# Strong law of large numbers for the capacity of the Wiener sausage in dimension four

Amine Asselah *  Bruno Schapira†  Perla Sousi‡

**Abstract**

We prove a strong law of large numbers for the Newtonian capacity of a Wiener sausage in the critical dimension four, where a logarithmic correction appears in the scaling. The main step of the proof is to obtain precise asymptotics for the expected value of the capacity. This requires a delicate analysis of intersection probabilities between two independent Wiener sausages.

*Keywords and phrases.* Capacity, Wiener sausage, Law of large numbers.
MSC 2010 *subject classifications.* Primary 60F05, 60G50.

## 1 Introduction

We denote by $(\beta_s, s \geq 0)$ a Brownian motion on $\mathbb{R}^4$, and for $r > 0$ and $0 \leq s \leq t < \infty$, the Wiener sausage of radius $r$ in the time period $[s,t]$ is defined as

$$W_r[s,t] = \{z \in \mathbb{R}^4 \ : \ \|z - \beta_u\| \leq r \ \text{ for some } s \leq u \leq t\}, \tag{1.1}$$

where $\|\cdot\|$ stands for the Euclidean norm. Let $\mathbb{P}_z$ and $\mathbb{E}_z$ be the law and expectation with respect to the Brownian motion started at $z \in \mathbb{R}^4$. Let $G$ denote Green's function ($G(z) = \|z\|^{-2}/(2\pi^2)$) and $H_A$ denote the hitting time of $A \subset \mathbb{R}^4$ by the Brownian motion. The Newtonian capacity of a compact set $A \subset \mathbb{R}^4$ may be defined through hitting time as

$$\mathrm{Cap}(A) = \lim_{\|z\| \to \infty} \frac{\mathbb{P}_z(H_A < +\infty)}{G(z)}. \tag{1.2}$$

A more classical definition through a variational expression reads

$$\mathrm{Cap}(A) = \Big(\inf\{\int\int G(x-y)d\mu(x)d\mu(y) : \ \mu \ \text{ prob. measure with support in } A\}\Big)^{-1}.$$

Our central object is the capacity of the Wiener sausage, and formula (1.2), with $A = W_1[0,t]$ (sampled independently of the Brownian motion inherent to the law $\mathbb{P}_z$), casts the problem into an intersection event for two independent sausages.

Our main result is the following law of large number for the capacity of the Wiener sausage.

*Université Paris-Est Créteil; amine.asselah@u-pec.fr
†Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France; bruno.schapira@univ-amu.fr
‡University of Cambridge, Cambridge, UK; p.sousi@statslab.cam.ac.uk

**Theorem 1.1.** *In dimension four, for any radius $r > 0$, almost surely and in $L^p$, for any $p \in [1, \infty)$, we have*

$$\lim_{t\to\infty} \frac{\log t}{t} \mathrm{Cap}(W_r[0,t]) \;=\; \pi^2. \tag{1.3}$$

The proof of (1.3) presents some similarities with the proof in the discrete case, which is given in our companion paper [4], but also substantial differences. The main difference concerns the computation of the expected capacity, which in the discrete setting had been essentially obtained by Lawler, see [3] for details, whereas in our context it requires new delicate analysis.

It may seem odd that the fluctuations result we obtain in the discrete model [4] are not directly transposable in the continuous setting. However, it was noticed some thirty years ago by Le Gall [13] that *it does not seem easy to deduce Wiener sausage estimates from random walks estimates*, and vice-versa. Let us explain one reason for that. The capacity of a set $A$ can be represented as the integral of the *equilibrium measure* of the set $A$, very much as in the discrete formula for the capacity of the range $\mathcal{R}[0,n]$ of a random walk:

$$\mathrm{Cap}(\mathcal{R}[0,n]) = \sum_{x\in\mathcal{R}[0,n]} \mathbb{P}_x\left(H^+_{\mathcal{R}[0,n]} = \infty\right),$$

where on the right-hand side $H^+_A$ stands for the first return time to a set $A$ for a random walk with law $\mathbb{P}_x$, and $\mathcal{R}[0,n]$ is the range of another independent random walk. Whereas Lawler [12] has established deep non-intersection results for two random walks in dimension four, the corresponding results for the equilibrium measure of $W_1(0,t)$ are still missing.

As noted in [4], the scaling in Theorem 1.1 is analogous to that of the law of large numbers for the volume of the Wiener sausage in dimension 2 (see [14]).

**Remark 1.2.** Our result is a result about non-intersection probabilities for two independent Wiener sausages, and the asymptotic result (1.3) reads as follows. For any $\varepsilon > 0$, almost surely, for $t$ large enough,

$$(1-\varepsilon)\frac{t}{2\log t} \le \lim_{\|z\|\to\infty} \|z\|^2 \cdot \mathbb{P}_{0,z}\left(W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing \;\middle|\; \beta\right) \le (1+\varepsilon)\frac{t}{2\log t}. \tag{1.4}$$

Estimates, up to constants, have been obtained in a different regime (where $z$ and $t$ are related as $z = \sqrt{t}x$) by Pemantle, Peres and Shapiro [19], but cannot be used to obtain our strong law of large numbers.

One delicate part in Theorem 1.1 is establishing convergence for the scaled expected capacity. This is Proposition 3.1 of Section 3. From (1.2), the expected capacity of a Wiener sausage is equivalent to the probability that two Wiener sausages intersect. Estimating such a probability has a long tradition: pioneering works were produced by Dvoretzky, Erdös and Kakutani [6] and Aizenman [1]; Aizenman's results have been subsequently improved by Albeverio and Zhou [2], Peres [20], Pemantle, Peres and Shapiro [19] and Khoshnevisan [10] (and references therein). In the discrete setting, the literature is even larger and older, and analogous results are presented in Lawler's comprehensive book [12].

As a byproduct of our arguments, we improve a large deviation estimate of Erhard and Poisat [8], and obtain a *nearly correct* estimate of the variance, which will have to be improved for studying the fluctuations.

**Proposition 1.3.** *There is a constant $c > 0$, such that for any $0 < \varepsilon < 1$, there exists $\kappa = \kappa(\varepsilon)$ such that for any $t$ large enough*

$$\mathbb{P}\Big(\mathrm{Cap}(W_1[0,t]) - \mathbb{E}[\mathrm{Cap}(W_1[0,t])] \geq \varepsilon \frac{t}{\log t}\Big) \leq \exp\big(-c\,\varepsilon^2 t^{\kappa}\big). \tag{1.5}$$

*Moreover, there exists a constant $C > 0$, such that for $t$ large enough,*

$$\mathrm{var}\,\big(\mathrm{Cap}(W_1[0,t])\big) \;\leq\; C\,(\log\log t)^9 \frac{t^2}{(\log t)^4}. \tag{1.6}$$

**Remark 1.4.** We do not know what is the correct speed in the large deviation estimate (1.5). The analogous result for the volume of the sausage in $d = 2$ (or even the size of the range of a random walk) is not known. On the other hand, the correct order for the variance should be $t^2/(\log t)^4$, as was proved in the discrete setting [4]. Thus our bound in (1.6) is off only by a $(\log\log t)^9$ term. Note that (1.6) is proved only at the end of the paper, as a byproduct of the proof of Theorem 1.1.

One key step of our investigation is a simple formula for the capacity of the sausage which is neither asymptotic nor variational. In Section 2.2, we deduce a decomposition formula for the capacity of the union of two sets in terms of the sum of capacities and a cross-term: for any two compact sets $A$ and $B$, and for any $r > 0$ with $A \cup B \subset \mathcal{B}(0,r)$,

$$\mathrm{Cap}(A \cup B) = \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi_r(A,B) - \varepsilon_r(A,B), \tag{1.7}$$

with

$$\chi_r(A,B) = 2\pi^2 r^2 \cdot \frac{1}{|\partial\mathcal{B}(0,r)|} \int_{\partial\mathcal{B}(0,r)} \left(\mathbb{P}_z(H_A < H_B < \infty) + \mathbb{P}_z(H_B < H_A < \infty)\right) dz, \tag{1.8}$$

and

$$\varepsilon_r(A,B) = 2\pi^2 r^2 \cdot \frac{1}{|\partial\mathcal{B}(0,r)|} \int_{\partial\mathcal{B}(0,r)} \mathbb{P}_z(H_A = H_B < \infty)\; dz, \tag{1.9}$$

where we use the notation $\mathcal{B}(0,r)$ for the ball of radius $r$ and $\partial\mathcal{B}(0,r)$ for its boundary. In particular $\varepsilon_r(A,B) \leq \mathrm{Cap}(A \cap B)$. The decomposition formula (1.7) is of a different nature to the one presented in [4] for the discrete setting. As an illustration, a key technical estimate here concerns the cross term $\chi_r(A,B)$ where $A$ and $B$ are independent sausages. In order to bound its first moment, we prove an estimate on the probability of intersection of a Wiener sausage by two other independent Brownian motions.

**Proposition 1.5.** *Let $\beta$, $\gamma$ and $\widetilde{\gamma}$ be three independent Brownian motions. For any $\alpha > 0$ and $c \in (0,1)$, there exist positive constants $C$ and $t_0$, such that for all $t > t_0$ and all $z, z' \in \mathbb{R}^4$, with $\sqrt{t}\cdot(\log t)^{-\alpha} \leq \|z\|, \|z'\| \leq \sqrt{t}\cdot(\log t)^{\alpha}$,*

$$\mathbb{P}_{0,z,z'}(W_1[0,t] \cap \gamma[0,\infty) \neq \varnothing,\; W_1[0,t] \cap \widetilde{\gamma}[0,\infty) \neq \varnothing) \;\leq C\, \frac{(\log\log t)^4}{(\log t)^2}\,(1 \wedge \frac{t}{\|z'\|^2})\,(1 \wedge \frac{t}{\|z\|^2}), \tag{1.10}$$

*where $\mathbb{P}_{0,z,z'}$ means that $\beta$, $\gamma$ and $\widetilde{\gamma}$ start from $0$, $z$ and $z'$ respectively.*

We note that the problem of obtaining a law of large numbers for the capacity of the Wiener sausage has been raised recently by van den Berg, Bolthausen and den Hollander [22] in connection with the torsional rigidity of the complement of the Wiener sausage on a torus.

The paper is organised as follows. Section 2 contains preliminary results: in Section 2.1 we gather some well-known facts about Brownian motion and Green's function, and in Section 2.2 we prove (1.7) and compare the capacity of a Wiener sausage to its volume. In Section 3 we prove the asymptotic for the expected capacity. In Section 4, we deduce our large deviation bounds, Proposition 1.3. In Section 5 we provide some intersection probabilities of a Wiener sausage by another Brownian motion, and deduce a second moment bound of the cross-terms $\chi_r$ appearing in the decomposition (1.7). Finally, we prove Theorem 1.1 in Section 6.

# 2 Preliminaries

## 2.1 Notation and basic estimates

We denote by $\mathbb{P}_z$ the law of a Brownian motion starting from $z$, and simply write $\mathbb{P}$ when $z$ is the origin. Likewise $\mathbb{P}_{z,z'}$ denotes the law of two independent Brownian motions starting respectively from $z$ and $z'$, and similarly for $\mathbb{P}_{z,z',z''}$. We denote by $\|\cdot\|$ the Euclidean norm, and for any $x \in \mathbb{R}^4$ and $r > 0$, by $\mathcal{B}(x,r)$ the closed Euclidean ball of radius $r$ centered at $x$. For $u, v \in \mathbb{R}$, we use the standard notation $u \wedge v$ and $u \vee v$ for $\min(u,v)$ and $\max(u,v)$ respectively. We write $|A|$ for the Lebesgue measure of a Borel set $A$, and let $p_s(x,y)$ be the transition kernel of the Brownian motion:

$$p_s(x,y) = \frac{1}{4\pi^2 s^2} e^{-\frac{\|x-y\|^2}{2s}} = p_s(0, y-x). \tag{2.1}$$

Green's function is defined by

$$G(z) := \int_0^\infty p_s(0,z)\, ds, \quad \text{and for any } t > 0 \text{ we define} \quad G_t(z) := \int_0^t p_s(0,z)\, ds. \tag{2.2}$$

The occupation time formula reads, for any $t \geq 0$ and any bounded measurable function $\varphi$,

$$\int_0^t \mathbb{E}[\varphi(\beta_s)]\, ds = \int_{\mathbb{R}^4} \varphi(x) G_t(x)\, dx. \tag{2.3}$$

We further recall, see Theorem 3.33 in [18], that for all $z \neq 0$,

$$G(z) \;=\; \frac{1}{2\pi^2} \cdot \frac{1}{\|z\|^2}\ . \tag{2.4}$$

The following regularized version of Green's function plays a key role:

$$G^*(z) := \int_{\mathcal{B}(0,1)} G(z-y)\, dy = \int_0^\infty \mathbb{P}(\|\beta_s - z\| \leq 1)\, ds, \tag{2.5}$$

(with the second inequality following from (2.3)).

Furthermore, Green's function is harmonic on $\mathbb{R}^4 \backslash \{0\}$, and thus satisfies the mean-value property on this domain. In particular one has for $\|z\| > 1$, using (2.4) and that $|\mathcal{B}(0,1)| = \pi^2/2$,

$$G^*(z) = |\mathcal{B}(0,1)| \cdot G(z) = \frac{1}{4\|z\|^2}. \tag{2.6}$$

Moreover, there exists a positive constant $C$ so that for all $z \in \mathbb{R}^4$ we have

$$G^*(z) \leq \frac{C}{\|z\|^2 \vee 1}. \tag{2.7}$$

The gambler's ruin estimate states that for any $z \in \mathbb{R}^4$, with $\|z\| > r$ (see Corollary 3.19 in [18]),

$$\mathbb{P}_z\big(H_{\mathcal{B}(0,r)} < \infty\big) = \frac{r^2}{\|z\|^2}. \tag{2.8}$$

We also need the following well-known estimates (see Remark 2.22 in [18] for the first one and use the scaling property of the Brownian motion together with either Exercice (3.10) in [21] or the remark after (3.40) below, for the second one): there exist positive constants $c$ and $C$, such that for any $t > 0$ and $r > 0$,

$$\mathbb{P}\Big(\sup_{s\le t} \|\beta_s\| > r\Big) \;\le\; C \cdot \exp(-c\, r^2/t), \tag{2.9}$$

and

$$\mathbb{P}\Big(\sup_{s\le t} \|\beta_s\| \le r\Big) \;\le\; C \cdot \exp(-c\, t/r^2). \tag{2.10}$$

Using (2.9), we get for some positive constants $c$ and $C$,

$$\mathbb{P}\left(\sup_{\frac{t}{(\log t)^3}\le s\le t} \frac{\|\beta_s\|^2}{s} > (\log t)^{1/5}\right) \le C\exp(-c(\log t)^{1/10}). \tag{2.11}$$

Indeed, to deal with the time $s$ in $\|\beta_s\|^2/s$, it is enough to divide the time period $[t/(\log t)^3, t]$ into a finite number of intervals $[t/(\log t)^{k/10}, t/(\log t)^{(k-1)/10}]$, $k = 1, \dots, 30$, and use the left boundary of each interval to bound time $s$. It also follows from (2.8) that

$$\mathbb{P}\left(\inf_{s\ge \frac{t}{(\log t)^3}} \|\beta_s\|^2 \le \frac{t}{(\log t)^{10}}\right) \;\le\; \frac{C}{(\log t)^4}. \tag{2.12}$$

Indeed, either the Brownian motion starts at time $t/(\log t)^3$ inside a ball of radius $\sqrt{t}/(\log t)^{5/2}$ centered at the origin, or it starts outside such a ball, and hits the ball of radius $\sqrt{t}/(\log t)^5$ afterwards: both events satisfy the desired bound (for the first one, this can be seen by integrating the density (2.1) over the ball, and the second one follows from (2.8)).

Finally, we recall a basic result (see Corollary 8.12 and Theorem 8.27 in [18]). For a set $A \subset \mathbb{R}^4$, let $d(z, A) := \inf\{\|z - y\| \,:\, y \in A\}$.

**Lemma 2.1.** *Let $A$ be a compact set in $\mathbb{R}^4$. Then, for any $z \in \mathbb{R}^4 \backslash A$,*

$$\mathbb{P}_z(H_A < \infty) \;\le\; \frac{\mathrm{Cap}(A)}{2\pi^2\, d(z,A)^2}.$$

## 2.2 On capacity

We first give a representation formula for the capacity of a set, which has the advantage of not being given as a limit. If $A$ is a compact subset of $\mathbb{R}^4$, with $A \subset \mathcal{B}(0, r)$ for some $r > 0$, then

$$\begin{aligned}\mathrm{Cap}(A) &= \lim_{\|z\|\to\infty} \frac{\mathbb{P}_z(H_A < \infty)}{G(z)} = \lim_{\|z\|\to\infty} \frac{\mathbb{P}_z\big(H_{\partial\mathcal{B}(0,r)} < \infty\big)}{G(z)} \cdot \int_{\partial\mathcal{B}(0,r)} \mathbb{P}_y(H_A < \infty)\; d\rho_z(y)\\ &= 2\pi^2\, r^2 \cdot \int_{\partial\mathcal{B}(0,r)} \mathbb{P}_y(H_A < \infty)\; d\lambda_r(y),\end{aligned} \tag{2.13}$$

where $\rho_z$ is the law of the Brownian motion starting from $z$ at time $H_{\partial\mathcal{B}(0,r)}$, conditioned on this hitting time being finite, and $\lambda_r$ is the uniform measure on $\partial\mathcal{B}(0,r)$. The second equality above follows from the Markov property, and the last equality follows from (2.8) and the fact that the harmonic measure of a ball (seen from infinity), which by Theorem 3.46 in [18] is also the weak limit of $\rho_z$ as $z$ goes to infinity, is the uniform measure on the boundary of the ball.

The decomposition formula (1.7) for the capacity of the union of two sets follows immediately using (2.13) and ordering of $H_A$ and $H_B$.

Now we state a lemma which bounds the capacity of the intersection of two Wiener sausages by the volume of the intersection of larger sausages.

