Anisotropic cyclic cosmologies

Chandrima Ganguly

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

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“The story so far: In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.”

-Douglas Adams, *The Restaurant at the End of the Universe*
Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

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Abstract

Standard models of cosmology use inflation as a mechanism to resolve the isotropy and homogeneity problem of the universe as well as the flatness problem. However, due to various well known problems with the inflationary paradigm, there has been an ongoing search for alternatives. Perhaps the most famous among these are the cyclic universe scenarios which incorporate bounces. As these scenarios have a contracting phase in the evolution of the universe, anisotropies and inhomogeneities would be expected to blow up on approach to the bounce. Thus, it is reasonable to ask whether the problems of homogeneity and isotropy can still be resolved in these scenarios. In this thesis, I will focus on the problem of the resolution of the isotropy problem.

I begin with a brief review of anisotropic, spatially homogeneous geometries of cosmological interest. Next, I review the existing literature on bouncing cosmologies, and discuss the mechanism of bounce studied in previously proposed models, as well as their theoretical and observational advantages and disadvantages.

I then discuss the process of isotropisation in the contracting phase of each bounce. In this phase of the evolution, the mechanism of ekpyrosis is used in most cosmological scenarios which incorporate a contracting phase to mitigate the problem of anisotropies blowing up on approaching the bounce. I start by studying anisotropic universes and I then examine the effect of the addition of ultra-stiff anisotropic pressures on the ekpyrotic phase.

I then consider evolving such anisotropic universes through several cycles with increasing expansion maxima at each successive bounce. This eventually leads to flatness in the isotropic case. My aim is to see if the resolution of the flatness problem also leads to a simultaneous resolution of the isotropy problem.

In the next chapter, I consider the effect of non comoving velocities on the shape of this anisotropic bouncing universe.

In the final section of my thesis, I consider anisotropic cosmological models within the context of canonical quantum cosmology and investigate the quantum behaviour of anisotropies.
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Chapter 1

Introduction

The standard model of cosmology has been subjected to detailed scrutiny by recent WMAP and Planck mission data [117, 80, 108–110]. It predicts an almost isotropic, homogeneous and flat expanding universe, an approximately scale-invariant inhomogeneity spectrum with some level of statistical non-gaussianity, and observational parameters linked by an underlying inflationary model for its very early history [102]. This inflationary model requires an early period of accelerated expansion to account for the horizon and flatness problems, and to generate density perturbations that seed the formation of galaxies. Despite the success of the inflationary paradigm, it has various theoretical inconsistencies. An ever-expanding universe as postulated in Big Bang cosmology, when extrapolated backwards must end in a singularity. Inflation says little about this and only pushes this singularity further back into the past. Large field inflation suffers from the $\eta$ problem, which essentially says that the inflaton would need to travel distances greater than the Planck mass $M_{Pl}$ in field space, giving rise to non-renormalisable quantum corrections in its action. This would effectively spoil inflation in the absence of fine-tuning. Small field inflation does not suffer from this drawback, but these models also require fine tuning, for example to account for the amplitude of the observed power spectrum. The question of the exit from inflation is still contested. Almost all inflationary models give rise to eternal inflation and the subsequent lack of a definition of a unique measure might lead to loss of predictability of results. Inflation also does not provide a satisfactory theory of initial conditions or explain why the inflaton starts out so high in the potential.

Problems such as these have prompted the search for alternatives to inflation or natural initial conditions that lead to inflation. The philosophy behind these searches is that although inflation is a very successful theory, it is important to search for alternative theories which can provide similar predictions to inflation, yet which might be distinguished by some decisive observations. If no such alternative theories can be found, then inflation is a more credible
theory of the early universe. One of the oldest alternatives to inflation is that of a non-singular bounce.

The existence of a non-singular bounce which facilitates the transition from an initially contracting universe to an expanding one was first hypothesized in general-relativistic cosmology by Tolman [125], and was updated to include more general aspects of general relativistic cosmology and the presence of a cosmological constant by Barrow and Dąbrowski [17], also see the review[28]. It also regained popularity in the context of pre-big-bang scenarios [64], which although not successful, led to developments in theories which could possess a non-singular bounce. Early efforts in this direction were made by Steinhardt and Turok and their collaborators [53, 79, 119, 120, 131, 84]. These are usually produced by the addition to standard cosmology of an effective field which violates the null energy condition (NEC). For example, cosmologies with ghost condensates or Galilean genesis take this approach [50]. This has also been used effectively in quantum editions of cosmology, especially in loop quantum cosmology [114], in theories involving canonical quantization of gravity [43, 1], or classical theories with varying constants [21, 23] and ghost fields [25].

One of the most important attributes of any viable model of the early universe must be its ability to reproduce observations of the current day universe. This brings up the question of whether alternatives to inflation are able to solve the isotropy and homogeneity problem. The problem of isotropising a contracting universe is an old one, as anisotropies blow up in the contracting phase with decreasing volume. The behaviour of anisotropic models near a singularity have been studied historically. One such study is the work of Belinsky, Khalatnikov and Lifshitz, regarding the characterisation of a ‘generic’ singularity. This work involved the study of Bianchi VIII and IX models, which are spatially homogeneous, but anisotropic models. The justification in the BKL work for disregarding inhomogeneities was the so-called assumption of ‘ultra-locality’, which says that spatial gradients become rapidly irrelevant compared to time derivatives in a contracting universe on approach to the bounce. BKL showed in their work that such universes on approach to the singularity ended up in a highly anisotropic (Kasner) state, with the scale factors undergoing infinite oscillations, and jumping from one Kasner epoch to the next. This chaotic behaviour, which is an attractor for these models, is often called the ‘Mixmaster’ chaotic attractor in the literature.

With respect to isotropy, the main question that faces models which introduce contraction, then, is of how to avoid the Mixmaster behaviour in cases where the anisotropy is not confined to just expansion anisotropy but also anisotropy in the 3-curvature. One proposal that has been put forward by the ekpyrotic model is to introduce a scalar field with effective equation of state, such that this field grows much faster than the anisotropies on approach to the
contracting phase. This has also been claimed to mitigate the chaotic Mixmaster oscillations, and avoid the BKL highly anisotropic singularity.

This work explores whether the most general forms of anisotropy can be mitigated by this ekpyrosis mechanism. So far, ekpyrosis is able to dominate over expansion anisotropies, because the effective equation of state of the ekpyrotic field is larger than the effective equation of state of these anisotropies. In the case of anisotropic 3-curvature, the infinite chaotic oscillations proceed in eras which have as their end and starting points Kasner-like behaviour. This consists of only expansion anisotropy and is considered to be taken care of by ekpyrosis. Moreover, in the presence of a stiff field, the oscillating behaviour, characteristic of these cosmologies ceases to give rise to an asymptotic power law solution. However, these works have not explored the possibility of anisotropic stress being present in the system; especially in the ekpyrotic field. This would be the case of anisotropic pressures which were ultra-stiff on average. At first glance, it may seem that the average ultra-stiff equation of state should be enough to allow ekpyrosis to work and dominate over the anisotropies. However, anisotropic stress sources the shear tensor directly, and this causes the anisotropies to grow and dominate over the fields governing the mechanism of the bounce. Thus, as we shall see, the bounce itself will not take place, if the universe becomes too anisotropic. On the question of extending these results to inhomogeneities, the standard assumption made in [31] is that the time derivatives are much larger than the spatial gradients of the metric functions in the contraction towards a singularity, and hence can be neglected.

This thesis also shows, by explicitly solving the Einstein equations, what the shape of an anisotropic, closed bouncing universe is. It is seen that increasing the entropy of the constituents results in a corresponding increase in volume of the universe as in the isotropic case. However, isotropisation does not take place in the absence of ekpyrosis, with increasing volume, and neither is there a corresponding approach to flatness. I also consider the conditions under which this bouncing behaviour can cease: that is, when a cosmological constant is added, and the behaviour asymptotes to de Sitter expansion. To complete the analysis of the behaviour of this closed, anisotropic, bouncing universe, I include vorticity in addition to shear and anisotropic 3-curvature. Physically this means that the matter content of the model is not comoving with respect to the fixed triad vectors. Respecting the conservation laws of conservation of angular momentum, we see that an increase in entropy leads to a corresponding decrease in the magnitude of the 3-velocities in the tetrad frame.

Finally, the question of the effect of anisotropies on the probability of the universe to tunnel into existence is investigated. If it is highly unlikely for the universe to come into existence in a very anisotropic state, it may be that the difficulties that bouncing cosmologies face to mitigate these anisotropies are not such a serious impediment after all. The canonical
quantum cosmology analogy of tunnelling through a potential barrier is used. Thus cases of anisotropic universes, flat and closed, which admit such tunnelling barriers are studied. The Wheeler de Witt equation is solved in the case of small and large anisotropies, and the corresponding tunnelling probability is calculated.
Chapter 2

Review of bouncing models

In this chapter, I review some aspects of the physics behind bouncing cosmologies. These include, the basic mechanism and required energy conditions for a bounce to take place, how the basic problems of modern cosmology, including the isotropy, horizon and flatness problems, are tackled in the context of bouncing cosmology. It also includes a brief overview of some of the models which incorporate a bounce. This list is by no means exhaustive; however, non-singular bouncing models and ekpyrotic models are discussed in more detail than other models as they are directly relevant to the remainder of the thesis.

