Automorphism Groups of Quadratic Modules and Manifolds

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the acknowledgements and specified in the text. It is not substantially the same as any that I have submitted, or is being concurrently submitted, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the acknowledgements and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the acknowledgements and specified in the text.

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Summary

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In this thesis we prove homological stability for both general linear groups of modules over a ring with finite stable rank and unitary groups of quadratic modules over a ring with finite unitary stable rank. In particular, we do not assume the modules and quadratic modules to be well-behaved in any sense: for example, the quadratic form may be singular. This extends results by van der Kallen and Mirzaii–van der Kallen respectively. Combining these results with the machinery introduced by Galatius–Randal-Williams to prove homological stability for moduli spaces of simply-connected manifolds of dimension $2n \geq 6$, we get an extension of their result to the case of virtually polycyclic fundamental groups. We also prove the corresponding result for manifolds equipped with tangential structures.

A result on the stable homology groups of moduli spaces of manifolds by Galatius–Randal-Williams enables us to make new computations using our homological stability results. In particular, we compute the abelianisation of the mapping class groups of certain 6-dimensional manifolds. The first computation considers a manifold built from $\mathbb{R}P^6$ which involves a partial computation of the Adams spectral sequence of the spectrum $MT\mathrm{Pin}^-(6)$. For the second computation we consider Spin 6-manifolds with $\pi_1 \cong \mathbb{Z}/2^k\mathbb{Z}$ and $\pi_2 = 0$, where the main new ingredient is an analysis of the Atiyah–Hirzebruch spectral sequence for $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$. Finally, we consider the similar manifolds with more general fundamental groups G, where $K_1(\mathbb{Q}[G^{\mathrm{ab}}])$ plays a role.

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Contents

Introduction

We say that a sequence $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \cdots$ of groups satisfies homological stability if the induced maps $(f_n)_*\colon H_k(G_n) \longrightarrow H_k(G_{n+1})$ are isomorphisms for $k < An+B$ for some constants A and B. In most cases where homological stability is known it is extremely hard to compute any particular $H_k(G_n)$. However, there are several techniques to compute the stable homology groups $H_k(G_\infty)$ and homological stability can therefore be used to give many potentially new homology groups.

There are various groups which are known to satisfy homological stability. These include symmetric groups Σ_n [**40**], braid groups B_n [**29**], and general linear groups $\mathrm{GL}_n(R)$ for certain rings R [**46**]. More geometric examples appear as automorphism groups of certain topological spaces, e.g. diffeomorphism groups of compact, simply-connected manifolds of dimension $2n \geq 6$ containing a copy of $\#_g(\mathbb{S}^n \times \mathbb{S}^n) \setminus \mathrm{int}(\mathbb{D}^{2n})$ for a sufficiently big number g [**24**].

The general strategy for proving theorems in homological stability for $G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots$ is to find complexes $(X_n)_{n\geq 0}$ with a G_n-action on X_n such that

(1) the complex X_n is $f(n)$-connected for a function f such that $f(n) \to \infty$ for $n \to \infty$,
(2) the group G_n acts transitively on the k-simplices of $X_\bullet$,
(3) the stabilizer $\mathrm{Stab}_{G_n}(\sigma^k)$ of a k-simplex $\sigma^k \in X_k$ is isomorphic to G_{n-k-1}.

The hardest of these properties to show is the first, i.e. showing that the complex is highly connected. Once these properties are shown, deducing homological stability for the sequence of groups G_n is a fairly standard argument using spectral sequences and is formalised in Randal-Williams–Wahl [**43**].

General Linear Groups. In [**46**], van der Kallen proves homological stability for the group $\mathrm{GL}_n(R)$ of R-module automorphisms of R^n. For the special case where R is a PID, Charney [**15**] had earlier shown homological stability. In Chapters 1 and 4 of this dissertation we consider the analogous homological stability problem for groups of automorphisms of general R-modules M; we write $\mathrm{GL}(M)$ for these groups. In order to phrase our stability range we define the rank $\mathrm{rk}(M)$ of an R-module M to be the biggest number n so that R^n is a direct summand of M. The stability range then says that the rank of M has to be big compared to the so-called stable rank $\mathrm{sr}(R)$ of R. In particular, the stable rank of R needs to be finite which holds for example for Dedekind domains and more generally algebras that are finite as a module over a commutative Noetherian ring of finite Krull dimension.

Theorem A. *The map*

$$H_k(\mathrm{GL}(M);\mathbb{Z}) \to H_k(\mathrm{GL}(M \oplus R);\mathbb{Z}),$$

induced by the inclusion $\mathrm{GL}(M) \hookrightarrow \mathrm{GL}(M \oplus R)$*, is an epimorphism for* $k \leq \frac{\mathrm{rk}(M)-\mathrm{sr}(R)}{2}$ *and an isomorphism for* $k \leq \frac{\mathrm{rk}(M)-\mathrm{sr}(R)-1}{2}$.

For the commutator subgroup $\mathrm{GL}(M)'$ *the map*

$$H_k(\mathrm{GL}(M)';\mathbb{Z}) \to H_k(\mathrm{GL}(M \oplus R)';\mathbb{Z})$$

is an epimorphism for $k \leq \frac{\mathrm{rk}(M)-\mathrm{sr}(R)-1}{3}$ *and an isomorphism for* $k \leq \frac{\mathrm{rk}(M)-\mathrm{sr}(R)-3}{3}$.

We emphasise that M is allowed to be any module over R. For example over the integers, M could be $\mathbb{Z}/100\mathbb{Z} \oplus \mathbb{Z}^{100}$. We also get statements for polynomial and abelian coefficients. The full statement of our theorem is given in Theorem 4.2.

This part of the dissertation can be seen as a warm up for the heart of the algebraic part, which is homological stability for the automorphism groups of quadratic modules.

Unitary Groups. A quadratic module is a tuple (M, λ, μ) consisting of an R-module M, a sesquilinear form $\lambda\colon M \times M \to R$, and a function μ on M into a quotient of R, where λ measures how far μ is from being linear. The precise definition is given in Chapter 2. The basic example of a quadratic module is the *hyperbolic module* H, which is given by

$$\left(R^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}; \mu \text{ determined by } \mu(e) = \mu(f) = 0 \right).$$

For a quadratic module M we write $U(M)$ for its unitary group, i.e. the group of all automorphisms that fix the quadratic structure on M. Mirzaii–van der Kallen **[39]** have shown homological stability for the unitary groups $U(H^n)$ and our Theorem B below extends this to general quadratic modules.

We write $g(M)$ for the Witt index of M as a quadratic module, which is defined to be the maximal number n so that H^n is a direct summand of M. In our stability range we use the notion of unitary stable rank, $\mathrm{usr}(R)$, of R which is at least as big as the stable rank and also requires a certain transitivity condition on unimodular vectors of fixed length. Analogously to Theorem A the Witt index of M has to be big in relation to the unitary stable rank of R. In particular, $\mathrm{usr}(R)$ needs to be finite which is the case for both examples of rings with finite stable rank given above.

Theorem B. *The map*

$$H_k(U(M);\mathbb{Z}) \to H_k(U(M \oplus H);\mathbb{Z})$$

is an epimorphism for $k \leq \frac{g(M)-\mathrm{usr}(R)-1}{2}$ *and an isomorphism for* $k \leq \frac{g(M)-\mathrm{usr}(R)-2}{2}$.

For the commutator subgroup $U(M)'$ *the map*

$$H_k(U(M)';\mathbb{Z}) \to H_k(U(M \oplus H)';\mathbb{Z})$$

is an epimorphism for $k \leq \frac{g(M)-\mathrm{usr}(R)-1}{3}$ *and an isomorphism for* $k \leq \frac{g(M)-\mathrm{usr}(R)-3}{3}$.

We again emphasise that M can be an arbitrary quadratic module – in particular, it can be singular. As in the case for general linear groups, we get an analogous statement for abelian and polynomial coefficients. The full statement is given in Theorem 5.2.

To show homological stability for both the automorphism groups of modules and quadratic modules we use the machinery developed in Randal-Williams–Wahl [**43**]. The actual homological stability results are straightforward applications of that paper assuming that a certain semisimplicial set is highly connected. Showing that this assumption is indeed satisfied is the main goal of Chapters 4 and 5.

Moduli Spaces of Manifolds. Our theorem in the unitary case can also be used to extend the homological stability result for moduli spaces of simply-connected manifolds of dimension $2n \geq 6$ by Galatius–Randal-Williams [**24**] to certain non-simply-connected manifolds.

For a compact connected smooth $2n$-dimensional manifold W we write $\mathrm{Diff}_\partial(W)$ for the topological group of all diffeomorphisms of W that restrict to the identity near the boundary, and call its classifying space $B\mathrm{Diff}_\partial(W)$ the *moduli space of manifolds of type* W. As in the algebraic settings described previously there is a notion of rank: Define the *genus* of W as

$$g(W) := \sup\{g \in \mathbb{N} \mid \text{there are } g \text{ disjoint embeddings of } \mathbb{S}^n \times \mathbb{S}^n \setminus \mathrm{int}(\mathbb{D}^{2n}) \text{ into } W\}.$$

Let S denote the manifold $([0,1] \times \partial W)\#(\mathbb{S}^n \times \mathbb{S}^n)$. We get an inclusion

$$\mathrm{Diff}_\partial(W) \hookrightarrow \mathrm{Diff}_\partial(W \cup_{\partial W} S) \tag{0.1}$$

by extending diffeomorphisms by the identity on S. This gluing map then has an induced map on classifying spaces which we denote by s. Galatius–Randal-Williams have shown that for simply-connected manifolds of dimension $2n \geq 6$ the induced map

$$s_*\colon H_k(B\mathrm{Diff}_\partial(W)) \longrightarrow H_k(B\mathrm{Diff}_\partial(W \cup_{\partial W} S))$$

is an epimorphism for $k \leq \frac{g(W)-1}{2}$ and an isomorphism for $k \leq \frac{g(W)-3}{2}$. The following extends this result to certain non-simply-connected manifolds.

THEOREM C. *Let W be a compact connected manifold of dimension $2n \geq 6$. Then the above map s_* is an epimorphism for $k \leq \frac{g(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k \leq \frac{g(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$.*

For a virtually polycyclic fundamental group, e.g. a finitely generated abelian group, the unitary stable rank of its group ring is known to be finite by Crowley–Sixt [**17**].

Tangential Structures and Stable Homology. A *tangential structure* is a map $\theta\colon B \to BO(2n)$, where B is a path-connected space. Let $\gamma_{2n} \to BO(2n)$ denote the universal vector bundle. A *θ-structure* on a $2n$-dimensional manifold W is a bundle map (fibrewise linear isomorphism) $\hat{\ell}_W\colon TW \to \theta^*\gamma_{2n}$, with underlying map $\ell_W\colon W \to B$.

Using this notion we can get a version of Theorem C for $2n$-dimensional manifolds equipped with a θ-structure extending a fixed θ-structure on the boundary, see Theorem 7.3.

Here, we write $MT\theta$ for the Thom spectrum of the virtual vector bundle $-\theta^*\gamma_{2n}$ over B, and $\Omega^\infty MT\theta$ for its infinite loop space.

Let $\hat{\ell}_W : TW \to \theta^*\gamma$ be a θ-structure. We consider the n-stage of the Moore–Postnikov tower of the underlying map $\ell_W : W \to B$, given by an n-connected map $\ell'_W : W \to B'$ and an n-coconnected fibration $u\colon B' \to B$. We then define $\theta' := \theta \circ u\colon B' \to B\mathrm{O}(2n)$ as shown in the following diagram.

$$\begin{tikzcd}
\partial W \arrow[d, hook] \arrow[drr, "\ell_{\partial W}"] & & \\
W \arrow[rr, "\ell_W"] \arrow[dr, "\ell'_W"'] & & B \arrow[dr, "\theta"] & \\
& B' \arrow[ur, "u"] \arrow[rr, "\theta'"] & & B\mathrm{O}(2n)
\end{tikzcd}$$

We combine the homological stability result mentioned above with results in Galatius–Randal-Williams **[23]** to get the following statement about the stable homology groups. Analogous to **[24]** we define the *θ-genus* for compact connected manifolds with θ-structure as

$$g^\theta(M, \hat{\ell}_M) = \max \left\{ g \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there are } g \text{ disjoint copies of the manifold} \\ (\mathbb{S}^n \# \mathbb{S}^n) \setminus \mathrm{int}(\mathbb{D}^{2n}) \text{ in } M \text{, each with stand-} \\ \text{ard } \theta\text{-structure} \end{array} \right\}.$$

THEOREM D. *Let W be a compact connected manifold of dimension $2n \geq 6$ which is $(n-1)$-connected relative to its boundary. Then the map*

$$\alpha\colon B\mathrm{Diff}^\theta_\partial(W) \longrightarrow \Omega^\infty MT\theta'$$

is acyclic in degrees $k \leq \frac{g^\theta(W,\hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{3}$, and an isomorphism in integral homology in degrees $k \leq \frac{g^\theta(W,\hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 2}{2}$, onto the path component that it hits.

Examples and Applications. We use Theorem D to compute the abelianisation of the mapping class group $\pi_1(B\mathrm{Diff}_\partial(M))$ for certain 6-dimensional manifolds M. For this, let G be a group and M be a compact connected 6-dimensional manifold which fits into the following commutative diagram

$$\begin{tikzcd}
B\mathrm{Spin}(6) \times BG \arrow[r, "\theta"] & BO(6) \\
M \arrow[u, "\ell_M"] \arrow[ur, "\tau"'] &
\end{tikzcd},$$

where the tangential structure θ is given by first projecting to $B\mathrm{Spin}(6)$ and then applying the usual $\theta^{\mathrm{Spin}(6)} \colon B\mathrm{Spin}(6) \to BO(6)$, and ℓ_M is 3-connected. In particular, the first two homotopy groups of M are $\pi_1(M) \cong G$ and $\pi_2(M) = 0$. Using Theorem D we get the following theorem.

THEOREM E. *Let G be a group with finite unitary stable rank $\mathrm{usr}(\mathbb{Z}[G])$ and abelianisation G^{ab}, a finitely generated group with no 2-torsion, and M be a manifold corresponding* *to G. If $g(M) \geq \mathrm{usr}(\mathbb{Z}[G]) + 4$ and M is 2-connected relative to its boundary, then there is an isomorphism*

$$H_1(B\mathrm{Diff}_\partial(M)) \cong G^{\mathrm{ab}} \oplus \mathrm{ko}_7(BG).$$

For the case where the group G is of the form $\mathbb{Z}/2^k\mathbb{Z}$ for some $k \geq 1$, we get a partial computation of $H_1(B\mathrm{Diff}_\partial(M))$.

Following [**21**, Ch. 1.5] we also consider the following example. On $\mathbb{R}P^6$ we consider the standard Morse function

$$\begin{aligned} f\colon \mathbb{R}P^6 &\longrightarrow \mathbb{R} \\ [x_0 : x_1 : \ldots : x_6] &\longmapsto \sum_{i=0}^{6} i \cdot x_i^2. \end{aligned}$$

Since $0, 1, \ldots, 6$ are the only critical values of f we can define a submanifold M given by $f^{-1}((-\infty, 2.5])$. We write K for the 3-handle $f^{-1}([2.5, 3.5])$. The involution of $\mathbb{R}P^\infty$ that sends $[x_0 : x_1 : \ldots : x_6]$ to $[x_6 : x_5 : \ldots : x_0]$, sends the boundary $\partial_0 K = f^{-1}(2.5)$, which is equal to ∂M, to the boundary $\partial_1 K = f^{-1}(3.5)$ by a diffeomorphism. Thus, we define

$$X_k := M \cup \bigcup_{i=1}^{k} K.$$

Using Theorem D we get the following theorem.

THEOREM F. *For $k \geq 15$ we have $H_1(B\mathrm{Diff}_\partial(X_k)) \cong \mathbb{Z}/2\mathbb{Z}$.*

CHAPTER 1

The Complex for General Linear Groups

The aim of this chapter is to show that a certain complex associated to a module is highly connected. For the case of modules of the form R^n for some ring R there are several results available already, e.g. results by Charney [**15**] for R a Dedekind domain and by van der Kallen [**46**] for R with finite stable rank.

We consider the case of general modules over a ring with finite stable rank. The approach we use to show homological stability is what has become the standard strategy of proving results in this area. It has been introduced by Maazen [**33**] and shortly afterwards used in various contexts by Charney [**15**], Dwyer [**18**], van der Kallen [**46**], and Vogtmann [**50**]. For us it is convenient to use the formulation in Randal-Williams–Wahl [**43**]. This mainly involves showing the high connectivity of a certain semisimplicial set. In this chapter we generalise a complex introduced by van der Kallen and show its high connectivity. Even though this complex is not exactly the one needed for the machinery of Randal-Williams–Wahl, it is good enough to deduce the high connectivity of that semisimplicial set. We can then immediately extract a homological stability result for various coefficients systems which is the content of Chapter 4.

Following [**46**], for a set V we define $\mathcal{O}(V)$ to be the poset of ordered sequences of distinct elements in V of length at least one. The partial ordering on $\mathcal{O}(V)$ is given by refinement, i.e. we write $(w_1,\dots,w_m) \le (v_1,\dots,v_n)$ if there is a strictly increasing map $\phi\colon \{1,\dots,m\} \to \{1,\dots,n\}$ such that $w_i = v_{\phi(i)}$. We say that $F \subseteq \mathcal{O}(V)$ satisfies the *chain condition* if for every element $(v_1,\dots,v_n) \in F$ and every $(w_1,\dots,w_m) \le (v_1,\dots,v_n)$ we also have $(w_1,\dots,w_m) \in F$. For $v = (v_1,\dots,v_n) \in F$, we write F_v for the set of all sequences $(w_1,\dots,w_m) \in F$ such that $(w_1,\dots,w_m,v_1,\dots,v_n) \in F$. Note that if F satisfies the chain condition and $v, w \in F$ then $(F_v)_w = F_{vw}$. We write $F_{\le k}$ for the subset of F containing all sequences of length $\le k$.

We write $\mathrm{GL}(M)$ for the group of automorphisms of general R-modules M. A sequence $(v_1,\dots,v_n)$ of elements in M is called *unimodular* if there are R-module homomorphisms

$$f_1,\dots,f_n\colon R \to M \text{ and } \phi_1,\dots,\phi_n\colon M \to R$$

such that $f_i(1) = v_i$ and $\phi_j \circ f_i = \delta_{i,j} \cdot \mathbb{1}_R$. An element $v \in M$ is called *unimodular* if it is unimodular as a sequence in M of length 1. The condition $\phi_j \circ f_i = \delta_{i,j} \cdot \mathbb{1}_R$ holds if and only if the matrix $(\phi_j \circ f_i(1))_{i,j}$ is the identity matrix. In fact, for a sequence to be unimodular it is enough to find $\tilde{\phi}_1,\dots,\tilde{\phi}_n$ so that the matrix $\left(\tilde{\phi}_j \circ f_i(1)\right)_{i,j}$ is invertible as the following lemma shows.

Lemma 1.1. *Given a sequence* $(v_1, \dots, v_n)$ *in* M *and* R*-module homomorphisms*

$$f_1, \dots, f_n \colon R \to M \text{ and } \tilde{\phi}_1, \dots, \tilde{\phi}_n \colon M \to R$$

so that $f_i(1) = v_i$ *and the matrix* $\left(\tilde{\phi}_j \circ f_i(1)\right)_{i,j}$ *is invertible. Then* $(v_1, \dots, v_n)$ *is already unimodular.*

Proof. Let A^{-1} denote the inverse of the matrix $\left(\tilde{\phi}_j \circ f_i(1)\right)_{i,j}$. We define R-module homomorphisms $\phi_j \colon M \to R$ via

$$\phi_1 \oplus \cdots \oplus \phi_n \colon M \xrightarrow{\tilde{\phi}_1 \oplus \cdots \oplus \tilde{\phi}_n} R^n \xrightarrow{\cdot A^{-1}} R^n,$$

where $\phi_j(m)$ is the j-th entry of the vector $\phi_1 \oplus \cdots \oplus \phi_n(m)$. By construction we have $\phi_j(v_i) = \delta_{i,j}$, and therefore, the sequence $(v_1, \dots, v_n)$ is unimodular. $\square$

We write $\mathcal{U}(M)$ for the subposet of $\mathcal{O}(M)$ consisting of unimodular sequences in M. Let R^∞ denote the free R-module with basis $e_1, e_2, \dots$ and let M^∞ denote the R-module $M \oplus R^\infty$. Note that for $(v_1, \dots, v_n) \in M$ it is the same to say the sequence is unimodular in M or it is unimodular in M^∞.

Definition 1.2. We say that a ring R satisfies the *stable range condition* (S_n) if for every unimodular vector $(r_1, \dots, r_{n+1}) \in R^{n+1}$ there are elements $t_1, \dots, t_n \in R$ such that the vector $(r_1 + t_1 r_{n+1}, \dots, r_n + t_n r_{n+1}) \in R^n$ is unimodular. If n is the smallest such number we say R has *stable rank* n, $\mathrm{sr}(R) = n$, and it has $\mathrm{sr}(R) = \infty$ if such an n does not exist.

Note that the stable range in the sense of Bass [**7**], (SR_n), is the same as our stable range condition (S_{n-1}). The absolute stable rank $\mathrm{asr}(R)$ of a ring R as defined by Magurn–van der Kallen–Vaserstein in [**34**] is an upper bound for the stable rank, i.e. $\mathrm{sr}(R) \leq \mathrm{asr}(R)$ ([**34**, Lemma 1.2]). In the following we give some of the well-known examples of rings and their stable ranks.

Examples 1.3.

(1) A commutative Noetherian ring R of finite Krull dimension d satisfies $\mathrm{sr}(R) \leq d{+}1$. In particular, if R is a Dedekind domain then $\mathrm{sr}(R) \leq 2$ ([**26**, 4.1.11]) and for a field k, the polynomial ring $K = k[t_1, \dots, t_n]$ satisfies $\mathrm{sr}(K) \leq n{+}1$ ([**48**, Thm. 8]).

(2) More generally, any R-algebra A that is finitely generated as an R-module satisfies $\mathrm{sr}(A) \leq d{+}1$, for R again a commutative Noetherian ring of finite Krull dimension d. [**34**, Thm. 3.1] or [**26**, 4.1.15]

(3) Recall that a ring R is called *semi-local* if $R/J(R)$ is a left Artinian ring, for $J(R)$ the Jacobson radical of R. A semi-local ring satisfies $\mathrm{sr}(R) = 1$. [**26**, 4.1.17]

(4) Recall that a group G is called *virtually polycyclic* if there is a sequence of normal subgroups

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_{n-1} \triangleright G_n = 0$$

such that each quotient G_i/G_{i+1} is cyclic or finite. Its *Hirsch number* $h(G)$ is the number of infinite cyclic factors. For a virtually polycyclic group G we have

$\mathrm{sr}(\mathbb{Z}[G]) \leq h(G)+2$. [**17**, Thm. 7.3] For example, this class contains all finite groups, all finitely generated abelian groups, and even more generally, all finitely generated nilpotent groups (cf. remark below [**32**, 1.3.1]).

For an R-module M we define the *rank* of M as

$$\mathrm{rk}(M) := \sup\{n \in \mathbb{N} \mid \text{there is an } R\text{-module } M' \text{ such that } M \cong R^n \oplus M'\}.$$

Using this notion we can phrase the following theorem. Henceforth, we use the convention that the condition of a space to be n-connected for $n \leq -2$ (and so in particular for $n = -\infty$) is vacuous.

THEOREM 1.4. *Let M be an R-module.*

(1) $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is $(\mathrm{rk}(M) - \mathrm{sr}(R) - 1)$-connected,
(2) $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\dots,v_k)}$ is $(\mathrm{rk}(M) - \mathrm{sr}(R) - k - 1)$-connected for all sequences $(v_1,\dots,v_k) \in \mathcal{U}(M^\infty)$.

In [**46**, Thm. 2.6 (i), (ii)] van der Kallen has proven this theorem for the special case of modules of the form R^n. Our proof of Theorem 1.4 adapts the techniques and ideas that he has used. Note that the integer sdim used in [**46**] satisfies $\mathrm{sr}(R) = \mathrm{sdim} - 1$. Just as in van der Kallen's proof, we use the following technical lemma several times in the proof of Theorem 1.4.

LEMMA 1.5. *Let $F \subseteq \mathcal{U}(M^\infty)$ satisfy the chain condition. Let $X \subseteq M^\infty$ be a subset.*

(1) Assume that $\mathcal{O}(X) \cap F$ is d-connected and that $\mathcal{O}(X) \cap F_{(v_1,\dots,v_m)}$ is $(d-m)$-connected for all sequences $(v_1,\dots,v_m)$ in $F \setminus \mathcal{O}(X)$. Then F is d-connected.
(2) Assume that for all sequences $(v_1,\dots,v_m)$ in $F \setminus \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1,\dots,v_m)}$ is $(d-m+1)$-connected. Assume further that there is a sequence (y_0) of length 1 in F with $\mathcal{O}(X) \cap F \subseteq F_{(y_0)}$. Then F is $(d+1)$-connected.

OUTLINE OF THE PROOF. The proof of [**46**, Lemma 2.13] also works in this setting, where we use the obvious modification of [**46**, Lemma 2.12] to allow $F \subseteq \mathcal{U}(M^\infty)$ so that it fits into our framework. □

We are not the first ones that have the idea of showing homological stability for automorphism groups of modules more general than R^n: In [**46**, Rmk. 2.7 (2)] van der Kallen has suggested a possible generalisation of his results using the notion of "big" modules as defined in [**49**].

PROOF OF THEOREM 1.4. Analogous to the proof of [**46**, Thm. 2.6] we will also show the following statements.

(a) $\mathcal{O}(M \cup (M+e_1)) \cap \mathcal{U}(M^\infty)$ is $(\mathrm{rk}(M) - \mathrm{sr}(R))$-connected,
(b) $\mathcal{O}(M \cup (M+e_1)) \cap \mathcal{U}(M^\infty)_{(v_1,\dots,v_k)}$ is $(\mathrm{rk}(M) - \mathrm{sr}(R) - k)$-connected for all sequences $(v_1,\dots,v_k) \in \mathcal{U}(M^\infty)$.

Recall that e_1 denotes the first standard basis element of R^∞ in $M^\infty = M \oplus R^\infty$.

The proof is by induction on $g = \mathrm{rk}(M)$. Note that statements (1), (2), and (b) all hold for $g < \mathrm{sr}(R)$ so we can assume $g \geq \mathrm{sr}(R)$. Statement (a) holds for $g < \mathrm{sr}(R) - 1$, so we can assume $g \geq \mathrm{sr}(R) - 1$ when proving this statement. The structure of the proof is as follows. We start by proving (b) which enables us to deduce (2). We will then prove statements (1) and (a) simultaneously by applying statement (2).

We may suppose $M = R^g \oplus M'$ for an R-module M', since the posets in statements (1), (2), (a), and (b) only depend on the isomorphism class of M. We write $x_1, \dots, x_g$ for the standard basis of R^g.

Proof of (b). For $Y := M \cup (M + e_1)$ we write $F := \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1,\dots,v_k)}$. Let $d := g - \mathrm{sr}(R) - k$, so we have to show that F is d-connected.

In the case $g = \mathrm{sr}(R)$ we only have to consider $k = 1$. Then, we have to show that F is non-empty. The strategy for this part is as follows: We define a map $f \in \mathrm{GL}(M^\infty)$ so that Y is fixed under f as a set and the projection of $f(v_1)$ onto R^g, $f(v_1)|_{R^g}$, is unimodular. Then the sequence $(f(v_1)|_{R^g}, e_1)$ is unimodular in M^∞. We will show that, therefore, the sequence $(f(v_1), e_1)$ is also unimodular in M^∞ and so is the sequence $(v_1, f^{-1}(e_1))$. Since $e_1 \in Y$ and the automorphism f fixes Y setwise we get $f^{-1}(e_1) \in Y$, and thus, F is non-empty as it contains $f^{-1}(e_1)$.

We start by writing

$$v_1 = \sum_{i=1}^{g} x_i r_i + p + a,$$

where $r_i \in R$, $p \in M'$, and $a \in R^\infty$. Since v_1 is unimodular there is an R-module homomorphism $\phi\colon M^\infty \to R$ satisfying $\phi(v_1) = 1$. In particular,

$$1 = \phi(v_1) = \sum_{i=1}^{g} \phi(x_i) r_i + \phi(p + a),$$

which shows that $(r_1, \dots, r_g, \phi(p + a)) \in R^{g+1}$ is unimodular. As $g = \mathrm{sr}(R)$ there are elements $t_1, \dots, t_g \in R$ such that the sequence

$$(r_1 + t_1\phi(p + a), \dots, r_g + t_g\phi(p + a))$$

is unimodular. Now consider the map

$$\begin{aligned} M^\infty = R^g \oplus M' \oplus R^\infty &\xrightarrow{f} M^\infty = R^g \oplus M' \oplus R^\infty \\ (a_1, \dots, a_g, q, b) &\longmapsto (a_1 + t_1\phi(q + b), \dots, a_g + t_g\phi(q + b), q, b) \end{aligned}$$

which is invertible. The map f satisfies $f(Y) = Y$ and the projection of $f(v_1)$ onto R^g is unimodular. Thus, by definition there are homomorphisms $f_1\colon R \to M^\infty$ and $\phi_1\colon M^\infty \to R$ so that $f_1(1) = f(v_1)|_{R^g}$ and $\phi_1 \circ f_1 = \mathbb{1}_R$. Note that we can assume that ϕ_1 is zero away from R^g as otherwise we can restrict to R^g before we apply ϕ_1. This shows that the sequence $(f(v_1)|_{R^g}, e_1)$ is unimodular by choosing $\phi_2\colon M^\infty \to R$ to be the projection onto the coefficient of e_1. For the sequence $(f(v_1), e_1)$ we change f_1 to map 1 to $f(v_1)$ but keep all other homomorphisms the same then the matrix $\left(\tilde{\phi}_j \circ f_i(1)\right)_{i,j}$ is an upper triangular matrix with 1's on the diagonal. In particular, it is invertible, so the sequence $(f(v_1), e_1)$ is unimodular by Lemma 1.1. Since f is an automorphism of M^∞ the sequence $(v_1, f^{-1}(e_1))$ is also unimodular. By construction we have $f(Y) = Y$ and so in particular $f^{-1}(e_1) \in Y$. Hence, F is non-empty as it contains $f^{-1}(e_1)$.

Now consider the case $g > \mathrm{sr}(R)$. As in the case above there is an $f \in \mathrm{GL}(M^\infty)$ such that $f(Y) = Y$ and $f(v_1)|_{R^g}$ is unimodular. The group $\mathrm{GL}_g(R)$ acts transitively on the set of unimodular elements in R^g (by [**47**, Thm. 2.3 (c)]). This only holds in the case $g > \mathrm{sr}(R)$ so the case $g = \mathrm{sr}(R)$ had to be proven separately. Hence, there exists a map $\psi \in \mathrm{GL}_g(R) \leq \mathrm{GL}(M^\infty)$ such that $\psi(f(v_1)|_{R^g}) = x_g$. By applying $\psi \circ f$, considered as an automorphism of M^∞, to M^∞, without loss of generality we can assume that the projection of v_1 to R^g is x_g. We define

$$X := \{v \in Y \mid \text{the } x_g\text{-coordinate of } v \text{ vanishes}\} = (R^{g-1} \oplus M') \cup (R^{g-1} \oplus M' + e_1).$$

We now check that the assumptions of Lemma 1.5 (1) are satisfied. Notice that

$$\mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)} = \mathcal{U}(M^\infty)_{(v_1,v_2',\ldots,v_k')},$$

for $v_i' = v_i + v_1 \cdot r_i$ for $r_i \in R$, as the span of $v_1, v_2', \ldots, v_k'$ is the same as that of $v_1, v_2, \ldots, v_n$. As the projection of v_1 to R^g is x_g, we may choose the r_i so that the x_g-coordinate of each v_i' vanishes.

$$\begin{aligned}
\mathcal{O}(X) \cap F &= \mathcal{O}(X) \cap \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)} \\
&= \mathcal{O}(X) \cap \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1,v_2',\ldots,v_k')} \\
&= \mathcal{O}((R^{g-1} \oplus M') \cup (R^{g-1} \oplus M' + e_1)) \cap \mathcal{U}(M^\infty)_{(v_2',\ldots,v_k')}.
\end{aligned}$$

Therefore, by the induction hypothesis, $\mathcal{O}(X) \cap F$ is d-connected. Analogously, for a sequence $(w_1, \ldots, w_l) \in F \setminus \mathcal{O}(X)$ we get

$$\begin{aligned}
\mathcal{O}(X) \cap F_{(w_1,\ldots,w_l)} &= \mathcal{O}(X) \cap \mathcal{O}(Y) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k,w_1,\ldots,w_l)} \\
&= \mathcal{O}((R^{g-1} \oplus M' \cup (R^{g-1} \oplus M' + e_1))) \cap \mathcal{U}(M^\infty)_{(v_2',\ldots,v_k',w_1',\ldots,w_l')},
\end{aligned}$$

which is $(d-l)$-connected by the induction hypothesis. Therefore, Lemma 1.5 (1) shows that F is d-connected.

Proof of (2). Let us write

$$X := (R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')$$

and $d := g - \mathrm{sr}(R) - k$ as in the previous part. Then, we have

$$\begin{aligned}
&\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)}) \\
= \; &\mathcal{O}\Big((R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')\Big) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)},
\end{aligned}$$

which is $(d-k-1)$-connected by (b) after a change of coordinates.

Similarly, for $(w_1, \ldots, w_l) \in \mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)} \setminus \mathcal{O}(X)$ we have

$$\begin{aligned} & \mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)})_{(w_1,\ldots,w_l)} \\ = \; & \mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k,w_1,\ldots,w_l)}), \end{aligned}$$

which is $(d-k-l-1)$-connected by the computation in the previous paragraph applied to the sequence $(v_1, \ldots, v_k, w_1, \ldots, w_l)$. Applying Lemma 1.5 (1) to $F := \mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_1,\cdots,v_k)}$ and X, d as defined above, the claim follows.

Proof of (1) and (a). Recall that we now only assume $g \geq \mathrm{sr}(R) - 1$. By induction let us assume that statement (a) holds for $R^{g-1} \oplus M'$ and we want to deduce it for $M = R^g \oplus M'$. Before we finish the induction for (a) we will show that this already implies statement (1) for $M = R^g \oplus M'$. From now on until the end of the proof we consider

$$X := (R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')$$

as in the proof of (2) and $d := g - \mathrm{sr}(R)$. Then

$$\begin{aligned} & \mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty)) \\ = \; & \mathcal{O}\Big((R^{g-1} \oplus M') \cup ((R^{g-1} + x_g) \oplus M')\Big) \cap \mathcal{U}(M^\infty) \end{aligned}$$

is $(d-1)$-connected by (a) after a change of coordinates. The remaining assumption of Lemma 1.5 (1), i.e. that $\mathcal{O}(X) \cap (\mathcal{O}(M) \cap \mathcal{U}(M^\infty))_{(v_1,\ldots,v_m)}$ is $(d-m-1)$-connected, we have already shown in the proof of (2). Thus, $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is $(g - \mathrm{sr}(R) - 1)$-connected which proves statement (1).

To prove (a) we will apply Lemma 1.5 (2) for $F = \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)$, $X = M$, and $y_0 = e_1$. Consider

$$(v_1, \ldots, v_k) \in \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty) \setminus \mathcal{O}(X).$$

Without loss of generality we may suppose that $v_1 \notin X$ as otherwise we can permute the v_i. By definition of X the coefficient of the e_1-coordinate of v_1 is therefore 1. Analogous to the proof of (b) we have

$$\mathcal{O}(X) \cap \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)_{(v_1,\ldots,v_k)} \cong \mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v'_2,\ldots,v'_k)},$$

where $v'_i := v_i + v_1 r_i$ is chosen so that the e_1-coordinate of v'_i is 0 for all i. This is $(d-k)$-connected by (1) for $k = 1$, and by (2) for $k \geq 2$. By construction we have

$$\mathcal{O}(X) \cap \mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty) \subseteq (\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty))_{(e_1)}$$

and thus we can apply Lemma 1.5 (2) to show that $\mathcal{O}(M \cup (M + e_1)) \cap \mathcal{U}(M^\infty)$ is $(g - \mathrm{sr}(R))$-connected which proves (a).

When showing statement (a) for $M = R^g \oplus M'$ we only used statement (1) for the same $M = R^g \oplus M'$ which follows from statement (a) for $R^{g-1} \oplus M'$ so this is indeed a valid induction to show both statements (1) and (a). $\square$

The following propositions are consequences of the path-connectedness of the complex $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$, and therefore, by Theorem 1.4, hold in particular for R-modules M with $\mathrm{rk}(M) \geq \mathrm{sr}(R) + 1$. The statements and proofs are [**24**, Prop. 3.3] and [**24**, Prop. 3.4] respectively for the case of general R-modules.

PROPOSITION 1.6 (Transitivity). *If $\phi_0, \phi_1 \colon R \to M$ are split injective morphisms of R-modules and the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ is path-connected, then there is an automorphism f of M such that $\phi_1 = f \circ \phi_0$.*

PROOF. Note that an R-module map $R \to M$ is defined by where it sends the unit 1 of the ring R. Suppose first that $(\phi_1(1), \phi_2(1))$ is in $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$. This implies

$$M \cong \phi_1(R) \oplus \phi_2(R) \oplus M'$$

for some R-module M' and that there is an automorphism of M which interchanges the $\phi_i(R)$ and fixes M'. Consider the equivalence relation between morphisms $f \colon R \to M$ of differing by an automorphism of M. We have just shown that two morphisms corresponding to two adjacent vertices in $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ are equivalent. But the poset is path-connected by assumption, and hence, all vertices are equivalent. □

PROPOSITION 1.7 (Cancellation). *Let M and N be R-modules with $M \oplus R \cong N \oplus R$. If the poset $\mathcal{O}(M \oplus R) \cap \mathcal{U}(M^\infty)$ is path-connected, then there is also an isomorphism $M \cong N$.*

PROOF. Analogous to the proof of Proposition 1.6 we can assume that the isomorphism $\phi \colon M \oplus R \to N \oplus R$ satisfies $\phi|_R = \mathrm{id}_R$. Thus, by considering quotient modules we get

$$M \cong \frac{M \oplus R}{R} \cong \frac{\phi(M \oplus R)}{\phi(R)} = \frac{N \oplus R}{R} \cong N.$$ □

CHAPTER 2

The Complex for Unitary Groups

The aim of this chapter is to prove the analogue of Theorem 1.4 for the case of unitary groups of quadratic modules. Again, using the formulation in Randal-Williams–Wahl [**43**] we can deduce homological stability (see Chapter 5). In this setting we consider the complex of hyperbolic unimodular sequences in a quadratic module M. For the special case where M is a hyperbolic module, this has been considered by Mizaii–van der Kallen in [**39**], but the general case requires new ideas. In this chapter we prove its high connectivity. We deduce the assumptions for the machinery of Randal-Williams–Wahl in Chapter 5.

Following [**5**] and [**6**] let R be a ring with an anti-involution $^-\colon R \to R$, i.e. $\overline{\overline{r}} = r$ and $\overline{rs} = \overline{s}\,\overline{r}$. Fix a unit $\varepsilon \in R$ which is a central element of R and satisfies $\overline{\varepsilon} = \varepsilon^{-1}$. Consider a subgroup Λ of $(R, +)$ satisfying

$$\Lambda_{\min} := \{r - \varepsilon\overline{r} \mid r \in R\} \subseteq \Lambda \subseteq \{r \in R \mid \varepsilon\overline{r} = -r\} =: \Lambda_{\max}$$

and $\overline{r}\Lambda r \subseteq \Lambda$ for all $r \in R$. An (ε, Λ)-*quadratic module* is a triple (M, λ, μ), where M is a right R-module, $\lambda\colon M \times M \to R$ is a sesquilinear form (i.e. λ is R-antilinear in the first variable and R-linear in the second meaning that $\lambda(mr, ns) = \overline{r}\lambda(m, n)s$ for $m, n \in M$ and $r, s \in R$), and $\mu\colon M \to R/\Lambda$ is a function, satisfying

(1) $\lambda(x, y) = \varepsilon\overline{\lambda(y, x)}$,
(2) $\mu(x \cdot a) = \overline{a}\mu(x)a$ for $a \in R$,
(3) $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y) \mod \Lambda$.

The pair (ε, Λ) is called the *form parameter*. The direct sum of two quadratic modules (M_1, λ_1, μ_1) and (M_2, λ_2, μ_2) is given by the quadratic module $(M_1 \oplus M_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2)$, where

$$(\lambda_1 \oplus \lambda_2)((m_1, m_2), (m_1', m_2')) := \lambda_1(m_1, m_1') + \lambda_2(m_2, m_2'),$$
$$(\mu_1 \oplus \mu_2)(m_1, m_2) := \mu_1(m_1) + \mu_2(m_2),$$

for $m_i, m_i' \in M_i$. A morphism of quadratic modules (with the same form parameter) is a homomorphism $f\colon M \to N$ of R-modules such that $\lambda_N \circ (f \oplus f) = \lambda_M$ and such that $\mu_M = \mu_N \circ f$. The *unitary group* is defined as

$$U_M^\varepsilon(R, \Lambda) := U(M) := \{A \in \mathrm{GL}(M) \mid \lambda(Ax, Ay) = \lambda(x, y), \mu(Ax) = \mu(x) \text{ for all } x, y \in M\}.$$

The *hyperbolic module* H over R is the (ε, Λ)-quadratic module given by

$$\left(R^2 \text{ with basis } e, f; \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}; \mu(e) = \mu(f) = 0\right).$$

Using the properties of μ defined above this already defines μ on all of H. In particular, we have

$$\mu(ea, fb) = \mu(ea) + \mu(fb) + \lambda(ea, fb) = \overline{a}b.$$

We write H^g for the direct sum of g copies of the hyperbolic module H.

Let M^* consist of functions $f\colon M \to R$ that are R-linear, i.e. functions satisfying $f(mr) = \overline{r}f(m)$ for all $m \in M$ and $r \in R$. Note that this is a right R-module via $(f \cdot r)(m) := f(m) \cdot r$. We say the quadratic module (M, λ, μ) is *non-degenerate* if the map

$$\begin{aligned} M &\longrightarrow M^* \\ x &\longmapsto \lambda(-, x) \end{aligned}$$

is an isomorphism. If M is non-degenerate, any morphism $f\colon M \to N$ of quadratic modules is canonically split injective because

$$M \xrightarrow{f} N \longrightarrow N^* \xrightarrow{f^*} M^*$$

is an isomorphism, and in fact induces an isomorphism $N \cong M \oplus (f(M))^\perp$ of (ε, Λ)-quadratic modules. In particular, if H^g appears as a subspace of some quadratic module M, then we have $M \cong H^g \oplus (H^g)^\perp$

Let us consider the following examples of unitary groups for the quadratic module H^g, given in [**39**, Ex. 6.1].

(1) If $\varepsilon = -1$ and the involution is the identity map $\mathbb{1}_R$ then $\Lambda_{\max} = R$. Thus, we have

$$\begin{aligned} U^{-1}_{H^g}(R, \Lambda_{\max}) &= \{A \in \mathrm{GL}(H^g) \mid \lambda(Ax, Ay) = \lambda(x, y), x, y \in H^g\} \\ &= \mathrm{Sp}_{2g}(R), \end{aligned}$$

which is the usual symplectic group.

(2) If $\varepsilon = 1$ and the involution is the identity map $\mathbb{1}_R$ then $\Lambda_{\min} = 0$. Thus, we have

$$\begin{aligned} U^{1}_{H^g}(R, \Lambda_{\min}) &= \{A \in \mathrm{GL}(H^g) \mid \mu(Ax) = \mu(x), x \in H^g\} \\ &= \mathrm{O}_{g,g}(R), \end{aligned}$$

which is the orthogonal group of the symmetric form $\begin{pmatrix} 0 & \mathbb{1}_g \\ \mathbb{1}_g & 0 \end{pmatrix}$.

(3) If $\varepsilon = -1$ and the involution is not the identity map $\mathbb{1}_R$ we have

$$U^{-1}_{H^g}(R, \Lambda_{\max}) = \mathrm{U}_{2g}(R),$$

which is the classical unitary group corresponding to the involution.

Let σ be the permutation of the set of natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$. Writing $e_{i,j}(r)$ for the $2n \times 2n$-matrix with $r \in R$ in the (i, j) place and zero elsewhere, for $1 \leq i, j \leq 2n$, $i \neq j$, and every $r \in R$ we define

$$E_{i,j}(r) := \begin{cases} \mathbb{1}_{2n} + e_{i,j}(r) & i = 2k-1, j = 2k, r \in \Lambda \\ \mathbb{1}_{2n} + e_{i,j}(r) & i = 2k, j = 2k-1, \overline{r} \in \Lambda \\ \mathbb{1}_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\overline{r}) & i + j = 2k \\ \mathbb{1}_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(-\varepsilon^{-1}\overline{r}) & i = 2k-1, j = 2l, k \neq l \\ \mathbb{1}_{2n} + e_{i,j}(r) + e_{\sigma(j),\sigma(i)}(\varepsilon\overline{r}) & i = 2k, j = 2l-1, k \neq l \end{cases}$$

Since $E_{i,j}(r) \in U^{\varepsilon}_{M}(R, \Lambda)$ we define the elementary unitary group $EU^{\varepsilon}_{M}(R, \Lambda)$ as the group generated by the $E_{i,j}(r)$, $r \in R$.

Definition 2.1. A ring R satisfies the *transitivity condition* (T_n) if $EU^{\varepsilon}_{H^n}(R, \Lambda)$ acts transitively on the set

$$C^{\varepsilon}_{r}(R, \Lambda) := \{x \in H^n \mid x \text{ is unimodular}, \mu(x) = r \mod \Lambda\}$$

for every $r \in R$. The ring R has *unitary stable range* (US_n) if it satisfies the stable range condition (S_n), as defined in Definition 1.2, as well as the transitivity condition (T_{n+1}). We say that R has *unitary stable rank* n, $\mathrm{usr}(R) = n$, if n is the least number such that (US_n) holds, and $\mathrm{usr}(R) = \infty$ if such an n does not exist.

The transitivity condition (T_n) and, hence, the unitary stable range (US_n) are conditions on the triple $(R, \varepsilon, \Lambda)$ and not just on R. However, to make our notation consistent with the literature we write $\mathrm{usr}(R)$ as introduced above which drops both ε and Λ.

As remarked in [**39**, Rmk. 6.4] we have $\mathrm{usr}(R) \leq \mathrm{asr}(R) + 1$ for the absolute stable rank of Magurn–van der Kallen–Vaserstein [**34**]. In the special case, where the involution on R is the identity map (which implies that R is commutative), we have $\mathrm{usr}(R) \leq \mathrm{asr}(R)$. We now give some well-known examples of rings and their unitary stable rank.

Examples 2.2. The following examples work for any anti-involution on R and every choice of ε and Λ.

(1) Let R be a commutative Noetherian ring of finite Krull dimension d. Then any R-algebra A that is finitely generated as an R-module satisfies $\mathrm{usr}(A) \leq d+2$. [**34**, Thm. 3.1]
(2) A semi-local ring satisfies $\mathrm{usr}(R) \leq 2$. [**34**, Thm. 2.4]
(3) For a virtually polycyclic group G we have $\mathrm{usr}(\mathbb{Z}[G]) \leq h(G) + 3$, where $h(G)$ is the Hirsch number as defined in Example 1.3 (4). [**17**, Thm. 7.3]

A sequence $(v_1, \ldots, v_k)$ of elements in the quadratic module (M, λ, μ) is called *unimodular* if the sequence is unimodular in M considered as an R-module (see Chapter 1). We say that the sequence is λ-*unimodular* if there are elements $w_1, \ldots, w_k$ in M such that $\lambda(w_i, v_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker delta. We write $\mathcal{U}(M)$ and $\mathcal{U}(M, \lambda)$ for the subposet of unimodular and λ-unimodular sequences in M respectively.

Note that every λ-unimodular sequence is in particular unimodular. The following lemma shows that there are cases where the converse is also true.

Lemma 2.3. *Let the sequence $(v_1, ..., v_k)$ be unimodular in M. If there is a submodule $N \subseteq M$ containing the v_i such that $\lambda|_N$ is non-degenerate, then the sequence $(v_1, ..., v_k)$ is λ-unimodular in N.*

Proof. Let $(v_1, \dots, v_k)$ be a unimodular sequence in M. This means that there are maps $f_1, \dots, f_k \colon R \to M$ with $f_i(1) = v_i$ and maps $\phi_1, \dots, \phi_k \colon M \to R$ with $\phi_j \circ f_i = \delta_{i,j} \cdot \mathbb{1}_R$. Note that this implies that $\phi_j(v_i) = \delta_{i,j}$. Now, λ being non-degenerate on N means that the map

$$\begin{aligned} N &\longrightarrow N^* \\ v &\longmapsto \lambda(-, v) \end{aligned}$$

is an isomorphism. Hence, there are $w'_1, \dots, w'_k \in N$ such that $\lambda(-, w'_i) = \phi_i(-)$ on N. Defining $w_i := w'_i \varepsilon$ then yields

$$\lambda(w_i, v_j) = \lambda(w'_i \varepsilon, v_j) = \varepsilon \overline{\lambda(v_j, w'_i)\varepsilon} = \varepsilon \overline{\varepsilon} \overline{\phi_i(v_j)} = \delta_{i,j}. \qquad \square$$

We call a subset S of a quadratic module (M, λ, μ) *isotropic* if $\mu(x) = 0$ and $\lambda(x, y) = 0$ for all $x, y \in S$. Let $\mathcal{IU}(M)$ denote the set of λ-unimodular sequences $(x_1, \dots, x_k)$ in M such that $x_1, \dots, x_k$ span an isotropic direct summand of M. We write $\mathcal{HU}(M)$ for the set of sequences $((x_1, y_1), \dots, (x_k, y_k))$ such that $(x_1, \dots, x_k)$, $(y_1, \dots, y_k) \in \mathcal{IU}(M)$, and $\lambda(x_i, y_j) = \delta_{i,j}$. This can also be thought of as the set of quadratic module maps $H^k \to M$. We call $\mathcal{IU}(M)$ the *poset of isotropic λ-unimodular sequences* and $\mathcal{HU}(M)$ the *poset of hyperbolic λ-unimodular sequences*. We say that $x = ((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{HU}(M)$ is of length $|x| = k$.

Let $\mathcal{MU}(M)$ be the set of sequences $((x_1, y_1), \dots, (x_k, y_k)) \in \mathcal{O}(M \times M)$ satisfying

(1) $(x_1, \dots, x_k) \in \mathcal{IU}(M)$,
(2) for each i we have either $y_i = 0$ or $\lambda(x_j, y_i) = \delta_{j,i}$,
(3) the span $\langle y_1, \dots, y_k \rangle$ is isotropic.

