

# Stability and optimality of distributed secondary frequency control schemes in power networks

Andreas Kasis, Nima Monshizadeh, Eoin Devane, and Ioannis Lestas

**Abstract**—We present a systematic method for designing distributed generation and demand control schemes for secondary frequency regulation in power networks such that stability and an economically optimal power allocation can be guaranteed. We consider frequency dynamics given by swing equation along with generation, controllable demand, and a secondary control scheme that makes use of local frequency measurements and a locally exchanged signal. A decentralized dissipativity condition is imposed on net power supply variables to provide stability guarantees. Furthermore, economic optimality is achieved by explicit steady state conditions on the generation and controllable demand. A distinctive feature of the proposed stability analysis is the fact that it can cope with generation and demand dynamics that are of general higher order. Moreover, we discuss how the proposed framework captures various classes of power supply dynamics used in recent studies. In case of linear dynamics, the proposed dissipativity condition can be efficiently verified using an appropriate linear matrix inequality. Moreover, it is shown how the addition of a suitable observer layer can relax the requirement for demand measurements in the secondary controller. The efficiency and practicality of the proposed results are demonstrated with simulations on the Northeast Power Coordinating Council (NPCC) 140-bus and a 9-bus system.

## NOMENCLATURE

Main symbols used within the paper.

### Functions

- $\dot{x}$  time derivative of function of time  $x$   
 $\hat{h}(s)$  the Laplace transform of a signal  $h(t)$ ,  $h : \mathbb{R} \rightarrow \mathbb{R}$   
 $\mathbf{1}_{a \leq b}$  a function that takes the value of 1 when  $a \leq b$ , for  $a, b \in \mathbb{R}$ , and of 0 otherwise  
 $f'(q)$  first derivative of function  $f(q)$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$   
 $f^{-1}(\cdot)$  inverse of function  $f(q)$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$

### Indices

- $[q]_a^b$   $\max\{\min\{q, b\}, a\}$  for some  $a, b \in \mathbb{R}$ ,  $a \leq b$   
 $\bigoplus_{j=1}^n B_j$  direct sum of input/output systems  $B_j$ ,  $j = 1, \dots, n$   
 $\mathcal{G}$  graph index  
 $\mathbf{0}_n$   $n \times 1$  vector with all elements equal to 0  
 $\mathbf{0}_{n \times m}$   $n \times m$  matrix with all elements equal to 0  
 $x^*$  equilibrium point of variable  $x$

### Sets

- $\mathbb{R}$  set of real numbers  
 $\mathbb{R}^n$  set of  $n$ -dimensional vectors with real entries

- $\tilde{E}$  set of communication lines  
 $E$  set of transmission lines  
 $G$  set of generation buses  
 $i : i \rightarrow j$  set of buses preceding bus  $j$   
 $k : j \rightarrow k$  set of buses succeeding bus  $j$   
 $L$  set of load buses  
 $N$  set of buses

### Variables

- $\eta_{ij}$  power angle difference between bus  $i$  and bus  $j$   
 $\omega_j$  frequency deviation at bus  $j$   
 $\psi_{ij}$  integral of power command difference between bus  $i$  and bus  $j$   
 $B_{ij}$  line susceptance between bus  $i$  and bus  $j$   
 $d_j^c$  controllable load at bus  $j$   
 $d_j^u$  uncontrollable frequency dependent load at bus  $j$   
 $M_j$  generator inertia at bus  $j$   
 $p_j^c$  power command at bus  $j$   
 $p_j^L$  step change in uncontrollable demand at bus  $j$   
 $p_j^M$  mechanical power injection at bus  $j$   
 $p_{ij}$  power transfer from bus  $i$  to bus  $j$   
 $s_j$  net power supply at bus  $j$   
 $x_j^M$  internal states of generation dynamics at bus  $j$   
 $x_j^c$  internal states of controllable load dynamics at bus  $j$   
 $x_j^u$  internal states of uncontrollable frequency dependent load dynamics at bus  $j$

## I. INTRODUCTION

Motivation: Renewable sources of energy are expected to grow in penetration within power networks over the next years [1], [2]. Moreover, it is anticipated that controllable loads will be incorporated within power networks in order to provide benefits such as fast response to changes in power generated from renewable sources and the ability for peak demand reduction [3]. Such changes will greatly increase power network complexity revealing a need for highly distributed schemes that will guarantee its stability when ‘plug and play’ devices are incorporated. In the recent years, research attention has increasingly focused on such distributed schemes with studies regarding both primary (droop) control as in [4], [5], [6] and secondary control as in [7], [8].

An issue of economic optimality in the power allocation is raised if highly distributed schemes are to be used for frequency control. Recent studies attempted to address this issue by crafting the equilibrium of the system such that it coincides with the optimal solution of a suitable network optimization problem. To establish optimality of an equilibrium in a distributed fashion, it is evident that a synchronising variable is required. While in the primary control, frequency is used as

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the synchronising variable (e.g. [6], [9], [10], [11]), in the secondary control a different variable is synchronized by making use of information exchanged between buses [7], [8], [12], [13].

Literature survey: Over the last few years many studies have attempted to address issues regarding stability and optimization in secondary frequency control. An important feature in many of those is that the dynamics considered follow from a primal/dual algorithm associated with some optimal power allocation problem [7], [14], [15], [16]. This is a powerful approach that reveals the information structure needed to achieve optimality and satisfy the constraints involved. Nevertheless, when higher order generation dynamics need to be considered, these do not necessarily follow as gradient dynamics of a corresponding optimization problem and therefore alternative approaches need to be employed.

Another trend in the secondary frequency control is the use of distributed averaging proportional integral (DAPI) controllers [8], [17], [18], [19], [20]. Advantages of DAPI controllers lie in their simplicity as they only measure local frequency and exchange a synchronization signal in a distributed fashion without requiring load and power flow measurements. On the other hand, it is not easy to accommodate line and power flow constraints, and higher-order generation and controllable demand dynamics in this setting. Moreover the existing results in this context are limited to the case of proportional active power sharing and quadratic cost functions.

*Contribution:* One of our aims in this paper is to present a methodology that allows to incorporate general classes of higher order generation and demand control dynamics while ensuring stability and optimality of the equilibrium points. Our analysis borrows ideas from our previous work in [6] and adapts those to secondary frequency control, by incorporating the additional communication layer needed in this context. In particular, we consider general classes of aggregate power supply dynamics at each bus and impose two conditions; a dissipativity condition that ensures stability, and a steady-state condition that ensures optimality of the power allocation. An important feature of these conditions is that they are decentralized. Furthermore, in the case of linear supply dynamics, the proposed dissipativity condition can be efficiently verified by means of a linear matrix inequality (LMI). Various examples are also described to illustrate the significance of our approach and the way it could facilitate a systematic analysis and design. Finally, we discuss how an appropriately designed observer, allows to relax the requirement of an explicit knowledge of the uncontrollable demand, and show that the stability and optimality guarantees remain valid in this case.

For clarity, we summarize the main contributions of the paper below: (i) We present a stability analysis framework that allows to incorporate general classes of higher order generation and demand control dynamics. (ii) An optimal power allocation is ensured at steady state by means of conditions on the input-output properties of suitable subsystems. (iii) The proposed stability and optimality conditions are decentralized which makes them applicable to large scale networks and highly distributed frequency control schemes. (iv) We relax the requirement of an explicit knowledge of the uncontrollable

demand needed in primal/dual secondary control schemes, by adding an appropriate observer layer.

Paper structure: Section II provides some basic notation and preliminaries. In section III we present the power network model, the classes of generation and controllable demand dynamics and the optimization problem to be considered. Sections IV and V include our main assumptions and results. In Section VI we discuss how the results apply to various dynamics for generation and demand, provide intuition regarding our analysis and show how the controller requirements may be relaxed by incorporating an appropriate observer. In section VII, we demonstrate our results through simulations on the NPCC 140-bus and a 9-bus system. Finally, conclusions are drawn in section VIII. The proofs of the main results can be found in the Appendix.

## II. NOTATION AND PRELIMINARIES

Real numbers are denoted by  $\mathbb{R}$ , and the set of  $n$ -dimensional vectors with real entries is denoted by  $\mathbb{R}^n$ . For a function  $f(q)$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we denote its first derivative by  $f'(q) = \frac{d}{dq}f(q)$ , its inverse by  $f^{-1}(\cdot)$ . A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be positive semidefinite if  $f(x) \geq 0$ . It is positive definite if  $f(0) = 0$  and  $f(x) > 0$  for every  $x \neq 0$ . We say that  $f$  is positive definite with respect to component  $x_j$  if  $f(x) = 0$  implies  $x_j = 0$ , and  $f(x) > 0$  for every  $x_j \neq 0$ . For  $a, b \in \mathbb{R}$ ,  $a \leq b$ , the expression  $[q]_a^b$  will be used to denote  $\max\{\min\{q, b\}, a\}$  and we write  $\mathbf{0}_n$  and  $\mathbf{0}_{n \times m}$  to denote the  $n \times 1$  vector and  $n \times m$  matrix respectively with all elements equal to 0. Furthermore, for matrices  $D_1 \in \mathbb{R}^{n_1 \times m_1}$ ,  $D_2 \in \mathbb{R}^{n_2 \times m_2}$ , the matrix  $D = \text{blockdiag}(D_1, D_2)$  is given by

$$D = \begin{bmatrix} D_1 & \mathbf{0}_{n_1 \times m_2} \\ \mathbf{0}_{n_2 \times m_1} & D_2 \end{bmatrix}.$$

We use  $\mathbb{1}_{a \leq b}$  to denote a function that takes the value of 1 when  $a \leq b$ , for  $a, b \in \mathbb{R}$ , and of 0 otherwise. The Laplace transform of a signal  $h(t)$ ,  $h : \mathbb{R} \rightarrow \mathbb{R}$ , is denoted by  $\hat{h}(s) = \int_0^\infty e^{-st}h(t) dt$ . Finally, for input/output systems  $B_j$ ,  $j = 1, \dots, n$ ,  $n > 1$ , with respective inputs  $u_j$  and outputs  $y_j$ , their direct sum, denoted by  $\bigoplus_{j=1}^n B_j$ , represents a system with input  $[u_1^T, u_2^T, \dots, u_n^T]^T$  and output  $[y_1^T, y_2^T, \dots, y_n^T]^T$ . For a graph  $\mathcal{G} = (N, E)$  we define the directed incidence matrix  $\tilde{D}$  to be the  $|N| \times |E|$  matrix such that the element  $\tilde{D}_{i,j} = -1$  if the edge  $j$  leaves node  $i$ ,  $\tilde{D}_{i,j} = 1$  if the edge  $j$  enters node  $i$  and 0 otherwise.

## III. PROBLEM FORMULATION

### A. Network model

We describe the power network model by a connected graph  $(N, E)$  where  $N = \{1, 2, \dots, |N|\}$  is the set of buses and  $E \subseteq N \times N$  the set of transmission lines connecting the buses. There are two types of buses in the network, buses with inertia and buses without inertia. In the former type, the presence of inertia typically follows from the rotation of machines that produce torque and a detailed mathematical description of it can be found in [22, Sec. 5.1]. Since generators have inertia, it is reasonable to assume that only buses with inertia have non-trivial generation dynamics. We

define  $G = \{1, 2, \dots, |G|\}$  and  $L = \{|G| + 1, \dots, |N|\}$  as the sets buses with and without inertia respectively such that  $|G| + |L| = |N|$ . Moreover, the term  $(i, j)$  denotes the link connecting buses  $i$  and  $j$ . The graph  $(N, E)$  is assumed to be directed with an arbitrary direction, so that if  $(i, j) \in E$  then  $(j, i) \notin E$ . Additionally, for each  $j \in N$ , we use  $i : i \rightarrow j$  and  $k : j \rightarrow k$  to denote the sets of buses that precede and succeed bus  $j$  respectively. It should be noted that the form of the dynamics in (1)–(4) below is not affected by changes in graph ordering, and our results are independent of the choice of direction. We make the following assumptions for the network:

- 1) Bus voltage magnitudes are  $|V_j| = 1$  p.u. for all  $j \in N$ .
- 2) Lines  $(i, j) \in E$  are lossless and characterized by their susceptances  $B_{ij} = B_{ji} > 0$ .
- 3) Reactive power flows do not affect bus voltage phase angles and frequencies.

