Some Geometrical Results in Algebraic Topology

by

Hugh Reynolds Morton

of

Queens' College

THE BOARD OF GRADUATE STUDIES
APPROVED THIS DISSERTATION
FOR THE Ph. D. DEGREE ON 9 MAY 1967

Presented for the degree of Ph.D at the
University of Cambridge.
December 1966
I declare that no part of this thesis has been or is being submitted for any degree, diploma or other qualification at any other university.

Except where stated in the text the results proved are original, and except for theorem 1 of section 2, they are not the result of joint work.
Introduction

The two parts of this thesis belong to different parts of algebraic topology. Each has its own introduction, but we give here a brief description of both, and mention some of the geometry involved.

The first part deals with certain spaces of topological homeomorphisms of 2-manifolds. Some of these spaces were considered by H.-B. Hamstrom [4,5], who has calculated their homotopy groups. Here we prove either the contractibility or homotopy equivalence of the spaces, by geometrical construction of a contracting map or homotopy equivalence.

The use of geometrical technique appears again in the second part, in the field of combinatorial topology. The main geometrical tool in the first two sections is the link of a point. Results are obtained using this about compact polyhedra regarded as joins of other polyhedra, and about PL homeomorphisms of polyhedra. The final section, still with the underlying idea of the link of a point, is taken up mainly with calculating the symmetric products of a circle.
Acknowledgements

I am grateful to the Ministry of Education of Northern Ireland for the award of a Research Studentship which has supported this work. I must also express my thanks to M.A. Armstrong and R.L.E. Schwarzenberger for several helpful conversations, and to the University of Warwick, for providing the opportunity for such talks. Most particularly though, I am indebted to my supervisor D.B.A. Epstein, and it is a pleasure to thank him here for all his suggestions, criticism and advice over the past three years.
M.-E. Hamstrom and E. Dyer [4], showed that the identity component of the space of homeomorphisms of an annulus onto itself, keeping the boundary pointwise fixed, is contractible. In [5], Hamstrom showed that, denoting by $H_{n+1}$ the identity component of the space of homeomorphisms of a 2-disc with $n$ holes onto itself, keeping the boundary pointwise fixed, we have $\pi_1(H_{n+1}) = 0$ for all $i > 0$.

The main result of this section is that $H_{n+1}$ is contractible for all $n$. The techniques used also show that, for any closed 2-manifold $M$, if $H(M_n)$ is the similar space of homeomorphisms of $M$ with $n$ holes, then $H(M_n)$ and $H(M_1)$ are homotopy equivalent, for all $n \geq 1$.

The results follow from two lemmas, which are stated here with sketch proofs. The detailed proofs are given later.

Define $H^*(M_n)$ to be the identity component of the space of homeomorphisms of a closed 2-manifold $M$, with $n$ holes, onto itself, keeping the boundary, and also one interior point $e$, pointwise fixed.

**Lemma 1** $H(M_n)$ deformation retracts onto $H^*(M_n)$ for $n \geq 1$.

To every homeomorphism $h$ in $H(M_n)$ we assign continuously a point $\epsilon(h)$ in the interior of the universal cover of $M_n$ which
lies above the point $h(e)$.

To every point of the interior of the universal cover we assign a canonically defined path to a chosen base point lying above $e$, and also a canonical isotopy of the manifold, keeping the boundary fixed. This isotopy starts at the identity, and makes the projection of $\epsilon(h)$ trace out the projection of the canonical path, to finish at the point $e$. By following a homeomorphism $h$ with this isotopy for the point $\epsilon(h)$ we get a canonical isotopy of $h$ to a homeomorphism $h^*$ which keeps $e$ fixed. This $h^*$ lies in $H^*(\mathcal{M}_n)$.

**Lemma 2** $H^*(\mathcal{M}_n)$ is homotopy equivalent to $H(\mathcal{M}_{n+1})$, for $n \geq 1$.

We define an inclusion $i: H(\mathcal{M}_{n+1}) \rightarrow H^*(\mathcal{M}_n)$ by filling in one of the holes in $\mathcal{M}$, and extending $h \in H(\mathcal{M}_{n+1})$ over this by the identity. Choosing $e$ as the centre of this filled-in disc, the extended map is in $H(\mathcal{M}_n)$.

The reverse map $r: H^*(\mathcal{M}_n) \rightarrow H(\mathcal{M}_{n+1})$ is constructed using a technique of H. Kneser [7]. This relies on the existence and uniqueness of certain conformal maps of annuli to produce canonical homeomorphisms of a standard annulus onto any other, given the map on the boundary.

**Theorem** $H(\mathcal{M}_n)$ is homotopy equivalent to $H(\mathcal{M}_1)$ for any closed
2-manifold $M$.

**Proof** Immediate from the lemmas.

**Corollary** $H^1_{n+1}$ is contractible for all $n > 0$.

**Proof** $H^1_{n+1}$ is $H(S^2_{n+1})$, which is homotopy equivalent to $H(S^1_1)$. Now this is the space of deformations of a 2-disc keeping the boundary fixed, and by Alexander's theorem [1], it is contractible.

**Remark** On all function spaces we shall use the compact-open topology, and refer to Hu [6] for results. In particular, for the space $X^Y$ where $X$ and $Y$ are both compact metric, the induced metric topology agrees with the compact-open topology, [6] p.102.
Construction of the standard manifold with \( n \) holes, \( M_n \).

The manifold is constructed as a ribbon of unit width about a central core \( C \). This core consists of linear segments, arranged so that at each vertex of \( C \) the segments meet at right angles to form either an \( L \) or a \( T \). Fig. 1 shows such an arrangement for a double torus with 1 hole, and a disc with \( n \) holes. It is possible to construct any other surface with holes similarly, twisting one of the ribbons for a non-orientable surface.

For each point \( c \) of \( C \) there is a linear embedding \( B(c) \) of the unit square into the manifold, with the centre of the square at \( c \). The images of these squares will then cover the whole manifold. Choose some deformation retraction \( r \) of the interior of \( M_n \) onto the core \( C \), so that the path \( r_x(x) \) traced out by the point \( x \in M_n \) under the deformation lies entirely inside the square centred on \( r(x) \).

Such a deformation can be defined by orthogonal projection along straight paths to the core for all points of \( M \) not lying in the squares centred on the vertices of \( C \). For the other points it is defined using the paths to be traced out by each point, as indicated in figs. 2 and 3, for the squares round an \( L \)-vertex and a \( T \)-vertex.
FIGURE 1

The double torus with one hole.

The disc with n holes.
The paths around a T-vertex.

Figure 5

The paths around an L-vertex.
Then, corresponding to each $x \in \text{int} \, M_n$, there is an isotopy through homeomorphisms $h_t$ with support $S(r(x))$ from the identity to a homeomorphism $h_1$ with $h_1(x) = r(x)$. On $S(r(x))$, the homeomorphism $h_t$ is defined by joining $x$ and $r_t(x)$ to the boundary of the square, and mapping the four triangles formed from $x$ linearly to those formed from $r_t(x)$. This gives a continuous map

$$R : \text{int} \, M_n \times I \longrightarrow H(M_n)$$

such that

$$R(x,0) = \text{identity}$$
$$R(x,1)(x) = r(x).$$

Suppose $f : I \longrightarrow M_n$ is a path. Then we say that a continuous family of homeomorphisms $h_t$ in $H(M_n)$ follows $f$ from $f_0$ to $f_1$ if

$$h_0 = \text{identity}$$
$$h_t(f_0) = f_t, \text{ for all } t \in I.$$

If two paths can be composed, then this defines uniquely a composition of the two families to give a family which follows the new path.

**Proof of lemma 1**

Choose a base point $e$ on the core $C$, and an arc $J$ joining $e$ to a point $b$ on the boundary. Choose a point $\tilde{b}$ above $b$ in the universal cover of the manifold, $\tilde{M}_n$. Then lift the arc $J$ to $\tilde{M}_n$ to define a base point $\tilde{e}$ in $C$, above $e$. 
For any \( h \in H(M_n) \), the arc \( h \) joins \( b \) to \( h(e) \). Lifting this to \( \tilde{M}_n \) gives an arc joining \( \tilde{b} \) to a point \( \epsilon(h) \) above \( h(e) \). Under the compact-open topology, \( \epsilon \) gives a continuous map

\[
\epsilon: H(M_n) \rightarrow \text{int } \tilde{M}_n
\]

For each homeomorphism \( h \) we shall define a path from \( h(e) \) to \( e \), and then along \( C \) to \( e \), with homeomorphisms to follow this path. We have already done this for the first part of the path. We shall now do it locally in \( C \), and then find homeomorphisms corresponding continuously to each point \( x \in \tilde{C} \) to follow a path from \( x \) to \( e \).