We identify the poset $\mathcal{IU}(M)$ with $\mathcal{MU}(M) \cap \mathcal{O}(M \times \{0\})$ and the poset $\mathcal{HU}(M)$ with $\mathcal{MU}(M) \cap \mathcal{O}(M \times (M \setminus \{0\}))$.

In order to phrase the main theorem of this section we introduce the following notion. For an (ε, Λ)-quadratic module (M, λ, μ) define the *Witt index* as

$$g(M) := \sup\{g \in \mathbb{N} \mid \text{there is a quadratic module } P \text{ such that } M \cong P \oplus H^g\}.$$

Theorem 2.4. *The poset $\mathcal{HU}(M)$ is $\lfloor \frac{g(M) - \mathrm{usr}(R) - 3}{2} \rfloor$-connected and for every element $x \in \mathcal{HU}(M)$ the poset $\mathcal{HU}(M)_x$ is $\lfloor \frac{g(M) - \mathrm{usr}(R) - |x| - 3}{2} \rfloor$-connected.*

For the special case where the quadratic module M is a direct sum of hyperbolic modules H^n, Theorem 2.4 has been proven by Mirzaii–van der Kallen in [**39**, Thm. 7.4]. Galatius–Randal-Williams have treated the case of general quadratic modules over the integers in [**24**, Thm. 3.2].

In order to prove the above theorem we need the following lemma which extends [**39**, Lemma 6.6] to the case of general quadratic modules. Note, however, that the proof is not an extension of the proof of [**39**, Lemma 6.6] but rather uses techniques of Vaserstein [**48**]. A similar statement has been given by Petrov in [**41**, Prop. 6]. However, Petrov considers hyperbolic modules which are defined over rings with a pseudoinvolution and only allows $\varepsilon = -\overline{1}$. He also states his connectivity range using a different rank, called the Λ-*stable rank*, which we shall not discuss.

LEMMA 2.5. *Let $P \oplus H^g$ be a quadratic module. If $g \geq \operatorname{usr}(R) + k$ and $(v_1, \ldots, v_k)$ is a sequence in $\mathcal{U}(P \oplus H^g, \lambda)$ then there is an automorphism $\phi \in U(P \oplus H^g)$ such that $\phi(v_1, \ldots, v_k) \subseteq P \oplus H^k$ and the projection of $\phi(v_1, \ldots, v_k)$ to the hyperbolic H^k is λ-unimodular.*

The following section contains the necessary foundations and the proof of this lemma.

2.1. Proof of Lemma 2.5

Following [**48**] an $(n+k) \times k$-matrix A is called *unimodular* if it has a left inverse. Note that the matrix A is unimodular if and only if the matrix CA is unimodular for any invertible matrix $C \in \mathrm{GL}_{n+k}(R)$. A ring R is said to satisfy the condition (S_n^k) if for every unimodular $(n+k) \times k$-matrix A, there exists an element $r \in R^{n+k-1}$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_{n+k-1} & r^\top \\ \hline 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the matrix B is unimodular and u is the last row of A.

Note that condition (S_n^1) is the same as condition (S_n). Furthermore, the conditions (S_n^k) and (S_n) are in fact equivalent as the following theorem shows.

THEOREM 2.6. [**48**, Thm. 3′] *For every $k, n \geq 1$ the condition (S_n) is equivalent to the condition (S_n^k).*

2.1.1. $n \times k$-Blocks. Given a quadratic R-module M we define an *$n \times k$-block A for M* to be an $n \times k$-matrix $(r_{i,j})_{i,j}$ with entries in the ring R together with k anti-linear maps $f_1, \ldots, f_k \colon M \to R$, i.e. maps satisfying $f_i(mr) = \overline{r} f_i(m)$ for $m \in M$, $r \in R$. We will write this data as

$$A = \begin{pmatrix} r_{1,1} & \ldots & r_{1,k} \\ \vdots & & \vdots \\ r_{n,1} & \ldots & r_{n,k} \\ f_1 & \ldots & f_k \end{pmatrix}.$$

Note that with this notation an $n \times k$-block has in fact $n+1$ rows. We refer to the row of maps $(f_1, \ldots, f_k)$ as the *last row* of A. Given an $(n+1) \times (n+1)$-matrix of the form

$$\left(\begin{array}{ccc|c} s_{1,1} & \ldots & s_{1,n} & m_1 \\ \vdots & & \vdots & \vdots \\ s_{n,1} & \ldots & s_{n,n} & m_{2g} \\ \hline 0 & \ldots & 0 & s \end{array}\right),$$

where $s, s_{i,j} \in R$, $m_i \in M$, we can act with it from the left on an $n \times k$-block A by matrix multiplication, where we define

$$m_i \cdot f_j := f_j(m_i) \text{ and } s \cdot f_j := f_j(-) \cdot \overline{s}.$$

We can act from the right on the block A with a $k \times k$-matrix with entries in R again by matrix multiplication, where we define $f_j \cdot r$ to send an element $m \in M$ to $f_j(m) \cdot r$ for $r \in R$.

Definition 2.7. We say that an $n \times k$-block A is *unimodular* if there is a $k \times (n+1)$-matrix A_L of the form

$$\begin{pmatrix} r'_{1,1} & \dots & r'_{1,n} & m'_1 \\ \vdots & & \vdots & \vdots \\ r'_{k,1} & \dots & r'_{k,n} & m'_k \end{pmatrix}$$

with $r'_{i,j} \in R$ and $m'_i \in M$, such that $A_L \cdot A = \mathbb{1}_k$, where the multiplication is again given by matrix multiplication, with $m'_i \cdot f_j$ as defined above.

Note that the $n \times k$-block A is unimodular if and only if any of the following blocks is unimodular:

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \\ \vdots & A \\ 0 & \\ f & \end{array}\right), \quad \left(\begin{array}{c|c} \mathbb{1}_n & v^\top \\ \hline 0 & 1 \end{array}\right) \cdot A, \quad \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A, \quad \text{or } A \cdot \left(\begin{array}{c|c} 1 & v \\ \hline 0 & \mathbb{1}_n \end{array}\right),$$

for an anti-linear map $f\colon M \to R$, a vector $v \in R^n$, and a matrix $C \in \mathrm{GL}_n(R)$.

Definition 2.8. An $n \times k$-block A for M is *matrix reducible* if there is a vector $m \in M^n$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_n & m^\top \\ \hline 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the $n \times k$-matrix B is unimodular and u is the last row of the block A.

Proposition 2.9. *If $k + \mathrm{sr}(R) \leq n + 1$ then every unimodular $n \times k$-block A is matrix reducible.*

Before we prove this proposition we show that matrix reducibility is preserved under certain operations as the following proposition shows (cf. proof of [**48**, Thm. 3′]).

Proposition 2.10. *Let A be an $n \times k$-block for M. Then A is matrix reducible if and only if the block obtained from A by doing any of the following moves is matrix reducible.*

(1) Multiply on the left by a matrix of the form

$$\left(\begin{array}{c|c} \mathbb{1}_n & v^\top \\ \hline 0 & 1 \end{array}\right),$$

for an element $v \in M^n$.

(2) Multiply on the left by a matrix of the form

$$\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right),$$

for a matrix $C \in \mathrm{GL}_n(R)$.

(3) Multiply on the right by a matrix $D \in \mathrm{GL}_k(R)$.

Proof. Note that each of the above moves may be inverted by a move of the same type. It is therefore enough to show that if A is matrix reducible then so is the block obtained from A by doing one of the above moves. Let $m \in M^n$ be the sequence showing that the block A is matrix reducible, i.e. we have

$$\left(\begin{array}{c|c} \mathbb{1}_n & m^\top \\ \hline 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the $n \times k$-matrix B is unimodular.

Statement (1) follows from the fact that multiplying two of these matrices with last column $(v_1, 1)$ and $(v_2, 1)$ respectively yields another matrix of this form whose last column is given by $(v_1 + v_2, 1)$.

To show (2) we can write

$$\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A = \left[\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbb{1}_n & -m^\top \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} C^{-1} & 0 \\ \hline 0 & 1 \end{array}\right)\right] \cdot \left[\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot \begin{pmatrix} B \\ u \end{pmatrix}\right],$$

where the product of the first three matrices is

$$\left(\begin{array}{c|c} \mathbb{1}_n & -Cm^\top \\ \hline 0 & 1 \end{array}\right)$$

and the product of the last two matrices is $\begin{pmatrix} CB \\ u \end{pmatrix}$. Note that multiplying a unimodular matrix by an invertible matrix on either side yields again a unimodular matrix. Thus, $Cm^\top$ is the corresponding sequence for the block

$$\left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A.$$

For (3) note that multiplying the matrix $\begin{pmatrix} B \\ u \end{pmatrix}$ on the right by D yields a matrix $\begin{pmatrix} BD \\ u' \end{pmatrix}$. As noted in part (2), the matrix BD is also unimodular, so m is also the sequence to show that the block AD is matrix reducible. ☐

Proof of Proposition 2.9. Let us write the unimodular $n \times k$-block as

$$A = \begin{pmatrix} r_{1,1} & \dots & r_{1,k} \\ \vdots & & \vdots \\ r_{n,1} & \dots & r_{n,k} \\ f_1 & \dots & f_k \end{pmatrix}.$$

The proof is by induction on k.

Let $k = 1$. Since the block A is unimodular, there is a left inverse

$$A_L := ((r_1')^\top, \ldots, (r_n')^\top, (m')^\top)$$

of A for vectors $r_i' \in R^k$ and $m' \in M^k$. Hence, the sequence $(r_{1,1}, \ldots, r_{n,1}, f_1(m_1')) \in R^{n+1}$ is unimodular by construction and since $n + 1 > \operatorname{sr}(R)$ there are elements $v_1, \ldots, v_n \in R$ such that the sequence

$$(r_{1,1} + v_1 f_1(m_1'), \ldots, r_{n,1} + v_n f_1(m_1'))$$

is unimodular. Defining $m_i := m' \cdot \overline{v}_i$ then yields the base case.

Let us assume that the statement is true for $k - 1$ and consider the case $k > 1$. Since A is a unimodular block, in particular the first column $(r_1, f_1)^\top$ is unimodular having a left inverse $(r'_{1,1}, \ldots, r'_{1,n}, m_1')$ which is the first row of the left inverse A_L of A. Hence, the sequence $(r_{1,1}, \ldots, r_{n,1}, f_1(m_1'))$ is unimodular. By assumption we have $n + 1 > \operatorname{sr}(R)$, so there is a vector $v := (v_1, \ldots, v_n) \in R^n$ such that the sequence

$$r_1' := r_{1,1} + v_1 f_1(m_1'), \ldots, r_{n,1} + v_n f_1(m_1') \in R^n$$

is unimodular. In fact, as $k > 1$, we have $n > \operatorname{sr}(R)$, and hence, by Proposition 1.6 there is an $C \in \mathrm{GL}_n(R)$ such that $Cr_1' = (1, 0, \ldots, 0)$. Consider the block

$$A_1 := \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbb{1}_n & (m_1'\overline{v}_1, \ldots, m_1'\overline{v}_n)^\top \\ \hline 0 & 1 \end{array}\right) \cdot A.$$

Then A_1 is of the form

$$\left(\begin{array}{c|c} 1 & u' \\ \hline 0 & \\ \vdots & A' \\ 0 & \\ f_1 & \end{array}\right)$$

for an $(n - 1) \times (k - 1)$-block A' for M. Now, by Proposition 2.10 the block A is matrix reducible if and only if the block A_1 is matrix reducible. Proposition 2.10 also implies that this is equivalent to the block

$$A_2 := A_1 \cdot \left(\begin{array}{c|c} 1 & -u' \\ \hline 0 & \mathbb{1}_{k-1} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \\ \vdots & A'' \\ 0 & \\ f_1 & \end{array}\right)$$

being matrix reducible. Therefore, it is enough to show that the block A_2 is matrix reducible. Since the block A is unimodular, so is A_2 as remarked above. This implies that the block A''

is unimodular as well. Hence, by the induction hypothesis there is a vector $m \in M^{n-1}$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_{n-1} & m^\top \\ \hline 0 & 1 \end{array}\right) \cdot A'' = \begin{pmatrix} \tilde{B} \\ \tilde{u} \end{pmatrix},$$

where the matrix $\tilde{B}$ is unimodular and $\tilde{u}$ is the last row of A''. Thus,

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \mathbb{1}_{n-1} & m^\top \\ \hline 0 & 0 & 1 \end{array}\right) \cdot A_2 = \begin{pmatrix} 1 & 0 \\ * & \tilde{B} \\ * & \tilde{u} \end{pmatrix},$$

where the matrix $\begin{pmatrix} 1 & 0 \\ * & \tilde{B} \end{pmatrix}$ is unimodular since $\tilde{B}$ is unimodular. $\square$

The next proposition is an extension of [**48**, Thm. 1].

PROPOSITION 2.11. *Let $k + \mathrm{sr}(R) = n + 1$ and $l > 0$. For any unimodular $(n + l) \times k$-block A there is a vector $m \in M^n$ such that*

$$\left(\begin{array}{c|c|c} \mathbb{1}_n & 0 & m^\top \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the $(n + l) \times k$-matrix B is unimodular and u is the last row of the block A.

PROOF. Since A is a unimodular $(n+l)\times k$-block, by Proposition 2.9 there is an element $\tilde{m} \in M^{n+l}$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_{n+l} & \tilde{m}^\top \\ \hline 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B_1 \\ u_1 \end{pmatrix},$$

where the $(n + l) \times k$-matrix B_1 is unimodular and $u_1 = u$ is the last row of the block A. Since $l > 0$ and $n+l-k \geq \mathrm{sr}(R)$ we can now apply the condition (S^k_{n+l-k}) to the unimodular matrix B_1 to get an element $v \in R^{n+l-1}$ such that

$$\left(\begin{array}{c|c} \mathbb{1}_{n+l-1} & v^\top \\ \hline 0 & 1 \end{array}\right) \cdot B_1 = \begin{pmatrix} B_2 \\ u_2 \end{pmatrix},$$

where the $(n + l - 1) \times k$-matrix B_2 is unimodular and u_2 is the last row of the matrix B_1. Together we get

$$\left(\begin{array}{c|c|c} \mathbb{1}_{n+l-1} & v^\top & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbb{1}_{n+l} & \tilde{m}^\top \\ \hline 0 & 1 \end{array}\right) \cdot A = \begin{pmatrix} B_2 \\ u_2 \\ u_1 \end{pmatrix}.$$

Notice that the product of the first two matrices can be written in the form

$$\left(\begin{array}{c|c|c} \mathbb{1}_{n+l-1} & * & * \\ \hline 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array}\right),$$

where the last column has entries in the module M and the rest of the matrix has entries in the ring R. Iterating this yields a matrix

$$C := \left(\begin{array}{c|ccc|c} \mathbb{1}_n & & * & & * \\ \hline & 1 & * & * & \\ 0 & 0 & \ddots & * & * \\ & 0 & 0 & 1 & \\ \hline 0 & & 0 & & 1 \end{array}\right)$$

and $C \cdot A$ is a matrix of the form $\begin{pmatrix} B' \\ B'' \end{pmatrix}$, where B' is an $n \times k$-matrix and B'' is an $l \times k$-block. The matrix B' is unimodular by construction. Note that row operations involving only the rows of B'' do not change the matrix B'. Hence, we can change the above matrix C to be of the form

$$C' := \left(\begin{array}{c|c|c} \mathbb{1}_n & * & * \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

Again, $C' \cdot A$ is a matrix of the form $\begin{pmatrix} B' \\ \tilde{B}'' \end{pmatrix}$, where B' is the same matrix as above and hence unimodular. Instead of dividing this matrix into the first n and the last $l+1$ rows, let us now divide it into the first $n+l$ and the last row, written as $\begin{pmatrix} B''' \\ u \end{pmatrix}$, where u is by construction the last row of the matrix A. Since the matrix B' is unimodular, so is the matrix B'''. Row operations on B''' correspond to multiplying B''' on the left by invertible matrices, which keeps the matrix unimodular. Hence, we can perform row operations on C' using all but the last row to get a matrix of the form

$$\left(\begin{array}{c|c|c} \mathbb{1}_n & 0 & m^\top \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array}\right).$$

This finishes the proof. □

We immediately get the following corollary.

Corollary 2.12. *Let $k+\mathrm{sr}(R)=n+1$ and $l>0$. For any unimodular $(n+l)\times k$-block A there is a vector $m\in M^n$ and an $n\times l$-matrix Q with entries in R such that*

$$\left(\begin{array}{c|c|c} \mathbb{1}_n & Q & m^\top \\ \hline 0 & \mathbb{1}_l & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\cdot A=\begin{pmatrix} B_1 \\ B_2 \\ u \end{pmatrix},$$

where the $n\times k$-matrix B_1 is unimodular and $\begin{pmatrix} B_2 \\ u \end{pmatrix}$ are the last $l+1$ rows of the block A.

Proof. The matrix C' constructed in the proof of Proposition 2.11 is the required matrix. $\square$

2.1.2. Orthogonal Transvections. Following [**34**, Ch. 7] let e and u be elements in the quadratic module (M,λ,μ) satisfying $\mu(e)=0$, $\lambda(e,e)=0$, and $\lambda(e,u)=0$. For $x\in\mu(u)$ we define an automorphism $\tau(e,u,x)$ of the quadratic module M by

$$\tau(e,u,x)(v)=v+u\lambda(e,v)-e\bar{\varepsilon}\lambda(u,v)-e\bar{\varepsilon}x\lambda(e,v).$$

If e is λ-unimodular, the map $\tau(e,u,x)$ is called an *orthogonal transvection.*

Note that compared to [**34**, Ch. 7] we have added $\lambda(e,e)=0$ to the assumptions. Adapting [**8**, page 91/35, Prop. 5.2] we see that orthogonal transvections are an element of the unitary group $U(M)$. Without this extra assumption, we do not see how to show that this formula defines a unitary transformation. Bass defines a quadratic module as an equivalence class of sesquilinear forms, from which one does extract μ and λ satisfying our axioms of quadratic modules, and in this case one can prove that $\mu(e)=0$ implies $\lambda(e,e)=0$. Our definition of quadratic modules is more general, and hence, we have to further assume this.

We find the above formula difficult to motivate, but in the symplectic case over $\mathbb{Z}$, i.e. $\varepsilon=-1$ and the involution is the identity map, for an orientable surface Σ, the group $H_1(\Sigma;\mathbb{Z})$ is then a $(-1,\mathbb{Z})$-quadratic module. Under this correspondence, a Dehn twist around a simple closed curve representing the homology class $\gamma\in H_1(\Sigma;\mathbb{Z})$ is given by the orthogonal transvection $\tau(\gamma,\gamma,-1)$.

The following is the last ingredient required to prove Lemma 2.5.

Proposition 2.13. ([**43**, Prop. 5.13]) *Let M be a quadratic module and $M\oplus H\cong H^{g+1}$. If $g\geq\mathrm{usr}(R)$ then $M\cong H^g$.*

Proof of Lemma 2.5. In the following we adapt the ideas of Step 1 in the proof of [**34**, Thm. 8.1]. Let $(v_1,\ldots,v_k)$ be a λ-unimodular sequence in the quadratic module $P\oplus H^g$ with $g\geq\mathrm{usr}(R)+k$. Recall that we want to show that there is an automorphism $\phi\in U(P\oplus H^g)$ such that $\phi(v_1,\ldots,v_k)\subseteq P\oplus H^k$ and the projection of $\phi(v_1,\ldots,v_k)$ to the hyperbolic H^k

is λ-unimodular. Denoting the basis of H^g by $e_1, f_1, \ldots, e_g, f_g$ we can write

$$v_i = p_i + \sum_{l=1}^{g} e_l A_l^i + \sum_{l=1}^{g} f_l B_l^i \quad \text{for } p_i \in P \text{ and } A_l^i, B_l^i \in R.$$

As the sequence $(v_1, \ldots, v_k)$ is λ-unimodular, there are

$$w_i = q_i + \sum_{l=1}^{g} e_l a_l^i + \sum_{l=1}^{g} f_l b_l^i \quad \text{for } q_i \in P \text{ and } a_l^i, b_l^i \in R$$

satisfying

$$\begin{aligned}
\delta_{i,j} &= \lambda(w_i, v_j) = (q_i, a_1^i, b_1^i, \ldots, b_g^i) \left(\begin{array}{c|cc|c} \lambda|_P & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & \cdots \\ 0 & \varepsilon & 0 & \\ \hline \vdots & & \vdots & \ddots \end{array}\right) \begin{pmatrix} p_j \\ A_1^j \\ B_1^j \\ \vdots \\ B_g^j \end{pmatrix} \\
&= \lambda(q_i, p_j) + \sum_{l=1}^{g} a_l^i B_l^j + \varepsilon \sum_{l=1}^{g} b_l^i A_l^j
\end{aligned}$$

Note that a sequence $(v_1 \ldots, v_k)$ is λ-unimodular if and only if its associated block

$$A_{(v_1,\ldots,v_k)} := \begin{pmatrix} A_1^1 & \ldots & A_1^k \\ B_1^1 & \ldots & B_1^k \\ \vdots & & \vdots \\ B_g^1 & \ldots & B_g^k \\ \lambda(-, p_1) & \ldots & \lambda(-, p_k) \end{pmatrix}$$

is unimodular. Since $g - k + 1 > \mathrm{sr}(R)$ by Proposition 2.11 there are $\tilde{p}_1, \ldots, \tilde{p}_g \in P$ such that

$$\left(\begin{array}{c|c} & \tilde{p}_1 \\ & 0 \\ \mathbb{1}_{2g} & \vdots \\ & \tilde{p}_g \\ & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A_{(v_1,\ldots,v_k)} = \begin{pmatrix} B \\ u \end{pmatrix},$$

where the matrix B is unimodular. Strictly speaking this is not of the form of Proposition 2.11 but we can reorder the basis of the matrix part to get the above statement. Now, for $y_i \in \mu(\tilde{p}_i)$ consider the following composition of transvections

$$\tilde{\phi} := \tau(e_g, -\tilde{p}_g\bar{\varepsilon}, y_g) \circ \ldots \circ \tau(e_1, -\tilde{p}_1\bar{\varepsilon}, y_1).$$

Then, by induction we have

$$\tilde{\phi}(v_i) = v_i + \sum_{j=1}^{g} \left(-\tilde{p}_j \bar{\varepsilon} B_j^i + e_j \lambda(\tilde{p}_j, p_i) - \left(e_j \bar{\varepsilon} \sum_{l=1}^{j-1} \lambda(\tilde{p}_j, \tilde{p}_l) B_l^i \right) - e_j y_j \bar{\varepsilon} B_j^i \right),$$

where we have used the identity $\bar{\varepsilon}\varepsilon = 1$ several times.

Next, we show that the projection of $\tilde{\phi}(v_1, \ldots, v_k)$ to H^g is λ-unimodular. For this we explain how the block $A_{\tilde{\phi}(v_1,\ldots,v_k)}$ is obtained from the block $A_{(v_1,\ldots,v_k)}$ and show that the matrix part of the block $A_{\tilde{\phi}(v_1,\ldots,v_k)}$ is unimodular. Adding $\sum_{j=1}^{g} -\tilde{p}_j\bar{\varepsilon}B_j^i$ to v_i for each i corresponds to changing only the last row of the block $A_{(v_1,\ldots,v_k)}$ and so doesn't affect its matrix part. Adding $\sum_{j=1}^{g} e_j\lambda(\tilde{p}_j, p_i)$ to v_i for $1 \leq i \leq k$ corresponds to the following multiplication on the level of blocks.

$$\left(\begin{array}{c|c} & \tilde{p}_1 \\ & 0 \\ \mathbb{1}_{2g} & \vdots \\ & \tilde{p}_g \\ & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A_{(v_1,\ldots,v_k)}$$

As we have seen above this is $\begin{pmatrix} B \\ u \end{pmatrix}$ with B a unimodular matrix. Adding the terms

$$\sum_{j=1}^{g} -e_j\bar{\varepsilon}\sum_{l=1}^{j-1}\lambda(\tilde{p}_j, \tilde{p}_l)B_l^i \text{ and } \sum_{j=1}^{g} -e_j y_j \bar{\varepsilon}B_j^i$$

corresponds to multiplying the block $A_{(v_1,\ldots,v_k)}$ from the left by matrices of the forms

$$\left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & 1 \end{array}\right) \text{ and } \left(\begin{array}{c|c} C_2 & 0 \\ \hline 0 & 1 \end{array}\right)$$

respectively, where C_1 is a lower triangular matrix with 1's on the diagonal and C_2 is an upper triangular matrix with 1's on the diagonal. In particular, both C_1 and C_2 are invertible. Note that all of the three above steps only change the coefficient of the e_i, by adding on multiples of the coefficients of the f_i and the last row. Therefore, applying $\tilde{\phi}$ to $(v_1, \ldots, v_k)$ corresponds to multiplying $A_{(v_1,\ldots,v_k)}$ from the left by the product of the above matrices:

$$\left(\begin{array}{c|c} C_2 & 0 \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} & \tilde{p}_1 \\ & 0 \\ \mathbb{1}_{2g} & \vdots \\ & \tilde{p}_g \\ & 0 \\ \hline 0 & 1 \end{array}\right) \cdot A_{(v_1,\ldots,v_k)} = \begin{pmatrix} C_2C_1B \\ u \end{pmatrix}.$$

Since B is unimodular so is C_2C_1B. This corresponds to the projection of $\tilde{\phi}(v_1, \ldots, v_k)$ to H^g which is therefore also unimodular.

Now, applying [**39**, Lemma 6.6] yields a hyperbolic basis $\{\tilde{e}_1, \tilde{f}_1, \ldots, \tilde{e}_g, \tilde{f}_g\}$ of H^g such that

$$\tilde{\phi}(v_1)|_{H^g}, \ldots, \tilde{\phi}(v_k)|_{H^g} \in \langle \tilde{e}_1, \tilde{f}_1, \ldots, \tilde{e}_k, \tilde{f}_k \rangle =: U.$$

Note that this does not need to be the standard basis of H^g, hence, we need to find an automorphism ψ of H^g that sends the above basis $\tilde{e}_1, \tilde{f}_1, \dots, \tilde{e}_g, \tilde{f}_g$ to the standard basis in H^g. Then $\phi := (\mathbb{1}_P \oplus \psi) \circ \tilde{\phi}$ will be the required automorphism.

Let V denote an orthogonal complement of U in H^g, i.e. $U \oplus V \cong H^g$. By assumption we have $g - k \geq \mathrm{usr}(R)$ and, hence, Proposition 2.13 implies $V \cong H^{g-k}$. Let ψ denote the automorphism of H^g which sends U to the first k copies of H in H^g, and V to the last $g - k$ copies. Using the above definition of ϕ we then have $\phi(v_1, \dots, v_k) \subseteq P \oplus H^k$, and the projection of $\phi(v_1, \dots, v_k)$ to H^k is unimodular. □

We get the following version of [**34**, Thm. 8.1], but phrased in terms of the unitary stable rank instead of the absolute stable rank. Note that the condition $g(M) \geq \mathrm{asr}(R) + 2$ used in the statement of [**34**, Thm. 8.1] in particular implies $g(M) \geq \mathrm{usr}(R) + 1$, making [**34**, Thm. 8.1] a special case of the following corollary.

Corollary 2.14. *Let (M, λ, μ) be a quadratic module satisfying $g(M) \geq \mathrm{usr}(R) + 1$ and $r \in R$. Then $U(M)$ acts transitively on the set of all λ-unimodular elements v in M satisfying $\mu(v) = r + \Lambda$.*

Proof. For $g = g(M)$ there is a quadratic module P such that $M \cong P \oplus H^g$. We write $e_1, f_1, \dots, e_g, f_g$ for the basis of H^g. We show that we can map a λ-unimodular element v with $\mu(v) = r + \Lambda$ to $e_1 + f_1 r$.

By Lemma 2.5 there is an automorphism $\phi \in U(P \oplus H^g)$ such that $\phi(v) \subseteq P \oplus H$ and the projection of $\phi(v)$ to the hyperbolic H is unimodular. Hence, by the transitivity condition (T_g) we can map the projection of $\phi(v)$ (considered in H^g) to $e_1 + f_1 b'$ mapped by μ to the same element as the projection of $\phi(v)$. Thus, we have mapped v to the element $p + e_1 + f_1 b'$ for some element $p \in P$. Applying the orthogonal transvection $\tau(f_1\varepsilon, -p, x)$ for some $x \in \mu(p)$ to the element $w = p + e_1 + f_1 b'$ we get

$$\begin{aligned}
& \tau(f_1\varepsilon, -p, x)(w) \\
= \quad & w - p\lambda(f_1\varepsilon, w) - f_1\varepsilon\bar{\varepsilon}\lambda(-p, w) - f_1\varepsilon\bar{\varepsilon}x\lambda(f_1\varepsilon, w) \\
= \quad & p + e_1 + f_1 b' - p\lambda(f_1\varepsilon, e_1) - f_1\lambda(-p, p) - f_1 x\lambda(f_1\varepsilon, e_1) \\
= \quad & p + e_1 + f_1 b' - p\lambda(f_1\varepsilon, e_1) - f_1\lambda(-p, p) - f_1 x\lambda(f_1\varepsilon, e_1) \\
= \quad & p + e_1 + f_1 b' - p + f_1\lambda(p, p) - f_1 x \\
= \quad & e_1 + f_1(b' + \lambda(p, p) - x)
\end{aligned}$$

where we have used $\lambda(f_1\varepsilon, e_1) = \bar{\varepsilon}\lambda(f_1, e_1) = \bar{\varepsilon}\varepsilon = 1$. Thus, the element $w = p + e_1 + f_1 b'$ gets mapped to $e_1 + f_1 b$, with $b = b' + \lambda(p, p) - x$, under the above transvection. We have

$$r + \Lambda = \mu(v) = \mu(e_1 + f_1 b) = b + \Lambda$$

and

$$\begin{aligned}\tau(f_1\varepsilon, 0, b-r)(e_1 + f_1 b) &= e_1 + f_1 b - f_1\varepsilon\overline{\varepsilon}(b-r)\lambda(f_1\varepsilon, e_1 + f_1 b)\\ &= e_1 + f_1 b - f_1(b-r)\lambda(f_1\varepsilon, e_1)\\ &= e_1 + f_1 r.\end{aligned}$$

□

By Lemma 2.3 this is a generalisation of [**24**, Prop. 3.3] which treats the special case of quadratic modules over the integers. Note that our bound is slightly better than the bound given in the special case.

Adapting the proof of [**34**, Cor. 8.3], using Corollary 2.14 instead of [**34**, Thm. 8.1] yields the following improvement to Proposition 2.13. Note that Step 6 of [**34**, Thm. 8.1] still works in our setting.

Corollary 2.15. *Let M and N be quadratic modules satisfying $M \oplus H \cong N \oplus H$. If $g(M) \geq \mathrm{usr}(R)$ then $M \cong N$.*

In contrast to Proposition 2.13, both M and N can be general quadratic modules and, in particular, both can be non-hyperbolic modules. As in the previous corollary, this bound is slightly better than the bound given in [**24**, Prop. 3.4] which only treats the case $R = \mathbb{Z}$.

2.2. Proof of Theorem 2.4

For the proof of Theorem 2.4 we follow a strategy similar to the proof of [**39**, Thm. 7.4]. As we have seen in Lemma 2.3, in the hyperbolic case every unimodular sequence is already λ-unimodular. In the case of general quadratic modules, however, a unimodular sequence of length 1, (v_1), need not be λ-unimodular and more generally, $(v_1, \ldots, v_k, u_1, \ldots, u_l)$ is not necessarily λ-unimodular, even if the individual sequences $(v_1, \ldots, v_k)$ and $(u_1, \ldots, u_l)$ are λ-unimodular. The following lemma, however, shows that in certain circumstances this implication is still valid.

Lemma 2.16. *Let $(v_1, \ldots, v_k) \in \mathcal{U}(M, \lambda)$ be a λ-unimodular sequence in M and let $w_1, \ldots, w_k \in M$ be such that $\lambda(w_i, v_j) = \delta_{i,j}$.*

(1) We have $M = \langle v_1, \ldots, v_k\rangle \oplus \langle w_1, \ldots, w_k\rangle^{\perp}$ as a direct sum of R-modules (i.e. the summands are not necessarily orthogonal with respect to λ).

(2) If $(u_1, \ldots, u_l) \in \mathcal{U}(M, \lambda)$ is a λ-unimodular sequence with $\lambda(w_i, u_j) = 0$ for all i, j then the sequence $(v_1, \ldots, v_k, u_1, \ldots, u_l)$ is λ-unimodular.

(3) Let $u_i = x_i + y_i$ for elements $x_i \in \langle v_1, \ldots, v_k\rangle$ and $y_i \in \langle w_1, \ldots, w_k\rangle^{\perp}$. Then $(v_1, \ldots, v_k, u_1, \ldots, u_l)$ is λ-unimodular if and only if $(v_1, \ldots, v_k, y_1, \ldots, y_l)$ is λ-unimodular.

Proof. For (1) consider the map

$$\bigoplus_{i=1}^{k} \lambda(w_i, -)\colon M \longrightarrow R^k$$

which sends v_i to the i-th basis vector in R^k. The v_i define a splitting, and hence

$$M = \langle v_1, \dots, v_k \rangle \oplus \operatorname{Ker}\left(\bigoplus_{i=1}^{k} \lambda(w_i, -)\right) = \langle v_1, \dots, v_k \rangle \oplus \langle w_1, \dots, w_k \rangle^{\perp}.$$

For (2) let $z_1, \dots, z_l \in M$ such that $\lambda(z_i, u_j) = \delta_{i,j}$. Replacing z_i by $z_i - \sum_{n=1}^{k} w_n \overline{\lambda(z_i, v_n)}$ shows that the sequence $(v_1, \dots, v_k, u_1, \dots, u_l)$ is λ-unimodular since we have

$$\lambda(w_i, v_j) = \delta_{i,j} \qquad \lambda\left(z_i - \sum_{n=1}^{k} w_n \overline{\lambda(z_i, v_n)}, v_j\right) = \lambda(z_i, v_j) - \lambda(z_i, v_j) = 0$$
$$\lambda(w_i, u_j) = 0 \qquad \lambda\left(z_i - \sum_{n=1}^{k} w_n \overline{\lambda(z_i, v_n)}, u_j\right) = \lambda(z_i, u_j) = \delta_{i,j}.$$

To prove (3) we first assume that the sequence $(v_1, \dots, v_k, y_1, \dots, y_l)$ is λ-unimodular. Hence, there are $w'_1, \dots, w'_k, z_1, \dots, z_l \in M$ such that

$$\lambda(w'_i, v_j) = \delta_{i,j} \qquad \lambda(z_i, v_j) = 0$$
$$\lambda(w'_i, y_j) = 0 \qquad \lambda(z_i, y_j) = \delta_{i,j}.$$

Note that since $y_i \in \langle w_1, \dots, w_k \rangle^{\perp}$ we could choose w'_i to be w_i for all i. Since we have $x_j \in \langle v_1, \dots, v_k \rangle$ and $\lambda(z_i, v_j) = 0$ for all i, j we have $\lambda(z_i, u_j) = \lambda(z_i, y_j) = \delta_{i,j}$. Replacing w'_i by $w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, u_n)}$ shows that the sequence $(v_1, \dots, v_k, u_1, \dots, u_l)$ is λ-unimodular:

$$\lambda\left(w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, u_n)}, v_j\right) = \lambda(w'_i, v_j) = \delta_{i,j} \qquad \lambda(z_i, v_j) = 0$$
$$\lambda\left(w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, u_n)}, u_j\right) = \lambda(w'_i, u_j) - \lambda(w'_i, u_j) = 0 \qquad \lambda(z_i, u_j) = \delta_{i,j}.$$

Now, assuming that the sequence $(v_1, \dots, v_k, u_1, \dots, u_l)$ is λ-unimodular we have elements $w'_1, \dots, w'_k, z_1, \dots, z_l \in M$ satisfying

$$\lambda(w'_i, v_j) = \delta_{i,j} \qquad \lambda(z_i, v_j) = 0$$
$$\lambda(w'_i, u_j) = 0 \qquad \lambda(z_i, u_j) = \delta_{i,j}.$$

As above we have $\lambda(z_i, y_j) = \lambda(z_i, u_j) = \delta_{i,j}$. Replacing w'_i by $w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, y_n)}$ yields

$$\lambda\left(w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, y_n)}, v_j\right) = \lambda(w'_i, v_j) = \delta_{i,j} \qquad \lambda(z_i, v_j) = 0$$
$$\lambda\left(w'_i - \sum_{n=1}^{l} z_n \overline{\lambda(w'_i, y_n)}, y_j\right) = \lambda(w'_i, y_j) - \lambda(w'_i, y_j) = 0 \qquad \lambda(z_i, y_j) = \delta_{i,j},$$

which shows that the sequence $(v_1, \dots, v_k, y_1, \dots, y_l)$ is λ-unimodular. □

To prove Theorem 2.4 we need an analogue of Theorem 1.4 for the complex of λ-unimodular sequences in a quadratic module. For this we use the following notation. Let $S \subseteq M$ be a subset of a quadratic module M. We write $\mathcal{I}(S, \mu)$ for the set of all elements $v \in S$ satisfying $\mu(v) = 0$.

THEOREM 2.17. *Let $M = P \oplus H^g$ and N be quadratic modules with $M \oplus H \subseteq N$.*

(1) $\mathcal{O}(\mathcal{I}(P \oplus \langle e_1, \ldots, e_g \rangle, \mu)) \cap \mathcal{U}(N, \lambda)$ *is* $(g - \operatorname{usr}(R) - 1)$*-connected,*

(2) $\mathcal{O}(\mathcal{I}(P \oplus \langle e_1, \ldots, e_g \rangle, \mu)) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k)}$ *is* $(g - \operatorname{usr}(R) - k - 1)$*-connected for every sequence* $(v_1, \ldots, v_k) \in \mathcal{U}(N, \lambda)$*.*

This is the natural generalisation of Theorem 1.4 to the case of quadratic modules. (Only considering N's of the form $M \oplus H^\infty$ is not sufficient for our proof of Lemma 2.21, see Remarks 2.22.) We can write N as $Q \oplus H^{g(M)} \oplus H^n$ for some $n \geq 1$, where $H^{g(M)}$ is the hyperbolic part of M and Q is some quadratic module. With this notation we have $P \subseteq Q \oplus H^{n-1}$, where H^{n-1} denotes the last $n-1$ copies of H in $H^n \subseteq N$. In particular, P is not necessarily contained in Q.

The proof of the above theorem is an adaptation of the proof of Theorem 1.4 for which we use the following results.

PROPOSITION 2.18. *Let N be a quadratic module with $H^k \subseteq N$ for $k \geq \operatorname{usr}(R)$. For a λ-unimodular element $v \in N$ there is an automorphism $\phi \in U(N)$ such that the projection of $\phi(v)$ to $H^k \subseteq N$ is λ-unimodular, and for every subset $S \subseteq (H^k)^\perp$ the automorphism ϕ fixes $S \oplus \langle e_1, \ldots, e_k \rangle$ as a set.*

Note that in comparison with Lemma 2.5 the bound for k in the above proposition is slightly lower than in the lemma. Hence, we get a weaker conclusion here, having to restrict to more copies of the hyperbolic module H to get λ-unimodularity, and we cannot guarantee that the image of v under ϕ lands outside certain copies of H.

Saying that the automorphism ϕ fixes $S \oplus \langle e_1, \ldots, e_k \rangle$ as a set for every $S \subseteq (H^k)^\perp$ is the same as saying that it is the identity on the associated graded for the filtration $0 \leq \langle e_1, \ldots, e_k \rangle \leq (H^k)^\perp \oplus \langle e_1, \ldots, e_k \rangle$. However, we prefer the above formulation as this is of the form we use later on.

PROOF. We adapt the ideas of the first part of the proof of Lemma 2.5. We can write $N = Q \oplus H^k$ for some quadratic module Q. Then

$$v = p + \sum_{i=1}^{k} e_i A_i + f_i B_i \quad \text{for } p \in Q \text{ and } A_i, B_i \in R.$$

As v is λ-unimodular, there is

$$w = q + \sum_{i=1}^{k} e_i a_i + f_i b_i \quad \text{for } q \in Q \text{ and } a_i, b_i \in R$$

satisfying

$$
\begin{aligned}
1 &= \lambda(w,v) = (q,a_1,b_1,\dots,b_k)\left(\begin{array}{c|cc|c} \lambda|_Q & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & \\ 0 & \varepsilon & 0 & \cdots \\ \hline \vdots & & \vdots & \ddots \end{array}\right)\begin{pmatrix} p \\ A_1 \\ B_1 \\ \vdots \\ B_k \end{pmatrix} \\
&= \lambda(q,p) + \sum_{i=1}^{k} a_i B_i + \varepsilon b_i A_i.
\end{aligned}
$$

Hence, using the notation from the proof of Lemma 2.5, the $2k \times 1$-block for Q associated to v

$$
A_v = \begin{pmatrix} A_1 \\ B_1 \\ \vdots \\ B_k \\ \lambda(-,p) \end{pmatrix}
$$

is unimodular. Since $k \geq \operatorname{usr}(R)$ by Proposition 2.11 there are $p_1,\dots,p_k \in Q$ such that for $m = (p_1,0,\dots,p_k,0)$ we get

$$
\left(\begin{array}{c|c} \mathbb{1}_{2k} & m^\top \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} A_1 \\ B_1 \\ \vdots \\ B_k \\ \lambda(-,p) \end{pmatrix} = \begin{pmatrix} b \\ u \end{pmatrix},
$$

where the vector $b \in H^k$ is unimodular. As in the proof of Lemma 2.5 this application of Proposition 2.11 involves reordering the basis of the matrix part. Using the correspondence between elements in quadratic modules and their associated blocks explained in the proof of Lemma 2.5, multiplication with the above matrix is the required automorphism ϕ. Note that the bottom entry of v stays fixed under ϕ and thus, for any $S \subseteq (H^k)^\perp = Q$ the automorphism ϕ fixes $S \oplus \langle e_1,\dots,e_g\rangle$ as a set. $\square$

LEMMA 2.19. *Let N be a quadratic module with $H^k \subseteq N$ for some k and $v \in N$ so that the projection of v to $H^{k-1} \oplus 0 \subseteq H^k \subseteq N$ is λ-unimodular. There are $w \in \langle e_k, f_k\rangle$, $u \in H^{k-1}$, and $x \in \mu(u)$ such that $\lambda(w,\tau(e_k,u,x)(v)) = 1$ and for every subset $S \subseteq (H^k)^\perp$ the transvection $\tau(e_k,u,x)$ fixes $S \oplus \langle e_1,\dots,e_k\rangle$ as a set.*

PROOF. Since the projection of v to H^{k-1} is λ-unimodular there is an element $z \in H^{k-1}$ such that $\lambda(z,v) = 1$. For $u := z(\overline{\lambda(f_k,v)-1})\overline{\varepsilon}$ and any $x \in \mu(u)$ we have

$$
\begin{aligned}
\tau(e_k,u,x)(v) &= v + u\lambda(e_k,v) - e_k\overline{\varepsilon}\lambda(u,v) - e_k\overline{\varepsilon}x\lambda(e_k,v) \\
&= v + u\lambda(e_k,v) + e_k(1-\lambda(f_k,v) - \overline{\varepsilon}x\lambda(e_k,v)).
\end{aligned}
$$

Since u is contained in H^{k-1} the second summand does not affect the coefficients of e_k and f_k. The third summand changes the coefficient of e_k to be $1-\overline{\varepsilon}x\lambda(e_k,v)$ and leaves all other coefficients fixed. Defining $w := e_k\overline{x}\varepsilon + f_k\varepsilon$ we get

$$\begin{aligned}\lambda(w,\tau(e_k,u,x)(v)) &= \lambda(e_k\overline{x}\varepsilon + f_k\varepsilon, e_k(1-\overline{\varepsilon}x\lambda(e_k,v)) + f_k\lambda(e_k,v))\\ &= \lambda(e_k\overline{x}\varepsilon, \lambda(e_k,v)f_k) + \lambda(f_k\varepsilon,(1-\overline{\varepsilon}x\lambda(e_k,v))e_k)\\ &= \overline{\varepsilon}x\lambda(e_k,v) + 1 - \overline{\varepsilon}x\lambda(e_k,v)\\ &= 1.\end{aligned}$$

Thus, choosing u, x, and w as above shows the claim since the constructed transvection fixes $S \oplus \langle e_1,\dots,e_k\rangle$ as a set for every subset $S \subseteq (H^k)^\perp$. □

Proof of Theorem 2.17. Analogous to the proof of Theorem 1.4 we will also show the following statements.

(a) $\mathcal{O}\Big(\mathcal{I}\big(P\oplus(\langle e_1,\dots,e_g\rangle \cup \langle e_1,\dots,e_g\rangle + e_{g+1}),\mu\big)\Big) \cap \mathcal{U}(N,\lambda)$ is $(g-\operatorname{usr}(R))$-connected.

(b) $\mathcal{O}\Big(\mathcal{I}\big(P\oplus(\langle e_1,\dots,e_g\rangle \cup \langle e_1,\dots,e_g\rangle + e_{g+1}),\mu\big)\Big) \cap \mathcal{U}(N,\lambda)_{(v_1,\dots,v_k)}$ is $(g-\operatorname{usr}(R)-k)$-connected for all $(v_1,\dots,v_k)$ in $\mathcal{U}(N,\lambda)$.

Here, we write $N = Q \oplus H^g \oplus H$ for some quadratic module Q and (e_{g+1}, f_{g+1}) for the basis of the last copy of the hyperbolic H in N.

The proof is by induction on g. Note that statements (1), (2), and (b) all hold for $g < \operatorname{usr}(R)$ so we can assume $g \geq \operatorname{usr}(R)$. Statement (a) holds for $g < \operatorname{usr}(R) - 1$ so we can assume $g \geq \operatorname{usr}(R) - 1$ when proving this statement. The structure of the proof is the same as in the proof of Theorem 1.4: We start by proving (b) which enables us to deduce (2). We will then prove statements (1) and (a) simultaneously by applying statement (2).

In the following we write $E_g = \langle e_1,\dots,e_g\rangle$.

Proof of (b). For $Y := P\oplus(E_g \cup (E_g + e_{g+1}))$ we write $F := \mathcal{O}(\mathcal{I}(Y,\mu)) \cap \mathcal{U}(N,\lambda)_{(v_1,\dots,v_k)}$. Let $d := g - \operatorname{usr}(R) - k$, so we have to show that F is d-connected.

For $g = \operatorname{usr}(R)$ the only case to consider is $k = 1$, where we have to show that F is non-empty. By Proposition 2.18 there is an automorphism $\phi \in U(N)$ such that the projection of $\phi(v_1)$ to $H^g \subseteq N$ is λ-unimodular and ϕ fixes Y as a set. Then the sequence $(\phi(v_1)|_{H^g}, e_{g+1})$ is λ-unimodular in N. In particular, there is an element $w_1 \in H^g$ such that $\lambda(w_1, \phi(v_1)|_{H_g}) = 1$ and $\lambda(w_1, e_{g+1}) = 0$. Now Lemma 2.16 (2) applied to $u_1 = e_{g+1}$ shows that the sequence $(\phi(v_1), e_{g+1})$ is λ-unimodular. Hence, the sequence $(v_1, \phi^{-1}(e_{g+1}))$ is also λ-unimodular. By construction we have $\phi^{-1}(e_{g+1}) \in Y$ and thus, F is non-empty as it contains the element $\phi^{-1}(e_{g+1})$.

Now consider the case $g > \operatorname{usr}(R)$. By Proposition 2.18 there is an automorphism ϕ of N such that the projection of $\phi(v_1)$ to H^{g-1} is λ-unimodular. Using Lemma 2.19 we get elements $u \in H^{g-1}$, $x \in \mu(u)$, and $w_1 \in \langle e_g, f_g\rangle$ such that $\lambda(w_1, \tau(e_g,u,x)(\phi(v_1))) = 1$. By construction, both ϕ and $\tau(e_g,u,x)$ fix $P \oplus (E_g \cup (E_g + e_{g+1}))$ as a set. Hence, the automorphism $\psi := \tau(e_g,u,x)\circ\phi$ defines an isomorphism

$$F = \mathcal{O}(\mathcal{I}(Y,\mu)) \cap \mathcal{U}(N,\lambda)_{(v_1,\dots,v_k)} \xrightarrow{\cong} \mathcal{O}(\mathcal{I}(Y,\mu)) \cap \mathcal{U}(N,\lambda)_{(\psi(v_1),\dots,\psi(v_k))} = \psi(F).$$

Writing $u_i := \psi(v_i)$ we have $\lambda(w_1, u_1) = 1$ with $w_1 \in \langle e_g, f_g\rangle$. This argument only works if $g > \operatorname{usr}(R)$, so we had to treat the case $g = \operatorname{usr}(R)$ separately.

We want to use Lemma 1.5 (1) to show that $\psi(F)$, and hence F, is d-connected. We define

$$X := \mathcal{I}(\{v \in Y \mid v|_{\langle e_g, f_g\rangle} = 0\}, \mu) = \mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)$$

and $u_i' := u_i - u_1\lambda(w_1, u_i)$ for $i > 1$, forcing $\lambda(w_1, u_i') = 0$. We have

$$\begin{aligned}
\mathcal{O}(X) \cap \psi(F) &= \mathcal{O}(X) \cap \mathcal{U}(N, \lambda)_{(u_1, \ldots, u_k)} \\
&= \mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(u_1, u_2', \ldots, u_k')} \\
&= \mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(u_2', \ldots, u_k')},
\end{aligned}$$

where the second equality holds as the span of $u_1, u_2', \ldots, u_k'$ is the same as the span of $u_1, u_2, \ldots, u_k$ and the third equality can be seen as follows: The inclusion $\subseteq$ of the second line into the third is obvious. For the other inclusion, $\supseteq$, let $(x_1, \ldots, x_l)$ be an element of $\mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(u_2', \ldots, u_k')}$. We have $\lambda(w_1, x_i) = 0$ since $w_1 \in \langle e_g, f_g\rangle$ and $\lambda(w_1, u_i') = 0$ by construction of the u_i'. Thus, by Lemma 2.16 (2) the sequence $(x_1, \ldots, x_l, u_1, u_2', \ldots, u_k')$ is λ-unimodular. In particular, the sequence $(x_1, \ldots, x_l)$ is an element of $\mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(u_1, u_2', \ldots, u_k')}$.