**Lemma 2.2.** *Let $W$ and $\widetilde{W}$ be two independent Wiener sausages. Then, almost surely, for all $t > 0$,*

$$\mathrm{Cap}(W_1[0,t]) \;\le\; C_1 \cdot |W_{4/3}[0,t]|, \tag{2.14}$$

*and*

$$\mathrm{Cap}(W_1[0,t] \cap \widetilde{W}_1[0,t]) \;\le\; C_1 \cdot |W_4[0,t] \cap \widetilde{W}_4[0,t]|. \tag{2.15}$$

*with $C_1 = \mathrm{Cap}(\mathcal{B}(0,4))/|\mathcal{B}(0,4/3)|$. Moreover, there is a constant $C_2 > 0$, such that for all $t \ge 2$,*

$$\mathbb{E}\Big[\mathrm{Cap}^2(W_1[0,t] \cap \widetilde{W}_1[0,t])\Big] \;\le\; C_2\,(\log t)^2. \tag{2.16}$$

**Proof.** We start with inequality (2.14). Let $(\mathcal{B}(x_i, 4/3), i \le M)$ be a finite covering of $W_1[0,t]$ by open balls of radius $4/3$ whose centers are all assumed to belong to $\beta[0,t]$, the trace of the Brownian motion driving $W_1[0,t]$. Then, by removing one by one some balls if necessary, one can obtain a sequence of disjoint balls $(\mathcal{B}(x_i, 4/3), i \le M')$, with $M' \le M$, such that the enlarged balls $(\mathcal{B}(x_i, 4), i \le M')$ still cover $W_1[0,t]$. Since the capacity is subadditive, one has on one hand

$$\mathrm{Cap}(W_1[0,t]) \;\le\; M' \cdot \mathrm{Cap}(\mathcal{B}(0,4)),$$

and on the other hand since the balls $\mathcal{B}(x_i, 4/3)$ are disjoint and are all contained in $W_{4/3}[0,t]$,

$$M'|\mathcal{B}(0,4/3)| \;\le\; |W_{4/3}[0,t]|.$$

Inequality (2.14) follows. Inequality (2.15) is similar: start with $(\mathcal{B}(x_i, 4/3), i \le M)$ a finite covering of $W_1[0,t] \cap \widetilde{W}_1[0,t]$ by balls of radius one whose centers are all assumed to belong to $\beta[0,t]$. Then, by removing one by one some balls if necessary, one obtain a sequence of disjoint balls $(\mathcal{B}(x_i, 4/3), i \le M')$, such that the enlarged balls $(\mathcal{B}(x_i, 4), i \le M')$ cover the set $W_1[0,t] \cap \widetilde{W}_1[0,t]$, and such that all of them intersect $W_1[0,t] \cap \widetilde{W}_1[0,t]$. But since the centers $(x_i)$ also belong to $\beta[0,t]$, all the balls $\mathcal{B}(x_i, 4/3)$ belong to the enlarged intersection $W_4[0,t] \cap \widetilde{W}_4[0,t]$. So as before one has on one hand

$$\mathrm{Cap}(W_1[0,t] \cap \widetilde{W}_1[0,t]) \;\le\; M' \cdot \mathrm{Cap}(\mathcal{B}(0,4)),$$

and on the other hand

$$|W_4[0,t] \cap \widetilde{W}_4[0,t]| \;\ge\; M'|\mathcal{B}(0,4/3)|.$$

We now prove (2.16). We start with a first moment bound (see [9] for more precise asymptotics):

$$\mathbb{E}\Big[|W_1[0,t] \cap \widetilde{W}_1[0,t]|\Big] \le C \log t. \tag{2.17}$$

This estimate is easily obtained. Indeed, by definition

$$\mathbb{E}\Big[|W_1[0,t] \cap \widetilde{W}_1[0,t]|\Big] = \int_{\mathbb{R}^4} \mathbb{P}\left(H_{\mathcal{B}(z,1)} < t\right)^2 \, dz, \tag{2.18}$$

and then we use the bounds (2.8) inside $\mathcal{B}(0,t)$ and (2.9) outside. For the second moment, we write similarly

$$\mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|^2\Big] = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \mathbb{P}\big(H_{\mathcal{B}(z,1)}<t, H_{\mathcal{B}(z',1)}<t\big)^2\, dz\, dz'. \tag{2.19}$$

When $\|z-z'\|\le 2$, one uses the trivial bound

$$\mathbb{P}\big(H_{\mathcal{B}(z,1)}<t, H_{\mathcal{B}(z',1)}<t\big) \le \mathbb{P}(H_{\mathcal{B}(z,1)}<t). \tag{2.20}$$

When $\|z-z'\|>2$, we first use that

$$\mathbb{P}\big(H_{\mathcal{B}(z,1)}<t, H_{\mathcal{B}(z',1)}<t\big) = \mathbb{P}\big(H_{\mathcal{B}(z,1)}<H_{\mathcal{B}(z',1)}<t\big) + \mathbb{P}\big(H_{\mathcal{B}(z',1)}<H_{\mathcal{B}(z,1)}<t\big)\,,$$

and hence taking the square on both sides gives

$$\mathbb{P}\big(H_{\mathcal{B}(z,1)}<t, H_{\mathcal{B}(z',1)}<t\big)^2 \le 2\mathbb{P}\big(H_{\mathcal{B}(z,1)}<H_{\mathcal{B}(z',1)}<t\big)^2 + 2\mathbb{P}\big(H_{\mathcal{B}(z',1)}<H_{\mathcal{B}(z,1)}<t\big)^2. \tag{2.21}$$

Let $\nu_z^t$ denote the hitting distribution of the ball $\mathcal{B}(z,1)$ by a Brownian motion starting from 0, conditioned to hit this ball before time $t$. Then by the strong Markov property we get

$$\mathbb{P}\big(H_{\mathcal{B}(z,1)}<H_{\mathcal{B}(z',1)}<t\big) \ \le\ \mathbb{P}\big(H_{\mathcal{B}(z,1)}<t\big)\ \mathbb{P}_{\nu_z^t}\big(H_{\mathcal{B}(z',1)}<t\big)\,.$$

Substituting this, (2.20) and (2.21) into (2.19) gives by symmetry

$$\begin{aligned}\mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|^2\Big] \ &\le 4\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \mathbb{P}\big(H_{\mathcal{B}(z,1)}<t\big)^2\, \mathbb{P}_{\nu_z^t}\big(H_{\mathcal{B}(z',1)}<t\big)^2\, dz\, dz' \\ &\quad + |\mathcal{B}(0,2)|\int_{\mathbb{R}^4} \mathbb{P}\big(H_{\mathcal{B}(z,1)}<t\big)^2\, dz.\end{aligned}$$

Using (2.18) and translation invariance of the Brownian motion, we now obtain for all $z$,

$$\int_{\mathbb{R}^4} \mathbb{P}_{\nu_z^t}\big(H_{\mathcal{B}(z',1)}<t\big)^2\, dz' = \mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|\Big]\,.$$

Recalling (2.17), the proof concludes from the bound

$$\mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|^2\Big] \ \le\ 4\mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|\Big]^2 + |\mathcal{B}(0,2)|\mathbb{E}\Big[|W_1[0,t]\cap \widetilde{W}_1[0,t]|\Big]\,.$$

□

# 3 On the Expected Capacity

## 3.1 Statement of the result and sketch of proof

The principal result of this section gives the precise asymptotics for the expected capacity.

**Proposition 3.1.** *In dimension four, and for any radius $r>0$,*

$$\lim_{t\to\infty} \frac{\log t}{t}\, \mathbb{E}[\mathrm{Cap}\,(W_r[0,t])] \ = \ \pi^2. \tag{3.1}$$

**Remark 3.2.** The scale invariance of Brownian motion yields in dimension four, for any $r > 0$,

$$\mathbb{E}[\mathrm{Cap}\,(W_r[0,t])] \;=\; r^2\,\mathbb{E}\big[\mathrm{Cap}\,\big(W_1[0,t/r^2]\big)\big]\,.$$

Thus, it is enough to prove (3.1) for $r = 1$.

The proof is inspired by the approach of Lawler in the random walk setting [11, Chapter 3], to obtain an upper bound for the probability that two random walks meet. This approach is based on the observation that the number of times when two random walks meet, conditionally on one of them, is concentrated. One interesting point is that by pushing his method further, and taking advantage of the continuous setting where some computation can be done explicitly and directly, we obtain a true equivalent of the probability that two Wiener sausages meet. Before giving the proof, let us explain its rough ideas and introduce the main notation.

The first step is to cast the expected capacity of $W_1[0,t]$ into a probability of intersection of this Wiener sausage by another Brownian motion $\widetilde{\beta}$, starting from infinity. More precisely we show in Section 3.3 that

$$\mathbb{E}[\mathrm{Cap}\,(W_1[0,t])] \;=\; \lim_{\|z\|\to\infty} \frac{1}{G(z)} \cdot \mathbb{P}_{0,z}\Big(W_1[0,t] \cap \widetilde{\beta}[0,\infty) \neq \varnothing\Big)\,. \tag{3.2}$$

This representation holds for deterministic sets (1.2), and here we need to justify the interchange of limit and expectation. We next introduce the following stopping time

$$\tau \;=\; \inf\{s \geq 0 \;:\; \widetilde{\beta}_s \in W_1[0,t]\}, \tag{3.3}$$

and note that $\{W_1[0,t] \cap \widetilde{\beta}[0,\infty) \neq \varnothing\} = \{\tau < \infty\}$. Then, we introduce a counting measure of the pairs of times at which the two trajectories come within distance 1: for $s \leq t$, let

$$R[s,t] \;:=\; \int_0^\infty du \int_s^t \mathbf{1}(\|\widetilde{\beta}_u - \beta_v\| \leq 1)\, dv. \tag{3.4}$$

Observe that, almost surely, $\{\tau < \infty\} = \{R[0,t] > 0\}$, and the following equality holds

$$\mathbb{P}_{0,z}(\tau < \infty) \;=\; \frac{\mathbb{E}_{0,z}[R[0,t]]}{\mathbb{E}_{0,z}[R[0,t] \mid \tau < \infty]}. \tag{3.5}$$

The estimate of the numerator in (3.5) is established by direct and explicit computations. More precisely we prove (see Section 3.3) that for all $t > 0$,

$$\lim_{\|z\|\to\infty} \frac{\mathbb{E}_{0,z}[R[0,t]]}{G(z)} \;=\; \frac{\pi^2}{2} t. \tag{3.6}$$

The estimate of the denominator in (3.5) is more intricate. Consider the random time

$$\sigma \;=\; \inf\Big\{s \geq 0 \,:\, \|\beta_s - \widetilde{\beta}(\tau)\| \leq 1\Big\}\,. \tag{3.7}$$

A crucial observation is that $\sigma$ is not a stopping time (with respect to any natural filtration), since $\tau$ depends on the whole Wiener sausage $W_1[0,t]$. In particular conditionally on $\tau$ and $\sigma$, one cannot consider the two trajectories $\widetilde{\beta}[\tau,\infty)$ and $W_1[\sigma,t]$, as being independent [1], and neither can be $\widetilde{\beta}[\tau,\infty)$ and $W_1[0,\sigma]$. To overcome this difficulty, the main idea (following Lawler) is to use that both $\mathbb{E}_{0,z}[R[\sigma,t] \mid \beta, (\widetilde{\beta}_s)_{s\leq\tau}]$ and $\mathbb{E}_{0,z}[R[0,\sigma] \mid \beta, (\widetilde{\beta}_s)_{s\leq\tau}]$ are concentrated around their mean values, which are of order $\log t$, at least when $\sigma$ and $t - \sigma$ are large enough. As a consequence, for typical values of $\sigma$, they are close to their mean values. The main part of the proof is then to estimate the probability that $\sigma$ is not typical with this respect.

[1] a mistake that Erdös and Taylor implicitly made in their pioneering work [7], and that Lawler corrected about twenty years later [11].

### 3.2 Proof of Proposition 3.1

Denote by $(\mathcal{F}_s)_{s\geq 0}$ and $(\widetilde{\mathcal{F}}_s)_{s\geq 0}$ the natural filtrations of $\beta$ and $\widetilde{\beta}$ respectively. Recall the definition (3.3) of $\tau$, and then define the sigma-field $\mathcal{G}_\tau := \widetilde{\mathcal{F}}_\tau \vee (\mathcal{F}_s)_{s\geq 0}$. Then by taking conditional expectation with respect to $\mathcal{G}_\tau$, we get using the strong Markov property for $\widetilde{\beta}$ at time $\tau$, and (2.5), that on event $\{\tau < \infty\}$, with $X := \beta(\sigma) - \widetilde{\beta}(\tau)$,

$$\begin{aligned}\mathbb{E}_{0,z}[R[0,t] \mid \mathcal{G}_\tau] &= \int_0^t G^*(\beta_u - \widetilde{\beta}(\tau))\, du \\ &= \int_0^\sigma G^*(\beta_u - \beta(\sigma) - X)\, du + \int_\sigma^t G^*(\beta_u - \beta(\sigma) - X)\, du. \end{aligned} \tag{3.8}$$

We shall see next that these last two integrals above are asymptotically of the same order. As already mentioned, to deal with the difficulty of $\sigma$ not being a stopping time, the main idea is to introduce the notion of *good* $\sigma$, when both integrals are close to their typical values. Then by using a *trick* of Lawler, one is led to estimate only the probability for a deterministic time not to be *good*, which can be done using the estimates gathered in the next section. Then we separate the proof of the proposition in two parts, one for the lower bound in (3.1) and the other one for the upper bound, and define in fact two notions of good $\sigma$ accordingly.

#### 3.2.1 Further notation and preliminary estimates

We introduce here some additional notation, and then state a few lemmas with all the basic estimates we shall need. Before this, we need to extend $\beta$ to negative times and therefore consider a two-sided Brownian motion $(\beta_u)_{u\in\mathbb{R}}$. Then for $s \leq t$, and $x \in \mathbb{R}^4$, set

$$D_x[s,t] := \int_s^t G^*(\beta_u - \beta_s - x)\, du, \quad \text{and} \quad \widetilde{D}_x[s,t] := \int_s^t G^*(\beta_u - \beta_t - x)\, du. \tag{3.9}$$

Note that with $X = \beta(\sigma) - \widetilde{\beta}(\tau)$, we get from (3.8),

$$\mathbb{E}_{0,z}[R[0,t] \mid \mathcal{G}_\tau] = \widetilde{D}_X[0,\sigma] + D_X[\sigma,t]. \tag{3.10}$$

In the following lemmas we gather all the estimates we need on $D_x$ and $\widetilde{D}_x$. The first one deals with the first and second moments of $D_0[0,t]$.

**Lemma 3.3.** *One has*

$$\lim_{t\to\infty} \frac{1}{\log t}\, \mathbb{E}[D_0[0,t]] \;=\; \frac{1}{4},$$

*and there exists a constant $C > 0$, such that for all $t \geq 2$,*

$$\mathbb{E}\big[D_0[0,t]^2\big] \;\leq\; C\,(\log t)^2. \tag{3.11}$$

The second result shows that $D_x[0,t]$ is uniformly close to $D_0[0,t]$, when $\|x\| \leq 1$. Define

$$\zeta = \int_0^\infty \frac{1}{\|\beta_u\|^3 \vee 1}\, du. \tag{3.12}$$

**Lemma 3.4.** *The following assertions hold. There exists a constant $C > 0$, so that for all $t > 0$, almost surely,*

$$\sup_{\|x\|\leq 1} |D_x[0,t] - D_0[0,t]| \;\leq\; C\,\zeta. \tag{3.13}$$

*Moreover, there exists a constant $\lambda > 0$, such that $\mathbb{E}[\exp(\lambda\,\zeta)] \;<\; \infty$.*

The third lemma gives a control of the fluctuations of $D_x[s,t]$ and $\widetilde{D}_x[0,s]$, as $s$ varies over $[0,t]$.

**Lemma 3.5.** *Let for $0 \le s \le s'$,*

$$\zeta_s := \int_s^\infty \frac{1}{\|\beta_u - \beta_s\|^3 \vee 1}\, du, \quad \widetilde{\zeta}_s := \int_{-\infty}^s \frac{1}{\|\beta_u - \beta_s\|^3 \vee 1}\, du, \quad \text{and} \quad M_{s,s'} := 1 + \sup_{s \le u \le v \le s'} \|\beta_u - \beta_v\|.$$

*Define further for $s \ge 0$ and $r \ge 0$,*

$$\xi_s(r) := \sup_{\|y\| \le r} \int_s^\infty \frac{\mathbf{1}(\|\beta_u - \beta_s - y\| \le r)}{\|\beta_u - \beta_s - y\|^2 \vee 1}\, du, \quad \text{and} \quad \widetilde{\xi}_s(r) := \sup_{\|y\| \le r} \int_{-\infty}^s \frac{\mathbf{1}(\|\beta_u - \beta_s - y\| \le r)}{\|\beta_u - \beta_s - y\|^2 \vee 1}\, du.$$

*(i) For any $s$, $\zeta_s$ and $\widetilde{\zeta}_s$ are equal in law to $\zeta_0$. Likewise, $\xi_s(r)$ and $\widetilde{\xi}_s(r)$ are equal in law to $\xi_0(r)$, for any $r \ge 0$.*

*(ii) There exists a constant $\lambda > 0$, such that $\mathbb{E}\big[\exp(\lambda M_{0,1}^2)\big] < \infty$. Moreover, there exist positive constants $c$ and $C$, so that for all $r \ge 2$, $\mathbb{E}[\xi_0(r)^2] \le C r^4$, and*

$$\mathbb{P}(\xi_0(r) > C r \log r) \le C \exp(-cr), \tag{3.14}$$

*(iii) There exists a constant $C > 0$, so that for all $0 \le s' \le s \le t$, almost surely,*

$$\sup_{\|x\| \le 1} \big| D_x[s',t] - D_x[s,t] \big| \;\le\; C \left( s - s' + M_{s',s} \zeta_s + \xi_s(R) \right),$$

*with $R = \|\beta_{s'} - \beta_s\|$, and likewise, for all $0 \le s \le s' \le t$, almost surely*

$$\sup_{\|x\| \le 1} \left| \widetilde{D}_x[0,s] - \widetilde{D}_x[0,s'] \right| \;\le\; C \left( s' - s + M_{s,s'} \widetilde{\zeta}_s + \widetilde{\xi}_s(R) \right).$$

The next result gives some large deviation bounds for $D_0[0,t]$, and shows that it is concentrated.