2.1 Mechanisms of the bounce

In order to get a bounce the Hubble rate $H$ coming out of the contracting phase with a negative value must reach a positive value during the expanding phase[28]. In order to do this, the time derivative of the Hubble rate $\dot{H}$ must be positive. The evolution equation for $\dot{H}$ is given by

$$\dot{H} = \frac{\mathcal{K}}{a^2} - \frac{1}{2}(\rho + p)$$  \hspace{1cm} (2.1)

where the curvature parameter $\mathcal{K} = -1, 0, 1$ depending on whether it describes an open, flat or closed universe. In the flat case, where $\mathcal{K} = 0$ we see that we require $\rho + p < 0$ for the positivity of $\dot{H}$ to be satisfied. This is an explicit violation of the Null energy condition (NEC) which often leads to the formation of negative kinetic energy states. For positively curved spaces, i.e., with $\mathcal{K} = 1$, it is possible to create a bounce without the violation of NEC, but with the violation of the Strong energy condition (SEC), which is given by $\rho + 3p > 0$. One of the main theoretical challenges facing bouncing cosmology today is that of creating a bounce scenario which is not plagued by ghosts (a NEC violation). Non-singular bounces which allow for non-zero curvature, and hence do not require a violation of the NEC, will
then have the problem of having curvature left over in the expansion phase and would need a phase of inflation to dilute out the curvature.

There is also the possibility of having a singular bounce but as the question of matching conditions for perturbations across a singularity is not clear, I will not pursue this topic further.

### 2.2 Requirements that a bouncing model must satisfy

In order to be a successful alternative to inflation, the bouncing model must reproduce the standard predictions of inflation. Among these are the resolution of the horizon, flatness and isotropy problems\[28, 34\].

#### 2.2.1 Horizon problem

The uniformity in the temperature of causally disconnected parts of the cosmic microwave background radiation can be resolved in the context of cosmological inflation. At the time of last scattering, the size of the horizon was \[105\],

\[
d_H \approx \frac{1}{H(1 + z_{\text{LSS}})^{3/2}}
\]  

(2.2)

where \(z_{\text{LSS}}\) is the redshift of the last scattering surface. The angular diameter distance is,

\[
da \approx \frac{1}{H(1+z_{\text{LSS}})}
\]  

(2.3)

The horizon of last scattering thus subtends an angle,

\[
\frac{d_H}{da} \approx \frac{1}{(1+z_{\text{LSS}})^{1/2}}
\]  

(2.4)

For \(z_{\text{LSS}} = 1100\), this angle is 1.6 degrees. Thus regions which are separated by an angular distance larger than this would not have been in causal contact, and there is no justification why they should be at the same temperature today. In the bouncing model, we consider a phase of contraction that occurs from \(t_{\text{initial}} < 0\) to \(t_{\text{end}} < 0\), dominated by a perfect fluid with constant equation of state with parameter \(w\), so that the scale factor is given by \(a(t) \propto (-t)^{2/3(1+w)}\). Assuming that the bounce occurs at \(t = 0\), the contribution of the
2.2 Requirements that a bouncing model must satisfy

The contraction phase to the size of the horizon is given by\[^{106}\],

\[
\begin{aligned}
d_{H}^{\text{contraction}} &= \frac{3(1+w)}{1+3w} t_{\text{end}} \left\{ 1 - \left( \frac{t_{\text{initial}}}{t_{\text{end}}} \right)^{(1+3w)/(3(1+w))} \right\} \\
\end{aligned}
\]  \hspace{1cm} (2.5)

If \( w > -1/3 \), this quantity can be made arbitrarily large for \( |t_{\text{initial}}| \gg |t_{\text{end}}| \).

As for fluctuations, the modes that start out deep inside the horizon can remain smaller than the horizon as it is rapidly growing and can grow faster than the scale factor. For a phase of slow contraction, we find that the modes can become super-Hubble as the Hubble radius is rapidly shrinking as the bounce is approached but are still sub-horizon. Thus the mechanism for generating fluctuations remains causal and the horizon problem is resolved.

### 2.2.2 Isotropy problem

Anisotropies grow with decreasing volume and are therefore one of the main theoretical challenges facing bouncing models and their attempts to explain the large scale isotropy of the universe today. To demonstrate the problem, we can look at a flat, anisotropic universe [83, 79]. The metric is as follows,

\[
ds^2 = -dt^2 + a^2(t) \left[ e^{2\theta_x(t)} dx^2 + e^{2\theta_y(t)} dy^2 + e^{2\theta_z(t)} dz^2 \right] \hspace{1cm} (2.6)
\]

such that \( \sum_i \theta_i = \theta_x + \theta_y + \theta_z = 0 \). The Einstein’s equations become,

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \rho + \frac{1}{6} \sum_i \dot{\theta}^2 = \frac{1}{3}(\rho + \rho_\theta) \hspace{1cm} (2.7)
\]

where \( \rho \) is the energy density of any fluid content of the model and \( \rho_\theta \) is the energy density in the anisotropies. The equation governing the rate of change of the Hubble rate is,

\[
\dot{H} = -\frac{1}{2}(\rho + p) - \frac{1}{2} \sum_i \dot{\theta}_i^2 \hspace{1cm} (2.8)
\]

And the evolution equation for the anisotropies is,

\[
\ddot{\theta}_i + 3H \dot{\theta}_i = 0 \hspace{1cm} (2.9)
\]

This coupled with the fact that \( \sum_i \theta_i = 0 \), implies that the energy density of the anisotropies grows as \( \rho_\theta \propto a^{-6} \). For ideal fluids the energy density evolves with scale factor as \( \rho \propto a^{-3(1+w)} \) which for dust or radiation with \( w = 0, 1/3 \) respectively, will not be able to domi-
nate over the primordial classical shear. The effective solution to the problem of growing anisotropies within the context of ekpyrosis is the introduction of a scalar field controlled bounce that acts like a fluid with an ideal, ultra-stiff equation of state ($p \gg \rho$). An example of this can be demonstrated by using a scalar field rolling down a potential hill where the potential is,

$$V(\phi) \approx -V_0 e^{-c\phi}$$  \hspace{1cm} (2.10)

where $c = \sqrt{2/p} \gg 1$, $p \ll 1$ and $V_0 > 1$. The condition for scale invariance is

$$\varepsilon \equiv \left( \frac{V_\phi}{V} \right)^2 \ll 1$$  \hspace{1cm} (2.11)

where $V_\phi = dV/d\phi$. Using the Klein Gordon equation,

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{dV}{d\phi}$$  \hspace{1cm} (2.12)

and the Friedmann equations,

$$3H^2 = V + \frac{1}{2} \dot{\phi}^2$$  \hspace{1cm} (2.13)

$$\dot{H} = -\frac{1}{2} \dot{\phi}^2$$  \hspace{1cm} (2.14)

we find the solution for the scalar field and scale factor to be,

$$a(t) = (-t)^p, \quad H = -\frac{p}{t}$$  \hspace{1cm} (2.15)

and

$$\phi(t) = \sqrt{2p} \ln \left[ -\sqrt{\frac{V_0}{p(1-3p)}} t \right]$$  \hspace{1cm} (2.16)

where $t$ is negative and increases towards zero. The equation of state parameter for the scalar field is given by,

$$w_\phi = \frac{p_\phi}{\rho_\phi} = -1 + \frac{2}{3p}$$  \hspace{1cm} (2.17)

Under the limit of $p \ll 1$, this becomes,

$$w_\phi = \frac{2}{3p} \gg 1$$  \hspace{1cm} (2.18)
2.2 Requirements that a bouncing model must satisfy

Thus the ekpyrotic field energy density would evolve as $a^{-n}$ such that $n \gg 6$ and would dominate over the anisotropies in the contracting phase and on approach to the bounce. This would on the face of things be able to deal with expansion anisotropy. However, the most general, spatially homogeneous and anisotropic universes also involve anisotropic parts of the 3-curvature. This gives rise to the infinite chaotic Mixmaster oscillations on approach to the singularity that are characteristic of the closed anisotropic universe called the Bianchi IX. Indeed, in a contracting phase, anisotropies and inhomogeneities grow in certain regions. However, studies claim that the ratio of proper volume of these regions to the regions which remain smooth and isotropic, when evaluated along time slices of constant mean curvature decay exponentially. These studies, however, do not account for the presence of anisotropic pressures in the ekpyrotic fluid itself. A more detailed discussion of this is included in Chapter 3 and the question of whether ekpyrosis is able to mitigate this problem is dealt with.