Suppose the metric on the manifold \( M_n \) is now such that the length of each linear segment of \( C \) is 1. Then for each linear segment we have two distance preserving linear maps \( \lambda: I \rightarrow C \), one for each orientation. Take the rectangle \( R_\lambda \) in \( \tilde{M}_n \) surrounding \( \lambda I \), \( R_\lambda = \bigcup S(x) \) for \( x \in \lambda I \). For any two points \( x \) and \( y \in I \), define a homeomorphism \( h_\lambda(x,y) \) in \( H(M_n) \) with support \( R_\lambda \) by joining \( \lambda x \rightarrow \lambda y \) to the four corners of \( R_\lambda \) and giving a piecewise linear map as before. Define the continuous map

\[
g_\lambda: \lambda I \times I \rightarrow H(M_n)
\]

by \( g_\lambda(\lambda x,t) = h_\lambda(x,(1-t)x) \). Then \( g_\lambda(\lambda x,t) \) are homeomorphisms which follow the linearly parametrised path from \( \lambda x \) to \( \lambda 0 \).
The linear and local metric structures of $O$ lift to $\mathcal{S}$ in an obvious way. For any two points $x$ and $y$ in $\mathcal{S}$, there is a unique piecewise linear arc joining them, of length $\rho(x,y)$. In particular, for each $x \in \mathcal{S}$, we have the unique arc

$$g^\mathcal{S} : [0,\rho(\mathcal{S},x)] \rightarrow \mathcal{S}$$

such that $g^\mathcal{S}(0) = \hat{x}$, $g^\mathcal{S}(\rho(\mathcal{S},x)) = x$.

Then $g^\mathcal{S}_{[i,i+1]}$ is a component of the inverse image of $\lambda I$ in $\mathcal{S}$ for some $\lambda$, where $i$ is an integer.

Let $\rho(\mathcal{S},x) = m + \delta(x)$ where $0 < \delta(x) < 1$. Then the arc $g^\mathcal{S}$ splits up into the sum of $m + 1$ linear segments each lying above some $\lambda I$. By using the appropriate function $g^\lambda$, there is, for each segment, a family of homeomorphisms which follows the projection of the segment from its initial point to its final point. These segments can be combined to form the path $g^\mathcal{S}$, with the parametrisation of arc length, and then normalised so that the total length of path is 1. We can then similarly combine the families of homeomorphisms to form one family following the projection of $g^\mathcal{S}$ from $x$ to $e$.

Since the function $h^\lambda(x,y)$ is continuous in $x$ and $y$, and becomes the identity when $x$ and $y$ coincide, the definition of the family is the same when $\rho(\mathcal{S},x) = r + 1$, whether we take $m = r$ and $\delta = 1$, or $m = r + 1$ and $\delta = 0$. Hence the families above define globally a continuous function $G : \mathcal{S} \times I \rightarrow \mathbf{H}(\mathcal{M}^n)$, where $G(\mathcal{X},t)$ is the family defined by the path $g^\mathcal{S}$. 
The deformation retraction \( r : \text{int } M_n \to \mathcal{C} \) lifts to \( \tilde{r} : \text{int } \tilde{M}_n \to \tilde{\mathcal{C}} \).

For each \( \tilde{x} \in \tilde{M}_n \), the map \( R : \text{int } \tilde{M}_n \times I \to H(\tilde{M}_n) \) gives a family of homeomorphisms following a path from \( x \) to \( r(x) \), and the map \( \int \tilde{M}_n \times I \xrightarrow{\tilde{r} \times I} \tilde{\mathcal{C}} \times I \xrightarrow{G} H(\tilde{M}_n) \) gives a family following \( r(x) \) to \( e \). Compose these two paths, and combine the two families to give homeomorphisms following a path from \( x \) to \( e \), by a map \( K : \text{int } \tilde{M}_n \times I \to H(\tilde{M}_n) \).

Note that if \( \tilde{x} = \tilde{e} \), then the paths are both constant, and \( K(\tilde{e}, t) = \text{identity for all } t \).

We use the map \( K \) to make \( h(e) \) return canonically to \( e \) for each \( h \). The following composition \( d \) will give the required deformation retraction \( d_1 \) by \( d_1 = d_{\mathcal{H}(\tilde{M}_n)} \times I \).

\[
\begin{align*}
\mathcal{H}(\tilde{M}_n) \times I &\xrightarrow{\Delta \times I} \mathcal{H}(\tilde{M}_n) \times \mathcal{H}(\tilde{M}_n) \times I \\
&\xrightarrow{1 \times \tilde{r} \times I} \mathcal{H}(\tilde{M}_n) \times \text{int } \tilde{M}_n \times I \\
&\xrightarrow{1 \times K} \mathcal{H}(\tilde{M}_n) \times \mathcal{H}(\tilde{M}_n) \\
&\xrightarrow{m} \mathcal{H}(\tilde{M}_n),
\end{align*}
\]

where \( m \) is the composition \( m(f,g) = g \circ f \).

Then \( d(h,t) = K(\epsilon(h), t) \circ h \). The map \( d_1 \) is the identity, for \( d(h,0) = K(\epsilon(h), 0) \circ h = i \circ h = h \).

\( K(\epsilon(h), t) \) was defined to follow a path from the projection of \( \epsilon(h) \) to \( e \), and since \( \epsilon(h) \) projects to \( h(e) \), then \( K(\epsilon(h), 1)(h(e)) = e \).
So \( d(h,1)(e) = e \), and then \( \delta_1 H(\tilde{M}_n) \) are homeomorphisms which keep \( e \) fixed.

Now \( \epsilon(h) = \delta \) for all \( h \in H(\tilde{M}_n) \), since \( hJ \cong J \) for such \( h \), and so \( d(h,t) = h \) for all \( t \), i.e. \( \delta_t H(\tilde{M}_n) \) is the identity. Hence \( \delta_1 \) will provide the deformation retraction when we show that its image lies in \( H^{\bullet}(\tilde{M}_n) \). This follows at once, since \( \delta_1 \) is continuous and \( H(\tilde{M}_n) \) is connected. Then the image lies in one component of the homeomorphisms keeping the boundary and \( e \) fixed. But the identity lies in the image, since \( d_1(1) = 1 \), and so the image is the component of the identity, \( H^{\bullet}(\tilde{M}_n) \).

For the second lemma we require a theorem about uniqueness and convergence of certain maps of annuli, similar to one quoted in [4].

It is a standard result in complex variable theory, see R.Courant [3]p.58, that, given two non-intersecting Jordan curves \( g_1 \) and \( g_2 \) in the plane, with \( g_1 \) inside \( g_2 \), then there is an annulus \( B(r) \), \( 1 < |z| < r \), and a homeomorphism \( \psi \) taking the annulus \( B(r) \) onto the annulus \( G \) defined by \( g_1 \) and \( g_2 \), which is conformal on the interior of \( B(r) \), and is uniquely determined by the orientation of the boundary and the image of one boundary point.

An important extension of this result is the following continuity property. Suppose that \( G_n \) is a sequence of annuli,
point sets bounded by two Jordan curves, whose boundaries, \( G_n \), converge to the boundary \( g \) of an annulus \( G \) in the sense of Fréchet. This means that the boundaries converge to \( g \) as sets, and if the points \( P_n \) and \( Q_n \) on \( G_n \) tend to \( P \) and \( Q \) on \( g \), then the whole arc \( P_n Q_n \) must tend to one of the two arcs \( PQ \). Then the values of \( r_n \) for \( G_n \) converge to the value \( r \) for \( G \), and if the image of a convergent sequence of boundary points, one from each \( B(r_n) \), is prescribed as a convergent sequence of boundary points of the corresponding \( G_n \), then the resulting homeomorphisms \( w_n \) will converge uniformly to \( w \).

**Proof** The Riemann mapping theorem, which gives the similar result for discs certainly has this continuity property. For, if \( g_n \) are curves tending to the curve \( g \) in the way above, then there are conformal maps \( f_n \), and \( f \), from the unit disc to the discs defined by \( g_n \), which are uniquely determined by specifying the images of certain points. Then, see [2] p.191, a subsequence of these functions \( f_n \) converges uniformly to \( f \). So if we take the inverse functions \( w_n = f_n^{-1} \), these converge uniformly on any closed subset of the disc defined by \( g \).

We can now use such maps, first on the outside sequence of boundary curves, and then on the inside, to give a sequence of functions transforming the annuli \( G_n \), \( G \), to ones \( F_n \), \( F \), whose
inner curve is $|z| = 1$ and whose outer curve is analytic, still with the property that the annuli $F_n$ tend to $F$. It is now enough to prove the continuity property for such annuli $F_n$.