**Lemma 3.6.** *For any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$, such that for $t$ large enough,*

$$\mathbb{P}(|D_0[0,t] - d(t)| > \varepsilon\, d(t)) \;\le\; \exp\big( - c(\log t)^{1/3} \big), \quad \text{with} \quad d(t) := \mathbb{E}[D_0[0,t]].$$

Dealing with another starting point than the origin can also be obtained as a corollary of the previous lemmas:

**Lemma 3.7.** *There exist positive constants $c$ and $C$, such that for all $t \ge 2$, and all $b \in \mathbb{R}^4$,*

$$\mathbb{E}_b \left[ \left( \sup_{\|x\| \le 1} D_x[0,t] + r_t \zeta_0 + \xi_0(r_t) \right)^2 \right] \le C (\log t)^2,$$

*and,*

$$\mathbb{P}_b \left( \zeta_0 > \frac{1}{4} \log t \right) + \mathbb{P}_b \left( \xi_0(r_t) > \frac{1}{4} r_t (\log r_t)^2 \right) \le C \exp(-c(\log t)^{1/3}).$$

*Moreover, for any $\varepsilon > 0$, there exists a constant $c > 0$, such that for all $t$ large enough, and all $b$ such that $t/(\log t)^{10} \le \|b\|^2 \le t(\log t)^{1/5}$*

$$\mathbb{P}_b \left( \sup_{\|x\| \le 1} D_x[0,t] > \varepsilon \log t \right) \le \exp(-c(\log t)^{1/3}).$$

Finally the last preliminary result we should need is the following elementary fact.

**Lemma 3.8.** *There exists a constant $C > 0$, so that for all $k, i \in \mathbb{N}$, and $z \in \mathbb{R}^4$,*

$$\mathbb{P}_{0,z}\left(\inf_{k \le u \le k+1} \inf_{i \le s \le i+1} \|\widetilde{\beta}_u - \beta_s\| \le 1\right) \;\le\; C \int_k^{k+2} \int_i^{i+2} \mathbb{P}_{0,z}\left(\|\widetilde{\beta}_u - \beta_s\| \le 3\right) du\, ds.$$

The proofs of these lemmas (together with the proof of (3.2), and (3.6)) are postponed to Subsections 3.3, 3.4, and 3.5, and assuming them one can now start the proof of Proposition 3.1.

### 3.2.2 Proof of the lower bound in (3.1)

We fix some $\varepsilon > 0$, and define a time $s \in [0, t]$ to be *good* if

$$\sup_{\|x\| \le 1} \widetilde{D}_x[0, s] \le (1 + \varepsilon) d(t) \quad \text{and} \quad \sup_{\|x\| \le 1} D_x[s, t] \le (1 + \varepsilon) d(t),$$

where $d(t)$ is defined in Lemma 3.6. Otherwise we say that $s$ is *bad*. The estimates gathered so far (see in particular Lemmas 3.4 and 3.6) show that the probability for a fixed time $s$ to be bad decays like a stretched exponential in $\log t$. However, this is far from being sufficient for getting the lower bound in (3.1). Indeed a subtle and difficult point here is that $\sigma$ (and $\beta_\sigma$) depend on the whole trajectory $(\beta_u)_{u \le t}$, and as a consequence it is actually not possible to obtain directly good estimates for the probability of $\sigma$ being bad. So the idea of Lawler, see [12, page 101], in the random walk setting, was to decompose the event $\{\sigma \text{ bad}\}$ into all the possible values for $\sigma$ and $\tau$, and replace the event $\{\sigma = i,\, \tau = k\}$ by the event that the two walks are at the same position at times $i$ and $k$ respectively. The event $\{S_i = \widetilde{S}_k\}$ for two independent walks $S$ and $\widetilde{S}$ is independent of the event $\{i \text{ bad}\}$, and hence the probability factorises. One can then use the estimate for the probability that a deterministic time is bad. What remains is a double sum which is equal to the expected number of pairs of times the two walks coincide, but which is negligible compared to the probability that a time $i$ is bad.

In our case, a number of new (technical) difficulties arise, mainly due to the continuous time setting. Indeed one is first led to discretise $\tau$ and $\sigma$. For $\tau$ this is not a serious problem, but doing it for $\sigma$ requires to relate the event $\{i \le \sigma \le i+1\} \cap \{\sigma \text{ bad}\}$ to the events that $i$ or $i+1$ are bad (more precisely we relate them to the events $\{i \text{ bad}_-\}$ or $\{i+1 \text{ bad}_+\}$, see below for a definition). For this we use Lemma 3.5 which relates $D_x[s, t]$ with $D_x[s', t]$ when $s$ and $s'$ are close (and similarly for $\widetilde{D}_x[0, s]$ and $\widetilde{D}_x[0, s']$).

Let us now start with the details. We first express the event $\{s \text{ bad}\}$ in terms of other events which are conditionally independent of $(\beta_u)_{[s] \le u \le [s]+1}$ (with $[s]$ the integral part of $s$). Set $r_t := \frac{2}{\sqrt{\lambda}} \sqrt{\log t}$, with the constant $\lambda$ as in Lemma 3.5. Then for an integer $i$, and using the notation of Lemma 3.5, define the events

$$\{i \text{ bad}_-\} := \left\{\sup_{\|x\| \le 1} \widetilde{D}_x[0, i] > (1 + \frac{\varepsilon}{2})\, d(t)\right\} \cup \left\{\widetilde{\zeta}_i > (\log t)^{1/3}\right\} \cup \left\{\widetilde{\xi}_i(r_t) > r_t (\log r_t)^2\right\},$$

$$\{i \text{ bad}_+\} := \left\{\sup_{\|x\| \le 1} D_x[i, t] > (1 + \frac{\varepsilon}{2})\, d(t)\right\} \cup \left\{\zeta_i > (\log t)^{1/3}\right\} \cup \left\{\xi_i(r_t) > r_t (\log r_t)^2\right\}.$$

From Lemma 3.3 we get that $d(t)$ is of order $\log t$ for large $t$. Using this and Lemma 3.5 (part (iii)) we obtain that for all $t$ large enough and for any $s \in [0, t-1]$, letting $i = [s]$, on the event $\{\sup_{s \le t} M_{s,s+1} \le r_t\}$,

$$\{s \text{ bad}\} \quad \subseteq \quad \{i \text{ bad}_-\} \cup \{i+1 \text{ bad}_+\}. \tag{3.15}$$

Note that all these events depend in fact on $t$ and $\varepsilon$, but since they are kept fixed in the rest of the proof this should not cause any confusion.

Recall that for $s \geq 0$, $\mathcal{F}_s = \sigma((\beta_u)_{u \leq s})$, and define $\mathcal{F}_s^+ := \sigma((\beta_u)_{u \geq s})$.

Now we first need to estimate the probability that an integer $i$ is bad$_+$, conditionally on $\mathcal{F}_i$. For this observe that for any $i < t$, one has by the Markov property, conditionally on $\mathcal{F}_i$,

$$\sup_{\|x\| \leq 1} D_x[i,t] \leq \sup_{\|x\| \leq 1} D_x[i,t+i] \overset{\text{(law)}}{=} \sup_{\|x\| \leq 1} D_x[0,t].$$

Therefore using Lemmas 3.4, 3.5 and 3.6 we get that there exists $c > 0$ so that for all $t$ large enough and all integers $i \in [0,t]$, almost surely

$$\mathbb{P}(i \text{ bad}_+ \mid \mathcal{F}_i) \leq \exp(-c(\log t)^{1/3}). \tag{3.16}$$

Using in addition Lemma 3.3 we also obtain that there exists $C > 0$ so that for all $t \geq 2$, almost surely

$$\mathbb{E}\left[\left(\sup_{\|x\| \leq 1} D_x[i,t] + r_t\zeta_i + \xi_i(r_t)\right)^2 \;\Big|\; \mathcal{F}_i\right] \leq C(\log t)^2. \tag{3.17}$$

The corresponding estimates for the event $\{i \,\text{bad}_-\}$ are harder to obtain, since now the law of the Brownian path between times 0 and $i$, conditionally on $\mathcal{F}_i^+$, is a Brownian bridge, and it is no more possible to obtain estimates which are valid almost surely. One need now to assume that $i$ is sufficiently large, and to work on an event where $\|\beta_i\|$ is neither too small nor too large. More precisely, we define for an integer $i$,

$$\mathcal{E}_i = \left\{ \frac{\sqrt{t}}{(\log t)^5} \leq \|\beta_i\| \leq \sqrt{i}(\log t)^{1/10} \right\}.$$

Then we gather the analogues of (3.16) and (3.17) in the following lemma whose proof is postponed to Section 3.2.3.

**Lemma 3.9.** *There exist positive constants $c$ and $C$, so that for all $t$ large enough, almost surely on the event $\mathcal{E}_i$,*

$$\mathbb{P}(i \text{ bad}_- \mid \mathcal{F}_i^+) \leq \exp(-c(\log t)^{1/3}), \tag{3.18}$$

*and*

$$\mathbb{E}\left[\left(\sup_{\|x\| \leq 1} \widetilde{D}_x[0,i] + r_t\widetilde{\zeta}_i + \widetilde{\xi}_i(r_t)\right)^2 \;\Big|\; \mathcal{F}_i^+\right] \leq C\exp(2(\log t)^{1/5}). \tag{3.19}$$

We resume now the proof of the lower bound. Since the event $\{\sigma \text{ good}\}$ is $\mathcal{G}_\tau$-measurable, one has from (3.10) and the definition of $\{\sigma \text{ good}\}$

$$\begin{aligned} \mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\tau < \infty, \sigma \text{ good})] &= \mathbb{E}_{0,z}\left[\mathbb{E}_{0,z}[R[0,t] \mid \mathcal{G}_\tau] \cdot \mathbf{1}(\tau < \infty, \sigma \text{ good})\right] \\ &\leq 2(1+\varepsilon)\, d(t)\, \mathbb{P}_{0,z}(\tau < \infty, \sigma \text{ good}). \end{aligned} \tag{3.20}$$

Thus, we can write

$$\mathbb{P}_{0,z}(\tau < \infty) \geq \mathbb{P}_{0,z}(\tau < \infty, \sigma \text{ good}) \geq \frac{1}{2(1+\varepsilon)d(t)} \cdot \mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\tau < \infty, \sigma \text{ good})]. \tag{3.21}$$

The last term above is estimated through

$$\mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\tau<\infty,\sigma \text{ good})] \geq \mathbb{E}_{0,z}[R[0,t]] - \mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})] - \mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\mathcal{E}^c)],$$

with

$$\mathcal{E} := \bigcap_{t/(\log t)^3 \leq i \leq t} \mathcal{E}_i \cap \left\{ \sup_{s\leq t} M_{s,s+1} \leq r_t \right\}.$$

Using (2.11), (2.12) and Lemma 3.5 (ii), we get

$$\mathbb{P}(\mathcal{E}^c) \leq \frac{C}{(\log t)^4} \tag{3.22}$$

From (2.7) we obtain for $\|z\| \geq s$,

$$\mathbb{E}\big[G^*(\beta_s - z)^2\big] \leq \mathbb{E}\left[\frac{C}{\|\beta_s - z\|^4 \vee 1}\right] \leq \frac{C}{\|z\|^4} + C(\log\|z\|)e^{-\|z\|^2/(8s)} \leq \frac{C}{\|z\|^4}.$$

Therefore for $\|z\| \geq t$ using Cauchy-Schwarz we obtain

$$\mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\mathcal{E}^c)] = \int_0^t \mathbb{E}[G^*(\beta_s - z)\mathbf{1}(\mathcal{E}^c)]ds \leq C\mathbb{P}(\mathcal{E}^c)^{1/2}\frac{t}{\|z\|^2} \leq C\frac{t}{\|z\|^2(\log t)^2}. \tag{3.23}$$

Next we estimate the expectation of $R[0,t]$ on $\{\sigma \text{ bad}\}$, using Lawler's approach. This is also the part requiring (3.16), (3.17) and Lemma 3.9. The first step is, as before, to take the conditional expectation with respect to $\mathcal{G}_\tau$ and use (3.8), which gives (with $X = \widetilde{\beta}(\tau) - \beta(\sigma)$)

$$\begin{aligned}&\mathbb{E}_{0,z}[R[0,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})] \\ &= \mathbb{E}_{0,z}\left[\widetilde{D}_X[0,\sigma]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right] + \mathbb{E}_{0,z}\left[D_X[\sigma,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right].\end{aligned} \tag{3.24}$$

Let us start with the second term, which is slightly easier to handle. Notice that on the event $\mathcal{E}$, using Lemma 3.5, we get (assuming $\sigma \leq t-1$),

$$\sup_{\|x\|\leq 1} D_x[\sigma,t] \leq \sup_{\|x\|\leq 1} D_x[[\sigma]+1,t] + C(1 + r_t\zeta_{[\sigma]+1} + \xi_{[\sigma]+1}(r_t)) =: H_{[\sigma]+1}.$$

Note also that when $t-1 \leq \sigma \leq t$, one can bound the left-hand side above just by a constant, since $G^*$ is bounded (recall the definition (3.9)). Also we use the convention that $D_x[s,t] = 0$ when $s > t$. Then using (3.15), we get

$$\begin{aligned}\mathbb{E}_{0,z}\left[D_X[\sigma,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right] \;&\leq\; \mathbb{E}_{0,z}\left[\left(\sup_{\|x\|\leq 1} D_x[\sigma,t]\right)\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right] \\ &\leq \sum_{k=0}^{\infty}\sum_{i=0}^{[t]} \mathbb{E}_{0,z}\left[H_{i+1}\mathbf{1}([\tau]=k,[\sigma]=i,\sigma \text{ bad},\mathcal{E})\right] \\ &\leq \sum_{k=0}^{\infty}\sum_{i=0}^{[t]} \mathbb{E}_{0,z}\left[H_{i+1}\mathbf{1}([\tau]=k,[\sigma]=i,i \text{ bad}_-,\mathcal{E})\right] + \mathbb{E}_{0,z}\left[H_{i+1}\mathbf{1}([\tau]=k,[\sigma]=i,i+1 \text{ bad}_+)\right] \\ &\leq \sum_{k=0}^{\infty}\sum_{i=0}^{[t]} \mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i \text{ bad}_-,\mathcal{E})\right] + \mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i+1 \text{ bad}_+)\right],\end{aligned} \tag{3.25}$$

where we set for any $k, i \in \mathbb{N}$,

$$I_{k,i} := \mathbf{1}\left(\inf_{k\le u\le k+1}\inf_{i\le v\le i+1}\|\widetilde{\beta}_u - \beta_v\| \le 1\right).$$

Note that (3.17) gives in fact that almost surely

$$\mathbb{E}[H_{i+1}^2 \mid \mathcal{F}_{i+1}] \le C(\log t)^2. \tag{3.26}$$

Therefore by conditioning first with respect to $\mathcal{F}_{i+1}\vee\widetilde{\mathcal{F}}_\infty$, and then using Cauchy-Schwarz and (3.16), we get for any $0\le i\le t$,

$$\mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i+1 \text{ bad}_+)\right] \le C\exp(-c(\log t)^{1/3})\mathbb{E}_{0,z}(I_{k,i}).$$

Similarly by conditioning first with respect to $\mathcal{F}_{i+1}\vee\widetilde{\mathcal{F}}_\infty$ and then with respect to $\mathcal{F}_i^+\vee\widetilde{\mathcal{F}}_\infty$, we get using (3.26) and (3.18), for any $\frac{t}{(\log t)^3}\le i\le t$

$$\mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i \text{ bad}_-,\mathcal{E})\right] \le \mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i \text{ bad}_-,\mathcal{E}_i)\right] \le C\exp(-c(\log t)^{1/3})\mathbb{E}_{0,z}\left[I_{k,i}\right].$$

On the other hand, for $i\le t/\log^3 t$, one can just use (3.26), which gives using Jensen's inequality,

$$\mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\mathbf{1}(i \text{ bad}_-,\mathcal{E})\right]\le\mathbb{E}_{0,z}\left[H_{i+1}I_{k,i}\right]\le C(\log t)\mathbb{E}_{0,z}[I_{k,i}].$$

Finally using Lemma 3.8 and (3.25), we deduce that

$$\mathbb{E}_{0,z}\left[D_X[\sigma,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right] \le Ce^{-c(\log t)^{1/3}}\cdot\mathbb{E}_{0,z}[R_3[0,t+1]]+C\log t\cdot\mathbb{E}_{0,z}\left[R_3[0,\frac{t}{(\log t)^3}+1]\right],$$

where for any $T>0$

$$R_3[0,T] := \int_0^T\int_0^\infty \mathbf{1}(\|\widetilde{\beta}_s-\beta_u\|\le 3)\,ds\,du.$$

By scaling $R_3[0,T]$ is equal in law to $81R[0,T/9]$. Therefore (3.6) gives for $\|z\|$ large enough,

$$\mathbb{E}_{0,z}\left[D_X[\sigma,t]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right]\le C\frac{t}{\|z\|^2(\log t)^2}. \tag{3.27}$$

The analogous estimate for the other expectation in (3.24) is similar. The only change is that one can rule out the case of small indices $i$ from the beginning. For this we use that by (3.6), for $\|z\|$ large enough,

$$\begin{aligned}\mathbb{E}_{0,z}\left[\widetilde{D}_X[0,\sigma]\mathbf{1}\left(\tau<\infty,\sigma\le\frac{t}{(\log t)^2}\right)\right] &= \mathbb{E}_{0,z}\left[R[0,\sigma]\mathbf{1}\left(\tau<\infty,\sigma\le\frac{t}{(\log t)^2}\right)\right]\\ &\le \mathbb{E}_{0,z}\left[R\left[0,\frac{t}{(\log t)^2}\right]\right]\le C\frac{t}{\|z\|^2(\log t)^2}.\end{aligned}$$

One then follows the same argument as for (3.27), summing over indices $i\ge t/(\log t)^2$ and using (3.19) this time to obtain

$$\mathbb{E}_{0,z}\left[\widetilde{D}_X[0,\sigma]\mathbf{1}(\tau<\infty,\sigma \text{ bad},\mathcal{E})\right]\le C\frac{t}{\|z\|^2(\log t)^2}. \tag{3.28}$$

Inserting (3.27) and (3.28) into (3.24), and then using (3.21), (3.22) together with (3.23) we obtain for all $t$ large enough.

