components in the last two equations in (3.1).

Using the fact that $\kappa > 0$, we will be able to deduce from (3.6) the bound

$$\left\| x \mathrm{e}^{-\mathrm{i}t\triangle} \phi(t) \right\|_{L^2_x} \leqslant C\left(\|\phi(0)\|_{\Sigma^1}\right) ,$$

which imply in particular that the solution ϕ decays in time essentially like a free Schrödinger wave,

$$\|\phi(t)\|_{L^p_x} \leqslant C\left(p, \|\phi(0)\|_{\Sigma^1}\right) t^{-\left(1-\frac{2}{p}\right)} , \quad 2 \leqslant p < \infty .$$

This allows us to immediately close the estimates on all the source terms on the right-hand side of the first equation of (3.1), except for the cubic null term $-2A_x \cdot \nabla\phi - \mathrm{i}A_0'\phi$. Exploiting the cubic null structure to estimate the cubic null term is slightly tricky, but the essential ideas are already contained in the heuristic computations above.

We end this section with a comment on the proof of Theorem 1.3. The key here is that of uncovering a very remarkable pseudo-conformal symmetry of the system (3.1). Indeed, the function $\psi = \psi(s,y) : (0,\infty) \times \mathbb{R}^2 \to \mathbb{C}$, defined by

$$\psi(s,y) := \frac{1}{2s} \mathrm{e}^{\mathrm{i}\frac{|y|^2}{4s}} \overline{\phi}\left(\frac{1}{4s}, -\frac{y}{2s}\right) ,$$

turns out to satisfy a PDE system which is almost identical to (3.1) and in particular is also globally well-posed. The solution of the initial-value problem for ψ then gives the desired wave operator.

3.2 Notations and preliminaries

As in the previous chapter, we will need to perform Fourier transforms over the spatial variable x, but never over the time variable t. Our convention for the Fourier transform is

$$\widehat{u}(\xi) := \mathcal{F}u(\xi) := \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}x\cdot\xi} u(x)\,\mathrm{d}x .$$

Using the Fourier transform we may define the inhomogeneous Littlewood-Paley decomposition. We denote the set of dyadic frequencies by

$$\mathfrak{D} := \left\{ 2^k \,\middle|\, k \in \mathbb{Z}_{\geqslant 0} \right\} .$$

Fix, once and for all, a smooth radial non-increasing function $\varphi_1 : \mathbb{R}^2_\xi \to \mathbb{R}$ such that $\varphi_1(\xi) \equiv 1$ on $|\xi| \leqslant 1$ and $\varphi_1(\xi) \equiv 0$ on $|\xi| \geqslant 2$. Then, for $\lambda \in \mathfrak{D}$, $\lambda \geqslant 2$, define

$$\varphi_\lambda(\xi) := \varphi_1\left(\tfrac{1}{\lambda}\xi\right) - \varphi_2\left(\tfrac{2}{\lambda}\xi\right) .$$

Now, we define the Littlewood-Paley frequency projectors as $\mathrm{P}_\lambda := \varphi_\lambda(\mathrm{D}_x)$. Equivalently,

$$\mathrm{P}_\lambda u(x) := \int_{\mathbb{R}^2} \widecheck{\varphi_\lambda}(x-y) u(y)\,\mathrm{d}y .$$

In this chapter, we reserve the letters λ, μ to denote elements of $\mathfrak{D}$. In particular, when summing over λ or μ, the summation is implicitly over $\mathfrak{D}$ unless otherwise stated.

Finally, we define the frequency projection operators

$$\mathrm{P}_{\leqslant\lambda} := \sum_{\mu\leqslant\lambda} \mathrm{P}_\mu \ , \quad \mathrm{P}_{<\lambda} := \mathrm{P}_{\leqslant\frac{1}{2}\lambda} \ , \quad \mathrm{P}_{\geqslant\lambda} := 1 - \mathrm{P}_{<\lambda} \ .$$

3.3 The pseudo-conformal energy conservation law

In this section we prove the pseudo-conformal energy conservation law (3.6). We shall deduce it as a special case of the following computation.

Proposition 3.1. *Consider, on $\mathbb{R}_t \times \mathbb{R}^n_x$, the gauged Schrödinger equation*

$$D_t u - \mathrm{i} \sum_{i=1}^n D_i D_i u = -2\mathrm{i} V'(|u|^2) u \tag{3.7}$$

where

$$D_t := \partial_t + \mathrm{i} A_0 \ ,$$

$$D_i := \partial_i + \mathrm{i} A_i \ , \quad i = 1, \ldots, n \ ,$$

and $A_0, A_1, \ldots, A_n : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{R}$ and $V : [0, \infty) \to \mathbb{R}$ are sufficiently smooth functions. Denote

$$F_{0i} := \partial_t A_i - \partial_i A_0 \ , \quad i = 1, \ldots, n \ ,$$

$$F_{ij} := \partial_i A_j - \partial_j A_i \ , \quad i, j = 1, \ldots, n \ .$$

Then, for a sufficiently smooth solution u to (3.7), the pseudo-conformal energy

$$H(t) := \frac{1}{2} \left\| x u(t) + 2\mathrm{i} t D_x u(t) \right\|^2_{L^2_x} + 4t^2 \int_{\mathbb{R}^n} V\left(|u(t,x)|^2\right) \mathrm{d}x$$

satisfies

$$\begin{aligned} \partial_t H = 2t \Bigg[& (2n+4) \int_{\mathbb{R}^n} V\left(|u|^2\right) \mathrm{d}x - 2n \int_{\mathbb{R}^n} V'\left(|u|^2\right) |u|^2 \,\mathrm{d}x \\ & - 2 \sum_{i,j=1}^n \int_{\mathbb{R}^n} F_{ij} x_j \mathrm{Im}\left(\overline{u}\, D_i u\right) \mathrm{d}x - \sum_{i=1}^n \int_{\mathbb{R}^n} F_{0i} x_i |u|^2 \,\mathrm{d}x \Bigg] \ . \end{aligned} \tag{3.8}$$