Following the proof [3] p. 58 of the existence of the map to a standard annulus we define harmonic functions $u_n$ on the annulus $F_n$ which are zero on $|z| = 1$ and 1 on the outer boundary. Since these boundaries are analytic, each $u_n$ can be extended across the boundary of $F_n$. Then, since $u$ is continuous and the boundaries of $F_n$ tend to the boundary of $F$, given $\epsilon > 0$ there is an $N$ such that $|u_n - u| < \epsilon$ on the boundary of $F_n$ for all $n > N$. Since $u_n - u$ is harmonic it satisfies a maximum and minimum modulus principle, and $|u_n - u| < \epsilon$ throughout $F_n$. Then, given any closed subset of $F$, the functions $u_n$ converge uniformly to $u$ on it. Hence the derivatives of $u_n$ converge uniformly to the derivatives of $u$ [3] p. 11. This shows that the conjugate harmonic functions $v_n$ converge uniformly to $v$, and hence, by their construction from $u_n$ and $v$, that the values $r_n$ tend to $r$, and that the analytic functions $w_n$ on $F_n$ tend uniformly to $w$ on $F$, when a convergent sequence of boundary points has been chosen.

To define a canonical map - no longer conformal - which takes prescribed boundary values on an annulus we proceed as follows.
Suppose $R$ is the annulus $1 < |x| < 2$, and $h_1, h_2$ are homeomorphisms of the inner and outer boundary curves onto themselves which have the same orientation. Then these extend to a homeomorphism of $R$, given by

$$(r, \theta) \mapsto (r, (2-r)(h_1(\theta + 2\pi n_1)) + (r-1)(h_2(\theta + 2\pi n_2))).$$

This is uniquely determined by $h_1$ and $h_2$, and by $n_1$ and $n_2$, or rather the difference $n_1 - n_2$. This can also be given by the angle change along the image of an arc in $R$ joining the boundary components, as seen from the centre of $R$, for example along the image of the real axis $1 < x < 2$.

To map the annulus $R$ onto a given annulus $G$ with prescribed similarly oriented homeomorphisms $f_1$ and $f_2$ on the boundary, we find $r$, and $w : B(r) \mapsto G$ as above, which restricts to $w_1$ and $w_2$ on the boundary. Choose $h_i = w_i^{-1} \circ f_i$, $i = 1, 2$, and extend to the homeomorphism $F(n)$ of $R$ to itself, where $n = n_1 - n_2$. Shrink $R$ radially to $B(r)$ and follow this with $w$. The combined map is a homeomorphism of $R$ to $G$ which is uniquely determined on choosing $n$. This can be done by choosing the angle change along, say, the image in $G$ of the arc $\tau$, $1 < x < 2$, in $R$.

The continuity property above now shows that if $f_1$ and $g_1$ are sequences of similarly oriented disjoint homeomorphisms of $S^1$ into the plane which converge uniformly to $f$ and $g$, and if $F_1, F_2$ are
homeomorphisms of $\mathbb{R}$ extending $f_1, f_2, f_3, f_4$ as above, then if the angle change along the arcs $F_i(\tau)$ converges to that along $F(\tau)$, the sequence $F_i$ will converge uniformly to $F$.

Since, for metric spaces, sequential continuity implies continuity, we have a continuous map from the (space of similarly oriented embeddings of a pair of disjoint circles into the plane, with one inside the other) $X$ (real line), representing the angle change, into the space of embeddings of an annulus in the plane, if we regard such embeddings as having the metric topology.

**Proof of lemma 2** Consider the manifold $M_{n+1}$ being locally embedded in the plane so that one of the holes has $|z| = \frac{1}{2}$ as a boundary curve, and that $\frac{1}{2} < |z| < 2$ lies in $M_{n+1}$, with the point $(2,0)$ lying on another boundary curve. Here we assume that $n \geq 1$. Let $M_n$ be $M_{n+1}$ with the disc $|z| < \frac{1}{2}$ filled in, and choose the origin as the point $e$. Then the inclusion $i : H(M_{n+1}) \to H(M_n)$ is defined simply by extending $g \in H(M_{n+1})$ by the identity over $|z| < \frac{1}{2}$. We consider some fixed covering of $M_n$ by the plane, so that the image of the disc $|z| < 1$ under any $h \in H^*(M_n)$ can be represented as lying in this plane.

Let $h \in H^*(M_n)$. Then $h$ maps the circle $|z| = 1$ into a
Jordan curve $g_1(h)$ around $e$. Let $\rho(h)$ be the distance from $e$ to this curve, and choose $\epsilon(h) = \min(\frac{1}{2}, \frac{1}{2}\rho(h))$. Then the circle $g_2(h)$, $|z| = \epsilon(h)$, lies inside the Jordan curve $g_1(h)$ and depends continuously on $h$.

Let the angle change under $h$ along the arc $1 \leq x \leq 2$ be $\theta(h)$ as seen from $e$, and then specify that the angle change along $\frac{1}{2} \leq x \leq 1$ shall be $-\theta(h)$. This defines uniquely a homeomorphism from the annulus $\frac{1}{2} \leq |z| \leq 1$ to the plane, taking $|z| = 1$ to $g_1(h)$ and $|z| = \frac{1}{2}$ radially to $g_2(h)$. Since $\theta$ depends continuously on $h$ then by the continuity property above, so does this homeomorphism. We can now define a map from the whole of $M_{n+1}$ using this one on $\frac{1}{2} \leq |z| \leq 1$, and $h$ on the rest of $M_{n+1}$. Call this map $f(h)$.

Following it with the map shrinking the annulus $\epsilon(h) \leq |z| \leq 1$ linearly onto $\frac{1}{2} \leq |z| \leq 1$, gives a homeomorphism $r(h)$ from $M_{n+1}$ to itself which keeps the boundary pointwise fixed, and depends continuously on $h \in H^*(M_n)$. When $h$ is the identity we have $\epsilon(h) = \frac{1}{2}$, $\theta(h) = 0$ so that the map on the annulus, and hence the whole map $r(h)$, is the identity. Since $H^*(M_n)$ is connected and $r$ is continuous, $r(h)$ lies in the component of the identity, $H(M_{n+1})$, for all $h$.

We want to prove $r \circ i \cong 1 : H(M_{n+1}) \rightarrow H(M_{n+1})$ and $i \circ r \cong 1 : H^*(M_n) \rightarrow H^*(M_n)$. 
The annulus \( \frac{1}{2} < |z| < 1 \) is a collar of a boundary component of \( M_{n+1} \), and hence we can choose a continuous family of homeomorphisms \( h_t \), shrinking \( M_{n+1} \) off this collar, with \( h_0 \) being the identity, and \( h_t \) mapping the boundary \( |z| = \frac{1}{2} \) linearly to \( |z| = \frac{1}{2} + it \). This provides a homotopy \( l \circ \alpha : H(M_{n+1}) \to H(M_{n+1}) \) by taking a homeomorphism \( g \) on \( M_{n+1} \) into \( h M_{n+1} \), extended by the identity over \( \frac{1}{2} < |z| < \frac{1}{2} + it \). Then \( \alpha \) sends \( g \) to a homeomorphism which is the identity on \( \frac{1}{2} < |z| < 1 \), and such are unaltered by \( r \circ l \).

So \( r \circ l \circ \circ l = \circ l \).

This proves the first part of the lemma.

Define a map \( f_1 \) from \( H(M_n) \) to itself, by extending the map \( f(h) \) to send the disc \( |z| < \frac{1}{2} \) radially to \( |z| < \epsilon(h) \). There is an obvious homotopy \( f_1 \circ i \circ r \).

Now define a family \( f_t \), \( 0 < t < 1 \), of such maps, defined as for \( f_1 \) using the circle \( |z| = t \) instead of \( |z| = 1 \). (see fig.4)

By the continuity property of the maps of annuli, this gives a continuous map \( (0,1] \times H(M_n) \to H(M_n) \).

Taking \( f_0 \) to be the identity gives a map \( F : I \times H(M_n) \to H(M_n) \).

It remains to show that \( F \) is continuous at points \( (0,h) \).