Thus, by the induction hypothesis the poset $\mathcal{O}(X) \cap \psi(F)$ is d-connected. Analogously, for $(w_1, \ldots, w_l) \in \psi(F) \setminus \mathcal{O}(X)$ we get

$$\begin{aligned}
\mathcal{O}(X) \cap \psi(F)_{(w_1, \ldots, w_l)} &= \mathcal{O}(X) \cap \mathcal{U}(N, \lambda)_{(u_1, \ldots, u_k, w_1, \ldots, w_l)} \\
&= \mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(u_2', \ldots, u_k', w_1', \ldots, w_l')},
\end{aligned}$$

which is $(d-l)$-connected by the induction hypothesis. Therefore, Lemma 1.5 (1) shows that $\psi(F)$ is d-connected. Since F and $\psi(F)$ are isomorphic, F is therefore also d-connected.

Proof of (2). Let us write

$$X := \mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_g)), \mu).$$

Then we have

$$\begin{aligned}
&\mathcal{O}(X) \cap (\mathcal{O}(\mathcal{I}(P \oplus E_g, \mu)) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k)}) \\
= \; &\mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_g)), \mu)) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k)},
\end{aligned}$$

which is $(d-1)$-connected by (b).

Similarly, for $(w_1, \ldots, w_l) \in \mathcal{O}(\mathcal{I}(P \oplus E_g, \mu)) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k)} \setminus \mathcal{O}(X)$ we have

$$\begin{aligned}
&\mathcal{O}(X) \cap (\mathcal{O}(\mathcal{I}(P \oplus E_g, \mu) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k)})_{(w_1, \ldots, w_l)} \\
= \; &\mathcal{O}(X) \cap \mathcal{U}(N, \lambda)_{(v_1, \ldots, v_k, w_1, \ldots, w_l)},
\end{aligned}$$

which is $(d-l-1)$-connected by the above. Hence, by Lemma 1.5 (1) the claim follows.

Proof of (1) and (a). Note that we now only assume $g \geq \operatorname{usr}(R) - 1$. By induction let us assume that statement (a) holds for $P \oplus (E_{g-1} \cup (E_{g-1} + e_g))$ and we want to show it for $P \oplus (E_g \cup (E_g + e_{g+1}))$. Before we finish the induction for (a) we will show that this already implies statement (1) for $P \oplus E_g$. For this let X be as in the proof of (2) and $d := g - \operatorname{usr}(R)$. Then

$$\begin{aligned} & \mathcal{O}(X) \cap (\mathcal{O}(\mathcal{I}(P \oplus E_g, \mu)) \cap \mathcal{U}(N, \lambda)) \\ = \quad & \mathcal{O}(\mathcal{I}(P \oplus (E_{g-1} \cup (E_{g-1} + e_g)), \mu)) \cap \mathcal{U}(N, \lambda) \end{aligned}$$

is $(d-1)$-connected by (a). The complex $\mathcal{O}(X) \cap (\mathcal{O}(\mathcal{I}(P \oplus E_g, \mu)) \cap \mathcal{U}(N, \lambda))_{(v_1,\ldots,v_m)}$ is $(d-m-1)$-connected as we have already shown in the proof of (2). Thus, the complex $\mathcal{O}(\mathcal{I}(P \oplus E_g, \mu)) \cap \mathcal{U}(N, \lambda)$ is $(g - \operatorname{usr}(R) - 1)$-connected by Lemma 1.5 (1) which proves statement (1).

To prove (a) we will apply Lemma 1.5 (2) for $X = \mathcal{I}(P \oplus E_g, \mu)$ and $y_0 = e_{g+1}$. Consider

$$(v_1, \ldots, v_k) \in \mathcal{O}(\mathcal{I}(P \oplus (E_g \cup (E_g + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda) \setminus \mathcal{O}(X).$$

Without loss of generality we may suppose that $v_1 \notin X$, as otherwise we can permute the v_i. By definition of X the coefficient of the e_{g+1}-coordinate of v_1 is therefore 1. Using Lemma 2.16 (2) as in part (b) above we have

$$\mathcal{O}(X) \cap \mathcal{O}(\mathcal{I}(P \oplus (E_g \cup (E_g + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)_{(v_1,\ldots,v_k)} = \mathcal{O}(X) \cap \mathcal{U}(N, \lambda)_{(v'_2,\ldots,v'_k)},$$

where $v'_i := v_i - v_1\lambda(f_{g+1}, v_i)$ is chosen so that the coefficient of the e_{g+1}-coordinate of v'_i is 0 for all $i > 1$. This is $(d-k)$-connected by (1) for $k = 1$ and by (2) for $k \geq 2$. By construction we have

$$\begin{aligned} & \mathcal{O}(X) \cap \mathcal{O}(\mathcal{I}(P \oplus (E_g \cup (E_g + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda) \\ \subseteq \; & (\mathcal{O}(\mathcal{I}(P \oplus (E_g \cup (E_g + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda))_{(e_{g+1})} \end{aligned}$$

and thus, Lemma 1.5 (2) implies that $\mathcal{O}(\mathcal{I}(P \oplus (E_g \cup (E_g + e_{g+1})), \mu)) \cap \mathcal{U}(N, \lambda)$ is $(g - \operatorname{usr}(R))$-connected which proves (a).

Note that when showing statement (a) for $P \oplus (E_g \cup (E_g + e_{g+1}))$ we only used statement (1) for $P \oplus E_g$ which follows from (a) for $P \oplus (E_{g-1} \cup (E_{g-1} + e_g))$, so this is indeed a valid induction to show both statements (1) and (a). □

In the following we write $\mathcal{U}(M, \lambda, \mu) := \mathcal{O}(\mathcal{I}(M, \mu)) \cap \mathcal{U}(M, \lambda)$.

Corollary 2.20. *Let M and N be quadratic modules with $M \oplus H \subseteq N$.*

(1) $\mathcal{O}(M) \cap \mathcal{U}(N, \lambda, \mu)$ is $(g(M) - \operatorname{usr}(R) - 1)$-connected,
(2) $\mathcal{O}(M) \cap \mathcal{U}(N, \lambda, \mu)_v$ is $(g(M) - \operatorname{usr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}(N, \lambda, \mu)$,
(3) $\mathcal{O}(M) \cap \mathcal{U}(N, \lambda, \mu) \cap \mathcal{U}(N, \lambda)_v$ is $(g(M) - \operatorname{usr}(R) - |v| - 1)$-connected for every $v \in \mathcal{U}(N, \lambda)$.

For the special case where the quadratic module M is a direct sum of hyperbolic modules H^n, Corollary 2.20 has been proven by Mirzaii–van der Kallen in [**39**, Lemma 6.8].

Proof. We write $g = g(M)$ and $M = P \oplus H^g$.

For (1) let $W := \mathcal{I}(P \oplus \langle e_1, \ldots, e_g \rangle, \mu)$ and $F := \mathcal{O}(M) \cap \mathcal{U}(N, \lambda, \mu)$. Then we have

$$\mathcal{O}(W) \cap F = \mathcal{O}(W) \cap \mathcal{U}(N, \lambda) \text{ and } \mathcal{O}(W) \cap F_u = \mathcal{O}(W) \cap \mathcal{U}(N, \lambda)_u,$$

for every $u \in \mathcal{U}(M, \lambda, \mu)$. Thus, by Theorem 2.17 the poset $\mathcal{O}(W) \cap F$ is $(g - \operatorname{usr}(R) - 1)$-connected and $\mathcal{O}(W) \cap F_u$ is $(g - \operatorname{usr}(R) - |u| - 1)$-connected. Now, by Lemma 1.5 (1) the poset F is $(g - \operatorname{usr}(R) - 1)$-connected.

To show (3) we choose W as above and F as the complex $\mathcal{O}(M) \cap \mathcal{U}(N, \lambda, \mu) \cap \mathcal{U}(N, \lambda)_v$. As before, using Lemma 1.5 (1) yields the claim. Note that statement (2) is a special case of statement (3). □

LEMMA 2.21. *The poset $\mathcal{O}(\langle v_1, \ldots, v_k\rangle^{\perp}) \cap \mathcal{U}(M, \lambda, \mu)_{(v_1,\ldots,v_k)}$ is $(g(M) - \operatorname{usr}(R) - k - 1)$-connected for $(v_1, \ldots, v_k) \in \mathcal{U}(M, \lambda, \mu)$, where $\perp$ denotes the orthogonal complement with respect to λ.*

For the special case where M is a sum of hyperbolic modules H^n this has been done by Mirzaii–van der Kallen in [**39**, Lemma 6.9].

PROOF. Let $g = g(M)$ and $M = P \oplus H^g$. By Lemma 2.5 we can assume without loss of generality that $v_1, \ldots, v_k \in P \oplus H^k$ and the projection to the hyperbolic H^k is λ-unimodular. In particular, there are $w_1, \ldots, w_k \in H^k$ such that $\lambda(w_i, v_j) = \delta_{i,j}$. Defining

$$W := \mathcal{I}(\langle v_1, \ldots, v_k, w_1, \ldots, w_k\rangle^{\perp}, \mu) \text{ and } F := \mathcal{O}(\langle v_1, \ldots, v_k\rangle^{\perp}) \cap \mathcal{U}(M, \lambda, \mu)_{(v_1,\ldots,v_k)}$$

we have

$$\mathcal{O}(W) \cap F = \mathcal{O}(W) \cap \mathcal{U}(M, \lambda, \mu)_{(v_1,\ldots,v_k)} = \mathcal{O}(W) \cap \mathcal{U}(M, \lambda, \mu),$$

where the second equality holds by Lemma 2.16 (2) as $W \subseteq \langle w_1, \ldots, w_k\rangle^{\perp}$. By construction we have $H^{g-k} \subseteq W$. Hence, $\mathcal{O}(W) \cap F$ is $(g - k - \operatorname{usr}(R) - 1)$-connected by Lemma 2.20 (1). By Lemma 2.16 (1) we can write $M = \langle v_1, \ldots, v_k\rangle \oplus \langle w_1, \ldots, w_k\rangle^{\perp}$, where we mean a direct sum of R-modules. Consider $(u_1, \ldots, u_l) \in F \setminus \mathcal{O}(W)$. We can write $u_i = x_i + y_i$ for $x_i \in \langle v_1, \ldots, v_k\rangle$ and $y_i \in \langle w_1, \ldots, w_k\rangle^{\perp}$. Note that $(y_1, \ldots, y_l)$ is in $\mathcal{U}(M, \lambda)$ but not necessarily in $\mathcal{U}(M, \lambda, \mu)$. Using Lemma 2.16 (2) and (3) we have

$$\mathcal{O}(W) \cap F_{(u_1,\ldots,u_l)} = \mathcal{O}(W) \cap \mathcal{U}(M, \lambda, \mu) \cap \mathcal{U}(M, \lambda)_{(y_1,\ldots,y_l)}$$

which is $(g - k - \operatorname{usr}(R) - l - 1)$-connected by Lemma 2.20 (3). Using Lemma 1.5 (1) now finishes the proof. □

REMARKS 2.22.

(1) We could apply Corollary 2.20 (2) directly to $\mathcal{O}(\langle v_1, \ldots, v_k\rangle^\perp) \cap \mathcal{U}(M, \lambda, \mu)_{(v_1,\ldots,v_k)}$ using that $H^{g(M)-k} \subseteq \langle v_1, \ldots, v_k\rangle^\perp$. However, this would only imply that the complex is $(g(M) - \mathrm{usr}(R) - 2k - 1)$-connected.

(2) In the proof of Lemma 2.21 we cannot assume that the y_i's are contained in $\langle v_1, \ldots, v_k, w_1, \ldots, w_k\rangle^\perp \oplus H^\infty$. Hence, we need Theorem 2.17 in the generality it is stated.

Following [**39**, Ch. 7], for $1 \leq k \leq n$, let $\mathcal{IU}(M, k)$ and $\mathcal{HU}(M, k)$ be the set of all elements of length k of $\mathcal{IU}(M)$ and $\mathcal{HU}(M)$ respectively.

LEMMA 2.23. *If $g(M) \geq \mathrm{usr}(R) + k$ then the group $U(M)$ acts transitively on the sets $\mathcal{IU}(M, k)$ and $\mathcal{HU}(M, k)$.*

The case of hyperbolic modules has already been shown in [**39**, Lemma 7.1].

PROOF. The proof is by induction on k. The case $\mathcal{HU}(M, 1)$ follows by adapting the proof of [**34**, Cor. 8.3], using Corollary 2.14 instead of [**34**, Thm. 8.1]. Note that Step 6 of [**34**, Thm. 8.1] still works in our setting. For $\mathcal{HU}(M, k)$ this is an easy induction. For $\mathcal{IU}(M, k)$ note that for every isotropic λ-unimodular sequence $(x_1, \ldots, x_k)$ there is a hyperbolic complement $(y_1, \ldots, y_k)$. Hence, the statement for $\mathcal{IU}(M, k)$ follows from the one for $\mathcal{HU}(M, k)$. $\square$

Let V be a set and $F \subseteq \mathcal{O}(V)$. For a non-empty set S we define the poset $F\langle S\rangle$ as

$$F\langle S\rangle := \{((v_1, s_1), \ldots, (v_k, s_k)) \in \mathcal{O}(V \times S) \mid (v_1, \ldots, v_k) \in F\}.$$

LEMMA 2.24. *Let $g(M) \geq \mathrm{usr}(R) + k$. For $((v_1, w_1), \ldots, (v_k, w_k)) \in \mathcal{HU}(M)$ we define $V := \langle v_1, \ldots, v_k\rangle$, $W := \langle w_1, \ldots, w_k\rangle$, and $Y := V^\perp \cap W^\perp$. Then*

(1) $\mathcal{IU}(M)_{(v_1,\ldots,v_k)} \cong \mathcal{IU}(Y)\langle V\rangle$,

(2) $\mathcal{HU}(M) \cap \mathcal{MU}(M)_{((v_1,0),\ldots,(v_k,0))} \cong \mathcal{HU}(X)\langle V \times V\rangle$,

(3) $\mathcal{HU}(M)_{((v_1,w_1),\ldots,(v_k,w_k))} \cong \mathcal{HU}(Y)$.

For the case of hyperbolic modules this has been done in [**39**, Lemma 7.2].

PROOF. We follow the proofs of [**16**, Lemma 3.4] and [**16**, Thm. 3.2].

For (1) note that $\mathcal{IU}(M)_{(v_1,\ldots,v_k)} \subseteq \mathcal{O}(V^\perp)$. Let $(u_1, \ldots, u_l) \in \mathcal{O}(V^\perp)$. We have $V^\perp = V \oplus Y$ by Lemma 2.16 (1), and therefore, $u_i = x_i + y_i$ for some $x_i \in V$ and $y_i \in Y$. By Lemma 2.16 (3) the sequence $(u_1, \ldots, u_l, v_1, \ldots, v_k)$ is λ-unimodular if and only if the sequence $(y_1, \ldots, y_l, v_1, \ldots, v_k)$ is λ-unimodular, which holds if and only if the sequence $(y_1, \ldots, y_l)$ is λ-unimodular by Lemma 2.16 (2). Furthermore, we have $\mu(u_i) = \mu(y_i)$ and $\lambda(u_i, u_j) = \lambda(y_i, y_j)$ since $(v_1, \ldots, v_k) \in \mathcal{IU}(M)$. Therefore, $\langle u_1, \ldots, u_l, v_1, \ldots, v_k\rangle$ is isotropic if and only if $\langle u_1, \ldots, u_l\rangle$ is isotropic and we get an isomorphism

$$\begin{array}{rcl} \mathcal{IU}(M)_{(v_1,\ldots,v_k)} & \longrightarrow & \mathcal{IU}(Y)\langle V\rangle \\ (u_1, \ldots, u_l) & \longmapsto & ((y_1, x_1), \ldots, (y_l, x_l)). \end{array}$$

A similar argument to the above for $\mathcal{HU}(M) \cap \mathcal{MU}(M)_{((v_1,0),\ldots,(v_k,0))} \subseteq \mathcal{O}(V^\perp \times V^\perp)$ shows (2). Statement (3) holds by construction of Y. $\square$

The proof of [**39**, Thm. 7.4] uses the connectivity of the poset of isotropic λ-unimodular sequences in the hyperbolic module H^n, $\mathcal{IU}(H^n)$, given in [**39**, Thm. 7.3]. The following result is the analogous statement for general quadratic modules.

THEOREM 2.25. *The poset $\mathcal{IU}(M)$ is $\lfloor \frac{g(M)-\mathrm{usr}(R)-2}{2} \rfloor$-connected and for every element $x \in \mathcal{IU}(M)$ the poset $\mathcal{IU}(M)_x$ is $\lfloor \frac{g(M)-\mathrm{usr}(R)-|x|-2}{2} \rfloor$-connected.*

PROOF. This is an adaptation of the proof of [**39**, Thm. 7.3]. Let $g = g(M)$. Since the statement is clear for $g \leq \mathrm{usr}(R)$, we can assume that $g > \mathrm{usr}(R)$. For $v \in \mathcal{U}(M,\lambda,\mu)$ let

$$S_v := \mathcal{IU}(M) \cap \mathcal{U}(M,\lambda,\mu)_v \cap \mathcal{O}(\langle v\rangle^\perp) \quad \text{and} \quad S := \bigcup_{v \in \mathcal{U}(M,\lambda,\mu)} S_v.$$

Using the transitivity statement for $\mathcal{IU}(M)$ in Lemma 2.23 for each $w \in \mathcal{IU}(M)_{\leq g-\mathrm{usr}(R)}$ we can find an element $v \in \mathcal{U}(M,\lambda,\mu)$ such that $w \in S_v$. In particular, this implies $\mathcal{IU}(M)_{\leq g-\mathrm{usr}(R)} \subseteq S$. Recall that $\mathcal{IU}(M)_{\leq g-\mathrm{usr}(R)}$ denotes the set of all sequences in $\mathcal{IU}(M)$ of length $\leq g - \mathrm{usr}(R)$. Thus, we get an inclusion

$$\mathcal{IU}(M)_{\leq g-\mathrm{usr}(R)} \subseteq S \subseteq \mathcal{IU}(M)$$

and, hence, showing that the poset S is $\lfloor \frac{g-\mathrm{usr}(R)-2}{2} \rfloor$-connected implies the first part of the statement since the inclusion $\mathcal{IU}(M)_{\leq g-\mathrm{usr}(R)} \subseteq \mathcal{IU}(M)$ is $(g - \mathrm{usr}(R) - 1)$-connected.

We first prove that the poset S_v is $\lfloor \frac{g-\mathrm{usr}(R)-|v|-2}{2} \rfloor$-connected for every $v \in \mathcal{U}(M,\lambda,\mu)$ using descending induction on $|v|$. If $|v| > g - \mathrm{usr}(R)$ there is nothing to prove.

In the case $g - \mathrm{usr}(R) - 1 \leq |v| \leq g - \mathrm{usr}(R)$ we must prove that S_v is non-empty. This follows from Lemma 2.5.

Now assume $|v| \leq g - \mathrm{usr}(R) - 2$. For $l = \lfloor \frac{g-\mathrm{usr}(R)-|v|-2}{2} \rfloor$ we have $g - \mathrm{usr}(R) - |v| \geq l+2$. Let $F := \mathcal{U}(M,\lambda,\mu)_v \cap \mathcal{O}(\langle v\rangle^\perp)$ and for $w \in F$ let

$$X_w := \mathcal{IU}(M) \cap \mathcal{U}(M,\lambda,\mu)_{wv} \cap \mathcal{O}(\langle wv\rangle^\perp) \quad \text{and} \quad X := \bigcup_{w \in F} X_w.$$

By Lemma 2.5 we have $(S_v)_{\leq g-\mathrm{usr}(R)-|v|} \subseteq X$ and, hence, by the same reasoning as above it suffices to prove that X is l-connected. By Lemma 2.21 the poset F is l-connected. By induction, the poset X_w is $\lfloor \frac{g-\mathrm{usr}(R)-|w|-|v|-2}{2} \rfloor$-connected since $X_w = S_{wv}$. Note that

$$\min\{l-1, l-|w|-1\} \leq \left\lfloor \frac{g - \mathrm{usr}(R) - |w| - |v| - 2}{2} \right\rfloor$$

and, hence, X_w is $\min\{l-1, l-|w|-1\}$-connected. For every $x \in X$, we have

$$\mathcal{A}_x := \{w \in F \mid x \in X_w\} \cong \mathcal{U}(M,\lambda,\mu)_{vx} \cap \mathcal{O}(\langle vx\rangle^\perp).$$

By Lemma 2.21 this is $(l - |x| + 1)$-connected. Let $w \in F$ with $|w| = 1$. For every $z \in X_w$ we have $wz \in S_v$, so X_w is contained in a cone inside S_v, which we denote by C_w. Define $C(X_w) := X_w \cup (C_w)_{\leq g-\mathrm{usr}(R)-|v|}$. Thus, $C(X_w) \subseteq X$, the equality

$$C(X_w)_{\leq g-\mathrm{usr}(R)-|v|} = (C_w)_{\leq g-\mathrm{usr}(R)-|v|},$$

and the fact that C_w is a cone imply that the poset $C(X_w)$ is l-connected. Now, [**39**, Thm. 4.7] applied to $V, T = M$ and $Y_w = C(X_w)$ yields that X is l-connected and so is S_v by the above. In a similar way (pretending that $|v| = 0$) one can show that S is $\lfloor \frac{g-\mathrm{usr}(R)-2}{2} \rfloor$-connected and therefore $\mathcal{IU}(M)$ is as well.

Now let us consider the poset $\mathcal{IU}(M)_x$ for an $x = (x_1, \dots, x_k) \in \mathcal{IU}(M)$. The proof is by induction on g. If $g < \mathrm{usr}(R) + |x|$ there is nothing to show and so we can assume that $g \geq \mathrm{usr}(R) + |x|$. Let $l = \lfloor \frac{g-\mathrm{usr}(R)-|x|-2}{2} \rfloor$. By Lemma 2.23 we can assume without loss of generality that $x_i = e_i$ for all i. Applying Lemma 2.24 (1) to the set $V := \langle x_1, \dots, x_k \rangle$ yields $\mathcal{IU}(M)_x \cong \mathcal{IU}(P)\langle V \rangle$ with $g(P) \geq g - |x|$ by construction. From the first part of the proof we know that the poset $\mathcal{IU}(P)$ is l-connected. Also, we know that by the induction hypothesis the poset $\mathcal{IU}(P)_y$ is $\lfloor \frac{g-|x|-\mathrm{usr}(R)-|y|-2}{2} \rfloor$-connected for every $y \in \mathcal{IU}(P)$. But we have $l - |y| \leq \lfloor \frac{g-|x|-\mathrm{usr}(R)-|y|-2}{2} \rfloor$ and, hence, [**39**, Lemma 4.1 (ii)] applied to $F = \mathcal{IU}(P)$ yields that the poset $\mathcal{IU}(P)\langle V \rangle$ is l-connected and, hence, so is $\mathcal{IU}(M)_x$. $\square$

Proof of Theorem 2.4. Following the proof of [**39**, Thm. 7.4] we use induction on the Witt index $g = g(M)$. If $g \leq \mathrm{usr}(R)$ then there is nothing to prove. Thus, we can assume that $g > \mathrm{usr}(R)$.

For $v \in \mathcal{IU}(M)$ let

$$X_v := \mathcal{HU}(M) \cap \mathcal{MU}(M)_v \quad \text{and} \quad X := \bigcup_{v \in \mathcal{IU}(M)} X_v.$$

By Lemma 2.23 we have $\mathcal{HU}(M)_{\leq g-\mathrm{usr}(R)} \subseteq X$ and, hence, it is enough to show that X is $\lfloor \frac{g-\mathrm{usr}(R)-3}{2} \rfloor$-connected. Let $l = \lfloor \frac{g-\mathrm{usr}(R)-3}{2} \rfloor$ and for $v = (v_1, \dots, v_k) \in \mathcal{IU}(M)$ let $V := \langle v_1, \dots, v_k \rangle$. If $g < \mathrm{usr}(R) + |v|$ then X_v is trivially $\min\{l-1, l-|v|+1\}$-connected as $l - |v| + 1 \leq -2$. In the case $g \geq \mathrm{usr}(R) + |v|$, by Lemma 2.24 there is an isomorphism

$$X_v \cong \mathcal{HU}(P)\langle V \times V \rangle,$$

where $P := V^\perp \cap W^\perp$ for a hyperbolic complement W of V. Note that by Lemma 2.23 we can assume without loss of generality that $V \oplus W$ is contained in the first $|v|$ copies of H in M. Hence, we have $g(P) \geq g - |v|$. By the induction hypothesis, the poset $\mathcal{HU}(P)$ is $\lfloor \frac{g-|v|-\mathrm{usr}(R)-3}{2} \rfloor$-connected and the poset $\mathcal{HU}(P)_y$ is $\lfloor \frac{g-|v|-\mathrm{usr}(R)-|y|-3}{2} \rfloor$-connected for every $y \in \mathcal{HU}(P)$. But $l - |y| \leq \lfloor \frac{g-|v|-\mathrm{usr}(R)-|y|-3}{2} \rfloor$ and, hence, [**39**, Lemma 4.1] applied to the poset $F = \mathcal{HU}(P)$ yields that the poset $\mathcal{HU}(P)\langle V \times V \rangle$ is $\lfloor \frac{g-|v|-\mathrm{usr}(R)-3}{2} \rfloor$-connected and, hence, so is X_v. In particular, X_v is $\min\{l-1, l-|v|+1\}$-connected. For $x = ((x_1, y_1), \dots, (x_m, y_m)) \in X$ we have

$$\mathcal{A}_x := \{v \in \mathcal{IU}(M) \mid x \in X_v\} \cong \mathcal{IU}(M)_{(x_1, \dots, x_m)}.$$

By Theorem 2.25 $\mathcal{A}_x$ is $\lfloor \frac{g-\mathrm{usr}(R)-m-2}{2} \rfloor$-connected. But $l - |x| + 1 \leq \lfloor \frac{g-\mathrm{usr}(R)-m-2}{2} \rfloor$ and, hence, $\mathcal{A}_x$ is in particular $l - |x| - 1$ connected. Let $v = (v_1) \in \mathcal{IU}(M)$ and define

$$D_v := \mathcal{HU}(M)_{(v_1, w_1)},$$

where $w_1 \in M$ is a hyperbolic dual of v_1. As above, using Lemma 2.23 and Lemma 2.24 we can assume that $D_v \cong \mathcal{HU}(Q)$ with $g(Q) \geq g - 1$. By induction the complex D_v is

$\lfloor \frac{g-1-\mathrm{usr}(R)-3}{2} \rfloor$-connected and therefore also $(l-1)$-connected. We have $D_v \subseteq X_v$, where D_v is contained in a cone in $\mathcal{HU}(M)$, which we denote by C_v. Define

$$C(D_v) := D_v \cup (C_v)_{\leq g-\mathrm{usr}(R)} \quad \text{and} \quad Y_v := X_v \cup C(D_v).$$

In order to apply [**39**, Lemma 4.7] we need to show that the poset Y_v is l-connected. Since $C(D_v)_{\leq g-\mathrm{usr}(R)} = (D_v)_{\leq g-\mathrm{usr}(R)}$, the poset $C(D_v)$ is l-connected. Hence, the Meyer–Vietoris Theorem yields the exact sequence

$$\tilde{H}_l(D_v;\ \mathbb{Z}) \xrightarrow{(i_v)_*} \tilde{H}_l(X_v;\ \mathbb{Z}) \longrightarrow \tilde{H}_l(Y_v;\ \mathbb{Z}) \longrightarrow 0,$$

where $i_v \colon D_v \to X_v$ is the inclusion. By the induction hypothesis, the poset $(D_v)_w$ for $w \in D_v$ is $\lfloor \frac{g-1-\mathrm{usr}(R)-|w|-3}{2} \rfloor$-connected, and hence, in particular is $(l-|w|)$-connected. By Lemma 2.24 (1) and [**39**, Lemma 4.1 (i)] the induced map $(i_v)_*$ is an isomorphism, and hence, by exactness of the above sequence we get $\tilde{H}_l(Y_v;\ \mathbb{Z}) \cong 0$. If $l \geq 1$ by the Seifert–van Kampen Theorem we get $\pi_1(Y_v, x) \cong \pi_1(X_v, x)/N$, where $x \in D_v$ and N is the normal subgroup generated by the image of the map $(i_v)_* \colon \pi_1(D_v, x) \to \pi_1(X_v, x)$. Now, by [**39**, Lemma 4.1 (ii)] the group $\pi_1(X_v, x)$ is trivial and, hence, so is the group $\pi_1(Y_v, x)$. Hence, by the Hurewicz Theorem, the poset Y_v is l-connected and, hence, so is X by [**39**, Thm. 4.7]. The fact that $\mathcal{HU}(M)_x$ is $\lfloor \frac{g-\mathrm{usr}(R)-|x|-3}{2} \rfloor$-connected follows from the above and Lemma 2.24. $\square$

CHAPTER 3

An Axiomatic Approach to Homological Stability

In this chapter we recapitulate the main constructions and theorems from Randal-Williams–Wahl [**43**]. We use this framework in Chapters 4 and 5 to deduce homological stability for general linear groups over modules, and unitary groups over quadratic modules respectively.

In the following we consider (strict) monoidal categories $(\mathcal{C}, \oplus, 0)$ in which the unit 0 is initial. Recall that a strict monoidal category satisfies the equalities $A \oplus 0 = A = 0 \oplus A$ and $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ for all objects A, B, and C instead of just asking for them to be isomorphisms. Thus, for every pair of objects A and B in such a category, we have a preferred morphism

$$\iota_A \oplus \mathbb{1}_B \colon B = 0 \oplus B \to A \oplus B,$$

where $\iota_A \colon 0 \to A$ denotes the unique morphism in the category $\mathcal{C}$ from the initial object 0 to A. We denote

$$\mathrm{Fix}(B, A \oplus B) := \{\phi \in \mathrm{Aut}(A \oplus B) | \phi \circ (\iota_A \oplus \mathbb{1}_B) = \iota_A \oplus \mathbb{1}_B\}.$$

3.1. The Axioms

We define axioms $LH1$, $LH2$, and $LH3$ which are the main assumptions for homological stability results in the next section.

Definition 3.1. [**43**, Def. 1.2] A monoidal category $(\mathcal{C}, \oplus, 0)$ is *locally homogeneous* at a pair of objects (A, X) if 0 is initial in $\mathcal{C}$ and if it satisfies the following two axioms.

LH1 For all $0 \leq p < n$, $\mathrm{Hom}(X^{\oplus p+1}, A \oplus X^{\oplus n})$ is a transitive $\mathrm{Aut}(A \oplus X^{\oplus n})$-set under postcomposition.

LH2 For all $0 \leq p < n$, the map $\mathrm{Aut}(A \oplus X^{\oplus n-p-1}) \to \mathrm{Aut}(A \oplus X^{\oplus n})$ taking f to $f \oplus \mathbb{1}_{X^{\oplus p+1}}$ is injective with image $\mathrm{Fix}(X^{\oplus p+1}, A \oplus X^{\oplus n-p-1} \oplus X^{\oplus p+1})$.

We give an example of a locally homogeneous category following [**43**, p. 9]. For a field k we consider the category $\mathcal{V}_k$ of finite-dimensional k-vector spaces and split injective linear maps. Hence, a morphism in $\mathcal{V}_k$ from V to W is a pair (f, W_f), where $f \colon V \to W$ is an injective homomorphism and $W_f \subseteq W$ is a subspace such that $W \cong W_f \oplus f(V)$. The monoidal structure on $\mathcal{V}_k$ is given by the direct sum and, hence, the zero vector space is the unit. The automorphism group of W is its associated general linear group $\mathrm{GL}(W)$.

Let V, W be objects in $\mathcal{V}_k$ and $(f, W_f), (g, W_g) \in \mathrm{Hom}(V, W)$. Since f and g are split injections we have $f(V) \cong g(V)$. This implies $W_f \cong W_g$ since complements of isomorphic subspaces of W are isomorphic. Hence, there is an automorphism $\phi \in \mathrm{GL}(W)$ such that

$\phi \circ f = g$. Hence, $\mathrm{GL}(W)$ acts transitively on $\mathrm{Hom}(V, W)$. Since V and W were arbitrary objects in $\mathcal{V}_k$ axiom LH1 is satisfied for every pair of k-vector spaces (A, X).

For V, W as above we have that the map $\iota_V \oplus \mathbb{1}_W \colon W \to V \oplus W$ is the same as the inclusion of W into $V \oplus W$ that sends w to $(0, w)$. Hence, automorphisms of $V \oplus W$ fixing $\iota_V \oplus W$ are exactly the ones fixing V setwise and W pointwise which corresponds to the automorphisms of V. Thus, the image of the map $\mathrm{GL}(V) \to \mathrm{GL}(V \oplus W)$ is $\mathrm{Fix}(W, V \oplus W)$. This implies the axiom LH2.

In fact, we have just shown that $\mathcal{V}_k$ is also an example of a so-called homogeneous monoidal category (cf. [**43**, Def. 1.3]).

Definition 3.2. [**43**, Def. 1.5] Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category with 0 initial. We say that $\mathcal{C}$ is *pre-braided* if its underlying groupoid is braided and for each pair of objects A and B in $\mathcal{C}$, the groupoid braiding $b_{A,B} \colon A \oplus B \to B \oplus A$ satisfies

$$b_{A,B} \circ (\mathbb{1}_A \oplus \iota_B) = \iota_B \circ \mathbb{1}_A \colon A \to B \oplus A.$$

Note that braided monoidal categories are in particular pre-braided.

The following definition introduces Quillen's construction of a category $\langle \mathcal{G}, \mathcal{G} \rangle$ for a given monoidal groupoid $\mathcal{G}$.

Definition 3.3. [**43**, p. 11] Let $(\mathcal{G}, \oplus, 0)$ be a monoidal groupoid. Following [**25**, p. 219] for the special case $S = \mathcal{G} = X$ we define a category $\langle \mathcal{G}, \mathcal{G} \rangle$, also denoted as $U\mathcal{G}$, which has the same objects as $\mathcal{G}$. A morphism in $\langle \mathcal{G}, \mathcal{G} \rangle$ from A to B is an equivalence class of pairs (X, f), where X is an object of $\mathcal{G}$ and $f \colon X \oplus A \to B$ is a morphism in $\mathcal{G}$. We define $(X, f) \sim (X', f')$ if there exists an isomorphism $g \colon X \to X'$ in $\mathcal{G}$ such that the diagram

$$\begin{tikzcd} X \oplus A \arrow[r, "f"] \arrow[d, "g \oplus \mathbb{1}_A"'] & B \\ X' \oplus A \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.

For example, as noted in [**43**, p. 12], if $\mathcal{G}$ is the monoidal groupoid fR-Mod of finitely generated free modules over a ring R and monoidal structure given by direct sum, then $\langle \mathcal{G}, \mathcal{G} \rangle$ is the category of finitely generated free R-modules and free split injections, that is split injections $f \colon M \to N$ equipped with a choice of free submodule $F \leq N$ such that $N = \mathrm{Im}(f) \oplus F$.

This proposition shows some of the close correlations between a monoidal groupoid $\mathcal{G}$ and its corresponding category $\langle \mathcal{G}, \mathcal{G} \rangle$.

Proposition 3.4. [**43**, Prop. 1.7] *Let $(\mathcal{G}, \oplus, 0)$ be a monoidal groupoid and $U\mathcal{G} := \langle \mathcal{G}, \mathcal{G} \rangle$. Then*

(1) 0 is initial in $U\mathcal{G}$,

(2) if $\mathcal{G}$ is braided monoidal then $U\mathcal{G}$ is a pre-braided monoidal category,

(3) if $\mathcal{G}$ is symmetric monoidal then $U\mathcal{G}$ is a symmetric monoidal category.

Moreover, in the latter two cases, the monoidal structure of $U\mathcal{G}$ is such that the map $\mathcal{G} \to U\mathcal{G}$ taking an isomorphism f to $[0, f]$ is monoidal.

This shows, for example, that $(UfR\text{-Mod}, \oplus, 0)$ is a symmetric monoidal category.

We show in Theorem 3.6 that the axiom LH1 on $U\mathcal{G}$ is equivalent to the following cancellation property of $\mathcal{G}$.

Definition 3.5. [**43**, Def. 1.9] For a pair of objects (A, X) in a monoidal groupoid $(\mathcal{G}, \oplus, 0)$ we say that $\mathcal{G}$ satisfies *local cancellation at (A, X)* if it satisfies

LC For all $0 \leq p < n$, if $Y \in \mathcal{G}$ is such that $Y \oplus X^{\oplus p+1} \cong A \oplus X^{\oplus n}$ then we have $Y \cong A \oplus X^{\oplus n-p-1}$.

The following theorem is the main result used to show that a pre-braided monoidal category $U\mathcal{G}$ is locally homogeneous.

Theorem 3.6. [**43**, Thm. 1.10 (a) and (b)] *Let $(\mathcal{G}, \oplus, 0)$ be a braided monoidal groupoid and $U\mathcal{G} := \langle \mathcal{G}, \mathcal{G} \rangle$ its associated pre-braided category.*

(1) $U\mathcal{G}$ satisfies LH1 at (A, X) if and only if $\mathcal{G}$ satisfies LC at (A, X).

(2) If the map $\mathrm{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n-p-1}) \to \mathrm{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n})$ taking f to $f \oplus \mathbb{1}_{X^{\oplus p+1}}$ is injective for all $0 \leq p < n$, then $U\mathcal{G}$ satisfies LH2 at (A, X).

In particular, if (1) and (2) are both satisfied, then $U\mathcal{G}$ is locally homogeneous at (A, X).

We again consider the category fR-Mod as an example. For M an R-module with $\mathrm{rk}(M) \geq \mathrm{sr}(R)$ the pair (M, R) satisfies the axiom LC by Proposition 1.7. Applying Theorem 3.6 now shows that the category UfR-Mod satisfies both LH1 and LH2 and, hence, is locally homogeneous at (M, R).

Definition 3.7. [**43**, Def. 2.1] Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category with 0 initial and (A, X) a pair of objects in $\mathcal{C}$. Define $W_n(A, X)_\bullet$ to be the semisimplicial set with set of p-simplices

$$W_n(A, X)_p := \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$$

and with face map

$$d_i \colon \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p}, A \oplus X^{\oplus n})$$

defined by precomposing with $X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}$.

Postcomposition in $\mathcal{C}$ defines a simplicial action of the group $\mathrm{Aut}(A \oplus X^{\oplus n})$ on the semisimplicial set $W_n(A, X)_\bullet$.

We now define one final axiom which is a connectivity property for a pair of objects in a monoidal category.

Definition 3.8. [**43**, Def. 2.2] Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category and (A, X) a pair of objects in $\mathcal{C}$. We say that $\mathcal{C}$ satisfies *LH3 at (A, X) with slope l* if

LH3 For all $n \geq 1$, $|W_n(A,X)_\bullet|$ is $\lfloor \frac{n-2}{l} \rfloor$-connected.

We show in Lemma 4.1 that the category UfR-Mod satisfies LH3 at (M,R) with slope 2 for R-modules M satisfying $\mathrm{rk}(M) \geq \mathrm{sr}(R)$.

3.2. The Homological Stability Theorems

The next three theorems which are the main results from **[43]** all imply homological stability with different types of coefficients for certain automorphism groups.

For the case of integral coefficients we get the following theorem.

THEOREM 3.9. [**43**, Thm. 3.1] *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair of objects (A,X). Suppose that $\mathcal{C}$ satisfies LH3 at (A,X) with slope $l \geq 2$. Then the map*

$$H_k(\mathrm{Aut}(A \oplus X^{\oplus n}); \mathbb{Z}) \to H_k(\mathrm{Aut}(A \oplus X^{\oplus n+1}); \mathbb{Z})$$

is an epimorphism if $k \leq \frac{n}{l}$ and an isomorphism if $k \leq \frac{n-1}{l}$.

Note that this implies homological stability with coefficients in any abelian group by the Universal Coefficient Theorem ([**27**, Thm. 3A.3]).

For a pair of objects in a pre-braided category $\mathcal{C}$ we define $\mathrm{Aut}(A \oplus X^{\oplus\infty})$ to be the colimit of the sequence

$$\cdots \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+1}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+2}) \xrightarrow{-\oplus X} \cdots.$$

In the following we use the notation $G_n := \mathrm{Aut}(A \oplus X^{\oplus n})$ and $G_\infty := \mathrm{Aut}(A \oplus X^{\oplus\infty})$. Let M be a G_∞-module. We can consider M as a G_n-module for any n, by restriction. We say that M is *abelian* if the action of G_∞ on M factors through the abelianisation of G_∞. This is equivalent to the condition that the derived subgroup G'_∞ acts trivially on M.

THEOREM 3.10. [**43**, Thm. 3.4] *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair of objects (A,X). Suppose that $\mathcal{C}$ satisfies LH3 at (A,X) with slope $l \geq 3$ (!). Then for any abelian $\mathrm{Aut}(A \oplus X^{\infty})$-module M the map*

$$H_k(\mathrm{Aut}(A \oplus X^{\oplus n}); M) \to H_k(\mathrm{Aut}(A \oplus X^{\oplus n+1}); M)$$

is an epimorphism if $k \leq \frac{n-l+2}{l}$ and an isomorphism if $k \leq \frac{n-l}{l}$.

The last of these three main theorems show homological stability for polynomial coefficient systems which we introduce now.

DEFINITION 3.11. [**43**, Def. 4.1] Let $\mathcal{C}$ be a pre-braided category and let A, X be two objects in $\mathcal{C}$. We write $\mathcal{C}_{A,X}$ for the full subcategory of $\mathcal{C}$ whose objects are $A \oplus X^n$ for all $n \geq 0$. A *coefficient system for $\mathcal{C}$ at (A,X)* is a functor

$$F \colon \mathcal{C}_{A,X} \to \mathcal{A}$$

from $\mathcal{C}_{A,X}$ to an abelian category $\mathcal{A}$.

In order to give the definition of a polynomial coefficient system we need the following construction. Consider the functor

$$\Sigma^X := - \oplus X \colon \mathcal{C}_{A,X} \to \mathcal{C}_{A,X}$$

that sends an object B to $B \oplus X$, and a morphism $f\colon A \to B$ to $f \oplus \mathbb{1}_X\colon A \oplus X \to B \oplus X$. Then the *upper suspension by* X

$$\sigma^X := \mathbb{1}_{A\oplus X^{\oplus n}} \oplus \iota_X \colon A \oplus X^{\oplus n} \longrightarrow A \oplus X^{\oplus n} \oplus X = A \oplus X^{\oplus n+1}$$

is a natural transformation from the identity to the functor Σ^X.

Analogously, we define the *lower suspension by* X to be

$$\sigma_X := (b_{X,A} \oplus \mathbb{1}_{X^{\oplus n}}) \circ (\iota_X \oplus \mathbb{1}_{A\oplus X^{\oplus n}})\colon A \oplus X^{\oplus n} \longrightarrow A \oplus X \oplus X^{\oplus n}.$$

Since the category $\mathcal{C}$ is pre-braided, this is related to the upper suspension by X via

$$\sigma_X = (b_{X,A} \oplus \mathbb{1}_{X^{\oplus n}}) \circ (b_{A\oplus X^{\oplus n},X}) \circ \sigma^X.$$

As before, this defines a natural transformation from the identity to the functor

$$\Sigma_X \colon \mathcal{C}_{A,X} \longrightarrow \mathcal{C}_{A,X}$$

which maps an object $A \oplus X^{\oplus n}$ to $A \oplus X \oplus X^{\oplus n}$, and a morphism $f\colon A \oplus X^{\oplus n} \to A \oplus X^{\oplus k}$ to

$$\begin{aligned}\Sigma_X(f)\colon A \oplus X^{\oplus n+1} &\xrightarrow{b^{-1}_{X,A}\oplus \mathbb{1}_{X^{\oplus n}}} X \oplus A \oplus X^{\oplus n}\\ &\xrightarrow{\mathbb{1}_X \oplus f} X \oplus A \oplus X^{\oplus k}\\ &\xrightarrow{b_{X,A}\oplus \mathbb{1}_{X^{\oplus k}}} A \oplus X^{\oplus k+1}.\end{aligned}$$

Note that the functors Σ^X and Σ_X commute and that they are related via

$$\Sigma_X(f) = (b_{X,A} \oplus \mathbb{1}_{X^{\oplus n}}) \circ (b_{A\oplus X^{\oplus n},X}) \circ \Sigma^X(f) \circ (b^{-1}_{A\oplus X^{\oplus n},X}) \circ (b^{-1}_{X,A} \oplus \mathbb{1}_{X^{\oplus n}})$$

for $f \in \mathrm{Aut}(A \oplus X^{\oplus n})$.

For a coefficient system $F\colon \mathcal{C}_{A,X} \to \mathcal{A}$ we define the *suspension* of F with respect to X by $\Sigma F := F \circ \Sigma_X$. Note that both the kernel and the cokernel of the natural transformation $\sigma_X\colon F \to \Sigma F$ are again coefficient systems.

Definition 3.12. [**43**, Def. 4.10] A coefficient system $F\colon \mathcal{C}_{A,X} \to \mathcal{A}$ has *degree* $r < 0$ *at* 0 *with respect to* X if $F(A \oplus X^n) = 0$ for all $n \geq 0$.

For $r \geq 0$, we define inductively that F *has degree* r *at* 0 if

(1) the kernel of the suspension map $F \to \Sigma F$ is a coefficient system of degree -1 at 0, and
(2) the cokernel of the suspension map $F \to \Sigma F$ is a coefficient system of degree $(r-1)$ at 0.

Furthermore, we say that F is a *split* coefficient system of degree r at 0 if

(1) the suspension map $F \to \Sigma F$ is split injective in the category of coefficient systems,

(2) the cokernel of the suspension map $F \to \Sigma F$ is a split coefficient system of degree $(r-1)$ at 0.

Using the above notation we write $s_n \colon G_n \to G_\infty$ for the canonical homomorphism, $G_\infty^{ab} := H_1(G_\infty;\mathbb{Z})$ for the abelianisation, and $\mathbb{Z}[G_\infty^{ab}]$-Mod for the (abelian) category of left G_∞^{ab}-modules. Given a coefficient system

$$F \colon \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}\text{-Mod}],$$

for each n we get a left G_∞^{ab}-module $F_n := F(A \oplus X^{\oplus n})$. The module F_n has a left action of G_n via G_∞^{ab}-module maps. It therefore has commuting left G_n and G_∞^{ab} actions which we denote by $\cdot$ and $*$ respectively. Using the canonical map $G_n \xrightarrow{s_n} G_\infty \to G_\infty^{ab}$ this gives a modified G_n-module structure on F_n, given by

$$\begin{array}{ccc} G_n \times F_n & \longrightarrow & F_n \\ (g,x) & \longmapsto & g \cdot (s_n(g) * x). \end{array}$$

We write F_n° for this left G_n-module and call it the module obtained by *internalising* F_n.

Example 3.13. [**43**, Ex. 4.6] For a G_∞^{ab}-module M the functor $F_M \colon \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]$-Mod is defined to be the constant functor mapping all objects to M, and all morphisms to the identity on M. The corresponding internalised coefficient system F_M° is just the abelian coefficient system M as considered above.

Before we can state the final main theorem of [**43**] we need to define relative homology groups for groups with coefficients given by modules over the respective group. We follow [**43**, pp. 35-36] which in turn follows [**46**, 3.9]. Let $\mathcal{R}ep$ denote the category of pairs (G,M), where G is a group and M is a left G-module, whose morphisms $(\phi,f)\colon (G,M) \to (G',M')$ consist of a group homomorphism $\phi\colon G \to G'$ and a ϕ-linear map $f\colon M \to M'$. We write $\mathcal{R}el\mathcal{R}ep$ for the arrow category of $\mathcal{R}ep$, i.e. the category with objects the morphisms of $\mathcal{R}ep$, and morphisms the commutative squares in $\mathcal{R}ep$.

Given an object $(\phi,f)\colon (G,M) \to (G',M')$ in $\mathcal{R}el\mathcal{R}ep$, and projective resolutions P_* and P'_* of M and M' respectively. Then there is a (ϕ-linear) map of resolutions $P_* \to P'_*$ covering f. By definition, the mapping cone of the chain map

$$\mathbb{Z} \otimes_{\mathbb{Z}G} P_* \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G'} P'_*$$

computes the relative homology groups $H_*(G',G;M',M)$. These groups fit into a long exact sequence

$$\cdots \longrightarrow H_k(G,M) \longrightarrow H_k(G',M') \longrightarrow H_k(G',G;M',M) \longrightarrow H_{k-1}(G;M) \to \cdots .$$

This defines functors $H_k(-)\colon \mathcal{R}el\mathcal{R}ep \to \mathbb{Z}$-Mod which associates to a morphism

$$\begin{tikzcd} (G_0,M_0) \arrow[r] \arrow[d, "(\phi_0,f_0)"] & (G_1,M_1) \arrow[d, "(\phi_1,f_1)"] \\ (G_0',M_0') \arrow[r] & (G_1',M_1') \end{tikzcd}$$

in $\mathcal{R}el\mathcal{R}ep$ an induced map

$$H_*(G_0', G_0; M_0', M_0) \longrightarrow H_k(G_1', G_1; M_1', M_1).$$

Let (A, X) be a pair of objects in a pre-braided monoidal category $\mathcal{C}$, and $F\colon \mathcal{C}_{A,X} \to \mathbb{Z}$-Mod a coefficient system. Using the above notation we write

$$Rel_*^F(A, n) := H_*(G_{n+1}, G_n; F_{n+1}, F_n)$$

for the relative groups associated to the upper suspension maps $\Sigma^X\colon G_n \to G_{n+1}$ and $\sigma^X\colon F_n \to F_{n+1}$. If we have $F\colon \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]$-Mod instead, then by [**43**, Lemma 4.9] we can define

$$Rel_*^{F^\circ}(A, n) := H_*(G_{n+1}, G_n; F_{n+1}^\circ, F_n^\circ).$$

We can now state the third of the main theorems of [**43**].

Theorem 3.14. [**43**, Thm. 4.20] *Let $\mathcal{C}$ be a category and (A, X) a pair of objects in $\mathcal{C}$. Suppose that $\mathcal{C}$ is locally homogeneous at (A, X) and satisfies LH3 at (A, X) with slope $k \geq 2$.*

If $F\colon \mathcal{C}_{A,X} \to \mathbb{Z}$-Mod is a coefficient system of degree r at 0, then

(1) $Rel_i^F(A, n)$ vanishes for $n \geq \max(1, k(i + r))$, and

(2) if F is split then $Rel_i^F(A, n)$ vanishes for $n \geq \max(1, ki + r)$.

If $F\colon \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]$-Mod is a coefficient system of degree r at 0 and $k \geq 3$, then

(3) $Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(1, k(i + r) + k - 2)$, and

(4) if F is split then $Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(1, ki + 2r + k - 2)$.