Proof. We follow the strategy of [8], Section 7.2, the only difference being that we account for the fact that the covariant derivatives do not commute, but instead,

$$[D_t, D_i] = \mathrm{i} F_{0i} \ , \quad i = 1, \ldots, n \ ,$$

and

$$[D_i, D_j] = \mathrm{i} F_{ij} \ , \quad i, j = 1, \ldots, n \ .$$

Note that the solution u to (3.7) also satisfies the energy conservation law,

$$E(t) := \frac{1}{2} \left\| D_x u(t) \right\|^2_{L^2_x} + \int_{\mathbb{R}^n} V\left(|u(t,x)|^2\right) \mathrm{d}x \equiv E \ .$$

Firstly, we note the identity

$$\begin{aligned}\|xu(t)+2\mathrm{i}tD_xu(t)\|_{L_x^2}^2 &= \|xu(t)\|_{L_x^2}^2+4t\sum_{i=1}^n \mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u(t)}x_iD_iu(t)\,\mathrm{d}x\right)+4t^2\,\|D_xu(t)\|_{L_x^2}^2\\ &= \|xu(t)\|_{L_x^2}^2+4t\sum_{i=1}^n \mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u(t)}x_iD_iu(t)\,\mathrm{d}x\right)\\ &\quad +4t^2\left(2E-2\int_{\mathbb{R}^n}V(|u(t,x)|^2)\,\mathrm{d}x\right)\,.\end{aligned}$$

From this, we may rewrite $H(t)$ as

$$H(t)=\frac{1}{2}\,\|xu(t)\|_{L_x^2}^2+2t\sum_{i=1}^n \mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u(t)}x_iD_iu(t)\,\mathrm{d}x\right)+4Et^2\,.$$

On differentiating, we find

$$\begin{aligned}\partial_tH &= \mathrm{Re}\left(\int_{\mathbb{R}^n}|x|^2\overline{u}D_tu\,\mathrm{d}x\right)+2\sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_iu\,\mathrm{d}x\right)\\ &\quad+2t\sum_{i=1}^n\partial_t\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_iu\,\mathrm{d}x\right)+8Et\,.\end{aligned}\tag{3.9}$$

Using (3.7) we observe

$$\begin{aligned}\mathrm{Re}\left(\int_{\mathbb{R}^n}|x|^2\overline{u}D_tu\,\mathrm{d}x\right) &= \sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}|x|^2\overline{u}D_iD_iu\,\mathrm{d}x\right)\\ &= -2\sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}x_i\overline{u}D_iu\,\mathrm{d}x\right)\,.\end{aligned}$$

Therefore, the first two terms on the right-hand side of (3.9) cancel out, and we are left with

$$\partial_tH=2t\sum_{i=1}^n\partial_t\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_iu\,\mathrm{d}x\right)+8Et\,.\tag{3.10}$$

We now compute

$$\begin{aligned}\sum_{i=1}^n\partial_t\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_iu\,\mathrm{d}x\right) &= \sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{D_tu}x_iD_iu\,\mathrm{d}x\right)+\sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_tD_iu\,\mathrm{d}x\right)\\ &= \sum_{i=1}^n\mathrm{Re}\left(-\mathrm{i}\int_{\mathbb{R}^n}x_i\,\overline{D_iu}\,D_tu\,\mathrm{d}x\right)+\sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_iD_iD_tu\,\mathrm{d}x\right)\\ &\quad-\sum_{i=1}^n\int_{\mathbb{R}^n}x_iF_{0i}|u|^2\,\mathrm{d}x\\ &= -2\sum_{i=1}^n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}x_i\,\overline{D_iu}\,D_tu\,\mathrm{d}x\right)-n\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}D_tu\,\mathrm{d}x\right)\\ &\quad-\sum_{i=1}^n\int_{\mathbb{R}^n}x_iF_{0i}|u|^2\,\mathrm{d}x\,.\end{aligned}\tag{3.11}$$

For the first term on the right-hand side of (3.11), we compute

$$\begin{aligned}
\sum_{i=1}^{n}\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n} x_i\,\overline{D_i u}\,D_t u\,\mathrm{d}x\right) &= -\sum_{i,j=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n} x_i\,\overline{D_i u}\,D_j D_j u\,\mathrm{d}x\right) + 2\sum_{i=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n} x_i V'(|u|^2)\overline{u} D_i u\,\mathrm{d}x\right)\\
&= \sum_{i,j=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n}\overline{D_j u}\,D_j\left(x_i D_i u\right)\,\mathrm{d}x\right) + \sum_{i=1}^{n}\int_{\mathbb{R}^n} x_i V'(|u|^2)\partial_i\left(|u|^2\right)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + \sum_{i,j=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n}\overline{D_j u}\,x_i D_j D_i u\,\mathrm{d}x\right) - n\int_{\mathbb{R}^n} V(|u|^2)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + \sum_{i,j=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n}\overline{D_j u}\,x_i D_i D_j u\,\mathrm{d}x\right)\\
&\quad + \sum_{i,j=1}^{n}\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{D_j u}\,x_i F_{ji} u\,\mathrm{d}x\right) - n\int_{\mathbb{R}^n} V(|u|^2)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^n} x_i\partial_i\left(|D_x u|^2\right)\,\mathrm{d}x\\
&\quad + \sum_{i,j=1}^{n}\int_{\mathbb{R}^n} F_{ij}x_j\mathrm{Im}\left(\overline{u}\,D_i u\right)\,\mathrm{d}x - n\int_{\mathbb{R}^n} V(|u|^2)\,\mathrm{d}x\\
&= \left(1-\frac{n}{2}\right)\int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + \sum_{i,j=1}^{n}\int_{\mathbb{R}^n} F_{ij}x_j\mathrm{Im}\left(\overline{u}D_i u\right)\mathrm{d}x - n\int_{\mathbb{R}^n} V(|u|^2)\,\mathrm{d}x
\end{aligned}$$

while for the second term in (3.11), we have

$$\begin{aligned}
\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}\,D_t u\,\mathrm{d}x\right) &= -\sum_{i=1}^{n}\mathrm{Re}\left(\int_{\mathbb{R}^n}\overline{u}\,D_i D_i u\,\mathrm{d}x\right) + \int_{\mathbb{R}^n} 2V'(|u|^2)|u|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} 2V'(|u|^2)|u|^2\,\mathrm{d}x\,.
\end{aligned}$$