*Remark 1:* The assumptions above are generally valid at medium to high voltages, and are standard in secondary frequency control studies [23]. In particular, voltage control often takes place at a fast timescale, and the voltage variables are approximated with their steady-state values relative within the secondary frequency control dynamics. A possible way to incorporate the time-varying nature of voltages in our analysis is to include voltage-dependent terms in the storage function [8], [24], [25], [26]. However, considering simultaneously time-varying voltages and high-order turbine governor and demand dynamics substantially complicates the analysis. Addressing these complications is of independent interest, and requires further investigation which goes beyond the scope of this work.

The assumption of lossless transmission lines, or essentially dominantly inductive lines, is motivated by the medium to high voltages and large output impedances of synchronous generators. This assumption is often made for the construction of valid energy functions that verify the stability of the power system [27]. However, while our theoretical treatment relies on the lossless assumption, the numerical investigation of realistic models in Section VII shows robustness of the considered secondary frequency control schemes against the losses in the transmission lines.

Swing equations can then be used to describe the rate of change of frequency at generation buses. Power must also be conserved at each of the load buses. This motivates the following system dynamics (e.g. [23]),

$$\dot{\eta}_{ij} = \omega_i - \omega_j, \quad (i, j) \in E, \quad (1a)$$

$$M_j \dot{\omega}_j = -p_j^L + p_j^M - (d_j^c + d_j^u) - \sum_{k:j \rightarrow k} p_{jk} + \sum_{i:i \rightarrow j} p_{ij}, \quad j \in G, \quad (1b)$$

$$0 = -p_j^L - (d_j^c + d_j^u) - \sum_{k:j \rightarrow k} p_{jk} + \sum_{i:i \rightarrow j} p_{ij}, \quad j \in L, \quad (1c)$$

$$p_{ij} = B_{ij} \sin \eta_{ij}, \quad (i, j) \in E. \quad (1d)$$

In system (1), the time-dependent variable  $\omega_j$  represents the deviation of the frequency at bus  $j$  from its nominal value, namely 50Hz (or 60Hz). The time dependent variables  $d_j^c$

and  $p_j^M$  represent, respectively, the controllable load at bus  $j$  and the mechanical power injection to the generation bus  $j$ . The quantity  $d_j^u$  represents the uncontrollable frequency-dependent load and generation damping present at bus  $j$ . While the uncontrollable frequency dependent loads are in general nonlinear, their first order approximations can be given by  $d_j^u = \lambda_j \omega_j + c$  where  $\lambda_j > 0$  and  $c$  is constant [28], [22, Sec. 9.1.2]. Note that the constant term  $c$  can be absorbed in  $p_j^L$ , and thus we have  $d_j^u = \lambda_j \omega_j$  in this case. The time-dependent variables  $\eta_{ij}$  and  $p_{ij}$  represent, respectively, the power angle difference<sup>1</sup> and the power transferred from bus  $i$  to bus  $j$ . The constant  $M_j > 0$  denotes the generator inertia. The response of the system (1) will be studied, when a step change  $p_j^L, j \in N$  occurs in the uncontrollable demand.

*Remark 2:* Note that three types of loads are considered in (1), namely *i*) uncontrollable frequency-independent loads  $p_j^L$ , *ii*) uncontrollable frequency-dependent loads  $d_j^u$ , and *iii*) controllable loads  $d_j^c$ . The first type is constant and reflects a step change in the demand, the second one depends statically or dynamically to the frequency deviation, and the third type refers to the loads that can be exploited in frequency regulation and distributed load-side participation schemes [7], [9]. The distinction between (frequency-dependent) uncontrollable and controllable demand amounts to the fact that the former is typically dictated by the physics of the systems, whereas the latter is treated as a part of the design and can be leveraged to improve stability and performance.

In order to investigate broad classes of generation and demand dynamics and control policies, we will consider general dynamical systems of the form

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= g(x, u), \end{aligned} \quad (2)$$

with input<sup>2</sup>  $u(t) \in \mathbb{R}^m$ , state  $x(t) \in \mathbb{R}^n$ , and output  $y(t) \in \mathbb{R}^k$ . In addition the maps  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$  are locally Lipschitz continuous. We assume in (2) that given any constant input  $u(t) \equiv \bar{u}$ , there exists a unique<sup>3</sup> locally asymptotically stable equilibrium point  $\bar{x} \in \mathbb{R}^n$ , i.e.  $f(\bar{x}, \bar{u}) = 0$ . The region of attraction<sup>4</sup> of  $\bar{x}$  is denoted by  $\Omega(\bar{x})$ . We also define the static input-state characteristic map  $k_x : \mathbb{R}^m \rightarrow \mathbb{R}^n$  as

$$k_x(\bar{u}) := \bar{x},$$

and the static input-output characteristic map  $k_y : \mathbb{R}^m \rightarrow \mathbb{R}^k$ ,

$$k_y(\bar{u}) := g(k_x(\bar{u}), \bar{u}). \quad (3)$$

Let  $\Sigma : (u, x, y)$  denote in short the input-state-output system (2). Then, we consider the case where scalar variables  $p_j^M$ ,

<sup>1</sup>The quantities  $\eta_{ij}$  represent the phase differences between buses  $i$  and  $j$ , given by  $\theta_i - \theta_j$ , i.e.  $\eta_{ij} = \theta_i - \theta_j$ . The angles themselves must also satisfy  $\dot{\theta}_j = \omega_j$  at all  $j \in N$ . This equation is omitted in (1) since the power transfers are functions of the phase differences only.

<sup>2</sup>All the state components, input and output variables considered throughout the paper are time dependent. However, their explicit dependence on time is dropped for the sake of notational simplicity.

<sup>3</sup>The uniqueness assumption on the equilibrium point for a given input could be relaxed to having isolated equilibrium points, but it is used here for simplicity in the presentation.

<sup>4</sup>That is, for the constant input  $\zeta_j = \bar{\zeta}_j$ , any solution  $x(t)$  of (4) with initial condition  $x(0) \in \Omega(\bar{x})$  must satisfy  $x(t) \rightarrow \bar{x}$  as  $t \rightarrow \infty$ .

$d_j^c$ , and  $d_j^u$  are generated by dynamical systems of form (2), namely

$$\Sigma_j^M(\zeta_j, x_j^M, p_j^M), \quad j \in G, \quad (4a)$$

$$\Sigma_j^c(\zeta_j, x_j^c, -d_j^c), \quad j \in N, \quad (4b)$$

$$\Sigma_j^u(-\omega_j, x_j^u, -d_j^u), \quad j \in N, \quad (4c)$$

where the input  $\zeta_j$  is defined as  $\zeta_j = [-\omega_j \ p_j^c]^T$  with  $p_j^c$  representing the power command signal. Notice that in the case of uncontrollable demand, the input is given in terms of the local frequency deviation  $\omega_j$  only, and is decoupled from the power command signal as expected.

For notational convenience, we collect the variables in (4) into the vectors  $x^M = [x_j^M]_{j \in G}$ ,  $x^c = [x_j^c]_{j \in N}$ , and  $x^u = [x_j^u]_{j \in N}$ . These quantities represent the internal states of the dynamical systems used to update the outputs  $p_j^M$ ,  $d_j^c$ , and  $d_j^u$ . Moreover, it will be useful to consider the net supply variables  $s_j$ , defined as

$$s_j = p_j^M - d_j^c, \quad j \in G, \quad s_j = -d_j^c, \quad j \in L. \quad (5)$$

The variables defined in (5) evolve according to the dynamics described in (4a) - (4b). Therefore,  $s_j$  are outputs from these combined controlled dynamical systems with inputs  $\zeta_j$ .

### B. Power Command Dynamics

We consider a communication network described by a connected graph  $(N, \tilde{E})$ , where  $\tilde{E}$  represents the set of communication lines among the buses, i.e.,  $(i, j) \in \tilde{E}$  if buses  $i$  and  $j$  communicate. We will study the behavior of the system (1)–(4) under the following dynamics for the power command signal  $p_j^c$  which has been used in literature (e.g. [7], [14]),

$$\gamma_{ij} \dot{\psi}_{ij} = p_i^c - p_j^c, \quad (i, j) \in \tilde{E} \quad (6a)$$

$$\gamma_j \dot{p}_j^c = -(s_j - p_j^c) - \sum_{k:j \rightarrow k} \psi_{jk} + \sum_{i:i \rightarrow j} \psi_{ij}, \quad j \in N \quad (6b)$$

where  $\gamma_j$  and  $\gamma_{ij}$  are positive constants,  $p_i^c$  and  $p_j^c$  are variables shared between communicating buses  $i$  and  $j$ , and the variable  $\psi_{ij}$  is a state of the controller that integrates the power command difference of communicating buses  $i$  and  $j$ . We should highlight that the set of communication lines  $\tilde{E}$  and power lines  $E$  can be the same, e.g. when power line communication is used, or different when a separate communication network is used.

Although the dynamics in (6) do not directly integrate frequency, we will see later that under a weak condition on the steady state behavior of  $d^u$ , they guarantee convergence to the nominal frequency for a broad class of supply dynamics. The dynamics in (6), often referred as ‘virtual swing equations’, are frequently used in the literature<sup>5</sup> as they achieve both the synchronization of the communicated variable  $p^c$ , something that can be exploited to guarantee optimality of the equilibrium

<sup>5</sup>In this paper we use for simplicity a single communicating variable. It should be noted that more advanced communication structures (e.g. [7]) can allow additional constraints to be satisfied in the optimization problem posed.

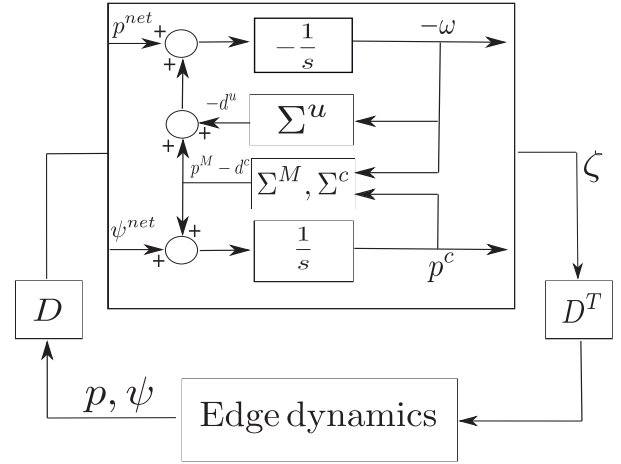


Fig. 1: Schematic overview of the system described by (1)–(6).

point reached, and also the convergence of frequency to its nominal value<sup>6</sup>.