CHAPTER 4

Homological Stability for General Linear Groups

We now prove homological stability of general linear groups over modules, which induces in particular Theorem A, using the machinery of Randal-Williams–Wahl [**43**]. As before, we write $(fR\text{-Mod}, \oplus, 0)$ for the groupoid of finitely generated free right R-modules and their isomorphisms. As we have shown in the previous chapter, the corresponding category UfR-Mod is locally homogeneous at (M, R) for an R-module M satisfying $\mathrm{rk}(M) \geq \mathrm{sr}(R)$. The following lemma verifies the axiom LH3 from the connectivity of the complex considered in Theorem 1.4.

LEMMA 4.1. *The semisimplicial set $W_n(M,R)_\bullet$ is $\lfloor \frac{n+\mathrm{rk}(M)-\mathrm{sr}(R)-2}{2} \rfloor$-connected.*

The proof adapts the ideas of the proof of [**43**, Lemma 5.10]. Here, we just comment on the changes that have to be made to the proof of [**43**, Lemma 5.10] required to prove the above lemma.

OUTLINE OF THE PROOF. Let $X(M)_\bullet$ be the semisimplicial set with p-simplices the split injective R-module homomorphisms $f\colon R^{p+1} \to M$, and with i-th face map given by precomposing with the inclusion $R^i \oplus 0 \oplus R^{p-i} \to R^{p+1}$. We write $U(M)$ for the simplicial complex with vertices the R-module homomorphisms $v\colon R \to M$ which are split injections (without a choice of splitting), and where a tuple $(v_0, \ldots, v_p)$ spans a p-simplex if and only if the sum $v_0 \oplus \ldots \oplus v_p\colon R^{p+1} \to M$ is a split injection.

Note that the poset of simplices of $X(M)_\bullet$ is equal to the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)$ and that, given a p-simplex $\sigma = \langle v_0, \ldots, v_p \rangle \in U(M)$, the poset of simplices of the complex $(\mathrm{Link}_{U(M)}(\sigma))_\bullet^{ord}$, which has a p-simplex for every p-simplex in $(\mathrm{Link}_{U(M)}(\sigma))_\bullet$ and every choice of ordering of its vertices, equals the poset $\mathcal{O}(M) \cap \mathcal{U}(M^\infty)_{(v_0,\ldots,v_p)}$. Hence, by applying Theorem 1.4 and arguing as in the proof of [**43**, Lemma 5.10] we get that $U(M \oplus R^n)$ is weakly Cohen–Macaulay (as defined in [**24**, Sec. 2.1]) of dimension $n + \mathrm{rk}(M) - \mathrm{sr}(R)$.

As in the proof of [**43**, Lemma 5.10] we want to show that the assumptions of [**29**, Thm. 3.6] are satisfied. The complex $S_n(M,R)$ is a join complex over $U(M \oplus R^n)$ by the same reasoning as in the proof in [**43**]. In order to show that $\pi(\mathrm{Link}_{S_n(M,R)}(\sigma))$ is weakly Cohen–Macaulay of dimension $n + \mathrm{rk}(M) - \mathrm{sr}(R) - p - 2$ for each p-simplex $\sigma \in S_n(M,R)$ we apply Proposition 1.7 instead of [**43**, Prop. 5.9] in the proof of [**43**, Lemma 5.10]. This shows that the remaining assumptions of [**29**, Thm. 3.6] are satisfied. Applying this and [**43**, Thm. 2.10] then yields the claim. □

Applying Theorems 3.9, 3.10 and 3.14 to $(UfR\text{-Mod}, \oplus, 0)$ yields the following Theorem which directly implies Theorem A.

THEOREM 4.2. *Let* $F\colon UfR\text{-Mod} \to \mathbb{Z}\text{-Mod}$ *be a coefficient system of degree* r *at* 0. *Then for* $s = \mathrm{rk}(M) - \mathrm{sr}(R)$ *the map*

$$H_k(\mathrm{GL}(M); F(M)) \to H_k(\mathrm{GL}(M \oplus R); F(M \oplus R))$$

is

(1) *an epimorphism for* $k \leq \frac{s}{2}$ *and an isomorphism for* $k \leq \frac{s-1}{2}$, *if* F *is constant,*
(2) *an epimorphism for* $k \leq \frac{s-r}{2}$ *and an isomorphism for* $k \leq \frac{s-2-r}{2}$, *if* F *is split polynomial,*
(3) *an epimorphism for* $k \leq \frac{s}{2} - r$ *and an isomorphism for* $k \leq \frac{s-2}{2} - r$.

For the commutator subgroup $\mathrm{GL}(M)'$ *we get that the map*

$$H_k(\mathrm{GL}(M)'; F(M)) \to H_k(\mathrm{GL}(M \oplus R)'; F(M \oplus R))$$

is

(4) *an epimorphism for* $k \leq \frac{s-1}{3}$ *and an isomorphism for* $k \leq \frac{s-3}{3}$, *if* F *is constant,*
(5) *an epimorphism for* $k \leq \frac{s-1-2r}{3}$ *and an isomorphism for* $k \leq \frac{s-4-2r}{3}$, *if* F *is split polynomial,*
(6) *an epimorphism for* $k \leq \frac{s-1}{3} - r$ *and an isomorphism for* $k \leq \frac{s-4}{3} - r$.

CHAPTER 5

Homological Stability for Unitary Groups

We now show homological stability for unitary groups over quadratic modules. This induces in particular Theorem B. As in the previous chapter we use the machinery of Randal-Williams–Wahl [**43**]. Let $(R, \varepsilon, \Lambda)$-Quad be the groupoid of quadratic modules over $(R, \varepsilon, \Lambda)$ and their isomorphisms. We write $f(R, \varepsilon, \Lambda)$-Quad for the full subcategory on those quadratic modules which are finitely generated as R-modules. Since this is a braided monoidal category it has an associated pre-braided category $Uf(R, \varepsilon, \Lambda)$-Quad by Proposition 3.4 (2).

By Corollary 2.15 the pair (M, H) in $f(R, \varepsilon, \Lambda)$-Quad satisfies the axiom LC in the case $g(M) \geq \operatorname{usr}(R)$. In particular, Theorem 3.6 (1) shows that $Uf(R, \varepsilon, \Lambda)$-Quad satisfies LH1 at (M, H) for $g(M) \geq \operatorname{usr}(R) + 1$. As noted in [**43**], $Uf(R, \varepsilon, \Lambda)$-Quad satisfies LH2 for all pairs, and hence, it is locally homogeneous at (M, H) for $g(M) \geq \operatorname{usr}(R)$. Axiom LH3 is verified by the following Lemma which for the special case of hyperbolic modules is shown in [**43**, Lemma 5.14].

LEMMA 5.1. *Let M be a quadratic module with $g(M) \geq \operatorname{usr}(R) + 1$. Then the semisimplicial set $W_n(M, H)_\bullet$ is $\lfloor \frac{n+g(M)-\operatorname{usr}(R)-3}{2} \rfloor$-connected.*

PROOF. As in the proof of [**43**, Lemma 5.14], the poset of simplices of the semisimplicial set $W_n(M, H)_\bullet$ is equal to the poset $\mathcal{HU}(M \oplus H^n)$ considered in Chapter 2. Hence, they have homeomorphic geometric realisations. The claim now follows from Theorem 2.4. $\square$

THEOREM 5.2. *Let $F\colon Uf(R, \varepsilon, \Lambda)\text{-Quad} \to \mathbb{Z}\text{-Mod}$ be a coefficient system of degree r at 0. Then for $s = g(M) - \operatorname{usr}(R)$ the map*

$$H_k(U(M); F(M)) \to H_k(U(M \oplus H); F(M \oplus H))$$

is

(1) *an epimorphism for $k \leq \frac{s-1}{2}$ and an isomorphism for $k \leq \frac{s-2}{2}$, if F is constant,*
(2) *an epimorphism for $k \leq \frac{s-r-1}{2}$ and an isomorphism for $k \leq \frac{s-r-3}{2}$, if F is split polynomial,*
(3) *an epimorphism for $k \leq \frac{s-1}{2} - r$ and an isomorphism for $k \leq \frac{s-3}{2} - r$.*

For the commutator subgroup $U(M)'$ we get that the map

$$H_k(U(M)'; F(M)) \to H_k(U(M \oplus H)'; F(M \oplus H))$$

is

(4) *an epimorphism for $k \leq \frac{s-1}{3}$ and an isomorphism for $k \leq \frac{s-3}{3}$, if F is constant,*
(5) *an epimorphism for $k \leq \frac{s-2r-1}{3}$ and an isomorphism for $k \leq \frac{s-2r-4}{3}$, if F is split polynomial,*

(6) an epimorphism for $k \leq \frac{s-1}{3} - r$ *and an isomorphism for* $k \leq \frac{s-4}{3} - r$.

This result in particular implies Theorem B.

Outline of the proof. The theorem is vacuous for $s < 1$. In the case $s \geq 1$ we have $g(M) \geq \mathrm{usr}(R) + 1$. In particular, we can write M as $M' \oplus H^{g(M)-\mathrm{usr}(R)-1}$, where the genus of M' is $g(M') = \mathrm{usr}(R) + 1$. Applying Lemma 5.1 shows that $H_n(M', H)$ is $\lfloor \frac{n-2}{2} \rfloor$-connected, and hence, LH3 holds at (M', H) with slope 2. Applying Theorems 3.9, 3.10 and 3.14 to the pair (M', H), with $n = g(M) - \mathrm{usr}(R) - 1$ yields the theorem. $\square$

CHAPTER 6

Moduli Spaces of High Dimensional Manifolds

We now use the results of Chapter 2 to derive a result about homological stability of moduli spaces of high dimensional manifolds (Theorem 6.3). In this chapter we follow Galatius–Randal-Williams [**24**].

Following [**24**, Ch. 6] we describe a point-set model for the classifying space $B\mathrm{Diff}_\partial(X)$ of the topological group of diffeomorphisms of the manifold X restricting to the identity near its boundary.

Definition 6.1. [**24**, Def. 6.1] For a $2n$-dimensional manifold X with boundary P and collar $c\colon (-\infty,0]\times P \hookrightarrow X$, and an $\varepsilon>0$, let $\mathrm{Emb}_\varepsilon(X,(-\infty,0]\times\mathbb{R}^\infty)$ denote the space, in the C^∞-topology, of those embeddings $e\colon X\hookrightarrow(-\infty,0]\times\mathbb{R}^\infty)$ that satisfy $e\circ c(t,x)=(t,x)$ as long as $t\in(-\varepsilon,0]$, and let

$$\mathcal{E}(X) := \operatorname*{colim}_{\varepsilon\to 0} \mathrm{Emb}_\varepsilon(X,(-\infty,0]\times\mathbb{R}^\infty).$$

The space $\mathcal{E}(X)$ has a (free) action of $\mathrm{Diff}_\partial(X)$ by precomposition, and we write

$$\mathcal{M}(X)=\mathcal{E}(X)/\mathrm{Diff}_\partial(X).$$

Two elements of $\mathcal{E}(X)$ are in the same orbit if and only if they have the same image, so as a set, $\mathcal{M}(X)$ is the set of submanifolds $M\subset(-\infty,0]\times\mathbb{R}^\infty$ such that

(1) $M\cap(\{0\}\times\mathbb{R}^\infty)=\{0\}\times P$ and M contains $(-\varepsilon,0]\times P$ for some $\varepsilon>0$,
(2) the boundary of M is precisely $\{0\}\times P$, and
(3) M is diffeomorphic to X relative to P.

(The underlying set of $\mathcal{M}(X)$ depends on the specific identification $\partial X\cong P\subset\mathbb{R}^\infty$.)

By [**9**] the quotient map

$$\mathcal{E}(X)\to\mathcal{E}(X)/\mathrm{Diff}_\partial(X)$$

has slices and, hence, is a principal $\mathrm{Diff}_\partial(X)$-bundle. By Whitney's embedding theorem $\mathcal{E}(X)$ is weakly contractible, and hence, the quotient space is a model for the classifying space $B\mathrm{Diff}_\partial(X)$.

We now describe what the map in Theorem C looks like in this model for $B\mathrm{Diff}_\partial(X)$. The map (0.1) can be generalised by gluing on any cobordism K. In $\mathcal{M}(X)$ this is modelled using a choice of collared embedding $K\subset[-1,0]\times\mathbb{R}^\infty$, i.e. K agrees with $[-1,0]\times P$ near $\{-1\}\times\mathbb{R}^\infty$ and $\{0\}\times\mathbb{R}^\infty$, such that $K\cap(\{-1\}\times\mathbb{R}^\infty)=P$. Then the gluing map is given by

$$\begin{aligned} - \cup K \colon \mathcal{M}(X) &\longrightarrow \mathcal{M}(X \cup_P K) \\ M &\longmapsto (M - e_1) \cup K, \end{aligned}$$

that is, translation by one unit in the first coordinate direction followed by union of submanifolds of $(-\infty, 0] \times \mathbb{R}^\infty$.

Let P be a closed non-empty $(2n-1)$-dimensional manifold, and let W and M be compact path-connected $2n$-dimensional manifolds with identified boundaries $\partial W = P = \partial M$. We say that W and M are *stably diffeomorphic relative to* P if there is a diffeomorphism

$$W \# W_g \cong M \# W_h$$

relative to P, for some $g, h \geq 0$, where $W_g := \#_g(\mathbb{S}^n \times \mathbb{S}^n)$.

Definition 6.2. [**24**, Def. 6.2] Let $P \subset \mathbb{R}^\infty$ be a closed non-empty $(2n-1)$-dimensional manifold, and let W be a compact manifold, with a specific identification $\partial W = P$. Let

$$\mathcal{M}^{\mathrm{st}}(W) = \bigsqcup_{[T]} \mathcal{M}(T)$$

where the union is taken over the set of compact manifolds T with boundary $\partial T = P$, which are stably diffeomorphic to W relative to P, one in each diffeomorphism class relative to P. The space $\mathcal{M}^{\mathrm{st}}(W)$ depends on $P \subset \mathbb{R}^\infty$ and the stable diffeomorphism class of W relative to $P = \partial W$, but we shall suppress that from the notation.

For the analogue of Theorem C in this model, we choose a submanifold $S \subset [-1, 0] \times \mathbb{R}^\infty$ with collared boundary $\partial S = \{-1, 0\} \times P = S \cap (\{-1, 0\} \times \mathbb{R}^\infty)$, such that S is diffeomorphic to $([-1, 0] \times P) \# W_1$ relative to its boundary. If P is not path-connected, we also choose in which path component to perform the connected sum. Gluing $K = S$ then induces the self-map

$$\begin{aligned} s = - \cup S \colon \mathcal{M}^{\mathrm{st}}(W) &\longrightarrow \mathcal{M}^{\mathrm{st}}(W) \\ M &\longmapsto (M - e_1) \cup S, \end{aligned} \tag{6.1}$$

Note that by construction we have $M \cup_P S \cong M \# W_1$ relative to P, and hence $M \cup_P S$ is stably diffeomorphic to W if and only if M is.

As in the previous chapters, we have a notion of genus: Writing $W_{g,1} := W_g \setminus \mathrm{int}(\mathbb{D}^{2n})$, the *genus* of a compact connected $2n$-dimensional manifold W is

$$g(W) := \max\{g \in \mathbb{N} \mid \text{there is an embedding } W_{g,1} \hookrightarrow W\}$$

and the *stable genus* of W is

$$\overline{g}(W) := \max_{k \geq 0} \{g(W \# W_k) - k \mid k \in \mathbb{N}\}.$$

Note that since the map $k \mapsto g(W \# W_k) - k$ is non-decreasing and bounded above by $\frac{b_n(W)}{2}$, where $b_n(W)$ is the n-th Betti number of W, the above maximum is well-defined. We write $\mathcal{M}^{\mathrm{st}}(W)_g \subset \mathcal{M}^{\mathrm{st}}(W)$ for the subspace of manifolds of stable genus precisely g, i.e. those manifolds $M \in \mathcal{M}^{\mathrm{st}}(W)$ satisfying $\overline{g}(M) = g$. Note that by definition of the stable genus, the map s defined in (6.1) restricts to a map $s\colon \mathcal{M}^{\mathrm{st}}(W)_g \to \mathcal{M}^{\mathrm{st}}(W)_{g+1}$.

THEOREM 6.3. *Let $2n \geq 6$ and W be a compact connected manifold. Then the map*

$$s_*\colon H_k(\mathcal{M}^{\mathrm{st}}(W)_g) \longrightarrow H_k(\mathcal{M}^{\mathrm{st}}(W)_{g+1})$$

is an epimorphism for $k \leq \frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k \leq \frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$.

For the case of simply-connected compact manifolds Galatius–Randal-Williams have shown homological stability for the spaces $\mathcal{M}^{\mathrm{st}}(W)_g$ in [**24**, Thm. 6.3].

In particular, this theorem implies that for any manifold W with boundary P, the restriction

$$s\colon \mathcal{M}(W) \longrightarrow \mathcal{M}(W \cup_P S)$$

induces an epimorphism on homology in degrees $k \leq \frac{\overline{g}(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism in degrees $k \leq \frac{\overline{g}(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$. Since $g(W) \leq \overline{g}(W)$ this implies Theorem C.

Using Example 2.2 (3) we get the following special case of Theorem 6.3.

COROLLARY 6.4. *Let $2n \geq 6$ and W be a compact connected manifold whose fundamental group is virtually polycyclic with Hirsch number h. Then the map*

$$s_*\colon H_k(\mathcal{M}^{\mathrm{st}}(W)_g) \longrightarrow H_k(\mathcal{M}^{\mathrm{st}}(W)_{g+1})$$

is an epimorphism for $k \leq \frac{g-h-3}{2}$ and an isomorphism for $k \leq \frac{g-h-5}{2}$.

This theorem applies in particular to all compact connected manifolds with finite fundamental group and more generally with finitely generated abelian fundamental group.

Another consequence of the above theorem is the following cancellation result which in the case of simply-connected manifolds has been done in [**24**, Cor. 6.4]. The statement is closely related to [**17**, Thm. 1.1].

COROLLARY 6.5. *Let $2n \geq 6$ and P be a $(2n-1)$-dimensional manifold. Let W and W' be compact connected manifolds with boundary P such that $W \# W_g \cong W' \# W_g$ relative to P, for some $g \geq 0$. If $\overline{g}(W) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) + 2$, then $W \cong W'$ relative to P.*

PROOF. Analogous to the proof of [**24**, Cor. 6.4], where we apply Theorem 6.3 instead of [**24**, Thm. 6.3]. □

The proof of Theorem 6.3 is analogous to that of [**24**, Thm. 6.3] which treats the case of simply-connected manifolds. The idea is to consider the group of immersions of $W_{1,1}$ into a manifold. Equipping this with a bilinear form that counts intersections and a function that counts self-intersections we get a quadratic module. The precise construction is the content of the following section. The high connectivity shown in Chapter 2 then implies a connectivity statement for a complex of geometric data associated to the manifold. This is the crucial result required to show homological stability which we do in Section 6.2.

6.1. Associating a Quadratic Module to a Manifold

In order to relate the objects in this chapter to the algebraic objects considered in Chapter 2 we want to associate to each compact connected $2n$-dimensional manifold W a quadratic module $(\mathcal{I}_n^{\mathrm{tn}}(W), \lambda, \mu)$ with form parameter $((-1)^n, \Lambda_{\min})$. This will be a module over $\mathbb{Z}[\pi_1(W,*)]$ given by a version of the group of immersed n-spheres in W with trivial normal bundle, with pairing given by the intersection form, and quadratic form given by counting self-intersections, both considered over the group ring $\mathbb{Z}[\pi_1(W,*)]$. For the rest of this chapter we drop the basepoint $*$ from the notation and just write $\pi_1(W)$.

To make this construction precise we use the *standrad framing* $\xi_{\mathbb{S}^n\times\mathbb{D}^n}$ of $\mathbb{S}^n \times \mathbb{D}^n$ induced by the embedding

$$\mathbb{S}^n \times \mathbb{D}^n \longrightarrow \mathbb{R}^{n+1} \times \mathbb{R}^{n-1} = \mathbb{R}^{2n} \tag{6.2}$$
$$(x; y_1, y_2, \dots, y_n) \longmapsto (2e^{-\frac{y_1}{2}}x; y_2, y_3, \dots, y_n).$$

This standard framing at $(0,0,\dots,0,-1;0) \in \mathbb{S}^n \times \mathbb{D}^n$ gives a point $b_{\mathbb{S}^n\times\mathbb{D}^n} \in \mathrm{Fr}(\mathbb{S}^n \times \mathbb{D}^n)$.

We can now generalise [**24**, Def. 5.2], following the construction in the proof of [**51**, Thm. 5.2].

Definition 6.6. Let $2n \geq 6$ and W be a compact connected $2n$-dimensional manifold, equipped with a *framed basepoint*, i.e. a point $b_W \in \mathrm{Fr}(W)$, and an orientation compatible with b_W.

(1) We consider the ring $\mathbb{Z}[\pi_1(W)]$ with involution given by $\overline{g} := \omega_1(g)g^{-1} \in \mathbb{Z}[\pi_1(W)]$, where $\omega_1(g)$ is the first Stiefel–Whitney class of g. Recall that the first Stiefel–Whitney class can be viewed as the homomorphism $\pi_1(W) \to \mathbb{Z}^\times = \{-1, 1\}$ which sends a loop to 1 if and only if it is orientation preserving.

We define $\mathcal{I}_n^{\mathrm{fr}}(W)$ to be the set of regular homotopy classes of immersions $i\colon \mathbb{S}^n \times \mathbb{D}^n \looparrowright W$ equipped with a path in $\mathrm{Fr}(W)$ from $Di(b_{\mathbb{S}^n\times\mathbb{D}^n})$ to b_W. We write $\mathcal{I}_n(W)$ for the set of regular homotopy classes of immersions $\mathbb{S}^n \looparrowright W$ equipped with a path in W from a fixed basepoint in $\mathbb{S}^n$ to the basepoint $*$ in W. We define $\mathcal{I}_n^{\mathrm{tn}}(W)$ to be the image of the map $\mathcal{I}_n^{\mathrm{fr}}(W) \to \mathcal{I}_n(W)$ which is given by forgetting the framing. Since an immersion $\mathbb{S}^n \looparrowright W$ is frameable if and only if it has a trivial normal bundle, the set $\mathcal{I}_n^{\mathrm{tn}}(W)$ is given by regular homotopy classes of immersions with a trivial normal bundle.

Using Smale–Hirsch immersion theory we can identify $\mathcal{I}_n(W)$ with the n-th homotopy group of n-frames in W. This induces an (abelian) group structure on $\mathcal{I}_n(W)$. The $\pi_1(W)$-action is given by concatenating a loop in W with the path corresponding to an element in $\mathcal{I}_n(W)$ as described in [**51**, Thm. 5.2]. Now, $\mathcal{I}_n^{\mathrm{tn}}(W)$ is a $\mathbb{Z}[\pi_1(W)]$-submodule of $\mathcal{I}_n(W)$.

(2) Let $a, b \in \mathcal{I}_n^{\mathrm{tn}}(W)$ be two immersed spheres, which we may suppose meet in general position, i.e. transversely in a finite set of points. For a point p in a let $\gamma_a(p)$ denote the concatenation of the chosen path from the basepoint $*$ to the basepoint of a

with a path from the basepoint of a to p in a. Since $2n \geq 6$ such a path is canonical up to homotopy. For $p \in a \cap b$ we define $\gamma_{(a,b)}(p)$ to be the concatenation of $\gamma_a(p)$ followed by the inverse of $\gamma_b(p)$.

Let us fix an orientation of W at the basepoint $*$ and transport the orientation to p along $\gamma_a(p)$. We also get an orientation at p from $T_p a(\mathbb{S}^n) \oplus T_p b(\mathbb{S}^n)$. We define the sign of the intersection of a and b at p, denoted by $\varepsilon_{(a,b)}(p)$, to be 1 if these orientations at p agree, and -1 otherwise. Note that we have

$$\varepsilon_{(b,a)}(p) = (-1)^n \omega_1(a * b^{-1}) \varepsilon_{a,b}(p)$$

which shows that λ satisfies $\lambda(a,b) = \varepsilon \overline{\lambda(b,a)}$ using the involution $\overline{g} = \omega_1(g) g^{-1}$. Given these notions we define a map

$$\begin{aligned} \lambda \colon \mathcal{I}_n^{\mathrm{tn}}(W) \times \mathcal{I}_n^{\mathrm{tn}}(W) &\longrightarrow \mathbb{Z}[\pi_1(W)] \\ (a,b) &\longmapsto \sum_{p \in a \cap b} \varepsilon_{(a,b)}(p) \gamma_{(a,b)}(p). \end{aligned}$$

(3) Any $a \in \mathcal{I}_n^{\mathrm{tn}}(W)$ may be represented by an immersed sphere in general position, i.e. an immersed sphere that only has double points and is self-transversed, and let $p \in \mathbb{S}^n \times \{0\}$ be a point in a. We write $\gamma(p)$ for the path from the basepoint $*$ to p in the universal cover of the image of a in W.

At a self-intersection point of a two branches of a cross. By choosing an order of these branches we can define $\varepsilon(p,q)$ as above. Recall that $\Lambda_{\min}$ is given by the set $\{\gamma - \varepsilon\overline{\gamma} \mid \gamma \in \mathbb{Z}[\pi_1(W)]\}$. We define a map

$$\begin{aligned} \mu \colon \mathcal{I}_n^{\mathrm{tn}}(W) &\longrightarrow \mathbb{Z}[\pi_1(W)]/\Lambda_{\min} \\ a &\longmapsto \sum_{\substack{\{p,q\} \subset \mathbb{S}^n \\ i_a(p) = i_a(q) \\ p \neq q}} \varepsilon(p,q) \gamma(p,q), \end{aligned}$$

where i_a is an immersion of $\mathbb{S}^n$ corresponding to a and $\gamma(p,q)$ is the loop in a based at the basepoint $*$ given by the concatenation of $a(\gamma(p))$ and the inverse of $a(\gamma(q))$, see Figure 1. The definition of $\Lambda_{\min}$ guarantees that the order of the points p, q is not relevant, i.e. we have $\varepsilon(p,q)\gamma(p,q) \equiv \varepsilon(q,p)\gamma(q,p) \mod \Lambda_{\min}$.

Remarks 6.7.

(1) The (abelian) group structure on $\mathcal{I}_n^{\mathrm{tn}}(W)$ is given by forming the connected sum along the path as described in [**51**, Ch. 5].

(2) The proof of [**51**, Thm. 5.2 (i)] shows that both maps λ and μ are well-defined.

(3) We show that we can always change a by an isotopy so that every point in $a \cap b$ yields a summand in $\lambda(a,b)$, i.e. so that no two intersection points give summands that cancel. The idea is to pair up intersection points that give the same element in $\mathbb{Z}[\pi_1(W)]$ but with opposite signs, and to use the Whitney trick ([**44**, Sec. 1.5]) to kill these intersection points. Figure 2 shows a sector of a and b in W with two intersection points p and q. Both paths $\gamma_{(a,b)}(p)$ and $\gamma_{(a,b)}(q)$ correspond to g in

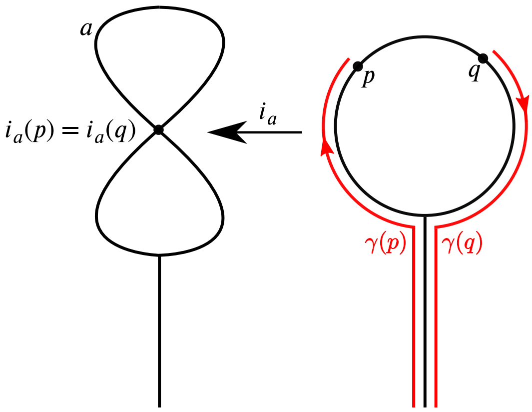

FIGURE 1. Definition of $\gamma(p,q)$.

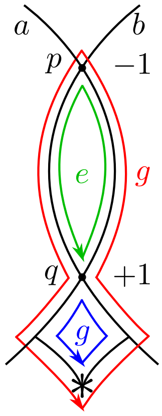

FIGURE 2. Using the Whitney trick.

$\pi_1(W)$ and the points p and q have opposite signs. If this is the case the loop e is contractible. Hence, we can fill in a 2-disc and use the Whitney trick to move a away from b in the sector shown in the picture.

The subsequent lemma generalises [**24**, Lemma 5.3]. The proof is analogous to the proof of [**24**, Lemma 5.3], again using [**51**, Thm. 5.2].

LEMMA 6.8. *The triple* $(\mathcal{I}_n^{\mathrm{tn}}(W), \lambda, \mu)$ *is a* $((-1)^n, \Lambda_{\min})$*-quadratic module.*

6.2. Proof of Theorem 6.3

As in [**24**, Ch. 5] it will be convenient to work with the following small modification of the manifold $W_{1,1}$. Consider $W_{1,1}$, where the removed disc lies in $\mathbb{D}^n_+ \times \mathbb{D}^n_+$, the product of the two upper hemispheres in W_1. We denote by H the manifold we obtain from $W_{1,1}$ by gluing $[-1,0]\times\mathbb{D}^{2n-1}$ onto $\partial W_{1,1}$ along an orientation preserving embedding

$$\{-1\}\times\mathbb{D}^{2n-1} \longrightarrow \partial W_{1,1}.$$

We choose this embedding once and for all. After smoothing corners, H is diffeomorphic to $W_{1,1}$ but contains a standard embedding of $[-1,0]\times\mathbb{D}^{2n-1}$. By an embedding of H into a manifold W we always mean an embedding that maps $\{0\}\times\mathbb{D}^{2n-1}$ into ∂W and the rest of H into the interior of W.

We now define the *core* C of H. To do so let $x_0\in\mathbb{S}^n$ be a basepoint. We may suppose that $\mathbb{S}^n\vee\mathbb{S}^n = (\mathbb{S}^n\times\{x_0\})\cup(\{x_0\}\times\mathbb{S}^n)\subset W_1$ is contained in the interior of $W_{1,1}$. Choose an embedded path γ in the interior of H from $(x_0,-x_0)$ to $(0,0)\in[-1,0]\times\mathbb{D}^{2n-1}$, whose interior does not intersect $\mathbb{S}^n\vee\mathbb{S}^n$, and whose image is $[-1,0]\times\{0\}$ inside $[-1,0]\times\mathbb{D}^{2n-1}$, and define

$$C := (\mathbb{S}^n\vee\mathbb{S}^n)\cup\gamma([-1,0])\cup(\{0\}\times\mathbb{D}^{2n-1})\subset H$$

as shown in Figure 3.

For the manifold $H = ([-1,0]\times\mathbb{D}^{2n-1})\cup_{\{-1\}\times\mathbb{D}^{2n-1}} W_{1,1}$ we choose the framed basepoint b_H given by the Euclidean framing of $T_{0,0}([-1,0]\times\mathbb{D}^{2n-1})$, i.e. the framing induced by the

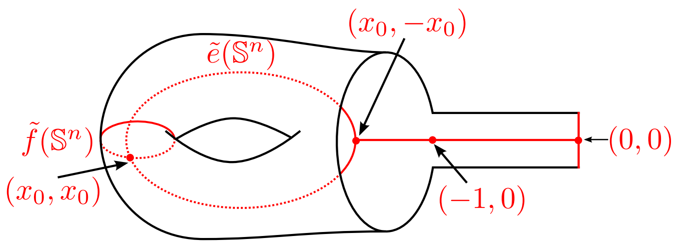

FIGURE 3. The manifold H in the case $2n = 2$, with the core C indicated in red. Figure from [**24**].

inclusion $[-1, 0] \times \mathbb{D}^{2n-1} \subset \mathbb{R}^{2n}$. We define canonical elements $e, f \in \mathcal{I}_n^{\mathrm{fr}}(H)$ in the following way. There are embeddings

$$\begin{aligned} \overline{e}\colon \mathbb{S}^n \times \mathbb{D}^n &\longrightarrow W_{1,1} \subset \mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ (x, y) &\longmapsto \left(x; \frac{y}{2}, -\sqrt{1 - |\frac{y}{2}|^2}\right) \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} \overline{f}\colon \mathbb{S}^n \times \mathbb{D}^n &\longrightarrow W_{1,1} \subset \mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \\ (x, y) &\longmapsto \left(-\frac{y}{2}, -\sqrt{1 - |\frac{y}{2}|^2}; x\right) \end{aligned} \tag{6.4}$$

which are orientation preserving. These may be considered as embeddings into H.

We define the embeddings $\tilde{e}$ and $\tilde{f}$ of $\mathbb{S}^n$ into H as the inclusion $\mathbb{S}^n \hookrightarrow \mathbb{S}^n \times \mathbb{D}^n$ given by $x \mapsto (x, 0)$ followed by the above maps $\overline{e}$ and $\overline{f}$ respectively.

The embedding $\tilde{e}$ together with a path in H from the basepoint of $\tilde{e}(\mathbb{S}^n)$ to the basepoint $(0, 0)$ in $[-1, 0] \times \mathbb{D}^{2n-1} \subseteq H$ defines an element $e \in \mathcal{I}_n^{\mathrm{tn}}(H)$. Since H is simply-connected the choice of path is unique up to isotopy. Analogously, we get an element $f \in \mathcal{I}_n^{\mathrm{tn}}(H)$. These elements satisfy

$$\lambda(e, e) - \lambda(f, f) = 0 \qquad \lambda(e, f) = 1 \qquad \mu(e) = \mu(f) = 0,$$

and so determine a morphism of quadratic modules $H \to (\mathcal{I}_n^{\mathrm{tn}}(H), \lambda, \mu)$. Note that from now, to avoid confusion, we write H_{hyp} for the hyperbolic module and H for the manifold defined above. Following the explanation of [**24**, (5.4)] we see that, in fact, disjoint embeddings of H into W give orthogonal submodules. This will translate to a map of simplicial complexes using the following complex of geometric data on the manifold W.

DEFINITION 6.9. [**24**, Def. 5.1] Let W be a compact manifold, equipped with (the germ of) an embedding $c\colon (-\delta, 0] \times \mathbb{R}^{2n-1} \to W$ for some $\delta > 0$, such that $c^{-1}(\partial W) = \{0\} \times \mathbb{R}^{2n-1}$. Two embeddings c and c' define the same germ if they agree after making δ smaller.

(1) Let $K_0(W) = K_0(W, c)$ be the space of pairs (t, ϕ), where $t \in \mathbb{R}$ and $\phi\colon H \to W$ is an embedding whose restriction to $(-1, 0] \times \mathbb{D}^{2n-1} \subset H$ satisfies that there exists an $\varepsilon \in (0, \delta)$ such that

$$\phi(s, p) = c(s, p + te_1)$$

for all $s \in (-\varepsilon, 0]$ and all $p \in \mathbb{D}^{2n-1}$. Here, $e_1 \in \mathbb{R}^{2n-1}$ denotes the first basis vector.

(2) Let $K_p(W) \subset (K_0(W))^{p+1}$ consist of those tuples $((t_0, \phi_0), \ldots, (t_p, \phi_p))$ satisfying that $t_0 < \ldots < t_p$ and that the embeddings ϕ_i have disjoint cores, i.e. the sets $\phi_i(C)$ are disjoint.

(3) Topologise $K_p(W)$ using the C^∞-topology on the space of embeddings and let $K_p^\delta(W)$ be the same set considered as a discrete topological space.

(4) The assignments $[p] \mapsto K_p(W)$ and $[p] \mapsto K_p^\delta(W)$ define semisimplicial spaces, where the face map d_i forgets (t_i, ϕ_i).

(5) Let $K^\delta(W)$ be the simplicial complex with vertices $K_0^\delta(W)$, and where the (unordered) set $\{(t_0, \phi_0), \ldots, (t_p, \phi_p)\}$ is a p-simplex if, when written with $t_0 < \ldots < t_p$, it satisfies $((t_0, \phi_0), \ldots, (t_p, \phi_p)) \in K_p^\delta(W)$.

Instead of (t, ϕ) we sometimes denote a vertex simply by ϕ, since t is determined by ϕ. There is a natural homeomorphism $|K_\bullet^\delta(W)| = |K^\delta(W)|$ since the simplices of $K_\bullet^\delta(W)$ are determined by its (unordered) set of vertices.

As explained above, we get a map of simplicial complexes

$$K^\delta(W) \longrightarrow \mathcal{HU}(\mathcal{I}_n^{fr}(W)),$$

where $\mathcal{HU}(\mathcal{I}_n^{fr}(W))$ is the simplicial complex defined in Chapter 2. In Theorem 6.12 below, we use this map to deduce the connectivity of $|K_\bullet^\delta(W)| = |K^\delta(W)|$ from the connectivity of $\mathcal{HU}(\mathcal{I}_n^{fr}(W))$ which we have shown in Theorem 2.4.

As in [**24**, Def. 5.9] let $\overline{K}_\bullet(W) \subset K_\bullet(W)$ denote the sub-semisimplicial space with p-simplices those tuples of embeddings which are disjoint. (Recall that in $K_\bullet(W)$ we only ask for the embeddings to have disjoint cores.)

The *stable Witt index* of a quadratic module M is

$$\overline{g}(M) := \sup_{k \geq 0} \{g(M \oplus H^k) - k\}.$$

By definition we have $g(M) \leq \overline{g}(M)$ and if the stable Witt index is big enough we in fact have equality, as the following corollary shows.

Lemma 6.10. *If* $\overline{g}(M) \geq \operatorname{usr}(R)$ *then we have* $g(M) \geq \overline{g}(M)$.

Proof. For $g = \overline{g}(M)$ we know that $M \oplus H^k \cong P \oplus H^g \oplus H^k$ for some k. If $k = 0$ we immediately get $g(M) \geq g$. If $k > 0$ we get $M \oplus H^{k-1} \cong P \oplus H^g \oplus H^{k-1}$ by Corollary 2.15. Applying this argument inductively then yields $g(M) \geq g$. ☐

Remark 6.11. We can also define the *stable rank* of an R-module M given by

$$\overline{\operatorname{rk}}(M) := \sup_{k \geq 0} \{\operatorname{rk}(M \oplus R^k) - k\}.$$

Analogous to the above this coincides with the rank of M if $\overline{\operatorname{rk}}(M) \geq \operatorname{sr}(R)$. This can be shown similarly to the proof of Lemma 6.10 by inductively applying Theorem 1.4 and Proposition 1.7. Using this we get a version of Theorem 1.4 in terms of the stable rank.

THEOREM 6.12. *Let $2n \geq 6$ and W be a compact connected $2n$-dimensional manifold. Then the following spaces are all $\lfloor \frac{\overline{g}(W)-\operatorname{usr}(\mathbb{Z}[\pi_1(W)])-3}{2} \rfloor$-connected:*

(1) $|K^\delta_\bullet(W)|$,
(2) $|K_\bullet(W)|$,
(3) $|\overline{K}_\bullet(W)|$.

For the case of simply-connected manifolds this has been done in [**24**, Lemma 5.5], [**24**, Thm. 5.6], and [**24**, Cor. 5.10] respectively.

Using the above correspondence between the Witt index and the stable Witt index we can now state Theorem 2.4 in terms of the stable Witt index which will be used in the proof of the above theorem.

COROLLARY 6.13. *The poset $\mathcal{HU}(M)$ is $\lfloor \frac{\overline{g}(M)-\operatorname{usr}(R)-3}{2} \rfloor$-connected and for every element $x \in \mathcal{HU}(M)$ the poset $\mathcal{HU}(M)_x$ is $\lfloor \frac{\overline{g}(M)-\operatorname{usr}(R)-|x|-3}{2} \rfloor$-connected.*

PROOF OF THEOREM 6.12. For $|K^\delta_\bullet(W)|$ the proof is analogous to the proof of [**24**, Lemma 5.5], and hence, we just comment on the changes we have to make required to show the above statement. Note that the complex $K^a(\mathcal{I}^{\mathrm{tn}}_n(W), \lambda, \mu)$ as defined in [**24**, Def. 3.1] is the same as $\mathcal{HU}(\mathcal{I}^{\mathrm{tn}}_n(W))$.

For $\overline{g} = \overline{g}(\mathcal{I}^{\mathrm{tn}}_n(W), \lambda, \mu)$ we have $\overline{g}(W) \leq \overline{g}$, and hence, it is sufficient to show that $|K^\delta_\bullet(W)|$ is $\lfloor \frac{\overline{g}-\operatorname{usr}(\mathbb{Z}[\pi_1(W)])-3}{2} \rfloor$-connected.

For $k \leq \frac{\overline{g}-\operatorname{usr}(\mathbb{Z}[\pi_1(W)])-3}{2}$ we consider a map $f\colon \partial I^{k+1} \to |K^\delta_\bullet(W)|$, which, as in [**24**], we may assume is simplicial with respect to some piecewise linear triangulation $\partial I^{k+1} \cong |L|$. By Corollary 6.13 and composing with the map constructed above we get a nullhomotopy $\overline{f}\colon I^{k+1} \to |\mathcal{HU}(\mathcal{I}^{\mathrm{tn}}_n(W))|$. We show that this lifts to a nullhomotopy $F\colon I^{k+1} \to |K^\delta_\bullet(W)|$ of f.

By Corollary 6.13 the complex $\mathcal{HU}(\mathcal{I}^{\mathrm{tn}}_n(W))$ is locally weakly Cohen–Macaulay (as defined in [**24**, Sec. 2.1]) of dimension $\left\lfloor \frac{\overline{g}-\operatorname{usr}(\mathbb{Z}[\pi_1(W)])}{2} \right\rfloor \geq k+1$. Hence, there is a triangulation $I^{k+1} \cong |K|$ extending L which satisfies the same properties as in [**24**].

We choose an enumeration of the vertices in K as $v_1, \dots, v_N$ such that the vertices in L come before the vertices in $K \setminus L$. We inductively pick lifts of each $\overline{f}(v_i) \in \mathcal{HU}(\mathcal{I}^{\mathrm{tn}}_n(W))$ to a vertex $F(v_i) \in K^\delta_\bullet(W)$ given by an embedding $j_i\colon H \to W$ satisfying the properties (i) and (ii) in the proof of [**24**, Lemma 5.5] which control how the images of every two such embeddings intersect. By construction, the vertices in L already satisfy the required properties (i) and (ii), so we can assume that $\overline{f}(v_1), \dots, \overline{f}(v_{i-1})$ have already been lifted to maps $j_1, \dots, j_{i-1}$, satisfying properties (i) and (ii). Then $v_i \in K \setminus L$ yields a morphism of quadratic modules $\overline{f}(v_i) = h\colon H_{hyp} \to \mathcal{I}^{\mathrm{tn}}_n(W)$, where H_{hyp} is the hyperbolic module defined in Chapter 2, which we want to lift to an embedding j_i satisfying properties (i) and (ii). The element $h(e)$ is represented by an immersion $x\colon \mathbb{S}^n \looparrowright W$ with trivial normal bundle satisfying $\mu(x) = 0$ and a path in W from the basepoint of $\mathbb{S}^n$ to the basepoint $*$ of W. By the Whitney trick (which works in our case, but we have to use it over the group ring

$\mathbb{Z}[\pi_1(W)]$ as described in Remark 6.7 (3)) we can replace x by an embedding $j(e)\colon \mathbb{S}^n \hookrightarrow W$. Similarly, $h(f)$ yields an embedding $j(f)\colon \mathbb{S}^n \hookrightarrow W$, along with another path in W.

Using the Whitney trick again, we can arrange for the embeddings $j(e)$ and $j(f)$ to intersect transversally in exactly one point. Hence, by picking a trivialisation of their normal bundles, this induces an embedding $W_{1,1} \hookrightarrow W$. To extend this map to an embedding of manifolds $H \hookrightarrow W$, note that both $h(e)$ and $h(f)$ come with a path to the basepoint. The proof in [**24**] forgets both paths and chooses a new one later on (which works since W is simply-connected and, hence, oriented). Instead, we can keep track of the path coming from $h(e)$. This can be viewed as an embedding $[-1,0] \times \{0\} \hookrightarrow W$. This then has a thickening by definition which gives an embedding $H \hookrightarrow W$. Analogous to the proof of [**24**, Lemma 5.5] we can show that the properties (i) and (ii) hold, and hence, conclude the connectivity range.

The proof for the case $|K_\bullet(W)|$ is an easy extension of the proof of [**24**, Thm. 5.6], where we use Corollary 6.13 instead of [**24**, Thm. 3.2], and hence, get a slightly weaker connectivity range.

The remaining case follows exactly as in [**24**, Cor. 5.10]. □

In the above proof, we have lifted the chosen nullhomotopy $\overline{f}\colon I^{k+1} \to |\mathcal{HU}(\mathcal{I}_n^{\mathrm{tn}}(W))|$ and do not have to use the "spin flip" argument as in [**24**]. Applying the above approach of keeping track of the path of $h(e)$ instead of forgetting both paths and choosing some path in the end would also make the "spin flip" argument in the proof of [**24**, Lemma 5.5] unnecessary.

Outline of the proof of Theorem 6.3. The proof of Theorem 6.3 is analogous to the proof of [**24**, Thm. 6.3]. The assumption of W being simply-connected is only used in [**24**, Lemma 6.8] so we just need to show that the map given in [**24**, Lemma 6.8] is $\lfloor \frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-1}{2} \rfloor$-connected for a compact connected manifold W of dimension $2n \geq 6$ that is not necessarily simply-connected. But this follows from the proof of [**24**, Lemma 6.8] by using Theorem 6.12 (3) instead of [**24**, Cor. 5.10]. □

Remark 6.14. We can combine the above results with the results from Kupers in [**31**] for homeomorphisms, PL-homeomorphisms and homeomorphisms as a discrete group of high-dimensional manifolds. Note that the machinery in Kupers' paper does not rely on the manifolds being simply-connected but rather the input does (i.e. the connectivity of a certain complex uses that the manifold is simply-connected). Therefore, by using our more general theorem (Theorem 2.4) as the input, we can replace the assumption of the manifold being simply-connected by the group ring of the fundamental group having finite unitary stable rank.

CHAPTER 7

Tangential Structures

In this chapter we extend Theorem 6.3 in two different ways. One is by considering moduli spaces of manifolds with some additional structure and the other is by taking homology with coefficients in certain twisted coefficient systems. We follow the approach of Galatius–Randal-Williams [**24**, Ch. 7].

7.1. Definition of Tangential Structures

Recall that given a tangential structure θ, a θ-structure on a $2n$-dimensional manifold W is a bundle map $\hat{\ell}_W\colon TW \to \theta^*\gamma_{2n}$, with underlying map $\ell_W\colon W \to B$, i.e. we have the commutative diagram

$$\begin{tikzcd}
 & B \arrow[d, "\theta"] \\
W \arrow[ur, "\ell_W"] \arrow[r, "\tau"] & B\mathrm{O}(2n) \\
TW \arrow[u] \arrow[r] & \gamma_{2n} \arrow[u]
\end{tikzcd}$$

where the bottom part is a pullback square. For example, for a $2n$-dimensional manifold W we can consider $B = B\mathrm{SO}(2n)$, where θ is the inclusion map $B\mathrm{SO}(2n) \hookrightarrow B\mathrm{O}(2n)$. We have $TW \cong \tau^*\gamma_{2n} = (\ell_W)^*\theta^*\gamma_{2n} \cong (\ell_W)^*\gamma_{2n}^{SO}$, and therefore, this encodes orientability. As this example shows, there might be no lifts ℓ_W (if W is non-orientable) or several lifts (connected orientable manifolds have two orientations) and it is therefore convenient to consider the space of all θ-structures $\mathrm{Bun}^\theta(W)$.

If W has boundary P equipped with a collar $(\varepsilon, 0] \times P \to W$, then the collar induces an isomorphism $\varepsilon^1 \oplus TP \cong TW_P$. By fixing a θ-structure $\hat{\ell}_P\colon \varepsilon^1 \oplus TP \to \theta^*\gamma$ we may consider the subspace $\mathrm{Bun}^\theta_\partial(W, \hat{\ell}_P) \subset \mathrm{Bun}^\theta(W)$ consisting of all bundle maps $\hat{\ell}_W\colon TW \to \theta^*\gamma$ that extend $\hat{\ell}_P$. Note that $\mathrm{Diff}_\partial(W)$ acts on $\mathrm{Bun}^\theta_\partial(W, \hat{\ell}_P)$ by precomposing with the derivative. Hence, we can define

$$\mathscr{M}^\theta(W, \hat{\ell}_P) := (E\mathrm{Diff}_\partial(W) \times \mathrm{Bun}^\theta_\partial(W, \hat{\ell}_P))/\mathrm{Diff}_\partial(W).$$

Analogous to the previous chapter we define a certain point-set model for the above space. Using the model $E\mathrm{Diff}_\partial(W) = \mathcal{E}(W)$ as defined in the previous chapter, we let

$$\mathcal{M}^\theta(W, \hat{\ell}_P) := (\mathcal{E}(W) \times \mathrm{Bun}^\theta_\partial(W, \hat{\ell}_P))/\mathrm{Diff}_\partial(W).$$

As a set this is given by the pairs $(M, \hat{\ell})$, where $M \in \mathcal{M}(W)$, and hence, in particular $\partial M = \partial W = \{0\} \times P$, and $\hat{\ell}\colon TM \to \theta^*\gamma_{2n}$ is a bundle map with $\hat{\ell}_{\partial M} = \hat{\ell}_P$.

As in the previous chapter it is convenient to consider all manifolds which are stably diffeomorphic to W at once, and so we start by adapting Definition 6.2 to our setting.

DEFINITION 7.1. [**24**, Def. 7.1] Let $P \subset \mathbb{R}^\infty$ be a $(2n-1)$-dimensional manifold, equipped with a θ-structure $\hat{\ell}_P$, and let W be a manifold with boundary P.

As in the previous section we define

$$\mathcal{M}^{\mathrm{st},\theta}(W, \hat{\ell}_P) := \bigsqcup_{[T]} \mathcal{M}^\theta(T, \hat{\ell}_P),$$

where the union is taken over the set of compact manifolds T with $\partial T = P$, which are stably diffeomorphic to W, one for each diffeomorphism class relative to P. This turns $\mathcal{M}^{\mathrm{st},\theta}(W, \hat{\ell}_P)$ into a space.

As a set this may be described as pairs of a submanifold $M \subset (-\infty, 0] \times \mathbb{R}^\infty$ of dimension $2n$ and a θ-structure $\hat{\ell}_M \colon TM \to \theta^*\gamma_{2n}$ such that

(1) $M \in \mathcal{M}^{\mathrm{st}}(W)$,
(2) $\hat{\ell}_M|_P = \hat{\ell}_P$.

Choose once and for all a bundle map $\tau \colon \mathbb{R}^{2n} \to \theta^*\gamma$ from the trivial $2n$-dimensional vector bundle over a point, or what is the same thing a basepoint $\tau \in \mathrm{Fr}(\theta^*\gamma)$. This determines a canonical θ-structure on any framed $2n$-manifold (or $(2n-1)$-manifold); if X is a framed manifold we denote this θ-structure by $\hat{\ell}^\tau_X$.