2.2.3 Flatness problem

Constraints from the CMB and the Type 1A supernovae data place the current value of $|\Omega_{K}| \ll 1$, where

$$\Omega_{K} = \frac{\mathcal{K}}{a^2 H^2}$$  \hspace{1cm} (2.19)

Since,

$$\frac{d|\Omega_{K}|}{dt} = -2 \mathcal{K} \frac{\dot{a}}{a^3}$$  \hspace{1cm} (2.20)

Inflation solves the flatness problem by including a period of accelerated expansion ($\ddot{a} > 0$ and $\dot{a} > 0$) such that $\frac{d\Omega_{K}}{dt} < 0$ and $|\Omega_{K}|$ evolves to smaller and smaller values. To reproduce the condition that $|\Omega_{K}| \ll 1$, roughly 60 e-folds of inflation are required. In the case of the contracting phase of the bouncing universe model, if we consider decelerated expansion $\ddot{a} < 0$ and $\dot{a} < 0$, then we see that $\frac{d\Omega_{K}}{dt} < 0$ and for a long enough contracting phase, the curvature parameter $|\Omega_{K}|$ can be made as small as desired. Furthermore, we can rewrite the Friedmann constraint in terms of the curvature parameter as follows,

$$1 + \frac{\mathcal{K}}{a^2 H^2} = \frac{\rho}{3H^2}$$  \hspace{1cm} (2.21)

The energy density on the RHS $\rho$ evolves as $a^{-3} H^2$ for dust, $a^{-4} H^2$ for radiation and $a^{-6} H^2$ for shear anisotropies, and hence in all cases is the dominant contribution over the curvature term on approach to the bounce, via a contracting phase. Thus, if the bounce is caused by a null-energy condition (NEC) violating field contained in the $\rho$ term, then the curvature might not blow up to dangerously large values at the bounce. Of course, this might all come at a
cost of instabilities being introduced in the theory by the NEC-violating field causing the bounce. In general, problems arising from non-negligible curvature coming out of the bounce in the expanding phase can be circumvented only by fine-tuning the bounce itself and making sure that it is symmetric. This is philosophically similar, in many respects to the fine-tuning introduced by requiring $\sim 60$ e-folds of inflation to solve the flatness problem.[28]

2.3 Cosmological perturbations

The other aspect of reproducing the predictions of the standard model of cosmology is to be able to produce a scale-invariant, slightly red spectrum on Gaussian perturbations to agree with Planck data. This places severe constraints on the bounce model. If the contracting phase is mediated by a single degree of freedom, then in order for the perturbations generated from the bounce to be scale-invariant, the effective ideal fluid equation of state of the bounce must be $w \sim 0$ or that of dust. Such bounces are called matter bounces although it does not imply the presence of actual dust, but rather that the scalar field in a potential creating such a bounce has the $p \approx 0$ equation of state.[34]

The question of perturbations is difficult in the context of bouncing cosmologies as they are strongly growing in general in the contracting phase. Thus it is not certain whether linear perturbation theory can still be trusted in this regard. In some gauges, such as the comoving gauge and the longitudinal, linear perturbation theory fails, but in others, such as the spatially flat gauge, it is still found to be valid. In the context of non-linear perturbations, one study[131], has found a harmonic gauge where the coordinates follow the equations,

$$\nabla_\alpha \nabla^\alpha x^\mu = 0$$  \hspace{1cm} (2.22)

and

$$g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} = 0$$  \hspace{1cm} (2.23)

The advantage of this gauge is that one can avoid the use of constant mean curvature slices which stop being spacelike in the presence of inhomogeneities in the transition to the bounce, and coordinate singularities are absent when $a$ and $H$ stop being monotonic in the bounce phase. This numerical study showed that the bounce occurs only when

$$\left( \frac{\rho_\sigma}{\rho_\chi} \right) < 1$$  \hspace{1cm} (2.24)
where $\rho_\sigma$ is the energy density in the anisotropies and $\rho_\chi$ is the energy density of the ghost field causing the non-singular bounce. When

$$\left( \frac{\rho_\sigma}{\rho_\chi} \right) > 1 \quad (2.25)$$

the anisotropies are always greater than the ghost field, and as the latter can never dominate over the anisotropies, it is unable to cause the bounce to happen and the universe collapses to a singularity.

Let us now compute the power spectra for scalar perturbations for ekpyrotic models. The perturbed FRW metric in conformal time $dt = a(\eta)d\eta$ is,

$$ds^2 = a^2 \left\{ - (1 + 2A)d\eta^2 + 2B_i d\eta dx^i + [(1 - 2\psi)\delta_{ij} + 2E_{ij}]dx^i dx^j \right\} \quad (2.26)$$

Fixing the gauge freedom requires a choice of gauge and this can be done in the Newtonian gauge by choosing $B = E = 0$ and the gauge invariant Bardeen potentials are $\Phi = A$ and $\Psi = \psi$. Given that the scalar field perturbation is $\delta \phi$, the curvature perturbation on hypersurfaces orthogonal to comoving worldlines is given by,

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}} \delta \phi \quad (2.27)$$

We define the Mukhanov-Sasaki variable as follows,

$$v_k = z\delta \phi_k \quad (2.28)$$

the subscript $k$ is the wavenumber and denotes the relevant quantities in Fourier space. And,

$$z = a\sqrt{\frac{2H}{H^2}} \quad (2.29)$$

The evolution equation for this Mukhanov-Sasaki variable can be written as

$$v_k'' + \left( k^2 - \frac{z''}{z} \right) v_k = 0 \quad (2.30)$$

Other variables that will be used in the other sections are the gauge-invariant Mukhanov-Sasaki variables,

$$Q_I = \delta \phi_I + \frac{\dot{\phi}_I}{H} \psi \quad (2.31)$$
where $I = 1, ..., \mathcal{N}$, $\mathcal{N}$ being the number of fields under consideration. This follows the equation in Fourier space given by,

$$\ddot{Q}_I + 3H\dot{Q}_I + \frac{k^2}{a^2}Q_I + \sum_J \left[ V_{IJ} - \frac{1}{a^3} \left( \frac{a}{H} \dot{\phi}_I \dot{\phi}_J \right) \right] Q_J = 0$$

(2.32)

The curvature perturbation is related to this quantity as

$$\mathcal{R} = \sum_I \left( \frac{\dot{\phi}_I}{\sum_J \dot{\phi}_J} \right) Q_I$$

(2.33)

The comoving curvature perturbation $\zeta$ is given in terms of the Newtonian potential $\Psi$ as,

$$-\zeta = \mathcal{R} + \frac{2\rho}{3(\rho + p)} \left( \frac{k}{aH} \right)^2 \Psi$$

(2.34)

The power spectrum of the curvature fluctuations is given by the Fourier transform of the 2-point function,

$$\langle \mathcal{R}(k_1)\mathcal{R}(k_2) \rangle = (2\pi)^3 \delta(k_1 + k_2)P_R(k_1)$$

(2.35)

where $P_R \sim |\mathcal{R}|^2$. From this, we can define a rescaled, dimensionless power spectrum as,

$$P_{\mathcal{R}} \equiv \frac{k^3}{2\pi^2}P_R(k) = \frac{4\pi k^3}{(2\pi)^3} |\mathcal{R}|^2$$

(2.36)

The power spectrum for the comoving curvature fluctuation $\zeta$ is given by,

$$P_\zeta = \sum_I (\delta N_I)^2 \mathcal{P}_{\delta \phi_I | \Psi = 0}$$

(2.37)

where

$$\mathcal{P}_{\delta \phi_I | \Psi = 0} \equiv \frac{4\pi k^3}{(2\pi)^3} \left| \frac{u_I}{a} \right|^2$$

(2.38)

and $N \equiv \int_0^{t_{end}} Hdt$, $N_I = \partial N / \partial \phi_I$ and $u_I \equiv a\dot{\phi}_I$. The scalar spectral index $n_s$ is given by,

$$n_s - 1 \equiv \frac{d\ln P_\zeta}{d\ln k}$$

(2.39)

The Planck measurement puts this value at $n_s = 0.9603 \pm 0.0073[?]$. Thus blue power spectra are ruled out by these constraints.
The transverse traceless tensor $h_{ij}$ for the tensor modes can be expressed in terms of the polarisation states,

$$h_{ij}(x, \eta) = h_+ (x, \eta) \varepsilon_{ij}^+ + h_X (x, \eta) \varepsilon_{ij}^X$$

(2.40)

where $\varepsilon_{ij}^+$ and $\varepsilon_{ij}^X$ are the two fixed polarisation tensors, and $h_+$ and $h_X$ are the amplitude functions for these modes. Putting this back into the Einstein action, we find that the variable in which the perturbed action has a canonical kinetic energy term is

$$\mu(x, \eta) \equiv a(\eta) h(x, \eta)$$

(2.41)

This has an evolution equation in Fourier space given by,

$$\mu'' + \left( k^2 - \frac{a''}{a} \right) \mu_k = 0$$

(2.42)

The power spectrum for the tensor modes, and this value is constrained by the tensor to scalar ratio $r$, which is given by,

$$r \equiv \frac{P_T}{P_R}$$

(2.43)

Planck constrains this value to be $r < 0.11$.