To further illustrate the described dynamics, we have provided a schematic representation on Fig. 1, representing equations (1)–(6). The figure clarifies the interconnection among the various control dynamics in the power network. Pure integrators are denoted by  $\frac{1}{s}$ , and  $D = \text{blockdiag}(D_p, D_c)$  is a block diagonal matrix with its blocks being the incidence matrices of the power and communication networks  $D_p$  and  $D_c$  respectively. Note that the vectors  $p^{net}$  and  $\psi^{net}$  represent the aggregate power transfer and summation of  $\psi$  variables at each bus and have respective  $j^{\text{th}}$  components  $\sum_{i:i \rightarrow j} p_{ij} - \sum_{k:j \rightarrow k} p_{jk}$  and  $\sum_{i:i \rightarrow j} \psi_{ij} - \sum_{k:j \rightarrow k} \psi_{jk}$ .

### C. Equilibrium analysis

We now describe what is meant by an equilibrium of the interconnected dynamical system (1)–(6). Note that this is a common notion of equilibrium for dynamical systems, and should not be confused with other equilibrium notions used in different contexts (e.g. in game theory).

*Definition 1:* The point  $\beta^* = (\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$  defines an equilibrium of the system (1)–(6) if all time derivatives of (1)–(6) are equal to zero at this point.

It should be noted that the static input-output maps  $k_{p_j^M}$ ,  $k_{d_j^c}$ , and  $k_{d_j^u}$ , as defined in (3), completely characterize the equilibrium behavior of (4). In our analysis, we shall consider conditions on these characteristic maps relating input  $\zeta_j = [-\omega_j \ p_j^c]^T$  and generation/demand such that their equilibrium values are optimal for (7), thus making sure that frequency will be at its nominal value at steady state.

Throughout the paper, it is assumed that there exists some equilibrium of (1)–(6) as defined in Definition 1.

*Remark 3:* The core of the assumption on the existence of some equilibrium of (1)–(6) is related to the presence of sinusoids in the expression of the active power transfer  $p_{ij}$ , which implies that the power transfer is bounded and cannot

<sup>6</sup>The primary control stage can be seen to occur at a timescale that is faster than the  $p^c$  dynamics such that  $p^c$  is approximately constant. Primary control was studied in [6] and the stability conditions in this paper imply those in [6] for a system with constant  $p^c$ .

tolerate an arbitrary mismatch between the supply and demand at the equilibrium (see (1b)). This roughly means that the mismatch between the demand and optimal supply<sup>7</sup> must be sufficiently small compared to the size of the line susceptances. Studying the existence of an equilibrium of the power network is rather an independent research venue [30], [31], and goes beyond the scope of this paper.

Any such equilibrium is denoted by  $\beta^* = (\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$ . Furthermore, we use  $(p^*, p^{M,*}, d^{c,*}, d^{u,*}, \zeta^*, s^*)$  to represent the equilibrium values of respective quantities in (1)–(6).

The power angle differences at the considered equilibrium are assumed to satisfy the following condition:

*Assumption 1:*  $|\eta_{ij}^*| < \frac{\pi}{2}$  for all  $(i, j) \in E$ .

The condition above on power angle differences is standard in power grid stability analysis, and is often referred to as a security constraint [19].

Moreover, the following assumption is related with the steady state values of variable  $d^u$ , describing uncontrollable demand and generation damping. It is a mild condition associated with having negative feedback from  $d^u$  to frequency.

*Assumption 2:* For each  $j \in N$ , the functions  $k_{d_j^u}$  relating the steady state values of frequency and uncontrollable loads satisfy  $\bar{u}_j k_{d_j^u}(\bar{u}_j) > 0$  for all  $\bar{u}_j \in \mathbb{R} - \{0\}$ .

*Remark 4:* Assumption 2 requires the static value of  $d_j^u$  to have the same sign as the frequency at an equilibrium of (4c). This can be interpreted as a requirement for positive droop gains. A simple model used within Section VI that satisfies Assumption 2 is  $d_j^u = \lambda_j \omega_j$ ,  $\lambda_j > 0$ . Note also that such loads do not contribute to secondary frequency control at equilibrium.

Although not required for stability, Assumption 2 guarantees that the frequency will be equal to its nominal value at equilibrium, i.e.  $\omega^* = \mathbf{0}_{|N|}$ , as stated in the following lemma.

*Lemma 1:* Let Assumption 2 hold. Then, any equilibrium point  $\beta^*$  given by Definition 1 satisfies  $\omega^* = \mathbf{0}_{|N|}$ .

The stability and optimality properties of such equilibria will be studied in the following sections.

### D. Additional conditions

Due to the fact that the frequency at the load buses is related with the system states by means of algebraic equations, additional conditions are needed for the system (1)–(4) to be well-defined. We use below the vector notation  $\omega^G = [\omega_j]_{j \in G}$  and  $\omega^L = [\omega_j]_{j \in L}$ .

*Assumption 3:* There exists an open neighborhood  $T$  of  $(\eta^*, \omega^{G,*}, x^{c,*}, x^{u,*}, p^{c,*})$  and a unique locally Lipschitz map  $f^L$  such that when  $(\eta, \omega^G, x^c, x^u, p^c) \in T$ ,  $\omega^L = f^L(\eta, \omega^G, x^c, x^u, p^c)$ .

*Remark 5:* Note that the model (1)–(6) constitutes a differential algebraic system, where the algebraic equations are associated with the load buses. The presence of algebraic equations introduces additional technical steps as standard stability analysis tools, like Lyapunov's direct method and LaSalle's

invariance principle, are not directly applicable. These complications arise from the fact the algebraic equations might not necessarily be solvable, or have no unique solutions for a given value of the system states. To address this issue additional technical conditions are needed. One such condition used in the literature involves the use of the implicit function theorem [32], [33]. This requires a mild nonsingularity assumption on the Jacobian of the load power flow at equilibrium. To avoid the aforementioned technicalities, we have imposed Assumption 3, which is implied by the condition mentioned above, and ensures that the system can be reduced to an ordinary differential equation with no algebraic constraints. In general, Assumption 3 is expected to be satisfied in most practically relevant settings, see also [6, Remark 6],[32].

### E. Optimal supply and load control

We aim to study how generation and controllable demand should be adjusted in order to meet the step change in frequency independent demand and simultaneously minimize the cost that comes from the deviation in the power generated and the disutility of loads. We now introduce an optimization problem, which we call the optimal supply and load control problem (OSLC), that can be used to achieve this goal.

A cost  $C_j(p_j^M)$  is supposed to be incurred when the generation output at bus  $j$  is  $p_j^M$ . Similarly, a cost of  $C_{dj}(d_j^c)$  is incurred when controllable demand is  $d_j^c$ . Note that the cost  $C_j$  is accounted for the generation cost at bus  $j$ , whereas  $C_{dj}$  is the cost associated with the disutility of the loads when these adjust their demand. Both cost functions can be any continuously differentiable strictly convex function as formalized in Assumption 4, see also [8], [19], [34].

The OSLC problem amounts to find the vectors  $p^M$  and  $d^c$  that minimize the total cost and simultaneously achieve power balance, while satisfying physical saturation constraints. More precisely, the following optimization problem is considered

OSLC:

$$\begin{aligned} & \min_{p^M, d^c} \sum_{j \in G} C_j(p_j^M) + \sum_{j \in N} C_{dj}(d_j^c), \\ & \text{subject to } \sum_{j \in G} p_j^M = \sum_{j \in N} (d_j^c + p_j^L), \\ & p_j^{M, \min} \leq p_j^M \leq p_j^{M, \max}, \forall j \in G, \\ & d_j^{c, \min} \leq d_j^c \leq d_j^{c, \max}, \forall j \in N, \end{aligned} \quad (7)$$

where  $p_j^{M, \min}$ ,  $p_j^{M, \max}$ ,  $d_j^{c, \min}$ , and  $d_j^{c, \max}$  are bounds for the minimum and maximum values for generation and controllable demand, respectively, at bus  $j$ . The equality constraint in (7) requires all the frequency-independent loads<sup>8</sup> to be matched by the total generation and controllable demand. This ensures that when system (1) is at equilibrium and Assumption 2 holds, the frequency will be at its nominal value.

It should be clear that we are interested in the equilibrium values  $p^{M,*}$  and  $d^{c,*}$  that solve the OSLC problem

<sup>7</sup>Note that eventually we are interested in a power supply which is optimal in the sense of OSLC (7).

<sup>8</sup>Note that frequency independent demand  $p^L$  represents the demand that results from consumer behavior. In the considered setting, we study the impact of an alteration in  $p^L$ , modeled by a step change at  $t = 0$ .

*Remark 6:* Note that the aim of the (static) OSLC problem (7) is to return the optimal values  $p^{M,*}$  and  $d^{c,*}$ . We specify properties on the control dynamics of  $p^M$  and  $d^c$ , described in (4a)–(4b), such that they solve the OSLC problem at the equilibrium. Moreover, in Theorem 2, we show that solutions of the power system asymptotically converge to the optimal solution of (7).

The assumption below allows the use of the KKT conditions to prove the optimality result in Theorem 1 in Section V.

*Assumption 4:* The cost functions  $C_j$  and  $C_{dj}$  are continuously differentiable and strictly convex, and the OSLC problem admits a solution.

#### IV. DISSIPATIVITY CONDITIONS ON GENERATION AND DEMAND DYNAMICS

Before we state our main results in Section V, it would be useful to provide a dissipativity definition, based on [35], for systems of the form (2). This notion will be used to formulate appropriate decentralized conditions on the uncontrollable demand and power supply dynamics (4c), (5).

*Definition 2:* The system (2) is said to be locally dissipative about the constant input values  $\bar{u}$  and corresponding equilibrium state values  $\bar{x}$ , with supply rate function  $W : \mathbb{R}^{n+k} \rightarrow \mathbb{R}$ , if there exist open neighborhoods  $U$  of  $\bar{u}$  and  $X$  of  $\bar{x}$ , and a continuously differentiable, positive definite function  $V : \mathbb{R}^m \rightarrow \mathbb{R}$  (called the storage function), with a strict local minimum at  $x = \bar{x}$ , such that for all  $u \in U$  and all  $x \in X$ ,

$$\dot{V}(x) \leq W(u, y). \quad (8)$$

Furthermore, when  $W(u, y) = u^T y$ , the system is said to be passive about the constants  $\bar{u}$  and  $\bar{x}$ .

We now assume that the systems with input  $\zeta_j = [-\omega_j \ p_j^c]^T$  and output the power supply variables and uncontrollable loads satisfy the following local dissipativity condition.

*Assumption 5:* The systems with inputs  $\zeta_j = [-\omega_j \ p_j^c]^T$  and outputs  $y_j = [s_j \ -d_j^u]^T$  described in (5) and (4c) satisfy a dissipativity condition about constant input values  $\zeta_j^*$  and corresponding equilibrium state values  $(x_j^{M,*}, x_j^{c,*}, x_j^{u,*})$  in the sense of Definition 2, with supply rate functions

$$W_j(\zeta_j, y_j) = [(s_j - s_j^*) \ (-d_j^u - (-d_j^{u,*}))] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\zeta_j - \zeta_j^*) - \phi_j(\zeta_j - \zeta_j^*), \quad j \in N, \quad (9)$$

and some function  $\phi_j : \mathbb{R}^2 \rightarrow \mathbb{R}$  which satisfies at least one of the following two properties

- (a) The function  $\phi_j$  is positive definite.
- (b) The function  $\phi_j$  is positive semidefinite and positive definite with respect to  $\omega_j$ . Also when  $\omega_j, s_j$  are constant for all times then  $p_j^c$  cannot be a nontrivial sinusoid<sup>9</sup>.

We shall refer to Assumption 5 when condition (a) holds for  $\phi_j$  as Assumption 5(a) (respectively Assumption 5(b) when (b) holds).