Plugging the above expressions into (3.11), we find

$$\begin{aligned}
\sum_{i=1}^{n}\partial_t\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\overline{u}x_i D_i u\,\mathrm{d}x\right) &= -2\int_{\mathbb{R}^n}|D_x u|^2\,\mathrm{d}x + 2n\int_{\mathbb{R}^n} V\left(|u|^2\right)\,\mathrm{d}x - 2n\int_{\mathbb{R}^n} V'\left(|u|^2\right)|u|^2\,\mathrm{d}x\\
&\quad - 2\sum_{i,j=1}^{n}\int_{\mathbb{R}^n} F_{ij}x_j\mathrm{Im}\left(\overline{u}D_i u\right)\,\mathrm{d}x - \sum_{i=1}^{n}\int_{\mathbb{R}^n} F_{0i}x_i|u|^2\,\mathrm{d}x\,.
\end{aligned}$$

Substituting this expression into (3.10) completes the proof of the identity (3.8) as desired. □

Corollary 3.2. *Let ϕ be a solution to (3.1) with $\phi(0) \in \Sigma^1$, so that $\mathcal{H}(0) < \infty$. Then the pseudo-conformal energy conservation law (3.6) holds.*

Moreover, we have a uniform bound

$$\left\|\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)\right\|_{\Sigma^1} \leqslant C\left(\|\phi(0)\|_{\Sigma^1}\right)\,, \tag{3.12}$$

as well as, for $t > 0$, the decay rate

$$\|\phi(t)\|_{L^p_x} \leqslant C\left(p, \|\phi(0)\|_{\Sigma^1}\right)\, t^{-\left(1-\frac{2}{p}\right)}\,, \quad 2 \leqslant p < \infty\,. \tag{3.13}$$

Proof. For a sufficiently smooth solution ϕ to (3.1), the right-hand side of (3.8) vanishes identically so that the pseudo-conformal energy (3.6) is conserved. For general $\phi(0)$, let $\{\phi_m\}_{m=1}^{\infty}$ be a sequence of solutions to (3.1) whose initial data $\phi_m(0)$ are Schwartz functions, and such that $\phi_m(0) \to \phi(0)$ in Σ^1

and also pointwise a.e.. By the well-posedness result, Theorem 1.1, for a fixed $t > 0$, we have that $\phi_m(t) \to \phi(t)$ in H^1 and, passing to a subsequence if necessary, also pointwise a.e.. Using Fatou's lemma, we find

$$\begin{aligned}
\mathcal{H}(t) &= \frac{1}{2}\left\|x\phi(t) + 2\mathrm{i}t\mathbf{D}_x\phi(t)\right\|_{L^2_x}^2 + \kappa t^2 \left\|\phi(t)\right\|_{L^4_x}^4 \\
&\leqslant \lim_{m\to\infty}\left(\frac{1}{2}\left\|x\phi_m(t) + 2\mathrm{i}t\left(\mathbf{D}_x\right)_m\phi_m(t)\right\|_{L^2_x}^2 + \kappa t^2 \left\|\phi_m(t)\right\|_{L^4_x}^4\right) \\
&= \lim_{m\to\infty}\frac{1}{2}\left\|x\phi_m(0)\right\|_{L^2_x}^2 \\
&= \mathcal{H}(0) \ .
\end{aligned}$$

By a similar argument, we can obtain $\mathcal{H}(0) \leqslant \mathcal{H}(t)$, and this establishes the conservation of the pseudo-conformal energy (3.6).

Since $\kappa > 0$, the conserved pseudo-conformal energy $\mathcal{H}$ provides the control

$$\left\|\phi(t)\right\|_{L^4_x} \leqslant C\left(\left\|\phi(0)\right\|_{\Sigma^1}\right) t^{-\frac{1}{2}} \ .$$

The second equation of (3.1) then gives, using the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}
\left\|A_x(t)\right\|_{L^4_x} &\leqslant C\left\||\phi(t)|^2\right\|_{L_x^{\frac{4}{3}}} \leqslant C\left\|\phi(t)\right\|_{L^2_x}\left\|\phi(t)\right\|_{L^4_x} \\
&\leqslant C\left(\left\|\phi(0)\right\|_{\Sigma^1}\right) t^{-\frac{1}{2}} \ .
\end{aligned}$$

Since $x_j + 2\mathrm{i}t\partial_j = \mathrm{e}^{\mathrm{i}t\triangle}x_j\mathrm{e}^{-\mathrm{i}t\triangle}$ as operators, we obtain

$$\begin{aligned}
\left\|x\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)\right\|_{L^2_x} &= \left\|x\phi(t) + 2\mathrm{i}t\nabla\phi(t)\right\|_{L^2_x} \\
&\leqslant \left\|x\phi(t) + 2\mathrm{i}t\mathbf{D}_x\phi(t)\right\|_{L^2_x} + 2t\left\|A_x(t)\phi(t)\right\|_{L^2_x} \\
&\leqslant C\left(\left\|\phi(0)\right\|_{\Sigma^1}\right) + Ct\left\|A_x(t)\right\|_{L^4_x}\left\|\phi(t)\right\|_{L^4_x}
\end{aligned}$$

which proves (3.12).