$$\liminf_{\|z\|\to\infty}\frac{\mathbb{P}_{0,z}(\tau<\infty)}{G(z)} \ge \frac{\pi^2}{4(1+\varepsilon)}\cdot\frac{t}{d(t)}\cdot\left(1-\frac{C}{\log t}\right).$$

Since the above estimate holds for all $\varepsilon>0$, we obtain, using in addition Lemma 3.3, (recall (3.2) and (3.3))

$$\liminf_{t\to\infty}\frac{\log t}{t}\cdot\mathbb{E}[\mathrm{Cap}(W_1[0,t])] \ge \pi^2.$$

### 3.2.3 Proof of Lemma 3.9

The proof is based on the fact that conditionally on $\mathcal{F}_i^+$, the process $(\beta_u)_{0\le u\le i}$ is a Brownian bridge. Denote by $\mathbb{Q}_{0,b}^{(i)}$ the law of a Brownian bridge starting from 0 and ending up in $b$ at time $i$ (and abusing notation let it also denote the expectation with respect to this law). It follows from Markov's property, that for any bounded (or nonnegative) measurable function $F$, one has for all $b\in\mathbb{R}^4$,

$$\mathbb{Q}_{0,b}^{(i)}[F(\beta_u, u\le i/2)] = \mathbb{E}\left[F(\beta_u, u\le i/2)\frac{p_{i/2}(\beta_{i/2}, b)}{p_i(0,b)}\right]. \tag{3.29}$$

Furthermore, using the explicit expression (2.1), we see that for any $b$ satisfying $\|b\|^2\le i(\log t)^{1/5}$, we have

$$\sup_{x\in\mathbb{R}^4}\frac{p_{i/2}(x,b)}{p_i(0,b)}\le 4\exp((\log t)^{1/5}).$$

One deduces first that for any $b$ as above, and any nonnegative functional $F$,

$$\begin{aligned}\mathbb{Q}_{0,b}^{(i)}[F(\beta_u-\beta_i, u\le i/2)] &\le 4\exp((\log t)^{1/5})\mathbb{E}\left[F(\beta_u-b, u\le i/2)\right]\\ &\le 4\exp((\log t)^{1/5})\mathbb{E}_{-b}\left[F(\beta_u, u\le i/2)\right].\end{aligned} \tag{3.30}$$

Note also that $u$ is in fact allowed to run over the whole interval $(-\infty, i/2]$ in (3.29) and (3.30). By using next that under $\mathbb{Q}_{0,b}^{(i)}$, the law of $(\beta_{i-u}-\beta_i)_{0\le u\le i}$ is just $\mathbb{Q}_{0,-b}^{(i)}$, one deduces as well that for $b$ as above,

$$\mathbb{Q}_{0,b}^{(i)}[F(\beta_{i-u}-\beta_i, 0\le u\le i/2)] \le 4\exp((\log t)^{1/5})\mathbb{E}\left[F(\beta_u, 0\le u\le i/2)\right]. \tag{3.31}$$

Therefore on the event $\mathcal{E}_i$ applying first (3.31) with $F=(\sup_{\|x\|\le 1}\widetilde{D}_x[i/2,i])^2$, and using Lemmas 3.3 and 3.4, we obtain that for any $i\le t$, almost surely

$$\mathbb{Q}_{0,\beta_i}^{(i)}\left[\left(\sup_{\|x\|\le 1}\widetilde{D}_x[i/2,i]\right)^2\right]\le 4\exp((\log t)^{1/5})\mathbb{E}\left[\left(\sup_{\|x\|\le 1}D_x[0,i/2]\right)^2\right]\le C\exp(2(\log t)^{1/5}).$$

Likewise, using this time (3.30), and Lemma 3.7, one has on $\mathcal{E}_i$

$$\mathbb{Q}_{0,\beta_i}^{(i)}\left[\left(\sup_{\|x\|\le 1}\int_0^{i/2}G^*(\beta_s-\beta_i-x)\,ds\right)^2\right]\le C\exp(2(\log t)^{1/5}).$$

Combining the last two displays we get that on $\mathcal{E}_i$

$$\mathbb{Q}_{0,\beta_i}^{(i)}\left[\left(\sup_{\|x\|\le 1}\widetilde{D}_x[0,i]\right)^2\right]\le C\exp(2(\log t)^{1/5}). \tag{3.32}$$

Similarly, using now Lemmas 3.4 and 3.6, on the event $\mathcal{E}_i$ we have

$$\begin{aligned}\mathbb{Q}_{0,\beta_i}^{(i)}\left[\sup_{\|x\|\le 1}\widetilde{D}_x[i/2,i]>\left(1+\frac{\varepsilon}{4}\right)d(t)\right] &\le 4\exp((\log t)^{1/5})\mathbb{P}\left(\sup_{\|x\|\le 1}D_x[0,i/2]>\left(1+\frac{\varepsilon}{4}\right)d(t)\right)\\ &\le C\exp(-c(\log t)^{1/3}),\end{aligned} \tag{3.33}$$

for some constant $c > 0$. Moreover, it follows from (3.30) and Lemma 3.7, that on $\mathcal{E}_i$, for $t$ large enough,

$$\mathbb{Q}^{(i)}_{0,\beta_i}\left[\sup_{\|x\|\le 1}\int_0^{i/2} G^*(\beta_s-\beta_i-x)\,ds > \frac{\varepsilon}{4}d(t)\right] \le 4\exp((\log t)^{1/5})\ \mathbb{P}_{-\beta_i}\left(\sup_{\|x\|\le 1} D_x[0,i/2] > \frac{\varepsilon}{4}d(t)\right)$$
$$\le \exp(-c(\log t)^{1/3}). \tag{3.34}$$

Combining (3.33) and (3.34), we get that on $\mathcal{E}_i$, for $t$ large enough,

$$\mathbb{Q}^{(i)}_{0,\beta_i}\left[\sup_{\|x\|\le 1}\widetilde{D}_x[0,i] > (1+\frac{\varepsilon}{2})d(t)\right] \ \le\ \exp(-c(\log t)^{1/3}). \tag{3.35}$$

The last two events involved in the definition of $\{i \text{ bad}_-\}$ are handled similarly. For instance, for the event concerning $\widetilde{\zeta}_i$, denoting by $\widetilde{\zeta}_{i,1}$ the integral on $(-\infty, i/2)$ and by $\widetilde{\zeta}_{i,2}$ the integral on $[i/2, i]$, one has using (3.30) and (3.31), using that $\beta$ is a two-sided Brownian motion for the third inequality, and Lemmas 3.4 and 3.7 for the last one

$$\begin{aligned}
\mathbb{Q}^{(i)}_{0,\beta_i}\left[\widetilde{\zeta}_i > (\log t)^{1/3}\right] \ &\le\ \mathbb{Q}^{(i)}_{0,\beta_i}\left[\widetilde{\zeta}_{i,1} > \frac{1}{2}(\log t)^{1/3}\right] + \mathbb{Q}^{(i)}_{0,\beta_i}\left[\widetilde{\zeta}_{i,2} > \frac{1}{2}(\log t)^{1/3}\right] \\
&\le\ \mathbb{P}_{-\beta_i}\left(\int_{-\infty}^{i/2}\frac{1}{\|\beta_s\|^3\vee 1}\,ds > \frac{1}{2}(\log t)^{1/3}\right) + \mathbb{P}\left(\zeta_0 > \frac{1}{2}(\log t)^{1/3}\right) \\
&\le\ 2\mathbb{P}_{-\beta_i}\left(\zeta_0 > \frac{1}{4}(\log t)^{1/3}\right) + \mathbb{P}\left(\zeta_0 > \frac{1}{2}(\log t)^{1/3}\right) \le\ C\exp(-c(\log t)^{1/3}).
\end{aligned}$$

Likewise, and as for (3.32), using also the last part of Lemma 3.4 one has almost surely on the event $\mathcal{E}_i$

$$\mathbb{Q}^{(i)}_{0,\beta_i}\left[\widetilde{\zeta}_i^2\right] \ \le\ C\exp(2(\log t)^{1/5}).$$

The corresponding estimates involving $\widetilde{\xi}_i(r_t)$ are entirely similar. Finally we obtain that for $t$ large enough, on the event $\mathcal{E}_i$

$$\mathbb{Q}^{(i)}_{0,\beta_i}\left[i \text{ bad}_-\right] \ \le\ C\exp(-c(\log t)^{1/3}),$$

and

$$\mathbb{Q}^{(i)}_{0,\beta_i}\left[\left(\sup_{\|x\|\le 1}\widetilde{D}_x[0,i] + r_t\widetilde{\zeta}_i + \widetilde{\xi}_i(r_t)\right)^2\right] \ \le\ C\exp(2(\log t)^{1/5}).$$

### 3.2.4 Proof of the upper bound in (3.1)

This part is similar to the lower bound, except that we work on a slightly longer time period, to avoid discussing the cases when $\sigma$ or $t-\sigma$ would not be of order $t$. So given $\varepsilon \in (0,1)$, which we fix for the moment, we define a time $s \in [0,t]$ to be *good* if

$$\inf_{\|x\|\le 1}\widetilde{D}_x[-\varepsilon t, s] \ge (1-\varepsilon)d(t) \quad \text{and} \quad \inf_{\|x\|\le 1} D_x[s,(1+\varepsilon)t] \ge (1-\varepsilon)d(t),$$

and otherwise we say that $s$ is *bad*.

We write next

$$\mathbb{P}_{0,z}(\tau<\infty)=\mathbb{P}_{0,z}(\tau<\infty,\,\sigma\text{ good})+\mathbb{P}_{0,z}(\tau<\infty,\,\sigma\text{ bad})\,.$$

Let us treat first the probability with the event $\{\sigma\text{ good}\}$. We have

$$\mathbb{P}_{0,z}(\tau<\infty,\,\sigma\text{ good})\le\frac{\mathbb{E}_{0,z}[R[-\varepsilon t,(1+\varepsilon)t]]}{\mathbb{E}_{0,z}[R[-\varepsilon t,(1+\varepsilon)t]\mid\tau<\infty,\sigma\text{ good}]}.$$

By conditioning first with respect to $\mathcal{G}_\tau$, we get by definition of $\sigma$ good, that

$$\mathbb{E}_{0,z}[R[-\varepsilon t,(1+\varepsilon)t]\mid\tau<\infty,\sigma\text{ good}]\ge 2(1-\varepsilon)\,d(t).$$

Together with (3.6) this provides the upper bound, at least for $t$ large enough,

$$\limsup_{\|z\|\to\infty}\frac{\mathbb{P}_{0,z}(\tau<\infty,\sigma\text{ good})}{G(z)}\ \le\ \pi^2(1+2\varepsilon)\frac{t}{\log t}.$$

Concerning the probability of the event $\{\sigma\text{ bad}\}$, one can argue as for the proof of the lower bound, by discretizing $\tau$ and $\sigma$, and summing over all possible values of $[\tau]$ and $[\sigma]$. Since this part is entirely similar to the arguments given in the proof of the lower bound in Section 3.2.2, and is actually even simpler since we do not have to deal with the additional factor $R[0,t]$, we omit the details. This completes the proof of Proposition 3.1. □

## 3.3 Proofs of (3.2), (3.6), and Lemmas 3.3 and 3.8

**Proof of** (3.2)**.** For any real $\rho>0$, with $d\lambda_\rho$ denoting the uniform probability measure on the boundary of $\mathcal{B}(0,\rho)$, we have shown in (2.13) that

$$\mathrm{Cap}\,(W_1[0,t]\cap\mathcal{B}(0,\rho))=\frac{1}{G(2\rho)}\int_{\partial\mathcal{B}(0,2\rho)}\mathbb{P}_{0,z}\Big(W_1[0,t]\cap\mathcal{B}(0,\rho)\cap\widetilde{\beta}[0,\infty)\neq\varnothing\ \Big|\ W_1[0,t]\Big)\ d\lambda_{2\rho}(z).$$

Taking expectation on both sides we obtain

$$\mathbb{E}[\mathrm{Cap}\,(W_1[0,t]\cap\mathcal{B}(0,\rho))]=\frac{1}{G(2\rho)}\int\mathbb{P}_{0,z}\Big(W_1[0,t]\cap\mathcal{B}(0,\rho)\cap\widetilde{\beta}[0,\infty)\neq\varnothing\Big)\ d\lambda_{2\rho}(z).$$

By rotational invariance of $\beta$ and $\widetilde{\beta}$, we get that the probability appearing in the integral above is the same for all $z\in\partial\mathcal{B}(0,2\rho)$. Writing $2\rho=(2\rho,0,\dots,0)$ we get

$$\begin{aligned}\mathbb{E}[\mathrm{Cap}\,(W_1[0,t]\cap\mathcal{B}(0,\rho))]&=\frac{1}{G(2\rho)}\mathbb{P}_{0,2\rho}\Big(W_1[0,t]\cap\mathcal{B}(0,\rho)\cap\widetilde{\beta}[0,\infty)\neq\varnothing\Big)\\&=\frac{1}{G(2\rho)}\mathbb{P}_{0,2\rho}\Big(W_1[0,t]\cap\widetilde{\beta}[0,\infty)\neq\varnothing\Big)+\mathcal{O}\Big(\frac{\mathbb{P}(W_1[0,t]\cap\mathcal{B}^c(0,\rho)\neq\varnothing)}{G(2\rho)}\Big).\end{aligned}$$

Using that the $\mathcal{O}$ term appearing above tends to 0 as $\rho\to\infty$ and invoking monotone convergence proves (3.2). □

**Proof of** (3.6)**.** Note first that by (2.5) and (3.4), one has $\mathbb{E}_{0,z}[R[0,t]\mid\beta]=D_z[0,t]$, and thus by (2.3) and (3.9),

$$\mathbb{E}_{0,z}[R[0,t]]=\int_{\mathbb{R}^4}G^*(z-x)G_t(x)\,dx. \tag{3.36}$$

Then, (2.6) shows that for any fixed $x$, $G^*(z-x)/G(z)$ converges to $\pi^2/2$, as $\|z\| \to \infty$. Moreover, using Fubini we get $\int G_t(x)\, dx = t$. We now explain why we can interchange the limit as $z$ goes to infinity and the integral in (3.36).

Set $F_z = \{x : \ \|z-x\| \le \|z\|/2\}$. Using that $G^*$ is bounded, (2.3) and (2.4), we obtain that for positive constants $C$ and $C'$, for all $z$ satisfying $\|z\| \ge 1$,

$$
\begin{aligned}
\int_{F_z} \frac{G^*(z-x)}{G(z)} G_t(x)\, dx \le \ C\,\|z\|^2 \int_{\|x\|\ge \|z\|/2} G_t(x)\, dx \ &= C\,\|z\|^2 \int_0^t \mathbb{P}(\|\beta_s\| \ge \|z\|/2)\, ds \\
&\le \ C 2^4\,\|z\|^2 \int_0^t \frac{\mathbb{E}[\|\beta_s\|^4]}{\|z\|^4}\, ds \ \le \ C' \frac{t^3}{\|z\|^2},
\end{aligned}
$$

using also the scaling property of the Brownian motion for the last inequality. On the other hand on $\mathbb{R}^4 \backslash F_z$, the ratio $G^*(z-x)/G(z)$ is upper bounded by a constant (recall (2.6)) and hence one can apply the dominated convergence theorem. We conclude that, for any $t>0$, (3.6) holds. $\square$

**Proof of Lemma 3.3.** One has recalling (3.9), and then (2.6),

$$
\begin{aligned}
\mathbb{E}[D_0[0,t]] = \int_0^t \mathbb{E}[G^*(\beta_s)]\, ds &= \frac{\pi^2}{2} \int_0^t \mathbb{E}[G(\beta_s)\mathbf{1}(\|\beta_s\|>1)]\, ds + \mathcal{O}\left(\int_0^t \mathbb{P}(\|\beta_s\| \le 1)\, ds\right) \\
&= \frac{\pi^2}{2} \int_0^t \int_{\|x\|>1} \frac{G(x)}{2\pi^2 s^2} e^{-\frac{\|x\|^2}{2s}}\, dx\, ds + \mathcal{O}(1) = \frac{1}{4\pi^2} \int_{\|x\|>1} \frac{1}{\|x\|^4} e^{-\frac{\|x\|^2}{2t}}\, dx + \mathcal{O}(1),
\end{aligned}
$$

using that $\mathbb{P}(\|\beta_s\| \le 1) \le 1 \wedge (C/s^2)$, for some constant $C>0$, at the second line, and applying Fubini at the last line. Then a change of variable gives

$$
\begin{aligned}
\frac{1}{4\pi^2} \int_{\|x\|>1} \frac{1}{\|x\|^4} e^{-\frac{\|x\|^2}{2t}}\, dx \ &= \ \frac{1}{4\pi^2} \int_1^\infty \frac{2\pi^2 \rho^3}{\rho^4} e^{-\frac{\rho^2}{2t}}\, d\rho \ = \ \frac{1}{2} \int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{r} e^{-\frac{r^2}{2}}\, dr \\
&= \ \frac{1}{2} \int_{\frac{1}{\sqrt{t}}}^1 \frac{1}{r}\, dr + \mathcal{O}(1) = \frac{\log t}{4} + \mathcal{O}(1).
\end{aligned}
$$

It remains to bound the second moment of $D_0[0,t]$. Recalling (3.9), and by using the Markov property, we get

$$
\begin{aligned}
\mathbb{E}[D_0[0,t]^2] \ &= \ \mathbb{E}\left[\int_0^t \int_0^t G^*(\beta_s) G^*(\beta_{s'})\, ds\, ds'\right] \\
&= \ 2 \int_{0\le s\le s'\le t} \mathbb{E}[G^*(\beta_s) G^*(\beta_{s'})]\, ds\, ds' \\
&\le \ 2 \int_0^t ds\, \mathbb{E}\left[G^*(\beta_s)\, \mathbb{E}\left[\int_0^t G^*(\beta_s + \widetilde{\beta}_{s'})\, ds' \mid \beta_s\right]\right] \\
&\le \ 2\mathbb{E}[D_0[0,t]] \cdot \left(\sup_{x\in\mathbb{R}^4} \mathbb{E}[D_x[0,t]]\right), \qquad (3.37)
\end{aligned}
$$

with $\widetilde{\beta}$ a standard Brownian motion independent of $\beta$. Using the simplest form of a rearrangement inequality (see for instance [17, Theorem 3.4]) shows that for any $x \in \mathbb{R}^4$ and all $t>0$,

$$
\mathbb{P}(\|\beta_t - x\| \le 1) \le \mathbb{P}(\|\beta_t\| \le 1).
$$

Using next that if $\beta$ and $\widetilde{\beta}$ are two independent standard Brownian motions, then $\beta_u - \widetilde{\beta}_s$ equals in law $\beta_{u+s}$, for any fixed positive $u$ and $s$, we deduce that also for any $x \in \mathbb{Z}^4$,

$$
\mathbb{P}_{0,x}(\|\beta_u - \widetilde{\beta}_s\| \le 1) \ = \ \mathbb{P}_{0,0}(\|\beta_u - \widetilde{\beta}_s - x\| \le 1) \ \le \ \mathbb{P}_{0,0}(\|\beta_u - \widetilde{\beta}_s\| \le 1).
$$