*Remark 7:* Assumption 5 is a decentralized condition that allows to incorporate a broad class of generation and load

dynamics, including various examples that have been used in the literature which are discussed in Section VI. Furthermore, for linear systems Assumption 5 can be formulated as the feasibility problem of a corresponding LMI (linear matrix inequality) [36], and it can therefore be verified by means of computationally efficient methods.

*Remark 8:* Condition (b) in Assumption 5 is a relaxation of condition (a) whereby  $\phi$  is not required to be positive definite. This permits the inclusion of a broader class of dynamics from  $p_j^c$  to  $s_j$  as it will be discussed in Section VI. However, it requires that the power command  $p^c$  cannot be a sinusoid if both  $s_j$  and  $\omega_j$  are constant. This additional condition is necessary as the dynamics in (6) allow  $p_j^c$  to be a sinusoid when  $s_j$  is constant. For linear systems, this condition is implied by the rather mild assumption that no imaginary axis zeros are present in the transfer function from  $p_j^c$  to  $s_j$ .

*Remark 9:* Further intuition on the dissipativity condition in Assumption 5 will be provided in Section VI-A. In particular, it will be discussed that when  $\phi_j = 0$  that this is a decentralized condition that is necessary and sufficient for the passivity of an appropriately defined multivariable system quantifying aggregate dynamics at each bus.

#### V. MAIN RESULTS

In this section we state our main results, with their proofs provided in the Appendix. Our first result provides sufficient conditions for the equilibrium points to be solutions<sup>10</sup> to the OSLC problem (7).

*Theorem 1:* Suppose that Assumption 4 is satisfied and the control dynamics in (4a) and (4b) are chosen such that

$$\begin{aligned} k_{p_j^M}([0 \ p_j^c]^T) &= [(C'_j)^{-1}(p_j^c)]_{p_j^{M, \min}}^{p_j^{M, \max}} \\ k_{d_j^c}([0 \ p_j^c]^T) &= [(C'_{dj})^{-1}(p_j^c)]_{d_j^{c, \min}}^{d_j^{c, \max}} \end{aligned} \quad (10)$$

holds. Then, the equilibrium values  $p^{M,*}$  and  $d^{c,*}$  are optimal solutions to the OSLC problem (7).

Our second result extends Theorem 1 by providing sufficient conditions for the local convergence of solutions to (1)–(6). In particular, it shows that the set of equilibria for the system described by (1)–(6) for which Assumptions 1–5 are satisfied is asymptotically attracting, the equilibria are global minima of the OSLC problem (7) and, as shown in Lemma 1, satisfy  $\omega^* = \mathbf{0}_{|N|}$ .

*Theorem 2:* Consider an equilibrium of (1)–(6) with respect to which Assumptions 1–5 are all satisfied and suppose that the control dynamics in (4a) and (4b) are chosen such that (10) holds. Then there exists an open region of the state space containing the equilibrium such that solutions of (1)–(6) asymptotically converge to a set of equilibria that solve the OSLC problem (7) with  $\omega^* = \mathbf{0}_{|N|}$ .

#### VI. DISCUSSION

In this section we discuss examples that fit within the framework presented in the paper, and also describe how the

<sup>9</sup>By nontrivial sinusoid, we mean functions of the form  $\sum_j A_j \sin(\omega_j t + \phi_j)$  that are not equal to a constant.

<sup>10</sup>Note that an equilibrium point is a solution to the OSLC problem when at that point the variables that appear in (7) are solutions to the problem.

dissipativity condition of Assumption 5 can be verified for linear systems via a linear matrix inequality.

We start by giving various examples of power supply dynamics that have been used in the literature that satisfy our proposed dissipativity condition in Assumption 5. Consider the load models used in [7], [13], and [14], where the power supply is a static function of  $\omega_j$  and  $p_j^c$ ,

$$s_j = (C_j')^{-1}(p_j^c - \omega_j), \quad j \in N, \quad (11)$$

where  $C_j$  is some convex cost function, and generation damping/uncontrollable frequency dependent demand is given by  $d_j^u = \lambda_j \omega_j$ ,  $\lambda_j > 0$ . It is easy to show that Assumption 5(a) holds for these widely used schemes.

Furthermore, Assumption 5(b) is satisfied when first order generation dynamics are used such as

$$\dot{s}_j = -\mu_j(C_j'(s_j) - (p_j^c - \omega_j)) \quad (12)$$

with  $d_j^u = \lambda_j \omega_j$  and  $\lambda_j, \mu_j > 0$ . Such first order models have often been used in the literature as in [16].

A significant aspect of the framework presented in this paper is that it also allows higher order dynamics for the power supply to be incorporated. As an example, we consider the following second-order model,

$$\begin{aligned} \dot{\alpha}_j &= -\frac{1}{\tau_{a,j}}(\alpha_j - K_j(p_j^c - \omega_j)), \\ \dot{z}_j &= -\frac{1}{\tau_{b,j}}(z_j - \alpha_j), \\ s_j - d_j^u &= z_j - \lambda_j \omega_j + \lambda_j^{PC} p_j^c, \end{aligned} \quad (13)$$

where  $\alpha_j$ ,  $z_j$  are states and  $\tau_{a,j}, \tau_{b,j} > 0$  time constants associated with the turbine-governor dynamics,  $\lambda_j > 0$  is a damping coefficient<sup>11</sup>, constant  $K_j > 0$  determines the strength of the feedback gain, and the term  $\lambda_j^{PC} p_j^c$  represents static dependence on power command due to either generation or controllable loads<sup>12</sup>. It can be shown that Assumption 5 is satisfied for all  $\tau_{a,j}, \tau_{b,j} > 0$  when<sup>13</sup>  $K_j < 8\lambda_j^{PC}$  and  $\lambda_j^{PC} \leq \lambda_j$ . A storage function for this case is  $V = \frac{\tau_{a,j}}{4}\alpha_j^2 + \tau_{b,j}z_j^2$ .

Another feature of Assumption 5 is that it can be efficiently verified for a general linear system by means of an LMI, i.e. a computationally efficient convex problem. In particular, it can be shown [36] that if the system in Assumption 5 is linear with a minimal state space realization

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \\ \tilde{y} &= Cx + D\tilde{u}, \end{aligned} \quad (14)$$

where  $\tilde{u} = \zeta - \zeta^*$  and  $\tilde{y} = y - y^*$ , and  $\phi_j$  is chosen as a quadratic function  $\phi_j = \epsilon_1(\omega_j - \omega_j^*)^2 + \epsilon_2(p_j^c - p_j^{c,*})^2$  with<sup>14</sup>

<sup>11</sup>Note that the term  $\lambda_j \omega_j$  can be incorporated in  $s_j$  or  $d_j^u$ .

<sup>12</sup>It should be noted that the term  $d_j^u$  can also include controllable demand and generation that depend on frequency only (i.e. not on power command). Therefore,  $d_j^u$  can be perceived to contain all frequency dependent terms that return to their nominal value at steady state and therefore do not contribute to secondary frequency control.

<sup>13</sup>A second order model was studied for a related problem in [15], with the stability condition requiring, roughly speaking, that the gain of the system is less than the damping provided by the loads. The LMI approach described in this section allows such conditions to be relaxed.

<sup>14</sup>We could also have  $\epsilon_2 = 0$  if (14) has no zeros on the imaginary axis, as stated in condition (b) for  $\phi$  in Assumption 5, and Remark 8.

$\epsilon_1, \epsilon_2 > 0$  then the dissipativity condition in Assumption 5 is satisfied if and only if there exists  $P = P^T \geq 0$  such that

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T Q \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \leq 0, \quad (15)$$

where the matrix  $Q$  is given by

$$Q = \begin{bmatrix} 0 & M \\ M & K \end{bmatrix}, \quad M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -\epsilon_1 & 0 \\ 0 & -\epsilon_2 \end{bmatrix}.$$

*Remark 10:* While finding storage functions satisfying Assumption 5 is in general a nontrivial task, the LMI in (15) provides an efficient test to verify this assumption in the case of linear power dynamics (14). In addition, the LMI above, which is a convex constraint, can be exploited to establish bounds on the minimum damping or maximum droop gains allowed for closed loop stability, by forming appropriate convex optimization problems. For example, one of the elements of the damping term  $D$  in (14) can be minimized with (15) as a constraint which is a semidefinite program.

To further demonstrate the applicability of our approach we consider a fifth order model for turbine governor dynamics provided by the Power System Toolbox [38]. The dynamics are described by the following transfer function relating the mechanical power supply<sup>15</sup>  $\hat{s}_j$  with the negative frequency deviation  $-\hat{\omega}_j$ ,

$$G_j(s) = K_j \frac{1}{(1 + sT_{s,j})} \frac{(1 + sT_{3,j})(1 + sT_{4,j})}{(1 + sT_{c,j})(1 + sT_{5,j})},$$

where  $K_j$  and  $T_{s,j}, T_{3,j}, T_{c,j}, T_{4,j}, T_{5,j}$  are the droop coefficient and time-constants respectively. Realistic values for these models are provided by the toolbox for the NPCC network<sup>16</sup>, with turbine governor dynamics implemented on 22 buses. The corresponding buses also have appropriate frequency damping  $\lambda_j$ . We examined the effect of incorporating a power command input signal in the above dynamics by considering the supply dynamics

$$\hat{s}_j - \hat{d}_j^u = (G_j + \lambda_j)(-\hat{\omega}_j) + (G_j + \lambda_j^{PC})(\hat{p}_j^c), \quad j \in N$$

where  $\lambda_j^{PC} > 0$ ,  $j \in N$  is a coefficient representing the static dependence on power command. For appropriate values of  $\lambda_j^{PC}$ , the condition in Assumption 5 was satisfied for 20 out of the 22 buses, while for the remaining 2 buses the damping coefficients  $\lambda_j$  needed to be increased by 37% and 28% respectively. Furthermore, filtering the power command signal with appropriate lead-lag compensators, allowed a significant decrease in the required value<sup>17</sup> for  $\lambda_j^{PC}$ . Power command and frequency compensation may also be used with alternative objectives, such as to improve the stability margins and system performance.

<sup>15</sup>Note that  $\hat{s}_j$  denotes the Laplace transform of  $s_j$ .

<sup>16</sup>The data were obtained from the Power System Toolbox [38] data file datanp48.

<sup>17</sup>By significant drop, we mean that the required values of  $\lambda_j^{PC}$  can be decreased by at least a factor of 100 in all cases. It should be noted though, that decreasing  $\lambda_j^{PC}$  might result in slower dynamics and thus a trade-off exists between the amount of static dependence on power command and the response speed.

The fact that our condition is satisfied at all but two buses<sup>18</sup>, demonstrates that it is not conservative in existing implementations. Note also that a main feature of this condition is the fact that it is decentralized, involving only local bus dynamics, which can be important in practical implementations.