Given \( \epsilon > 0 \), choose \( \delta \) such that \( \text{diam} \ h(|z| < \delta) < \epsilon/4 \).
FIGURE 4

The map $f_t(h)$

$|z| = t$

$|z| = \frac{1}{2} t$

$|z| = c(h, t)$

$|z| = h(|z| = t)$

Boundary curve

Boundary curve
Choose a neighbourhood of \( h \), radius \( \varepsilon/4 \), then \( \text{diam } h'(|z| < \delta) < 3\varepsilon/4 \) for \( h' \) in this neighbourhood. For \( x \) in \( M_n \) outside \( |z| = \delta \),

\[
\| F(0, h)(x) - F(t, h')(x) \| = \| h(x) - h'(x) \| < \varepsilon/4.
\]

For \( x \) inside \( |z| = \delta \),

\[
\| F(0, h)(x) - F(t, h')(x) \| < \| F(0, h)(x) - 0 \| + \| 0 - F(t, h')(x) \| < \varepsilon/4 + 3\varepsilon/4.
\]

So \( \| F(0, h) - F(t, h') \| < \varepsilon \). Hence the map \( F \) is continuous and so the lemma is proved.
PART II

Section 1  Joins of Polyhedra

In this section we analyse the structure of compact polyhedra when regarded as joins of other polyhedra. The result, proved jointly with M. A. Armstrong for use in [2], that two polyhedra are PL homeomorphic if their suspensions are PL homeomorphic, provided the initial stimulus.

The main result, theorems 3 and 4, is that any polyhedron is uniquely expressible as the join of a maximal ball or sphere with other factors which are indecomposable as joins. This nearly gives the unique decomposition of a polyhedron into indecomposable factors. The only uncertainty is whether to regard a ball as the repeated cone on a point, or as the repeated suspension of a point. The following cancellation law is an immediate corollary.

Let \( T \) be a point, \( Z \) be two points, and let \( \mu \) mean a PL homeomorphism. Then if \( A \star B = A \star C \), either

(i) \( A = T \star A' \), \( B \) and \( C \) are \( T \star X \) and \( Z \star X \) for some polyhedra \( A' \) and \( X \), or

(ii) \( B = C \).

The proofs rely on a knowledge of the link of points and simplexes in various simplicial complexes. To make these proofs
more readable the technical facts have been collected in an
appendix at the end of part II, and are referred to in the text
as, for example, result A.5.

Notation and definitions The terminology used is that of
Zeeman [9]. We shall assume all polyhedra to be compact, so that
they can be triangulated by a finite simplicial complex.

The join of the polyhedra A and B will be written A B.
For convenience, all triangulations of a polyhedron will be assumed
to contain the empty simplex. Then the natural triangulation of
the join of two polyhedra A and B, which have already been
triangulated, consists of the joins of all possible pairs of
simplexes, one from each of the two polyhedra. Call such a
triangulation a join triangulation.

A PL homeomorphism between two polyhedra will be denoted
by =. When two polyhedra are PL homeomorphic, we shall often
require them to be triangulated so that the homeomorphism is
simplicial, and the simplicial isomorphism will then be written as

For the join of two polyhedra, such a triangulation will always
be assumed to be the subdivision of some join triangulation.

Under the operation of join, the set of (PL homeomorphism
classes of) finite dimensional polyhedra forms an abelian
semigroup. The empty polyhedron, $\emptyset$, is admitted, and acts as the unit element in the semigroup.

We define an **indecomposable** to be one which is not the join of two non-trivial polyhedra.

A **prime** polyhedron is one which must divide one of the factors of a join if it divides the join.

The link of a simplex $a$ in the complex $A$ will be written as $\text{lk}(a, A)$. This link will usually be considered as an abstract polyhedron, rather than a subcomplex of $A$. When $A$ is the largest complex under consideration which contains $a$ it will simply be written as $\text{lk} a$.

The polyhedron consisting of one point will be called $T$, and that consisting of two points will be called $S$. Then the cone on $A$ is $TA$, and the suspension of $A$ is $SA$.

A **principal** simplex in a complex is one which is not the face of any larger simplex. Consequently its link in the complex is empty, and also every point of the complex is contained in some principal simplex.

The sign $\blacksquare$ will denote the end of a proof.
**Theorem 1**  
Maximal spheres cancel.

Suppose \( E^r X \cong E^k Y \), where \( X \) and \( Y \) are not suspensions.

Then \( r = k \) and \( X = Y \).

**Proof**  
Suppose \( r < k \), and let \( \xi \) be any point of \( E^r \).

Then (A.2) \( 1k \xi = E^{r-1} X \). Suppose that \( x \) is the image of \( \xi \) in \( E^k Y \) under the isomorphism.

If \( x \) lies in the factor \( E^k \), then \( 1k x = E^{k-1} Y \), and so \( E^{r-1} X = E^{k-1} Y \). The result now follows by induction on \( r \).

Otherwise \( x \) has the same link as some point of \( X, (A.4) \).

So \( 1k x = E^k 1k(x,Y) \). Hence \( E^{r-1} X = E^k 1k(x,Y) \).

Suppose \( 1k(x,Y) = E^s Z \) for some \( Z \), where \( s \) is maximal.

Then \( E^{r-1} X = E^{k+s} Z \), so by induction on \( r, r-1 = k+s \).

But \( r < k \), so this is impossible. \( \square \)

**Corollary**  
Any sphere factors will cancel.

**Proof**  
Suppose \( E X = E Y \).

Let \( X = E^r X' \) and \( Y = E^k Y' \), where \( r \) and \( k \) are maximal.

Then \( E^{r+1} X' = E^{k+1} Y' \), so \( r+1 = k+1 \) and \( X' = Y' \).

Hence \( X = Y \). \( \square \)
The fact that a ball can be taken as the join either of two lower-dimensional balls, or a ball and a sphere, makes it impossible that balls should always cancel. The best result is the following.

**Theorem 2** Maximal balls cancel.

Suppose \( T^r X = T^k Y \), where \( X \) and \( Y \) are not cones or suspensions. Then \( k = r \) and \( X = Y \).

**Proof** Suppose \( r < k \), and let \( \xi \) be any point in the interior of \( T^r \). Then, \( (A.2) \), \( \text{lk} \xi = \xi^{r-1} X \).

Suppose \( x \) is the image of \( \xi \) in \( T^k Y \). Then \( x \) lies either (i) in \( T^k \), or (ii) in \( Y \), or (iii) in neither factor.

(i) If \( x \) lies in the interior of \( T^k \), then \( \text{lk}(x,T^k) = \xi^{k-1} \).

So \( \xi^{r-1} X = \xi^{k-1} Y \), and theorem 1 gives the result.

If \( r = k \) it will be possible to choose a different point \( \xi \), so that the corresponding \( x \) does not lie on the boundary of \( T^k \).

Otherwise suppose \( r < k \) and that \( x \) is in the boundary of \( T^k \).

Then \( k \) is large enough to write \( T^{k-1} = \xi^{k-2} T \), to give \( \xi^{r-1} X = \xi^{k-2} Y \). By the corollary to theorem 1 we can cancel \( \xi^{r-1} \), leaving \( X = \xi^s T Y \) with \( s > 0 \). This is impossible, since \( X \) is not a cone.

(ii) If \( x \) lies in \( Y \), then \( \xi^{r-1} X = T^k \text{lk}(x,Y) \).
Now $T^k = 2^{k-1} T$, and again the factor $2^{r-1}$ cancels to show that $X$ is a cone.

(iii) If $x$ lies between the two factors then, $(A, k)$, there are points $t \in T^k$ and $y \in Y$, with

$$lk_x = E lk(t, y^k) lk(y, X).$$

Depending on whether $t$ lies in the interior or the boundary of $T^k$, either $2^{r-1} X = 2^k \text{lk}(y, X)$ or $2^{r-1} X = T^k \text{lk}(y, X)$.

This requires $X$ to be either a suspension or a cone, and so is impossible.

**Definition.** A reduced polyhedron is one which is not a cone or a suspension.

**Corollary to theorems 1 and 2.** Any polyhedron factors uniquely as a ball or sphere joined to a reduced polyhedron.

**Proof.** There are three cases to consider, (i) $E^r X = 2^k Y$, (ii) $T^r X = T^k Y$, and (iii) $2^r X = T^k Y$, where $X$ and $Y$ are reduced. Theorems 1 and 2 give uniqueness immediately for the first two cases. It remains to show that (iii) is impossible.

Take the cone on each side. Then $T E^r X = T^{r+1} X = T^{k+1} Y$, and so, by theorem 2, $r = k$ and $X = Y$.

Thus $E^r X = T^r X = 2^{r-1} T X$. The factor $2^{r-1}$ will now
cancel, by corollary to theorem 1, to give $ ZX = TX$.  

The property A.9 shows this to be impossible.  

**Lemma 1** Any indecomposable polyhedron except $T$ or $Z$ is prime.  

**Proof** Suppose $A$ is indecomposable, and $AX = YZ$.

Let $x$ be a principal simplex of $X$, and $x$ its image in $YZ$.