Then by combining (2.5) and (3.9) we deduce that

$$\mathbb{E}[D_x[0,t]] \;\leq\; \mathbb{E}[D_0[0,t]]\,,$$

for all $x \in \mathbb{R}^4$. Together with (3.37), this shows that

$$\mathbb{E}[D_0[0,t]^2] \;\leq\; 2\mathbb{E}[D_0[0,t]]^2,$$

which concludes the proof, using the first part of the lemma. □

**Proof of Lemma 3.8.** For $A \subset \mathbb{R}^4$, define $A^+ := \cup_{z\in A}\mathcal{B}(z,1)$. We claim that there exists a constant $C>0$, such that for any $z$, and $A$,

$$\mathbb{P}_z(H_A \leq 1) \leq C\int_0^2 \mathbb{P}_z(\beta_s \in A^+)\, ds. \tag{3.38}$$

Note that by applying this inequality first for $\widetilde{\beta}$ on the interval $[k,k+1]$, with $A = W_1[i,i+1]$, and then for $\beta$ on the interval $[i,i+1]$ with $A = \mathcal{B}(\widetilde{\beta}_s, 2)$, for every $s \in [k,k+2]$, we get the lemma. Thus only (3.38) needs to be proved. Consider $\beta$ a Brownian motion starting from some $z$ at time 0. Note first that almost surely,

$$\mathbf{1}\left(H_A \leq 1,\ \sup_{0\leq u\leq 1} \|\beta(H_A+u)-\beta(H_A)\| \leq 1\right) \;\leq\; \int_0^2 \mathbf{1}(\beta_s\in A^+)\, ds, \tag{3.39}$$

just because when the indicator function on the left-hand side equals 1, we know that $\beta$ remains within distance at most 1 from $A$ during a time period of length at least 1. Now, we can use the strong Markov property at time $H_A$ to obtain

$$\mathbb{P}\left(H_A \leq 1,\ \sup_{0\leq u\leq 1} \|\beta(H_A+u)-\beta(H_A)\| \leq 1\right) \;=\; \mathbb{P}(H_A\leq 1)\cdot \mathbb{P}\left(\sup_{0\leq u\leq 1}\|\beta_u\|\leq 1\right).$$

Thus, (3.38) follows after taking expectation in (3.39), with $C = 1/\mathbb{P}\big(\sup_{0\leq u\leq 1}\|\beta_u\|\leq 1\big)$, which is a positive and finite constant. □

## 3.4 Proofs of Lemmas 3.4, 3.5 and 3.7

For the proofs of these lemmas, it is convenient to introduce new notation. For $A \subset \mathbb{R}^4$ Borel-measurable, $\ell(A)$ denotes the total time spent in the set $A$ by the Brownian motion $\beta$:

$$\ell(A) := \int_0^\infty \mathbf{1}(\beta_s \in A)\, ds.$$

We also define the sets $A_0 = \mathcal{B}(0,1)$, and

$$A_i \;:=\; \mathcal{B}(0,2^i)\setminus \mathcal{B}(0,2^{i-1}), \quad \text{for } i\geq 1.$$

Note that for any $A$ and $k\geq 1$, one has using the Markov property

$$\mathbb{E}[\ell(A)^k] = k!\,\mathbb{E}\left[\int_{s_1\leq\cdots\leq s_k} \mathbf{1}(\beta_{s_1}\in A,\dots,\beta_{s_k}\in A)\, ds_1\dots ds_k\right] \leq k!\,\left(\sup_{x\in A}\mathbb{E}_x[\ell(A)]\right)^k. \tag{3.40}$$

In particular there is $\lambda>0$, such that $\mathbb{E}[\exp(\lambda\ell(A_0))]<\infty$. Using the scaling property of Brownian motion and Markov's inequality, this gives an alternative proof of (2.10) (in dimension $d\geq 3$).

**Proof of Lemma 3.4.** Let us start with the exponential moment of $\zeta$. Observe that

$$\zeta \;\le\; \sum_{i=0}^{\infty} \frac{\ell(A_i)}{2^{3i}}.$$

Using Jensen's inequality and that $\ell(A_i)$ equals in law to $2^{2(i-1)}\ell(A_1)$ for all $i \ge 1$, we obtain for some constant $C > 0$,

$$\mathbb{E}[\zeta^k] \;\le \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \frac{1}{2^{i+2}} \frac{\ell(A_i)}{2^{2(i-1)}}\right)^k\right] \le \sum_{i=0}^{\infty} \frac{1}{2^{i+2}}\, \mathbb{E}\left[\left(\frac{\ell(A_i)}{2^{2(i-1)}}\right)^k\right] \le 4^k\, \mathbb{E}\Big[\ell(A_0)^k\Big] + \mathbb{E}\Big[\ell(A_1)^k\Big] \le C^k\, k!$$

using (3.40) for the last inequality. Thus $\zeta$ has some exponential moments.

Now we prove (3.13). Suppose that $\|u\| > 2$ and $\|x\| \le 1$. Then, by (2.6)

$$|G^*(u+x) - G^*(u)| = \frac{1}{4}\left|\frac{1}{\|u+x\|^2} - \frac{1}{\|u\|^2}\right| \le \frac{1 + 2\,\|u\|}{\|u+x\|^2\,\|u\|^2} \le \frac{C}{\|u\|^3}.$$

Since $G^*$ is bounded on $\mathcal{B}(0,3)$, there exists $C > 0$, so that for all $u \in \mathbb{R}^4$,

$$\sup_{\|x\|\le 1} |G^*(u+x) - G^*(u)| \le \frac{C}{\|u\|^3 \vee 1}. \tag{3.41}$$

Then, (3.13) follows from (3.9). $\square$

**Proof of Lemma 3.5.** Part (i) follows from standard properties of the Brownian motion. We next prove (ii). The bound on $M_{0,1}$ follows from (2.9). For the rest of the proof, it is convenient to define for $s \ge 0$, $y \in \mathbb{R}^4$ and $r > 0$,

$$\xi_s(y,r) := \int_s^{\infty} \frac{\mathbf{1}(\|\beta_u - \beta_s - y\| \le r)}{\|\beta_u - \beta_s - y\|^2 \vee 1}\, du,$$

so that $\xi_s(r) = \sup_{\|y\|\le r} \xi_s(y,r)$. Then observe that $\xi_s(y,r)$ is equal in law to $\xi_0(y,r)$ for any $s \ge 0$. Moreover, for any given $y$, $\xi_0(0,r)$ stochastically dominates $\xi_0(y,r)$. Indeed in the integral defining $\xi_0(0,r)$ the part of the integral after the hitting time of the sphere of radius $\|y\|$ is equal in law to $\xi_0(y,r)$ by rotational and translation invariance of Brownian motion. Now using the scaling property of Brownian motion, and a change of variables we can see that

$$\xi_0(0,r) \overset{\text{(law)}}{=} \int_0^{\infty} \frac{\mathbf{1}(\|\beta_u\| \le 1)}{\|\beta_u\|^2 \vee \frac{1}{r^2}}\, du \le Z + \sum_{i=0}^{\log_2(r)} Z_i,$$

where

$$Z := \int_0^{\infty} \frac{\mathbf{1}(\|\beta_u\| \le 1/r)}{\|\beta_u\|^2 \vee \frac{1}{r^2}}\, du \overset{\text{(law)}}{=} \ell(A_0),$$

and for any $i \ge 0$,

$$Z_i := \int_0^{\infty} \frac{\mathbf{1}(2^{-i-1} \le \|\beta_u\| \le 2^{-i})}{\|\beta_u\|^2}\, du \overset{\text{(law)}}{=} \int_0^{\infty} \frac{\mathbf{1}(\frac{1}{2} \le \|\beta_u\| \le 1)}{\|\beta_u\|^2}\, du \le 4\ell(A_0).$$

We deduce by a union bound that

$$\mathbb{P}(\xi_0(0,r) > (\log_2(r) + 2)r) \le (\log_2(r) + 2)\mathbb{P}(\ell(A_0) \ge r/4) \le C\exp(-cr),$$

for some positive constants $c$ and $C$, using that $\ell(A_0)$ has a finite exponential moment (see (3.40)). The analogous result for $\xi_0(r)$ follows by a union bound. The bound on its second moment is immediate once we observe that $\xi_0(r) \leq \ell(\mathcal{B}(0,2r))$, since the latter is equal in law to $4r^2\ell(A_0)$.

It remains to prove (iii). Applying Lemma 3.4 to the standard Brownian motion $(\beta_u - \beta_s)_{u\geq s}$ or $(\beta_u - \beta_{s'})_{u\geq s'}$ we get that there exists a constant $C > 0$ so that almost surely

$$\sup_{\|x\|\leq 1} |D_x[s,t] - D_0[s,t]| \leq C\zeta_s \quad \text{and} \quad \sup_{\|x\|\leq 1} |D_x[s',t] - D_0[s',t]| \leq C\zeta_{s'}.$$

From (2.7) there exists a positive constant $C$ so that

$$|D_0[s',t] - D_0[s,t]| \leq C(s-s') + C\int_s^\infty \left| \frac{1}{\|\beta_u - \beta_s - Y\|^2 \vee 1} - \frac{1}{\|\beta_u - \beta_s\|^2 \vee 1} \right| du,$$

where $Y = \beta_{s'} - \beta_s$. We next divide the last integral in three pieces, one over times $u$ when $\|\beta_u - \beta_s - Y\| \leq \|Y\|$, one over times $u$ when $\|\beta_u - \beta_s\| \leq \|Y\|$ and $\|\beta_u - \beta_s - Y\| \geq \|Y\|$, and the last piece over the remaining times. Using again the same argument as in the proof of Lemma 3.4, one can bound each of these pieces respectively by $C\xi_s(Y, \|Y\|)$, $C\xi_s(0, \|Y\|)$ and $C\zeta_s$. Finally one can bound similarly $|\zeta_s - \zeta_{s'}|$, proving the bound concerning the $D_x$'s. The other bound concerning the $\widetilde{D}_x$'s is entirely similar. □

**Proof of Lemma 3.7.** Exactly for the same reason as the fact that $\xi_0(0,r)$ stochastically dominates $\xi_0(y,r)$, for any $y \in \mathbb{R}^4$ (see the argument given in the proof of the previous lemma), one can see that the law of $\xi_0(r)$ and $\zeta_0$ when the Brownian motion $\beta$ starts from some $b \in \mathbb{R}^4$, are stochastically dominated by these same random variables when $\beta$ starts from the origin. Therefore all the statements of Lemma 3.7 concerning these two quantities follow from Lemma 3.5.

Note that the law of $D_x[0,t]$ under $\mathbb{E}_b$ is the same as the law of $D_{x-b}[0,t]$ under $\mathbb{E}_0$. Moreover, using (2.6) and that $G^*$ is bounded one can see that for some $C > 0$ independent of $b$

$$\sup_{\|x\|\leq 1} D_{x-b}[0,t] \leq C D_{-b}[0,t],$$

Then, the proof of Lemma 3.3 reveals that for any $b \in \mathbb{R}^4$,

$$\mathbb{E}[D_b[0,t]^2] \leq 2\mathbb{E}[D_0[0,t]]^2.$$

Thus, the proof of the first statement follows from Lemma 3.3. Finally, we prove the last claim of the lemma. So assume that $b$ is such that $t/(\log t)^5 \leq \|b\|^2 \leq t(\log t)^{1/5}$. Note that by the above arguments, it just amounts to showing that for some constant $c > 0$ (possibly depending on $\varepsilon$, but not on $b$),

$$\mathbb{P}(D_b[0,t] > \varepsilon \log t) \leq \exp(-c(\log t)^{1/3}),$$

for $t$ large enough. For $b \in \mathbb{R}^4$ we write $H(b) = H_{\partial\mathcal{B}(0,\|b\|)}$. Then by standard properties of Brownian motion, and using rotational invariance of the function $G^*$, one has for any $b \in \mathbb{R}^4$,

$$D_b[0,t] = \int_0^t G^*(\beta_s - b)\, ds \overset{\text{(law)}}{=} \int_{H(b)}^{H(b)+t} G^*(\beta_s)\, ds.$$

Moreover, (2.9) and (2.10) show respectively that

$$\mathbb{P}(H(b) \leq t/(\log t)^6) \leq \exp(-c(\log t)), \quad \text{and} \quad \mathbb{P}(H(b) \geq t(\log t)^2) \leq \exp(-c(\log t)).$$

Note furthermore that

$$\int_{t/(\log t)^6}^{t(\log t)^2} G^*(\beta_s)\, ds = D_0[0, t(\log t)^2] - D_0[0, t/(\log t)^6].$$

Therefore Lemma 3.3 and 3.6 show that for $t$ large enough

$$\mathbb{P}(D_b[0,t] > \varepsilon \log t) \le \mathbb{P}\left(D_0[0, t(\log t)^2] > (1+\varepsilon)d(t)\right) + \mathbb{P}\left(D_0\left[0, \frac{t}{(\log t)^6}\right] < (1-\varepsilon)d(t)\right) + \exp(-c(\log t)) \ \le\ \exp(-c(\log t)^{1/3}).$$

This concludes the proof of the lemma. □

### 3.5 Proof of Lemma 3.6

The idea of the proof is to show that $D_0[0,t]$ is close to a sum of order $\log t$ terms which are i.i.d. with enough moments, and then apply standard concentration results. This idea was guiding Lawler's intuition in the discrete setting, as he explains in [12, page 98]. However, he showed, by direct computation, that the variance of $D_0[0,t]$ is of order $\log t$, as its mean. Here we obtain more precise estimate in the continuous setting.

Actually, we do not use the full strength of Lemma 3.6. However, having just a control of the variance would not be sufficient for the proof; we need at least a good control of the fourth centered moment. Since this is not more difficult nor longer to obtain, we prove the stronger result stated in Lemma 3.6.

First, let us define the sequence of stopping times $(\tau_i)_{i\ge 0}$ by

$$\tau_i := \inf\{s \ge 0 \ :\ \|\beta_s\| > 2^i\},$$

for all $i \ge 0$. Then, set for $i \ge 0$,

$$Y_i := \int_{\tau_i}^{\tau_{i+1}} G(\beta_s)\, ds, \qquad \text{and for } n \ge 0 \quad D_n \ := \ \sum_{i=0}^{n} Y_i.$$

Note that in dimension four, for any positive real $\lambda$ and $x \in \mathbb{R}^4$, one has $\lambda^2 G(\lambda x) = G(x)$. Therefore using the scaling property of the Brownian motion, we see that the $Y_i$'s are independent and identically distributed. The following lemma shows that $Y_0$ has sufficiently small moments, and as a consequence that $D_n$ is concentrated. We postpone its proof.

**Lemma 3.10.** *There exists a positive constant $\lambda$, such that $\mathbb{E}[e^{\lambda\sqrt{Y_0}}] < \infty$. As a consequence there exist positive constants $c$ and $C$, such that for all $\varepsilon > 0$ and $n \ge 1$,*

$$\mathbb{P}(|D_n - \mathbb{E}[D_n]| > \varepsilon\, \mathbb{E}[D_n]) \ \le\ C \exp(-c\,(\varepsilon n)^{1/3}).$$

Now we see that as $t$ goes to infinity, $D_0[0,t]$ is close to $D_{N_t}$, where $N_t$ is defined for all $t > 0$, by

$$N_t = \sup\{i \ :\ \tau_i \le t\},$$

if $\tau_0 \le t$, and $N_t = 0$ otherwise. Indeed, recall that $G^*(z) = G(z)$, whenever $\|z\| > 1$, so that

$$D_0[0,t] \ := \int_0^t G^*(\beta_s)\, ds = \ D_{N_t} - Z_1(t) - Z_2(t) + Z_3(t), \tag{3.42}$$

with

$$Z_1(t) = \int_{\tau_0}^{\tau_0 \vee t} \mathbf{1}(\|\beta_s\| \le 1) G(\beta_s)\, ds, \qquad Z_2(t) = \int_{\tau_0 \vee t}^{\tau_{N_t+1}} G(\beta_s)\, ds,$$

and

$$Z_3(t) = \int_0^t \mathbf{1}(\|\beta_s\| \le 1) G^*(\beta_s)\, ds.$$

Since, $G^*$ is bounded on $\mathcal{B}(0,1)$, we see that $Z_3(t) \le Z_3(\infty) \le C\,\ell(A_0)$, for some constant $C > 0$, with the notation introduced at the beginning of Section 3.4. Moreover, by definition $Z_2(t) \le Y_{N_t}$. These bounds together with (3.40) and the next lemma show that $Z_1(t)$, $Z_2(t)$ and $Z_3(t)$ are negligible in (3.42).

**Lemma 3.11.** *There exists $\lambda > 0$, such that*

$$\mathbb{E}[e^{\lambda\sqrt{Z_1(\infty)}}] \;<\; +\infty,$$

*and for any $\varepsilon > 0$, there exist $c > 0$ and $C > 0$, such that*

$$\mathbb{P}(Y_{N_t} \;\ge\; \varepsilon \log t) \;\le\; C \exp(-c\sqrt{\log t}).$$

*Moreover, $\mathbb{E}[Y_{N_t}] = o(\log t)$.*

Let us postpone the proof of this lemma and continue the proof of Lemma 3.6.