### A. System Representation

Although the condition in Assumption 5, which is the key assumption in our analysis, may seem ad hoc at first glance, it is in fact intimately related to passivity properties of certain subsystems of the network that constitute the aggregate dynamics at each bus. To see this, we note that the system (1)–(6) can be represented by a negative feedback interconnection of systems  $I$  and  $B = \bigoplus_{j=1}^{|N|} B_j$ , containing all interconnection and bus dynamics respectively. More precisely,  $I$  and  $B$  have respective inputs  $u^I$  and  $u^B$ , and respective outputs  $u^B$  and  $-u^I$ , defined as

$$u^I = \begin{bmatrix} \omega_1 \\ -p_1^c \\ \dots \\ \omega_{|N|} \\ -p_{|N|}^c \end{bmatrix}, \quad u^B = \begin{bmatrix} \sum_{k:1 \rightarrow k} p_{1k} - \sum_{i:i \rightarrow 1} p_{i1} \\ \sum_{i:i \rightarrow 1} \psi_{i1} - \sum_{k:1 \rightarrow k} \psi_{1k} \\ \dots \\ \sum_{k:|N| \rightarrow k} p_{|N|k} - \sum_{i:i \rightarrow |N|} p_{i|N|} \\ \sum_{i:i \rightarrow |N|} \psi_{i|N|} - \sum_{k:|N| \rightarrow k} \psi_{|N|k} \end{bmatrix}.$$

The subsystems  $B_j$ , representing the dynamics at bus  $j$ , have inputs  $u_j$  and outputs  $y_j$  described by,

$$u_j = \begin{bmatrix} \sum_{k:j \rightarrow k} p_{jk} - \sum_{i:i \rightarrow j} p_{ij} \\ \sum_{i:i \rightarrow j} \psi_{ij} - \sum_{k:j \rightarrow k} \psi_{jk} \end{bmatrix}, \quad y_j = \begin{bmatrix} \omega_j \\ -p_j^c \end{bmatrix}. \quad (16)$$

It can easily be shown that System  $I$  is locally passive<sup>19</sup>. The following theorem shows that Assumption 5 with  $\phi = 0$  is sufficient for the passivity of each individual subsystem  $B_j$ .

**Theorem 3:** Let Assumption 5 hold with  $\phi_j = 0$  about an equilibrium. Then the corresponding subsystem  $B_j$  with inputs and outputs given by (16) is passive about that equilibrium.

**Remark 11:** Note that the subsystems  $B_j$  are multivariable systems representing the aggregated bus dynamics including the frequency dynamics, power supply, and the power command dynamics, at bus  $j$ . The significance of the interpretation discussed above is that stability of the overall interconnected power system is guaranteed if the aggregated local bus dynamics  $B_j$  are passive. This is therefore a decentralized condition that is of value in highly distributed schemes and large scale networks where scalability and decentralization are important.

**Remark 12:** Note that Assumption 5 is also necessary for systems  $B_j$  to be passive, for general affine nonlinear dynamics, see [39, Appendix B]. Hence, Assumption 5 introduces no additional conservatism in this property for a large class of nonlinear systems. Note, however, that passivity of subsystems  $B_j$  is not necessary for stability of the overall power network.

<sup>18</sup>Note that this is satisfied at all buses with appropriate increase in damping.

<sup>19</sup>By a locally passive system we refer to a system satisfying the dissipativity condition in Definition 2 with the supply rate being  $W(u, y) = (u - u^*)^T(y - y^*)$ .

### B. Observing uncontrollable frequency independent demand

The power command dynamics in (6) involve the uncontrollable frequency independent demand  $p^L$ . We discuss in this section that the inclusion of appropriate observer dynamics allows to relax the requirement to have explicit knowledge of  $p^L$  without compromising the stability and optimality properties presented in Theorem 2.

A way to obtain  $p^L$ , could be by re-arranging equations (1b)–(1c). This approach would require knowledge of power supply and power transfers in load buses, which is realistic. However, knowledge of the frequency derivative would also be required for its estimation at generation buses, which might be difficult to obtain in noisy environments.

We therefore consider instead observer dynamics<sup>20</sup> for  $p_j^L$  that are incorporated within the power command dynamics. In particular the following dynamics are considered

$$\gamma_{ij} \dot{\psi}_{ij} = p_i^c - p_j^c, \quad (i, j) \in \tilde{E}, \quad (17a)$$

$$\gamma_j \dot{p}_j^c = -(s_j - \chi_j) - \sum_{k:j \rightarrow k} \psi_{jk} + \sum_{i:i \rightarrow j} \psi_{ij}, \quad j \in N, \quad (17b)$$

$$\tau_{\chi,j} \dot{\chi}_j = b_j - \omega_j - p_j^c - \chi_j, \quad j \in G, \quad (17c)$$

$$M_j \dot{b}_j = -\chi_j + s_j - d_j^u - \sum_{k:j \rightarrow k} p_{jk} + \sum_{i:i \rightarrow j} p_{ij}, \quad j \in G, \quad (17d)$$

$$0 = -\chi_j + s_j - d_j^u - \sum_{k:j \rightarrow k} p_{jk} + \sum_{i:i \rightarrow j} p_{ij}, \quad j \in L, \quad (17e)$$

where  $\tau_{\chi,j}$  are positive time constants and  $b_j$  and  $\chi_j$  are auxiliary variables associated with the observer.

**Remark 13:** The control scheme in (17) replaces the power command dynamics (6). The additional auxiliary variables  $b_j$  and  $\chi_j$  obviate the need to know  $p_j^L$  without affecting the stability results described within the paper. This is accomplished by adding an observer that mimics the swing equation, described by (17c)–(17e). The dynamics in (17d)–(17e) ensure that the variable  $\chi_j$  is equal at steady state to the value  $\chi_j^* = s_j^* - d_j^{u,*} - \sum_{k:j \rightarrow k} p_{jk}^* + \sum_{i:i \rightarrow j} p_{ij}^* = p_j^L$  for  $j \in N$ , with the second part of the equality following from (1b)–(1c) at equilibrium. Furthermore, the dynamics in (17c) are important to ensure the convergence of system's solutions, as shown in Proposition 1.

The equilibria of the system (1)–(5), (17) are defined in a similar way to Definition 1 and it is assumed that at least one such equilibrium exists. Note that the existence of an equilibrium of (1)–(6) implies the existence of an equilibrium of (1)–(5), (17).

We now provide a result analogous to Lemma 1 in the case where the observer dynamics are included. As shown in Lemma 2, proven in the Appendix, any equilibrium where Assumption 2 holds guarantees that the steady state value of the frequency will be equal to the nominal one.

**Lemma 2:** Let Assumption 2 hold. Then, any equilibrium point  $(\eta^*, \psi^*, \omega^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*}, b^*, \chi^*)$  of the system (1)–(5), (17) satisfies  $\omega^* = \mathbf{0}_{|N|}$ .

<sup>20</sup>See also the use of observer dynamics in [40] as a means of counteracting agent dishonesty.



The following proposition, proven in the Appendix, shows that the set of equilibria for the system described by (1) – (5), (17) for which Assumptions 1–5 are satisfied is asymptotically attracting and that these equilibria are also solutions to the OSLC problem (7).

*Proposition 1:* Consider equilibria of (1) – (5), (17) with respect to which Assumptions 1–5 are all satisfied and suppose that the control dynamics in (4a) and (4b) are chosen such that (10) is satisfied. Then there exists an open region of the state space containing the equilibrium such that solutions of (1) – (5), (17) asymptotically converge to a set of equilibria that solve the OSLC problem (7) with  $\omega^* = \mathbf{0}_{|N|}$ .

*Remark 14:* Note that in some cases there could be uncertainty in the knowledge of the  $d^u$  dynamics. This does not affect the optimality of the equilibrium points since at equilibrium we have  $d^u = \mathbf{0}_{|N|}$ . Numerical simulations with realistic data have demonstrated that network stability is also robust to variations in the  $d^u$  model used in (17d)–(17e).

## VII. SIMULATIONS

In this section we verify our analytic results with simulations on two realistic systems, the 140-bus Northeast Power Coordinating Council (NPCC) system and a smaller 9-bus system from [41]. In both cases, the simulation results demonstrate that frequency converges to its nominal value. Furthermore, by plotting the corresponding marginal costs we validate the optimality in the power allocation predicted by our analysis. The latter is in contrast with current implementations where issues of optimality are typically not incorporated in secondary control and are addressed in tertiary control at a slower timescale. As will be shown in the subsequent simulations, the presence of controllable loads, often employed in a decentralized fashion, results in an improved frequency response (see also [12]). This demonstrates the significance of our analysis as it provides decentralized stability and optimality guarantees in such schemes.

### A. Simulation on the NPCC 140-bus system

In this section we use the Northeast Power Coordinating Council (NPCC) 140-bus interconnection system, simulated using the Power System Toolbox [38], in order to illustrate our results. This model is more detailed and realistic than our analytical one, including line resistances, a DC12 exciter model, a subtransient reactance generator model, and higher order turbine governor models<sup>21</sup>.

The test system consists of 93 load buses serving different types of loads including constant active and reactive loads and 47 generation buses. The overall system has a total real power of 28.55GW. For our simulation, we added three loads on units 2, 9, and 17, each having a step increase of magnitude 1 p.u. (base 100MVA) at  $t = 1$  second.

Controllable demand was considered within the simulations, with loads controlled every 10ms. The disutility function for controllable loads in each bus was  $C_{d_j}(d_j^c) = \frac{1}{2}\alpha_j(d_j^c -$

<sup>21</sup>The details of the simulation models can be found in the Power System Toolbox [38] manual and data file datanp48.

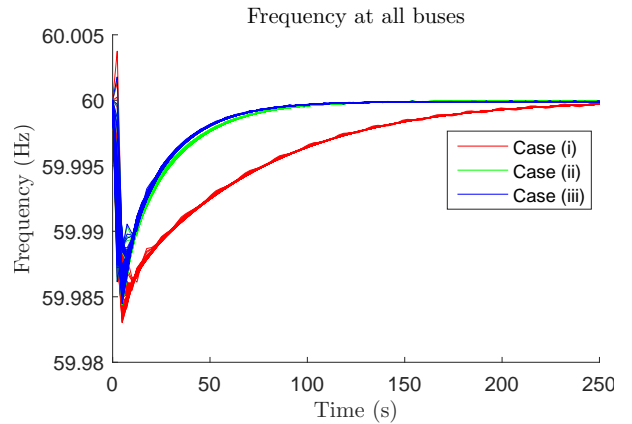


Fig. 2: Frequency at all buses of the NPCC network with: i) 10 generators, ii) 10 generators and 20 controllable loads, iii) 15 generators and 20 controllable loads, contributing to secondary frequency control.

$d_j^{c,\text{nom}})^2$ , where  $d_j^{c,\text{nom}}$  is a constant nominal value. The selected values for cost coefficients were  $\alpha_j = 1$  for load buses 1 – 5 and 11 – 15 and  $\alpha_j = 2$  for the rest. Similarly, the cost functions for generation were  $C_j(p_j^M) = \frac{1}{2}\kappa_j(p_j^M - p_j^{M,\text{nom}})^2$ , where  $p_j^{M,\text{nom}}$  is a constant nominal value<sup>22</sup> and  $\kappa_j$  were selected as the inverse of the generators droop coefficients, as suggested in (10). Note that the cost coefficients  $\alpha_j$  were selected such that the power allocated between total generation and controllable demand would be roughly equal, as suggested in [42]. Quadratic cost functions are frequently used in the literature, [34], [44], motivated by the fact that a convex function can be locally approximated by a quadratic one and also for convenience in the illustration.

Consider the static and first order dynamic schemes given by  $d_j^c = (C'_{d_j})^{-1}(\omega_j - p_j^c)$  and  $\dot{d}_j^c = -d_j^c + (C'_{d_j})^{-1}(\omega_j - p_j^c)$ ,  $j \in N$ , where  $p_j^c$  has dynamics as described in (6). We refer to the resulting dynamics as Static and Dynamic OSLC respectively since in both cases, steady state conditions that solve the OSLC problem were used. As discussed in Section VI, in the presence of arbitrarily small frequency damping, both schemes satisfy Assumption 5 and are thus included in our framework.