### 2.4 Ekpyrotic and cyclic models

In this section, I review the simplest ekpyrotic models and the way they suppress the dominant growth of anisotropies in a contracting universe as it approaches the singularity. The ekpyrotic models [79, 53] were originally based on a 5-dimensional braneworld scenario, where the fifth dimension ends at two boundary branes, one of them being our universe. The branes could interact with each other only gravitationally and are attracted by inter-brane tension during the phase of ekpyrosis. Thus, the universe underwent a phase of slow contraction before the collision and re-expansion of the branes, an event which was identified with the hot big bang. The branes were not completely uniform. Quantum fluctuations cause their collision to occur at different times in different places. Thus some parts of the universe end up hotter than others, giving rise to density and temperature fluctuations. This model has been criticized due to fine tuning problems [41], problems regarding the contracting phase seeming to end in a singularity [76], and also because of its failure to produce a scale-invariant spectrum of density fluctuations [5]. To circumvent such problems, some modifications were proposed in terms of a cyclic model [119, 120, 79], with alternating phases of contraction (when the branes approach each other) and expansion (when the
branes are pulled apart and the universe enters a phase of dark-energy domination) occurring simultaneously in a cycle. New versions still attract debate about fundamental issues [27, 41]. The turnaround from contraction to expansion was hypothesized to occur in the form of a non-singular bounce facilitated by a ghost condensation mechanism [36]. Furthermore, it was seen that a scale-invariant density fluctuation power-spectrum could be generated in the new ekpyrotic scenario which considered two-field ekpyrosis [36]. These possibilities have sustained interest in the ekpyrotic scenario as an alternative to inflation for the origin of structure in the universe. These entropic mechanisms predict large non-Gaussianities which are now heavily constrained by the Planck data. Thus, there is now an effort to create compensating mechanisms for this large non-Gaussianity, to match the prediction of $f_{NL} \sim \mathcal{O}(1)$. If primordial gravitational waves were reliably detected [2] then their amplitude could provide a decisive test between the two alternatives (and others [35]).

### 2.4.1 2-field ekpyrosis

The ekpyrotic scenario, despite its various successes predicts a blue spectrum for perturbations in direct tension with the Planck constraints. Hence a new iteration of this model was formed where the ekpyrosis occurred with 2 fields instead of one. The advantage of having 2 fields is that this can generate isocurvature perturbations, which can in turn generate curvature perturbations. If the isocurvature perturbations are scale invariant then the curvature perturbations they generate will also be scale invariant [103, 84].

We consider 2 scalar fields $\phi_I$ where $I = 1, 2$, which are placed in an external, uncoupled potential given by [103, 84],

$$V_I = -\sum V_I e^{-c_I \phi}$$  \hspace{1cm} (2.44)

We can decide to work with variables that are instead defined as,

$$\phi = \frac{c_2 \phi_1 + c_1 \phi_2}{\sqrt{c_1^2 + c_2^2}}$$  \hspace{1cm} (2.45)

and the isocurvature field perpendicular to it, given by,

$$\xi = \frac{c_1 \phi_1 - c_2 \phi_2}{\sqrt{c_1^2 + c_2^2}}$$  \hspace{1cm} (2.46)

The potential in these coordinates becomes,

$$V = -U(\xi) e^{-c\phi}$$  \hspace{1cm} (2.47)
2.4 Ekpyrotic models

where \( c^{-2} = c_1^{-2} + c_2^{-2} \) and

\[
U(\xi) = V_1 e^{-c_1/c_2}c_2\xi + V_2 e^{c_2/c_1}c_2\xi
\] (2.48)

This potential has a maximum at

\[
\xi_0 = \frac{1}{\sqrt{c_1^2 + c_2^2}} \ln \left( \frac{c_1^2 V_1}{c_2^2 V_2} \right)
\] (2.49)

There is an instability at \( \xi = \xi_0 \) which corresponds to the scalar field rolling down the potential slope and inducing power law contraction with effective equation of state \( w \gg 1 \).

This also gives rise to scale-invariant perturbations in \( \xi \) in the contracting phase of the universe. We assume that \( \xi \) stays close to \( \xi_0 \) such that \( \dot{\xi} \approx 0 \) and \( U(\xi) \approx U(\xi_0) = \text{constant} \).

The Friedmann equations lead to a power law contraction with \( a \propto (−t)^{p} \) and \( p = 2/c^2 \ll 1 \). In conformal time, this becomes, \( a \propto (−\eta)^{p/(1−p)} \propto (−\eta)^{1/(\bar{\epsilon}−1)} \), \( (\bar{\epsilon} = 3(1 + w)/2) \). We also obtain

\[ n_\phi - 1 \approx 2 + 4\epsilon \] (2.50)

\[ n_\xi - 1 \approx 4\epsilon \] (2.51)

where \( \epsilon \) is the fast-roll parameter, the smallness of which is satisfied by a steep, nearly exponential potential. This parameter is defined as

\[
\epsilon = \left( \frac{V}{V_\phi} \right)^2 \ll 1
\] (2.52)

In this specific model, the growing of the Bardeen potentials in the Newtonian gauge can be seen through the evolution equation of the potential \( \Phi \) as follows,

\[
\Phi + \frac{2 + p}{t} \Phi + \frac{k^2}{(t/t_0)^{2p}} \Phi = 0
\] (2.53)

The full normalised solution is,

\[
\Phi(\tau) = \frac{p\sqrt{\pi}}{2a(1−p)\sqrt{2k}} \left[ \frac{J_{1+p}}{\sqrt{-k\tau}} + i \frac{Y_{1+p}}{\sqrt{-k\tau}} \right] \] (2.54)
The late time solution of this then becomes,

$$
\Phi \sim k^{(-3/2-p)}t^{1-p} + k^{(-1/2+p)}t^p
$$

This means that the first term will diverge on approach to the bounce, and is, thus, just a suitable gauge to be used in the pre-bounce phase.

### 2.5 Matter bounces

Efforts to generate a scale-invariant spectrum through simple power law evolution led to the creation of the matter bounce scenario [61]. This simply refers to a scalar field dominated contraction phase such that the effective equation of state of the scalar field resembles that of dust. Perturbation theory in dust dominated contracting phases have been widely studied, for example in [95, 33, 56]. To see explicitly how power law contraction can give rise to scale invariance let us consider,

$$
a \propto (-t)^p
$$

we get

$$
n_s - 1 = 3 - 2|\nu|
$$

where

$$
\nu = \frac{1 - 3p}{2(1 - p)}
$$

Thus in order for scale invariance or $n_s \approx 1$, we need $\nu = 3/2$, which corresponds to $p = 2/3$ in the power law solution for the scale factor. This would imply the existence of a contracting phase dominated by a component that follows the dust equation of state. This also predicts a slightly red spectrum for the fluctuations. The matter bounce, however, in its first iteration is not able to resolve the problem of anisotropies or inhomogeneities blowing up on approach to the bounce.

### 2.6 Galileon bounces

Galileons are non-canonical scalar fields, whose Lagrangians obey the symmetry

$$
\phi(x) \rightarrow \phi(x) + c + b^\mu x_\mu
$$

where $\phi$ is the Galileon scalar field and $c$, $b^\mu$ are constants. These arise naturally within massive gravity theories. The bounce in these theories is effected by using Horndeski
2.7 Quantum bounces

Theories which are theories which allow higher derivatives in the action while requiring the equations of motion to be second order. The bounce obtained in this manner maintains the smooth nature of the contracting phase into the expanding phase without giving rise to further instabilities. Galileon bounces or “G-bounces” as they are called are useful in the context of bouncing cosmologies as they can give rise to NEC violation without giving rise to instabilities or ghosts. However, even such theories often take recourse to an ekpyrotic contraction phase to smooth out inhomogeneities and anisotropies [104]. It is worth noting here that several Galileon theories have been ruled out by the recent LIGO 170817 event [60]. It would be interesting to understand what kind of bouncing solutions are still possible among the theories that remain.

2.7 Quantum bounces

Loop quantum cosmology models are homogeneous and isotropic solutions to the non-perturbative, background-independent quantisation of gravity, called Loop Quantum Gravity [32, 115] (LQG is reviewed in[116, 112]). This theory naturally introduces a concept of a ‘minimum length’ [48] and hence claims to resolve the initial singularity. This makes this proposal attractive, although it suffers from various internal inconsistencies, such as it being unclear whether it stands up to current observational tests of Lorentz invariance. At the level of the Friedmann equation, bouncing loop quantum cosmology models introduce a $-\rho^2$ term, where $\rho$ is the energy density [115, 4]. This introduces naturally a field which effectively looks like a NEC-violating field. However, the quantisation of anisotropic loop quantum cosmological models is still debated.