If $x$ lies in one of the factors, say $Y$, then

$$A = 	ext{lk } x = 	ext{lk}(x,Y)Z$$

Now $A$ is indecomposable, so $\text{lk}(x,Y) = \emptyset$ and $A = Z$.

Otherwise $x$ lies between $Y$ and $Z$. Then, by the pushing lemma (A.8), there are simplexes $y$, $z$ in the original triangulation of $Y$ and $Z$ with $A = \text{lk}(y,Y) \text{lk}(z,Z)$.

$A$ is indecomposable, so one of these factors is empty, say $\text{lk}(y,Y)$. The pushing lemma also gives a simplex $\eta$ of $X$ such that $A \text{lk}(\eta,X) = \text{lk}(y,Y)Z$.

But $\text{lk}(y,Y)$ is empty, so $A$ divides $Z$.  

---

We now need some criterion for a polyhedron to be reduced.

This is given by the following lemma, whose proof uses most of the technical results from the appendix.

**Lemma 2** The join of two reduced polyhedra is also reduced.
Proof: Suppose $X$ is the join $\ast X_1$, where $X_1$ are indecomposable polyhedra which are not $\Sigma$ or $T$. We show that $X$ is reduced by induction on $\dim X$. This is obvious for $\dim X = 0$.

Suppose the result is true for all dimensions $< \dim X$, but that $\ast X_1 \cong \Sigma^p Y$ or $\Sigma^r Y$, with $Y$ reduced.

Write $Z$ for $\Sigma^p$ or $\Sigma^r$.

Suppose there is some principal simplex of $Z$, whose image $x$ in $X$ does not lie in any of the factors $X_1$. Then there must be one factor, $X_1$ say, such that $X$ lies between $X_1$ and $X_1'$, where $X_1'$ is the complement of $X_1$ in $X$, $= \ast_{i \neq 1} X_1$. Applying the pushing lemma $(A, 8)$ gives simplexes $x_1$, $x_1'$ in the join triangulations of $X_1$ and $X_1'$ such that $Y = \text{lk}(x_1, X_1) \text{lk}(x_1', X_1')$. It also gives a non-principal simplex $\mathfrak{F}_i$ of $Z$ so that

$$Y \text{lk}(\mathfrak{F}_i, Z) = \text{lk}(x_1, X_1) X_1'. $$

Now $\text{lk}(x_1, X_1)$ is reduced, since it divides $Y$, and the total dimension of this join is less than $\dim X$, so the induction shows that the r.h.s. is reduced. But $\text{lk}(\mathfrak{F}_i, Z)$ is a non-empty ball or sphere, and hence the l.h.s. is not reduced.

So we can assume that the image of each principal simplex of $Z$ lies in one of the factors $X_1$. The factors $X_1$ are
disjoint in \( X \), and \( Z \) is connected, apart from the case
\( Z = E \), which is considered separately, so that the image of
\( Z \) must all lie in the same factor, \( X_1 \) say.

Since \( X_1 \) and \( X'_1 \) are disjoint there is no simplex of
\( X'_1 \) whose image lies in \( Z \). Let \( \eta \) be a principal simplex of
\( X'_1 \). Then its image \( y \) in \( Z \) cannot lie in \( Z \). Nor can \( y \)
lie between \( Z \) and \( Y \), for then the pushing lemma will produce
a simplex \( \eta' \) of \( X'_1 \) which goes to \( Z \). So \( y \) must lie in \( Y \).
Then \( 1k \eta = X_1 = T^k 1k(y,Y) \) or \( E^k 1k(y,Y) \).

But \( X_1 \) is reduced, so this too is impossible.

The only remaining case to examine is \( Z = E \),
i.e. \( X = Z Y \) with \( Y \) reduced. If \( X \) has three or more
factors, then the image of \( Z \) will lie in at most two of them,
say \( X_1 \) and \( X_2 \). The argument used above will now still work,
using \( X_1 X_2 \) and its complement in \( X \) in place of \( X_1 \) and \( X'_1 \).
This leaves the final possibility \( X = X_1 X_2 = Z Y \).
Express \( Y \) as the join of indecomposable factors. \( Y \) is reduced,
so none of these are \( Z \) or \( T \). Then, by lemma 1, every factor
of \( X \) must divide \( X_1 \) or \( X_2 \), and so must be either \( X_1 \) or \( X_2 \).
Similarly, by lemma 1, \( X_1 \) and \( X_2 \) must both appear as factors
of \( Y \), so we must have \( Y = X_1 = X_2 \). This is impossible,
since then, \((A,2)\), \( X_1 \) must be \( E \), but \( X_1 \) is reduced.
Corollary  The set of reduced polyhedra form a semigroup, generated by all the indecomposable polyhedra except \( T \) and \( S \).

Theorem 3  The semigroup of polyhedra is the direct sum of the semigroup of balls and spheres with the semigroup of reduced polyhedra.

Proof  This follows directly from the corollary above, and the corollary to theorems 1 and 2.

Application  Zeeman [9] defines the intrinsic skeleton \( X_i \) of a polyhedron \( X \) as the intersection of the \( i \)-skeletons of all triangulations of \( X \).

Armstrong [2] defines the intrinsic dimension of a point \( x \) in \( X \) to be \( k \), if \( \text{lk}(x,X) = s^k A \), with \( k \) maximal. He shows that the intrinsic skeleton \( X_i \) consists of all points with intrinsic dimension less than or equal to \( i \). In this definition, \( A \) is not necessarily reduced, but could be the cone on a reduced polyhedron.

Definition  A polyhedron without boundary has no point whose link is a cone.

Then in the definition of intrinsic dimension \( A \) will
always be reduced.

Corollary to lemma 2 Suppose $X$ and $Y$ are polyhedra without boundary. If $Z = X \times Y$, then $Z_n = \bigcup_{i+j=n} X_i \times Y_j$.

Proof Suppose $x \in X_i$ and $y \in Y_j$. Then the link of $(x,y)$ in $Z = \text{lk}(x,X) \cup \text{lk}(y,Y)$ (see Zeeman [8]).

So the link of $(x,y)$ is $Z^i A Z^j B = Z^{i+j} A B$. But $A$ and $B$ are reduced, so by lemma 2 $A B$ is also reduced. Then the intrinsic dimension of $(x,y)$ is $i+j$.

We now prove a theorem which describes the semigroup of reduced polyhedra completely, and so, with theorem 3, the semigroup of all polyhedra.

Theorem 4 The semigroup of reduced polyhedra is free abelian on the indecomposable polyhedra.

So there is unique prime factorisation in this semigroup, for all indecomposables are prime.

Proof Let $Z$ be any reduced polyhedron which can be decomposed as the join \( \bigotimes_i X_i \bigotimes_j Y_j \) of indecomposable factors $X_i$ or $Y_j$.

Let $A$ be some indecomposable factor and write

\[ Z = A^r X_1 \bigotimes_i X_i = A^k Y_j \bigotimes_j Y_j, \]

where the factors
X_i or Y_j which were A have been collected together.
Rewrite this as \( Z = A^r X = A^k Y \). Now by lemma 1 A is prime, and since it does not occur among the factors of X or Y, it cannot divide X or Y.

Thus it is enough to prove the following lemma.

**Lemma 3** In the semigroup of reduced polyhedra, if A is indecomposable, and \( A^r B = A^k C \), with \( r, k \) maximal, then \( r = k \) and \( B = C \).

**Proof** Suppose \( r > k \), and use induction on \( \dim B = n \).
The induction starts for \( n = 1 \). For if \( A^r = A^k C \), then any prime factor of C must divide A. But \( k \) is maximal, so C is empty.

Suppose the result true for all \( B \) with \( \dim B < n-1 \).
Let \( \beta \) be a principal simplex of \( B \), and \( b \) its image in \( A^k C \).

(i) If \( b \) lies in \( A^k \) then \( A^r = \text{lk}(b, A^k) C \). Then any prime factor of C must divide A. So, as above, C is empty.

(ii) If \( b \) lies between \( A^k \) and C, then \((A, B) \) \( b \) lies in a c for some simplexes \( a \in A^k \) and \( c \in C \) of the join triangulation, such that \( A^r = \text{lk}(a, A^k) \text{lk}(c, C) \).
These simplexes \( a \) and \( c \) are both non-empty, so
\[ \text{lk}(a, A^k) = A^s \text{ for some } s < k. \]
The pushing lemma also gives a non-empty simplex \( \beta' \) of \( B \), whose
image lies in \( s \), such that \( A^s \operatorname{lk}(\beta', B) = \operatorname{lk}(a, A^k) \) \( s = A^s \) \( C \).