In (6.2) we have defined a specific embedding $\mathbb{S}^n \times \mathbb{D}^n \hookrightarrow \mathbb{R}^{2n}$, and hence obtained a framing $\xi_{\mathbb{S}^n\times\mathbb{D}^n}$ of $\mathbb{S}^n \times \mathbb{D}^n$. We will say that a θ-structure on $\mathbb{S}^n \times \mathbb{D}^n$ is *standard* if it is homotopic to $\hat{\ell}^\tau_{\mathbb{S}^n\times\mathbb{D}^n}$.

In (6.3) and (6.4) we defined embeddings $\overline{e}, \overline{f} \colon \mathbb{S}^n \times \mathbb{D}^n \to W_{1,1}$, and hence we obtained embeddings

$$\overline{e}_1, \overline{f}_1, \ldots, \overline{e}_g, \overline{f}_g \colon \mathbb{S}^n \times \mathbb{D}^n \longrightarrow W_{g,1}.$$

Let us say that a θ-structure $\hat{\ell} \colon TW_{g,1} \to \theta^*\gamma$ on $W_{g,1}$ is *standard* if all the pulled-back structures $\overline{e}_i^*\hat{\ell}$ and $\overline{f}_i^*\hat{\ell}$ on $\mathbb{S}^n \times \mathbb{D}^n$ are standard.

The embeddings $\overline{e}$ and $\overline{f}$ defined in Section 6.2 yield embeddings

$$\overline{e}_1, \overline{f}_1, \ldots, \overline{e}_g, \overline{f}_g \colon \mathbb{S}^n \longrightarrow W_{g,1}.$$

We say that a θ-structure $\hat{\ell} \colon TW_{g,1} \to \theta^*\gamma_{2n}$ on $W_{g,1}$ is *standard* if there is a trivialisation of the normal bundle of $\mathbb{S}^n$ (i.e. a framing on $\mathbb{S}^n$) such that the structures $\overline{e}_i^*\hat{\ell}$ and $\overline{f}_i^*\hat{\ell}$ on $\mathbb{S}^n \times \mathbb{D}^n$ are standard.

7.2. Homological Stability

We say that a θ-structure $\hat{\ell} \colon TW_{1,1} \to \theta^*\gamma_{2n}$ is *admissible* if there is a pair of orientation-preserving embeddings $e, f \colon \mathbb{S}^n \times \mathbb{D}^n \hookrightarrow W_{1,1} \subset \mathbb{S}^n \times \mathbb{S}^n$ whose cores $e(\mathbb{S}^n \times \{0\})$ and $f(\mathbb{S}^n \times \{0\})$ intersect transversely in a single point, such that each of the θ-structures $e^*\hat{\ell}$ and $f^*\hat{\ell}$ on $\mathbb{S}^n \times \mathbb{D}^n$ extend to $\mathbb{R}^{2n}$ for some orientation-preserving embeddings $\mathbb{S}^n \times \mathbb{D}^n \hookrightarrow \mathbb{R}^{2n}$. This is closely related to the notion of a standard θ-structure as shown in [**24**, Rmk. 7.3].

We say that a θ-structure $\hat{\ell}_S$ on the cobordism $S := ([-1,0]\times P)\#W_1$ is admissible if it is admissible in the sense above when restricted to $W_{1,1}\subset S$. Writing $\hat{\ell}_P$ for its restriction to $\{0\}\times P\subset S$, and $\hat{\ell}'_P$ for its restriction to $\{-1\}\times P$, we obtain the following map

$$\begin{aligned} s = -\cup(S,\hat{\ell}_S)\colon \mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}'_P) &\longrightarrow \mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P) \\ (M,\hat{\ell}_M) &\longmapsto ((M-e_1)\cup S, \hat{\ell}_M\cup\hat{\ell}_S). \end{aligned} \tag{7.1}$$

As in [**24**] we define the *θ-genus* for compact connected manifolds with θ-structure as

$$g^\theta(M,\hat{\ell}_M) := \max\left\{ g\in\mathbb{N} \;\middle|\; \begin{array}{l} \text{there are } g \text{ disjoint copies of } W_{1,1} \text{ in } M, \\ \text{each with admissible } \theta\text{-structure} \end{array} \right\}$$

and the *stable θ-genus* as

$$\overline{g}^\theta(M,\hat{\ell}_M) := \max\left\{ g^\theta((M,\hat{\ell}_M)\natural_k(W_{1,1},\hat{\ell}_{W_{1,1}})) - k \mid k\in\mathbb{N}\right\},$$

where the boundary connected sum is formed with k copies of $W_{1,1}$ each equipped with an admissible θ-structure $\hat{\ell}_{W_{1,1}}$. As in Chapter 6, we define $\mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P)_g\subset\mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P)$ as the subspace of those manifolds with stable genus precisely g. With this notation, the stabilisation map s defined above then restricts to a map

$$s\colon \mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}'_P)_g \to \mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P)_{g+1}.$$

We will now introduce a class of twisted coefficient systems. Since the spaces considered here are usually disconnected and do not have a preferred basepoint, twisted coefficients can be considered as a functor from the fundamental groupoid to the category of abelian groups. Note that this is closely related to the corresponding definitions in [**43**]. Then, an *abelian coefficient system* is a twisted coefficient system which has trivial monodromy along all nullhomologous loops.

For the theorem below we use the following definition.

Definition 7.2. [**24**, Def. 7.4] A tangential structure $\theta\colon B\to B\mathrm{O}(2n)$ is *spherical* if any θ-structure on $\mathbb{D}^{2n}$ extends to $\mathbb{S}^{2n}$.

Theorem 7.3. *Let $2n\geq 6$, W be a compact connected θ-manifold, and $\mathcal{L}$ be a twisted coefficient system on $\mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P)$. Considering twisted homology with coefficients in $\mathcal{L}$ we get a map*

$$s_*\colon H_k(\mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}'_P)_g; s^*\mathcal{L}) \longrightarrow H_k(\mathcal{M}^{\mathrm{st},\theta}(W,\hat{\ell}_P)_{g+1};\mathcal{L}).$$

(1) If $\mathcal{L}$ is abelian then s_* is an epimorphism for $k\leq\frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{3}$ and an isomorphism for $k\leq\frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-3}{3}$.

(2) If θ is spherical and $\mathcal{L}$ is constant, then s_* is an epimorphism for $k\leq\frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2}$ and an isomorphism for $k\leq\frac{g-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$.

For the case of simply-connected compact manifolds Galatius–Randal-Williams have shown in [**24**, Thm. 7.5] that the above stabilisation map s_* is an isomorphism in a range. Analogous to Theorems 4.2 and 5.2 there is a notion of coefficient system of degree d for moduli spaces. This can be found in [**30**] and the corresponding homological stability result is given in [**30**, Thm. H].

Given a pair $(W, \hat{\ell}_W) \in \mathcal{M}^\theta(W, \hat{\ell}'_P)$, we write $\mathcal{M}^\theta(W, \hat{\ell}_W) \subset \mathcal{M}^\theta(W, \hat{\ell}'_P)$ for the path component containing $(W, \hat{\ell}_W)$. By Theorem 7.3 the map

$$s \colon \mathcal{M}^\theta(W, \hat{\ell}_W) \longrightarrow \mathcal{M}^\theta(W \cup_P S, \hat{\ell}_W \cup \hat{\ell}_S)$$

is an isomorphism on homology with (abelian) coefficients in a range of degrees depending on $\overline{g}^\theta(W, \hat{\ell}_W)$.

The proof of Theorem 7.3 is analogous to the proof of [**24**, Thm. 7.5] and closely related to the proof of Theorem 6.3. We define a quadratic module for a pair $(W, \hat{\ell}_W) \in \mathcal{M}^\theta(W, \hat{\ell}'_P)$ as follows. Let $\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W) \subseteq \mathcal{I}_n^{\mathrm{tn}}(W)$ be the subgroup of those regular homotopy classes of immersions $i \colon \mathbb{S}^n \looparrowright W$ (together with a path in W) that have a trivialisation of the normal bundle of $\mathbb{S}^n$ such that the θ-structure $i^*\hat{\ell}_W$ on $\mathbb{S}^n \times \mathbb{D}^n$ is standard. The bilinear form λ and the quadratic function μ on $\mathcal{I}_n^{\mathrm{tn}}(W)$ restrict to the subgroup $\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W)$, and hence, define a quadratic module $(\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W), \lambda, \mu)$.

Following [**23**, Def. 7.14] we get an analogous version of Definition 6.9 for manifolds with tangential structures. Elements in $K_0(M, \hat{\ell}_M)$ are tuples (t, ϕ, ν), where $(t, \phi) \in K_0(M)$ and ν is a path in $\mathrm{Bun}^\theta(H)$ from $\varphi^*\hat{\ell}_M$ to a fixed θ-structure on H which is constant over $\{0\} \times \mathbb{D}^{2n-1} \subset H$. Then $K_p(M, \hat{\ell}_M)$ consist of tuples of elements in $(K_0(M, \hat{\ell}_M))$ which give an element of $K_p(M)$ after forgetting the paths ν. We topologise this as a subspace of $K_p(M) \times (\mathrm{Bun}^\theta(H)^I)^{p+1}$, and we write $K_p^\delta(M, \hat{\ell}_M)$ for the same set considered as a discrete space. The collection $K_\bullet(M, \hat{\ell}_M)$ forms a semisimplicial space. This is a simplicial complex by forgetting the paths and using Definition 6.9. The semi-simplicial space $\overline{K}_\bullet(M, \hat{\ell}_M)$ is defined analogous to Definition 6.9.

As in the previous section using the above notation we get a map of simplicial complexes

$$K^\delta(W, \hat{\ell}_W) \longrightarrow \mathcal{HU}(\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W)).$$

The following proposition is the analogue of Theorem 6.12 (3). For the case of simply-connected manifolds this has been shown in [**24**, Prop. 7.15].

Proposition 7.4. *Let $2n \geq 6$, W be a compact connected $2n$-dimensional manifold, and $\hat{\ell}_W$ be a θ-structure on W. Then $|\overline{K}_\bullet(W, \hat{\ell}_W)|$ is $\lfloor \frac{\overline{g}(W, \hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{2} \rfloor$-connected.*

Outline of the proof. We have already seen in the previous section that an embedding $i \colon W_{g,1} \hookrightarrow W$ yields elements $e_1, f_1, \ldots, e_g, f_g \in \mathcal{I}_n^{\mathrm{tn}}(W)$. If there is a trivialisation of the normal bundle such that the θ-structure $i^*\hat{\ell}_W$ is standard these elements are also contained in $\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W)$. In particular, we get $\overline{g}(\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W)) \geq \overline{g}(W, \hat{\ell}_W)$ and the complex $\mathcal{HU}(\mathcal{I}_n^{\mathrm{tn}}(W, \hat{\ell}_W))$ is locally weakly Cohen–Macaulay of dimension $\lfloor \frac{\overline{g}(W, \hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2} \rfloor$ – meaning that the link of any p-simplex is $(\lfloor \frac{\overline{g}(W, \hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)])}{2} \rfloor - p - 2)$-connected – by Corollary 6.13.

We first show that the complex $|K^\delta(W, \hat{\ell}_W)|$ is $\lfloor \frac{\overline{g}(W, \hat{\ell}_W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{2} \rfloor$-connected by arguing as in the proof of Theorem 6.12 (1). There we have described how to get a lift $F \colon I^{k+1} \to |K^\delta(M)|$ of the map

$$\overline{f} \colon I^{k+1} \to |\mathcal{HU}(I_n^{\mathrm{tn}}(W, \hat{\ell}_W))| \to |\mathcal{HU}(\mathcal{I}_n^{\mathrm{tn}}(W))|.$$

As shown in the proof of [**24**, Prop. 7.15] we can turn this into a lift $I^{k+1} \to |K^\delta(W, \hat{\ell}_W)|$.

The connectivity of $|\overline{K}_\bullet(W, \hat{\ell}_W)|$ now follows as in Theorem 6.12. $\square$

Outline of the proof of Theorem 7.3. This proof is based on the proof of [**24**, Thm. 7.5] and we therefore just describe the changes that we have to make to that proof. Note that the simply-connected assumption is only used in [**24**, analogue of Lemma 6.8] so we only have to show that the map considered in that statement is $\lfloor \frac{\overline{g}^\theta(W,\hat{\ell}_W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-1}{2} \rfloor$-connected for a compact and connected manifold W of dimension $2n \geq 6$. But this follows analogously to the proof of [**24**, analogue of Lemma 6.8] by applying Proposition 7.4 instead of [**24**, Prop. 7.15] as in the original proof. This also explains the slightly lower bound in our case. Throughout this proof we need to replace [**24**, Thm. 6.3] in the proof of [**24**, Thm. 7.5] by Theorem 6.3. $\square$

7.3. Stable Homology Groups

In this section we combine our homological stability results from this and the previous chapter with the results on stable homology groups by Galatius–Randal-Williams [**23**]. Let θ be a tangential structure. Using the Grassmannian $Gr_d(\mathbb{R}^\infty)$ of d-planes in $\mathbb{R}^\infty$ as a model of the classifying space of the orthogonal group $O(d)$ we can define a map θ_N as follows, where the right part of the diagram is a pullback square and γ_{N-d} is the orthogonal complement of the universal bundle.

$$\begin{tikzcd} \theta_N^*\gamma_{N-d} \arrow[r] & B_N \arrow[r] \arrow[d, "\theta_N"] & B \arrow[d, "\theta"] \\ \gamma_{N-d} \arrow[r] & Gr_d(\mathbb{R}^N) \arrow[r, hook] & Gr_d(\mathbb{R}^\infty) \end{tikzcd}$$

Using this notation we define the *Madsen–Tillmann spectrum of* θ, denoted by $MT\theta$, as the following Thom spectrum. The spaces are given by the Thom spaces

$$(MT\theta)_N := \mathrm{Th}(\theta_N^*\gamma_{N-d})$$

and the maps $\sigma_N \colon \Sigma\mathrm{Th}(\theta_N^*\gamma_{N-d}) \to \mathrm{Th}(\theta_{N+1}^*\gamma_{N+1-d})$ are induced by the pullback square

$$\begin{tikzcd} \gamma_{N-d} \oplus \varepsilon^1 \arrow[r] \arrow[d] & \gamma_{N+1-d} \arrow[d] \\ Gr_d(\mathbb{R}^N) \arrow[r, hook] & Gr_d(\mathbb{R}^{N+1}) \end{tikzcd}$$

and the fact that the Thom space of $\theta_N^*\gamma_{N-d} \oplus \varepsilon^1$ is the reduced suspension of $\mathrm{Th}(\theta_N^*\gamma_{N-d})$. We then define its infinite loop space by

$$\Omega^\infty MT\theta := \operatorname*{hocolim}_{N\to\infty} \Omega^N \mathrm{Th}(\theta_N^*\gamma_{N-d}),$$

where the maps are given by

$$\Omega^N \mathrm{Th}(\Omega_N^*\gamma_{N-d}) \xrightarrow{\Omega^N\hat{\sigma}_N} \Omega^{N+1}\mathrm{Th}(\theta_{N+1}^*\gamma_{N+1-d}),$$

where $\hat{\sigma}_N$ is the adjoint of σ_N.

$$\begin{tikzcd}
\partial W \arrow[d, hook] \arrow[drr, "\ell_{\partial W}"] & & & \\
W \arrow[rr, "\ell_W"] \arrow[dr, "\ell'_W"'] & & B \arrow[dr, "\theta"] & \\
& B' \arrow[ur, "u"] \arrow[rr, "\theta'"] & & BO(2n)
\end{tikzcd}$$

FIGURE 4. The Moore–Postnikov n-stage of the map $\ell_W \colon W \to B$.

Analogously, we define the spectrum $M\theta$ by

$$M\theta_N := \mathrm{Th}(\theta_N^*(\gamma_N))$$

with maps $\sigma_N \colon \Sigma M\theta_N \to M\theta_{N+1}$ induced by the pullback square

$$\begin{tikzcd}
\gamma_N \oplus \varepsilon^1 \arrow[r] \arrow[d] & \gamma_{N+1} \arrow[d] \\
Gr_d(\mathbb{R}^N) \arrow[r, hook] & Gr_d(\mathbb{R}^{N+1})
\end{tikzcd}$$

This is the more usual version of a Thom spectrum associated to a tangential structure θ. We will use this construction in later chapters.

Let $\hat{\ell}_W \colon TW \to \theta^*\gamma$ be a θ-structure. We consider the n-stage of the Moore–Postnikov tower of the underlying map $\ell_W \colon W \to B$, given by an n-connected map $\ell'_W \colon W \to B'$ and an n-coconnected fibration $u \colon B' \to B$. Without loss of generality we can assume that the map ℓ'_W is a cofibration. We define $\theta' := \theta \circ u \colon B' \to BO(2n)$ as shown in Figure 4.

Let $\mathrm{hAut}(u, \ell_{\partial W})$ be the space of weak equivalences $B' \to B'$ that make the following diagram commute.

$$\begin{tikzcd}
\partial W \arrow[r, "\ell'_{\partial W}"] \arrow[d, "\ell'_{\partial W}"'] & B' \arrow[d, "u"] \\
B' \arrow[r, "u"] \arrow[ur, dashed] & B
\end{tikzcd}$$

As remarked below [**23**, Def. 9.1] composition of maps makes $\mathrm{hAut}(u, \ell_{\partial W})$ into a grouplike topological monoid.

In [**21**, Rmk. 1.11] there is a construction of the map

$$\alpha \colon \mathcal{M}^\theta(W, \hat{\ell}_W) \longrightarrow \Omega^\infty MT\theta' /\!\!/ \mathrm{hAut}(u, \ell_{\partial W}),$$

where "$/\!\!/$" denotes the homotopy orbit space. Combining our homological stability theorem from the previous section with the stable homology results and techniques in [**23**] we show that the above map induces an isomorphism on homology in a certain range.

Theorem 7.5. *Let $2n \geq 6$ and W be a compact connected θ-manifold. The map α is acyclic in degrees $k \leq \frac{\overline{g}^\theta(W,\hat{\ell}_W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-3}{3}$, and if θ is spherical, the map is an isomorphism in integral homology in degrees $k \leq \frac{\overline{g}^\theta(W,\hat{\ell}_W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$, considered as a map to the path component that it hits.*

For the special case of simply-connected manifolds see [**23**, Cor. 1.9] and [**23**, Thm. 9.5]. Since $\overline{g}(W) \geq g(W)$ this implies Theorem D. For the trivial tangential structure $\theta = \mathbb{1}_{BO(2n)}$ we have $\mathcal{M}^\theta(W, \hat{\ell}_W) \simeq B\mathrm{Diff}_\partial(W)$. Also, the θ-genus and the ordinary genus of a manifold coincide. This yields the following corollary.

Corollary 7.6. *Let $2n \geq 6$ and W be a compact connected manifold with tangential structure $\theta = \mathbb{1}_{\mathrm{BO}(2n)}$. Then the map*

$$\alpha\colon B\mathrm{Diff}_\partial(W) \longrightarrow \Omega^\infty MT\theta' /\!\!/ \mathrm{hAut}(u, \ell_{\partial W})$$

is acyclic in degrees $k \leq \frac{\overline{g}(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-3}{3}$, and an isomorphism in integral homology in degrees $k \leq \frac{\overline{g}(W)-\mathrm{usr}(\mathbb{Z}[\pi_1(W)])-2}{2}$, onto the path component that it hits.

Analogous to [**23**, p. 6] we can consider $\theta\colon B\mathrm{SO}(2n) \to B\mathrm{O}(2n)$ to be the orientation cover. For an *oriented* map $\ell_W\colon W \to B\mathrm{SO}(2n)$ we again write

$$W \xrightarrow{\ell'_W} B' \xrightarrow{u^+} B\mathrm{SO}(2n)$$

for its Moore–Postnikov n-stage factorisation. Then the space $\mathcal{M}^\theta(W, \hat{\ell}_W)$ is a model for $B\mathrm{Diff}^+(W)$, the orientation preserving diffeomorphisms. For $2n \geq 6$ the map

$$\alpha\colon B\mathrm{Diff}^+(W) \longrightarrow \Omega^\infty MT\theta' /\!\!/ \mathrm{hAut}(u^+, \ell_{\partial W})$$

is a homology isomorphism in a range, onto the path component that it hits.

Lemma 7.7. *Let $2n \geq 6$ and W be a compact connected manifold which is $(n-1)$-connected relative to its boundary ∂W. Then* $\mathrm{hAut}(W, \ell_{\partial W})$ *is weakly contractible.*

Proof. We first show that $\mathrm{hAut}(u, \ell_{\partial W})$ is 0-connected. For this we show that any element $f \in \mathrm{hAut}(u, \ell_{\partial W})$ is homotopic to the identity $\mathbb{1}_{B'}$. This is equivalent to showing that there is a lift

$$\begin{tikzcd}
(I \times \partial W) \cup (\{0,1\} \times B') \arrow[rr, "\ell_{\partial W} \cup (\mathbb{1}_{B'} \sqcup f)"] \arrow[d, hook] & & B' \arrow[d, "u"] \\
I \times B' \arrow[rru, dashed] \arrow[r, two heads] & B' \arrow[r, "u"] & B
\end{tikzcd}$$

extending the given lift on $(I \times \partial W) \cup (\{0,1\} \times B')$, which is given by $I \times \partial W \to \partial W \to B'$, where the first map is the projection to ∂W and the second map is the map $\ell_{\partial W}$, and $\{0,1\} \times B' \to B'$ which maps $\{0\} \times B'$ via $\mathbb{1}_{B'}$ and $\{1\} \times B'$ via f. Note that by construction of f this map agrees on the intersection $\{0,1\} \times \partial W$. By Obstruction Theory ([**27**, Ch. 4.3, Prob. 24]) this lift exists if the groups

$$H^{k+1}(I \times B', (I \times \partial W) \cup (\{0,1\} \times B'); \pi_k(F))$$

vanish for all k, where F is the fibre of the fibration $u\colon B' \to B$. The map u is n-coconnected by construction, and hence, the homotopy groups of the fibre F vanish in degrees $k \geq n$. In particular, the above cohomology groups vanish for $k \geq n$. On the other hand, the inclusion $\partial W \hookrightarrow W$ is $(n-1)$-connected by assumption, and the map $\ell'_W\colon W \to B'$ is n-connected by construction, hence, the composition $\partial W \to B'$ is $(n-1)$-connected. Thus, the pair $(B', \partial W)$ is $(n-1)$-connected, and B' can be obtained from ∂W (up to weak equivalence) by attaching cells of dimension n and higher. Thus, the pair $(I \times B', (I \times \partial W) \cup \{0,1\} \times B')$ is a relative cell complex with relative cells of dimension at least $n+1$. In particular, the pair is n-connected, and hence, the above cohomology group vanishes in degrees $k+1 \leq n$. Thus, the above cohomology group vanishes in all degrees.

To show that $\pi_i(\mathrm{hAut}(u, \ell_{\partial W}))$ vanishes for $i \geq 1$, we need to show that the following dashed map exists.

$$\begin{tikzcd}
(\mathbb{D}^{i+1} \times \partial W) \cup (\mathbb{S}^i \times B') \arrow[rr, "\ell_{\partial W} \cup (\mathbb{1}_{B'} \sqcup f)"] \arrow[d, hook] & & B' \arrow[d, "u"] \\
\mathbb{D}^{i+1} \times B' \arrow[r, two heads] \arrow[urr, dashed] & B' \arrow[r, "u"] & B
\end{tikzcd}$$

The obstructions are now in

$$H^{k+1}(\mathbb{D}^{i+1} \times B', (\mathbb{D}^{i+1} \times \partial W) \cup (\mathbb{S}^i \times B'); \pi_k(F)).$$

The homotopy groups $\pi_k(F)$ still vanish in degrees $k \geq n$ and the relative cell complex $(\mathbb{D}^{i+1} \times B', (\mathbb{D}^{i+1} \times \partial W) \cup (\mathbb{S}^i \times B'))$ now is $(n+i)$-connected. In particular, the above cohomology group vanishes in all degrees. $\square$

Combining this lemma with Corollary 7.6 we get the following corollary.

Corollary 7.8. *Let $2n \geq 6$ and W be a compact connected manifold which is $(n-1)$-connected relative to its boundary with tangential structure $\theta = \mathbb{1}_{BO(2n)}$. Then the map*

$$\alpha\colon B\mathrm{Diff}_\partial(W) \longrightarrow \Omega^\infty MT\theta'$$

is acyclic in degrees $k \leq \frac{\overline{g}(W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 3}{3}$, and an isomorphism in integral homology in degrees $k \leq \frac{\overline{g}(W) - \mathrm{usr}(\mathbb{Z}[\pi_1(W)]) - 2}{2}$, onto the path component that it hits.

As in the previous chapter we can use Example 2.2 (3) to get versions of Theorem 7.5, Corollary 7.6, and Corollary 7.8 for the special case, where the fundamental group of the manifold is virtually polycyclic. The respective bounds can be obtained from the inequality

$$\mathrm{usr}(\mathbb{Z}[\pi_1(W)]) \leq h(\pi_1(W)) + 3$$

with $h(\pi_1(W))$ the Hirsch number of the fundamental group of W.

CHAPTER 8

Example:
A $\mathrm{Pin}^{-}(6)$-Manifold

In this chapter we compute the abelianisation of the mapping class group $\Gamma_c(X_\infty)$, i.e. the group of isotopy classes of compactly supported diffeomorphisms of X_∞, where X_∞ is the direct limit of a certain sequence of manifolds X_k with non-trivial boundary and fundamental group $\pi_1(X_\infty) = \mathbb{Z}/2\mathbb{Z}$. Using the Hurewicz Theorem, this is isomorphic to the first stable homology group of the moduli spaces of X_k, i.e. $H_1(B\mathrm{Diff}_c(X_\infty))$. The manifolds X_k have already been considered by Galatius–Randal-Williams in [**21**, Ch. 1.5], where they have talked about the stable homology. However, since the manifolds X_k are non-simply-connected, it was previously not possible to show homological stability in this setting. Now, we can use results from the previous chapters, in particular Corollary 7.8, to deduce homological stability. The construction of the manifolds X_k is as follows.

Let us consider $\mathbb{R}P^6$ as a quotient space of $\mathbb{S}^6$ by identifying antipodal points. Consider the Morse function

$$\begin{aligned} f\colon \mathbb{R}P^6 &\longrightarrow \mathbb{R} \\ [x_0 : x_1 : \ldots : x_6] &\longmapsto \sum_{i=0}^{6} i \cdot x_i^2. \end{aligned}$$

As shown in Figure 5. All its critical points are of the form $[0 : \ldots : 0 : 1 : 0 : \ldots : 0]$. The degree is i for the critical point for which the 1 is in the i-th position. This shows that $\mathbb{R}P^6$ has a handle decomposition with one 0-handle, one 1-handle, ..., and one 6-handle. Since $0, 1, \ldots, 6$ are the only critical values of f we can define submanifolds M and N given by $f^{-1}((-\infty, 2.5])$ and $f^{-1}([3.5, \infty))$ respectively.

The map $\sigma\colon \mathbb{R}P^6 \to \mathbb{R}P^6$ sending $[x_0 : x_1 : \ldots : x_6]$ to $[x_6 : x_5 : \ldots : x_0]$ is an involution on $\mathbb{R}P^6$. Precomposing this with the Morse function f yields the map $6 - f$, i.e. this turns f upside down. In particular, it identifies M with N. We write P for the boundary ∂M which under the above identification corresponds to ∂N, and K for the 3-handle $f^{-1}([2.5, 3.5])$.

We then define

$$X_k := M \cup \bigcup_{i=1}^{k} K.$$

8.1. Genus Estimation

We want to find copies of $W_{1,1}$ in X_k that intersect trivially with respect to the sesquilinear form λ.

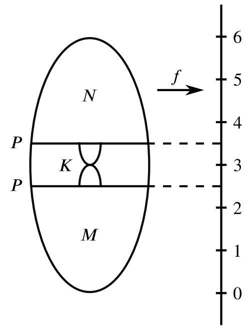

FIGURE 5. The Morse function f.

PROPOSITION 8.1. *The genus of the space* X_k *satisfies*

$$g(X_k) \geq \left\lfloor \frac{k-1}{2} \right\rfloor.$$

PROOF. We start by showing that the standard inclusion formed by three copies of K as shown in red in Figure 6 forms the first copy of $W_{1,1}$ with the standard framing. By

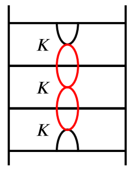

FIGURE 6. The first copy of $W_{1,1}$.

construction it is already a copy of $\mathbb{S}^3 \vee \mathbb{S}^3$. Therefore, we have to show that each copy of $\mathbb{S}^3$ has a trivial normal bundle. We do this by showing that it has a framing. Given two copies of K, K_1 and K_2 say, we choose a framing on the lower hemisphere of the copy of $\mathbb{S}^3$, which is a disc in K_1. Using the reflection σ described above we can define a reflection τ that swaps the copies of K along their intersection line as shown in Figure 7. This yields a framing of

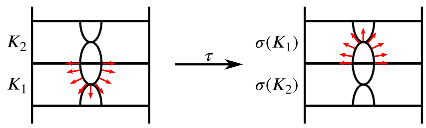

FIGURE 7. The reflection τ.

the upper hemisphere of $\mathbb{S}^3$. Since the reflection τ is the identity on the intersection of K_1 and K_2, and the framing at the points of K_1 is in this line, the two framings agree on their intersection. Thus, by gluing them together we get a framing of $\mathbb{S}^3$.

Now, given five copies of K, we want to find a second copy of $W_{1,1}$ whose intersection with the above copy of $W_{1,1}$ is trivial with respect to λ. Note that we cannot just pick the copy of $W_{1,1}$ that comes from adding two more copies of K as this intersects non-trivially with the copy of $W_{1,1}$ that we have described above. Instead, we start by considering a copy of the bottom-most $\mathbb{S}^3$. We have shown above that $\mathbb{S}^3$ has a trivial normal bundle so we can slide the copy off the original $\mathbb{S}^3$. Note that the intersection points with the sphere above and the disc below are preserved. We now take the connected sum of the third sphere from the bottom with the copy that we just moved. Note that the orientation and the connecting tube itself can be chosen so that the two intersection points of the two copies of $W_{1,1}$ have opposite signs and, hence, cancel. This comes from the fact that the boundary of a thickening of $W_{1,1}$ is simply-connected. Thus, we are in the situation as indicated in blue in Figure 8.

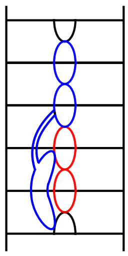

FIGURE 8. Intermediate step to achieve two copies of $W_{1,1}$.

Since the two copies of $W_{1,1}$ now have trivial intersection, we can slide the second copy of $W_{1,1}$ off the first one to make them disjoint. The first copy of $W_{1,1}$ now only intersects the bottom-most disc, and hence, we can drag it so that it is away from the rest. Now we can deform the bottom half of the second copy of $W_{1,1}$ so that it looks like an ordinary copy of $\mathbb{S}^3$, see Figure 9.

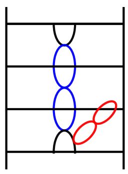

FIGURE 9. Two copies of $W_{1,1}$.

If we have more copies of K we can apply this procedure iteratively. We start at the bottom and choose the first and second copy of $W_{1,1}$ as described above. Note that during this process we do not change any intersection other than those of the two copies of $W_{1,1}$ involved. After dragging off the first copy as described in Figure 9 we can apply the procedure again to get the next two copies of $W_{1,1}$ to be disjoint. Again, this does not change any other intersections and in particular keeps the first copy of $W_{1,1}$ disjoint from all others. $\square$

8.2. Outline of the Computation of $\Gamma_\partial(X_\infty)^{\mathrm{ab}}$

As we have seen above $\Gamma_\partial(X_\infty)^{\mathrm{ab}}$ is isomorphic to the first stable homology group $H_1(B\mathrm{Diff}_\partial(X_\infty))$, where X_∞ is the direct limit of the manifolds X_k. Note that X_k is

2-connected relative to its boundary P since we can built X_k from P by attaching only handles of dimension 3 and higher. Using Corollary 7.8 we get an isomorphism

$$H_1(B\mathrm{Diff}_\partial(X_\infty)) \longrightarrow H_1(\Omega_0^\infty MT\theta'),$$

where the index 0 in $\Omega_0^\infty MT\theta'$ indicates that we only consider the path component of the basepoint. The same result also yields an isomorphism

$$H_1(B\mathrm{Diff}_\partial(X_k)) \longrightarrow H_1(\Omega_0^\infty MT\theta')$$

for k such that $1 \leq \frac{\overline{g}(X_k)-\mathrm{usr}(\mathbb{Z}[\pi_1(X_k)])-2}{2}$ which is equivalent to

$$\overline{g}(X_k) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(X_k)]) + 4.$$

Since the fundamental group $\pi_1(X_k) \cong \mathbb{Z}/2\mathbb{Z}$ is in particular finite, its Hirsch number vanishes, and hence, by Example 2.2 (3) we have $\mathrm{usr}(\mathbb{Z}[\pi_1(X_k)]) \leq 3$. Using the genus estimation from Proposition 8.1 the above inequality holds for $k \geq 15$, where we consider the tangential structure $\theta = \mathbb{1}_{BO(6)}$.

Let $W = X_1$ and $\theta = \mathbb{1}_{BO(6)}$. In the following section we show that B', corresponding to the theta structure θ' coming from the Moore–Postnikov tower as described in Chapter 7, is weakly homotopy equivalent to $B\mathrm{Pin}^-(6)$ defined below. Thus, in order to compute $H_1(\Omega_0^\infty MT\theta')$, which in the following we denote as $H_1(\Omega_0^\infty MT\mathrm{Pin}^-(6))$, we observe that

$$H_1(\Omega_0^\infty MT\mathrm{Pin}^-(6)) \cong \pi_1(\Omega_0^\infty MT\mathrm{Pin}^-(6)) \cong \pi_1(MT\mathrm{Pin}^-(6)), \tag{8.1}$$

where the first isomorphism holds by the Hurewicz Theorem and the fact that the first homotopy group of an infinite loop space of a spectrum is abelian by the correspondence $\pi_1(\Omega_0^\infty X) \cong \pi_2(\Omega_0^{\infty-1} X)$ for a spectrum X, and the second isomorphism holds because of $\pi_*(\Omega_0^\infty X) \cong \pi_*(X)$ in every degree $*$ by the definition of homotopy of a spectrum and the fact that colimits commute with taking homotopy groups.

To compute $\pi_1(MT\mathrm{Pin}^-(6))$ we first show that it is torsion. We do this by showing that $H^*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q})$ vanishes in low degrees, hence, so does $H_*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q})$ by the Universal Coefficient Theorem ([**27**, Thm. 3.2]). Therefore, by the Hurewicz Theorem the group $\pi_1(MT\mathrm{Pin}^-(6)) \otimes \mathbb{Q}$ is trivial which implies that the first homotopy group of $MT\mathrm{Pin}^-(6)$ is torsion.

We then show that $\pi_1(MT\mathrm{Pin}^-(6))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ by first computing the 2-torsion of this homotopy group and later showing that this group does not have any odd torsion. To compute the 2-torsion we compute $H^*(B\mathrm{Pin}^-(6);\mathbb{F}_2)$ as a module over the Steenrod algebra $\mathcal{A}$. Using the Thom isomorphism we get

$$H^*(B\mathrm{Pin}^-(6);\mathbb{F}_2) \xrightarrow{\cong} H^{*-6}(MT\mathrm{Pin}^-(6);\mathbb{F}_2)$$

which we use to compute $H^*(MT\mathrm{Pin}^-(6);\mathbb{F}_2)$ as a module over $\mathcal{A}$. This is the necessary input for Robert Bruner's programme [**12**] to compute the E_2-page of the Adams spectral sequence ([**36**, Thm. 9.1]) for $MT\mathrm{Pin}^-(6)$. We deduce from this (using a long exact sequence involving $MT\mathrm{Spin}(6)$, $MT\mathrm{Pin}^-(6)$, and another spectrum C as well as the E_2-page of the Adams spectral sequence for the spectrum C) that the 2-torsion of $\pi_1(MT\mathrm{Pin}^-(6))$ is $\mathbb{Z}/2\mathbb{Z}$.

8.3. Pin^{-}-Structures

Consider the tangential structure $\theta = \mathbb{1}_{BO(6)}$. We show that in this setting, the space B', given by the 3-stage of the Moore–Postnikov tower as described in the previous chapter, is weakly homotopy equivalent to $B\mathrm{Pin}^{-}(6)$ which, snalogous to the definitions of $B\mathrm{SO}(6)$ and $B\mathrm{Spin}(6)$, is defined as the fibre of the map

$$B\mathrm{O}(6) \xrightarrow{\omega_1^2+\omega_2} K(\mathbb{Z}/2\mathbb{Z}, 2),$$

where ω_1 and ω_2 are the first and second Stiefel–Whitney class of the universal bundle γ_6 on $B\mathrm{O}(6)$. For the space X_1 associated with a $\theta = \mathbb{1}_{BO(6)}$-structure we want to compute the associated quantities B' and θ'. Using that the 3-stage of the Moore–Postnikov tower factors the map $\ell_{X_1}\colon X_1 \to B\mathrm{O}(6)$ as a 3-connected map followed by a 3-coconnected map, $X_1 \to B' \to B\mathrm{O}(6)$, we get

$$\pi_i(B') \cong \begin{cases} \pi_i(X_1) & 0 \le i \le 2 \\ \pi_i(B\mathrm{O}(6)) & i \ge 4 \end{cases}$$

and an injection $\pi_3(B') \hookrightarrow \pi_3(B\mathrm{O}(6))$. Using $\pi_3(B\mathrm{O}(6)) = 0$ and $X_1 = M \cup K \simeq \mathbb{R}P^3$, since X_1 is a neighbourhood of the 3-skeleton of $\mathbb{R}P^6$, we can therefore compute the homotopy groups of B' up to degree 4 as

$$\pi_i(B') \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 4 \\ 0 & \text{otherwise} \end{cases}$$

By the long exact sequence in homotopy the fibre F of the map $\theta'\colon B' \to B\mathrm{O}(6)$ then satisfies $\pi_1(F) = \mathbb{Z}/2\mathbb{Z}$ and all other homotopy groups vanish. Hence, the fibre F is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$. Using the fact that $\pi_2(B')$ vanishes this long exact sequence also shows that the boundary homomorphism $\partial\colon \pi_2(B\mathrm{O}(6)) \to \pi_1(F)$ is an isomorphism.

Note that for $X_1 \subset \mathbb{R}P^6$ the total Stiefel–Whitney class is given by

$$\omega(TX_1) = (1+x)^7 = 1 + x + x^2 + x^3,$$

where $x \in H_1(\mathbb{R}P^6; \mathbb{F}_2)$, and elements of degree at least 4 vanish since $X_1 \simeq \mathbb{R}P^3$. Thus, for $\mathbb{F}_2$-coefficients we get $\omega_2(TX_1) = \omega_1(TX_1)^2$ in X_1. Since the map $X_1 \to B'$ coming from the Moore–Postnikov tower is 3-connected, this equality also holds in B'. This implies that the composition

$$B' \longrightarrow B\mathrm{O}(6) \xrightarrow{\omega_1^2+\omega_2} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

is trivial. In particular, we get a map $B' \to B\mathrm{Pin}^{-}(6)$ by picking a lift. Therefore, we get a map of fibrations as shown in the following diagram, where the map f is defined so that the left square commutes.

$$\begin{tikzcd}
K(\mathbb{Z}/2\mathbb{Z},1) \arrow[r] & B\mathrm{Pin}^-(6) \arrow[r] & B\mathrm{O}(6) \arrow[r, "\omega_1^2+\omega_2"] & K(\mathbb{Z}/2\mathbb{Z},2) \\
K(\mathbb{Z}/2\mathbb{Z},1) \arrow[r] \arrow[u, "f"'] & B' \arrow[r] \arrow[u] & B\mathrm{O}(6) \arrow[u, equal] &
\end{tikzcd}$$

We consider the induced diagram on homotopy groups. By construction, the induced map $\pi_k(B\mathrm{O}(6)) \to \pi_k(B\mathrm{O}(6))$ is the identity for all degrees k. Since the fibre F is a $K(\mathbb{Z}/2\mathbb{Z},1)$ all homotopy groups other than the fundamental group are trivial. Thus, for $k \neq 1$ the induced map f_* on the k-th homotopy group is in particular an isomorphism. We show that the map f_* is also an isomorphism on π_1, and hence, by the Five Lemma the induced vertical maps $\pi_k(B') \to \pi_k(B\mathrm{Pin}^-(6))$ are isomorphisms in all degrees.

We saw above that the boundary homomorphism $\partial\colon \pi_2(B\mathrm{O}(6)) \to \pi_1(F)$ of the bottom long exact sequence is an isomorphism. Thus, we get the following diagram.

$$\begin{tikzcd}
\pi_2(B\mathrm{O}(6)) \arrow[r, "\partial_{\mathrm{Pin}^-(6)}"] & \pi_1(K(\mathbb{Z}/2\mathbb{Z},1)) \\
\pi_2(B\mathrm{O}(6)) \arrow[r, "\cong"] \arrow[u, equal] & \pi_1(K(\mathbb{Z}/2\mathbb{Z},1)) \arrow[u, "f_*"]
\end{tikzcd}$$

We now show that the top boundary homomorphism $\partial_{\mathrm{Pin}^-(6)}$ is also an isomorphism. This implies that f_* is an isomorphism on π_1. As described above we then get a weak homotopy equivalence $B' \simeq B\mathrm{Pin}^-(6)$.

To show that $\partial_{\mathrm{Pin}^-(6)}$ is an isomorphism we consider the composition of isomorphisms

$$\pi_2(B\mathrm{O}(6)) \xleftarrow{\cong} \pi_2(B\mathrm{SO}(6)) \xrightarrow{\cong} H_2(B\mathrm{SO}(6);\mathbb{Z}),$$

where there first map is an isomorphism since $B\mathrm{SO}(6)$ is the universal cover of $B\mathrm{O}(6)$, and the second map is the Hurewicz homomorphism which is an isomorphism because $\pi_1(B\mathrm{SO}(6))$ is trivial, as well as the Bockstein long exact sequence [**45**, Ch. 5.2, Thm. 7]

$$\cdots \longrightarrow H_2(B\mathrm{SO}(6);\mathbb{Z}) \longrightarrow H_2(B\mathrm{SO}(6);\mathbb{Z}) \longrightarrow H_2(B\mathrm{SO}(6);\mathbb{Z}/2\mathbb{Z}) \longrightarrow \cdots,$$

induced by the short exact sequence

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

These maps fit into the following commutative diagram.

$$\begin{tikzcd}
& \pi_2(B\mathrm{O}(6)) \arrow[r, "\omega_1^2+\omega_2"] & \pi_1(K(\mathbb{Z}/2\mathbb{Z},1)) \cong \mathbb{Z}/2\mathbb{Z} \\
& \pi_2(B\mathrm{SO}(6)) \arrow[u, "\cong"] \arrow[d, "\cong"] & \\
H_2(B\mathrm{SO}(6);\mathbb{Z}) \arrow[r, "\cdot 2"] & H_2(B\mathrm{SO}(6);\mathbb{Z}) \arrow[r] & H_2(B\mathrm{SO}(6);\mathbb{Z}/2\mathbb{Z}) \arrow[uu, "\omega_2"']
\end{tikzcd}$$

Since $\pi_2(B\mathrm{O}(6))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so is $H_2(B\mathrm{SO}(6);\mathbb{Z})$ by the above isomorphism. In particular, the leftmost map in the above diagram is trivial. Thus, the next map in the Bockstein long exact sequence is injective, and hence, by

$$H_*(B\mathrm{SO}(6);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\omega_2,\omega_3,\dots,\omega_6]$$

(cf. [**38**, Thm. 12.4]) it is also an isomorphism. This representation of the $\mathbb{Z}/2\mathbb{Z}$-homology of $B\mathrm{SO}(6)$ also implies that the rightmost vertical map is an isomorphism. By commutativity of the diagram, the top horizontal map, which is the boundary homomorphism $\partial_{\mathrm{Pin}^-(6)}$, is an isomorphism.

8.4. Rational Cohomology of $\Omega_0^\infty MT\mathrm{Pin}^-(6)$

In order to compute the rational cohomology of $\Omega_0^\infty MT\mathrm{Pin}^-(6)$ we first express it as a free graded commutative algebra on the positive part of the cohomology ring of $B\mathrm{Pin}^-(6)$ with twisted coefficients, and we will show this is a free module over $H^*(BO(6);\mathbb{Q})$ on one generator. We start by defining twisted coefficients.

Given a based space (X,x_0) and a $\mathbb{Z}[\pi_1(X,x_0)]$-module M, we define

$$\begin{aligned} H_*(X;M) &:= H_*\left(C_*(\tilde{X})\otimes_{\mathbb{Z}[\pi_1(X)]} M\right)\\ H^*(X;M) &:= H_*\left(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}\left(C_*(\tilde{X}),M\right)\right)\end{aligned}$$

In the following we write $\mathbb{Z}^{\omega_1}$ for $\mathbb{Z}$ considered as a $\mathbb{Z}[\pi_1(X,x_0)]$-module, where a loop $g\in\pi_1(X,x_0)$ acts by multiplication by its first Stiefel–Whitney class $\omega_1(g)\in\{\pm1\}$.

Example 8.2. We compute the twisted rational cohomology $H^*(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1})$, where we define $\mathbb{Q}^{\omega_1}$ as the product $\mathbb{Z}^{\omega_1}\otimes\mathbb{Q}$. The rational cohomology groups of $B\mathrm{O}(6)$ and $B\mathrm{SO}(6)$ are given by

$$\begin{aligned} H^*(B\mathrm{O}(6);\mathbb{Q}) &\cong \mathbb{Q}[p_1,p_2,p_3]\\ H^*(B\mathrm{SO}(6);\mathbb{Q}) &\cong \mathbb{Q}[p_1,p_2,e],\end{aligned}$$

where the p_i are the Pontryagin classes of degree $|p_i| = 4i$ and e denotes the Euler class which is in degree 6 (cf. [**38**, Prob. 15-B] and [**38**, Thm. 15.9]). We start by computing $H^*(B\mathrm{O}(6);\mathbb{Q}^{\omega_1})$ which we show later agrees with $H^*(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1})$. Writing π for the fundamental group $\pi_1(B\mathrm{O}(6))$ and t for its generator, the sequence

$$\begin{aligned} 0\longrightarrow \mathbb{Q} \longrightarrow &\mathbb{Q}\pi \longrightarrow \mathbb{Q}^{\omega_1}\longrightarrow 0\\ 1\longmapsto &1+t\\ &1\longmapsto 1\\ &t\longmapsto -1\end{aligned}$$

is exact. The map $\mathbb{Q}\pi\to\mathbb{Q}$ which sends both 1 and t to $1/2$ shows that this sequence is in fact split exact. This short exact sequence then induces a long exact sequence on cohomology groups of $B\mathrm{O}(6)$ with the three different modules as coefficients. Since $B\mathrm{SO}(6)$ is the universal, finite cover of $B\mathrm{O}(6)$, we have $H^*(B\mathrm{O}(6);\mathbb{Q}\pi)\cong H^*(B\mathrm{SO}(6);\mathbb{Q})$. Using this and the description of the cohomology given at the beginning of the example, we get the following exact sequence.

$$\begin{aligned}\mathbb{Q}[p_1,p_2,p_3] &\hookrightarrow \mathbb{Q}[p_1,p_2,e] \longrightarrow H^*(B\mathrm{O}(6);\mathbb{Q}^{\omega_1})\\ p_1 &\longmapsto p_1\\ p_2 &\longmapsto p_2\\ p_3 &\longmapsto e^2\end{aligned}$$

The injectivity of the first map comes from the splitting of the above short exact sequence. By exactness we get the isomorphism

$$H^{*+6}(B\mathrm{O}(6);\mathbb{Q}^{\omega_1}) \cong H^*(B\mathrm{O}(6);\mathbb{Q})\langle e\rangle$$

as modules, i.e. elements of $H^*(B\mathrm{O}(6);\mathbb{Q})\langle e\rangle$ are of the form αe for $\alpha \in H^*(B\mathrm{O}(6);\mathbb{Q})$. The next step is to identify $H^*(B\mathrm{O}(6);\mathbb{Q}^{\omega_1})$ with $H^*(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1})$. For this we again consider the fibration

$$K(\mathbb{Z}/2\mathbb{Z},1) \xrightarrow{i} B\mathrm{Pin}^-(6) \xrightarrow{\phi} B\mathrm{O}(6).$$

Given a coefficient system $\mathcal{F}$ on $B\mathrm{O}(6)$ we get a coefficient system $\phi^*\mathcal{F}$ on $B\mathrm{Pin}^-(6)$ and by exactness a trivial coefficient system $i^*\phi^*\mathcal{F}$ on $K(\mathbb{Z}/2\mathbb{Z},1)$. This gives a Serre spectral sequence

$$H^p(B\mathrm{O}(6);H^q(K(\mathbb{Z}/2\mathbb{Z},1);\mathbb{Q})\otimes\mathcal{F}) \Rightarrow H^{p+q}(B\mathrm{Pin}^-(6);\phi^*\mathcal{F}).$$

The only non-trivial rational cohomology group of the Eilenberg–Mac Lane space $K(\mathbb{Z}/2\mathbb{Z},1)$ is a copy of the rational numbers in degree 0. Thus, we have

$$H^p(B\mathrm{O}(6);\mathcal{F}) \cong H^p(B\mathrm{Pin}^-(6);\phi^*\mathcal{F}).$$

For $\mathcal{F} = \mathbb{Q}^{\omega_1}$ we then get $\phi^*\mathbb{Q}^{\omega_1} = \mathbb{Q}^{\omega_1}$ since Stiefel–Whitney classes pull back to Stiefel–Whitney classes, and hence

$$H^p(B\mathrm{O}(6);\mathbb{Q}^{\omega_1}) \cong H^p(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1}).$$

For the next part we follow [**28**, Thm. C.1]. Analogous to this result by Hatcher we get the following theorem which is a standard result among researchers in this area.

Theorem 8.3. *Let X be a connective spectrum with finitely generated cohomology groups in each degree. There is a map*

$$S(H^*(X;\mathbb{Q})) \longrightarrow H^*(\Omega^\infty_0 X;\mathbb{Q})$$

which is an isomorphism, where S denotes the free graded commutative algebra on a given set of generators.