The system was tested on three different cases. In case (i) 10 generators were employed to perform secondary frequency control by having frequency and power command as inputs. In case (ii) controllable loads were included on 20 load buses in addition to the 10 generators. Controllable load dynamics in 10 buses were described by Static OSLC and in the rest by Dynamic OSLC. Finally, in case (iii), all controllable loads of case (ii) and 15 generators were used for secondary frequency control. Note that the 15 generators used for secondary frequency control had third, fourth and fifth order turbine governor dynamics. Furthermore, note that the stability and optimality properties of the system, demonstrated below, are retained when controllable demand is considered at different buses, although transient performance might be affected.

The frequency at all buses for the three tested cases is shown

<sup>22</sup>The nominal values  $d_j^{c,\text{nom}}$ ,  $p_j^{M,\text{nom}}$  for the loads and generation were used initially in the simulation, before the step change in load was applied.

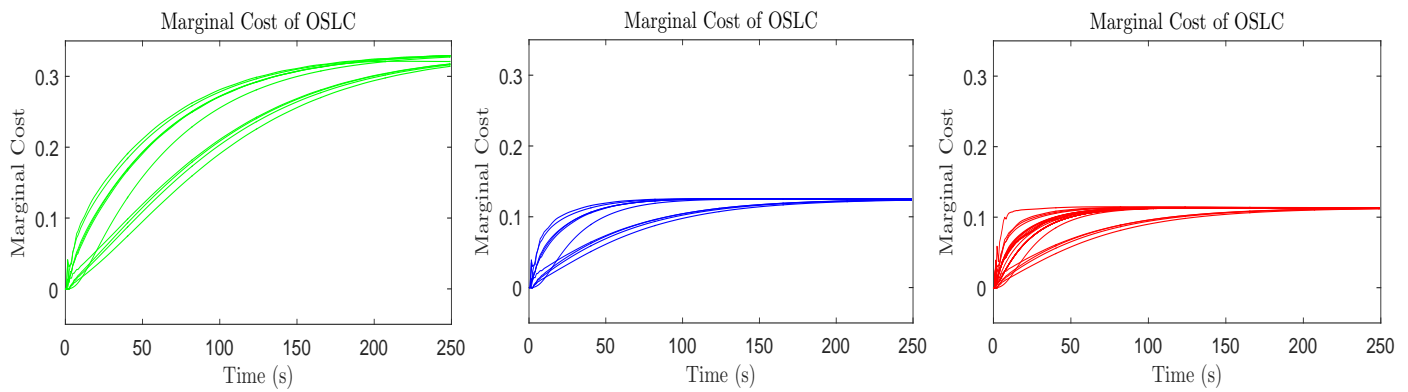


Fig. 3: Marginal costs for the three cases where secondary control is performed by: (i)(left) 10 generators, (ii) (center) 10 generators and 20 loads, (iii) (right) 15 generators and 20 loads.

in Fig. 2. From this figure, we observe that in all cases the frequency returns to its nominal value. However, the presence of controllable loads makes the frequency return much faster and with a smaller overshoot.

Furthermore, from Fig. 3, it is observed that the marginal costs at all controlled loads and generators that contribute to secondary frequency control, converge to the same value. This illustrates the optimality in the power allocation among generators and loads, since equality in the marginal cost is necessary to solve (7) when the power generated does not saturate to its maximum/minimum value.

To explore the case when controllable loads outputs saturate, we repeated case (iii) of the simulation setting upper and lower bounds on the controllable loads outputs at buses 3, 7, 11, 13 and 17. The simulation results demonstrated a similar frequency response as the one depicted on Fig. 2, with frequency converging to its nominal value. However, as a result of the saturation bounds, not all marginal costs converged to the same value, as demonstrated in Fig. 4. In particular, this figure shows convergence of the marginal costs of all non-saturated loads and generators to the same value, which is a property satisfied by an optimal allocation. As expected, the presence of bounds slightly increases the marginal cost of the non-saturated generators/loads, as can be seen by comparing Figures 3(c) and 4. Note also that the marginal costs do not need to be equal for loads that saturate, as follows analytically from the KKT conditions.

Additional simulations were also carried out to test the robustness of the scheme to communication delays and measurement noise. In particular communication delays up to  $0.5s$  were introduced in the power command signals without compromising stability. Also a 10% error in the uncontrollable demand  $p^L$  that appears in the control law had a very small effect in the steady state value of the system frequency (remains within 0.002 Hz of its nominal value).

### B. Simulation on a 9-bus system

To further validate our stability and optimality results we simulated a smaller, 9-bus network [41, p.70] using the Power System Toolbox [38]. The simulated model is more realistic

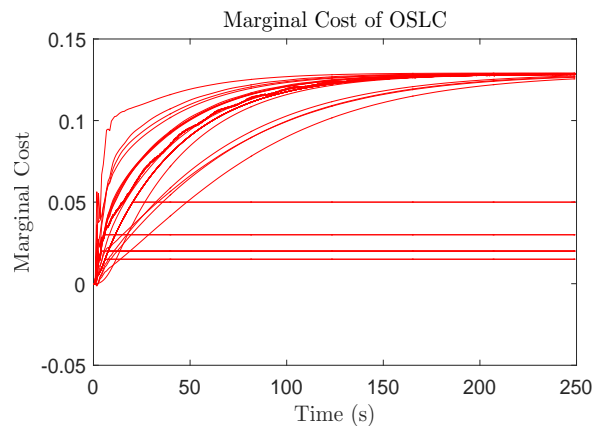


Fig. 4: Marginal costs for the 15 generators and 20 loads of case (iii) with bounds on 5 controllable loads.

than our analytic and includes line resistances, varying voltages, reactive power flows and uses transient and subtransient electro-mechanical machine models<sup>23</sup>. The overall system consists of three generation and six load buses and has total real power of 248MW.

Controllable demand was considered at all six load buses with loads controlled every 10ms. Moreover, we assumed a quadratic disutility function for loads described by  $C_{dj} = \frac{1}{2}\alpha_j(d_j^c - d_j^{c,nom})^2$  where  $d_j^{c,nom}$  is a constant nominal value. The cost coefficients were selected to be  $\alpha_j = 1$  for load buses 4 – 6 and  $\alpha_j = 2$  for the rest, arbitrarily choosing the buses with low and high cost coefficients. Furthermore, the Static and Dynamic OSLC schemes described in Section VII-A were used to describe controllable demand dynamics in buses 4 – 6 and 7 – 9 respectively.

The system was tested under a step change disturbance of magnitude 2.5MW at buses 5 and 9. The frequency response for this tested case is depicted in Fig. 5. This figure demonstrates that the addition of the power command control scheme does not destabilize the system and also manages to take frequency back to its nominal value. Moreover, Fig. 6

<sup>23</sup>The analytic models for the 9-bus system are provided in the Power System Manual [38] and the data in the file d3m9bm.m.

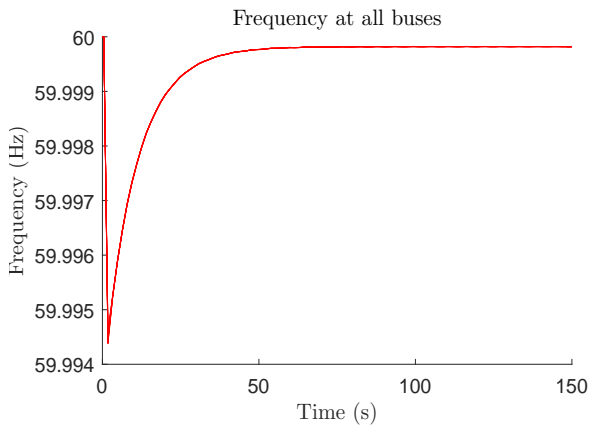


Fig. 5: Frequency at all buses of the 9-bus network.

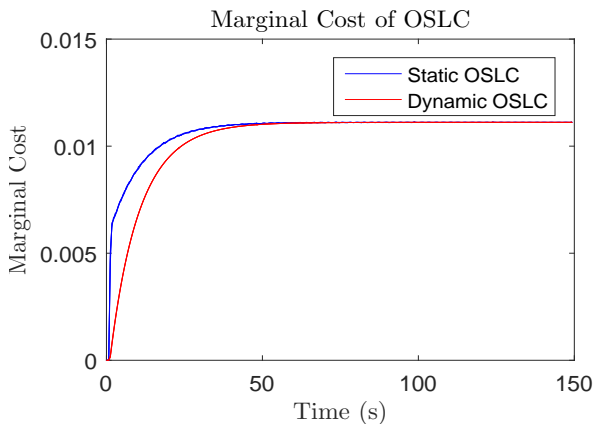


Fig. 6: Marginal costs of controllable loads with dynamics described by the Static and Dynamic OSLC schemes, within the 9-bus network.

depicts that marginal costs among controllable loads converge to the same value which demonstrates optimality in allocation.

*Remark 15:* It should be noted that the computation time for the NPCC network simulation is about 3.5 times longer than the corresponding one for the 9-bus network. This reflects the different sizes of the two networks, 140 buses compared to 9 buses, and the slower convergence rate in the former.

## VIII. CONCLUSIONS

We have considered the problem of designing distributed schemes for secondary frequency control in power networks such that stability and optimality of the power allocation can be guaranteed. In particular, we have proposed a systematic analysis framework that allows for higher order generation and demand control dynamics and captures various schemes that have been studied in the literature. We have shown that a dissipativity condition in conjunction with appropriate decentralized steady state conditions can guarantee stability of the overall power network, recover the frequency to its nominal value, and provide an optimal allocation of the power supply. We have also discussed that for linear systems the dissipativity condition can be easily verified by solving a corresponding LMI, i.e. a computationally efficient convex

problem. In addition, we have shown that the requirement to have knowledge of demand may be relaxed by incorporating an appropriate observer. Our results have been illustrated with simulations on the NPCC 140-bus system and a 9-bus system. An interesting problem for future research is to consider loads with switching behavior. The passivity approach taken in this paper seems promising, and a preliminary result on secondary frequency control with switching loads is reported in [47]. Another research direction is to consider different secondary control schemes, such as the distributed averaging proportional integral controllers [18], together with high order turbine governor and demand dynamics, and see how the stability and optimality results compare to the controllers considered in this work. Other interesting extensions include incorporating voltage dynamics, excitation control [29], [43], and more advanced communication structures, as well as investigating possible applications to cyber-physical security [40], [45].

## APPENDIX

In this appendix we prove our main results, Theorems 1 - 2, and also Lemmas 1-2, Theorem 3 and Proposition 1.