All the factors are reduced, and \( \dim \operatorname{lk}(\beta', B) < n \) so we can apply the induction step to show that \( r+1 = s \), where \( l \) is the maximum power of \( A \) which divides \( \operatorname{lk}(\beta', B) \). This is impossible, since \( s < k < r \).

(iii) So, if \( C \) is non-empty, \( b \) must lie in \( C \), and so must the image of every principal simplex of \( B \). Thus the whole image of \( B \) lies in \( C \).

Now examine the image of \( A^k \) under the reverse homeomorphism. Since \( A^k \) and \( C \) are disjoint, no simplex of \( A^k \) can go to \( B \). If any principal simplex of \( A^k \) lies between \( B \) and \( A^r \), then, because \( C \) is reduced, the pushing lemma gives a simplex of \( A^k \) which goes to \( B \). So every simplex must go to \( A^r \).

Let \( \alpha \) be the image of a principal simplex of \( A^k \) in \( A^r B \).

Then \( C = \operatorname{lk} \alpha = \operatorname{lk}(\alpha, A^r) B = A'B \).

\( C \) has now split into two factors, so we can look at how the simplex \( b \) lies in \( C \).

1. If \( b \) lies in \( A' \), then \( A^r = \operatorname{lk} b = A^k \operatorname{lk}(b, A') B \), and so \( B \) is empty.

2. If \( b \) lies between \( A' \) and \( B \), then it lies between \( A^r B \), so, by the pushing lemma, there is a non-empty simplex \( \beta' \) in \( B \) which goes to \( b' \) in \( A' \).

Then \( A^r \operatorname{lk}(\beta', B) = A^k B \operatorname{lk}(b', A') \).
All these factors are reduced, and since $\dim \text{lk}(\beta',B) < n$, the induction will show that $B$ divides $\text{lk}(\beta',B)$, which is impossible.

3. So $b$ must lie in the second factor $B$.

Then $A' = A' A' \text{lk}(b,B)$. This shows that $A'$ is a power of $A$, and hence is empty, since it divides $C$.

Thus $C = B$.

The proof of the lemma, and so of theorem 4, is now complete. ☒

**Corollary** If $A B = A C$, for any polyhedra $A$, $B$, $C$, then either (i) $B = C$ or (ii) $B = TX$, $C = EX$ and $A = TY$ for some $X$ and $Y$.

**Proof** By theorem 3, $A$, $B$ and $C$ split up uniquely as a sphere or ball joined to the reduced polyhedra $A'$, $B'$ and $C'$, and $A' B' = A' C'$. Theorem 4 now shows that $B' = C'$. Hence, either $B = C$, or $B = T^k B'$, $C = T^k B'$. Then take $X$ to be $T^{r-1} B'$. $A$ must be $T^s A'$ for $s > 0$, so $Y$ is $T^{s-1} A'$. ☒
Section 2  Homeomorphisms of polyhedra

The previous section has shown something of the rigidity of structure of a polyhedron which is the join of a number of factors. We can now say precisely what the images of the various factors of a reduced polyhedron $X$ are under any PL homeomorphism of $X$ to itself, and so find out how many different ways there are to regard $X$ as the given join.

**Theorem**  If $X$ is the join $\bigast_i X_i^{r_i}$ of reduced indecomposable factors $X_i$ with $X_i \neq X_j$ for $i \neq j$, and $g$ is a PL homeomorphism from $X$ to itself, then $g$ sends each factor $X_i^{r_i}$ to itself, and $g|_{X_i^{r_i}}$ permutes the factors $X_i$.

**Proof**  By the proof of lemma 3, section 1, $g$ does send each factor $X_i^{r_i}$ to itself. It remains to prove that, if $Y = A^r$, with $A$ reduced and indecomposable then any homeomorphism of $Y$ to itself permutes the factors $A$.

For any $s < r$, split up $Y$ as $A^r = A^s A^{r-s}$. Then we show that any simplex of this factor $A^s$ goes to a simplex of some other factor $A^{s'}$ under the homeomorphism $h$.

For let $\Delta$ be a principal simplex of $A^s$. Then its
image $h \alpha$ is contained in the join of simplices $a_1, \ldots, a_r$ of the join triangulation, one from each factor $A$ of $A^r$, where some of the $a_i$ may be empty. We can assume that the $a_i$ are chosen so that $A^{r-s} = \text{lk} \alpha = \text{lk}(a_1, A) \text{lk}(a_2, A) \ldots \text{lk}(a_r, A)$.

Now by the unique factorisation of reduced polyhedra exactly $s$ of these $r$ factors are empty, and the others, suppose the last $r-s$, are $A$. Then $\text{lk}(a_i, A) = A$ for $i > s$, so $a_i = \emptyset$. Thus the image of $\alpha$ lies in the join of the first $s$ factors $A$.

So, for each $s$, the sets $UA^s$ are invariant under $h$. In particular, the set $UA$ is invariant under $h$, so if $A$ is connected, then $h$ must permute the disjoint factors $A$.

If $A$ is not connected, suppose that there are two points $a_1$ and $a_2$ in different components of one of the factors $A_0$, say, whose images $\alpha_1$ and $\alpha_2$ under $h$ lie in two different factors, $A_1$ and $A_2$. Consider the line $\alpha_1 \alpha_2$ in $A_1A_2$. This lies in $UA^2$, but only the points $\alpha_1$ and $\alpha_2$ lie in $UA$, so its image under $h^{-1}$ is a line joining $a_1$ and $a_2$ in $(UA^2 - UA_{A \neq A_0}) = \mathbb{Z}$.

But since $a_1$ and $a_2$ lie in different components of $A_0$, they also lie in different components of $Z$. This is impossible, and so $ha_1$ and $ha_2$ must lie in the same factor for all points $a_1, a_2$ of $A_0$. Hence, for any $A$, $h$ permutes the factors $A$. 

Corollary. In the join $X$ above, $g$ sends any subjoin $\mathbf{X} = \bigvee_{\alpha \in \mathcal{I}} X_{\alpha}$ into the subjoin $\bigvee_{\alpha \in \mathcal{I}} (gX_{\alpha})$.

Proof. Let $A'$ denote the complementary factor of $A$ in the join $X$. Then, for an indecomposable factor $X_{\beta}$, $g(X_{\beta}') = (gX_{\beta})'$.

Let $x$ be a principal simplex of $X'$. Then since $X_{\beta}$, $X_{\beta}'$ are disjoint, $gX$ cannot lie in $X_{\beta}$, nor, by a pushing lemma argument, can it lie between $gX_{\beta}$ and its complement. So $gX \in (gX_{\beta})'$.

Now the join $\mathbf{X}$ can be written as $\bigvee_{\beta \in \mathcal{I}} X_{\beta}$.

Then $gX = \bigvee_{\beta \in \mathcal{I}} g(X_{\beta}') = \bigvee_{\beta \in \mathcal{I}} (gX_{\beta})' = \bigvee_{\alpha \in \mathcal{I}} (gX_{\alpha})$.

On the polyhedron $X = A^r$ there are some standard homeomorphisms, induced by permuting the factors, which give an action of the symmetric group $S_r$ on $X$. Call the quotient space of $X$ by this action the symmetric join of $A$.

Suppose $Y$ is a polyhedron, and $lk(y, Y) = A$, for some point $y \in Y$. Then the link of the point $(y, \ldots, y)$ in the diagonal of $Y \times Y \times \cdots \times Y$ is just $A^r = X$. So the link of this point in the $r$th symmetric product $Sym^rY$ will be the $r$th symmetric join of $A$.

A knowledge of all the symmetric joins of the links of points in $Y$ would enable us to work out the link of any point in $Sym^r Y$. In particular, if $Y$ is an $n$-manifold, then
$A = S^{n-1}$ for all $y$, and we need only work out the symmetric joins of an $(n-1)$ sphere.

For $n > 2$ these symmetric joins will have singularities, but for the cases $n = 1$ and $n = 2$, the joins are balls and spheres respectively, and so the symmetric products of 1- or 2-manifolds are manifolds with boundary, or manifolds. This can be seen from the following section, which is concerned with the particular case of the symmetric products of 1-manifolds.
Section 3  Symmetric products of the circle

Define the nth symmetric product, \( \text{Sym}^n X \), of a topological space \( X \) to be the quotient of the cartesian product \( X^n \) by the action of the symmetric group which permutes the factors. Even if \( X \) is a manifold, this product is not in general a manifold. In this section we determine these products when \( X \) is the circle, \( S^1 \), and show that they are manifolds with boundary. Then the symmetric joins of a pair of points, \( S^0 \), are balls.