The other side of the coin is canonical quantum cosmology, which attempts to quantise gravity directly, by promoting the metric and its derivatives to operators, and then restricting the resulting superspace, dependent on the symmetries of the problem (such as homogeneity and isotropy) [127, 107]. The resulting Hamiltonian for the system when canonically quantised gives the Wheeler de Witt equation. Bouncing solutions from this have been constructed, either by solving the Wheeler de Witt equation directly, under certain approximations, or by computing the quantum path integral [42]. There have been studies, however, that have shown, that bouncing solutions this way are unstable with respect to quantum decay to zero size [99].
2.7.1 Singular bounces

Bouncing cosmologies in general are attractive due to their avoidance of the singularity which is present in Big Bang cosmology. However, there are models which consider the evolution of universe trajectories through the classical singularity. One such example, is the ‘perfect quantum cosmological’ bounce scenario [65]. In this model, an isotropic, homogeneous FRW universe containing radiation is considered. The Einstein-Hilbert action for this model containing radiation and $M$ conformally coupled scalars is written in a Weyl-invariant manner. In these variables, there is the existence of the radiation dominated classical region, an ‘anti-gravity’ region (named so as the gravitational constant in the Einstein frame becomes negative, when written in terms of the fields introduced to make the action Weyl invariant), followed by another classical radiation dominated region. The path integral, because of conformal symmetry, can be solved exactly to have trajectories pass smoothly through the singularity back out into the classical region. Thus this model is named a ‘quantum bouncing’ model.
Chapter 3

Spatially homogeneous and anisotropic models

This chapter gives a brief overview of the basic features of the Bianchi Class of spatially homogeneous and anisotropic models[128, 57].

3.1 Overview

Homogeneous spaces are characterised by a group of transformations that identify a specified point in that space with any other point in the same space. The spaces of cosmological interest are three dimensional and must hence admit three parameters characterising this group of transformations. A non-Euclidean three-dimensional homogeneous space thus admits a group of motions whose transformations leave invariant three differential forms, which can be expressed as,

\[ e^\alpha_\alpha dx^\alpha \]  

where the latin index refers to three independent vectors and the greek index refers to the components of each such vector. Using these forms then, we can construct a spatial metric that is invariant under this group of motions as follows,

\[ dl^2 = \gamma_{ab}(e^a_\alpha dx^\alpha)(e^b_\beta dx^\beta) \]  

The spatial metric tensor is hence,

\[ \gamma_{\alpha\beta} = \gamma_{ab} e^a_\alpha e^b_\beta \]
The inverse of the one form is given by,

\[ e_a^\alpha e_a^\beta = \delta^\alpha_\beta \quad \text{and} \quad e_a^\alpha e_a^b = \delta^a_b \]  

(3.4)

The invariance of the differential forms under group transformations indicates,

\[ e_a^\alpha (x) dx^\alpha = e_a^\alpha (x') dx'^\alpha \]  

(3.5)

where the \( e_a^\alpha \) is the same function of old and new coordinates, \( x \) and \( x' \). We multiply the equation by \( e_b^\beta (x') \). We also note that \( dx'^\beta = (\partial_\alpha x'^\beta) dx^\alpha \). Thus we obtain,

\[ \partial_\alpha x'^\beta = e_a^\beta (x') e_a^\alpha (x) \]  

(3.6)

These form evolution equations for the quantities \( x'^\beta (x) \) and in order to be integrable, they must satisfy,

\[ \partial_\alpha \partial_\gamma x'^\beta = \partial_\gamma \partial_\alpha x'^\beta \]  

(3.7)

Using the evolution equation for \( x'^\beta (x) \), (3.6), we obtain, after multiplying with \( e_d^\gamma (x)e_c^\gamma (x)e_b^\gamma (x') \),

\[ e_f^\beta \left[ e_c^\delta (x') \partial_\delta x'^\beta (x') - e_d^\delta (x') \partial_\delta x'^\beta (x') \right] = e_b^\beta (x') e_c^\delta (x') \left[ \partial_\delta x'^\beta (x') - \partial_\delta x'^\beta (x') \right] \]  

(3.8)

As \( x \) and \( x' \) are arbitrary, we have,

\[ e_a^\alpha e_b^\beta \left( \partial_\gamma e^c_\delta - \partial_\delta e^c_\beta \right) = C^c_{ab} \]  

(3.9)

The \( C^c_{ab} \) are the structure constants of the Lie algebra of the Killing vector fields of the group of isometries that exist for the Bianchi cosmologies. They are antisymmetric in the lower 2 indices,

\[ C^c_{ab} = -C^c_{ba} \]  

(3.10)

They also obey the Jacobi identity,

\[ C^f_{ab} C_f^d c + C^f_{bc} C_f^d a + C^f_{ca} C_f^d b = 0 \]  

(3.11)

This can be seen by noting that equation (3.8) can be written as,

\[ [X_a, X_b] = X_a X_b - X_b X_a = C^c_{ab} X_c \]  

(3.12)
where $X_a = \epsilon_{(a)}^{\alpha} \frac{\partial}{\partial x^\alpha}$. And the Jacobi identity is also true for these vectors,

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0 \quad (3.13)$$

The 3-index quantity can be expressed in terms of the 2-index quantity $C_{ab}$ as follows,

$$C^c_{ab} = \epsilon_{abd} C^{dc} \quad (3.14)$$

where $\epsilon_{abc} = \epsilon^{abc}$ is the unit anti-symmetric symbol. Using this anti-symmetric symbol, the relation (3.12) can be written as,

$$\epsilon^{abc} X_b X_c = C^{ad} X_d \quad (3.15)$$

and the Jacobi identity can be written as,

$$\epsilon_{bcd} C^{cd} C^{ba} = 0 \quad (3.16)$$

The new tensor $C_{ab}$ can further be broken up into a symmetric and an anti-symmetric part as follows,

$$C^{ab} = n^{ab} + \epsilon^{abc} a_c \quad (3.17)$$

Substituting the structure constants in equation (3.16), we get,

$$n^{ab} a_b = 0 \quad (3.18)$$

Further by linear transformations with constant(real) coefficients given by,

$$\epsilon^{\alpha}_{a} = A^b_{a} \epsilon^{\alpha}_{b} \quad (3.19)$$

the matrix $n_{ab}$ can be diagonalised with its principal, diagonal values $n_i$ where $i = 1, 2, 3$. From the equation (3.18) we can then conclude that the vector $a_b$ can be written as $(a, 0, 0)$. When the parameter $a = 0$ we classify the cosmologies as Class A and Class B otherwise.
3.2 Einstein’s equations

In order for the Einstein’s equations for homogeneous spaces to be functions of time only, all the three dimensional vectors and tensors must be expanded in terms of the triad of reference vectors. Thus we have,

\[ R_{ab} = R_{\alpha\beta} e^\alpha_a e^\beta_b, \quad R_{0\alpha} = R_{0\alpha} e^\alpha_a, \quad u^a = u^\alpha e^\alpha_a \]  \hspace{1cm} (3.20)

We express Einstein’s equations in this tetrad frame as follows,

\[ R^0_0 = -\frac{1}{2} \frac{\partial x^\alpha_\alpha}{\partial t} - \frac{1}{4} x^\beta_\alpha x^\alpha_\beta \]  \hspace{1cm} (3.21)

\[ R^\beta_\alpha = -\left( \frac{1}{2} \sqrt{-g} \right) \frac{\partial}{\partial t} \left( \sqrt{-g} x^\beta_\alpha \right) - P^\beta_\alpha \]  \hspace{1cm} (3.22)

\[ R^0_\alpha = \frac{1}{2} \left( x^\beta_\alpha,\beta - x^\beta_{\beta,\alpha} \right) \]  \hspace{1cm} (3.23)

where \( x^\alpha_\beta \) denotes the three dimensional tensor,

\[ x^\alpha_\beta = \frac{\partial x^\alpha_\beta}{\partial t} \]  \hspace{1cm} (3.24)

We can thus write the Einstein’s equations of motion in this frame as,

\[ \frac{1}{2} \dot{\kappa} + \frac{1}{4} \kappa^b_a \kappa^a_b = T_0^0 - \frac{1}{2} \left( T_0^0 + T_a^a \right) \]  \hspace{1cm} (3.25)

\[ \frac{1}{2} \kappa^b_c C^c_{ba} = T_a^0 \]  \hspace{1cm} (3.26)

\[ \frac{1}{2} \sqrt{\eta} \left( \sqrt{\eta} \kappa^b_a \right) + P_a^b = T_a^b - \frac{1}{2} \delta^b_a \left( T_0^0 + T_a^a \right) \]  \hspace{1cm} (3.27)
3.2 Einstein’s equations

where

\[ P^b_a = \frac{1}{2\eta} \left[ 4C^{bd}C^{ce}\eta_{dc}\eta_{ea} - 2C^{bd}C^{ce}\eta_{da}\eta_{ec} + \delta^b_a \left\{ (C^{ce}\eta_{ec})^2 - 2C^{fd}C^{cd}\eta_{dc}\eta_{ef} \right\} \right] \] (3.28)

For the diagonal metric defined in terms of the tetrad basis \( e^a_\alpha \),

\[ ds^2 = dt^2 - \gamma_{\alpha\beta} e^a_\alpha e^b_\beta dx^\alpha dx^\beta \] (3.29)

where \( \gamma_{\alpha\beta} = \text{diag}[a(t)^2, b(t)^2, c(t)^2] \), we find that the tensor \( x^\beta_\alpha \) is defined as,

\[ x^\beta_\alpha = \left( \frac{a}{\dot{a}} \right) e^\alpha_\beta + \left( \frac{b}{\dot{b}} \right) e^\alpha_\beta + \left( \frac{c}{\dot{c}} \right) e^\alpha_\beta \] (3.30)

The Einstein’s equations (EFEs) for the Bianchi models can be formulated in two main ways.