To prove (3.13), recall that for $p \geqslant 2$ the Schrödinger semigroup $\mathrm{e}^{\mathrm{i}t\triangle}$ has the continuity property

$$\left\|\mathrm{e}^{\mathrm{i}t\triangle}w\right\|_{L^p_x} \leqslant C\left\|w\right\|_{L^{p'}_x} t^{-\left(1-\frac{2}{p}\right)} \ .$$

Therefore, for $2 \leqslant p < \infty$, Hölder's inequality gives

$$\begin{aligned}
\left\|\phi(t)\right\|_{L^p_x} \leqslant C\left\|\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)\right\|_{L^{p'}_x} t^{-\left(1-\frac{2}{p}\right)} &\leqslant C\left\|(1+|x|)^{-1}\right\|_{L_x^{\frac{2p}{p-2}}}\left\|(1+|x|)\,\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)\right\|_{L^2_x} t^{-\left(1-\frac{2}{p}\right)} \\
&\leqslant C\left(p, \left\|\phi(0)\right\|_{\Sigma^1}\right) t^{-\left(1-\frac{2}{p}\right)}
\end{aligned}$$

as desired. □

3.4 Proof of Theorem 1.2

Our goal in the present section is to prove (3.2), which would immediately imply Theorem 1.2. To this end, we rewrite the first equation of (3.1) as

$$\left(\partial_t - \mathrm{i}\triangle\right)\phi = \mathcal{N}(\phi,\phi,\phi) - 2\left(\mathrm{P}_{\geqslant 2}A_x\right)\cdot\nabla\phi - \mathrm{i}\left(\mathrm{P}_{\geqslant 2}A_0'\right)\phi - \mathrm{i}A_0''\phi - \mathrm{i}\left|A_x\right|^2\phi - \mathrm{i}\kappa|\phi|^2\phi \ , \tag{3.14}$$

where

$$A_0' := \epsilon_{ij} \triangle^{-1} \partial_i \mathrm{Im} \left(\overline{\phi}\, \partial_j \phi \right) \ ,$$
$$A_0'' := \triangle^{-1} \mathrm{rot} \left(A_x |\phi|^2 \right) \ ,$$

are the quadratic and quartic parts of A_0 respectively, and

$$\mathcal{N}(u_1, u_2, u_3) := \epsilon_{ij} \left(\mathrm{P}_1 \triangle^{-1} \partial_i \left(u_1 \overline{u_2} \right) \right) \partial_j u_3 - \epsilon_{ij} u_1 \mathrm{P}_1 \triangle^{-1} \partial_i \left(\overline{u_2}\, \partial_j u_3 \right)$$

is such that $\mathcal{N}(\phi, \phi, \phi) = -2(\mathrm{P}_1 A_x) \cdot \nabla \phi - \mathrm{i}(\mathrm{P}_1 A_0')\phi$. The point here is that the long-range effects are only present in the low-frequency components of A_x and A_0', and at low-frequencies we can replace the L^∞_x norm by some L^p_x norm for $p < \infty$ using Bernstein's inequality, in order to circumvent the failure of the Hardy-Littlewood-Sobolev inequality for $p = \infty$.

We first dispense with all the easy terms on the right-hand side of (3.14), namely every term except for the cubic null term $\mathcal{N}(\phi, \phi, \phi)$.

Proposition 3.3. *Let ϕ be a solution to (3.1) with $\phi(0) \in \Sigma^1$. Then*

$$\begin{aligned} \int_1^\infty \Big(2 \left\| \left((\mathrm{P}_{\geqslant 2} A_x) \cdot \nabla \phi \right)(t) \right\|_{L^2_x} + \left\| \left(\left(\mathrm{P}_{\geqslant 2} A_0' \right) \phi \right)(t) \right\|_{L^2_x} \\ + \left\| \left(A_0'' \phi \right)(t) \right\|_{L^2_x} + \left\| \left(|A_x|^2 \phi \right)(t) \right\|_{L^2_x} + \kappa \left\| \left(|\phi|^2 \phi \right)(t) \right\|_{L^2_x} \Big) \, \mathrm{d}t < \infty \ . \end{aligned} \tag{3.15}$$

Proof. Using the second equation of (3.1) and the Hölder and Bernstein inequalities, we have for $\lambda \geqslant 2$ that

$$\begin{aligned} \left\| \left((\mathrm{P}_\lambda A_x) \cdot \nabla \phi \right)(t) \right\|_{L^2_x} &\leqslant C \lambda^{-1} \left\| \mathrm{P}_\lambda \left(|\phi|^2 \right)(t) \right\|_{L^\infty_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \\ &\leqslant C \lambda^{-\frac{1}{2}} \left\| |\phi|^2(t) \right\|_{L^4_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \\ &\leqslant C \lambda^{-\frac{1}{2}} \left\| \phi(t) \right\|^2_{L^8_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \ . \end{aligned}$$

We sum over $\lambda \geqslant 2$ and recall Corollary 3.2 to deduce

$$\left\| \left((\mathrm{P}_{\geqslant 2} A_x) \cdot \nabla \phi \right)(t) \right\|_{L^2_x} \leqslant C \left\| \phi(t) \right\|^2_{L^8_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \leqslant C \left(\left\| \phi(0) \right\|_{\Sigma^1} \right) t^{-\frac{3}{2}} \ . \tag{3.16}$$

Next, using the Hölder and Bernstein inequalities, we have for $\lambda \geqslant 2$ that

$$\begin{aligned} \left\| \left(\left(\mathrm{P}_\lambda A_0' \right) \phi \right)(t) \right\|_{L^2_x} &\leqslant C \lambda^{-1} \left\| \mathrm{P}_\lambda \left(\overline{\phi}\, \nabla \phi \right)(t) \right\|_{L^{\frac{8}{3}}_x} \left\| \phi(t) \right\|_{L^8_x} \\ &\leqslant C \lambda^{-\frac{1}{2}} \left\| \left(\overline{\phi}\, \nabla \phi \right)(t) \right\|_{L^{\frac{8}{5}}_x} \left\| \phi(t) \right\|_{L^8_x} \\ &\leqslant C \lambda^{-\frac{1}{2}} \left\| \phi(t) \right\|^2_{L^8_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \ . \end{aligned}$$

We sum over $\lambda \geqslant 2$ and recall Corollary 3.2 to deduce

$$\left\| \left(\left(\mathrm{P}_{\geqslant 2} A_0' \right) \phi \right)(t) \right\|_{L^2_x} \leqslant C \left\| \phi(t) \right\|^2_{L^8_x} \left\| \nabla \phi(t) \right\|_{L^2_x} \leqslant C \left(\left\| \phi(0) \right\|_{\Sigma^1} \right) t^{-\frac{3}{2}} \ . \tag{3.17}$$