Proof. Let $\{b_i\}_{i\in I}$ be a basis for $H^*(X;\mathbb{Q})$. Since cohomology is represented by maps to Eilenberg–Mac Lane spaces, these generators correspond to a map

$$f = \bigvee_{i\in I} b_i\colon X \longrightarrow \bigvee_{i\in I}\Sigma^{|b_i|}H\mathbb{Q},$$

where we write $H\mathbb{Q}$ for the Eilenberg–Mac Lane spectrum which has $K(\mathbb{Q},n)$ as its n-th space. By definition of the b_i's the map

$$f^*\colon \prod_{i\in I}\mathbb{Q}\{b_i\} \longrightarrow H^*(X;\mathbb{Q}),$$

where $\mathbb{Q}\{b_i\}$ denotes a copy of the rational numbers in degree $|b_i|$, is an isomorphism. Dually, the induced map f_* on homology is also an isomorphism. Thus, the map f is an isomorphism on rational homotopy of spectra because rational homotopy is equal to rational homology in spectra. Therefore, the map

$$\Omega^\infty X \xrightarrow{\Omega^\infty f} \prod_{i\in I} K(\mathbb{Q},|b_i|)$$

is an isomorphism on rational homotopy since in positive degrees the rational homotopy groups of a spectra and those of its infinite loop space are the same. Hence, by the Hurewicz Theorem mod $\mathcal{C}$ [**38**, Thm. 18.3] it is also an isomorphism on rational cohomology of spaces. $\square$

Given a spectrum X we have a fibration

$$\tau_{>0}X \longrightarrow X \longrightarrow \tau_{\leq 0}X,$$

where $\tau_{\leq 0}X$ denotes the spectrum which in non-positive degrees has the same homotopy groups as X and is 0 otherwise. This can be obtained by attaching cells to X in order to kill the higher homotopy groups. The spectrum $\tau_{>0}X$ is called the connective cover of X and has homotopy groups

$$\pi_i(\tau_{>0}X) \cong \begin{cases} \pi_i(X) & i>0 \\ 0 & i\leq 0 \end{cases}$$

Since rational homotopy groups are isomorphic to rational homology groups on spectra, we have the following commutative diagram for $k\geq 1$, induced by the inclusion $\tau_{>0}X \hookrightarrow X$.

$$\begin{tikzcd} \pi_k(\tau_{>0}X)\otimes\mathbb{Q} \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_k(X)\otimes\mathbb{Q} \arrow[d, "\cong"] \\ H_k(\tau_{>0}X;\mathbb{Q}) \arrow[r] & H_k(X;\mathbb{Q}) \end{tikzcd}$$

Thus, the rational homology groups of the connective cover $\tau_{>0}X$ agree with those of X in positive degrees. In the situation $X = MT\mathrm{Pin}^-(6)$ we get the following corollary.

COROLLARY 8.4. *There is an isomorphism*

$$H^*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q}) \cong S(H^*(BO(6);\mathbb{Q})_{>0}),$$

where the subscript ">0" indicates that we only take the positive degree part.

Proof. Let $\tau_{>0}MT\mathrm{Pin}^-(6)$ be the connective cover of $MT\mathrm{Pin}^-(6)$. By Theorem 8.3 we get

$$\begin{aligned} H^*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q}) &\cong H^*(\Omega^\infty \tau_{>0}MT\mathrm{Pin}^-(6);\mathbb{Q}) \\ &\cong S(H^*(\tau_{>0}MT\mathrm{Pin}^-(6);\mathbb{Q})) \\ &\cong S(H^*(MT\mathrm{Pin}^-(6);\mathbb{Q})_{>0}). \end{aligned}$$

Applying the (inverse of the) Thom isomorphism with twisted coefficients and the results of the previous section then yields

$$\begin{aligned} H^*(MT\mathrm{Pin}^-(6);\mathbb{Q}) &\cong \varprojlim_N H^{*+N}\left(\mathrm{Th}((\theta')_N^*(\gamma_{N-6}^{\mathrm{Pin}^-}));\mathbb{Q}\right) \\ &\cong \varprojlim_N H^{*+6}(B'_N;\mathbb{Q}^{\omega_1}) \\ &\cong H^{*+6}(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1}). \end{aligned}$$

Combining this with the computation of $H^*(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1})$ we made in Example 8.2 we get

$$\begin{aligned} H^*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q}) &\cong S(H^{*+6}(B\mathrm{Pin}^-(6);\mathbb{Q}^{\omega_1})_{>0}) \\ &\cong S(H^{*+6}(B\mathrm{O}(6);\mathbb{Q}^{\omega_1})_{>0}) \\ &\cong S((H^{*+6}(B\mathrm{O}(6);\mathbb{Q})\langle e\rangle)_{>0}) \\ &\cong S((H^*(\Sigma^{-6}B\mathrm{O}(6);\mathbb{Q})\langle e\rangle)_{>0}) \\ &\cong S(H^*(B\mathrm{O}(6);\mathbb{Q})_{>0}), \end{aligned}$$

where the last isomorphism sends an element αe for $\alpha \in H^*(\Sigma^{-6}B\mathrm{O}(6);\mathbb{Q})$ to the element in $H^*(B\mathrm{O}(6);\mathbb{Q})$ corresponding to α under the suspension isomorphism. □

An element in $H^*(B\mathrm{O}(6);\mathbb{Q})$ is a sum of monomials of the form $p_1^a p_2^b p_3^c$. Thus, by the above corollary we get

$$H^*(\Omega_0^\infty MT\mathrm{Pin}^-(6);\mathbb{Q}) \cong \mathbb{Q}[\tau_{a,b,c} \mid a+b+c>0],$$

where we write $\tau_{a,b,c}$ for the monomial $p_1^a p_2^b p_3^c$.

8.5. Computation of $\pi_1(MT\mathrm{Pin}^-(6))$

The computation above implies that the first three rational cohomology groups of the infinite loop space $\Omega_0^\infty MT\mathrm{Pin}^-(6)$ vanish. Hence, by the Hurewicz Theorem, its fundamental group has to be torsion. In this section we compute $\pi_1(MT\mathrm{Pin}^-(6))$ using the Adams spectral sequence of the spectrum $MT\mathrm{Pin}^-(6)$. For this we need the cohomology groups with $\mathbb{F}_2$-coefficients up to degree 2 (note that the first cohomology class is in degree -6) and its Steenrod square structure. For the latter we use of the Wu Formula ([**35**, p. 197])

$$\mathrm{Sq}^i(\omega_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} \omega_{i-t}\omega_{j+t}$$

which computes the Steenrod squares of the Stiefel–Whitney classes. Afterwards we show that there cannot be any odd torsion in $\pi_1(MT\mathrm{Pin}^-(6))$. The aim of this section is to prove the following theorem.

THEOREM 8.5. $\pi_1(MT\mathrm{Pin}^-(6)) \cong \mathbb{Z}/2\mathbb{Z}$.

As explained in Section 8.2 this implies Theorem F.

8.5.1. $H^*(B\mathrm{Pin}^-(6);\mathbb{F}_2)$ and its Steenrod Squares. In order to compute the Adams spectral sequence we first need to know the generators of the first eight cohomology groups with $\mathbb{F}_2$-coefficients, and the Steenrod squares between them. For the computation of the cohomology of $B\mathrm{Pin}^-(6)$ we consider the fibration

$$K(\mathbb{Z}/2\mathbb{Z},1) \longrightarrow B\mathrm{Pin}^-(6) \longrightarrow B\mathrm{O}(6)$$

which we described earlier in this chapter. This yields a Serre spectral sequence in cohomology ([**36**, Thm. 5.2]) and computing the E_{10}-page gives the required generators and their relations. Using the ring isomorphisms

$$H^*(BO(6);\mathbb{F}_2) \cong \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6]$$
$$H^*(K(\mathbb{Z}/2\mathbb{Z},1);\mathbb{F}_2) \cong \mathbb{F}_2[x],$$

where x is a generator of $H^1(K(\mathbb{Z}/2\mathbb{Z},1);\mathbb{F}_2)$, the E_2-page is given by

$$E_2 = \mathbb{F}_2[x] \otimes \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6].$$

By definition of the fibration we have $d_2(x) = \omega_1^2 + \omega_2$. Thus, when passing on to the next page this differential kills x and induces the relation $\omega_1^2 = \omega_2$, i.e. we have

$$E_3 = \mathbb{F}_2[x^2] \otimes \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6]/(\omega_1^2+\omega_2).$$

For the computation of $d_3(x^2)$ we use $\mathrm{Sq}^1(x) = x^2$ and the fact that Steenrod squares commute with transgressions in the Serre spectral sequence ([**36**, Prop. 6.5]). By the Cartan formula and the Wu formula we get

$$d_3(x^2) = d_3(\mathrm{Sq}^1(x)) = \mathrm{Sq}^1(d_2(x)) = \mathrm{Sq}^1(\omega_1^2+\omega_2) = \omega_1^3+\omega_3,$$

where we have used the relation $\omega_1^2 = \omega_2$. For a general element $(x^2)^k \otimes p$ in the E_3-page we have $d_3((x^2)^k \otimes p) = k(x^2)^{k-1} \otimes (\omega_1^3+\omega_3)p$. Since this is trivial for k even, the image and kernel of d_3 are given by

$$\mathrm{Im}(d_3) = \left\{ x^{4l} \otimes (\omega_1^3+\omega_3)\cdot p \;\middle|\; l \in \mathbb{N}, p \in \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6]/(\omega_1^2+\omega_2) \right\}$$
$$\mathrm{Ker}(d_3) = \left\{ x^{4l} \otimes p \;\middle|\; l \in \mathbb{N}, p \in \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6]/(\omega_1^2+\omega_2) \right\}$$

respectively. In particular, the element x^2 is not a cycle. Using this computation, the E_4-page looks like

$$E_4 = \mathbb{F}_2[x^4] \otimes \mathbb{F}_2[\omega_1,\omega_2,\dots,\omega_6]/(\omega_1^2+\omega_2, \omega_1^3+\omega_3).$$

The next differential to consider is $d_5(x^4)$. Since $\mathrm{Sq}^2(x^2) = x^4$ we can use the same techniques as above, as well as the identifications $\omega_1^2 = \omega_2$ and $\omega_1^3 = \omega_3$ to get $d_5(x^4) = \omega_1\omega_4 + \omega_5$ and hence

$$E_6 = \mathbb{F}_2[x^8] \otimes \mathbb{F}_2[\omega_1, \omega_2, \ldots, \omega_6]/(\omega_1^2 + \omega_2, \omega_1^3 + \omega_3, \omega_1\omega_4 + \omega_5).$$

Analogous to the above and using the additional identification $\omega_1\omega_4 = \omega_5$, we can compute $d_9(x^8) = 0$, and hence, the E_{10}-page is isomorphic to the E_6-page. Thus, the cohomology with $\mathbb{F}_2$-coefficients is given by

$$H^k(B\mathrm{Pin}^-(6); \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & k = 0 \\ \mathbb{F}_2\langle\omega_1\rangle & k = 1 \\ \mathbb{F}_2\langle\omega_1^2\rangle & k = 2 \\ \mathbb{F}_2\langle\omega_1^3\rangle & k = 3 \\ \mathbb{F}_2\langle\omega_1^4, \omega_4\rangle & k = 4 \\ \mathbb{F}_2\langle\omega_1^5, \omega_1\omega_4\rangle & k = 5 \\ \mathbb{F}_2\langle\omega_1^6, \omega_1^2\omega_4, \omega_6\rangle & k = 6 \\ \mathbb{F}_2\langle\omega_1^7, \omega_1^3\omega_4, \omega_1\omega_6\rangle & k = 7 \\ \mathbb{F}_2\langle\omega_1^8, \omega_1^4\omega_4, \omega_1^2\omega_6, \omega_4^2, x_8\rangle & k = 8 \end{cases}$$

where x_8 denotes the element corresponding to x^8. Note that it is not a power of a generator of lower degree anymore, hence, the change of notation to avoid confusion.

In order to compute the Steenrod squares we use the multiplicativity of the total Steenrod square and the Wu formula. For example, for the generator $\omega_1\omega_4$ in degree 5 we get

$$\begin{aligned} \mathrm{Sq}(\omega_1\omega_4) &= \mathrm{Sq}(\omega_1)\mathrm{Sq}(\omega_4) \\ &= (\omega_1 + \omega_1^2)(\omega_4 + \omega_6 + \omega_1^2\omega_4 + \omega_1\omega_6 + \omega_4^2) \\ &= \omega_1\omega_4 + \omega_1^2\omega_4 + \omega_1^3\omega_4 + \omega_1\omega_6 + \omega_1^4\omega_4 \end{aligned}$$

so the only non-trivial Steenrod squares of $\omega_1\omega_4$ up to degree 8 are

$$\begin{aligned} \mathrm{Sq}^0(\omega_1\omega_4) &= \omega_1\omega_4, \\ \mathrm{Sq}^1(\omega_1\omega_4) &= \omega_1^2\omega_4, \\ \mathrm{Sq}^2(\omega_1\omega_4) &= \omega_1^3\omega_4 + \omega_1\omega_6 \\ \mathrm{Sq}^3(\omega_1\omega_4) &= \omega_1^4\omega_4. \end{aligned}$$

Note that the above ingredients enable us to compute all Steenrod squares up to degree 8 since the only Steenrod squares that involve x_8 are those starting in x_8, and hence, have their target in degree at least 9. Table 1 in Appendix A shows the complete Steenrod square structure in this range.

8.5.2. $H^*(MT\mathrm{Pin}^-(6);\mathbb{F}_2)$ and its Steenrod Squares. Similar to the previous section we have a Thom isomorphism

$$H^{*+6}(B\mathrm{Pin}^-(6);\mathbb{F}_2) \xrightarrow{\cong} H^*(MT\mathrm{Pin}^-(6);\mathbb{F}_2)$$

which is induced by taking the cup product with the Thom class u. Note that since we now work with $\mathbb{F}_2$-coefficients we can use the ordinary Thom isomorphism instead of the version with twisted coefficients. Using this, we immediately get the following cohomology groups of $MT\mathrm{Pin}^-(6)$ with $\mathbb{F}_2$-coefficients.

$$H^k(MT\mathrm{Pin}^-(6);\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\langle u\rangle & k=-6 \\ \mathbb{F}_2\langle \omega_1 u\rangle & k=-5 \\ \mathbb{F}_2\langle \omega_1^2 u\rangle & k=-4 \\ \mathbb{F}_2\langle \omega_1^3 u\rangle & k=-3 \\ \mathbb{F}_2\langle \omega_1^4 u, \omega_4 u\rangle & k=-2 \\ \mathbb{F}_2\langle \omega_1^5 u, \omega_1\omega_4 u\rangle & k=-1 \\ \mathbb{F}_2\langle \omega_1^6 u, \omega_1^2\omega_4 u, \omega_6 u\rangle & k=0 \\ \mathbb{F}_2\langle \omega_1^7 u, \omega_1^3\omega_4 u, \omega_1\omega_6 u\rangle & k=1 \\ \mathbb{F}_2\langle \omega_1^8 u, \omega_1^4\omega_4 u, \omega_1^2\omega_6 u, \omega_4^2 u, x_8 u\rangle & k=2 \end{cases}$$

In order to compute the Steenrod squares we use the multiplicativity of the total Steenrod square and the identity $\mathrm{Sq}(u) = \omega\left(-\gamma_6^{\mathrm{Pin}^-}\right)u$. Note that $-\gamma_6^{\mathrm{Pin}^-}$ is the complement of the Pin^--bundle $\gamma_6^{\mathrm{Pin}^-}$. For two complementary bundles V and W, i.e. $V \oplus W = \varepsilon^n$, we have $\omega(V)\omega(W) = \omega(V \oplus W) = 1$, and hence, their total Stiefel–Whitney classes are inverses to each other. In our case, up to degree 8 we therefore get

$$\begin{aligned}\omega\left(-\gamma_6^{\mathrm{Pin}^-}\right) &= \frac{1}{\gamma_6^{\mathrm{Pin}^-}} \\ &= \frac{1}{1+\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6} \\ &= \frac{1}{1+\omega_1+\omega_1^2+\omega_1^3+\omega_4+\omega_1\omega_4+\omega_6} \\ &= 1+\omega_1+\omega_1^4+\omega_4+\omega_1^5+\omega_1\omega_4+\omega_1^2\omega_4+\omega_6+\omega_1^3\omega_4+\omega_1^8+\omega_1^2\omega_6+\omega_4^2+\cdots,\end{aligned}$$

where we have used the relations $\omega_2 = \omega_1^2$, $\omega_3 = \omega_1^3$, and $\omega_5 = \omega_1\omega_4$ computed in Section 8.5.1.

Using the Steenrod squares we have computed in the previous section for the cohomology of $B\mathrm{Pin}^-(6)$, we can now compute all Steenrod squares for the cohomology of $MT\mathrm{Pin}^-(6)$ up to degree 2 since the only Steenrod squares that involve x_8 are those starting in x_8, and hence, have their target in higher degrees. For example, for the element $\omega_1\omega_4 u$ we then get

$$\begin{aligned}\mathrm{Sq}(\omega_1\omega_4 u) &= \mathrm{Sq}(\omega_1\omega_4)\omega\left(-\gamma_6^{\mathrm{Pin}^-}\right)u \\ &= (\omega_1\omega_4+\omega_1\omega_6+\omega_1^2\omega_6)u\end{aligned}$$

which shows that the only non-trivial Steenrod squares of $\omega_1\omega_4 u$ up to degree 2, where the lowest class is in degree -6, are

$$\begin{aligned}\mathrm{Sq}^0(\omega_1\omega_4 u) &= \omega_1\omega_4 u,\\ \mathrm{Sq}^2(\omega_1\omega_4 u) &= \omega_1\omega_6 u,\\ \mathrm{Sq}^3(\omega_1\omega_4 u) &= \omega_1^2\omega_6 u.\end{aligned}$$

The Steenrod squares up to degree 2 can be found in Table 2 in Appendix A.

8.5.3. Two Adams Spectral Sequences. Given both the $\mathbb{F}_2$-cohomology groups of the spectrum $MT\mathrm{Pin}^-(6)$ and its Steenrod square structure we can now use Robert Bruner's programme [**12**] to get a partial E_2-page of the Adams spectral sequence which converges to the 2-primary homotopy groups of the spectrum $MT\mathrm{Pin}^-(6)$ as shown in Figure 10. Note that the diagram is complete to the left of the dotted line. In the figure, vertical lines correspond to multiplication by $2 \in S^0$, where S^0 is the sphere spectrum, lines of slope 1 correspond to multiplication by $\eta \in \pi_1(S^0)$. The code for the module definition is given in Table 1 in Appendix B.

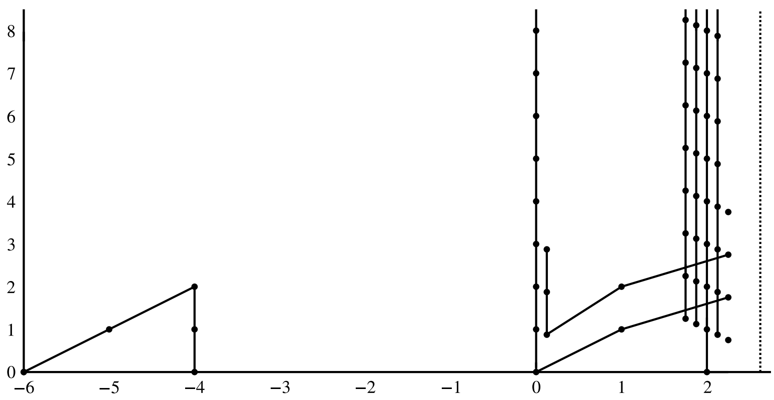

FIGURE 10. The Adams spectral sequence for $MT\mathrm{Pin}^-(6)$.

On the E_2-page of the Adams spectral sequence in degree 1, there is a copy of $\mathbb{Z}/2\mathbb{Z}$ both in filtrations 1 and 2 without any additive extensions. Neither of the elements in degree 1 can have an outgoing differential as this would force a differential from the element in degree 0 that corresponds to this element under multiplication of η. For degree reasons there are also no incoming differentials for the element in degree 1 and filtration 1, and therefore, $\pi_1(MT\mathrm{Pin}^-(6))_{(2)}$ is either $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. This argument also implies

$$\pi_0(MT\mathrm{Pin}^-(6))_{(2)} \cong \mathbb{Z}_{(2)} \oplus \mathbb{Z}/8\mathbb{Z}.$$

Note that both the integers and the 2-adic integers are represented as an infinite tower in the Adams spectral sequence, so we cannot tell them apart from Figure 10 alone. However, since the homology groups of $B\mathrm{Pin}^-(6)$ are finitely generated, the homotopy groups are also finitely generated. Therefore, $\pi_0(MT\mathrm{Pin}^-(6))_{(2)}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.

We now show that there is an injective map $\pi_1(MT\mathrm{Pin}^-(6))_{(2)} \hookrightarrow \mathbb{Z}/2\mathbb{Z}$. For this we consider the following cofibration sequence which is a special case of the one described in the proof of [**20**, Prop. 3.1] (cf. [**42**, p. 28]).

$$MT\mathrm{Spin}(6) \longrightarrow MT\mathrm{Pin}^-(6) \longrightarrow \mathrm{Th}\left(-\gamma_6^{\mathrm{Pin}^-} \oplus \gamma_1^{\pm} \to B\mathrm{Pin}^-(6)\right) =: C$$

Here, $\gamma_1^{\pm}$ denotes the unique non-trivial real line bundle over $B\mathrm{Pin}^-(6)$. This induces a long exact sequence on homotopy groups which looks as follows.

$$\cdots \longrightarrow \pi_1(MT\mathrm{Spin}(6))_{(2)} \longrightarrow \pi_1(MT\mathrm{Pin}^-(6))_{(2)} \longrightarrow \pi_1(C)_{(2)} \longrightarrow \cdots \tag{8.2}$$

By [**22**, Lemma 5.4] and [**22**, Lemma 5.6] the groups $\pi_1(MT\mathrm{Spin}(6))$ and $\pi_7(M\mathrm{Spin})$ are isomorphic, using $M\mathrm{O}\langle 3\rangle = M\mathrm{Spin}$. By [**37**] the group $\pi_7(M\mathrm{Spin})$ vanishes, and thus, so does $\pi_1(MT\mathrm{Spin}(6))$. This implies that the above map to $\pi_1(C)_{(2)}$ is injective. The following lemma is the remaining ingredient to show that $\pi_1(MT\mathrm{Pin}^-(6))_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$.

LEMMA 8.6. *$\pi_1(C)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$.*

PROOF. As for the spectrum $MT\mathrm{Pin}^-(6)$, we consider the Adams spectral sequence of the spectrum C by computing its cohomology ring over the Steenrod algebra. The Thom isomorphism shows

$$H^*(C;\mathbb{F}_2) \cong H^{*-5}(B\mathrm{Pin}^-(6);\mathbb{F}_2).$$

We proceed analogously to the computation above, now with total Stiefel–Whitney class

$$\begin{aligned}\omega\left(-\gamma_6^{\mathrm{Pin}^-} \oplus \gamma_1^{\pm}\right) &= \omega\left(-\gamma_6^{\mathrm{Pin}^-}\right)(1+\omega_1)\\ &= 1+\omega_1^2+\omega_1^4+\omega_4+\omega_1^6+\omega_6+\omega_1\omega_6+\omega_1^8+\omega_1^4\omega_4+\omega_1^2\omega_6+\omega_4^2+\cdots.\end{aligned}$$

As before, we can now compute all Steenrod squares up to degree 3 (note that the lowest cohomology class is now in degree -5). For example, we get

$$\begin{aligned}\mathrm{Sq}(\omega_1\omega_4 u) &= \mathrm{Sq}(\omega_1\omega_4)\omega\left(-\gamma_6^{\mathrm{Pin}^-} \oplus \gamma_1^{\pm}\right)u\\ &= (\omega_1\omega_4+\omega_1^2\omega_4+\omega_1\omega_6)u.\end{aligned}$$

A full chart can be found in Table 3 in Appendix A.

We use Robert Bruner's programme again to get a partial E_2-page of the Adams spectral sequence which converges to the 2-primary homotopy groups of C as shown in Figure 11. Again, the diagram is complete to the left of the dotted line. The code for the module definition is given in Table 2 in Appendix B.

The element in degree 1 filtration 1 cannot have an outgoing differential since this would force a differential from the element in degree 0 filtration 0. By the multiplicative structure this cannot happen. For degree reasons, there are also no incoming differentials for this element. Hence, the copy of $\mathbb{Z}/2\mathbb{Z}$ in degree 1 survives the Adams spectral sequence. $\square$

The above discussion shows that $\pi_1(MT\mathrm{Pin}^-(6))_{(2)}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It remains to show that $\pi_1(MT\mathrm{Pin}^-(6))$ does not have any odd torsion. We do this by showing that there is a map $f\colon S^0\left[\frac{1}{2}\right] \to MT\mathrm{Pin}^-(6)\left[\frac{1}{2}\right]$ which induces an isomorphism on π_1. Since

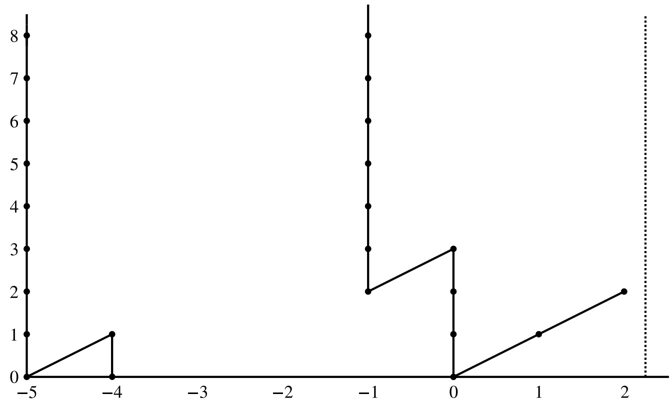

FIGURE 11. The Adams spectral sequence for C.

$\pi_1(S^0)\left[\frac{1}{2}\right]$ is trivial, the group $\pi_1(MT\mathrm{Pin}^-(6))\left[\frac{1}{2}\right]$ vanishes. This implies the there is no odd torsion in $\pi_1(MT\mathrm{Pin}^-(6))$.

Analogous to Example 8.2 we can compute

$$H^*(BO(6);\mathbb{Z}\left[\tfrac{1}{2}\right]^{\omega_1}) \cong (\mathbb{Z}\left[\tfrac{1}{2}\right])[p_1,p_2,p_3]\langle e\rangle,$$

which has no elements in degrees below 6. We consider the fibration

$$\mathrm{Pin}^-(6) \longrightarrow B\mathrm{O}(6) \xrightarrow{\omega_1^2+\omega_2} K(\mathbb{Z}/2\mathbb{Z},2).$$

Away from prime 2 the Eilenberg–Mac Lane space $K(\mathbb{Z}/2\mathbb{Z},2)$ has trivial homology groups in every degree, and thus, the homology groups of $\mathrm{Pin}^-(6)$ and $B\mathrm{O}(6)$ are isomorphic. By a Thom isomorphism argument as in the proof of Corollary 8.4 we then get the isomorphism

$$H^*(MT\mathrm{Pin}^-(6);\mathbb{Z}\left[\tfrac{1}{2}\right]) \cong H^{*+6}(B\mathrm{Pin}^-(6);\mathbb{Z}\left[\tfrac{1}{2}\right]^{\omega_1}).$$

In particular, the spectrum $MT\mathrm{Pin}^-(6)$ has no integral homology away from prime 2 in negative degrees. This implies that by the Hurewicz Theorem we have

$$\pi_0(MT\mathrm{Pin}^-(6)\left[\tfrac{1}{2}\right]) \cong \mathbb{Z}\left[\tfrac{1}{2}\right].$$

By definition, the element 1 in $\pi_0(MT\mathrm{Pin}^-(6)\left[\frac{1}{2}\right])$ defines a map from the sphere spectrum S^0 to the spectrum $MT\mathrm{Pin}^-(6)\left[\frac{1}{2}\right]$. By the universal property of the localisation of a spectrum this map factors through $S^0\left[\frac{1}{2}\right]$ as shown below.

$$\begin{tikzcd}
S^0 \arrow[rr] \arrow[dr] & & S^0\left[\tfrac{1}{2}\right] \arrow[dl, dashed, "f"] \\
& MT\mathrm{Pin}^-(6)\left[\tfrac{1}{2}\right] &
\end{tikzcd}$$

By construction, on the 0-th homology the induced map f_* sends the element $1 \in \mathbb{Z}\left[\frac{1}{2}\right]$ to itself, and hence, is an isomorphism. Note also that the first, second, and third homology groups of both the sphere spectrum S^0 – and hence, in particular of $S^0\left[\frac{1}{2}\right]$ – and the spectrum $MT\mathrm{Pin}^-(6)$ vanish. This implies that the cofibre C_f of the map f has trivial homology up to degree 3. By the Hurewicz Theorem the cofibre C_f also has trivial homotopy groups in this range. By the long exact sequence corresponding to this cofibration sequence, the map f is 3-connected.

With the above discussion this proves Theorem 8.5.

CHAPTER 9

Example: Spin(6)-Manifolds with Fundamental Group $\mathbb{Z}/2^k\mathbb{Z}$

In the previous chapter we started with a specific manifold and computed the abelianisation of its mapping class group by identifying its θ-structure and using a stable homology result from Chapter 7. In both this and the following chapter we start by fixing a θ-structure and consider manifolds with this θ-structure. Again, the aim is to compute the abelianisation of its mapping class group. For this, let G be a group such that the unitary stable rank $\mathrm{usr}(\mathbb{Z}[G])$ is finite, and M a 6-dimensional manifold which is 2-connected relative to its boundary and fits into the following commutative diagram

$$\begin{tikzcd} B\mathrm{Spin}(6) \times BG \arrow[r, "\theta"] & BO(6) \\ M \arrow[u, "\ell_M"] \arrow[ur, "\tau"'] & \end{tikzcd} ,$$

where the tangential structure θ is given by first projecting to $B\mathrm{Spin}(6)$ and then applying the usual $\theta^{\mathrm{Spin}(6)}\colon B\mathrm{Spin}(6) \to BO(6)$, and ℓ_M is 3-connected. In particular, the first two homotopy groups of M are $\pi_1(M) \cong G$ and $\pi_2(M) = 0$. We say that M is of *tangential 3-type* $B\mathrm{Spin}(6) \times BG$.

We use Corollary 7.8 to get an isomorphism from the first stable homology of the moduli space of this manifold to the first homology of $\Omega_0^\infty MT\theta'$, where $\theta'\colon B' \to B\mathrm{O}(6)$ comes from the Moore–Postnikov tower. Note that in our case, θ' is given by the above map $\theta\colon B\mathrm{Spin}(6) \times BG \to B\mathrm{O}(6)$ as Figure 4 is given by the following diagram.

$$\begin{tikzcd} \partial M \arrow[d, hook] \arrow[drr, "\ell_{\partial M}"] & & & \\ M \arrow[rr, "\tau"] \arrow[dr, "\ell_M"'] & & B\mathrm{O}(6) \arrow[dr, "\mathbb{1}_{B\mathrm{O}(6)}"] & \\ & B\mathrm{Spin}(6) \times BG \arrow[ur, "\theta"] \arrow[rr, "\theta"] & & B\mathrm{O}(6) \end{tikzcd}$$

To apply the corollary we need to ensure that the stable genus of M is big. In particular, to use the corollary in homological degree 1, the genus of M has to satisfy

$$\overline{g}(M) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(M)]) + 4.$$

Taking the connected sum of M with $W_g = \#_g \mathbb{S}^3 \times \mathbb{S}^3$ still satisfies all conditions above. Since the group ring of G has finite unitary stable rank by assumption, for $g \geq \mathrm{usr}[\mathbb{Z}[\pi_1(M)]] + 4$

we get

$$\begin{aligned} H_1(B\mathrm{Diff}_\partial(M\#W_g)) &\cong H_1(\Omega_0^\infty(MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+)) \\ &\cong \pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+), \end{aligned}$$

where we write $H_*(\Omega_0^\infty(MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+))$ for $H^*(\Omega_0^\infty MT\theta')$, and the second isomorphism above is as described in Equation (8.1). The results for $G=\mathbb{Z}/2\mathbb{Z}$ and $G=\mathbb{Z}/2^k\mathbb{Z}$ for $k\geq 2$ are as follows.

THEOREM 9.1. $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)\cong\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/16\mathbb{Z}$.

THEOREM 9.2. *The order of the group* $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ *is* 2^{3k+2}.

For the first theorem the structure of the proof is similar to that described in Section 8.2.

- Step 1: We compute $H^*(B\mathrm{Spin}(6)\times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)$ as a module over the Steenrod algebra $\mathcal{A}$ by using the Künneth theorem.
- Step 2: The Thom isomorphism relates $H^*(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{F}_2)$ as a module over $\mathcal{A}$ to the computation of Step 1.
- Step 3: The previous step enables us to compute a part of the E_2-page of the Adams spectral sequence. We use Robert Bruner's programme **[12]** to compute this. We work out the necessary differentials in this spectral sequence to show that the even torsion in $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ is as stated above.
- Step 4: We show that the group $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ has no odd torsion.

For the proof of the second theorem, the first two steps are the analogues of Step 1 and Step 2 for $\mathbb{Z}/2^k\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$, Step 3 gets replaced by

- Step 3′: To get the order of the even torsion in $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ we consider the Atiyah–Hirzebruch spectral sequence ([**1**, Ch. 7, Thm.])

$$H_p(\mathbb{Z}/2^k\mathbb{Z};\pi_q(MT\mathrm{Spin}(6)))\Longrightarrow\pi_{p+q}(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+),$$

and the final step is the analogue of Step 4 for $\mathbb{Z}/2^k\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$.

9.1. The Case $G=\mathbb{Z}/2\mathbb{Z}$

Step 1. We want to compute $H^*(B\mathrm{Spin}(6)\times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)$ as a module over the Steenrod algebra $\mathcal{A}$. By the Künneth Theorem we have

$$H^k(B\mathrm{Spin}(6)\times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)\cong\bigoplus_{i+j=k}H^i(B\mathrm{Spin}(6);\mathbb{F}_2)\otimes H^j(B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)$$

and, hence, we can compute both parts individually. To compute $H^*(B\mathrm{Spin}(6);\mathbb{F}_2)$ we consider the following fibration

$$B\mathrm{Spin}(6)\longrightarrow B\mathrm{SO}(6)\xrightarrow{\omega_2}K(\mathbb{Z}/2\mathbb{Z},2),$$

where the second map is given by mapping to the second Stiefel–Whitney class ω_2. The above fibration can also be viewed as a fibration

$$K(\mathbb{Z}/2\mathbb{Z},1)\longrightarrow B\mathrm{Spin}(6)\longrightarrow B\mathrm{SO}(6).$$

Using the ring isomorphisms

$$H^*(B\mathrm{SO}(6); \mathbb{F}_2) \cong \mathbb{F}_2[\omega_2, \omega_3, \ldots, \omega_6],$$
$$H^*(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{F}_2) \cong \mathbb{F}_2[x],$$

where ω_i is the i-th Stiefel–Whitney class and x is a generator in degree 1, the E_2-page of the Serre spectral sequence in cohomology corresponding to the above fibration is given by

$$E_2 = \mathbb{F}_2[x] \otimes \mathbb{F}_2[\omega_2, \omega_3, \ldots, \omega_6].$$

By definition of the fibration we have a differential $d_2(x) = \omega_2$. Passing on to the next page of the Serre spectral sequence this differential kills x and induces the relation $\omega_2 = 0$, i.e. we have

$$E_3 = \mathbb{F}_2[x^2] \otimes \mathbb{F}_2[\omega_2, \omega_3, \ldots, \omega_6]/(\omega_2).$$

As in the previous chapter, using the Cartan formula, the Wu formula and the relation $\omega_2 = 0$ we get

$$d_3(x^2) = d_3(\mathrm{Sq}^1(x)) = \mathrm{Sq}^1(d_2(x)) = \mathrm{Sq}^1(\omega_2) = \omega_1\omega_2 + \omega_3 = \omega_3$$

and, therefore, the E_4-page looks like

$$E_4 = \mathbb{F}_2[x^4] \otimes \mathbb{F}_2[\omega_2, \omega_3, \ldots, \omega_6]/(\omega_2, \omega_3).$$

Using the above techniques and $\mathrm{Sq}^2(x^2) = x^4$ we get $d_5(x^4) = \omega_5$, and hence

$$E_6 = \mathbb{F}_2[x^8] \otimes \mathbb{F}_2[\omega_2, \omega_3, \ldots, \omega_6]/(\omega_2, \omega_3, \omega_5).$$

Analogous to the above calculations we get $d_9(x^8) = 0$ and, thus, we have $E_{10} \cong E_6$. Up to degree 8 we therefore get

$$H^*(B\mathrm{Spin}(6); \mathbb{F}_2) \cong \mathbb{F}_2[\omega_4, \omega_6, x_8],$$

where x_8 denotes the generator coming from $K(\mathbb{Z}/2\mathbb{Z}, 1)$ in degree 8, i.e.

$$H^k(B\mathrm{Spin}(6); \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & k = 0 \\ 0 & k = 1 \\ 0 & k = 2 \\ 0 & k = 3 \\ \mathbb{F}_2\langle\omega_4\rangle & k = 4 \\ 0 & k = 5 \\ \mathbb{F}_2\langle\omega_6\rangle & k = 6 \\ 0 & k = 7 \\ \mathbb{F}_2\langle\omega_4^2, x_8\rangle & k = 8 \end{cases}$$

Using the multiplicativity of total Steenrod squares and the Wu formula, the Steenrod square structure is given as follows.

$$\begin{aligned}\mathrm{Sq}^2(\omega_4) &= \omega_6,\\ \mathrm{Sq}^4(\omega_4) &= \omega_4^2,\\ \mathrm{Sq}^2(\omega_6) &= 0.\end{aligned}$$

Note that these are all Steenrod squares up to degree 8 since the only Steenrod squares that involve x_8 are those starting in x_8, and hence, have their target in degree at least 9.

For the computation of $H^*(B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)$ note that the infinite real projective space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z},1)$ and so we get

$$H^*(B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2) \cong H^*(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[z],$$

for a generator z in degree 1. The Steenrod square structure is given by $\mathrm{Sq}^1(z) = z^2$ and the Cartan formula.

In total, we have

$$H^*(B\mathrm{Spin}(6) \times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2) \cong \mathbb{F}_2[z,\omega_4,\omega_6,x_8]$$

up to degree 8, i.e.

$$H^k(B\mathrm{Spin}(6) \times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & k=0\\ \mathbb{F}_2\langle z\rangle & k=1\\ \mathbb{F}_2\langle z^2\rangle & k=2\\ \mathbb{F}_2\langle z^3\rangle & k=3\\ \mathbb{F}_2\langle z^4,\omega_4\rangle & k=4\\ \mathbb{F}_2\langle z^5, z\omega_4\rangle & k=5\\ \mathbb{F}_2\langle z^6, z^2\omega_4,\omega_6\rangle & k=6\\ \mathbb{F}_2\langle z^7, z^3\omega_4, z\omega_6\rangle & k=7\\ \mathbb{F}_2\langle z^8, z^4\omega_4, z^2\omega_6, \omega_4^2, x_8\rangle & k=8 \end{cases}$$

and the Steenrod square structure can be computed by using the multiplicativity of the total Steenrod squares. This can be found in Table 4 in Appendix A.

Step 2. Using the Thom isomorphism as described in the proof of Corollary 8.4, we get an isomorphism

$$(-)\cup u\colon H^*(B\mathrm{Spin}(6) \times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2) \xrightarrow{\cong} H^{*-6}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{F}_2).$$

Combining this with the result from Step 1 then yields

$$H^*(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{F}_2) \cong \mathbb{F}_2[z,\omega_4,\omega_6,x_8]u,$$

up to degree 2, i.e. we have

$$H^k(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\langle u \rangle & k = -6 \\ \mathbb{F}_2\langle zu \rangle & k = -5 \\ \mathbb{F}_2\langle z^2 u \rangle & k = -4 \\ \mathbb{F}_2\langle z^3 u \rangle & k = -3 \\ \mathbb{F}_2\langle z^4 u, \omega_4 u \rangle & k = -2 \\ \mathbb{F}_2\langle z^5 u, z\omega_4 u \rangle & k = -1 \\ \mathbb{F}_2\langle z^6 u, z^2\omega_4 u, \omega_6 u \rangle & k = 0 \\ \mathbb{F}_2\langle z^7 u, z^3\omega_4 u, z\omega_6 u \rangle & k = 1 \\ \mathbb{F}_2\langle z^8 u, z^4\omega_4 u, z^2\omega_6 u, \omega_4^2 u, x_8 u \rangle & k = 2 \end{cases}$$

Analogous to Section 8.5.2 we have $\mathrm{Sq}(u) = \omega(-\gamma_6^{\mathrm{Spin}})u$. Since the bundle $-\gamma_6^{\mathrm{Spin}}$ is the complement of γ_6^{Spin} we can compute up to degree 8

$$\omega(-\gamma_6^{\mathrm{Spin}}) = \frac{1}{\omega(\gamma_6^{\mathrm{Spin}})} = \frac{1}{1 + \omega_4 + \omega_6} = 1 + \omega_4 + \omega_6 + \omega_4^2 + \cdots , \tag{9.1}$$

where we have used that ω_1, ω_2, ω_3, and ω_5 vanish. Therefore, the Steenrod square structure can be computed using the multiplicativity of the total Steenrod squares. This can be found in Table 5 in Appendix A.

Step 3. Using the $\mathbb{F}_2$-cohomology of the spectrum $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$ as a module over the Steenrod algebra $\mathcal{A}$ we can compute the E_2-page of the Adams spectral sequence which converges to the 2-primary homotopy groups of $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$. We do this by using Robert Bruner's programme [**12**] with the input given in Table 3 in Appendix B. The resulting E_2-page is given in Figure 12. As before, the diagram is complete to the left of the dotted line.

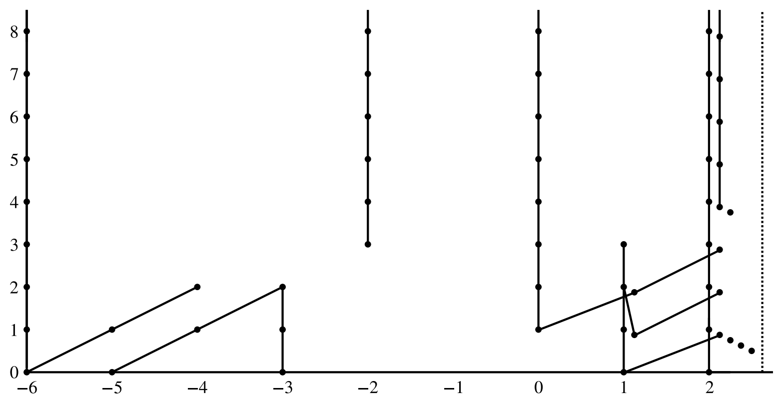

FIGURE 12. The Adams spectral sequence for $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$.

There cannot be any outgoing differentials in degree 1 as there is only an infinite tower in degree 0. For degree reasons there are also no incoming differentials into the elements in degree 1 and filtration 0 and 1. Hence, they survive the Adams spectral sequence. At least one of the elements in degree 1 and filtration 2 has to survive since they can only be killed by an element in degree 2 and filtration 0 but there is only one such element. We show that in fact the element in degree 1 and filtration 2 that is not part of the tower is killed this way.

We use $B\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{R}P^\infty$ to see that the based spectrum splits as

$$\begin{aligned} MT\mathrm{Spin}(6) \wedge \Sigma^\infty \mathbb{R}P^\infty_+ &\simeq MT\mathrm{Spin}(6) \wedge (\Sigma^\infty(\mathbb{S}^0 \vee \mathbb{R}P^\infty)) \\ &\simeq MT\mathrm{Spin}(6) \vee MT\mathrm{Spin}(6) \wedge \Sigma^\infty \mathbb{R}P^\infty. \end{aligned}$$

In particular, the Adams spectral sequence of the spectrum $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$ contains a copy of the Adams spectral sequence of the spectrum $MT\mathrm{Spin}(6)$. Using the computations of $H^*(B\mathrm{Spin}(6); \mathbb{F}_2)$ and $\omega(-\gamma_6^{\mathrm{Spin}})$ from the previous step we get

$$H^k(MT\mathrm{Spin}(6); \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2\langle u\rangle & k=-6 \\ 0 & k=-5 \\ 0 & k=-4 \\ 0 & k=-3 \\ \mathbb{F}_2\langle \omega_4 u\rangle & k=-2 \\ 0 & k=-1 \\ \mathbb{F}_2\langle \omega_6 u\rangle & k=0 \\ 0 & k=1 \\ \mathbb{F}_2\langle \omega_4^2 u, x_8 u\rangle & k=2 \end{cases}$$

with Steenrod square structure given by $\mathrm{Sq}(u) = (1+\omega_4+\omega_6+\omega_4^2+\cdots)u$ and $\mathrm{Sq}^2(\omega_4 u) = \omega_6 u$. This yields the module definition given in Table 4 in Appendix B. Using Robert Bruner's programme [**12**] we get the E_2-page of the Adams spectral sequence for $MT\mathrm{Spin}(6)$ up to degree 2 as shown in Figure 13.

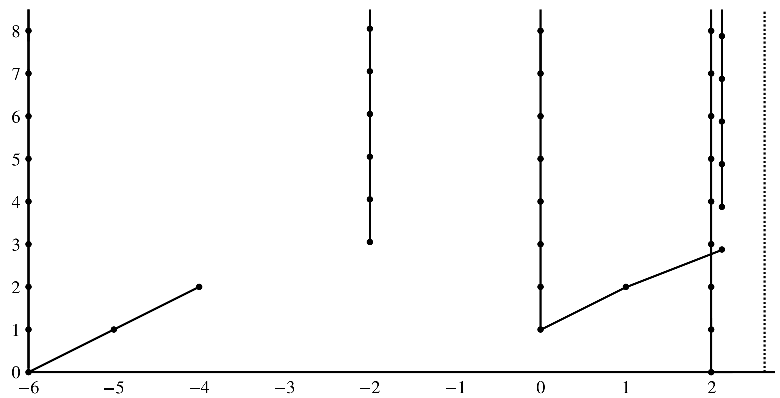

FIGURE 13. The Adams spectral sequence for $MT\mathrm{Spin}(6)$.

We have shown in Section 8.5.3 that $\pi_1(MT\mathrm{Spin}(6))$ vanishes by [**21**], and hence, the element in degree 1 must be killed in the Adams spectral sequence. This implies that there is a differential from the element in degree 2 and filtration 0. In particular, this differential must also be in the Adams spectral sequence for the spectrum $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$. Hence, the element in degree 1 and filtration 2 in Figure 12 that is not part of the tower is killed as claimed above. Therefore, we can conclude that $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)_{(2)}$ is either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}$, depending on whether the element in degree 1 and filtration 3 gets killed. We show that this element survives the Adams spectral sequence by showing that we can detect an element of order 16.

Consider the maps

$$\lambda\colon MT\mathrm{Spin}(6) \longrightarrow \Sigma^{-6}M\mathrm{Spin} \longrightarrow \Sigma^{-6}\mathrm{ko},$$

where the first map is induced by $B\mathrm{Spin}(6) \to B\mathrm{Spin}$ on Thom spectra and the second map is the Atiyah–Bott–Shapiro orientation [**3**], and

$$\mathrm{ko} \xrightarrow{\mu} H\mathbb{Z} \xrightarrow{\rho_2} H\mathbb{Z}/2\mathbb{Z},$$

where the first map is given by the 0-stage of the Moore–Postnikov tower, and the second map is reduction modulo 2. By [**11**, Ch. 3.1] we have

$$\pi_7(\mathrm{ko} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \cong \mathbb{Z}/16\mathbb{Z}.$$

We show that the map

$$f = (\lambda \wedge \mathbb{1}_{\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+})_*\colon \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \to \pi_7(\mathrm{ko} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$$

is surjective. Hence, this detects a copy of $\mathbb{Z}/16\mathbb{Z}$ in $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$. To show that f is surjective we construct an element $\alpha \in \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ that gets mapped to a generator in $\pi_7(\mathrm{ko}\wedge\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ under f. The map f fits into the commutative diagram

$$\begin{tikzcd}
\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \arrow[r, "f"] \arrow[ddr, "h"] & \pi_7(\mathrm{ko} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \arrow[d] \arrow[dd, bend left, "g"] \\
& \pi_7(H\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \arrow[d] \\
& \pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)
\end{tikzcd}$$

where g is given by $(\rho_2 \wedge \mathbb{1}_{\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+})_* \circ (\mu \wedge \mathbb{1}_{\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+})_*$ and h is the composition $g \circ f$. Using $B\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{R}P^\infty$ we have

$$\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) = H_7(\Sigma^\infty \mathbb{R}P^\infty_+; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Thus, if h is surjective, then so is g, and g therefore corresponds to the map $\mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ given by reduction modulo 2. Therefore, an element in $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ that gets mapped to the non-trivial element in $\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ under h must map to an odd number in $\mathbb{Z}/16\mathbb{Z} \cong \pi_7(\mathrm{ko} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$. Note that all odd numbers generate the group $\mathbb{Z}/16\mathbb{Z}$ so this implies that f is surjective. We have an isomorphism

$$\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \longrightarrow \mathrm{Hom}(H^7(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

coming from the correspondence $\pi_*(HA \wedge X) \cong H_*(X, A)$ for abelian groups A. The non-trivial element in the groups of homomorphisms $\mathrm{Hom}(H^7(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ is given by the map Ψ that sends z^7 to 1, where z^7 is the element in $H^7(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z})$ corresponding to z^7 in the calculation in Step 1. We write ψ for the element in the group $\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ corresponding to Ψ under the above isomorphism.

PROPOSITION 9.3. *There is an element $\alpha \in \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ that gets sent to ψ under h.*

The proposition shows that the map h is surjective. As shown above, this implies that the map f is also surjective, and hence, $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)_{(2)}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/16\mathbb{Z}$.

PROOF. Consider the following commutative diagram

$$\begin{tikzcd}
\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \arrow[r] \arrow[d, "h"] & \mathrm{Hom}_{\mathcal{A}}(H^*(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}[1]) \arrow[d, "\widetilde{h^*}"] \\
\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \arrow[r, "\cong"] & \mathrm{Hom}(H^{*+6}(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}[1])
\end{tikzcd}$$

where the top horizontal map is given by the Hurewicz homomorphism, the bottom horizontal map is as defined above, and the map $\widetilde{h^*}$ is given as follows: Since the map h is the induced map on homotopy groups of a map of spectra, there is a corresponding induced map on cohomology given by

$$h^* \colon H^*(\Sigma^{-6}H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}).$$

By construction, this is a map of $\mathcal{A}$-modules. Let Υ be an element in the group of $\mathcal{A}$-module homomorphisms $\mathrm{Hom}_{\mathcal{A}}(H^*(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}[1])$. By the Künneth Theorem, the composition $\Upsilon \circ h^*$ corresponds to a map

$$\mathcal{A} \otimes H^{*+6}(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{F}_2[1]$$

which is a map of $\mathcal{A}$-modules. By the shear isomorphism this corresponds to a linear map

$$H^{*+6}(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{F}_2[1],$$

i.e. a homology class. We define this to be $\widetilde{h^*}(\Upsilon)$.

