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FREGE'S BEGRIFFSSCHRIFT IS INDEED FIRST-ORDER COMPLETE

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1. It is widely taken that the first-order part of Frege's *Begriffsschrift* (*1879*) is complete. However, there does not seem to have been a formal verification of this received claim. The general concern is that Frege's system is one axiom short in the first-order predicate calculus comparing to, by now, standard first-order theory. Yet Frege has one extra inference rule in his system. Then the question is whether Frege's first-order calculus is still deductively sufficient as far as the first-order completeness is concerned. In this short note we confirm that the missing axiom is derivable from his stated axioms and inference rules, and hence the logic system in the *Begriffsschrift* is indeed first-order complete.

2. We provide a list of Frege's logical axioms and inference rules for first-order calculus. We use modern notations instead of Frege's two dimensional notations. The equation or section numbers at the end of each line refer to the original propositions and sections in the *Begriffsschrift*, where these axioms and rules appear.

Logical Axioms

Sentential calculus

F1. $A \to (B \to A)$ (1)

F2. $\big[A \to (B \to C)\big] \to \big[(A \to B) \to (A \to C)\big]$ (2)

F3. $\big[A \to (B \to C)\big] \to \big[B \to (A \to C)\big]$ (8)

F4. $(A \to B) \to (\neg B \to \neg A)$ (28)

F5. $\neg\neg A \to A$ (31)

F6. $A \to \neg\neg A$ (41)

Predicate calculus

F7. $x \equiv y \to \big[\varphi(x) \to \varphi(y)\big]$ (52)

F8. $x \equiv x$ (54)

F9. $\forall x \varphi(x) \to \varphi(t)$ if t is free for x in $\varphi(x)$ (58)

Inference Rules

i. Modus ponens (MP) (§6)

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}$$

ii. Universal generalization (UG) (§11)

$$\frac{\vdash \varphi(x)}{\vdash \forall x \varphi(x)}$$

iii. Frege's special rule of universal generalization (FUG) (§11)

$$\frac{\vdash \alpha \to \varphi(x)}{\vdash \alpha \to \forall x \varphi(x),}$$

provided that x is not free in α.

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3. The following logical axiom in standard first-order logic is absent from Frege's system:[1]

F10*. $\forall x\big(\varphi \to \psi(x)\big) \to \big(\varphi \to \forall x \psi(x)\big)$ if φ contains no free occurrences of x.

We show that F10* is derivable from the stated axioms F1–F9 and inference rules MP, UG, and FUG. To this end, we first establish the following lemma.

Lemma. Suppose that $\Gamma, \alpha \vdash \beta$ and that no application of FUG or UG to a wff has as its quantified variable a free variable of α.[2] Then $\Gamma \vdash \alpha \to \beta$.

Proof. Let $\beta_0, \ldots, \beta_{n-1}$ be a proof of β from Γ and α. It suffices to show that, for every $i < n$,

$$\Gamma \vdash \alpha \to \beta_i.$$

(1) It is easy to show, by choosing proper tautologies, that the lemma holds for the cases in which β_i is either an axiom or a member of $\Gamma \cup \{\alpha\}$, or can be derived from previous steps by MP.
(2) If β_i is obtained from β_j by FUG, where $\beta_j = \beta_r \to \beta_k(x)$ for some $r, k < j < i$, and x is not free in β_r, that is, $\beta_i = \beta_r \to \forall x \beta_k(x)$. By the assumption of the lemma we know that x is not free in α. We want to show $\Gamma \vdash \alpha \to \beta_i$, namely, $\Gamma \vdash \alpha \to \big(\beta_r \to \forall x \beta_k(x)\big)$. Note that, by the inductive hypothesis, $\Gamma \vdash \alpha \to \beta_r$ and $\Gamma \vdash \alpha \to \beta_k$. For the latter, we have that x is not free in α, then by FUG we get $\Gamma \vdash \alpha \to \forall x \beta_k(x)$. These together with the tautology $(A \to B) \to \big[(A \to C) \to \big(A \to (B \to C)\big)\big]$ give us $\Gamma \vdash \alpha \to \big(\beta_r \to \forall x \beta_k(x)\big)$, which is what we want.
(3) If β_i is obtained from β_j by UG, that is, $\beta_i = \forall x \beta_j$ for some $j < i$. By the inductive hypothesis $\Gamma \vdash \alpha \to \beta_j$, and by the assumption of the lemma that x is not free in α, we therefore have, by FUG, $\Gamma \vdash \alpha \to \forall x \beta_j$, and hence $\Gamma \vdash \alpha \to \beta_i$. □

Theorem. For any wffs φ and ψ, show that F10* holds.

Proof. (1) $\forall x\big(\varphi \to \psi(x)\big)$ Hyp
(2) φ Hyp
(3) $\varphi \to \psi(x)$ (1) and F9
(4) $\psi(x)$ (2), (3), and MP
(5) $\forall x \psi(x)$ (4) and UG
(6) $\forall x\big(\varphi \to \psi(x)\big), \varphi \vdash \forall x \psi(x)$ (1)–(5)
(7) $\vdash \forall x\big(\varphi \to \psi(x)\big) \to \big(\varphi \to \forall x \psi(x)\big)$ (6) and Lemma (twice)
□

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[1]Take, for instance, the complete first-order theory (A1–A7 + MP & Gen) in *Mendelson 2015* (§2). It is easy to verify that F1–F6 are provably equivalent to the sentential axioms A1–A3. For predicate calculus, F7–F9 correspond to A7, A6 and A4, respectively. The only missing axiom is A5, i.e., F10*.

[2]That is to say, no variable in a wff which becomes quantified by applying FUG or UG is a free variable in α. Actually, we can broaden the application of this lemma by considering the possibility that α does not play any role in proving β, then in this case the quantified variable can be a free variable in α. The proof for this case is easy.

References

Frege, G. 1879. 'Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought', in: J. van Heijenoort, ed. 1967. *From Frege to Gödel: a source book in mathematical logic, 1879-1931*, Cambridge, MA: Harvard University Press, pp. 1–82.

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