Actually the proof is almost finished. First, all the previous estimates and (3.42) show that $D_0[0,t]$ and $D_{N_t}$ have asymptotically the same mean, i.e.

$$\lim_{t\to\infty} \frac{1}{d(t)} \mathbb{E}[D_{N_t}] \;=\; 1.$$

Moreover, using the strong Markov property at times $\tau_i$, one obtains

$$\begin{aligned} \mathbb{E}[D_{N_t}] \;&=\; \sum_{i=0}^{\infty} \mathbb{E}[Y_i \mathbf{1}(i \le N_t)] \;=\; \sum_{i=0}^{\infty} \mathbb{E}[Y_i \mathbf{1}(\tau_i \le t)] \\ &=\; \sum_{i=0}^{\infty} \mathbb{E}[Y_i]\mathbb{P}(\tau_i \le t) \;=\; \mathbb{E}[Y_0]\mathbb{E}[N_t]. \end{aligned}$$

Then all that remains to do is to recall that $N_t$ is concentrated. Indeed, letting $n_t = \log t/(2\log 2)$, it follows from (2.9) that for any $\varepsilon > 0$,

$$\mathbb{P}(N_t \ge (1+\varepsilon)n_t) \;=\; \mathbb{P}\left(\sup_{s\le t} \|\beta_s\| > t^{(1+\varepsilon)/2}\right) \;\le\; C\, \exp(-ct^{\varepsilon}), \tag{3.43}$$

and it follows from (2.10) that

$$\mathbb{P}(N_t \le (1-\varepsilon)n_t) \;=\; \mathbb{P}\left(\sup_{s\le t} \|\beta_s\| \le t^{(1-\varepsilon)/2}\right) \;\le\; C\, \exp(-ct^{\varepsilon}), \tag{3.44}$$

for some positive constants $c$ and $C$. So for all $\varepsilon < 1$ we obtain $\mathbb{E}[N_t] \ge (1-\varepsilon)n_t$ for all $t$ sufficiently large. Therefore,

$$d(t) \;\sim\; \mathbb{E}[D_{N_t}] \ge c_0\,(1-\varepsilon)n_t,$$

with $c_0 = \mathbb{E}[Y_0]$. Note also that $\mathbb{E}[D_n] = c_0 n$, for all $n \geq 0$. Then with all the estimates obtained so far (in particular with Lemma 3.10 and (3.40)), we deduce that for $t$ large enough,

$$\begin{aligned}\mathbb{P}(D_0[0,t] \geq (1+\varepsilon)d(t)) &\leq \mathbb{P}\left(D_{(1+\frac{\varepsilon}{4})n_t} \geq \left(1+\frac{\varepsilon}{2}\right)d(t)\right) + \mathbb{P}\left(N_t \geq \left(1+\frac{\varepsilon}{4}\right)n_t\right)\\ &+ \mathbb{P}\left(Z_3(t) \geq \frac{\varepsilon}{2}d(t)\right) \leq C\exp(-c(\log t)^{1/3}),\end{aligned}$$

and likewise for the lower bound (using also Lemma 3.11):

$$\begin{aligned}\mathbb{P}(D_0[0,t] \leq (1-\varepsilon)d(t)) &\leq \mathbb{P}\left(D_{(1-\frac{\varepsilon}{4})n_t} \leq \left(1-\frac{\varepsilon}{2}\right)d(t)\right) + \mathbb{P}\left(N_t \leq (1-\frac{\varepsilon}{4})n_t\right)\\ &+ \mathbb{P}\left(Z_1(t)+Z_2(t) \geq \frac{\varepsilon}{2}d(t)\right) \leq C\exp(-c(\log t)^{1/3}),\end{aligned}$$

which concludes the proof of Lemma 3.6. □

At this point it just remains to prove Lemma 3.10 and 3.11.

**Proof of Lemma 3.10.** We first extend the definition of the $\tau_i$'s and $A_i$'s to negative indices:

$$\tau_{-i} := \inf\{s \geq \tau_0 \ : \ \beta_s \in \partial\mathcal{B}(0, 2^{-i})\}, \qquad \text{and} \qquad A_{-i} = \mathcal{B}(0, 2^{-i+1}) \setminus \mathcal{B}(0, 2^{-i}),$$

for $i \geq 1$. We also set

$$\ell_0(A_{-i}) := \int_{\tau_0}^{\infty} \mathbf{1}(\beta_s \in A_{-i})\, ds.$$

In particular, compare it with the notation introduced in Section 3.4, and note that $\ell_0(A_{-i}) \leq \ell(A_{-i})$ . Then similarly as in the proof of Lemma 3.4, one has

$$Y_0 \ = \ \int_{\tau_0}^{\tau_1} G(\beta_s)\, ds \ \leq \ \sum_{i\geq 1} \mathbf{1}(\tau_{-i+1} < \tau_1) 2^{2i} \ell_0(A_{-i}) + \tau_1.$$

Note that $\tau_1$ has an exponential tail by (2.10), so it suffices to bound the moments of the first sum. More precisely it amounts to proving that its $k$-th power is bounded by $C^k\,(k!)^2$. First,

$$\mathbb{E}\left[\left(\sum_{i\geq 1} \mathbf{1}(\tau_{-i+1} < \tau_1) 2^{2i} \ell_0(A_{-i})\right)^k\right] \ = \ \sum_{i_1,\dots,i_k} 4^{\sum_{j=1}^k i_j}\, \mathbb{E}\left[\prod_{j=1}^k \mathbf{1}(\tau_{-i_j+1} < \tau_1)\ell_0(A_{-i_j})\right].$$

Next, by Holder's inequality we get

$$\mathbb{E}\left[\prod_{j=1}^k \mathbf{1}(\tau_{-i_j+1} < \tau_1)\ell_0(A_{-i_j})\right] \ \leq \ \prod_{j=1}^k \mathbb{E}\Big[\mathbf{1}(\tau_{-i_j+1} < \tau_1)\ell_0(A_{-i_j})^k\Big]^{1/k}.$$

Now by scaling and rotational invariance of the Brownian motion, for any $x \in \partial\mathcal{B}(0, 2^{-i+1})$, and $y \in \partial\mathcal{B}(0,1)$,

$$\mathbb{E}_x[\ell(A_{-i})^k] \ = \ 4^{-k(i-1)}\, \mathbb{E}_y[\ell(A_{-1})^k] \ \leq \ 4^{-k(i-1)}\, \mathbb{E}[\ell(A_{-1})^k].$$

Therefore using the strong Markov property, we get

$$\mathbb{E}\Big[\mathbf{1}(\tau_{-i_j+1} < \tau_1)\ell_0(A_{-i_j})^k\Big] \ \leq \ \mathbb{P}(\tau_{-i_j+1} < \tau_1)\, 4^{-k\,(i_j-1)}\, \mathbb{E}[\ell(A_{-1})^k].$$

From (3.40) we deduce that there is a constant $C > 0$, such that

$$\mathbb{E}\left[\left(\sum_{i\geq 1}\mathbf{1}(\tau_{-i+1}<\tau_1)2^{2i}\ell_0(A_{-i})\right)^k\right] \leq C^k\, k! \sum_{i_1,\dots,i_k}\prod_{j=1}^k \mathbb{P}(\tau_{-i_j+1}<\tau_1)^{1/k}$$
$$= C^k\, k! \left(\sum_{i\geq 1}\mathbb{P}(\tau_{-i+1}<\tau_1)^{1/k}\right)^k \leq C^k\, k! \left(\sum_{i\geq 1}\frac{1}{2^{2i/k}}\right)^k \leq C^k\,(k!)^2,$$

using (2.8) for the second inequality. This concludes the proof of the first part of the lemma.

It remains to prove the second part. Let $\varepsilon > 0$ be fixed. Since $Y_0$ is integrable, there exists $L \geq 1$, such that $\mathbb{E}[Y_0 \mathbf{1}(Y_0 > L)] \leq \varepsilon/4$. Then using Bernstein's inequality (see for instance Theorems 3.4 and 3.5 in [5]) and the first part of the lemma at the third line, we obtain for some positive constants $C$ and $c$,

$$\mathbb{P}\left(\left|\sum_{i=0}^n (Y_i - \mathbb{E}[Y_i])\right| > \varepsilon(n+1)\right) \leq \mathbb{P}(\exists i\leq n\, :\, Y_i > L) + \mathbb{P}\left(\left|\sum_{i=0}^n (Y_i\mathbf{1}(Y_i<L) - \mathbb{E}[Y_i])\right| > \varepsilon(n+1)\right)$$
$$\leq (n+1)\mathbb{P}(Y_0 > L) + \mathbb{P}\left(\left|\sum_{i=0}^n (Y_i\mathbf{1}(Y_i<L) - \mathbb{E}[Y_i\mathbf{1}(Y_i<L)])\right| > \frac{\varepsilon}{2}(n+1)\right)$$
$$\leq C\left(n\exp(-\lambda\sqrt{L}) + \exp\left(-c\,\frac{\varepsilon^2 n}{\mathbb{E}\left[Y_0^2\right]+L\varepsilon}\right)\right).$$

The desired result follows by taking $L = (\varepsilon n)^{2/3}$, and $\varepsilon n$ large enough. □

**Proof of Lemma 3.11.** We start with the first part. Exactly as in the proof of Lemma 3.10, and using the same notation, one has

$$Z_1(\infty) = \int_{\tau_0}^\infty \mathbf{1}(\|\beta_s\|\leq 1)G(\beta_s)\,ds \leq C\sum_{i\geq 1}\mathbf{1}(\tau_{-i+1}<\infty)2^{2i}\ell(A_{-i}),$$

and the result follows exactly as in the previous lemma.

Concerning the second part, recall the notation introduced at the end of the proof of Lemma 3.6. Then using (3.43), (3.44) and Lemma 3.10, we get

$$\begin{aligned}
\mathbb{P}(Y_{N_t}\geq \varepsilon\log t) &\leq \mathbb{P}(|N_t - n_t|\geq \varepsilon\log t) + \mathbb{P}\left(\exists i\in[n_t-\varepsilon\log t, n_t+\varepsilon\log t]\, :\, Y_i\geq \varepsilon\log t\right)\\
&\leq C\,\exp(-c\,t^\varepsilon) + 2\varepsilon\log t\cdot\mathbb{P}(Y_0\geq \varepsilon\log t)\\
&\leq C\,\exp(-c\,t^\varepsilon) + C\varepsilon(\log t)\,\exp(-c\sqrt{\varepsilon\log t}).
\end{aligned}$$

Finally we compute the expectation of $Y_{N_t}$ as follows: for any fixed $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{E}[Y_{N_t}] &= \sum_{i\geq 0}\mathbb{E}[\mathbf{1}(\tau_i\leq t<\tau_{i+1})Y_i]\\
&\leq \sum_{i\leq n_t-\varepsilon\log t}\mathbb{E}[\mathbf{1}(t<\tau_{i+1})Y_i] + \sum_{i\geq n_t+\varepsilon\log t}\mathbb{E}[\mathbf{1}(\tau_i\leq t)Y_i] + 2\varepsilon(\log t)\mathbb{E}[Y_0],
\end{aligned}$$

where for indices $i$ between $n_t - \varepsilon\log t$ and $n_t + \varepsilon\log t$, we used the simple bound $\mathbb{E}[\mathbf{1}(\tau_i\leq t < \tau_{i+1})Y_i] \leq \mathbb{E}[Y_i] = \mathbb{E}[Y_0]$. Then using Cauchy-Schwarz and (3.44) for the first sum above, and the Markov property and (2.9) for the second sum, we get

$$\mathbb{E}[Y_{N_t}] \leq C\,n_t\exp(-ct^\varepsilon)\mathbb{E}\left[Y_0^2\right]^{1/2} + \mathbb{E}[Y_0]\sum_{j\geq\varepsilon\log t}\exp(-c2^j) + 2\varepsilon(\log t)\mathbb{E}[Y_0],$$

and the result follows. $\square$

## 4 Upward Large Deviation

Using our estimate on the expected capacity, we obtain a rough estimate on the upward large deviation, which we use in the next section when bounding the square of the cross-terms (recall their definition (1.8)). Our estimate improves a recent inequality of Erhard and Poisat: inequality (5.55) in the proof of their Lemma 3.7 in [8]. They estimated the probability that the capacity of the sausage exceeds by far its mean value and obtained polynomial bounds.

**Proposition 4.1.** *There exist positive constants $c$ and $t_0$, such that for any $a \in (0,1)$, there is $\kappa = \kappa(a) > 0$, satisfying*

$$\mathbb{P}\Big(\mathrm{Cap}\,(W_1[0,t]) - \mathbb{E}[\mathrm{Cap}\,(W_1[0,t])] > a\frac{t}{\log t}\Big) \;\le\; \exp\Big(-c\, a\, t^{\kappa}\min(1, \frac{a}{\log t})\Big),$$

*for all $t \geq t_0$. Moreover, there exists a constant $\kappa > 0$, such that the inequality holds true for any $a \geq 1$ and $t \geq 2$.*

**Remark 4.2.** The proposition shows in particular that the process $\Big(\frac{\log t}{t}\mathrm{Cap}(W_1[0,t]),\, t \geq 2\Big)$, is bounded in $L^p$, for all $p \geq 1$. It also implies (1.5) of Proposition 1.3, since for $t$ large enough $a/\log t < 1$, and the $\log t$ can be absorbed in $t^{\kappa}$ by choosing a smaller $\kappa$.

**Proof of Proposition 4.1.** Let $a > 0$ be fixed. Using that the capacity is subadditive, one has for any $t \geq 2$ and $L \geq 1$,

$$\mathrm{Cap}\,(W_1[0,t]) \;\le\; \sum_{k=0}^{2^L-1} \mathrm{Cap}\left(W_1\left[k\frac{t}{2^L}, (k+1)\frac{t}{2^L}\right]\right). \tag{4.1}$$

To simplify notation, we write

$$X = \mathrm{Cap}\,(W_1[0,t])\,, \quad \text{and} \quad X_k = \mathrm{Cap}\left(W_1\left[k\frac{t}{2^L}, (k+1)\frac{t}{2^L}\right]\right), \quad \text{for } k \geq 0.$$

Note that the $(X_k)$ are independent and identically distributed. Then choose $L$ such that $2^L = [t^{\kappa}]$, with $\kappa < 1$, some positive constant to be fixed later. For $t$ large enough, Proposition 3.1 gives

$$\mathbb{E}[X] \geq 4\pi^2(1-2^{-10}a)\frac{t}{\log t}, \quad \text{and} \quad \mathbb{E}[X_1] \leq 4\pi^2(1+2^{-10}a)\frac{t/2^L}{\log(t/2^L)}.$$

Plugging this into (4.1) we obtain

$$X - \mathbb{E}[X] \leq \sum_{k=0}^{2^L-1}(X_k - \mathbb{E}[X_k]) + 4\pi^2\frac{t}{\log t}\Big(\frac{(1+2^{-10}a)}{1-\log(2^L)/\log t} - (1-2^{-10}a)\Big).$$

Furthermore when $a \leq 1$, by choosing $\kappa$ small enough (depending on $a$), one can make the last term above smaller than $at/(2\log t)$, and when $a \geq 1$, it is easy to check that this is also true with $\kappa = 1/1000$. Thus for this choice of $\kappa$,

$$\mathbb{P}\Big(X - \mathbb{E}[X] \geq a\frac{t}{\log t}\Big) \;\le\; \mathbb{P}\left(\sum_{k=0}^{2^L-1}(X_k - \mathbb{E}[X_k]) \geq \frac{a}{2}\frac{t}{\log t}\right). \tag{4.2}$$

Now we claim that $X_1/(t/2^L)$ has a finite exponential moment. Indeed, thanks to Lemma 2.2, it suffices to compute the moments of the volume of a Wiener sausage. But this is easily obtained, using a similar argument as for the local time of balls, see (3.40). To be more precise, for $z \in \mathbb{R}^4$, set

$$\sigma_z := \inf\{s \geq 0 \ : \ \|\beta_s - z\| \leq 1\}.$$

Then for any $t \geq 1$ and $k \geq 1$, one has using the strong Markov property at times $\sigma_{z_i}$, and translation invariance of the Brownian motion,

$$\begin{aligned}
\mathbb{E}[|W_1[0,t]|^k] &= \int \cdots \int \mathbb{P}(\sigma_{z_1} \leq t, \dots, \sigma_{z_k} \leq t)\, dz_1 \dots dz_k \\
&= k! \int \cdots \int \mathbb{P}(\sigma_{z_1} \leq \cdots \leq \sigma_{z_k} \leq t)\, dz_1 \dots dz_k \\
&\leq k!\, \mathbb{E}[|W_1[0,t]|]^k.
\end{aligned}$$

Next recall a classical result of Kesten, Spitzer, and Whitman on the volume of the Wiener sausage, (see e.g. [15] or [16] and references therein).

$$\lim_{t\to\infty} \frac{1}{t} \cdot \mathbb{E}[|W_1(0,t)|] = \mathrm{Cap}(\mathcal{B}(0,1)) = 2\pi^2.$$

As a consequence, for some constant $C$, we have $\mathbb{E}[|W_1[0,t]|^k] \leq C^k k! t^k$, and there exists $\lambda_0 > 0$, such that

$$\sup_{t\geq 1} \mathbb{E}\left[\exp\left(\lambda_0 \frac{\mathrm{Cap}(W_1[0,t])}{t}\right)\right] \ < \ +\infty. \tag{4.3}$$

Now from (4.2) and (4.3) it is quite standard to deduce the result of the proposition. But let us give some details for the reader's convenience. First, using a Taylor expansion, one has for any $x \in \mathbb{R}$, and any integer $n \geq 0$,

$$\left| e^x - \sum_{i=0}^{n} \frac{x^i}{i!} \right| \ \leq \ e^{|x|} \frac{|x|^{n+1}}{(n+1)!}.$$

Applying this with $n = 2$, shows that for any $\lambda \geq 0$, and any nonnegative random variable $Y$ with finite mean,

$$\left| e^{\lambda(Y-\mathbb{E}[Y])} - \sum_{i=0}^{2} \frac{\lambda^i (Y-\mathbb{E}[Y])^i}{i!} \right| \ \leq \ \frac{\lambda^3}{3!} \,|Y - \mathbb{E}[Y]\,|^3\, e^{\lambda|Y-\mathbb{E}[Y]|}.$$

Therefore, if we assume in addition that $\mathbb{E}[e^Y]$ is finite and that $\lambda \leq 1/2$, we obtain

$$\mathbb{E}[e^{\lambda(Y-\mathbb{E}[Y])}] \ \leq \ 1 + \frac{\lambda^2}{2}\mathbb{E}[(Y-\mathbb{E}[Y])^2] + C_1\lambda^3 \ \leq \ e^{C_2\lambda^2},$$

for some constants $C_1$ and $C_2$ (that only depend on $\mathbb{E}[e^Y]$). Finally we apply the previous bound to $Y = \lambda_0 X_0/(t/2^L)$, with $\lambda_0$ as in (4.3). Using Chebychev's exponential inequality, we get for any $\lambda \in [0,1/2]$,

$$\begin{aligned}
\mathbb{P}\left( \sum_{k=0}^{2^L-1} \frac{(X_k - \mathbb{E}[X_k])}{t/2^L} \geq \frac{a}{2}\frac{2^L}{\log t} \right) &\leq \exp\big(-\frac{\lambda\lambda_0 a}{2\log t} 2^L\big) \prod_{k=0}^{2^L-1} \mathbb{E}\left[\exp\left(\lambda\lambda_0 \frac{X_k - \mathbb{E}[X_k]}{t/2^L}\right)\right] \\
&\leq \exp\left(-\left(\frac{\lambda\lambda_0 a}{2\log t} - C_2\lambda^2\right) 2^L\right),
\end{aligned}$$

and the result follows by optimizing in $\lambda$. □

## 5 Intersection of Sausages and Cross-terms

### 5.1 Intersection of Wiener sausages

Our aim in this section is to obtain some bounds on the probability of intersection of two Wiener sausages. Then, in the next section, we apply these results to bound the second moment of the cross-term in the decomposition (1.8) of the capacity of two Wiener sausages.