Throughout the proofs we will make use of the following equilibrium equations for the dynamics in (1)–(4),

$$0 = \omega_i^* - \omega_j^*, \quad (i, j) \in E, \quad (18a)$$

$$0 = -p_j^L + p_j^{M,*} - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j \rightarrow k} p_{jk}^* + \sum_{i:i \rightarrow j} p_{ij}^*, \quad j \in G, \quad (18b)$$

$$0 = -p_j^L - (d_j^{c,*} + d_j^{u,*}) - \sum_{k:j \rightarrow k} p_{jk}^* + \sum_{i:i \rightarrow j} p_{ij}^*, \quad j \in L, \quad (18c)$$

$$p_j^{M,*} = k_{p_j^M}(\zeta_j^*), \quad j \in G, \quad (18d)$$

$$d_j^{c,*} = k_{d_j^c}(\zeta_j^*), \quad \zeta_j^* = [-\omega_j^* p_j^{c,*}]^T, \quad j \in N. \quad (18e)$$

*Proof of Lemma 1:* In order to show that  $\omega^* = \mathbf{0}_{|N|}$ , we sum equations (6b) at equilibrium for all  $j \in N$ , resulting in  $\sum_{j \in N} s_j^* = \sum_{j \in N} p_j^L$ , which shows that  $\sum_{j \in N} d_j^{u,*} = 0$  (by summing (18b) and (18c) over all  $j \in G$  and  $j \in L$  respectively). Then, Assumption 2 implies that this equality holds only if  $\omega^* = \mathbf{0}_{|N|}$ . ■

*Proof of Theorem 1:* Due to Assumption 4,  $C'_j$  and  $C'_{d_j}$  are strictly increasing and hence invertible. Therefore all variables in (10) are well-defined. Also, Assumption 4 guarantees that the OSLC optimization problem (7) is convex and has a continuously differentiable cost function. Thus, a point  $(\bar{p}^M, \bar{d}^c)$  is a global minimum for (7) if and only if it satisfies the KKT conditions [46]

$$C'_j(\bar{p}_j^M) = \nu - \lambda_j^+ + \lambda_j^-, \quad j \in G, \quad (19a)$$

$$C'_{d_j}(\bar{d}_j^c) = -\nu - \mu_j^+ + \mu_j^-, \quad j \in N, \quad (19b)$$

$$\sum_{j \in G} \bar{p}_j^M = \sum_{j \in N} (\bar{d}_j^c + p_j^L), \quad (19c)$$

$$p_j^{M,min} \leq \bar{p}_j^M \leq p_j^{M,max}, \quad j \in G, \quad (19d)$$

$$d_j^{c,min} \leq \bar{d}_j^c \leq d_j^{c,max}, \quad j \in N, \quad (19e)$$

$$\lambda_j^+(\bar{p}_j^M - p_j^{M,max}) = 0, \quad \lambda_j^-(\bar{p}_j^M - p_j^{M,min}) = 0, \quad j \in G, \quad (19f)$$

$$\mu^+(\bar{d}_j^c - d_j^{c,max}) = 0, \quad \mu^-(\bar{d}_j^c - d_j^{c,min}) = 0, \quad j \in N, \quad (19g)$$

for some constants  $\nu \in \mathbb{R}$  and  $\lambda_j^+, \lambda_j^-, \mu_j^+, \mu_j^- \geq 0$ . It will be shown below that these conditions are satisfied by the equilibrium values  $(\bar{p}^M, \bar{d}^c) = (p^{M,*}, d^{c,*})$  defined by equations (18d), (18e) and (10).

Since  $C'_j$  and  $C'_{dj}$  are strictly increasing, we can uniquely define  $\beta_j^{M,max} := C'_j(p_j^{M,max})$ ,  $\beta_j^{M,min} := C'_j(p_j^{M,min})$ ,  $\beta_j^{c,max} := -C'_{dj}(d_j^{c,max})$ , and  $\beta_j^{c,min} := -C'_{dj}(d_j^{c,min})$ . We let  $\beta_0^* = p_j^{c,*}$  and note that  $p_j^c$  are equal  $\forall j$  at equilibrium, therefore  $\beta_0^*$  is the same at each bus  $j$ . We now define in terms of these quantities the nonnegative constants

$$\begin{aligned} \lambda_j^+ &:= (\beta_0^* - \beta_j^{M,max}) \mathbf{1}_{(\beta_0^* \geq \beta_j^{M,max})}, \\ \lambda_j^- &:= (\beta_j^{M,min} - \beta_0^*) \mathbf{1}_{(\beta_0^* \leq \beta_j^{M,min})}, \\ \mu_j^+ &:= (\beta_j^{c,max} - \beta_0^*) \mathbf{1}_{(\beta_0^* \leq \beta_j^{c,max})}, \\ \mu_j^- &:= (\beta_0^* - \beta_j^{c,min}) \mathbf{1}_{(\beta_0^* \geq \beta_j^{c,min})}. \end{aligned}$$

Then, since  $(C'_j)^{-1}(\beta_0^*) \geq p_j^{M,max} \Leftrightarrow \beta_0^* \geq \beta_j^{M,max}$ ,  $(C'_j)^{-1}(\beta_0^*) \leq p_j^{M,min} \Leftrightarrow \beta_0^* \leq \beta_j^{M,min}$ ,  $(C'_{dj})^{-1}(-\beta_0^*) \geq d_j^{c,max} \Leftrightarrow \beta_0^* \leq \beta_j^{c,max}$ , and  $(C'_{dj})^{-1}(-\beta_0^*) \leq d_j^{c,min} \Leftrightarrow \beta_0^* \geq \beta_j^{c,min}$ , it follows by (18d), (18e), and (10) that the complementary slackness conditions (19f) and (19g) are satisfied.

Now define  $\nu = \beta_0^*$ . Then  $(C'_j)^{-1}(\nu - \lambda_j^+ + \lambda_j^-) = (C'_j)^{-1}([\beta_0^*]_{\beta_j^{M,max}}^{\beta_j^{M,min}}) = [(C'_j)^{-1}(\beta_0^*)]_{p_j^{M,min}}^{p_j^{M,max}} = p_j^{M,*}$ , by the above definitions and equations (18d) and (10). Thus, the optimality condition (19a) holds. Analogously,  $(C'_{dj})^{-1}(-\nu - \mu_j^+ + \mu_j^-) = (C'_{dj})^{-1}([\beta_0^*]_{-\beta_j^{c,max}}^{-\beta_j^{c,min}}) = [(C'_{dj})^{-1}(-\beta_0^*)]_{d_j^{c,min}}^{d_j^{c,max}} = d_j^{c,*}$ , by (18e) and (10), satisfying (19b).

Summing equations (18b) and (18c) over all  $j \in G$  and  $j \in L$  respectively and using the fact that  $\sum_{j \in N} d_j^{u,*} = 0$  as shown in the proof of Lemma 1 shows that (19c) holds. Finally, the saturation constraints in (10) verify (19d) and (19e).

Hence, the values  $(\bar{p}^M, \bar{d}^c) = (p^{M,*}, d^{c,*})$  satisfy the KKT conditions (19). Therefore, the equilibrium values  $p^{M,*}$  and  $d^{c,*}$  define a global minimum for (7).  $\blacksquare$

*Proof of Theorem 2:* We will use the dynamics in (1)–(6) and the conditions of Assumption 5 to define a Lyapunov function for the system (1)–(6).

Firstly, let  $V_F(\omega^G) = \frac{1}{2} \sum_{j \in G} M_j(\omega_j - \omega_j^*)^2$ . The time-derivative of  $V_F$  along the trajectories of (1)–(4) is given by

$$\dot{V}_F = \sum_{j \in N} (\omega_j - \omega_j^*) \left( -p_j^L + s_j - d_j^u - \sum_{k:j \rightarrow k} p_{jk} + \sum_{i:i \rightarrow j} p_{ij} \right),$$

by substituting (1b) for  $\dot{\omega}_j$  for  $j \in G$  and adding extra terms for  $j \in L$ , which are equal to zero by (1c). Subtracting the product of  $(\omega_j - \omega_j^*)$  with each term in (18b) and (18c), this becomes

$$\begin{aligned} \dot{V}_F &= \sum_{j \in N} \left( (\omega_j - \omega_j^*)(s_j - s_j^*) + (\omega_j - \omega_j^*)(-d_j^u - (-d_j^{u,*})) \right) \\ &\quad + \sum_{(i,j) \in E} (p_{ij} - p_{ij}^*)(\omega_j - \omega_i), \end{aligned} \quad (20)$$

using the equilibrium condition (18a) for the final term.

Furthermore, let  $V_C(p^c) = \frac{1}{2} \sum_{j \in N} \gamma_j (p_j^c - p_j^{c,*})^2$ . Using (6b) the time derivative of  $V_C$  can be written as

$$\begin{aligned} \dot{V}_C &= \sum_{j \in N} (p_j^c - p_j^{c,*}) \left( (-s_j + s_j^*) \right. \\ &\quad \left. - \sum_{k:j \rightarrow k} (\psi_{jk} - \psi_{jk}^*) + \sum_{i:i \rightarrow j} (\psi_{ij} - \psi_{ij}^*) \right). \end{aligned} \quad (21)$$

Additionally, define  $V_P(\eta) = \sum_{(i,j) \in E} B_{ij} \int_{\eta_{ij}^*}^{\eta_{ij}} (\sin \theta - \sin \eta_{ij}^*) d\theta$ . Using (1a) and (1d), the time-derivative is given by

$$\begin{aligned} \dot{V}_P &= \sum_{(i,j) \in E} B_{ij} (\sin \eta_{ij} - \sin \eta_{ij}^*) (\omega_i - \omega_j) \\ &= \sum_{(i,j) \in E} (p_{ij} - p_{ij}^*) (\omega_i - \omega_j). \end{aligned} \quad (22)$$

Finally, consider  $V_\psi(\psi) = \frac{1}{2} \sum_{(i,j) \in \bar{E}} \gamma_{ij} (\psi_{ij} - \psi_{ij}^*)^2$  with time derivative given by (6a) as

$$\dot{V}_\psi = \sum_{(i,j) \in \bar{E}} (\psi_{ij} - \psi_{ij}^*) ((p_i^c - p_i^{c,*}) - (p_j^c - p_j^{c,*})). \quad (23)$$

Furthermore, from the dissipativity condition in Assumption 5 the following holds: There exist open neighborhoods  $U_j$  of  $\omega_j^*$  and  $U_j^c$  of  $p_j^{c,*}$  for each  $j \in N$ , open neighborhoods  $X_j^G$  of  $(x_j^{M,*}, x_j^{c,*}, x_j^{u,*})$  and  $X_j^L$  of  $(x_j^{c,*}, x_j^{u,*})$  for each  $j \in G$  and  $j \in L$  respectively, and continuously differentiable, positive semidefinite functions  $V_j^D(x_j^M, x_j^c, x_j^u)$ ,  $j \in G$  and  $V_j^D(x_j^c, x_j^u)$ ,  $j \in L$ , satisfying (8) with supply rate given by (9), i.e.,

$$\begin{aligned} \dot{V}_j^D &\leq [(s_j - s_j^*) \quad (-d_j^u - (-d_j^{u,*}))] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (\zeta_j - \zeta_j^*) \\ &\quad - \phi_j(\zeta_j - \zeta_j^*), \quad j \in N, \end{aligned} \quad (24)$$

for all  $\omega_j \in U_j$ ,  $p_j^c$  in  $U_j^c$  for  $j \in N$  and all  $(x_j^M, x_j^c, x_j^u) \in X_j^G$  and  $(x_j^c, x_j^u) \in X_j^L$  for  $j \in G$  and  $j \in L$  respectively.

Based on the above, we define the function

$$V(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) = V_F + V_P + \sum_{j \in N} V_j^D + V_C + V_\psi \quad (25)$$

which we aim to use in Lasalle's theorem. Using (20) - (23), the time derivative of  $V$  is given by

$$\begin{aligned} \dot{V} &= \sum_{j \in N} [(\omega_j - \omega_j^*)(s_j - s_j^*) + \dot{V}_j^D + (p_j^c - p_j^{c,*})(-s_j + s_j^*) \\ &\quad + (\omega_j - \omega_j^*)(-d_j^u - (-d_j^{u,*}))]. \end{aligned} \quad (26)$$

Using (24) it therefore holds that

$$\dot{V} \leq \sum_{j \in N} \left( -\phi_j(\zeta_j - \zeta_j^*) \right) \leq 0 \quad (27)$$

whenever  $\omega_j \in U_j$ ,  $p_j^c \in U_j^c$  for  $j \in N$ ,  $(x_j^M, x_j^c, x_j^u) \in X_j^G$  for  $j \in G$ , and  $(x_j^c, x_j^u) \in X_j^L$  for  $j \in L$ .