- the metric approach
- the orthonormal approach

The metric approach involves the metric \( g_{ab} \) with respect to a group-invariant, time-independent frame, being the basic variables. It was then discovered that the Bianchi models admitted an automorphism group, which made the identification of gauge-invariant variables easier. Expressing the Einstein field equations in terms of the these gauge-invariant variables simplifies them and leads to a complete description of the true degrees of freedom of the Bianchi models.

The orthonormal frame method, on the other hand, uses the commutation functions, \( \gamma_{ab} \), associated with a group-invariant orthonormal frame as the basic variables. We choose \( e_0 = n \) and these functions have a geometrical and physical interpretations in terms of the kinematical quantities of \( n \) and the spatial curvature of the group orbits.

The Einstein’s equations can further be analysed quantitatively to gain insight into the behaviour of these models. There are three main methods that are used to analyse these models which are:

- **piecewise approximation methods**: In this method, the cosmological evolution is approximated by a series of successive time periods, during which certain terms in the Einstein equations can be dropped on arguments of sub-leading orders of magnitude and the analysis becomes simpler. See, for example [31, 81]
• **Hamiltonian methods:** These methods were developed by [98] and other collaborators. This involves considering the Einstein field equation system as a time-dependent Hamiltonian system. The time-dependent potential is approximated by moving potential walls which reflect the ‘universe-point’.

• **dynamical systems methods:** The Einstein field equations are expressed as a system of autonomous, coupled, first-order differential equations, and a phase-plane analysis is performed on this. This method will be used in the remainder of the thesis.

All three methods have been used in conjunction with the metric approach. However, the ‘dynamical systems methods’ are the only ones that have been used with the orthonormal frame approach. In order to effectively use this approach, it is convenient to express the EFEs in terms of the expansion-normalised variables. These allow us to examine the behaviour of various physical quantities relative to the expansion of the universe, which is expressed in terms of the expansion scalar $\Theta = u^a_{,a}$, which is related to the Hubble scalar $H$ as,

$$H = \frac{1}{3} \Theta$$

(3.31)

In the first instance, I shall discuss non-tilted models where the 4-velocity, $u$ is orthogonal to the group orbits, or, the matter content is comoving with the triad frame, and is given by,

$$u = \frac{\partial}{\partial t}$$

(3.32)

Let $\{e_a\}$ be a group-invariant orthonormal frame, such that,

$$e_0 = u$$

(3.33)

This allows us to define a projection tensor

$$h_{ab} = g_{ab} + u_a u_b$$

(3.34)

We can write the expansion scalar and the shear tensor combined in the form of an expansion tensor,

$$\Theta_{ab} = \sigma_{ab} + \frac{1}{3} \Theta h_{ab}$$

(3.35)

We use the commutation functions,

$$[e_a, e_b] = \gamma_{ab} e_c$$

(3.36)
3.2 Einstein’s equations

These are constant on the group orbits and are thus functions of just time and are given by,

$$\gamma_{ab} = \{H, \sigma_{\alpha\beta}, n_{\alpha\beta}, \Omega_\alpha, a_\alpha\}$$  \hspace{1cm} (3.37)

The remaining freedom in the choice of orthonormal frame can now be eliminated by specifying the vector $\Omega_\alpha$, and thus the model can be expressed in terms of the state vector $\{H, x\}$ where the vector $x$ is just the reduced set of variables from the original $\gamma_{ab}$. The next step is to actually define the expansion-normalised variables, as a set of dimensionless variables contained in the vector $y$ which is given by,

$$y = \frac{x}{H}$$  \hspace{1cm} (3.38)

One can also describe the evolution of $H(t)$ in terms of the deceleration parameter $q$ which can be more generally defined for the Bianchi models. If one defines a length scale $l$, then the Hubble scalar can be expressed in terms of this as,

$$H = \frac{\dot{l}}{l}$$  \hspace{1cm} (3.39)

and the deceleration parameter $q$ is given by,

$$q = -\frac{\ddot{l}}{l^2}$$  \hspace{1cm} (3.40)

The evolution equation for $H(t)$ can then be written as,

$$\dot{H} = -(1 + q)H^2$$  \hspace{1cm} (3.41)

In order for the models to define a flow, then we can define a dimensionless time coordinate $\tau$ as follows,

$$l = l_0 e^{\tau}$$  \hspace{1cm} (3.42)

where $l + 0$ is the value of the length scale at some arbitrarily defined time instant. For ever-expanding cosmologies, $l$ can lie in the range $0 < l < \infty$ and hence the time coordinate $\tau \to -\infty$ at the initial singularity and $\tau \to \infty$ at late times. The evolution equations are then expressed in terms of this time coordinate as follows,

$$\frac{dt}{d\tau} = \frac{1}{H}$$  \hspace{1cm} (3.43)
The evolution equation for the Hubble scalar can now be written in terms of $\tau$ as

$$\frac{dH}{d\tau} = -(1 + q)H$$

(3.44)

The Raychaudhuri equation is,

$$\dot{H} = -H^2 - \frac{2}{3} \sigma^2 - \frac{1}{6}(3\gamma - 2)\rho$$

(3.45)

and the Friedmann equation is,

$$3H^2 = \sigma^2 - \frac{1}{2} R + \rho$$

(3.46)

where $\rho$ is the energy density of the matter content in the universe, $^3R$ is the spatial curvature of the $t =$constant hypersurfaces, and $\sigma^2$ is the shear scalar. We define these quantities in terms of the expansion normalised variables,

$$\Omega \equiv \frac{\rho}{3H^2}, \quad \Sigma^2 \equiv \frac{\sigma^2}{3H^2}, \quad K \equiv -\frac{^3R}{6H^2}.$$  

(3.47)

The Friedmann constraint is then given in terms of the expansion normalised variables as follows,

$$\Sigma^2 + K + \Omega = 1$$

(3.48)

The continuity equation for the density parameter $\rho$ is given by,

$$\rho = -3\gamma H \rho$$

(3.49)

In terms of the expansion normalised variables, this can be expressed as,

$$\frac{d\Omega}{d\tau} = [2q - (3\gamma - 2)]\Omega$$

(3.50)

### 3.3 Invariant sets and equilibrium points

From the equation (3.50), we see that the Bianchi models corresponding to $\Omega = 0$(vacuum Bianchi models) form an invariant set. This is important, as in qualitative analysis these vacuum states could be asymptotic states for late time models or for describing behaviour near the Big Bang. The $D$-dimensional state space must be compact so that relevant physical quantities do not diverge faster than an appropriate power of $H$ near the Big Bang and they do not go to zero faster than an appropriate power of $H$ at late times.
Using the evolution equation for $H(\tau)$ in terms of $\tau$, we see that it can be integrated to give the following expression,

$$H(\tau) = H_0 \exp \left( - \int (1 + \bar{q}(u))du \right)$$  \hspace{1cm} (3.51)

$H_0$ is arbitrarily specifiable and hence each non-singular orbit of the symmetry group corresponds to a one-parameter family of physical universes which can be related by a rescaling of the commutation functions, and hence of the metric and the orthonormal frame. At an equilibrium point $y^*$ which satisfies $f(y^*) = 0$ where $f(y)$ are the right hand side of the differential equation, for some state vector $y$, the function $q(y^*) = q^*$ is a constant, and hence $H(\tau)$ can be written as,

$$H(\tau) = H_0 \exp (1 + q(y^*)\tau)$$ \hspace{1cm} (3.52)

In terms of the cosmological time $t$,

$$Ht = \frac{1}{1 + q^*}$$ \hspace{1cm} (3.53)

In addition to this, all non-vacuum equilibrium points satisfy,

$$q^* = \frac{1}{2}(3\gamma - 2)$$ \hspace{1cm} (3.54)

And hence, in cosmological time, the Hubble function is given by,

$$H = \frac{2}{3}\gamma^{-1}$$ \hspace{1cm} (3.55)