Using the Hölder and Hardy-Littlewood-Sobolev inequalities, we have

$$\begin{aligned} \left\| \left(A_0'' \phi \right)(t) \right\|_{L^2_x} &\leqslant \left\| A_0''(t) \right\|_{L^4_x} \left\| \phi(t) \right\|_{L^4_x} \leqslant C \left\| \left(A_x |\phi|^2 \right)(t) \right\|_{L^{\frac{4}{3}}_x} \left\| \phi(t) \right\|_{L^4_x} \\ &\leqslant C \left\| A_x(t) \right\|_{L^4_x} \left\| \phi(t) \right\|^3_{L^4_x} \leqslant C \left\| |\phi|^2(t) \right\|_{L^{\frac{4}{3}}_x} \left\| \phi(t) \right\|^3_{L^4_x} \leqslant C \left\| \phi(t) \right\|_{L^2_x} \left\| \phi(t) \right\|^4_{L^4_x} \ . \end{aligned}$$

Then Corollary 3.2 gives

$$\left\| \left(A_0''\phi\right)(t)\right\|_{L_x^2} \leqslant C\left(\|\phi(0)\|_{\Sigma^1}\right) t^{-2} . \tag{3.18}$$

Next, again using the Hölder and Hardy-Littlewood-Sobolev inequalities, we have

$$\left\| \left(|A_x|^2\phi\right)(t)\right\|_{L_x^2} \leqslant \|A_x(t)\|_{L_x^6}^2 \|\phi(t)\|_{L_x^6} \leqslant C \left\| |\phi|^2(t)\right\|_{L_x^{\frac{3}{2}}}^2 \|\phi(t)\|_{L_x^6} \leqslant C\|\phi(t)\|_{L_x^2}^2 \|\phi(t)\|_{L_x^6}^3 .$$

Then Corollary 3.2 gives

$$\left\| \left(|A_x|^2\phi\right)(t)\right\|_{L_x^2} \leqslant C\left(\|\phi(0)\|_{\Sigma^1}\right) t^{-2} . \tag{3.19}$$

Finally, using Hölder's inequality and Corollary 3.2 we get

$$\left\| \left(|\phi|^2\phi\right)(t)\right\|_{L_x^2} \leqslant \|\phi(t)\|_{L_x^6}^3 \leqslant C\left(\|\phi(0)\|_{\Sigma^1}\right) t^{-2} . \tag{3.20}$$

The assertion (3.15) then follows from (3.16), (3.17), (3.18), (3.19), (3.20). □

We now turn to estimating the remaining cubic null term $\mathcal{N}(\phi,\phi,\phi)$. In order to be able to exploit the cancellation in the cubic null structure, we will need to know that the kernel of $\mathrm{P}_1\triangle^{-1}\partial_i$ has the following form.

Lemma 3.4. *The distributional kernel of the operator* $\mathrm{P}_1\triangle^{-1}\partial_i$ *is* $x_i K(|x|)$*, where* $K = K(r)$ *is a continuous function which satisfies the estimate*

$$K(r) \leqslant \frac{C}{(1+r)^2} . \tag{3.21}$$

Proof. We shall in fact prove the following exact formula

$$K(r) = \frac{1}{2\pi r^2}\int_{\{|y|<r\}} \widetilde{\varphi_1}(y)\,\mathrm{d}y$$

which clearly is continuous and satisfies (3.21). From the Biot-Savart law, the operator $\mathrm{P}_1\triangle^{-1}\partial_i$ has a kernel given by $\frac{1}{2\pi}\frac{x_i}{|x|^2} * \widetilde{\varphi_1}$. Since $\widetilde{\varphi_1}$ is a radial Schwartz function, it suffices to prove that if $\chi = \chi(r)$ is a radial Schwartz function then

$$\int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^2}\chi(|y|)\,\mathrm{d}y = \frac{x_i}{|x|^2}\int_{\{|y|<|x|\}} \chi(|y|)\,\mathrm{d}y . \tag{3.22}$$

By a simple scaling argument, it suffices to prove (3.22) in the special case where $|x| = 1$.

Let e be the unit vector in the direction of x. We effect a change of variables for the integral on the left-hand side of (3.22), setting $r := |y|$ and θ to be the angle between x and y. Then

$$\int_{\mathbb{R}^2} \frac{(x-y)\cdot e}{|x-y|^2}\chi(|y|)\,\mathrm{d}y = \int_0^\infty \chi(r) r \int_0^{2\pi} \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r . \tag{3.23}$$

The θ integral in (3.23) may, in fact, be evaluated exactly. Indeed, effecting the change of variables $z := \mathrm{e}^{\mathrm{i}\theta}$, we obtain

$$\int_0^{2\pi} \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}\,\mathrm{d}\theta = \frac{1}{2\mathrm{i}}\int_{\partial\mathbb{D}} \frac{z^2 - \frac{2}{r}z + 1}{z\,(z-r)\left(z - \frac{1}{r}\right)}\,\mathrm{d}z = 2\pi\mathbb{1}_{\{r<1\}}$$

where $\partial\mathbb{D}$ denotes the boundary of the closed unit disc in $\mathbb{C}$, and the second equality is a simple application of Cauchy's residue theorem. Plugging into (3.23), we have

$$\int_{\mathbb{R}^2} \frac{(x-y)\cdot e}{|x-y|^2}\chi(|y|)\,\mathrm{d}y = 2\pi\int_0^1 \chi(r)r\,\mathrm{d}r = \int_{\{|y|<1\}} \chi(|y|)\,\mathrm{d}y\ . \tag{3.24}$$

On the other hand, if $e^{\perp}$ is any unit vector perpendicular to the direction of x, then

$$\int_{\mathbb{R}^2} \frac{(x-y)\cdot e^{\perp}}{|x-y|^2}\chi(|y|)\,\mathrm{d}y = \int_0^{\infty}\chi(r)r\int_0^{2\pi}\frac{-r\sin\theta}{1+r^2-2r\cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r = 0\ . \tag{3.25}$$

Hence (3.22) follows from (3.24) and (3.25). $\square$

Definition 3.5. For $w \in \varSigma^1$ and $t>0$, we define the *Fraunhofer lift* of w at time t to be the function

$$\mathfrak{F}_t w(x) := \frac{1}{4\pi\mathrm{i}t}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\,\widehat{w}\left(\frac{x}{2t}\right)\ . \qquad \square$$

The following is the crucial estimate for the cubic null term $\mathcal{N}$.