The symmetric joins of a circle are spheres, for when \( X \) is the sphere \( S^2 \) the symmetric products of \( X \) are well known to be manifolds. The standard proof of this gives an interesting interpretation for the products of \( S^1 \).

Regard \( S^2 \) as the complex projective line. The cartesian product \( (S^2)^n \) represents ordered \( n \)-tuples of complex homogeneous parameters \( t_1 \ldots t_n \). The homogeneous polynomial of degree \( n \) having these as roots determines, by its coefficients, a point in \( n \)-dimensional complex projective space, \( P^n \). This gives a continuous map \( (S^2)^n \rightarrow P^n \). Permuting the roots \( t_1 \ldots t_n \) does not alter the polynomial, so this map factors through \( \text{Sym}^n S^2 \). The map from this symmetric product is certainly 1-1, and since every such complex polynomial has \( n \) complex roots, it is onto as
well. The fact that \((S^2)^n\) is compact, and \(P_n\) is Hausdorff now shows that \(\text{Sym}^n(S^2)\) is homeomorphic to \(P_n\).

A similar argument, regarding \(S^1\) as the real projective line, gives an embedding of \(\text{Sym}^n(S^1)\) in \(P_n\), \(n\)-dimensional real projective space, which is not onto. This embedding gives a homeomorphism of \(\text{Sym}^n(S^1)\) with the space of real homogeneous polynomials of degree \(n\) which have all their roots real.

Using this interpretation it is easy to determine the first couple of symmetric products from projective geometry, but for the general case a different approach is better.

The cartesian product \((S^1)^n\) is covered by \(R^n\). The group \(Z + Z + \ldots + Z = Z^n\) of covering translations, and the group \(S_n\)

acting on \(R^n\) by permuting the coordinates, both lie in the group of affine motions of \(R^n\). Let \(G\) be the subgroup of this group generated by \(S_n\) and \(Z^n\). Then \(Z^n\) is normal in \(G\) with quotient \(S_n\). Hence \(R^n/G\) is the quotient of \(R^n\) first by \(Z^n\) and then by \(S_n\), and so it is the product \(\text{Sym}^n(S^1)\).

The approach is now to break up this action of \(G\) in a different way. Regard \(R^n\) as the product \(R^{n-1} \times R^1\), where \(R^1\) is the diagonal, \(x_1 = x_2 = \ldots = x_n\), and \(R^{n-1}\) is the orthogonal hyperplane \(Ex_1 = 0\). Then the action of \(G\) preserves this
product structure. Let $G_0$ be the subgroup of $G$ which stabilises the first factor. Then $G_0$ is normal in $G$ and $G/G_0 \cong \mathbb{Z}$. Hence $\mathbb{R}^n/G \cong (\mathbb{R}^{n-1}/G_0 \times \mathbb{R}^1)/\mathbb{Z}$. Then $\mathbb{R}^n/G$ is a bundle over $S^1$ with fibre $\mathbb{R}^{n-1}/G_0$. This bundle is determined by the homeomorphism of $\mathbb{R}^{n-1}/G_0$ on itself induced by the operation of the generator of $\mathbb{Z}$.

Remark Other products can be defined for any space $X$ by factoring $X^n$ with some subgroup of $S_n$. For the circle, the preceding argument will still go through, replacing $G$ by the appropriate subgroup $\Gamma$, to give a bundle over $S^1$ with fibre $\mathbb{R}^{n-1}/\Gamma_0$.

These products, however, are not all manifolds with boundary. Let $\mathbb{Z}_2 \subset S_0$ act on $(S^1)^6$ by cycling the three factors $S^1 \times S^1$. Then the corresponding product is not a manifold. For consider the action on the link of a point on the main diagonal. Such a link is the join of three $S^1$s, and the action cycles the three factors. We can regard this link as the unit sphere in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$, with the same action. Then this action splits as before into the trivial action on the diagonal, and an action on the hyperplane $Z$, $z_1 + z_2 + z_3 = 0$. So the link in the quotient is the join of $S^1$ to the quotient of the unit sphere in $Z$ under the action. But on this sphere the action is fixed point free, and so its quotient
is a closed 3-manifold $M$, with $\pi_1(M) = \mathbb{Z}_2$. So the link in the quotient is the suspension $\mathbb{Z}_2^2 M$ which is not homeomorphic to either a ball or a sphere.

Returning to the case of the symmetric product, we shall work out the fibre $R^{n-1}/G_0$, by finding a subspace of $R^{n-1}$ which is homeomorphic to it. Such a subspace exists if the conditions of the following lemma are satisfied.

**Lemma 1** Let the finite group $\Gamma$ act continuously on a space $X$, and let $F$ be a closed subspace of $X$ such that

(i) No distinct points of $F$ belong to the same orbit.
(ii) The translates of $F$ cover $X$.

Then $F$ is homeomorphic to $X/\Gamma$.

**Proof.** Consider the continuous map $k : F \times X \to X/\Gamma$. Then, by (i), $k$ is 1-1, and by (ii), it is onto. It remains to show that $k$ is open.

Let $U$ be any open set in $F$. Then $V = (X - F) \cup U$ is open in $X$. By (i), $\bigcup_{g \in \Gamma} U_g \subset V$. Hence $\bigcap_{g \in \Gamma} U_g$ which is also open in $X$, since $\Gamma$ is finite.

Now $\bigcap_{g \in \Gamma} U_g \subset U = (\bigcup_{g \in \Gamma} U_g) \cap F$.

So by (ii), $\bigcap_{g \in \Gamma} U_g \subset \bigcup_{g \in \Gamma} U_g$. 
Hence $\bigcup_{g} U_{g}$ is open in $X$, and $p\left(\bigcup_{g} U_{g}\right) = kU$ is open in $X/\Gamma$. We can relax the condition that the group $\Gamma$ be finite to include a group such as $G_0$, which is the extension of a group which acts without fixed points by a finite group, $S_n$.

Lemma 2 The region $F$, $x_1 < x_2 \leq \ldots \leq x_n < x_1 + 1$, forms a suitable subspace of $\mathbb{R}^{n-1}$, $E x_i = 0$, for the action of $G_0$.

Proof Any point of $\mathbb{R}^{n-1}$ can be translated by $G$ to lie in $F$ by a standard procedure to be defined, which takes the whole orbit of a point to one point of $F$. This procedure maps $F$ to itself by the identity. The conditions of lemma 1 will then be satisfied.

Any given point $\xi'$ of $\mathbb{R}^{n-1}$ can be translated uniquely to a point $\xi$ of the half-open unit cube $-1 < x_1 \leq 0$ in $\mathbb{R}^n$ under the action of $Z^n$. If $\eta' = g' \xi'$ is any other point in the orbit of $\xi'$, then $\eta = g \xi$, for some $g \in G$. Since $Z^n$ is normal in $G$, we can write $G = S_n Z^n$ so $g = \pi \sigma$. But $Z^n$ maps the half-open cube outside itself, so $\sigma$ = identity, and $g = \pi$, some permutation.

Apply a permutation $p$ to $\xi$ to give $\xi = p \xi$ satisfying $-1 < \xi_1 < \xi_2 \leq \ldots \leq \xi_n < 0 < \xi_{n+1}$.

This point $\xi$ will be uniquely determined, although the permutation
\( \rho \) may not be, if some of the \( \xi_i \) are equal. Since \( \eta = \pi \xi \), the point \( \bar{\eta} \) will also give rise to the same point \( \xi \) under this treatment.

We now have \( \xi \) with \( \sum \xi_i = -r \) for some integer \( r \), with \( 0 < r < n \). Add 1 to each of the first \( r \) coordinates of \( \xi \), and then cycle them \( r \) times to return them to ascending order. This gives the required point \( \xi \in \mathbb{F} \).

Consider this process for a point \( \xi' \in \mathbb{F} \). Choose \( r \) so that exactly the last \( r \) coordinates of \( \xi' \) are \( > 0 \). Subtract 1 from each of these coordinates to give \( \xi'' \). Permuting these new coordinates into ascending order is done by cycling them \( r \) times. Obviously \( \sum \xi''_i = -r \), so we must add 1 to the first \( r \) coordinates. But these are just the original last \( r \) coordinates minus 1. We have now recovered the original coordinates, so placing them in ascending order returns us to the point \( \xi' \).

Thus \( \mathbb{R}^{n-1}/G_0 \cong \mathbb{F} \).

Now a bounded subspace of \( \mathbb{R}^{n-1} \) defined by \( n \) linear inequalities, and containing points at which all the inequalities are strict, is an \((n-1)\)-simplex. But \( \mathbb{F} \) satisfies these conditions, so \( \text{Sym}^n(S^1) \) is an \((n-1)\)-disc bundle over the circle. It only remains to determine whether it is orientable or non-orientable.