Clearly  $V_F$  has a strict global minimum at  $\omega^{G,*}$  and  $V_j^D$  has strict local minima at  $(x_j^{M,*}, x_j^{c,*}, x_j^{u,*})$  and  $(x_j^{c,*}, x_j^{u,*})$  for  $j \in G$  and  $j \in L$  respectively by Assumption 5 and Definition

2. Furthermore,  $V_C$  and  $V_\psi$  have strict global minima at  $p^{c,*}$  and  $\psi^*$  respectively. Furthermore, Assumption 1 guarantees the existence of some neighborhood of each  $\eta_{ij}^*$  in which  $V_P$  is increasing. Since the integrand is zero at the lower limit of the integration,  $\eta_{ij}^*$  this immediately implies that  $V_P$  has a strict local minimum at  $\eta^*$ . Thus,  $V$  has a strict local minimum at the point  $Q^* := (\eta^*, \psi^*, \omega^{G,*}, x^{M,*}, x^{c,*}, x^{u,*}, p^{c,*})$ . From Assumption 3, we know that, provided  $(\eta, \omega^G, x^c, x^u, p^c) \in T$ ,  $\omega^L$  can be uniquely determined from these quantities. Therefore, the states of the differential equation system (1)–(6) with  $(\eta, \omega^G, x^c, x^u, p^c)$  within the region  $T$  can be expressed as  $(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c)$ . We now choose a neighborhood in the coordinates  $(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c)$  about  $Q^*$  on which the following hold:

- 1)  $Q^*$  is a strict minimum of  $V$ ,
- 2)  $(\eta, \omega^G, x^c, x^u, p^c) \in T$ ,
- 3)  $\omega_j \in U_j$ ,  $p_j^c \in U_j^c$  for  $j \in N$ , and  $(x_j^M, x_j^c, x_j^u) \in X_j^G$ ,  $(x_j^c, x_j^u) \in X_j^L$  for  $j \in G$ ,  $j \in L$  respectively<sup>24</sup>,
- 4)  $x_j^M$ ,  $x_j^c$ , and  $x_j^u$  all lie within their respective neighborhoods  $\Omega(x_j^{M,*})$ ,  $\Omega(x_j^{c,*})$  and  $\Omega(x_j^{u,*})$  as defined in Section III-A.

Recalling now (27), it is easy to see that within this neighborhood,  $V$  is a nonincreasing function of all the system states and has a strict local minimum at  $Q^*$ . Consequently, the connected component of the level set  $\{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) : V \leq \epsilon\}$  containing  $Q^*$  is guaranteed to be both compact and positively invariant with respect to the system (1)–(6) for sufficiently small  $\epsilon > 0$ . Therefore, there exists a compact positively invariant set  $\Xi$  for (1)–(6) containing  $Q^*$ .

Lasalle's Invariance Principle can now be applied with the function  $V$  on the compact positively invariant set  $\Xi$ . This guarantees that all solutions of (1)–(6) with initial conditions  $(\eta(0), \psi(0), \omega^G(0), x^M(0), x^c(0), x^u(0), p^c(0)) \in \Xi$  converge to the largest invariant set within  $\Xi \cap \{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c) : \dot{V} = 0\}$ . We now consider this invariant set. If  $\dot{V} = 0$  holds at a point within  $\Xi$ , then (27) holds with equality, hence we must have  $\omega = \omega^*$  and  $p_j^c = p_j^{c,*}$  at all buses  $j$  where Assumption 5(a) holds. The fact that  $\omega$  is constant guarantees from (1a), (1d) that  $\eta$  and  $p$  are also constant. This is sufficient to deduce from (1b)–(1c) that  $s$  is also constant. If instead Assumption 5(b) holds at a bus  $j$  we have that  $\omega = \omega^*$  when  $\dot{V} = 0$ . Furthermore, we have the additional property that if  $\omega_j$  and  $s_j$  are constant then  $p_j^c$  cannot be a sinusoid. This latter property guarantees that  $p_j^c$  is also constant by noting that the dynamics for the power command (6) with constant  $s_j$ , allow  $p_j^c$  to be either a constant or a sinusoid within a compact invariant set. Hence, we have  $\omega = \omega^*$  and  $p^c = p^{c,*}$  in the invariant set considered.

Furthermore, note that  $\omega = \omega^*$ ,  $p^c = p^{c,*}$  within the invariant set implies by the definitions in Section II that  $(x^M, x^c, x^u)$  converge to the point  $(x^{M,*}, x^{c,*}, x^{u,*})$ , at which  $V_j^D$  take strict local minima from Assumption 5. Thus, from (24) and (26) it follows that the values of  $V_j^D$  must

decrease along all nontrivial trajectories within the invariant set, contradicting  $\dot{V}_j^D = 0$ . The fact that  $(p^c, s) = (p^{c,*}, s^*)$  is sufficient to show that  $\psi$  equals some constant  $\psi^*$ . Using the same argument, it can be shown that within the invariant set, the fact that  $\zeta = \zeta^*$  implies that  $(x^M, x^c, x^u, p^M, d^c, d^u)$  converges to  $(x^{M,*}, x^{c,*}, x^{u,*}, p^{M,*}, d^{c,*}, d^{u,*})$ . Therefore, we conclude by Lasalle's Invariance Principle that all solutions of (1)–(6) with initial conditions  $(\eta(0), \psi(0), \omega^G(0), x^M(0), x^c(0), x^u(0), p^c(0)) \in \Xi$  converge to the set of equilibrium points as defined in Definition 1. Finally, choosing for  $S$  any open neighborhood of  $Q^*$  within  $\Xi$  completes the proof for convergence. From Lemma 1 it can then be deduced that  $\omega^* = \mathbf{0}_{|N|}$ . Furthermore, noting that all conditions of Theorem 1 hold shows the convergence to an optimal solution of the OSLC problem (7). ■

*Remark 16:* It should be noted that for given  $p^{c,*}$  and  $\omega^*$  all  $(\eta^*, x^{M,*}, x^{c,*}, x^{u,*})$  are unique. The uniqueness of  $\eta^*$  can be seen by noting that  $\eta_{ij} = \theta_i - \theta_j$ ,  $(i, j) \in E$ , which requires  $\eta$  to lie in a space where a corresponding vector  $\theta$  exists. Furthermore, the value of  $p^{c,*}$  becomes unique when (10) holds. This follows from summing (18b)–(18c) over all buses and noting that the strict convexity of the cost functions and the monotonicity of  $f$  in (10) makes the static input output maps from  $p^{c,*}$  to  $s^*$  monotonically increasing. The values of  $\psi^*$  are non-unique for general network topologies.

*Proof of Theorem 3:* The proof follows from the fact that the function  $V_j^B$  defined as

$$V_j^B = \frac{1}{2}M_j(\omega_j - \omega_j^*)^2 + \frac{1}{2}\gamma_j(p_j^c - p_j^{c,*})^2 + V_j^D, \quad (28)$$

where  $V_j^D$  is as in (24) with  $\phi_j = 0$ , is a storage function for the system  $B_j$ . In particular, using arguments similar to those in the proof of Theorem 2, it can be shown that

$$\begin{aligned} \dot{V}_j^B \leq & (p_j^c - p_j^{c,*}) \left( \sum_{i:i \rightarrow j} (\psi_{ij} - \psi_{ij}^*) - \sum_{k:j \rightarrow k} (\psi_{jk} - \psi_{jk}^*) \right) \\ & + (-\omega_j - (-\omega_j^*)) \left( \sum_{k:j \rightarrow k} (p_{jk} - p_{jk}^*) - \sum_{i:i \rightarrow j} (p_{ij} - p_{ij}^*) \right) \end{aligned} \quad (29)$$

and therefore that system  $B_j$  is passive. ■

*Proof of Lemma 2:* Using (17d) at equilibrium, it can be deduced that  $\chi_j^* = s_j^* - d_j^{u,*} - \sum_{k:j \rightarrow k} p_{jk}^* + \sum_{i:i \rightarrow j} p_{ij}^*$ . Hence, it follows by summing (17b) at equilibrium over all buses that  $\sum_{j \in N} s_j^* = \sum_{j \in N} \chi_j^* = \sum_{j \in N} s_j^* - d_j^{u,*}$ , which results to  $\sum_{j \in N} d_j^{u,*} = 0$ . Hence, from Assumption 2, it follows that  $\omega^* = \mathbf{0}_{|N|}$ . ■

*Proof of Proposition 1:* We shall make use of the Lyapunov function in (25) to construct a new Lyapunov function for the system (1) – (5), (17).

First, consider the function

$$V_b(b, \chi, \omega) = \frac{1}{2} \sum_{j \in G} (M_j((b_j - b_j^*) - (\omega_j - \omega_j^*))^2 + \tau_{\chi,j}(\chi_j - \chi_j^*)^2),$$

and note that its time-derivative along the trajectories of (17)

<sup>24</sup>This is possible because  $\omega_j \in U_j$  for all  $j \in L$  corresponds, by Assumption 3 and the continuity of the equations in (1)–(6), to requiring the states  $(\eta, \omega^G, x^M, x^c, x^u, p^c)$  to lie in some open neighborhood about  $Q^*$ .

is given by

$$\dot{V}_b = \sum_{j \in N} \left( -(\chi_j - \chi_j^*)[(p_j^c - p_j^{c,*}) + (\chi_j - \chi_j^*)] \right), \quad (30)$$

noting that for  $j \in L$  it holds that  $\chi = \chi^*$ , and hence the added terms in (30) are equal to zero.

Furthermore, the time-derivative of  $V_C(p^c) = \frac{1}{2} \sum_{j \in N} \gamma_j (p_j^c - p_j^{c,*})^2$  under (17b) is given by

$$\begin{aligned} \dot{V}_C = & \sum_{j \in N} (p_j^c - p_j^{c,*}) \left( (-s_j + s_j^*) + (\chi_j - \chi_j^*) \right. \\ & \left. - \sum_{k:j \rightarrow k} (\psi_{jk} - \psi_{jk}^*) + \sum_{i:i \rightarrow j} (\psi_{ij} - \psi_{ij}^*) \right). \end{aligned} \quad (31)$$

Now consider the function  $V$  in (25) and note that its derivative is as in (26) with an extra term given by  $\sum_{j \in N} (p_j^c - p_j^{c,*}) (\chi_j - \chi_j^*)$ . Then consider the function

$$V_O(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) = V + V_b \quad (32)$$

which can be shown to have a time derivative given by

$$\dot{V}_O \leq \sum_{j \in N} \left( -\phi_j (\zeta_j - \zeta_j^*) - (\chi_j - \chi_j^*)^2 \right) \leq 0, \quad (33)$$

by similar arguments as in the proof of Theorem 2.

Now, in analogy to the proof of Theorem 2, it can be shown that an invariant compact set  $\Xi_O$  exists such that  $\{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) : V_O \leq \epsilon\}$ . Then, Lasalle's theorem can be invoked to show that all solutions of (1) – (5), (17) with initial conditions within  $\Xi_O$  will converge to the largest invariant set within  $\Xi_O \cap \{(\eta, \psi, \omega^G, x^M, x^c, x^u, p^c, b, \chi) : \dot{V} = 0\}$ . Within this invariant set, it holds that  $(\omega, \chi) = (\omega^*, \chi^*)$ . Applying the same arguments as in the proof of Theorem 2 shows that  $(\eta, \psi, x^M, x^c, x^u, p^M, d^c, d^u, p^c)$  converges to  $(\eta^*, \psi^*, x^{M,*}, x^{c,*}, x^{u,*}, p^{M,*}, d^{c,*}, d^{u,*}, p^{c,*})$  which implies the convergence of  $b$  to  $b^*$  from the dynamics in (17d). The optimality result follows directly from the proof of Theorem 1 since none of its arguments are affected from the dynamics in (17). ■

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