To complete the proof we show that there are elements $\phi \in \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ and a degree 0 $\mathcal{A}$-homomorphism $\Phi \in \mathrm{Hom}_{\mathcal{A}}(H^*(MT\mathrm{Spin}(6);\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}[1])$ such that ϕ is mapped to Φ which is mapped to Ψ in the above diagram. Since the bottom horizontal map is an isomorphism, this shows that ϕ is mapped to the element in $\pi_7(H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ which corresponds to Ψ under the above isomorphism. This element is ψ by construction.

We define a map

$$\begin{aligned} \Phi\colon H^1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}) &\longrightarrow \mathbb{F}_2. \\ z^7 u &\longmapsto 1 \\ z^3\omega_4 u &\longmapsto 0 \\ z\omega_6 u &\longmapsto 0 \end{aligned}$$

This map is an $\mathcal{A}$-module map since the Steenrod square decomposables in degree 1 involve only $\omega_4 z^3 u$ and $\omega_6 z u$ as shown in Table 5 in Appendix A. Thus, Φ defines a class in $\mathrm{Ext}^{0,1}_{\mathcal{A}}(H^1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}), \mathbb{F}_2)$. Since the element in degree 1 and filtration 0 in the Adams spectral sequence of $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$ cannot have any outgoing differentials (see Figure 12), this class must survive the spectral sequence. In particular, there is an element $\phi \in \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+)$ which gets mapped to Φ in the above commutative square.

The map Φ fits into the diagram

$$\begin{tikzcd} \mathbb{F}_2[1] & H^*(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\Phi"'] \\ & H^*(\Sigma^{-6} H\mathbb{Z}/2\mathbb{Z} \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}) \arrow[u, "h^*"'] \arrow[ul] \end{tikzcd}$$

where $\mathbb{F}_2[1]$ denotes that there is only one copy of $\mathbb{F}_2$ which is in degree 1, and the diagonal map is defined so that the diagram commutes. By construction, the vertical map is induced by the map $u \wedge \mathbb{1}_{\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+}$, where u is the Thom class. Thus, using the Künneth formula as described above, the vertical map sends $\mathrm{Sq}^i(\iota) \otimes z^n$ to $z^n \mathrm{Sq}^i(u)$. Since the element $z^7 u \in H^1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z})$ is indecomposable under the Steenrod squares, the only element that gets mapped to $z^7 u$ under the vertical map is $\iota \otimes z^7$. Therefore, by definition of Φ, the diagonal map sends $\iota \otimes z^7$ to 1 and everything else to 0. By the definition of $\widetilde{h^*}$, the homomorphism Φ gets sent to the homomorphism

$$H^7(\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z},$$

which sends z^7 to 1 and everything else to 0, i.e. $\widetilde{h^*}(\Phi) = \Psi$. As described above, this finishes the proof. $\square$

Step 4. To show that there is no odd torsion, recall from Section 8.5.3 that the homotopy group $\pi_1(MT\mathrm{Spin}(6))$ vanishes by **[21]**. Since the map $\Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+ \left[\frac{1}{2}\right] \to S^0 \left[\frac{1}{2}\right]$ induced by the constant map is a homotopy equivalence, we have

$$\begin{aligned} \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+) \left[\tfrac{1}{2}\right] &\cong \pi_1(MT\mathrm{Spin}(6) \wedge S^0 \left[\tfrac{1}{2}\right]) \\ &\cong \pi_1(MT\mathrm{Spin}(6)) \left[\tfrac{1}{2}\right] \\ &= 0 \end{aligned}$$

i.e. there is no odd torsion. This finishes the proof of Theorem 9.1.

9.2. The Case $G = \mathbb{Z}/2^k\mathbb{Z}$

Step 1. To compute $H^*(B\mathrm{Spin}(6) \times B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{F}_2)$ as a module over the Steenrod algebra $\mathcal{A}$ we use the Künneth Theorem as above to get

$$H^k(B\mathrm{Spin}(6) \times B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{F}_2) \cong \bigoplus_{i+j=k} H^i(B\mathrm{Spin}(6); \mathbb{F}_2) \otimes H^j(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{F}_2).$$

We have already computed $H^*(B\mathrm{Spin}(6); \mathbb{F}_2)$ as a module over the Steenrod algebra $\mathcal{A}$ in Step 1 of the case $G = \mathbb{Z}/2\mathbb{Z}$. For the computation of $H^*(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{F}_2)$ consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2^k} \mathbb{Z} \longrightarrow \mathbb{Z}/2^k\mathbb{Z} \longrightarrow 0.$$

This yields a fibration

$$B\mathbb{Z} \longrightarrow B\mathbb{Z} \longrightarrow B\mathbb{Z}/2^k\mathbb{Z}.$$

Using the identification $B\mathbb{Z} \cong \mathbb{S}^1$ this is a fibration $\mathbb{S}^1 \to \mathbb{S}^1 \to B\mathbb{Z}/2^k\mathbb{Z}$. We can deloop once to get the fibration $\mathbb{S}^1 \to B\mathbb{Z}/2^k\mathbb{Z} \to \mathbb{C}P^\infty$, where we have used the identification $B(B\mathbb{Z}) \cong \mathbb{C}P^\infty$. The corresponding Serre spectral sequence on cohomology with integer coefficients is as shown in Figure 14. The first differential has to be multiplication by 2^k since $H_1(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/2^k\mathbb{Z}$, which implies that the first cohomology group of $B\mathbb{Z}/2^k\mathbb{Z}$ vanishes and the second cohomology groups contains a copy of $\mathbb{Z}/2^k\mathbb{Z}$. Writing a for a generator in degree $(2, 0)$ we get

$$H^*(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}[a]/(2^k a).$$

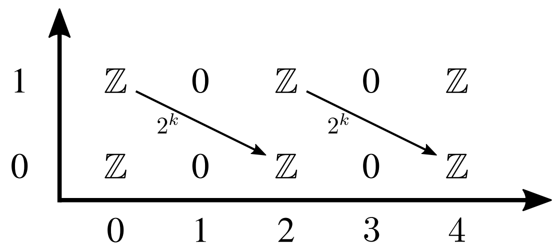

FIGURE 14. The Serre spectral sequence in cohomology with integral coefficients for the fibration $\mathbb{S}^1 \to B\mathbb{Z}/2^k\mathbb{Z} \to \mathbb{C}P^\infty$.

In the induced Serre spectral sequence with $\mathbb{F}_2$-coefficients, multiplication by 2^k is the trivial map. Hence, there are no differentials in this spectral sequence which is shown in Figure 15. Thus, the cohomology is generated by x and y, the elements in degree $(0, 1)$ and $(2, 0)$ respectively, with $x^2 = ay$ for some a. In fact, $a = 0$ as shown in Equation (9.2) below and thus we have (see [**10**, p. 382])

$$H^*(B\mathbb{Z}/2^k\mathbb{Z}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^2) \otimes \mathbb{F}_2[y].$$

$$\begin{array}{c|ccccc}
1 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

FIGURE 15. The Serre spectral sequence in cohomology with $\mathbb{F}_2$-coefficients for the fibration $\mathbb{S}^1 \to B\mathbb{Z}/2^k\mathbb{Z} \to \mathbb{C}P^\infty$.

The Steenrod square structure is as follows. For $\mathrm{Sq}^1(x)$ consider the following commuting diagram

$$\begin{tikzcd}
H^1(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\mathrm{Sq}^1=\beta"] \arrow[d, equal] & H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) & \\
H^1(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\tilde{\beta}"] & H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z}) \arrow[u, "\rho_2"] \arrow[r, "\cdot 2"] & H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z})
\end{tikzcd}$$

where the horizontal maps β and $\tilde{\beta}$ are the Bockstein homomorphism for the short exact sequences

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \text{ and } \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

respectively, and the vertical map ρ_2 is reduction modulo 2. Since the bottom horizontal maps are part of a long exact sequence the generator x of $H^1(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z}/2\mathbb{Z})$ gets send to 0 in $H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z})$ under the map $(\cdot 2) \circ \tilde{\beta}$. Since $H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2^k\mathbb{Z}$ by Figure 14, the image of x in $H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z})$ must be $2^{k-1}a$, for a the generator in $H^2(B\mathbb{Z}/2^k\mathbb{Z};\mathbb{Z})$ as above. Since ρ_2 sends every multiple of 2 to 0 we get

$$0 = \rho_2 \circ \tilde{\beta}(x) = \beta(x) = \mathrm{Sq}^1(x) = x^2. \tag{9.2}$$

Since y is pulled back from $\mathbb{C}P^\infty$, where Sq^1 vanishes, we have $\mathrm{Sq}^1(y) = 0$. Thus, the only non-trivial Steenrod squares are given by $\mathrm{Sq}^2(y) = y^2$ and the Cartan formula.

In total, we have

$$H^*(B\mathrm{Spin}(6) \times B\mathbb{Z}/2^k\mathbb{Z};\mathbb{F}_2) \cong \mathbb{F}_2[x, y, \omega_4, \omega_6, x_8]/(x^2)$$

up to degree 8, i.e.

$$H^k(B\mathrm{Spin}(6)\times B\mathbb{Z}/2^k\mathbb{Z};\mathbb{F}_2)\cong\begin{cases}\mathbb{F}_2 & k=0\\ \mathbb{F}_2\langle x\rangle & k=1\\ \mathbb{F}_2\langle y\rangle & k=2\\ \mathbb{F}_2\langle xy\rangle & k=3\\ \mathbb{F}_2\langle y^2,\omega_4\rangle & k=4\\ \mathbb{F}_2\langle xy^2,x\omega_4\rangle & k=5\\ \mathbb{F}_2\langle y^3,y\omega_4,\omega_6\rangle & k=6\\ \mathbb{F}_2\langle xy^3,xy\omega_4,x\omega_6\rangle & k=7\\ \mathbb{F}_2\langle y^4,y^2\omega_4,y\omega_6,\omega_4^2,x_8\rangle & k=8\end{cases}$$

and the Steenrod square structure can be computed by using the multiplicativity of the total Steenrod squares. They can be found in Table 6 in Appendix A.

Step 2. Applying the Thom isomorphism we get an isomorphism

$$(-)\cup u\colon H^*(B\mathrm{Spin}(6)\times B\mathbb{Z}/2^k\mathbb{Z};\mathbb{F}_2)\xrightarrow{\cong}H^{*-6}(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+).$$

Using the result from Step 1 we have

$$H^*(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+;\mathbb{F}_2)\cong\mathbb{F}_2[x,y,\omega_4,\omega_6,x_8]/(x^2)u,$$

up to degree 2, i.e. we have

$$H^k(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z};\mathbb{F}_2)\cong\begin{cases}\mathbb{F}_2u & k=-6\\ \mathbb{F}_2\langle xu\rangle & k=-5\\ \mathbb{F}_2\langle yu\rangle & k=-4\\ \mathbb{F}_2\langle xyu\rangle & k=-3\\ \mathbb{F}_2\langle y^2u,\omega_4u\rangle & k=-2\\ \mathbb{F}_2\langle xy^2u,x\omega_4u\rangle & k=-1\\ \mathbb{F}_2\langle y^3u,y\omega_4u,\omega_6u\rangle & k=0\\ \mathbb{F}_2\langle xy^3u,xy\omega_4u,x\omega_6u\rangle & k=1\\ \mathbb{F}_2\langle y^4u,y^2\omega_4u,y\omega_6u,\omega_4^2u,x_8u\rangle & k=2\end{cases}$$

As computed in Equation (9.1) we have $\mathrm{Sq}(u)=(1+\omega_4+\omega_6+\omega_4^2+\cdots)u$. Therefore, the Steenrod square structure can be computed using the multiplicativity of the total Steenrod squares as shown in Table 7 in Appendix A.

Step 3′. There is an Atiyah–Hirzebruch spectral sequence

$$H_p(B\mathbb{Z}/2^k\mathbb{Z};\pi_q(MT\mathrm{Spin}(6)))\Longrightarrow\pi_{p+q}(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$$

is shown in Figure 16 using the known homotopy groups of $MT\mathrm{Spin}(6)$ up to degree 1. To get the differentials note that there is a map $MT\mathrm{Spin}(6)\to\Sigma^{-6}M\mathrm{Spin}$ given by the induced

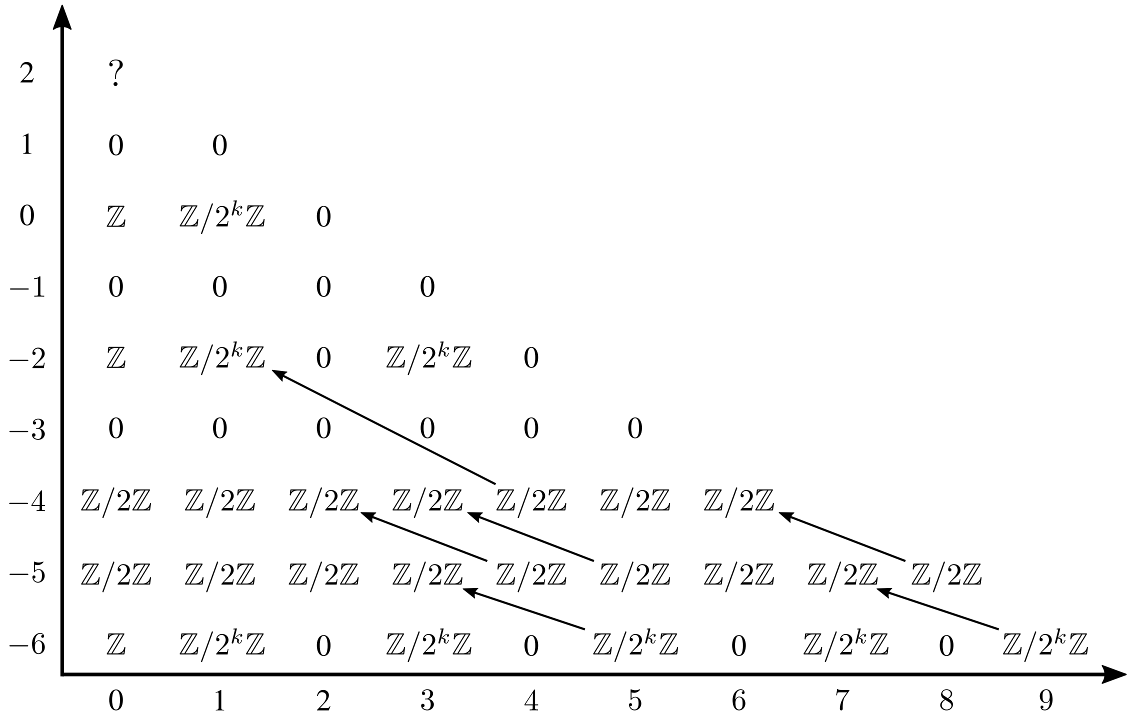

FIGURE 16. The Atiyah–Hirzebruch spectral sequence
$H_p(\mathbb{Z}/2^k\mathbb{Z}; \pi_q(MT\mathrm{Spin}(6))) \Longrightarrow \pi_{p+q}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$.

map on Thom spectra of the inclusion $B\mathrm{Spin}(6) \hookrightarrow B\mathrm{Spin}$, which can be composed with the Atiyah-Bott-Shapiro map [**2**, Thm. 2.2] $\Sigma^{-6}M\mathrm{Spin} \to \Sigma^{-6}\mathrm{ko}$. Both of these maps are 0-connected, hence, so is the composition. This stays true after smashing both sides with $\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$ since this only has non-vanishing homotopy groups in positive degrees. This induces a map of Atiyah–Hirzebruch spectral sequences from the spectral sequence shown in Figure 16 to the spectral sequence

$$H_p(B\mathbb{Z}/2^k\mathbb{Z}_+; \mathrm{ko}_q) \Longrightarrow \mathrm{ko}_{p+q}(B\mathbb{Z}/2^k\mathbb{Z}_+) = \pi_{p+q}(\mathrm{ko} \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$$

given in [**10**, Ch. 3]. In particular, the differentials in the bottom six rows coincide with the differentials described in [**10**, Ch. 3], and hence, are as drawn in Figure 16. All groups on the diagonals of total degree 0 and 2 get killed in the Atiyah–Hirzebruch spectral sequence apart from the $\mathbb{Z}$ in total degree 0 in the first column. Hence, there is no space for differentials involving the diagonal of total degree 1, and thus, all groups on this diagonal must survive the spectral sequence. The E_∞-page of the Atiyah–Hirzebruch spectral sequence is shown in Figure 17.

This shows that the order of the group $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)}$ is 2^{3k+2}.

Step 4. In order to complete the proof of Theorem 9.2 we need to show that the homotopy group $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ does not have any odd torsion. This follows analogous to Step 4 in the case $G = \mathbb{Z}/2\mathbb{Z}$.

2	?									
1	0	0								
0	$\mathbb{Z}$	$\mathbb{Z}/2^k\mathbb{Z}$	0							
−1	0	0	0	0						
−2	$\mathbb{Z}$	$\mathbb{Z}/2^{k-1}\mathbb{Z}$	0	$\mathbb{Z}/2^k\mathbb{Z}$	0					
−3	0	0	0	0	0	0				
−4	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	0			
−5	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	
−6	$\mathbb{Z}$	$\mathbb{Z}/2^k\mathbb{Z}$	0	$\mathbb{Z}/2^k\mathbb{Z}$	0	$\mathbb{Z}/2^{k-1}\mathbb{Z}$	0	$\mathbb{Z}/2^k\mathbb{Z}$	0	$\mathbb{Z}/2^{k-1}\mathbb{Z}$
	0	1	2	3	4	5	6	7	8	9

FIGURE 17. The E_∞-page of the Atiyah–Hirzebruch spectral sequence $H_p(\mathbb{Z}/2^k\mathbb{Z};\pi_q(MT\mathrm{Spin}(6))) \Longrightarrow \pi_{p+q}(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$.

9.2.1. The Adams Spectral Sequence for $MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$. The Atiyah–Hirzebruch spectral sequence in Figure 17 shows that $\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ is some extension of two copies of $\mathbb{Z}/2\mathbb{Z}$ and three copies of $\mathbb{Z}/2^k\mathbb{Z}$. However, it could be $\mathbb{Z}/2^{3k+2}\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2\oplus(\mathbb{Z}/2^k\mathbb{Z})^3$ or anything in between. In this section we look at the Adams spectral sequence of the spectrum $MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$ to exclude most of these extensions.

Using the $\mathbb{F}_2$-cohomology of the spectrum $MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$ as a module over the Steenrod algebra $\mathcal{A}$ as computed in Step 2, we can compute the E_2-page of the Adams spectral sequence converging to the 2-primary homotopy groups of the spectrum $MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$. For this we use Robert Bruner's programme [**12**] with the input given in Table 5 in Appendix B. The resulting E_2-page is given in Figure 18. As before the diagram is complete to the left of the dotted line. We use the E_∞-page of the Atiyah–Hirzebruch spectral sequence given in Figure 17 to make some conclusions about the differentials in this Adams spectral sequence.

In total degree -6 we have a copy of the integers in the Atiyah–Hirzebruch spectral sequence so the tower in degree -6 in the Adams spectral sequence survives. Hence, we have

$$\pi_{-6}(MT\mathrm{Spin}(6)\wedge\Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)}\cong\mathbb{Z}_{(2)}.$$

In the Adams spectral sequence in degree -5 the copy of $\mathbb{Z}/2\mathbb{Z}$ that is not part of the tower has to survive for degree reasons. The Atiyah–Hirzebruch spectral sequence therefore

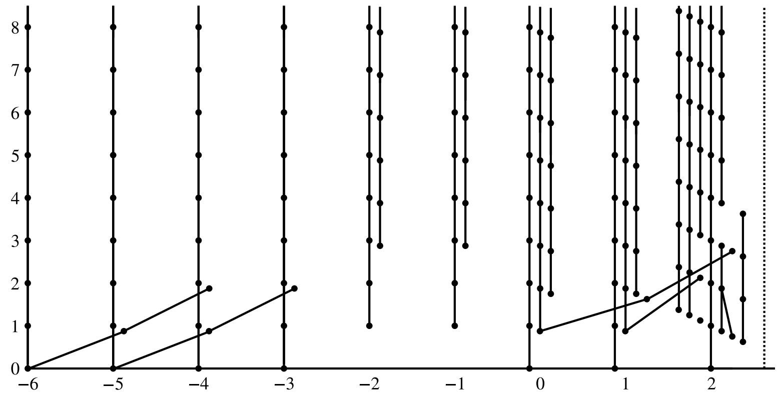

FIGURE 18. The Adams spectral sequence for $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$.

yields

$$\pi_{-5}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}.$$

In particular, there must be a differential in the Adams spectral sequence from the element in degree -4 and filtration 0 to the element in degree -5 and filtration k, leaving a copy of $\mathbb{Z}/2^k\mathbb{Z}$.

We already know that the tower in degree -4 dies. Since the Atiyah–Hirzebruch spectral sequence shows that a group of order 4 survives the spectral sequence both copies of $\mathbb{Z}/2\mathbb{Z}$ in the Adams spectral sequence survive and we get

$$\pi_{-4}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

The copy of $\mathbb{Z}/2\mathbb{Z}$ that is not part of the tower in degree -3 in the Adams spectral sequence must survive for degree reasons. Hence, the Atiyah–Hirzebruch spectral sequence implies that

$$\pi_{-3}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{k+1}\mathbb{Z}.$$

This implies that there is a differential from the bottom of one of the towers in degree -2 to the element in degree -3 and filtration $k+1$, making a copy of $\mathbb{Z}/2^{k+1}\mathbb{Z}$ survive the Adams spectral sequence.

Since one of the towers in degree -2 dies, the Atiyah–Hirzebruch spectral sequence implies that the other tower survives, implying

$$\pi_{-2}(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)} \cong \mathbb{Z}_{(2)}.$$

In total degree -1 in the Atiyah–Hirzebruch spectral sequence we get either $\mathbb{Z}/2^{2k-2}\mathbb{Z}$ or $\mathbb{Z}/2^{k-1}\mathbb{Z} \oplus \mathbb{Z}/2^{k-1}\mathbb{Z}$. This implies that there must be two differentials between columns -1 and 0 both killing a tower in degree 0. The Atiyah–Hirzebruch spectral sequence shows that the remaining tower in column 0 survives which implies

$$\pi_0(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)_{(2)} \cong \mathbb{Z}_{(2)}.$$

In particular, there are no differentials between columns 0 and 1 in the Adams spectral sequence.

As in the case $G = \mathbb{Z}/2\mathbb{Z}$, looking at the Adams spectral sequence for $MT\mathrm{Spin}(6)$ shows that there is a differential in the Adams spectral sequence from the element in degree 2 and filtration 0 to the element in degree 1 and filtration 2 that is not part of a tower. For degree reasons, all remaining elements in degree 1 and filtration at most 2 have to survive the Adams spectral sequence. Hence, $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ is a direct sum of exactly three non-zero summands, where one summand contains a copy of $\mathbb{Z}/8\mathbb{Z}$ and one of the remaining summands contains a copy of $\mathbb{Z}/4\mathbb{Z}$. From the Atiyah–Hirzebruch spectral sequence we know that $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+)$ is an extension of two copies of $\mathbb{Z}/2\mathbb{Z}$ and three copies of $\mathbb{Z}/2^k\mathbb{Z}$. Therefore, it must be one of the following groups

(1) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^{2k+1}\mathbb{Z}$,
(2) $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{k+1}\mathbb{Z} \oplus \mathbb{Z}/2^{2k}\mathbb{Z}$,
(3) $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^{2k}\mathbb{Z}$,
(4) $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^{k+2}\mathbb{Z}$,
(5) $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^{k+1}\mathbb{Z} \oplus \mathbb{Z}/2^{k+1}\mathbb{Z}$.

Note that in the case $k = 2$ the groups (3) and (4) coincide.

CHAPTER 10

Example: Spin(6)-Manifolds with Fundamental Group $\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$

Recall from the previous chapter that for a group G satisfying $\operatorname{usr}(\mathbb{Z}[G]) < \infty$, a manifold of 3-type $B\mathrm{Spin}(6) \times BG$ is a 6-dimensional manifold M which is 2-connected relative to its boundary and fits into the following commutative diagram

$$\begin{tikzcd} B\mathrm{Spin}(6) \times BG \arrow[r, "\theta"] & BO(6) \\ M \arrow[u, "\ell_M"] \arrow[ur, "\tau"'] & \end{tikzcd} ,$$

where the tangential structure θ is given by first projecting to $B\mathrm{Spin}(6)$ and then applying the usual $\theta^{\mathrm{Spin}(6)}\colon B\mathrm{Spin}(6) \to BO(6)$, and ℓ_M is 3-connected.

In the previous chapter we have computed the abelianisation of the mapping class group of M for the cases $G = \mathbb{Z}/2^k\mathbb{Z}$. In this chapter we consider the cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/p^k\mathbb{Z}$.

As in Section 9.2 we have a (co)fibration sequence of spectra

$$F \wedge \Sigma^\infty BG_+ \longrightarrow MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+ \longrightarrow \Sigma^{-6} M\mathrm{Spin} \wedge \Sigma^\infty BG_+. \tag{10.1}$$

By [**22**, Lemma 5.2] the first homotopy groups of the spectrum F are given as follows.

$$\pi_i(F) \cong \begin{cases} 0 & i < 0 \\ \mathbb{Z} & i = 0 \\ \mathbb{Z}/4\mathbb{Z} & i = 1 \end{cases}$$

Thus, by the Hurewicz and Künneth Theorem we get

$$\pi_0(F \wedge \Sigma^\infty BG_+) \cong H_0(F \wedge \Sigma^\infty BG_+) \cong H_0(F) \otimes H_0(BG) \cong \mathbb{Z}.$$

Using the Atiyah–Hirzebruch spectral sequence we get

$$\pi_1(F \wedge \Sigma^\infty BG_+) \cong H_0(G; \mathbb{Z}/4\mathbb{Z}) \oplus H_1(G; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}}$$

since we have $\Sigma^\infty BG_+ \simeq S^0 \vee \Sigma^\infty BG$, and hence, there are no differential or extensions in the relevant part of the spectral sequence.

We consider the two long exact sequences corresponding to the sequence in (10.1) for the trivial group $\{*\}$ and for an arbitrary group G respectively. We get a map between these sequences induced by the homomorphism $\{*\} \to G$ as shown in Figure 19, where we have used that

$$\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\{*\}_+) = \Omega_7^{\mathrm{Spin}} = 0,$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\pi_8(M\mathrm{Spin} \wedge \Sigma^\infty B\{*\}_+) \arrow[r] \arrow[d] & \pi_8(M\mathrm{Spin} \wedge \Sigma^\infty BG_+) \arrow[d] \\
\mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d, "0"] & \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \arrow[d] \\
\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\{*\}_+) \arrow[r] \arrow[d] & \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \arrow[d] \\
0 \arrow[r] \arrow[d] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+) \arrow[d] \\
\mathbb{Z} \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

FIGURE 19. A map of long exact sequences, induced by the inclusion $\{*\} \to G$.

by [**37**], and that, in the sequence corresponding to the trivial group $\{*\}$, the group $\mathbb{Z}/4\mathbb{Z}$ gets hit from behind, as shown in [**22**, Lemma 5.6]. In particular, the composition

$$\mathbb{Z}/4\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$$

is trivial, and by construction, this means that $\mathbb{Z}/4\mathbb{Z} \oplus 0 \subset \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}}$ gets hit from behind.

We get a splitting of the horizontal maps induced by the constant map $G \to \{*\}$. In particular, the left hand side sequence in Figure 19 is a direct summand of the right hand side sequence. Therefore, the bottom map of the right hand side sequence is injective since the corresponding map in the left hand side sequence is injective. This implies that the map

$$\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \longrightarrow \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)$$

is surjective. The aim of this chapter is to prove that the above map induces a splitting as in the following theorem.

THEOREM 10.1. *Let G be a group with abelianisation $G^{ab} \cong \bigoplus_{i=1}^m \mathbb{Z}/p_i^{k_i}\mathbb{Z} \oplus \mathbb{Z}^n$, where the p_i's are odd primes. There is an isomorphism*

$$\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \cong G^{\mathrm{ab}} \oplus \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+).$$

Note that $\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)$ is isomorphic to $\pi_7(\mathrm{ko} \wedge \Sigma^\infty BG_+) = \mathrm{ko}_7(BG)$ as the Atiyah–Bott–Shapiro orientation $M\mathrm{Spin} \to \mathrm{ko}$ is 8-connected by [**2**, Thm. 2.2], so we consider this to be known by [**13**].

The structure of this chapter is as follows. In the first section we describe some elements of the mapping class group of a 6-dimensional manifold M and their geometric interpretation. In particular, we define a map $\phi\colon \pi_1(M) \to \Gamma_\partial(M)$. In the following section we show that this map is split injective for manifolds of 3-type $B\mathrm{Spin}(6) \times BG$ for $G = \mathbb{Z}$ and $G = \mathbb{Z}/p^k\mathbb{Z}$, for an odd prime p. To show this we compose the map ϕ with a map from $\Gamma_\partial(M)$ to a subgroup of the group ring $R[\pi_1(M)]^\times$ for some ring R by using a modified version of the determinant on matrices. We show that this composition is an isomorphism by considering the image of the specific elements of $\Gamma_\partial(M)$ described earlier. This implies that the map ϕ is split injective. We then show the above theorem for these two examples. The final section of this chapter contains a proof of Theorem 10.1.

10.1. Some Elements in $\Gamma_\partial(M\#W_1)$

Let M be a 6-dimensional manifold, not necessarily of 3-type $B\mathrm{Spin}(6) \times BG$. In this chapter we describe specific elements in the mapping class group $\Gamma_\partial(M\#W_1)$, where we use $W_1 = \mathbb{S}^3 \times \mathbb{S}^3$ as introduced in Chapter 6.

Proposition 10.2. *For any manifold M there is a homomorphism*

$$\phi\colon \pi_1(M\#W_1) \longrightarrow \Gamma_\partial(M\#W_1)$$

that sends $\gamma \in \pi_1(M\#W_1)$ to the class of the diffeomorphism given by dragging W_1 around γ.

In the case of surfaces, this map is called the "point-pushing" map as defined in [**19**, Sec. 4.2.1]. Note that this is not exactly the same as in our case we need to keep track of rotation of frames as well as the loop that the disc gets dragged around. We later refer to the homomorphism ϕ as the *dragging homomorphism.*

Proof. The vertical maps in the following diagram form a fibration by [**14**, Ch. 2.2.2, Cor. 2].

$$\begin{tikzcd}
\mathrm{Diff}_\partial(M,\mathbb{D}) \arrow[d] & \\
\mathrm{Diff}_\partial(M) \arrow[d] \arrow[dr] & \\
\mathrm{Emb}(\mathbb{D},M) \arrow[r] & \mathrm{Fr}(TM)
\end{tikzcd}$$

Since the horizontal map $\mathrm{Emb}(\mathbb{D},M) \to \mathrm{Fr}(TM)$ is a weak homotopy equivalence, we get a map

$$\partial\colon \pi_1(\mathrm{Fr}(TM)) \cong \pi_1(\mathrm{Emb}(\mathbb{D},M)) \longrightarrow \pi_0(\mathrm{Diff}_\partial(M,\mathbb{D})),$$

where the second map is the connecting homomorphism of the long exact sequence on homotopy groups. By definition of the connecting homomorphism, a loop in the frame bundle $\mathrm{Fr}(TM)$ lifts to a path in $\pi_0(\mathrm{Diff}_\partial(M,\mathbb{D}))$ that drags the disc $\mathbb{D}$ around the corresponding loop in M (see Figure 20). The image of this loop under ∂ is given by the diffeomorphism at the endpoint of this path in $\pi_0(M,\mathbb{D})$.

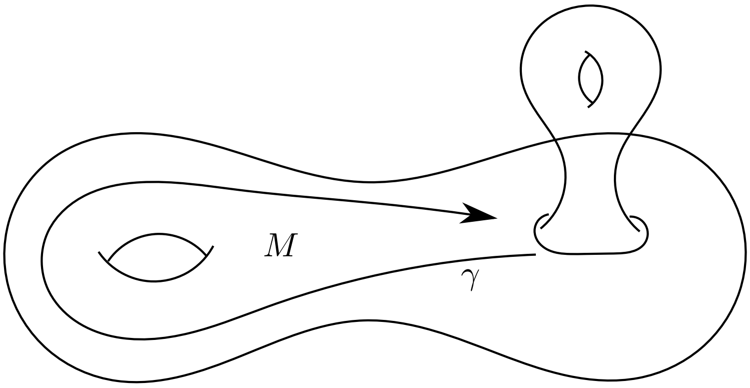

FIGURE 20. Dragging $\mathbb{D}\#W_1$ around the loop γ in M.

Taking the connected sum with W_1 in the fixed disc $\mathbb{D}$ defines a homomorphism

$$\pi_0(\mathrm{Diff}_\partial(M,\mathbb{D})) \xrightarrow{-\#\mathbb{1}_{W_1}} \pi_0(\mathrm{Diff}_\partial(M\#W_1, \mathbb{D}\#W_1))$$

given by extending by the identity on W_1. To get the homomorphism considered in the statement we use the homotopy fibre sequence $\mathrm{SO}(6) \to \mathrm{Fr}(TM) \to M$. The corresponding long exact sequence on homotopy groups then becomes

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1(\mathrm{Fr}(TM)) \longrightarrow \pi_1(M) \longrightarrow 0.$$

This fits into the following diagram,

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[d] & \pi_0(\mathrm{Diff}_\partial(M\#W_1)) \\
\pi_1(\mathrm{Fr}(TM)) \arrow[r] \arrow[d] \arrow[dr, "\partial"] & \pi_0(\mathrm{Diff}_\partial(M\#W_1, \mathbb{D}\#W_1)) \arrow[u] \\
\pi_1(M) \arrow[d] & \pi_0(\mathrm{Diff}_\partial(M,\mathbb{D})) \arrow[u, "-\#\mathbb{1}_{W_1}"'] \\
0 &
\end{tikzcd}$$

where the top right vertical map is induced by the inclusion that forgets that $\mathbb{D}\#W_1$ is fixed under the diffeomorphisms considered. We need the following lemma to finish this proof.

LEMMA 10.3. *The composition*

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1(\mathrm{Fr}(TM)) \longrightarrow \pi_0(\mathrm{Diff}_\partial(M\#W_1, \mathbb{D}\#W_1)) \longrightarrow \pi_0(\mathrm{Diff}_\partial(M\#W_1))$$

is trivial.

PROOF. The image in $\pi_1(\mathrm{Fr}(TM))$ of the non-zero element in $\mathbb{Z}/2\mathbb{Z}$ is given by the constant loop γ that just rotates a single frame at the basepoint. In $\pi_0(\mathrm{Diff}_\partial(M,\mathbb{D}))$ this corresponds to a small disc contained in a big disc, where the small disc gets rotated a full turn. Thus, we are in the situation of Figure 21, where the rightmost disc is this small disc, and considering the image of this rotation in $\pi_0(\mathrm{Diff}_\partial(M\#W_1))$ under the above map, this fixes everything outside a tubular neighbourhood of the boundary of the small disc. Thus, we

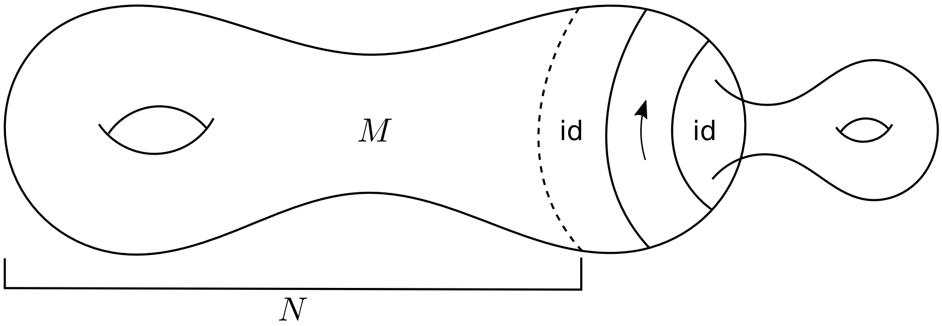

FIGURE 21. The image of the non-trivial element under the map $\mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathrm{Fr}(TM))$.

can consider the following commutative diagram, where N is defined as shown in Figure 21.

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & \pi_0(\mathrm{Diff}_\partial(M\#W_1, N)) \arrow[dr] & \\
\pi_1(\mathrm{Fr}(TM)) \arrow[r] & \pi_0(\mathrm{Diff}_\partial(M\#W_1, \mathbb{D}\#W_1)) \arrow[r] & \pi_0(\mathrm{Diff}_\partial(M\#W_1))
\end{tikzcd}$$

The diffeomorphism group $\mathrm{Diff}_\partial(M\#W_1, N)$ is isomorphic to $\mathrm{Diff}_\partial(M\#W_1 \setminus N)$. Since the manifold $M\#W_1 \setminus N$ is obtained from $W_1 \setminus \mathbb{D}$ by attaching a collar, these manifolds are diffeomorphic. In particular, we have $\pi_0(\mathrm{Diff}_\partial(M\#W_1 \setminus N)) \cong \pi_0(\mathrm{Diff}_\partial(W_1 \setminus \mathbb{D}))$. Using the same argument as before, this is also isomorphic to $\pi_0(\mathrm{Diff}_\partial(W_1, \mathbb{D}))$. Therefore, we get the following commutative diagram

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[dr] & \pi_0(\mathrm{Diff}_\partial(M\#W_1, N)) \cong \pi_0(\mathrm{Diff}_\partial(W_1, \mathbb{D})) \arrow[d] \\
& \pi_0(\mathrm{Diff}_\partial(W_1))
\end{tikzcd}$$

where the vertical map is induced by the inclusion $\mathrm{Diff}_\partial(W_1, \mathbb{D}) \hookrightarrow \mathrm{Diff}_\partial(W_1)$ that forgets that the disc $\mathbb{D}$ gets fixed. The diagonal map is trivial as we can "untwist" the disc $\mathbb{D}$ if we do not require it to be fixed pointwise. The vertical map is an isomorphism by a result of Kreck (cf. [**22**, Lemma 1.5]). Hence, the horizontal map is already trivial, which implies that the map considered in the statement is trivial, by the commutative diagram above. $\square$

Under the identification

$$\pi_1(\mathrm{Fr}(TM))/\mathbb{Z}/2\mathbb{Z} \cong \pi_1(M) \cong \pi_1(M\#W_1)$$

this defines a map $\phi\colon \pi_1(M\#W_1) \longrightarrow \Gamma_\partial(M\#W_1)$ which is a homomorphism by the lemma. $\square$

Let M be a manifold with boundary which is 1-connected relative to its boundary ∂M. Let $*$ be a basepoint of M which lies on the boundary ∂M. For simplicity we drop the basepoint from the notation. By assumption the map $\pi_1(\partial M) \to \pi_1(M)$ is surjective. An element in $\Gamma_\partial(M\#W_1)$ defines an automorphism of $\pi_3(M\#W_1)$. Since M is 1-connected relative to its boundary so is $M\#W_1$. This means that the action of $\phi \in \Gamma_\partial(M\#W_1)$ on $\pi_1(M\#W_1)$ is trivial since ϕ fixes the boundary of $M\#W_1$. In particular, an element in $\Gamma_\partial(M\#W_1)$ defines an automorphism of $\pi_3(M\#W_1)$ considered as a module over $\mathbb{Z}[\pi_1(M\#W_1)]$. Thus, we get a homomorphism

$$\psi\colon \Gamma_\partial(M\#W_1) \longrightarrow \mathrm{Aut}_{\mathbb{Z}[\pi_1(M\#W_1)]}(\pi_3(M\#W_1)).$$

Note that $\pi_3(M\#W_1)$ considered as a $\mathbb{Z}[\pi_1(M\#W_1)]$-module has a sesquilinear form coming from taking intersections of two elements in $\pi_3(M\#W_1)$ analogous to the sesquilinear form λ described in Definition 6.6. We will now calculate $\pi_3(M\#W_1)$ in terms of $\pi_3(M)$.

Removing the disc along which W_1 gets attached to M in the direct sum we write $\mathring{M} = M \setminus \mathbb{D}^6$. The inclusion $\mathring{M} \hookrightarrow M\#W_1$ yields a long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_k(\mathring{M}) \longrightarrow \pi_k(M\#W_1) \longrightarrow \pi_k(M\#W_1, \mathring{M}) \longrightarrow \cdots .$$

The map $M\#W_1 \to M$, which is given by collapsing $W_{1,1}$, gives a splitting of the maps $\pi_k(\mathring{M}) \to \pi_k(M\#W_1)$ in degrees $k \le 5$, as the skeleta of M and $\mathring{M}$ agree in this range, so the maps $\pi_k(\mathring{M}) \to \pi_k(M\#W_1)$ are in particular injective in this range. Therefore, the long exact sequence above collapses into short split exact sequences

$$0 \longrightarrow \pi_k(\mathring{M}) \longrightarrow \pi_k(M\#W_1) \longrightarrow \pi_i(M\#W_1, \mathring{M}) \longrightarrow 0$$

for $k < 5$. In particular, we get a splitting

$$\pi_3(M\#W_1) \cong \pi_3(M\#W_1, \mathring{M}) \oplus \pi_3(\mathring{M})$$

of $\mathbb{Z}[\pi]$-modules, where $\pi := \pi_1(M) = \pi_1(\mathring{M}) = \pi_1(M\#W_1)$. The group $\pi_3(\mathring{M})$ is isomorphic to $\pi_3(M)$ since we can arrange the map $\mathbb{S}^3 \to M$ to miss the disc $\mathbb{D}^6$.

We want to apply the relative Hurewicz Theorem [**27**, Thm. 4.32] to get

$$\pi_3(\widetilde{M\#W_1}, \widetilde{\mathring{M}}) \cong H_3(\widetilde{M\#W_1}, \widetilde{\mathring{M}}; \mathbb{Z}).$$

This is also isomorphic to $\pi_3(M\#W_1, \mathring{M})$ by the Five Lemma. Using the Excision Theorem, the group $H_3(\widetilde{M\#W_1}, \widetilde{\mathring{M}}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[\pi_1(M\#W_1)]^2$ by the Excision Theorem, which in total yields

$$\pi_3(M\#W_1) \cong \mathbb{Z}[\pi_1(M\#W_1)]^2 \oplus \pi_3(M)$$

and the sum is the orthogonal sum with respect to the bilinear map.

To apply the relative Hurewicz Theorem we need to show that the pair $(\widetilde{M\#W_1}, \widetilde{\mathring{M}})$ is 2-connected. Again, by the Five Lemma this is the same as showing that the pair $(M\#W_1, \mathring{M})$ is 2-connected. For this note that the map $\pi_k(\mathring{M}) \to \pi_k(M\#W_1)$ is surjective for $k \le 2$

since we can arrange the map $\mathbb{S}^2 \to M\#W_1$ to be in general position which implies that it misses W_1. Looking at the short exact sequences

$$0 \longrightarrow \pi_k(\mathring{M}) \longrightarrow \pi_k(M\#W_1) \longrightarrow \pi_k(M\#W_1, \mathring{M}) \longrightarrow 0$$

for $k \leq 2$ this implies that the pair $(M\#W_1, \mathring{M})$ is 2-connected.

Consider an elements in $\Gamma_\partial(M\#W_1)$ that lies in the image of the dragging homomorphism ϕ. Using the above splitting the map ψ is then given as follows. On $\mathring{M}$ we can fill in the missing disc $\mathbb{D}^6$ as $\pi_3(\mathring{M}) \cong \pi_3(M)$ and the boundary of the disc $\partial\mathbb{D}^6$ is fixed pointwise under ϕ. Thus, ϕ is the identity on $\mathring{M}$ which corresponds to $\mathbb{1}_{\pi_3(M)}$ under ψ. On W_1, for given e, f the map ϕ is given by concatenating the path of e and f with γ. In other words, the map ψ sends $\phi(\gamma)$ to the following automorphism of $\pi_3(M\#W_1)$.

$$\left(\begin{array}{c|c} \begin{matrix} \gamma & 0 \\ 0 & \gamma \end{matrix} & 0 \\ \hline 0 & \mathbb{1}_{\pi_3(M)} \end{array}\right)$$

10.2. Some Examples of Groups G

We now consider manifolds M of 3-type $B\mathrm{Spin}(6) \times BG$ for $G = \mathbb{Z}$ and $G = \mathbb{Z}/p^k\mathbb{Z}$, for an odd prime p. We also choose M so that its genus $g(M)$ is at least 1. Our aim is to compute the abelianised mapping class group $\Gamma_\partial(M)^{\mathrm{ab}}$ for two specific examples of manifolds M. Since we will refer to these specific manifolds later on, and to avoid confusion, we will denote them as $M_\mathbb{Z}$ and $M^g_{\mathbb{Z}/p^k\mathbb{Z}}$ respectively, where g denotes the genus of the manifold which will be important to keep track of in the case $G = \mathbb{Z}/p^k\mathbb{Z}$. If the genus of the latter manifold is not important we drop the genus from the notation.

In the following subsection we give a lower bound on $\Gamma_\partial(M_G)^{\mathrm{ab}}$, for the groups $G = \mathbb{Z}$ and $G = \mathbb{Z}/p^k\mathbb{Z}$ respectively, by showing that for the examples considered, the dragging homomorphism $\phi\colon \pi_1(M_G) \to \Gamma_\partial(M_G)$ is a split injection. This shows that $\pi_1(M_G) \cong G$ is a direct summand of $\Gamma_\partial(M_G)^{\mathrm{ab}}$.

If we additionally assume that the stable genus of M_G is big, in particular

$$\overline{g}(M_G) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(M_G)]) + 4,$$

we show that we get an upper bound on $\Gamma_\partial(M_G)^{\mathrm{ab}}$. In both examples, the manifold M_G satisfies the assumptions of the following proposition.

Proposition 10.4. *Let G be a group and M be a manifold of 3-type $B\mathrm{Spin}(6) \times BG$. If $\overline{g}(M) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(M)]) + 4$ then the map*

$$\alpha_*\colon \Gamma_\partial(M)^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$$

is an isomorphism.

Proof. Using Corollary 7.8, where θ', coming from the Moore–Postnikov tower, is given by the above map $\theta\colon B\mathrm{Spin}(6)\times BG\to BO(6)$ as described at the beginning of Chapter 9, the map

$$H_1(B\mathrm{Diff}_\partial(M))\longrightarrow H_1(\Omega^\infty MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+)$$

is an isomorphism. As in Equation (8.1) we get $\Gamma_\partial(M)^{\mathrm{ab}}\cong\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+)$. $\square$

We use this isomorphism, the right hand side long exact sequence in Figure 19, as well as the maps defined in Section 10.2.1 to construct an isomorphism

$$\pi_1(MT\mathrm{Spin}(6)\wedge\Sigma^\infty BG_+)\longrightarrow G^{\mathrm{ab}}\oplus\pi_7(M\mathrm{Spin}\wedge\Sigma^\infty BG_+)$$

for the examples of groups G considered above.

10.2.1. A lower bound for $\Gamma_\partial(M_G)^{\mathrm{ab}}$. In the next two subsections we consider the examples $G=\mathbb{Z}$ and $G=\mathbb{Z}/p^k\mathbb{Z}$ respectively. In both cases we pick manifolds M_G of 3-type $B\mathrm{Spin}(6)\times BG$ of genus $g(M)$ at least 1 and show that the dragging homomorphism $\phi\colon\pi_1(M_G)\to\Gamma_\partial(M_G)$ is split injective.

10.2.1.1. *The Case $G=\mathbb{Z}$.* We consider the manifold $M_{\mathbb{Z}}:=(\mathbb{S}^1\times\mathbb{D}^5)\#W_g$ with $g\geq 1$. This manifold is 2-connected relative to its boundary by Lefschetz Duality (see [**27**, Thm. 3.43]). The following theorem shows that the elements given by the dragging homomorphism for elements $\gamma\in\mathbb{Z}\cong\pi_1(M_{\mathbb{Z}})$, as described in the previous section, form a direct summand of $\Gamma_\partial(M_{\mathbb{Z}})$.

Theorem 10.5. *The dragging homomorphism*

$$\phi\colon\pi_1(M_{\mathbb{Z}})\longrightarrow\Gamma_\partial(M_{\mathbb{Z}})$$

is split injective.

Proof. Let γ be an element in $\pi_1(M_{\mathbb{Z}})\cong\mathbb{Z}$. By Proposition 10.2 this defines an element in $\Gamma_\partial(M_{\mathbb{Z}})$ by fixing a copy of W_1 which gets drags around. As we have seen at the end of the previous section, we get a homomorphism ψ to $\mathrm{Aut}_{\mathbb{Z}[\pi_1(M_{\mathbb{Z}})]}(\pi_3(M_{\mathbb{Z}}))$. For the universal cover $\tilde{M}_{\mathbb{Z}}$ of $M_{\mathbb{Z}}$ the first two homotopy groups vanish since $\pi_2(\tilde{M}_{\mathbb{Z}})\cong\pi_2(M_{\mathbb{Z}})=0$. Thus, by the Hurewicz isomorphism we have

$$\pi_3(M_{\mathbb{Z}})\cong\pi_3(\tilde{M}_{\mathbb{Z}})\cong H_3(\tilde{M}_{\mathbb{Z}}).$$

The universal cover of $M_{\mathbb{Z}}$ is $\mathbb{R}\times\mathbb{D}^5$ with $\mathbb{Z}$ many copies of W_g attached, where the action of $\pi_1(M_{\mathbb{Z}})$ is given by translation between the W_g's. To compute $H_3(\tilde{M}_{\mathbb{Z}})$ we consider the subspace $N\subset M_{\mathbb{Z}}$ obtained from $\mathbb{R}\times\mathbb{D}^5$ by cutting out $\mathbb{Z}$ many discs $\mathbb{D}^6$. Since $\mathbb{R}\times\mathbb{D}^5$ is contractible N is homotopic to $\bigvee_{\mathbb{Z}}\mathbb{S}^5$. Thus, the long exact sequence in homology of the pair $(\tilde{M}_{\mathbb{Z}},N)$ looks like

$$\cdots\longrightarrow H_3(N)\longrightarrow H_3(\tilde{M}_{\mathbb{Z}})\longrightarrow H_3(\tilde{M}_{\mathbb{Z}},N)\longrightarrow H_2(N)\longrightarrow\cdots$$

By construction, both $H_2(N)$ and $H_3(N)$ vanish. Using the Excision Theorem we get

$$H_3(\tilde{M}_{\mathbb{Z}})\cong H_3(\tilde{M}_{\mathbb{Z}},N)\cong\tilde{H}_3(\tilde{M}_{\mathbb{Z}}/N)\cong\tilde{H}_3(\bigvee_{\mathbb{Z}}W_g).$$

$$
\begin{tikzcd}
\gamma \arrow[d, mapsto] & \pi_1(M_{\mathbb{Z}}) \cong \mathbb{Z} = \langle t \rangle \arrow[d, "\phi"] \\
\phi(\gamma) \arrow[d, mapsto] & \Gamma_\partial(M_{\mathbb{Z}}) \arrow[d, "\psi"] \\
\left(\begin{array}{cc|c} t^{n_\gamma} & 0 & \multirow{2}{*}{0} \\ 0 & t^{n_\gamma} & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{1}_{2g-2} \end{array}\right) \arrow[dd, mapsto] & \mathrm{GL}_{2g}(\mathbb{Z}[t^{\pm 1}]) \arrow[d, "\det"] \\
& \mathbb{Z}[t^{\pm 1}]^\times \arrow[d, equal] \\
t^{2n_\gamma} & \{\pm t^n \mid n \in \mathbb{Z}\}
\end{tikzcd}
$$

FIGURE 22. The sequence of homomorphisms for the case $G = \mathbb{Z}$.