We consider two independent Brownian motions $(\beta_t, t \geq 0)$ and $(\widetilde{\beta}_t, t \geq 0)$ starting respectively from 0 and $z$, and denote their corresponding Wiener sausages by $W$ and $\widetilde{W}$. We estimate the probability that $W_{1/2}[0,t]$ intersects $\widetilde{W}_{1/2}[0,\infty)$, when $\|z\|$ is of order $\sqrt{t}$ up to logarithmic factors. Note that in Section 3 we also consider the same question but when $z$ is sent to infinity first. This section can be read independently of Section 3, and does not use its notation.

Such estimates have a long history in probability. Let us mention three occurrences of closely related estimates, which are however not enough to deduce ours. Aizenman in [1] obtained a bound for the Laplace transform integrated over space. Pemantle, Peres and Shapiro [19] obtained the existence of positive constants $c$ and $C$, such that for all $z \in \mathbb{R}^4$, almost surely, for all $t$ large enough,

$$\frac{ct}{\log t} \inf_{y \in \beta[0,t]} \|z-y\|^{-2} \leq \mathbb{P}_{0,z}\left( W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing \;\Big|\; \beta \right) \leq \frac{Ct}{\log t} \sup_{y \in \beta[0,t]} \|z-y\|^{-2}.$$

Lawler has obtained similar results for random walks. Finally, our result reads as follows.

**Proposition 5.1.** *For any $\alpha > 0$, there exist positive constants $C$ and $t_0$, such that for all $t > t_0$ and $z \in \mathbb{R}^4$, with $t/(\log t)^\alpha \leq \|z\|^2 \leq t \cdot (\log t)^\alpha$,*

$$\mathbb{P}_{0,z}\Big( W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing \Big) \;\leq\; C \cdot \left(1 \wedge \frac{t}{\|z\|^2}\right) \cdot \frac{(\log\log t)^2}{\log t}. \tag{5.1}$$

We divide the proof of Proposition 5.1 into two lemmas. The first one deals with $\|z\|$ *large.*

**Lemma 5.2.** *For any $\alpha > 0$, there exist positive constants $C$ and $t_0$, such that for all $t > t_0$ and all $z \in \mathbb{R}^4$ nonzero, with $\|z\| \leq \sqrt{t} \cdot (\log t)^\alpha$,*

$$\mathbb{P}_{0,z}\Big( W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing \Big) \;\leq\; C \cdot \frac{t}{\|z\|^2} \cdot \frac{\log\log t}{\log t}. \tag{5.2}$$

The second lemma improves on Lemma 5.2 in the region $\|z\|$ *small.*

**Lemma 5.3.** *For any $\alpha > 0$, there exist positive constants $C$ and $t_0$, such that for all $t > t_0$ and all $z \in \mathbb{R}^4$, with $t \cdot (\log t)^{-\alpha} \leq \|z\|^2 \leq t$,*

$$\mathbb{P}_{0,z}\Big( W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing \Big) \;\leq\; C \cdot \frac{(\log\log t)^2}{\log t}. \tag{5.3}$$

**Proof of Lemma 5.2.** Let $r := \sqrt{t/\log t}$. Assume that $\|z\| > 2r$, otherwise there is nothing to prove. Using (2.8), we see that estimating (5.2) amounts to bounding the term

$$\mathbb{P}_{0,z}\Big( W_{1/2}[0,t] \cap \widetilde{W}_{1/2}[0,\infty) \neq \varnothing,\, W_1[0,t] \cap \mathcal{B}(z,r) = \varnothing \Big).$$

Using now Proposition 4.1, we see that it suffices to bound the term

$$\mathbb{P}_{0,z}\left(W_{1/2}[0,t]\cap \widetilde{W}_{1/2}[0,\infty)\neq\varnothing,\ d(z,W_1[0,t])\geq r,\ \mathrm{Cap}(W_1[0,t])\leq 8\pi^2\frac{t}{\log t}\right).$$

By first conditioning on $W_1[0,t]$, and then applying Lemma 2.1, we deduce that the latter display is bounded, up to a constant factor, by

$$\mathbb{E}\left[\frac{\mathbf{1}(d(z,W_1[0,t])\geq r)}{d(z,W_1[0,t])^2}\right]\cdot\frac{t}{\log t}.$$

Furthermore, on the event $\{d(z,W_1[0,t])\geq r\}$, for $t$ sufficiently large we have

$$\frac{1}{2}d(z,\beta[0,t])\leq d(z,\beta[0,t])-1\leq d(z,W_1[0,t])\leq d(z,\beta[0,t]),$$

with $\beta[0,t]$ the trace of $\beta$ on the time interval $[0,t]$. Now by using again (2.8) and the bound $\|z\|\leq\sqrt{t}(\log t)^\alpha$, we get for some constant $C$ independent of $z$,

$$\begin{aligned}\mathbb{E}\left[\frac{\mathbf{1}(d(z,\beta[0,t])\geq r)}{d(z,\beta[0,t])^2}\right] &= 2\int_0^{1/r}u\cdot\mathbb{P}\left(d(z,\beta[0,t])\leq 1/u\right)\,du\\ &\leq 2\int_{1/\|z\|}^{1/r}u\cdot\mathbb{P}\left(d(z,\beta[0,t])\leq 1/u\right)\,du+\frac{1}{\|z\|^2}\\ &\leq C\frac{\log(\|z\|/r)}{\|z\|^2}\leq C(\alpha+\frac{1}{2})\frac{\log\log t}{\|z\|^2},\end{aligned}$$

which concludes the proof. □

**Proof of Lemma 5.3.** Set $t_1=0$, $t_2=\|z\|^2$ and for $k\geq 3$, denote $t_k=2t_{k-1}$. Let $K$ be the smallest integer such that $2^{K-1}\geq(\log t)^\alpha$. In particular $t\leq 2^{K-1}\|z\|^2=t_{K+1}$ by hypothesis. Then,

$$\mathbb{P}_{0,z}\left(W_{1/2}[0,t]\cap\widetilde{W}_{1/2}[0,\infty)\neq\varnothing\right)\ \leq\ \sum_{k=1}^{K}\mathbb{P}_{0,z}\left(W_{1/2}[t_k,t_{k+1}]\cap\widetilde{W}_{1/2}[0,\infty)\neq\varnothing\right).$$

We now bound each term of the sum on the right hand side. The first one (corresponding to $k=1$) is bounded using directly Lemma 5.2: for some positive constant $C$,

$$\mathbb{P}_{0,z}\left(W_{1/2}[0,\|z\|^2]\cap\widetilde{W}_{1/2}[0,\infty)\neq\varnothing\right)\ \leq\ C\cdot\frac{\log\log t}{\log t}.$$

Now for the other terms, we first observe that for some positive constant $C$, for all $z$, satisfying $\|z\|\geq 1$,

$$\mathbb{E}\left[\frac{1}{\|\beta_{t_k}-z\|^2}\right]\ \leq\ \frac{C}{t_k^2}\cdot\int\frac{1}{\|z-x\|^2}e^{-\frac{\|x\|^2}{2t_k}}\,dx\ \leq\ \frac{C}{t_k}. \tag{5.4}$$

Furthermore, it follows from (2.9), that for all $k\leq K$,

$$\mathbb{P}[\|\beta_{t_k}-z\|>\sqrt{t_k}(\log t_k)^\alpha]\leq\mathbb{P}[\|\beta_{t_k}\|>\sqrt{t_k}((\log t_k)^\alpha-1)]\leq C\exp(-c(\log t)^\alpha), \tag{5.5}$$

using that by hypothesis $\|z\|\leq\sqrt{t_k}$, and that $\log t_k$ and $\log t$ are of the same order. Then, we obtain, for some positive constant $C$, for $t$ large enough,

$$\mathbb{P}_{0,z}\left(W_{1/2}[t_k,t_{k+1}]\cap\widetilde{W}_{1/2}[0,\infty)\neq\varnothing\right)\ \leq\ \mathbb{E}\Big[\mathbb{P}_{0,z-\beta_{t_k}}\left(W_{1/2}[0,t_{k+1}-t_k]\cap\widetilde{W}_{1/2}[0,\infty)\neq\varnothing\right)\Big]$$

$$\leq \; C\,\mathbb{E}\Big[\frac{1}{\|\beta_{t_k}-z\|^2}\Big]\cdot\frac{t_k\cdot\log\log t}{\log t} \;\leq\; C\cdot\frac{\log\log t}{\log t},$$

using Lemma 5.2 and (5.5) for the second inequality and (5.4) for the third one. We conclude the proof recalling that $K$ is of order $\log\log t$. □

We now give the proof of Proposition 1.5.

**Proof of Proposition 1.5.** Define the stopping times

$$\sigma := \inf\{s \;:\; W_1[0,s]\cap\gamma[0,\infty)\neq\varnothing\},\quad\text{and}\quad \widetilde{\sigma} := \inf\{s \;:\; W_1[0,s]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing\}.$$

Note that

$$\mathbb{P}_{0,z,z'}\left(W_1[0,t]\cap\gamma[0,\infty)\neq\varnothing,\ W_1[0,t]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing\right) = \mathbb{P}_{0,z,z'}(\sigma<\widetilde{\sigma}\leq t)+\mathbb{P}_{0,z,z'}(\widetilde{\sigma}<\sigma\leq t).$$

By symmetry, we only need to deal with $\mathbb{P}_{0,z,z'}(\sigma<\widetilde{\sigma}\leq t)$. Now conditionally on $\gamma$, $\sigma$ is a stopping time for $\beta$. In particular, conditionally on $\sigma$ and $\beta_\sigma$, $W_1[\sigma,t]$ is equal in law to $\beta_\sigma+W_1'[0,t-\sigma]$, with $W'$ a Wiener sausage, independent of everything else. Therefore

$$\begin{aligned}
\mathbb{P}_{0,z,z'}(\sigma<\widetilde{\sigma}\leq t) &\leq \mathbb{E}_{0,z}\left[\mathbf{1}(\sigma\leq t)\,\mathbb{P}_{0,z,z'}(\sigma<\widetilde{\sigma}\leq t\mid\sigma,\,\gamma,\beta_\sigma)\right]\\
&\leq \mathbb{E}_{0,z}\left[\mathbf{1}(\sigma\leq t)\,\mathbb{P}_{0,z'-\beta_\sigma}(W_1'[0,t-\sigma]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing\mid\sigma)\right]\\
&\leq \mathbb{E}_{0,z}\left[\mathbf{1}(\sigma\leq t)\,\mathbb{P}_{0,z'-\beta_\sigma}(W_1'[0,t]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing)\right].
\end{aligned}$$

To simplify notation, write $D=\|z'-\beta_\sigma\|$. Note that one can assume $D>\sqrt{t}\cdot(\log t)^{-3\alpha-1}$, since by using (2.8) and the hypothesis on $\|z'\|$ we have

$$\mathbb{P}_{0,z}\left(\sigma\leq t, D\leq\sqrt{t}\cdot(\log t)^{-3\alpha-1}\right)\;\leq\;\frac{t}{\|z'\|^2\cdot(\log t)^{6\alpha+2}}\;\leq\;(\log t)^{-4\alpha-2},$$

and the right hand side in (1.10) is always larger than $(\log t)^{-4\alpha-2}$ by the hypothesis on $z$ and $z'$. Then by applying Proposition 5.1 we get for positive constants $C_1$ and $C_2$,

$$\begin{aligned}
&\mathbb{E}_{0,z}\;\left[\mathbf{1}\left(\sigma\leq t,\ D>\frac{\sqrt{t}}{(\log t)^{3\alpha+1}}\right)\ \mathbb{P}_{0,z'-\beta_\sigma}\left(W_1'[0,t]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing\right)\right]\\
&\leq C_1\ \mathbb{E}_{0,z}\left[\mathbf{1}(\sigma\leq t)\ \left(1\wedge\frac{t}{D^2}\right)\right]\cdot\frac{(\log\log t)^2}{\log t}\\
&\leq C_1\ \mathbb{P}_{0,z}\left(W_1[0,t]\cap\widetilde{\gamma}[0,\infty)\neq\varnothing\ \right)\cdot\left(1\wedge\frac{16t}{\|z'\|^2}\right)\cdot\frac{(\log\log t)^2}{\log t}+C_1\mathbb{P}_{0,z}\left(\sigma\leq t,\ D\leq\frac{\|z'\|}{4}\right)\cdot\frac{(\log\log t)^2}{\log t}\\
&\leq\ C_2\frac{(\log\log t)^4}{(\log t)^2}\cdot\left(1\wedge\frac{t}{\|z\|^2}\right)\cdot\left(1\wedge\frac{t}{\|z'\|^2}\right)+C_1\mathbb{P}_{0,z}\left(\sigma\leq t,\ D\leq\frac{\|z'\|}{4}\right)\cdot\frac{(\log\log t)^2}{\log t}.
\end{aligned}$$

Now define

$$\tau_{z,z'} := \begin{cases}\inf\{s\;:\;\beta_s\in\mathcal{B}(z',\|z'\|/4)\} & \text{if } \|z-z'\|>\|z'\|/2\\ \inf\{s\;:\;\beta_s\in\mathcal{B}(z',3\|z'\|/4)\} & \text{if } \|z-z'\|\leq\|z'\|/2.\end{cases}$$

Note that by construction $\|z-\beta_{\tau_{z,z'}}\|\geq\max(\|z-z'\|,\|z'\|)/4$, and that on the event $\{D\leq\|z'\|/4\}$, one has $\sigma\geq\tau_{z,z'}$. Therefore by conditioning first on $\tau_{z,z'}$ and the position of $\beta$ at this time, and then by using Proposition 5.1, we obtain for some positive constants $\kappa$, $C_3$ and $C_4$,

$$\begin{aligned}
\mathbb{P}_{0,z}\left(\sigma\leq t,\ D\leq\|z'\|/4\right) &\leq \mathbb{P}_{0,z}\left(\tau_{z,z'}\leq\sigma\leq t\right)\\
&\leq C_3\left(1\wedge\frac{t}{\|z-z'\|^2}\right)\cdot\frac{(\log\log t)^2}{\log t}\cdot\mathbb{P}(\tau_{z,z'}\leq t)
\end{aligned}$$

$$\leq \quad C_3 \left(1 \wedge \frac{t}{\|z - z'\|^2}\right) \cdot \frac{(\log\log t)^2}{\log t} \cdot e^{-\kappa \cdot \|z'\|^2/t}$$
$$\leq \quad C_4 \left(1 \wedge \frac{t}{\|z\|^2}\right) \left(1 \wedge \frac{t}{\|z'\|^2}\right) \cdot \frac{(\log\log t)^2}{\log t},$$

where we used (2.9) in the third line and considering two cases to obtain the last inequality: $\|z'\| \geq \|z\|/2$, in which case we bound the exponential term by the product and $\|z'\| < \|z\|/2$, in which case using the triangle inequality gives $\|z - z'\| \geq \|z\|/2$. This concludes the proof. □

### 5.2 A second moment estimate

Here we apply the results of the previous section to bound the second moment of the cross-term $\chi$ from the decomposition (1.7). Recall that for any compact sets $A$ and $B$ with $A \cup B \subset \mathcal{B}(0, r)$, we have defined

$$\chi_r(A, B) = 2\pi^2 \, r^2 \cdot \frac{1}{|\partial \mathcal{B}(0, r)|} \int_{\partial \mathcal{B}(0,r)} (\mathbb{P}_z[H_A < H_B < \infty] + \mathbb{P}_z[H_B < H_A < \infty]) \, dz,$$

**Proposition 5.4.** *Let $\beta$ and $\widetilde{\beta}$ be two independent Brownian motions and let $W$ and $\widetilde{W}$ be their corresponding Wiener sausages. Then, there is a constant $C$ such that for any $t > e$, with $r(t) = \sqrt{t} \cdot \log t$,*

$$\mathbb{E}\Big[\chi^2_{r(t)}(W_1[0, t], \widetilde{W}_1[0, t]) \, \mathbf{1}\big(W_1[0, t] \cup \widetilde{W}_1[0, t] \subset \mathcal{B}(0, r(t))\big)\Big] \;\leq\; C \, \frac{t^2 (\log\log t)^8}{(\log t)^4}. \qquad (5.6)$$

**Remark 5.5.** Note that on the event when $W_1[0, t]$ is not included in the ball $\mathcal{B}(0, r(t))$, one can use the deterministic bound $\chi_r(A, B) \leq 4\pi^2 r^2$, which directly follows from the definition (1.8) and holds for any sets $A$ and $B$. Thus by using (2.9), one can see that the upper bound in (5.6) also holds if one removes the indicator function on the left-hand side.