The self similar sets defined by equilibrium points are of physical significance as they provide an estimate of the asymptotic behaviour of the model as $\tau \to \pm \infty$. For example, if the equilibrium point $y^*$ forms the $\alpha$-limit set of the point $y$, then physically, the self-similar model represented by the point $y^*$ approximates the behaviour of the model $y$ as $\tau \to -\infty$, and will be called asymptotically self-similar into the past. If the model is asymptotically self similar in both the past and the future, it is called heteroclinic as the orbit shall join 2 equilibrium points. In this specific case, solutions for the state vector $(H, x)$ can be found, as follows,

$$H = \frac{1}{1 + q^*}t^{-1} + \mathcal{O}(t^{-2})$$ \hspace{1cm} (3.56)

and

$$x = \frac{y^*}{1 + q^*}t^{-1} + \mathcal{O}(t^{-2})$$ \hspace{1cm} (3.57)
Spatially homogeneous and anisotropic models

<table>
<thead>
<tr>
<th>Type of singularity</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&gt;0</td>
</tr>
<tr>
<td>cigar (cyclical on 1, 2, 3)</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&lt;0</td>
</tr>
<tr>
<td>pancake (cyclical on 1, 2, 3)</td>
<td>&gt;0</td>
<td>=0</td>
<td>=0</td>
</tr>
<tr>
<td>barrel (cyclical on 1, 2, 3)</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>=0</td>
</tr>
</tbody>
</table>

Table 3.1 Classification of singularity types

where $y^*$ is some equilibrium point of $x$ and the terms not including those that are higher order in $t^{-1}$ correspond to the exact asymptotic self similar model. If we now consider such an equilibrium point $y = y^*$ with the corresponding self similar model. Under a constant rotation, we can diagonalise the expansion tensor $\Theta_{\alpha\beta}$ so that we have only the diagonal principal values.

$$\Theta_{\alpha\beta} = \{\Theta_1, \Theta_2, \Theta_3, \}$$

(3.58)

We can write each of these principal values, which physically correspond to expansion rates, in terms of an expansion scale factor in each direction. Let this scale factor be $l_\alpha$, where $\alpha = 1, 2, 3$ corresponds to each spatial direction. Then,

$$\Theta_\alpha = \frac{l_\alpha}{l_\alpha}$$

(3.59)

Note that the above is simply a 3-dimensional version of (3.39) so that $H = (1/3)(\Theta_1 + \Theta_2 + \Theta_3)$. Since the commutation functions are proportional to $t^{-1}$, we can write,

$$p_\alpha = t\Theta_\alpha$$

(3.60)

The solution for the scale factors $l_\alpha$, is then $t^{p_\alpha}$. A singularity occurs when one or more of the $l_\alpha$ vanishes. Depending on the sign of $p_\alpha$ we can classify the nature of the singularities in Table 3.1. The various singularity types can be classified as,

- point singularity: all of the $l_\alpha \to 0$
- cigar singularity: 2 of the $l_\alpha \to 0$ while the third increases without bound
- pancake singularity: 1 of the $l_\alpha \to 0$ while the other 2 approach finite values
- barrel singularity: 2 of the $l_\alpha \to 0$ while the third approaches a finite value
3.4 Bianchi Class A models

As discussed before, we choose a group-invariant orthonormal frame, such that the only non-zero components are,

$$
\sigma_{\alpha\beta} = \text{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33}), \quad \text{diag}(n_1, n_2, n_3)
$$

(3.61)

As $\sigma_{\alpha\beta}$ is trace-free, there are only 2 independent components, which can be expressed as,

$$
\sigma_+ = \frac{1}{2}(\sigma_{22} + \sigma_{33}) \quad \sigma_- = \frac{1}{2\sqrt{3}}(\sigma_{22} - \sigma_{33})
$$

(3.62)

The shear tensor $\sigma_{\alpha\beta}$ and the trace-free Ricci tensor are related by,

$$
\dot{\sigma}_{\alpha\beta} = -3H\sigma_{\alpha\beta} + 2\varepsilon_{\mu\nu}^{(\alpha} \sigma_{\beta)\mu} \Omega_{\nu} - (3)S_{\alpha\beta}
$$

(3.63)

This also means that the trace-free Ricci tensor, $(3)S_{\alpha\beta}$ is also diagonal and the components of this can be written as,

$$
S_+ = \frac{1}{2}(S_{22} + S_{33}) \quad S_- = \frac{1}{2\sqrt{3}}(S_{22} - S_{33})
$$

(3.64)

The Einstein’s equations can thus be written in the diagonal frame as follows,

$$
\dot{H} = -H^2 - \frac{2}{3}\sigma^2 - \frac{1}{6}(\rho + 3p)
$$

(3.65)

$$
\dot{\sigma}_\pm = -3H\sigma_\pm - (3)S_\pm
$$

(3.66)

$$
\dot{n}_1 = (-H - 4\sigma_+)n_1
$$

(3.67)

$$
\dot{n}_2 = (-H + 2\sigma_+ + 2\sqrt{3}\sigma_-)n_2
$$

(3.68)

$$
\dot{n}_3 = (-H + 2\sigma_+ - 2\sqrt{3}\sigma_-)n_3
$$

(3.69)

$$
\dot{\rho} = -3\gamma H\rho
$$

(3.70)

where,

$$
(3)S_+ = \frac{1}{6H^2}[(n_2 - n_3)^2 - n_1(2n_1 - n_2 - n_3)],
$$

(3.71)

$$
(3)S_- = \frac{1}{2\sqrt{3}H^2}(n_3 - n_2)(n_1 - n_2 - n_3).
$$

(3.72)
and the equation of state of the perfect fluid matter content is given by,

\[ p = (\gamma - 1)\rho \]  \hspace{1cm} (3.73)

\( p \) and \( \rho \) being the pressure and energy density of the perfect fluid, respectively. The physical state of the models are described by \((H, x)\), where the state vector \( x \) is given by,

\[ x = (\sigma_+, \sigma_-, n_1, n_2, n_3) \]  \hspace{1cm} (3.74)

As we shall be working in expansion-normalised variables, the state vector that we are interested in becomes,

\[ y = (\Sigma_+, \Sigma_-, N_1, N_2, N_3) \]  \hspace{1cm} (3.75)

where

\[ \Sigma_{\pm} = \frac{\sigma_{\pm}}{H}, \quad N_\alpha = \frac{n_\alpha}{H} \quad \forall \alpha = 1, 2, 3 \]  \hspace{1cm} (3.76)

The expansion normalised Einstein equations then become,

\[ \Sigma_{\pm}' = -(2 - q)\Sigma_{\pm} - S_{\pm} \]  \hspace{1cm} (3.77)

and \( \Sigma^2 = \Sigma_+^2 + \Sigma_-^2 \). Where \( S_{\pm} \) is given by,

\[ S_+ = \frac{1}{6} \left[ (N_2 - N_3)^2 - N_1 (2N_1 - N_2 - N_3) \right] \]  \hspace{1cm} (3.78)

\[ S_- = \frac{1}{2\sqrt{3}} (N_3 - N_2) (N_1 - N_2 - N_3) \]

where the \( N_i \)s are expansion normalised curvature variables \( N_i = n_i/H \) and \( q \) is the deceleration parameter, given in terms of the expansion normalised shear and density variables.

\[ q = 2\Sigma^2 + \frac{1}{2} (3\gamma - 2)\Omega \]  \hspace{1cm} (3.79)

### 3.5 Equilibrium points and their stability

In order to perform stability analysis on the orthonormal frame reduced Einstein’s equations written in expansion normalised variables, we must expand to first order, this system of equations. Each of the equilibrium points correspond to a self-similar solution. The stability of each equilibrium point depends on the sign of the eigenvalues obtained from linearising
3.5 Equilibrium points and their stability

Table 3.2 List of self similar solutions and the corresponding expansion-normalised energy densities. FL in the following stands for the isotropic Friedmann Lemaitre point.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$\Omega$</th>
<th>Self-similar solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>1</td>
<td>flat FL</td>
</tr>
<tr>
<td>$P^\pm_{\alpha}(I)I$</td>
<td>$\frac{3}{16} (6 - \gamma)$</td>
<td>Collins-Stewart(I)</td>
</tr>
<tr>
<td>$P^\pm_{\alpha}(VI0)$</td>
<td>$\frac{3}{4} (2 - \gamma)$</td>
<td>Collins VI0</td>
</tr>
<tr>
<td>$\mathcal{K}$</td>
<td>$0$</td>
<td>Kasner vacuum</td>
</tr>
<tr>
<td>$L^\pm_\alpha$</td>
<td>$0$</td>
<td>Taub flat spacetime</td>
</tr>
<tr>
<td>$R^\pm_\alpha$</td>
<td>$1$</td>
<td>flat FL</td>
</tr>
<tr>
<td>$F^\pm(IIX)$</td>
<td>$\Omega &gt; 1$</td>
<td>closed FL</td>
</tr>
<tr>
<td>$\mathcal{J}$</td>
<td>$0 &lt; \Omega &lt; 1$</td>
<td>Jacobs stiff fluid</td>
</tr>
</tbody>
</table>

the evolution equations around that particular equilibrium point. The self similar solutions are listed in Table 3.2. The respective eigenvalues for each self-similar solution are given by,