Lemma 3.6. *For $t \geqslant 1$ and $w \in \varSigma^1$, we have the estimate*

$$\|\mathcal{N}(\mathfrak{F}_t w,\mathfrak{F}_t w,\mathfrak{F}_t w)\|_{L^2_x} \leqslant Ct^{-\frac{3}{2}}\,\|(1+|x|)\,w\|^3_{L^2_x}\ .$$

Proof. We first record the easy expression

$$\partial_j\mathfrak{F}_t w(x) = \frac{1}{8\pi\mathrm{i}t^2}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(\partial_{\xi_j}\widehat{w}\left(\frac{x}{2t}\right) + \mathrm{i}x_j\widehat{w}\left(\frac{x}{2t}\right)\right)\ . \tag{3.26}$$

Let $K=K(r)$ be the function given in Lemma 3.4. From the definition, we have

$$\begin{aligned}
&\mathcal{N}(\mathfrak{F}_t w,\mathfrak{F}_t w,\mathfrak{F}_t w)(x)\\
&= \int_{\mathbb{R}^2} K(|x-y|)\,\epsilon_{ij}(x_i-y_i)\left(\mathfrak{F}_t w(y)\partial_j\mathfrak{F}_t w(x) - \mathfrak{F}_t w(x)\partial_j\mathfrak{F}_t w(y)\right)\overline{\mathfrak{F}_t w(y)}\,\mathrm{d}y\\
&= \frac{1}{128\pi^3\mathrm{i}t^4}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\int_{\mathbb{R}^2} K(|x-y|)\,\epsilon_{ij}(x_i-y_i)\left(\widehat{w}\left(\frac{y}{2t}\right)\partial_{\xi_j}\widehat{w}\left(\frac{x}{2t}\right) - \widehat{w}\left(\frac{x}{2t}\right)\partial_{\xi_j}\widehat{w}\left(\frac{y}{2t}\right)\right)\overline{\widehat{w}\left(\frac{y}{2t}\right)}\,\mathrm{d}y
\end{aligned}$$

where, in the second equality, we have noted the exact cancellation of the contribution from the second term on the right-hand side of (3.26). Therefore we may write

$$\begin{aligned}
\mathcal{N}(\mathfrak{F}_t w,\mathfrak{F}_t w,\mathfrak{F}_t w)(x) &= \frac{1}{128\pi^3\mathrm{i}t^4}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\,\epsilon_{ij}\left(\mathrm{P}_1\triangle^{-1}\partial_i\left(|\widehat{w}|^2\left(\frac{\cdot}{2t}\right)\right)\right)(x)\partial_{\xi_j}\widehat{w}\left(\frac{x}{2t}\right)\\
&\quad - \frac{1}{128\pi^3\mathrm{i}t^4}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\,\widehat{w}\left(\frac{x}{2t}\right)\epsilon_{ij}\left(\mathrm{P}_1\triangle^{-1}\partial_i\left(\left(\overline{\widehat{w}}\partial_{\xi_j}\widehat{w}\right)\left(\frac{\cdot}{2t}\right)\right)\right)(x)\\
&=: \mathrm{I}+\mathrm{II}\ .
\end{aligned}$$

Using Hölder, Bernstein and Hardy-Littlewood-Sobolev inequalities, we may estimate

$$\begin{aligned}
\|\mathrm{I}\|_{L^2_x} &\leqslant Ct^{-3}\left\|\mathrm{P}_1\triangle^{-1}\nabla\left(|\widehat{w}|^2\left(\frac{\cdot}{2t}\right)\right)\right\|_{L^\infty_x}\|\nabla_\xi\widehat{w}\|_{L^2_\xi}\\
&\leqslant Ct^{-3}\left\|\triangle^{-1}\nabla\left(|\widehat{w}|^2\left(\frac{\cdot}{2t}\right)\right)\right\|_{L^4_x}\|xw\|_{L^2_x}\\
&\leqslant Ct^{-3}\left\||\widehat{w}|^2\left(\frac{\cdot}{2t}\right)\right\|_{L^{\frac{4}{3}}_x}\|(1+|x|)\,w\|_{L^2_x}\\
&\leqslant Ct^{-\frac{3}{2}}\|\widehat{w}\|^2_{L^{\frac{8}{3}}_x}\|(1+|x|)\,w\|_{L^2_x}\\
&\leqslant Ct^{-\frac{3}{2}}\|w\|^2_{L^{\frac{8}{5}}_x}\|(1+|x|)\,w\|_{L^2_x}\\
&\leqslant Ct^{-\frac{3}{2}}\|(1+|x|)\,w\|^3_{L^2_x}\ ,
\end{aligned}$$

and

$$\begin{aligned}
\|\mathrm{II}\|_{L^2_x} &\leqslant Ct^{-\frac{7}{2}} \|\widehat{w}\|_{L^4_\xi} \left\| \triangle^{-1} \nabla \left(\left(\overline{\widehat{w}}\, \nabla_\xi \widehat{w} \right) \left(\frac{\cdot}{2t} \right) \right) \right\|_{L^4_x} \\
&\leqslant Ct^{-\frac{7}{2}} \|\widehat{w}\|_{L^4_\xi} \left\| \left(\overline{\widehat{w}}\, \nabla_\xi \widehat{w} \right) \left(\frac{\cdot}{2t} \right) \right\|_{L^{\frac{4}{3}}_x} \\
&\leqslant Ct^{-2} \|\widehat{w}\|_{L^4_\xi} \left\| \overline{\widehat{w}} \right\|_{L^4_\xi} \|\nabla_\xi \widehat{w}\|_{L^2_\xi} \\
&\leqslant Ct^{-2} \|w\|^2_{L^{\frac{4}{3}}_x} \|xw_3\|_{L^2_x} \\
&\leqslant Ct^{-2} \|(1 + |x|)\, w\|^3_{L^2_x} \ .
\end{aligned}$$