**Proof.** For any compact sets $A$ and $B$ and any $r$ such that $A \cup B \subset \mathcal{B}(0, r)$, we bound $\chi_r(A, B)^2$ as follows. For some constant $C > 0$,

$$\begin{aligned}
\chi_r(A, B)^2 \;&\leq\; C \, \frac{r^4}{|\partial \mathcal{B}(0, r)|^2} \int_{\partial \mathcal{B}(0,r) \times \partial \mathcal{B}(0,r)} \Big(\mathbb{P}_{z,z'}(H_A < H_B < \infty, \, \widetilde{H}_A < \widetilde{H}_B < \infty) \\
+ \quad & \mathbb{P}_{z,z'}(H_B < H_A < \infty, \, \widetilde{H}_B < \widetilde{H}_A < \infty) + \mathbb{P}_{z,z'}(H_A < H_B < \infty, \, \widetilde{H}_B < \widetilde{H}_A < \infty) \\
+ \quad & \mathbb{P}_{z,z'}(H_B < H_A < \infty, \, \widetilde{H}_A < \widetilde{H}_B < \infty)\Big) \, dz \, dz',
\end{aligned} \qquad (5.7)$$

where $H$ and $\widetilde{H}$ refer to the hitting times of two independent Brownian motions $\gamma$ and $\widetilde{\gamma}$ starting respectively from $z$ and $z'$ in $\partial \mathcal{B}(0, r)$. To simplify notation, let $A = W_1[0, t]$, $B = \widetilde{W}_1[0, t]$, and $r = r(t)$. Also, with a slight abuse of notation, in the lines below we let $\mathbb{P}_{z,z'}$ be the law of $\gamma$ and $\widetilde{\gamma}$ conditionally on $W_1[0, t]$ and $\widetilde{W}_1[0, t]$. Then by using (2.8), we obtain

$$\begin{aligned}
&\mathbb{P}_{z,z'}(H_A < H_B < \infty, \, \widetilde{H}_A < \widetilde{H}_B < \infty) \\
&= \mathbb{P}_{z,z'} \left(H_A < H_B < \infty, \widetilde{H}_A < \widetilde{H}_B < \infty, H_{\mathcal{B}(0, \frac{\sqrt{t}}{(\log t)^3})} = \infty, \widetilde{H}_{\mathcal{B}(0, \frac{\sqrt{t}}{(\log t)^3})} = \infty\right) + \mathcal{O}\left(\frac{1}{(\log t)^8}\right).
\end{aligned}$$

Now, to bound the probability on the right-hand side, we use the Markov property at times $H_A$ and $\widetilde{H}_A$ for $\gamma$ and $\widetilde{\gamma}$ respectively. We then have using Proposition 1.5 twice, and for some constant $C$,

$$
\begin{aligned}
&\mathbb{P}_{z,z'}\left(H_A < H_B < \infty, \widetilde{H}_A < \widetilde{H}_B < \infty, H_{\mathcal{B}(0,\frac{\sqrt{t}}{(\log t)^3})} = \infty, \widetilde{H}_{\mathcal{B}(0,\frac{\sqrt{t}}{(\log t)^3})} = \infty\right) \\
&\le \mathbb{P}_{z,z'}(H_A < H_B < \infty,\ \widetilde{H}_A < \widetilde{H}_B < \infty, \|\gamma(H_A)\| \ge \frac{\sqrt{t}}{(\log t)^3}, \|\widetilde{\gamma}(\widetilde{H}_A)\| \ge \frac{\sqrt{t}}{(\log t)^3}) \\
&\le C\,\mathbb{P}_{z,z'}\left(H_A < \infty, \widetilde{H}_A < \infty\right) \frac{(\log\log t)^4}{(\log t)^2} + \mathcal{O}\big((\log t)^{-8}\big) \qquad (5.8) \\
&\le C\left(1 \wedge \frac{t}{\|z'\|^2}\right)\cdot\left(1 \wedge \frac{t}{\|z\|^2}\right)\frac{(\log\log t)^8}{(\log t)^4} + \mathcal{O}\big((\log t)^{-8}\big) = \mathcal{O}\left(\frac{(\log\log t)^8}{(\log t)^8}\right).
\end{aligned}
$$

Note that to apply Proposition 1.5 at the third line above, one also need the hypothesis that $\|\gamma(H_A)\|$ and $\|\widetilde{\gamma}(\widetilde{H}_A)\|$ are not larger than $\sqrt{t}(\log t)$ for instance. But these events have negligible probability by (2.9), so one can indeed apply the proposition. By symmetry, we get as well

$$
\mathbb{P}_{z,z'}(H_B < H_A < \infty,\ \widetilde{H}_B < \widetilde{H}_A < \infty) = \mathcal{O}\left(\frac{(\log\log t)^8}{(\log t)^8}\right). \qquad (5.9)
$$

The last two terms in (5.7) can be bounded as follows. One can first condition on $A = W_1[0,t]$ and $B = \widetilde{W}_1[0,t]$, and then using the inequality $ab \le a^2 + b^2$ for $a, b > 0$, together with (5.8) and (5.9), this gives

$$
\begin{aligned}
\mathbb{P}_{z,z'}(H_A < H_B < \infty, \widetilde{H}_B < \widetilde{H}_A < \infty) \le\ & \mathbb{P}_{z,z}(H_A < H_B < \infty,\ \widetilde{H}_A < \widetilde{H}_B < \infty) \\
& + \mathbb{P}_{z',z'}(H_B < H_A < \infty,\ \widetilde{H}_B < \widetilde{H}_A < \infty) = \mathcal{O}\left(\frac{(\log\log t)^8}{(\log t)^8}\right).
\end{aligned} \qquad (5.10)
$$

By symmetry it also gives

$$
\mathbb{P}_{z,z'}\left(H_{\widetilde{W}_1[0,t]} < H_{W_1[0,t]} < \infty,\ \widetilde{H}_{W_1[0,t]} < \widetilde{H}_{\widetilde{W}_1[0,t]} < \infty\right) = \mathcal{O}\left(\frac{(\log\log t)^8}{(\log t)^8}\right). \qquad (5.11)
$$

Then the proof follows from (5.7), (5.8), (5.9), (5.10), and (5.11). □

# 6 Proof of Theorem 1.1

The proof of the strong law of large number has four elementary steps: (i) the representation formula (2.13) of the capacity of the sausage in terms of a probability of intersection of two sausages, (ii) a decomposition formula as we divide the time period into two equal periods, and iterate the latter steps enough times (iii) an estimate of the variance of dominant terms of the decomposition, (iv) Borel-Cantelli's Lemma allows us to conclude along a subsequence, and the monotony of the capacity which yields the asymptotics along all sequence.

Since all the technicalities have been dealt before, we present a streamlined proof. We only give the proof when the radius of the sausage is equal to one, as the same proof applies for any radius.

**The decomposition.** We let $r = r(t) = \sqrt{t} \cdot \log t$. When dealing with the random set $W_1[0,t]$, (1.7) holds only on the event $\{W_1[0,t] \subset \mathcal{B}(0,r)\}$, and yields

$$\operatorname{Cap}\left(W_1[0,t]\right) = \operatorname{Cap}\left(W_1\left[0,\frac{t}{2}\right]\right) + \operatorname{Cap}\left(W_1\left[\frac{t}{2},t\right]\right) - \chi_r\left(W_1\left[0,\frac{t}{2}\right], W_1\left[\frac{t}{2},t\right]\right) - \varepsilon_r\left(W_1\left[0,\frac{t}{2}\right], W_1\left[\frac{t}{2},t\right]\right).$$

What is crucial here is that $\operatorname{Cap}(W_1[0,\frac{t}{2}])$ and $\operatorname{Cap}(W_1[\frac{t}{2},t])$ are independent. We iterate the previous decomposition $L$ times and center it, to obtain (with the notation $\overline{X} = X - \mathbb{E}[X]$), on the event $\{W_1[0,t] \subset \mathcal{B}(0,r)\}$,

$$\overline{\operatorname{Cap}(W_1[0,t])} = \overline{S(t,L)} - \overline{\Xi(t,L,r)} - \overline{\Upsilon(t,L,r)}, \tag{6.1}$$

where $S(t,L)$ is a sum of $2^L$ i.i.d. terms distributed as $\operatorname{Cap}(W_1[0,t/2^L])$, where

$$\Xi(t,L,r) = \sum_{\ell=1}^{L} \sum_{i=1}^{2^{\ell-1}} \chi_r\left(W_1\left[\frac{2i-2}{2^\ell}t, \frac{2i-1}{2^\ell}t\right], W_1\left[\frac{2i-1}{2^\ell}t, \frac{2i}{2^\ell}t\right]\right), \tag{6.2}$$

and

$$\Upsilon(t,L,r) = \sum_{\ell=1}^{L} \sum_{i=1}^{2^{\ell-1}} \varepsilon_r\left(W_1\left[\frac{2i-2}{2^\ell}t, \frac{2i-1}{2^\ell}t\right], W_1\left[\frac{2i-1}{2^\ell}t, \frac{2i}{2^\ell}t\right]\right). \tag{6.3}$$

In both (6.2) and (6.3), the second sum (with $\ell$ fixed) is made of independent terms.

**Variance Estimates.** We choose $L$ such that $(\log t)^4 \leq 2^L \leq 2(\log t)^4$, so that $L$ is of order $\log\log t$. Let now $\varepsilon > 0$ be fixed. By (2.9) and Chebychev's inequality, for $t$ large enough,

$$\begin{aligned}
\mathbb{P}\left(|\overline{\operatorname{Cap}(W_1[0,t])}| > \varepsilon \frac{t}{\log t}\right) &\leq \mathbb{P}(W_1[0,t] \not\subset \mathcal{B}(0,r)) + \mathbb{P}\left(|\overline{\Upsilon(t,L,r)}| > \frac{\varepsilon}{2}\frac{t}{\log t}\right) \\
&\quad + \mathbb{P}\left(|\overline{S(t,L)} - \overline{\Xi(t,L,r)}| > \frac{\varepsilon}{2}\frac{t}{\log t}\right) \\
&\leq e^{-c(\log t)^2} + \mathbb{P}\left(|\overline{\Upsilon(t,L,r)}| > \frac{\varepsilon}{2}\frac{t}{\log t}\right) \\
&\quad + 8(\log t)^2 \frac{\operatorname{var}(S(t,L)) + \operatorname{var}(\Xi(t,L,r))}{\varepsilon^2 t^2}.
\end{aligned} \tag{6.4}$$

Then we use the triangle inequality for the $L^2$-norm and the Cauchy-Schwarz inequality, as well as Proposition 5.4 (see also (5.5)), to obtain

$$\operatorname{var}(\Xi(t,L,r)) \leq CL \cdot \sum_{\ell=1}^{L} 2^{\ell-1} \frac{t^2 \cdot (\log\log t)^8}{2^{2\ell}(\log t)^4} \leq Ct^2 \cdot \frac{(\log\log t)^9}{(\log t)^4}. \tag{6.5}$$

To deal with $\operatorname{var}(S(t,L))$, we can use Proposition 4.1 which gives a constant $C > 0$, such that for any $t \geq 2$

$$\mathbb{E}\left[\operatorname{Cap}(W_1[0,t])^2\right] \leq C \frac{t^2}{(\log t)^2},$$

Thus there exists a constant $C' > 0$, such that for $t$ large enough,

$$\operatorname{var}(S_L(t)) \leq C' \, 2^L \frac{(t/2^L)^2}{\log^2(t/2^L)} \leq 2C' \frac{t^2}{(\log t)^6}. \tag{6.6}$$

The term $\Upsilon$ is controlled by invoking Lemma 2.2, and using that $\varepsilon_r(A,B) \leq \mathrm{Cap}(A\cap B)$. Since it is the sum of at most $L2^L$ such terms, we deduce

$$\mathrm{var}(\Upsilon(t,L,r)) \;\leq\; \mathbb{E}\big[\Upsilon(t,L,r)^2\big] \;=\; \mathcal{O}(L2^L(\log t)^2) \;=\; \mathcal{O}((\log t)^{10}), \tag{6.7}$$

so that

$$\mathbb{P}\left(\left|\overline{\Upsilon(t,L,r)}\right| > \frac{\varepsilon}{2}\frac{t}{\log t}\right) \;=\; \mathcal{O}\left(\frac{(\log t)^{12}}{t^2}\right). \tag{6.8}$$

Plugging (6.5) (6.6) and (6.8) into (6.4), we obtain

$$\mathbb{P}\left(|\mathrm{Cap}(W_1[0,t]) - \mathbb{E}[\mathrm{Cap}(W_1[0,t])]| \geq \varepsilon\frac{t}{\log t}\right) \;=\; \mathcal{O}\left(\frac{(\log\log t)^9}{(\log t)^2}\right).$$

**From Subsequences to SLLN.** Consider the sequence $a_n = \exp(n^{3/4})$, satisfying that $a_{n+1}-a_n$ goes to infinity but $a_{n+1}-a_n = o(a_n)$. Since the previous bound holds for all $\varepsilon > 0$, by using Borel-Cantelli's lemma and Proposition 3.1, we deduce that a.s.

$$\lim_{n\to\infty} \frac{\mathrm{Cap}(W_1[0,a_n])}{\mathbb{E}[\mathrm{Cap}(W_1[0,a_n])]} \;=\; 1. \tag{6.9}$$

Let now $t > 0$, and choose $n = n(t) > 0$, so that $a_n \leq t < a_{n+1}$. Using that the map $t \mapsto \mathrm{Cap}(W_1[0,t])$ is a.s. nondecreasing (since for any sets $A \subset B$, one has $\mathrm{Cap}(A) \leq \mathrm{Cap}(B)$), we can write

$$\frac{\mathrm{Cap}(W_1[0,a_n])}{\mathbb{E}[\mathrm{Cap}(W_1[0,a_{n+1}])]} \leq \frac{\mathrm{Cap}(W_1[0,t])}{\mathbb{E}[\mathrm{Cap}(W_1[0,t])]} \leq \frac{\mathrm{Cap}(W_1[0,a_{n+1}])}{\mathbb{E}[\mathrm{Cap}(W_1[0,a_n])]}. \tag{6.10}$$

Moreover, applying Proposition 3.1 again gives

$$\mathbb{E}[\mathrm{Cap}(W_1[a_n,a_{n+1}])] \;= \mathbb{E}[\mathrm{Cap}(W_1[0,a_{n+1}-a_n])] = \mathcal{O}\left(\frac{a_{n+1}-a_n}{\log(a_{n+1}-a_n)}\right) = \; o\left(\frac{a_n}{\log a_n}\right).$$

Then using that for any sets $A$ and $B$, one has $\mathrm{Cap}(A) \leq \mathrm{Cap}(A\cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B)$, we deduce that

$$\lim_{n\to\infty} \frac{\mathbb{E}[\mathrm{Cap}(W_1[0,a_{n+1}])]}{\mathbb{E}[\mathrm{Cap}(W_1[0,a_n])]} = 1,$$

which, together with (6.9) and (6.10), proves the almost sure convergence.

The convergence in $L^p$ follows from the boundedness result proved in Section 4, see Remark 4.2. □

Finally we note that the bound on the variance (1.6) follows from (6.1), (6.5), (6.6) and (6.7).


## Acknowledgements

We warmly thank the referee for his/her very careful reading and insightful comments, which greatly helped improve, clarify and correct the paper.

## References


[1] M. Aizenman. The intersection of Brownian paths as a case study of a renormalization group method for quantum field theory. *Comm. Math. Phys.*, 97(1-2):91–110, 1985.

[2] S. Albeverio and X. Y. Zhou. Intersections of random walks and Wiener sausages in four dimensions. *Acta Appl. Math.*, 45(2):195–237, 1996.

[3] A. Asselah, B. Schapira, and P. Sousi. Capacity of the range of random walk on $\mathbb{Z}^d$. *Trans. Amer. Math. Soc.* to appear.

[4] A. Asselah, B. Schapira, and P. Sousi. Capacity of the range of random walk on $\mathbb{Z}^4$. 2016. arXiv:1611.04567.

[5] F. Chung and L. Lu. Concentration inequalities and martingale inequalities: a survey. *Internet Math.*, 3(1):79–127, 2006.

[6] A. Dvoretzky, P. Erdös, and S. Kakutani. Double points of paths of Brownian motion in $n$-space. *Acta Sci. Math. Szeged*, 12(Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars B):75–81, 1950.

[7] P. Erdös and S. J. Taylor. Some intersection properties of random walk paths. *Acta Math. Acad. Sci. Hungar.*, 11:231–248, 1960.

[8] D. Erhard and J. Poisat. Asymptotics of the critical time in Wiener sausage percolation with a small radius. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(1):417–445, 2016.

[9] R. K. Getoor. Some asymptotic formulas involving capacity. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 4:248–252 (1965), 1965.

[10] D. Khoshnevisan. Intersections of Brownian motions. *Expo. Math.*, 21(2):97–114, 2003.

[11] G. F. Lawler. The probability of intersection of independent random walks in four dimensions. *Comm. Math. Phys.*, 86(4):539–554, 1982.

[12] G. F. Lawler. *Intersections of random walks.* Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1991.

[13] J.-F. Le Gall. Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection. *Comm. Math. Phys.*, 104(3):471–507, 1986.

[14] J.-F. Le Gall. Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.*, 14(4):1219–1244, 1986.

[15] J.-F. Le Gall. Fluctuation results for the Wiener sausage. *Ann. Probab.*, 16(3):991–1018, 1988.

[16] J.-F. Le Gall. Some properties of planar Brownian motion. In *École d'Été de Probabilités de Saint-Flour XX—1990*, volume 1527 of *Lecture Notes in Math.*, pages 111–235. Springer, Berlin, 1992.

[17] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, second edition, 2001.

[18] P. Mörters and Y. Peres. *Brownian Motion.* Cambridge University Press, 2010.

[19] R. Pemantle, Y. Peres, and J. W. Shapiro. The trace of spatial Brownian motion is capacity-equivalent to the unit square. *Probab. Theory Related Fields*, 106(3):379–399, 1996.

[20] Y. Peres. Intersection-equivalence of Brownian paths and certain branching processes. *Comm. Math. Phys.*, 177(2):417–434, 1996.

[21] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[22] M. van den Berg, E. Bolthausen, and F. den Hollander. Torsional rigidity for regions with a Brownian boundary. 2016. arXiv:1604.07007.