This is isomorphic to $\mathbb{Z}[t^{\pm 1}]^{2g}$ as a module over $\mathbb{Z}[\pi_1(M_{\mathbb{Z}})]$. Therefore, the automorphism group $\mathrm{Aut}_{\mathbb{Z}[\pi_1(M_{\mathbb{Z}})]}(\pi_3(M_{\mathbb{Z}}))$ is given by $\mathrm{GL}_{2g}(\mathbb{Z}[\pi_1(M_{\mathbb{Z}})])$. Note that alternatively we could have inductively applied the formula

$$\pi_3(M \# W_1) \cong \mathbb{Z}[\pi_1(M \# W_1)]^2 \oplus \pi_3(M)$$

coming from the computation at the end of Section 10.1, where M would be $M_{\mathbb{Z}}$ with gradually more copies of W_1 removed.

As shown at the end of Section 10.1, under the homomorphism ψ, the dragging homomorphism around γ gets mapped to

$$\left(\begin{array}{cc|c} \gamma & 0 & \multirow{2}{*}{0} \\ 0 & \gamma & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{1}_{2g-2} \end{array}\right) \in \mathrm{GL}_{2g}(\mathbb{Z}[\pi_1(M_{\mathbb{Z}})])^{\mathrm{ab}}.$$

Let t^{n_γ} be the image of γ under the isomorphism $\pi_1(M_{\mathbb{Z}}) \cong \mathbb{Z} = \langle t \rangle$. Using this identification we can consider the above matrix as a matrix in $\mathrm{GL}_{2g}(\mathbb{Z}[t^{\pm 1}])$ by replacing γ by t^{n_γ}. Considering the determinant of this matrix defines a homomorphism to $\mathbb{Z}[t^{\pm 1}]^\times$. The units in the group ring $\mathbb{Z}[t^{\pm 1}]^\times$ are given by $\{\pm t^n \mid n \in \mathbb{Z}\}$. In total, we get a sequence of homomorphisms as shown in Figure 22. Thus, we have shown that $\{t^{2n} \mid n \in \mathbb{Z}\}$ is contained in the image $\mathrm{Im}(\det \circ \psi)$. The following lemma shows that each element in $\mathrm{Im}(\det \circ \psi)$ is of the form t^{2n} for some $n \in \mathbb{Z}$.

LEMMA 10.6. $\mathrm{Im}(\det \circ \psi) = \{t^{2n} \mid n \in \mathbb{Z}\}$.

PROOF. Let $A(t)$ be a matrix in $\mathrm{GL}_{2g}(\mathbb{Z}[t^{\pm 1}])$ which, under the map ψ, is the image of an element in $\Gamma_\partial(M_{\mathbb{Z}})^{\mathrm{ab}}$. By construction of ψ the matrix $A(t)$ is an automorphism of a hyperbolic module. Using the examples of automorphism groups of hyperbolic modules in Chapter 2, the matrix $A(t)$ is in particular a symplectic matrix over $\mathbb{Z}[t^{\pm 1}]$. The determinant of a matrix in $\mathrm{Sp}_{2g}(\mathbb{Z}[t^{\pm 1}])$ can be written as εt^n for some $\varepsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}$. Since the determinant of symplectic matrices with real entries is 1, substituting $t = 1$ yields $1 = \det(A(t)|_{t=1}) = \varepsilon$. Using this, for $t = -1$ we get

$$1 = \det(A(t)|_{t=-1}) = (-1)^n$$

which implies $n \in 2\mathbb{Z}$. This shows that each matrix in the image of ψ has determinant t^{2n}, and hence, finishes the proof. $\square$

Using Lemma 10.6 we can define a function $f\colon \Gamma_\partial(M_\mathbb{Z}) \to \mathbb{Z}$ via

$$\det(\psi(\eta)) = t^{2f(\eta)}$$

for an element $\eta \in \Gamma_\partial(M_\mathbb{Z})$. Thus, we are in the following situation.

$$\begin{tikzcd}
\pi_1(M_\mathbb{Z}) \cong \mathbb{Z} \arrow[d, "\phi"] \arrow[dr] & \\
\Gamma_\partial(M_\mathbb{Z}) \arrow[r, "f"] & \mathbb{Z}
\end{tikzcd}$$

By construction, the composition $f \circ \phi$ is surjective. Since $\pi_1(M_\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, the map $f \circ \phi$ is an isomorphism as $\mathbb{Z}$ is Hopfian. Thus, ϕ is a split injection. $\square$

10.2.1.2. *The Case $G = \mathbb{Z}/p^k\mathbb{Z}$.* Let p be an odd prime, k a natural number, and G the group $\mathbb{Z}/p^k\mathbb{Z}$. Let L denote the lens space $L(p^k; 1) = \mathbb{S}^3/(\mathbb{Z}/p^k\mathbb{Z})$. We define 6-dimensional manifolds $M^g_{\mathbb{Z}/p^k\mathbb{Z}} := (L \times \mathbb{D}^3)\# W_g$ for $g \geq 1$. As in the case $G = \mathbb{Z}$ above, this manifold is 2-connected relative to its boundary by the Lefschetz Theorem (see [**27**, Thm. 3.43]). As before, we show that in this setting the dragging homomorphism ϕ is split injective. To prove this we need the following lemma.

LEMMA 10.7. *For $g \geq 7$ the abelianisation of the mapping class group $\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}}$ is finite.*

PROOF. Since the Hirsch number of finite groups is trivial we have an upper bound $\mathrm{usr}(\mathbb{Z}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]) \leq 3$, and hence

$$\overline{g}(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \geq 7 \geq \mathrm{usr}(\mathbb{Z}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]) + 4.$$

By Proposition 10.4 we get an isomorphism

$$\alpha_*\colon \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}}))^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+)$$

for $g \geq 7$. Since $\mathbb{Q}$ is a flat module, the right hand side exact sequence in Figure 19 for $G = \mathbb{Z}/p^k\mathbb{Z}$ gives an exact sequence

$$0 \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \otimes \mathbb{Q} \longrightarrow \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \otimes \mathbb{Q} \longrightarrow 0,$$

where we have used that $(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}) \otimes \mathbb{Q}$ vanishes. In particular, this gives an isomorphism between the two rational homotopy groups. By the Atiyah–Hirzebruch spectral sequence we have

$$\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \otimes \mathbb{Q} \cong H_7(B\mathbb{Z}/p^k\mathbb{Z};\mathbb{Q}) \oplus H_3(B\mathbb{Z}/p^k\mathbb{Z};\mathbb{Q}) = 0.$$

By the above isomorphism this implies that $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \otimes \mathbb{Q}$ vanishes. In particular, as $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+)$ is a finitely generated abelian group, it is finite. □

THEOREM 10.8. *For $g \geq 1$ the dragging homomorphism*

$$\phi\colon \pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \longrightarrow \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$$

is split injective.

PROOF. Let us first assume that $g \geq 7$. Analogous to the previous section, we consider the sequence of homomorphisms as shown in Figure 23, with details about the individual maps explained below. The first map ϕ^{ab} is the dragging homomorphism ϕ followed by the projection to the abelianisation of $\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$. Consider the map

$$\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \xrightarrow{\psi} \mathrm{Aut}_{\mathbb{Z}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]}(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}))^{\mathrm{ab}} \longrightarrow \mathrm{Aut}_{\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]}(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q})^{\mathrm{ab}},$$

where ψ is the map defined at the end of Section 10.1 which describes the action of the group $\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}}$ on $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$, and the second map is given by rationalising, i.e. by sending an automorphism f of $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$ to $f \otimes \mathbb{1}_{\mathbb{Q}}$. We denote this composition by $\tilde{\psi}$. Since $\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}}$ is finite by Lemma 10.7, the map $\tilde{\psi}$ lands in the torsion subgroup $\mathrm{Tor}(\mathrm{Aut}_{\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]}(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q})^{\mathrm{ab}})$. In particular, as shown at the end of Section 10.1, $\tilde{\psi}$ sends $\phi(\gamma)$ to

$$\left(\begin{array}{cc|c} \gamma & 0 & \\ 0 & \gamma & 0 \\ \hline & & \\ \multicolumn{2}{c|}{0} & \mathbb{1}_{\pi_3(M^{g-1}_{\mathbb{Z}/p^k\mathbb{Z}})} \end{array}\right),$$

where we fix a copy of $W_1 = \mathbb{S}^3 \times \mathbb{S}^3$ and write $M^g_{\mathbb{Z}/p^k\mathbb{Z}}$ as $W_1 \# M^{g-1}_{\mathbb{Z}/p^k\mathbb{Z}}$. Since $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$ is a semisimple ring every $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$-module is projective, and hence, so is $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q}$. Thus, there is a $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$-module P such that $(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q}) \oplus P$ is free. The homotopy group $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$ is given by $\mathbb{Z} \oplus \mathbb{Z}[\mathbb{Z}/p^k\mathbb{Z}]^{2g}$ since the universal cover of $M^g_{\mathbb{Z}/p^k\mathbb{Z}}$ is given by $(\mathbb{S}^3 \times \mathbb{D}^3) \#_{\mathbb{Z}/p^k\mathbb{Z}} W_g$. In particular, this is finitely generated over $\mathbb{Z}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$. Hence, $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q}$ is finitely generated over $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$. Since $(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q}) \oplus P$ is free, it is, therefore, isomorphic to $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]^N$ for some $N \in \mathbb{N}$. Note that this does not affect the fixed copy of W_1 since this already corresponds to a free $\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]$-module. Writing a^{n_γ} for the element in $\mathbb{Z}/p^k\mathbb{Z} = \langle a \rangle$ that corresponds to the loop γ in $\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})$ under the isomorphism $\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \cong \mathbb{Z}/p^k\mathbb{Z}$, the above map p is induced by $\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q} \hookrightarrow (\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q}) \oplus P$. Using the determinant this gets mapped to a^{2n_γ}.

$$\begin{tikzcd}
\gamma \arrow[d, mapsto] & \pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \cong \mathbb{Z}/p^k\mathbb{Z} = \langle a \rangle \arrow[d, "\phi^{\mathrm{ab}}"] \\
\phi^{\mathrm{ab}}(\gamma) \arrow[d, mapsto] & \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \arrow[d, "\tilde{\psi}"] \\
\left(\begin{array}{cc|c} \gamma & 0 & \multirow{2}{*}{0} \\ 0 & \gamma & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{1}_{\pi_3(M^{g-1}_{\mathbb{Z}/p^k\mathbb{Z}})} \end{array}\right) \arrow[d, mapsto] & \mathrm{Tor}(\mathrm{Aut}_{\mathbb{Q}[\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}})]}(\pi_3(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \otimes \mathbb{Q})^{\mathrm{ab}}) \arrow[d, "p"] \\
\left(\begin{array}{cc|c} a^{n_\gamma} & 0 & \multirow{2}{*}{0} \\ 0 & a^{n_\gamma} & \\ \hline \multicolumn{2}{c|}{0} & \mathbb{1}_{N-2} \end{array}\right) \arrow[d, mapsto] & \mathrm{Tor}(\mathrm{GL}_N(\mathbb{Q}[\mathbb{Z}/p^k\mathbb{Z}])^{\mathrm{ab}}) \arrow[d, "\det"] \\
a^{2n_\gamma} \arrow[d, mapsto] & \mathrm{Tor}(\mathbb{Q}[\mathbb{Z}/p^k\mathbb{Z}]^\times) \arrow[d, "\pi_{\mathrm{top}}"] \\
\xi_{p^k}^{2n_\gamma} \arrow[d, mapsto] & \mathrm{Tor}(\mathbb{Q}(\xi_{p^k})^\times) \arrow[d, "\cong"] \\
\xi_{p^k}^2 & \{\pm 1\} \times \mathbb{Z}/p^k\mathbb{Z}
\end{tikzcd}$$

FIGURE 23. The sequence of homomorphisms for the case $G = \mathbb{Z}/p^k\mathbb{Z}$.

Let ξ_{p^i} be a p^i-th root of unity, and $\mathbb{Q}(\xi_{p^i})$ the field extension of ξ_{p^i}. There is a map $\mathbb{Q}[\mathbb{Z}/p^k\mathbb{Z}] \to \bigoplus_{i=1}^k \mathbb{Q}(\xi_{p^i})$ given by sending a^n to $\bigoplus_{i=1}^k \xi_{p^i}^n$, where a is the generator of $\mathbb{Z}/p^k\mathbb{Z}$. By [**4**, Prop. 1] this map is an isomorphism. Projecting to the top summand in $\bigoplus_{i=1}^k \mathbb{Q}(\xi_{p^i})$ yields a surjective map $\pi_{\mathrm{top}} \colon \mathbb{Q}[\mathbb{Z}/p^k\mathbb{Z}] \to \mathbb{Q}(\xi_{p^k})$.

By [**52**, Exer. 2.3] we have $\mathrm{Tor}(\mathbb{Q}(\xi_{p^k})^\times) = \langle \pm\xi_{p^k} \rangle$. The bottom map in the above sequence is induced by sending $\xi_{p^k}^{n_\gamma}$ to ξ_{p^k}, a generator of $\mathbb{Z}/p^k\mathbb{Z}$ considered as the group of p-th roots of unity. Mapping $\{\pm 1\} \times \mathbb{Z}/p^k\mathbb{Z}$ to $\mathbb{Z}/p^k\mathbb{Z}$ by forgetting the sign, the above sequence can be abbreviated to

$$\begin{tikzcd}
\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \cong \mathbb{Z}/p^k\mathbb{Z} \arrow[d, "\phi^{\mathrm{ab}}"] \arrow[dr] & \\
\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \arrow[r, "f"] & \mathbb{Z}/p^k\mathbb{Z}
\end{tikzcd} \tag{10.2}$$

Since p is an odd prime, $\xi_{p^k}^2$ generates $\langle \xi_{p^k} \rangle$ and the diagonal map is surjective, and hence, an isomorphism. Therefore, ϕ^{ab} is split injective for $g \geq 7$.

It is left to show that the statement holds for $1 \leq g < 7$. In this case we consider the following commutative diagram.

$$\begin{tikzcd}
\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \arrow[r, "\phi^{\mathrm{ab}}_{M^g_{\mathbb{Z}/p^k\mathbb{Z}}}"] \arrow[dr, "\cong"'] & \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \arrow[r] & \Gamma_\partial(M^7_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \arrow[r, "f"] & \mathbb{Z}/p^k\mathbb{Z} \\
& \pi_1(M^7_{\mathbb{Z}/p^k\mathbb{Z}}) \arrow[ur, "\phi^{\mathrm{ab}}_{M^7_{\mathbb{Z}/p^k\mathbb{Z}}}"] \arrow[urr] & &
\end{tikzcd}$$

The homomorphisms $\phi^{\mathrm{ab}}_{M^g_{\mathbb{Z}/p^k\mathbb{Z}}}$ and $\phi^{\mathrm{ab}}_{M^7_{\mathbb{Z}/p^k\mathbb{Z}}}$ are the homomorphisms given by the abelianisation of the dragging construction for $M^g_{\mathbb{Z}/p^k\mathbb{Z}}$ and $M^7_{\mathbb{Z}/p^k\mathbb{Z}}$ respectively. The right triangle is the triangle above, and the two remaining maps are induced by the inclusion $M^g_{\mathbb{Z}/p^k\mathbb{Z}} \hookrightarrow M^7_{\mathbb{Z}/p^k\mathbb{Z}}$. In particular, the map

$$\pi_1(M^g_{\mathbb{Z}/p^k\mathbb{Z}}) \longrightarrow \pi_1(M^7_{\mathbb{Z}/p^k\mathbb{Z}}) \longrightarrow \mathbb{Z}/p^k\mathbb{Z}$$

is the composition of two isomorphisms. Thus, the map $\phi^{\mathrm{ab}}_{M^g_{\mathbb{Z}/p^k\mathbb{Z}}}$ is split injective for all $g \geq 1$. By definition of the map ϕ^{ab}, this implies that the map ϕ is split injective for all $g \geq 1$. $\square$

10.2.2. An upper bound for $\Gamma_\partial(M_G)^{\mathrm{ab}}$. We have just shown that for the cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/p^k\mathbb{Z}$, for an odd prime p, the group $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$ contains a copy of G as a direct summand. Using the right hand side long exact sequence in Figure 19 for the two examples of groups G, we determine $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$ for these two examples. For this we use the following result.

Lemma 10.9. *Let G be an abelian group, and M a manifold with boundary of 3-type $B\mathrm{Spin}(6) \times BG$. Given an embedding $e\colon M_{\mathbb{Z}} \hookrightarrow M$, where $M_{\mathbb{Z}}$ is the manifold as defined in Section 10.2.1.1, the diagram*

$$\begin{tikzcd}
\Gamma_\partial(M_{\mathbb{Z}})^{\mathrm{ab}} \arrow[r, "\tau_{\mathbb{Z}}"] \arrow[d, "e_*"] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}_+) \arrow[d, "e_*"] \\
\Gamma_\partial(M)^{\mathrm{ab}} \arrow[r, "\tau_G"] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)
\end{tikzcd}$$

commutes, where τ_G and $\tau_{\mathbb{Z}}$ are the compositions

$$\Gamma_\partial(M)^{\mathrm{ab}} \xrightarrow{\alpha_*} \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \longrightarrow \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)$$

and

$$\Gamma_\partial(M_{\mathbb{Z}})^{\mathrm{ab}} \xrightarrow{\alpha_*} \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+) \longrightarrow \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}_+)$$

respectively, with α_* as in Proposition 10.4 and the second map comes from the long exact sequence in Figure 19.

PROOF. Elements in $\pi_7(M\mathrm{Spin}\wedge\Sigma^\infty BG_+)$ are represented by a 7-dimensional manifold equipped with a Spin-structure and a map to BG.

Using the embedding e we can write M_G as $M_\mathbb{Z} \cup_{\partial M_\mathbb{Z}} K$ for some manifold K with boundary $\partial M_\mathbb{Z} \cup \partial M_G$ as shown in Figure 24. The left hand side vertical map sends a dif-

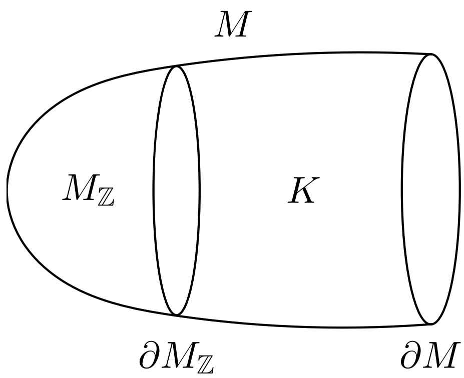

FIGURE 24. The embedding $e\colon M_\mathbb{Z} \hookrightarrow M = M_\mathbb{Z} \cup_{\partial M_\mathbb{Z}} K$.

feomorphism f representing an element in $\Gamma_\partial(M_\mathbb{Z})^{\mathrm{ab}}$ to the class of the diffeomorphism $g = f \cup \mathbb{1}_K$ in $\Gamma_\partial(M)^{\mathrm{ab}}$ which extends f via the identity on K. The right hand side vertical map is given by extending the map that sends the manifold to $B\mathbb{Z}$ to a map to BG. The horizontal maps send a diffeomorphism φ of M' – which is one of M and $M_\mathbb{Z}$ – to T_φ, which is given by $M'_\varphi \cup_{\mathbb{S}^1\times\partial M'} \mathbb{S}^1 \times \overline{M}'$, where M'_φ is the mapping torus of φ and $\overline{M}'$ denotes the manifold M' with the opposite orientation. Note that this lands in $\pi_7(M\mathrm{Spin}\wedge\Sigma^\infty BG_+)$ and $\pi_7(M\mathrm{Spin}\wedge\Sigma^\infty B\mathbb{Z}_+)$ respectively, as the spin structure can be extended from $\mathbb{S}^1\times\overline{M}'$ using that $(T_\varphi, \mathbb{S}^1\times\overline{M}')$ is 2-connected and the map $T_\varphi \to BG$ is extended analogously.

To show that the above diagram commutes we need to show that T_f is cobordant to $e_*(T_g)$ as manifolds over BG. For this, note that $K\cup\overline{K}$ is cobordant to the cylinder $\partial M_\mathbb{Z}\times I$ as shown in Figure 25. Here, the first second manifold comes from deleting and adding corners

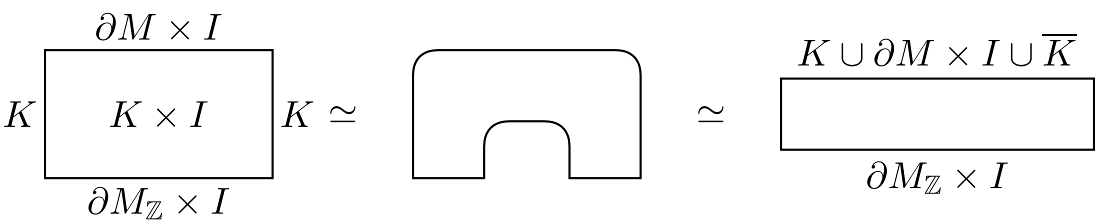

FIGURE 25. $K\cup\overline{K}$ is cobordant to the cylinder $\partial M_\mathbb{Z}\times I$.

to the first manifold, and the third manifold is diffeomorphic to the second. The top of the rectangle then becomes $K\cup\partial M\times I\cup\overline{K}$ which is diffeomorphic to $K\cup\overline{K}$ by stretching collars, and the bottom of the rectangle is a cylinder. By the composition

$$K\times I \longrightarrow K \hookrightarrow M \longrightarrow B\mathrm{Spin}\times BG$$

this gives a Spin-cobordism over BG. Thus, the manifolds T_f and T_g are Spin-cobordant as manifolds over BG as shown in Figure 26. Hence, the above diagram commutes. $\square$

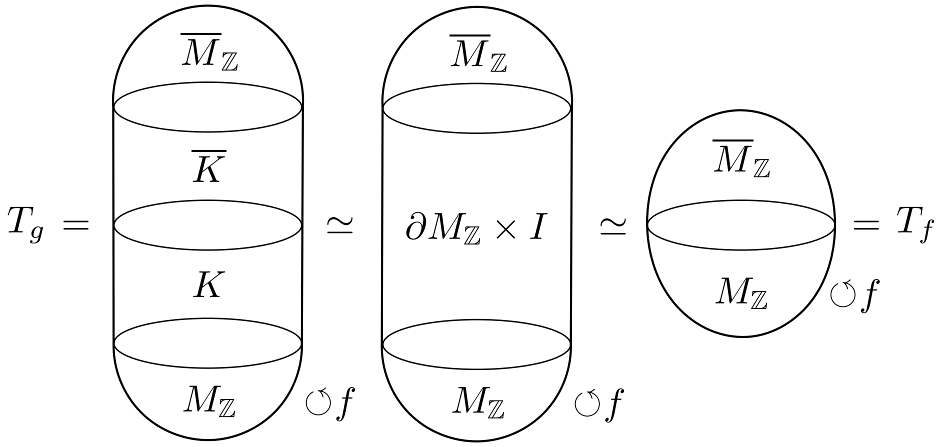

FIGURE 26. The manifold T_g is cobordant to T_f as (Spin-)manifolds over BG.

COROLLARY 10.10. *Let G be an abelian group, and M a manifold with boundary of 3-type $B\mathrm{Spin}(6) \times BG$. The composition $\tau_G \circ \phi$ is trivial, where ϕ is the dragging homomorphism.*

PROOF. For an element $g \in G$ find an embedding of $\mathbb{S}^1 \times \mathbb{D}^5$ into M with core $g \in G = \pi_1(M)$ in M. Find the copy of $W_{1,1}$ in M that gets dragged around by ϕ and a tube from this to the embedding of $\mathbb{S}^1 \times \mathbb{D}^5$ representing g and glue it in. This gives an embedding e_g of $(\mathbb{S}^1 \times \mathbb{D}^5)\# W_{1,1} = M_{\mathbb{Z}}$ into M. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\phi"] \arrow[d, "\pi_1(e)"] & \Gamma_\partial(M_{\mathbb{Z}}) \arrow[r, "\tau_{\mathbb{Z}}"] \arrow[d, "e_*"] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}_+) \arrow[d, "e_*"] \\
G \arrow[r, "\phi"] & \Gamma_\partial(M) \arrow[r, "\tau_G"] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)
\end{tikzcd}$$

where the left hand square is the diagram from Lemma 10.9. Using $B\mathbb{Z} \simeq \mathbb{S}^1$ we have

$$\begin{aligned}
\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}_+) &= \Omega_7^{\mathrm{Spin}}(B\mathbb{Z}) \\
&\cong \Omega_7^{\mathrm{Spin}}(*) \oplus \Omega_7^{\mathrm{Spin}}(B\mathbb{Z}/*) \\
&\cong \Omega_7^{\mathrm{Spin}}(*) \oplus \Omega_6^{\mathrm{Spin}}(*) \\
&= 0,
\end{aligned}$$

where $\Omega_7^{\mathrm{Spin}}(B\mathbb{Z}/*) \cong \Omega_6^{\mathrm{Spin}}(*)$ holds by the suspension isomorphism, and the last equation is shown in [**37**]. Thus, by commutativity of the above diagram, the composition $\tau_G \circ \phi \circ \pi_1(e_g)$ vanishes. Since since is holds for all elements $g \in G$, the composition $\tau_G \circ \phi$ also vanishes. $\square$

10.2.2.1. *The Case $G = \mathbb{Z}$.*

THEOREM 10.11. $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+) \cong \mathbb{Z}$.

Proof. As in the proof of Corollary 10.10 we have $\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}_+) = 0$. Considering the right hand side sequences in Figure 19 for $G = \mathbb{Z}$, this becomes

$$\cdots \longrightarrow \pi_8(M\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+) \longrightarrow \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+) \longrightarrow 0.$$

As we have seen at the beginning of this chapter, $\mathbb{Z}/4\mathbb{Z}$ gets hit from behind and, thus, $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+)$ is isomorphic to a quotient of $\mathbb{Z}$ (this could be all of $\mathbb{Z}$).

Let $M_\mathbb{Z}$ be the manifold considered in Section 10.2.1.1. Note that for $g \geq 8$ the manifold $M_\mathbb{Z}$ satisfies $\overline{g}(M_\mathbb{Z}) \geq \mathrm{usr}(\mathbb{Z}[\pi_1(M_\mathbb{Z})]) + 4$ since we have

$$\mathrm{usr}(\mathbb{Z}[\pi_1(M_\mathbb{Z})]) \leq 3 + h(\pi_1(M_\mathbb{Z})) = 4.$$

By construction, the manifold $M_\mathbb{Z}$ is 2-connected relative to its boundary. Thus, by Proposition 10.4 we get an isomorphism

$$\alpha_*\colon \Gamma_\partial(M_\mathbb{Z})^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+)$$

By Theorem 10.5 we know that $\Gamma_\partial(M_\mathbb{Z})^{\mathrm{ab}}$ contains a copy of $\mathbb{Z}$. The long exact sequence considered above therefore implies

$$\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}_+) \cong \mathbb{Z}. \qquad \square$$

10.2.2.2. *The Case $G = \mathbb{Z}/p^k\mathbb{Z}$.* Recall that for manifolds $M^g_{\mathbb{Z}/p^k\mathbb{Z}} = (L \times \mathbb{D}^3)\#W_g$ as in Section 10.2.1.2, $\Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}}$ contains a copy of $\mathbb{Z}/p^k\mathbb{Z}$ as a direct summand. Recall from the proof of Lemma 10.7 that for $g \geq 7$ there is an isomorphism

$$\alpha_*\colon \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+).$$

Using the right hand side long exact sequence in Figure 19 for $G = \mathbb{Z}/p^k\mathbb{Z}$ we get the following commutative diagram,

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z} \arrow[d, "s"] & \mathbb{Z}/p^k\mathbb{Z} \arrow[dl, hook', "\tilde{\phi}"'] \arrow[d, hook, "\phi"'] \arrow[dr, "\cong"] & \\
\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \arrow[d, "t"] & \Gamma_\partial(M^g_{\mathbb{Z}/p^k\mathbb{Z}})^{\mathrm{ab}} \arrow[l, "\cong", "\alpha_*"'] \arrow[r, two heads, "f"] & \mathbb{Z}/p^k\mathbb{Z} \\
\pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \arrow[d] & & \\
0 & &
\end{tikzcd}$$

where the rightmost triangle is the one in (10.2), and $\tilde{\phi}$ is defined so that the diagram commutes. We show that the group $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+)$ is isomorphic to the group $\mathbb{Z}/p^k\mathbb{Z} \oplus \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+)$ via the following map.

THEOREM 10.12. *The map*

$$\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+) \xrightarrow{f\circ\alpha_*^{-1}\oplus t} \mathbb{Z}/p^k\mathbb{Z} \oplus \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty B\mathbb{Z}/p^k\mathbb{Z}_+)$$

is an isomorphism.

PROOF. By Corollary 10.10 the composition $t \circ \tilde{\phi}$ vanishes. In particular, the map $\tilde{\phi}$ factors through $\mathrm{Ker}(t)$. Since $\tilde{\phi}$ is injective, so is the map to $\mathrm{Ker}(t)$, and hence, $\mathrm{Ker}(t)$ contains a copy of $\mathbb{Z}/p^k\mathbb{Z}$. By exactness of the vertical sequence, the kernel of t is the same as the image of the image of s. Recall that $\mathbb{Z}/4\mathbb{Z} \oplus 0 \subset \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/p^k\mathbb{Z}$ gets hit from behind. Hence, the image of s is a quotient of $\mathbb{Z}/p^k\mathbb{Z}$. Thus, $\mathrm{Ker}(t)$ must be the whole of $\mathbb{Z}/p^k\mathbb{Z}$. In particular, the map in the statement is an isomorphism since the map t is surjective and the kernel of t is mapped isomorphically to $\mathbb{Z}/p^k\mathbb{Z}$ under $f \circ \alpha_*^{-1}$. □

10.3. Proof of Theorem 10.1

We now combine the examples considered in the previous section to prove Theorem 10.1. Let G be a group as considered in the theorem. Analogous to Section 10.2 we first show the following proposition.

PROPOSITION 10.13. *Let G be a group with abelianisation $G^{ab} \cong \bigoplus_{i=1}^m \mathbb{Z}/p_i^{k_i}\mathbb{Z} \oplus \mathbb{Z}^n$, where the p_i's are odd primes. The composition*

$$s\colon G^{\mathrm{ab}} \hookrightarrow \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$$

is split injective, where the first map is given by the standard inclusion that sends g to $(0, g)$, and the second map is the map in the long exact sequence on homotopy groups for the above fibration.

PROOF. The projection $G \to G^{\mathrm{ab}}$ induces a map between two long exact sequences as shown in Figure 27. Here, we write A_i for the direct summands of G^{ab}, i.e.

$$A_i \cong \begin{cases} \mathbb{Z}/p_i^{k_i}\mathbb{Z} & i \leq m \\ \mathbb{Z} & i > m \end{cases}$$

and pr_i for the map induced by the projection $G^{\mathrm{ab}} \to A_i$. For each A_i we get a 6-dimensional manifold M_{A_i} with boundary, corresponding to A_i as described in the previous section. We assume that each M_{A_i} is 2-connected relative to its boundary, that its genus is at least 1, and that its stable genus is at least $\mathrm{usr}(\mathbb{Z}[\pi_1(M_{A_i})]) + 4$. For each A_i Proposition 10.4 yields an isomorphism

$$\alpha_{A_i *}\colon \Gamma_\partial(M_{A_i})^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BA_{i+}).$$

The map $(f \circ \alpha_*^{-1})_i$ in Figure 27 denotes the composition of ${\alpha_{A_i *}}^{-1}$ and the homomorphism f as defined in the proofs of Theorem 10.8 and Theorem 10.5 for $i \leq m$ and $i > m$ respectively. The dashed map g in Figure 27 is defined so that the diagram commutes.

$$
\begin{tikzcd}
\mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \arrow[r] \arrow[d, equal] & \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \arrow[r, two heads] \arrow[d] & \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+) \\
\mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \arrow[r] & \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG^{\mathrm{ab}}_+) \arrow[r] \arrow[d, "\bigoplus pr_i"] & \cdots \\
G^{\mathrm{ab}} \arrow[u, hook] \arrow[ddr, dashed, "g"'] & \bigoplus_i \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BA_{i+}) \arrow[d, "\bigoplus_i (f\circ\alpha_*^{-1})_i"] & \\
& \bigoplus_i A_i \arrow[d, equal] & \\
& G^{\mathrm{ab}} &
\end{tikzcd}
$$

FIGURE 27. The diagram for a general group G.

We show that the map g is surjective. Since finitely generated abelian groups are Hopfian, this implies that the map g is an isomorphism. In particular, this shows that the map

$$s\colon G^{\mathrm{ab}} \hookrightarrow \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$$

is split injective.

From the previous section we know that the projection of the image $g(A_i)$ in G^{ab} to A_i is surjective. It remains to show that the projection to any other summand of G^{ab} is trivial. For this we consider the maps $g_{ij}\colon A_i \hookrightarrow G^{\mathrm{ab}} \twoheadrightarrow A_j$. For $i \neq j$ this is the trivial map. This induces a map

$$(g_{ij})_*\colon \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BA_{i+}) \longrightarrow \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BA_{j+})$$

which for $i \neq j$ factors through $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\{*\}_+)$. Since this group is trivial, the map $(g_{ij})_*$ is trivial for $i \neq j$. Thus, the map g is surjective. $\square$

PROOF OF THEOREM 10.1. We consider the composition $t \circ s$, where the map s is as in the previous proof and t is the map $\pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \to \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+)$ coming from the long exact sequence on homotopy groups in Figure 19. We have shown at the beginning of this chapter that the map t is surjective. The map s is split injective by Proposition 10.13. Since $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}}$ gets hit from behind in the long exact sequence in Figure 19, the image of s is the same as the image of the map

$$\mathbb{Z}/4\mathbb{Z} \oplus G^{\mathrm{ab}} \to \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+)$$

that shows up in the long exact sequence. Thus, we get a split short exact sequence

$$0 \longrightarrow G^{\mathrm{ab}} \xrightarrow{s} \pi_1(MT\mathrm{Spin}(6) \wedge \Sigma^\infty BG_+) \xrightarrow{t} \pi_7(M\mathrm{Spin} \wedge \Sigma^\infty BG_+) \longrightarrow 0.$$

In particular, we get the isomorphism as claimed in the theorem. $\square$

Combining Theorem 10.1 with Proposition 10.4 yields Theorem E.

APPENDIX A

Tables of Steenrod Squares

The tables of Steenrod squares given in this appendix are computed in two different ways. For the spaces $B\mathrm{Pin}^{-}(6)$, $B\mathrm{Spin}(6) \times B\mathbb{Z}/2\mathbb{Z}$, and $B\mathrm{Spin}(6) \times B\mathbb{Z}/2^k\mathbb{Z}$, for $k \geq 2$, in Tables 1, 4, and 6 respectively, we have computed the Steenrod squares by hand, using the multiplicativity of the total Steenrod squares, the Cartan formula, and the Wu formula. Examples of these computations can be found in Section 8.5.1. For the spectra $MT\mathrm{Pin}^{-}(6)$, C, $MT\mathrm{Spin}(6) \wedge \Sigma^{\infty} B\mathbb{Z}/2\mathbb{Z}$, and $MT\mathrm{Spin}(6) \wedge \Sigma^{\infty} B\mathbb{Z}/2^k\mathbb{Z}$, for $k \geq 2$, in Tables 2, 3, 5, and 7 respectively, we have used Maple to compute the total Steenrod square of the respective Thom class u. After applying the relations between the elements this has given the results shown in Sections 8.5.2, 8.5.3, and 9.1, Step 2. We have then used Maple to compute the total Steenrod squares of the above spectra by using the results we got for the spaces above and multiplying it by the total Steenrod square of the respective Thom class u.

Sq^i	ω_1	ω_1^2	ω_1^3	ω_1^4	ω_4	ω_1^5	$\omega_1\omega_4$	ω_1^6	$\omega_1^2\omega_4$	ω_6	ω_1^7	$\omega_1^3\omega_4$	$\omega_1\omega_6$
0	ω_1	ω_1^2	ω_1^3	ω_1^4	ω_4	ω_1^5	$\omega_1\omega_4$	ω_1^6	$\omega_1^2\omega_4$	ω_6	ω_1^7	$\omega_1^3\omega_4$	$\omega_1\omega_6$
1	ω_1^2	0	ω_1^4	0	0	ω_1^6	$\omega_1^2\omega_4$	0	$\omega_1\omega_6$	$\omega_1\omega_6$	ω_1^8	$\omega_1^4\omega_4$	0
2	0	ω_1^4	ω_1^5	0	$\omega_1^2\omega_4$ $+\omega_6$	0	$\omega_1^3\omega_4$ $+\omega_1\omega_6$	ω_1^8	0	$\omega_1^2\omega_6$			
3	0	0	ω_1^6	0	$\omega_1\omega_6$	0	$\omega_1^4\omega_4$						
4	0	0	0	ω_1^8	ω_4^2								
5	0	0	0										
6	0	0											
7	0												

TABLE 1. The Steenrod square structure of $B\mathrm{Pin}^{-}(6)$.

Sq^i	u	$\omega_1 u$	$\omega_1^2 u$	$\omega_1^3 u$	$\omega_1^4 u$	$\omega_4 u$	$\omega_1^5 u$	$\omega_1\omega_4 u$	$\omega_1^6 u$	$\omega_1^2\omega_4 u$	$\omega_6 u$	$\omega_1^7 u$	$\omega_1^3\omega_4 u$	$\omega_1\omega_6 u$
0	u	$\omega_1 u$	$\omega_1^2 u$	$\omega_1^3 u$	$\omega_1^4 u$	$\omega_4 u$	$\omega_1^5 u$	$\omega_1\omega_4 u$	$\omega_1^6 u$	$\omega_1^2\omega_4 u$	$\omega_6 u$	$\omega_1^7 u$	$\omega_1^3\omega_4 u$	$\omega_1\omega_6 u$
1	$\omega_1 u$	0	$\omega_1^3 u$	0	$\omega_1^5 u$	$\omega_1\omega_4 u$	0	0	$\omega_1^7 u$	$\omega_1^3\omega_4 u$	0	0	0	$\omega_1^2\omega_6 u$
2	0	$\omega_1^3 u$	$\omega_1^4 u$	0	0	$\omega_1^2\omega_4 u$ $+\omega_6 u$	$\omega_1^7 u$	$\omega_1\omega_6 u$	$\omega_1^8 u$	$\omega_1^2\omega_6 u$	0			
3	0	0	$\omega_1^5 u$	0	0	$\omega_1^3\omega_4 u$	0	$\omega_1^2\omega_6 u$						
4	$\omega_1^4 u$ $+\omega_4 u$	$\omega_1^5 u$ $+\omega_1\omega_4 u$	$\omega_1^6 u$ $+\omega_1^2\omega_4 u$	$\omega_1^3\omega_4 u$	$\omega_1^4\omega_4 u$	$\omega_1^4\omega_4 u$ $+\omega_1^2\omega_6 u$								
5	$\omega_1^5 u$ $+\omega_1\omega_4 u$	0	$\omega_1^7 u$ $+\omega_1^3\omega_4 u$	0										
6	$\omega_1^2\omega_4 u$ $+\omega_6 u$	$\omega_1^7 u$ $+\omega_1\omega_6 u$	$\omega_1^8 u$ $+\omega_1^2\omega_6 u$											
7	$\omega_1^3\omega_4 u$	$\omega_1^2\omega_6 u$												
8	$\omega_1^8 u$ $+\omega_1^2\omega_6 u$ $+\omega_4^2 u$													

TABLE 2. The Steenrod square structure of the spectrum $MT\mathrm{Pin}^-(6)$.

Sq^i	u	$\omega_1 u$	$\omega_1^2 u$	$\omega_1^3 u$	$\omega_1^4 u$	$\omega_4 u$	$\omega_1^5 u$	$\omega_1\omega_4 u$	$\omega_1^6 u$	$\omega_1^2\omega_4 u$	$\omega_6 u$	$\omega_1^7 u$	$\omega_1^3\omega_4 u$	$\omega_1\omega_6 u$
0	u	$\omega_1 u$	$\omega_1^2 u$	$\omega_1^3 u$	$\omega_1^4 u$	$\omega_4 u$	$\omega_1^5 u$	$\omega_1\omega_4 u$	$\omega_1^6 u$	$\omega_1^2\omega_4 u$	$\omega_6 u$	$\omega_1^7 u$	$\omega_1^3\omega_4 u$	$\omega_1\omega_6 u$
1	0	$\omega_1^2 u$	0	$\omega_1^4 u$	0	0	$\omega_1^6 u$	$\omega_1^2\omega_4 u$	0	0	$\omega_1\omega_6 u$	$\omega_1^8 u$	$\omega_1^4\omega_4 u$	0
2	$\omega_1^2 u$	$\omega_1^3 u$	0	0	$\omega_1^6 u$	$\omega_6 u$	$\omega_1^7 u$	$\omega_1\omega_6 u$	0	$\omega_1^4\omega_4 u$ $+\omega_1^2\omega_6 u$	0			
3	0	$\omega_1^4 u$	0	0	0	$\omega_1\omega_6 u$	$\omega_1^8 u$	0						
4	$\omega_1^4 u$ $+\omega_4 u$	$\omega_1^5 u$ $+\omega_1\omega_4 u$	$\omega_1^2\omega_4 u$	$\omega_1^3\omega_4 u$	$\omega_1^4\omega_4 u$	$\omega_1^2\omega_6 u$								
5	0	$\omega_1^6 u$ $+\omega_1^2\omega_4 u$	0	$\omega_1^4\omega_4 u$										
6	$\omega_1^6 u$ $+\omega_6 u$	$\omega_1^7 u$ $+\omega_1\omega_6 u$	$\omega_1^4\omega_4 u$ $+\omega_1^2\omega_6 u$											
7	$\omega_1\omega_6 u$	$\omega_1^8 u$												
8	$\omega_1^8 u$ $+\omega_1^4\omega_4 u$ $+\omega_1^2\omega_6 u$ $+\omega_4^2 u$													

TABLE 3. The Steenrod square structure of the spectrum C.

Sq^i	z	z^2	z^3	z^4	ω_4	z^5	$z\omega_4$	z^6	$z^2\omega_4$	ω_6	z^7	$z^3\omega_4$	$z\omega_6$
0	z	z^2	z^3	z^4	ω_4	z^5	$z\omega_4$	z^6	$z^2\omega_4$	ω_6	z^7	$z^3\omega_4$	$z\omega_6$
1	z^2	0	z^4	0	0	z^6	$z^2\omega_4$	0	0	0	z^8	$z^4\omega_4$	$z^2\omega_6$
2	0	z^4	z^5	0	ω_6	0	$z\omega_6$	z^8	$z^4\omega_4$ $+z^2\omega_6$	0			
3	0	0	z^6	0	0	0	$z^2\omega_6$						
4	0	0	0	z^8	ω_4^2								
5	0	0	0										
6	0	0											
7	0												

TABLE 4. The Steenrod square structure of $B\mathrm{Spin}(6) \times B\mathbb{Z}/2\mathbb{Z}$.

Sq^i	u	zu	z^2u	z^3u	z^4u	ω_4u	z^5u	$z\omega_4u$	z^6u	$z^2\omega_4u$	ω_6u	z^7u	$z^3\omega_4u$	$z\omega_6u$
0	u	zu	z^2u	z^3u	z^4u	ω_4u	z^5u	$z\omega_4u$	z^6u	$z^2\omega_4u$	ω_6u	z^7u	$z^3\omega_4u$	$z\omega_6u$
1	0	z^2u	0	z^4u	0	0	z^6u	$z^2\omega_4u$	0	0	0	z^8u	$z^4\omega_4u$	$z^2\omega_6u$
2	0	0	z^4u	z^5u	0	ω_6u	0	$z\omega_6u$	z^8u	$z^4\omega_4u$ $+z^2\omega_6u$	0			
3	0	0	0	z^6u	0	0	0	$z^2\omega_6u$						
4	ω_4u	$z\omega_4u$	$z^2\omega_4u$	$z^3\omega_4u$	z^8u $+z^4\omega_4u$	0								
5	0	$z^2\omega_4u$	0	$z^4\omega_4u$										
6	ω_6u	$z\omega_6u$	$z^4\omega_4u$ $+z^2\omega_6u$											
7	0	$z^2\omega_6u$												
8	ω_4^2u													

TABLE 5. The Steenrod square structure of the spectrum $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$.

Sq^i	x	y	xy	y^2	ω_4	xy^2	$x\omega_4$	y^3	$y\omega_4$	ω_6	xy^3	$xy\omega_4$	$x\omega_6$
0	x	y	xy	y^2	ω_4	xy^2	$x\omega_4$	y^3	$y\omega_4$	ω_6	xy^3	$xy\omega_4$	$x\omega_6$
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	y^2	xy^2	0	ω_6	0	$x\omega_6$	y^4	$y^2\omega_4$ $+y\omega_6$	0			
3	0	0	0	0	0	0	0						
4	0	0	0	y^4	ω_4^2								
5	0	0	0										
6	0	0											
7	0												

TABLE 6. The Steenrod square structure of $B\mathrm{Spin}(6) \times B\mathbb{Z}/2^k\mathbb{Z}$.

Sq^i	u	xu	yu	xyu	y^2u	ω_4u	xy^2u	$x\omega_4u$	y^3u	$y\omega_4u$	ω_6u	xy^3u	$xy\omega_4u$	$x\omega_6u$
0	u	xu	yu	xyu	y^2u	ω_4u	xy^2u	$x\omega_4u$	y^3u	$y\omega_4u$	ω_6u	xy^3u	$xy\omega_4u$	$x\omega_6u$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	y^2u	xy^2u	y^4u	ω_6u	0	$x\omega_6u$	y^4u	$y^2\omega_4u$ $+y\omega_6u$	0			
3	0	0	0	0	0	0	0	0						
4	ω_4u	$x\omega_4u$	$y\omega_4u$	$xy\omega_4u$	$y^2\omega_4u$	0								
5	0	0	0	0										
6	ω_6u	$x\omega_6u$	$y^2\omega_4u$ $+y\omega_6u$											
7	0	0												
8	ω_4^2u													

TABLE 7. The Steenrod square structure of the spectrum $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$.

APPENDIX B

Tables of Module Definitions

The module definitions given in this appendix form the input for Robert Bruner's programme [**12**] to compute the E_2-page of the Adams spectral sequence. A documentation for the format of module definition files can be found at `http://www.math.wayne.edu/~rrb/cohom/modfmt.html`.

19																		
0	1	2	3	4	4	5	5	6	6	6	7	7	7	8	8	8	8	8
0	1	1	1															
0	4	2	4	5														
0	5	2	6	7														
0	6	2	9	10														
0	7	1	12															
0	8	3	14	16	17													
1	2	1	3															
1	4	2	6	7														
1	6	2	11	13														
1	7	1	16															
2	1	1	3															
2	2	1	4															
2	3	1	6															
2	4	2	8	9														
2	5	2	11	12														
2	6	2	14	16														
3	4	1	12															
4	1	1	6															
4	4	1	15															
5	1	1	7															
5	2	2	9	10														
5	3	1	12															
5	4	2	15	16														
6	2	1	11															
7	2	1	13															
7	3	1	16															
8	1	1	11															
8	2	1	14															
9	1	1	12															
9	2	1	16															
13	1	1	16															

TABLE 1. Module definition for $MT\mathrm{Pin}^{-}(6)$.

19																		
0	1	2	3	4	4	5	5	6	6	6	7	7	7	8	8	8	8	8
0	2	1	2															
0	4	2	4	5														
0	6	2	8	10														
0	7	1	13															
0	8	4	14	15	16	17												
1	1	1	2															
1	2	1	3															
1	3	1	4															
1	4	2	6	7														
1	5	2	8	9														
1	6	2	11	13														
1	7	1	14															
2	4	1	9															
2	6	2	15	16														
3	1	1	4															
3	4	1	12															
3	5	1	15															
4	2	1	8															
4	4	1	15															
5	2	1	10															
5	3	1	13															
5	4	1	16															
6	1	1	8															
6	2	1	11															
6	3	1	14															
7	1	1	9															
7	2	1	13															
9	2	2	15	16														
10	1	1	13															
11	1	1	14															
12	1	1	15															

TABLE 2. Module definition for the cofibre C.

19																		
0	1	2	3	4	4	5	5	6	6	6	7	7	7	8	8	8	8	8
0	4	1	5															
0	6	1	10															
0	8	1	17															
1	1	1	2															
1	4	1	7															
1	5	1	9															
1	6	1	13															
1	7	1	16															
2	2	1	4															
2	4	1	9															
2	6	2	15	16														
3	1	1	2															
3	2	1	6															
3	3	1	8															
3	4	1	12															
3	5	1	15															
4	4	2	14	15														
5	2	1	10															
6	1	1	8															
7	1	1	9															
7	2	1	13															
7	3	1	16															
8	2	1	14															
9	2	2	15	16														
11	1	1	14															
12	1	1	15															
13	1	1	16															

TABLE 3. Module definition for $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2\mathbb{Z}_+$.

5				
0	4	6	8	8
0	4	1	1	
0	6	1	2	
0	8	1	3	
1	2	1	2	

Table 4. Module definition for $MT\mathrm{Spin}(6)$.

19																		
0	1	2	3	4	4	5	5	6	6	6	7	7	7	8	8	8	8	8
0	4	1	5															
0	6	1	10															
0	8	1	17															
1	4	1	7															
1	6	1	13															
2	2	1	4															
2	4	1	9															
2	6	2	15	16														
3	2	1	6															
3	4	1	12															
4	4	2	14	15														
5	2	1	10															
7	2	1	13															
8	2	1	14															
9	2	2	15	16														

Table 5. Module definition for $MT\mathrm{Spin}(6) \wedge \Sigma^\infty B\mathbb{Z}/2^k\mathbb{Z}_+$.

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