Regard \( \mathbb{R}^n/G_0 \) as \( \mathbb{F} \times \mathbb{R}^1 \), where this is given as
$x_1 \leq \ldots \leq x_n \leq x_1 + 1$. The product structure is described by $F \times a \cong F \times 0$, where the isomorphism is given by adding $a$ to each coordinate of the original $F$, which is taken as $F \times 0$.

The action of $\mathbb{Z}$ on this product can be described by adding $1$ to the first coordinate, and then acting by $G_0$ to return each fibre to the standard form. This gives a map $F \times 0 \rightarrow F \times 1/n$. In this case, having added $1$ to the first coordinate, it is only necessary to cycle the coordinates once in order to return them to the standard form $x_1 \leq \ldots \leq x_n \leq x_1 + 1$ in $F \times 1/n$. So the map of $F$ to itself induced by this action has its orientation determined by this cycling of the coordinates. The map is thus orientation reversing or preserving as $n$ is even or odd respectively.

This gives rise to the following description.

**Theorem**

$\text{Sym}^n(s^1) \cong S^1 \times D^{n-1}$ - the non-orientable bundle (n even)

$\cong S^1 \times D^{n-1}$ (n odd).

The symmetric product is thus homeomorphic to the tubular neighbourhood of a projective line in $P_n$.

Returning to the representation of the symmetric product as a subset of $P_n$, it would be interesting to see if there is an actual projective line lying in this subset.
Consider the cross-section of the disc bundle above consisting of the barycentre of each simplex. These barycentres represent a set of \( n \) points equally spaced around the circle, and, in \( \mathbb{P}^n \), polynomials having these points as roots, where the point \( a \) corresponds to the root \( \tan \). The locus of such polynomials in \( \mathbb{P}^n \) is the projective line \( x_0 = -x_2 = x_4 = \ldots, x_1 = -x_3 = \ldots \).

So the symmetric product can be thought of as a tubular neighbourhood of this projective line in \( \mathbb{P}^n \).
Appendix

This appendix contains the facts about links of points and simplexes which are used in section 1, and the 'pushing lemma', a technical lemma where these facts are also strongly used. Apart from the pushing lemma, most of these results, together with the required definitions, can be found in [8].

Property 1  The link of a point \( x \) in a polyhedron is well defined as the link of \( x \) in any triangulation of the polyhedron having \( x \) as a vertex. For any two such triangulations have a common subdivision. It is then enough to show that the link of a vertex in a triangulation and in a subdivision are \( \mathbb{H} \) homeomorphic. This follows, using the technique of pseudo-radial projection of the one link to the other.

The join of two polyhedra is assumed to be triangulated by a subdivision of some join triangulation. Hence we need to know the link of simplexes in these triangulations.

Property 2  In a join triangulation of \( A \cup B \), the link of the simplex \( a \cup b \) is \( \text{lk}(a,A) \cup \text{lk}(b,B) \), with the convention that if \( a = \emptyset \), then \( \text{lk}(a,A) = A \).
Property 3  If a point $x$ lies in the interior of a simplex $a$ in some triangulation of $A$, then $\text{lk}(x,A) = \Sigma^r \text{lk}(a,A)$, where $r = \dim a$.

This follows by stellar subdivision at the point $x$ of all simplexes which contain $x$.

Property 4  Any point $x$ in the join $A \vee B$ which does not lie on $A$ or $B$ lies in the join of points $a$ and $b$ in $A$ and $B$. Then $\text{lk} x = \Sigma \text{lk}(a,A) \text{lk}(b,B)$.

These four properties are all that are required for theorem 1, so that theorem can be used to simplify the proof of the following properties.

Property 5  Let $x$ be a simplex of $X$, and $x'$ a simplex in some subdivision, such that the interior of $x'$ lies in the interior of $x$. Then $\text{lk} x' = \Sigma^k \text{lk} x$, where $k$ is the codimension of $x'$ in $x$.

Proof  Let $y$ be a point in the interior of $x'$, and let $n = \dim x'$. Then $\text{lk} y = \Sigma^n \text{lk} x' = \Sigma^{n+k} \text{lk} x$ (property 3).

Hence, by theorem 1, $\text{lk} x' = \Sigma^k \text{lk} x$.

Property 6  This gives a description of the link of a simplex in a subdivided join, and is used constantly in the text.

If $x$ is a simplex in a subdivision of $A \vee B$, then there
are simplexes \(a, b\) of the join triangulation such that interior of \(x\) lies in interior of \(a \cup b\). Then \(\text{lk } x = \sum k \text{lk}(a, A) \text{lk}(b, B)\), where \(k\) is the codimension of \(x\) in \(a \cup b\). This follows from property 2 and property 5.

**Property 7** The pushing lemma.

Suppose \(A \cup B \cong X \cup Y\) and \(A\) is reduced. Let \(x\) and \(y\) be simplexes of \(X\) and \(Y\) in the join triangulation. Suppose that the simplex \(xy\) of the join has been subdivided into simplexes \(b_i\) for the simplicial isomorphism between \(A \cup B\) and \(X \cup Y\), with \(\text{dim } b_i = \text{dim } xy\). Then if one of these \(b_i\) corresponds to a principal simplex of \(B\), so do all the others, hence the whole simplex \(xy\) has come from \(B\).

**Proof** Suppose \(b_1\) and \(b_2\) are two simplexes of \(xy\) which have a common top-dimensional face \(b\), and suppose \(a_1\) and \(a_2\) are their respective vertices opposite \(b\). Suppose \(b_1\) comes from a principal simplex \(\beta_i\) in \(B\). Now \(A = \text{lk } \beta_i = \text{lk } b_1 = \text{lk } (x \cup y)\). Then \(\text{lk } b_2 = A\) as well, and by property 5, \(\text{lk } b = \Sigma A\).

Regarding \(\text{lk } b\) as an actual subcomplex of \(X \cup Y\), the two suspension points must correspond to \(a_1\) and \(a_2\). For these both lie in \(\text{lk } b\), and their link in \(\text{lk } b\) is \(A\), which is reduced, while the suspension points in \(\Sigma A\) are the only points with
reduced link.

If \( b \) corresponds to the simplex \( \beta \) in \( B \), then

\[
\text{lk}(\beta) = \text{lk}(\beta, B) A, \quad \text{and} \quad \Sigma A = \text{lk}(\beta, B) A. \quad \text{Now, by property 9,}
\]

\[
\text{lk}(\beta, B) = \Sigma, \quad \text{and the suspension points are preserved in the}
\]

homeomorphism. Thus the point \( a_2 \) corresponds to a point of \( \text{lk}(\beta, B) \) and so the whole simplex \( b_2 \) corresponds to a principal simplex of \( B \).

Since any two top-dimensional simplexes of \( x y \) can be joined by a chain of simplexes such as \( b_1 \) and \( b_2 \), this shows that the whole of \( x y \) comes from \( B \). In particular, there must be one of the \( b_1 \) which has a face \( b' \) lying in \( x \) with the same dimension as \( x \).

This gives the following version of the pushing lemma, which is used in the text.

**Property 3** Suppose \( A B \cong x y \), with \( A \) reduced, and suppose some principal simplex of \( B \) lies between \( x \) and \( y \) under the isomorphism. Then there are non-empty simplexes \( x \) and \( y \) of the join triangulation such that \( A = \text{lk}(x, x) \text{lk}(y, y) \) and a simplex \( \beta' \) of \( B \), whose image lies in \( x \), such that

\[
A \text{lk}(\beta', B) = \text{lk}(x, x) X.
\]

**Proof** The principal simplex \( \beta \) of \( B \) determines simplexes \( x \) and \( y \) so that its interior lies in the interior of \( x y \).
Then, by property 6, \( A = \text{lk} \beta' = \varepsilon^r \text{lk}(x,y) \), where \( r \) is the codimension of \( \beta \) in \( x,y \). Now \( A \) is reduced, so \( r \) is zero. Then \( \beta \) corresponds to a simplex \( b_1 \) as in property 7. The simplex \( \beta' \) corresponds to \( b' \) in property 7, and the result follows.

**Property 9** If \( E A \cong X A \), and \( A \) is reduced, then \( X = \varepsilon \), and the isomorphism preserves the suspension points.

**Proof** The dimension of \( X \) must be zero, so it consists of a finite number of points. Look at the points on each side which have a reduced link. There are exactly two such points in \( E A \).
If \( X = T \), then this cone point is the only such point in \( X A \).
If \( X \) has three or more points, then at least these points of \( X A \) have reduced link.

So \( X \) is \( \varepsilon \) and the isomorphism preserves the suspension points.
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