This completes the proof of the Lemma. □

Proposition 3.7. *Let ϕ be a solution to (3.1) with $\phi(0) \in \Sigma^1$. Then*

$$\int_1^\infty \|\mathcal{N}(\phi(t), \phi(t), \phi(t))\|_{L^2_x} \, \mathrm{d}t < \infty \ .$$

Proof. Note that

$$\phi(t) = \frac{1}{4\pi \mathrm{i} t} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}} \left(\mathrm{e}^{-\mathrm{i}t\triangle} \phi(t) \right)(y) \, \mathrm{d}y = \mathfrak{F}_t \left(\mathrm{e}^{\mathrm{i}\frac{|\cdot|^2}{4t}} \mathrm{e}^{-\mathrm{i}t\triangle} \phi(t) \right) \ .$$

Thus, by Lemma 3.6 and Corollary 3.2, we have for $t \geqslant 1$ that

$$\|\mathcal{N}(\phi(t), \phi(t), \phi(t))\|_{L^2_x} \leqslant Ct^{-\frac{3}{2}} \left\| (1 + |x|)\, \mathrm{e}^{-\mathrm{i}t\triangle} \phi(t) \right\|^3_{L^2_x} \leqslant C\left(\|\phi(0)\|_{\Sigma^1} \right) t^{-\frac{3}{2}} \ ,$$

which completes the proof. □

Theorem 1.2 follows immediately from Propositions 3.3 and 3.7. □

3.5 Proof of Theorem 1.3

Let $\phi_\infty \in \Sigma^1$ be given. Consider on $(s, y) \in \mathbb{R} \times \mathbb{R}^2$ the following initial value problem

$$\begin{cases}
(\partial_s - \mathrm{i}\triangle_y)\, \psi = -2B_y \cdot \nabla_y \psi - \mathrm{i}B_0 \psi - \mathrm{i}|B_y|^2 \psi - \mathrm{i}\kappa |\psi|^2 \psi \ , \\
\partial_{y_1} B_2 - \partial_{y_2} B_1 = \frac{1}{2} |\psi|^2 \ , \\
\partial_s B_i - \partial_{y_i} B_0 = \epsilon_{ij} \mathrm{Im}(\overline{\psi}(\partial_{y_j} \psi + \mathrm{i}B_j \psi)) \ , \\
\partial_{y_1} B_1 + \partial_{y_2} B_2 = 0
\end{cases} \tag{3.27}$$

which, after straightforward manipulations, is equivalent to

$$\begin{cases}
(\partial_s - \mathrm{i}\triangle_y)\, \psi = -2B_y \cdot \nabla_y \psi - \mathrm{i}B_0 \psi - \mathrm{i}|B_y|^2 \psi - \mathrm{i}\kappa |\psi|^2 \psi \ , \\
-\triangle B_i = \frac{1}{2} \epsilon_{ij} \partial_{y_j} \left(|\psi|^2 \right) \ , \\
-\triangle B_0 = \mathrm{Im} \left(\nabla_y \overline{\psi} \wedge \nabla_y \psi \right) + \mathrm{rot}_y \left(B_y |\psi|^2 \right) \ .
\end{cases} \tag{3.28}$$

We impose initial conditions $\psi(0) = \psi^{\mathrm{in}}$ where

$$-\frac{1}{2\pi \mathrm{i}} \widehat{\psi^{\mathrm{in}}} := \phi_\infty \ . \tag{3.29}$$

Since $\phi_\infty \in \Sigma^1$, we also have $\psi^{\text{in}} = \psi^{\text{in}}(y) \in \Sigma^1$.

By the same iteration scheme as that used in Chapter 2 to prove Theorem 1.1, there exists a unique global solution $\psi \in C(\mathbb{R}_s, H^1_y)$ to (3.28) with initial datum ψ^{in} given by (3.29). Moreover, by Proposition 3.1, the system (3.28) enjoys conservation of the pseudo-conformal energy, and the proof of Corollary 3.2 also shows that $\mathrm{e}^{-\mathrm{i}s\triangle_y}\psi(s) \in L^\infty(\mathbb{R}_s, \Sigma^1_y)$. The crucial observation is the following.

Lemma 3.8. *On $(t,x) \in (0,\infty) \times \mathbb{R}^2$, set*

$$\begin{aligned}
\phi(t,x) &:= \frac{1}{2t}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\overline{\psi}\left(\frac{1}{4t}, -\frac{x}{2t}\right) ,\\
A_i(t,x) &:= \frac{1}{2t}B_i\left(\frac{1}{4t}, -\frac{x}{2t}\right) ,\\
A_0(t,x) &:= \frac{1}{4t^2}\left(B_0\left(\frac{1}{4t}, -\frac{x}{2t}\right) - 2x_jB_j\left(\frac{1}{4t}, -\frac{x}{2t}\right)\right) .
\end{aligned}$$

Then (ϕ, A) satisfies the Chern-Simons-Schrödinger system (3.1).

Proof. This is really a direct but somewhat tedious computation. We assist the reader by providing some of the details below.

We first record for convenience that

$$\partial_i\phi(t,x) = -\frac{1}{4t^2}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(\partial_{y_i}\overline{\psi} - \mathrm{i}x_i\overline{\psi}\right) . \tag{3.30}$$

Now, we have

$$\partial_1A_2 - \partial_2A_1 = -\frac{1}{4t^2}\left(\partial_{y_1}B_2 - \partial_{y_2}B_1\right) = -\frac{1}{8t^2}|\psi|^2 = -\frac{1}{2}|\phi|^2 .$$

Since the last equation of (3.27) implies the Coulomb gauge condition $\partial_1A_1 + \partial_2A_2 = 0$, we immediately obtain the second equation of (3.1).

Next, we compute

$$\begin{aligned}
\partial_tA_i - \partial_iA_0 &= -\frac{1}{8t^3}\left(\partial_sB_i - \partial_{y_i}B_0\right) + \frac{x_j}{4t^3}\left(\partial_{y_j}B_i - \partial_{y_i}B_j\right)\\
&= -\frac{1}{8t^3}\left(\partial_sB_i - \partial_{y_i}B_0\right) - \frac{1}{4t^3}\epsilon_{ij}x_j\left(\partial_{y_1}B_2 - \partial_{y_2}B_1\right)\\
&= \frac{1}{8t^3}\epsilon_{ij}\mathrm{Im}\left(\psi\left(\partial_{y_j}\overline{\psi} - \mathrm{i}B_j\overline{\psi}\right)\right) - \frac{1}{8t^3}\epsilon_{ij}x_j|\psi|^2\\
&= \frac{1}{8t^3}\epsilon_{ij}\mathrm{Im}\left(\psi\left(\partial_{y_j}\overline{\psi} - \mathrm{i}x_j\overline{\psi}\right)\right) - \frac{1}{8t^3}\epsilon_{ij}B_j|\psi|^2\\
&= -\epsilon_{ij}\mathrm{Im}\left(\overline{\phi}\partial_j\phi\right) - \epsilon_{ij}A_j|\phi|^2 .
\end{aligned}$$

Then the Coulomb gauge condition $\partial_1A_1 + \partial_2A_2 = 0$ gives the third equation of (3.1).

It remains to derive the first equation of (3.1). We leave it to the reader to verify, with the help of

(3.30), that

$$\begin{aligned}
\partial_t\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(\partial_s\overline{\psi} - 2x_j\partial_{y_j}\overline{\psi} + \left(\mathrm{i}\,|x|^2+4t\right)\overline{\psi}\right)\ ,\\
-\mathrm{i}\,\triangle\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(\mathrm{i}\,\triangle_y\overline{\psi} + 2x_j\partial_{y_j}\overline{\psi} - \left(\mathrm{i}\,|x|^2+4t\right)\overline{\psi}\right)\ ,\\
2A_x\cdot\nabla\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(2B_y\cdot\nabla_y\overline{\psi} - 2\mathrm{i}x_jB_j\overline{\psi}\right)\ ,\\
\mathrm{i}A_0\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(-\mathrm{i}B_0\overline{\psi} + 2\mathrm{i}x_jB_j\overline{\psi}\right)\ ,\\
\mathrm{i}|A_x|^2\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(-\mathrm{i}|B_y|^2\overline{\psi}\right)\ ,\\
\mathrm{i}\kappa|\phi|^2\phi &= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left(-\mathrm{i}\kappa|\psi|^2\overline{\psi}\right)\ .
\end{aligned}$$

On adding these up, we obtain

$$\begin{aligned}
&(\partial_t - \mathrm{i}\triangle)\phi + 2A_x\cdot\nabla\phi + \mathrm{i}A_0\phi + \mathrm{i}|A_x|^2\phi + \mathrm{i}\kappa|\phi|^2\phi\\
&= -\frac{1}{8t^3}\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\left((\partial_s+\mathrm{i}\triangle_y)\overline{\psi} + 2B_y\cdot\nabla_y\overline{\psi} - \mathrm{i}B_0\overline{\psi} - \mathrm{i}|B_y|^2\overline{\psi} - \mathrm{i}\kappa|\psi|^2\overline{\psi}\right) = 0\ ,
\end{aligned}$$

thus giving the first equation of (3.1). □

Therefore, we solve the initial value problem (3.28), with initial datum (3.29), uniquely to $s = \frac{1}{2}$. Then, applying the transformation in Lemma 3.8, we obtain $\phi(t = \frac{1}{2})$ with which we may solve (3.1) uniquely backwards to find $\phi(0)$. This constructs the desired wave operator.

It remains to show that the solution ϕ so constructed does indeed scatter to ϕ_∞. We calculate

$$\begin{aligned}
\left(\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)\right)(x) &= -\frac{1}{4\pi\mathrm{i}t}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}\frac{|x-x'|^2}{4t}}\frac{1}{2t}\mathrm{e}^{\mathrm{i}\frac{|x'|^2}{4t}}\overline{\psi}\left(\frac{1}{4t}, -\frac{x'}{2t}\right)\mathrm{d}x'\\
&= -\frac{1}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4t}}\int_{\mathbb{R}^2}\mathrm{e}^{-\mathrm{i}x\cdot z}\overline{\psi}\left(\frac{1}{4t}, z\right)\mathrm{d}z\\
&= -\frac{1}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4t}}\widehat{\overline{\psi}}\left(\frac{1}{4t}, x\right)\ .
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\left|\left(\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)-\phi_\infty\right)(x)\right| &\leqslant \frac{1}{2\pi}\left|\widehat{\overline{\psi}}\left(\frac{1}{4t}, x\right) - \widehat{\overline{\psi}}(0,x)\right| + \left|1-\mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4t}}\right|\left|\phi_\infty(x)\right|\\
&\leqslant \frac{1}{2\pi}\left|\widehat{\overline{\psi}}\left(\frac{1}{4t}, x\right) - \widehat{\overline{\psi}}(0,x)\right| + C\frac{|x|}{t^{\frac{1}{2}}}\left|\phi_\infty(x)\right|\ ,
\end{aligned}$$

which shows

$$\left\|\mathrm{e}^{-\mathrm{i}t\triangle}\phi(t)-\phi_\infty\right\|_{L^2_x} \leqslant C\left\|\psi\left(\frac{1}{4t}\right)-\psi(0)\right\|_{L^2_y} + Ct^{-\frac{1}{2}}\left\||x|\phi_\infty\right\|_{L^2_x}\ .$$

This completes the proof of Theorem 1.3. □

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