- **flat FL**: $\lambda_1 = \lambda_2 = -\frac{3}{2} (2 - \gamma)$, $\lambda_3 = \lambda_4 = \lambda_5 = \frac{1}{2} (3\gamma - 2)$

- **Collins-Stewart(II)**: $\lambda_{1,2} = -\frac{3}{4} (2 - \gamma) \left[ 1 \pm (1 - b^2)^{1/2} \right]$, $\lambda_3 = -\frac{3}{2} (2 - \gamma)$, $\lambda_4 = \lambda_5 = \frac{3}{4} (3\gamma - 2)$ where $b^2 = \frac{(3\gamma - 2)(6 - \gamma)}{2(2 - \gamma)}$

- **Collins VI0**: $\lambda_{1,2} = -\frac{3}{4} (2 - \gamma) \left[ 1 \pm (1 - r^2)^{1/2} \right]$, $\lambda_{3,4} = -\frac{3}{4} (2 - \gamma) \left[ 1 \pm (1 - s^2)^{1/2} \right]$, $\lambda_5 = \frac{3}{4} (3\gamma - 2)$

- **Kasner vacuum**: Circle of equilibrium points given by $\lambda_1 = 6p_1$, $\lambda_2 = 6p_2$, $\lambda_3 = 6p_3$, $\lambda_4 = 3(2 - \gamma)$ and $\lambda_5 = 0$ such that $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$

- **Taub flat spacetime**: line of equilibrium points emanating from the Taub point $T_1$ on $\mathcal{K}$

- **flat FL**: for $\gamma = \frac{2}{3}$ line of equilibrium points emanating from the flat FL equilibrium point

- **closed FL**: for $\gamma = \frac{2}{3}$ line of equilibrium points emanating from the flat FL equilibrium point

- **Jacobs stiff fluid**: disc of equilibrium points forming the interior of the Kasner circle
3.6 Bianchi VIII and IX models

We now focus on the behaviour of the model in the vacuum case on approach to the singularity \cite{29}. For this purpose a new time coordinate is introduced defined by,

\[ dt = -\sqrt{\eta} d\tau \] (3.80)

The equations of motion then become (still in the triad frame chosen)

\[ 2\eta R^b_a = \frac{d}{d\tau} \left( \eta^{bc} \frac{d\eta_{ca}}{d\tau} \right) + 4C^{bd} C^{ce} \eta_{de} \eta_{ea} - 2C^{bd} C^{ce} \eta_{da} \eta_{ec} \] (3.81)

\[ + \delta^b_a \left[ (C^{ce} \eta_{ec})^2 - 2C^{f'd} C^{ce} \eta_{de} \eta_{ef} \right] = 0 \]

\[ 2\eta (R^0_0 - R^a_a) = \frac{1}{2} \eta^{bc} \eta^{af} \frac{d\eta_{ea}}{d\tau} \frac{d\eta_{fb}}{d\tau} - \frac{1}{2} \left( \eta^{ab} \frac{d\eta_{ab}}{d\tau} \right)^2 \] (3.82)

\[ + 2C^{f'd} C^{ce} \eta_{de} \eta_{ef} - (C^{ce} \eta_{ec})^2 = 0 \]

Let us now introduce the matrix \( C \) which has components given by the structure constants \( C^{ab} \) and a matrix which is its square root, \( B \).

\[ B^2 = C, \quad B = \bar{B} \] (3.83)

For Bianchi IX, \( B \) is real because \( C \) is positive definite, but can have complex components for Bianchi VIII, as the determinant of \( C \) is negative. We construct a symmetric matrix \( U \),

\[ U = BgB \] (3.84)

where \( g \) is the metric of the frame we have chosen. The Einstein equations thus become,

\[ \frac{d}{d\tau} \left( U^{-1} \frac{d}{d\tau} U \right) + 4U^2 - 2U Tr(U) + I(Tr(U))^2 - 2ITr(U^2) = 0 \] (3.85)

\[ \frac{1}{2} Tr \left[ \left( U^{-1} \frac{d}{d\tau} U \right)^2 \right] - \frac{1}{2} \left[ Tr \left( U^{-1} \frac{d}{d\tau} U \right) \right]^2 + 2Tr(U^2) - (Tr(U))^2 = 0 \] (3.86)

The matrix \( U \) can be diagonalised by an orthogonal matrix \( O \) as follows,

\[ U = O^T \Gamma O, \quad O^T O = 1 \] (3.87)
where \( \Gamma \) is the diagonal matrix consisting of the scale factors relative to the rotating frame,

\[
\Gamma = \text{diag}(\Gamma_1, \Gamma_2, \Gamma_3)
\]  

(3.88)

The orthogonal matrix \( O \) is given by,

\[
O = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(3.89)

The matrix of angular velocities \( \Omega \) is defined as,

\[
\Omega = \left( \frac{d}{d\tau} O \right) O^{-1} = \begin{pmatrix}
0 & \Omega_3 & -\Omega_2 \\
-\Omega_3 & 0 & \Omega_1 \\
\Omega_2 & -\Omega_1 & 0
\end{pmatrix}
\]  

(3.90)

These angular velocities can be written in terms of the Euler angles of the rotation matrix as follows,

\[
\Omega_1 = \frac{d\phi}{d\tau} \sin \psi \sin \theta + \frac{d\theta}{d\tau} \cos \psi \\
\Omega_2 = \frac{d\phi}{d\tau} \cos \psi \sin \theta - \frac{d\theta}{d\tau} \sin \psi \\
\Omega_3 = \frac{d\phi}{d\tau} \cos \theta + \frac{d\psi}{d\tau}
\]  

(3.91, 3.92, 3.93)

Substituting the form of the matrix \( U \), (3.84) in the matrix Einstein equations, multiplying from the left by \( O \) and from the right by \( \tilde{O} \), and splitting the resulting equation into its symmetric and antisymmetric parts, we get,

\[
\frac{d}{d\tau} (K - \tilde{K}) = 0
\]  

(3.94)

where \( K = U^{-1} \frac{d}{d\tau} U \). On simplifying, this part gives the Euler top equations,

\[
\frac{d}{d\tau} (I_1 \Omega_1) + (I_3 - I_2) |\Omega_2| \Omega_3 = 0
\]  

(3.95)

\[
\frac{d}{d\tau} (I_2 \Omega_2) + (I_1 - I_3) |\Omega_1| \Omega_3 = 0
\]  

(3.96)

\[
\frac{d}{d\tau} (I_1 \Omega_3) + (I_2 - I_1) |\Omega_1| \Omega_2 = 0
\]  

(3.97)
The constraint equation in this notation becomes,

\[ I_1 = \frac{(\Gamma_2 - \Gamma_3)^2}{\Gamma_2 \Gamma_3}, \quad I_2 = \frac{(\Gamma_1 - \Gamma_3)^2}{\Gamma_1 \Gamma_3}, \quad I_3 = \frac{(\Gamma_1 - \Gamma_2)^2}{\Gamma_1 \Gamma_2} \]  

(3.98)

The time evolution of the scale factors are given by,

\[ \frac{d^2 \ln \Gamma_1}{d \tau^2} + \Gamma_1^2 - (\Gamma_2 - \Gamma_3)^2 - \frac{\Gamma_1 \Gamma_2 (\Gamma_1 + \Gamma_2)}{(\Gamma_1 - \Gamma_2)^3} (I_3 \Omega_3)^2 - \frac{\Gamma_1 \Gamma_3 (\Gamma_1 + \Gamma_3)}{(\Gamma_1 - \Gamma_3)^3} (I_2 \Omega_2)^2 = 0 \]  

(3.99)

\[ \frac{d^2 \ln \Gamma_2}{d \tau^2} + \Gamma_2^2 - (\Gamma_1 - \Gamma_3)^2 - \frac{\Gamma_1 \Gamma_2 (\Gamma_1 + \Gamma_2)}{(\Gamma_1 - \Gamma_2)^3} (I_3 \Omega_3)^2 - \frac{\Gamma_2 \Gamma_3 (\Gamma_2 + \Gamma_3)}{(\Gamma_2 - \Gamma_3)^3} (I_1 \Omega_1)^2 = 0 \]  

(3.100)

\[ \frac{d^2 \ln \Gamma_3}{d \tau^2} + \Gamma_3^2 - (\Gamma_1 - \Gamma_2)^2 - \frac{\Gamma_1 \Gamma_3 (\Gamma_1 + \Gamma_3)}{(\Gamma_1 - \Gamma_3)^3} (I_2 \Omega_2)^2 - \frac{\Gamma_2 \Gamma_3 (\Gamma_2 + \Gamma_3)}{(\Gamma_2 - \Gamma_3)^3} (I_1 \Omega_1)^2 = 0 \]  

(3.101)

The symmetric part is,

\[ \mathbf{O} \left[ \frac{d}{d \tau} \left( \mathbf{K} + \mathbf{\tilde{K}} \right) \right] \mathbf{\tilde{O}} + 8 \Gamma^2 - 4 \Gamma Tr(\Gamma) + 2I(Tr(\Gamma))^2 - 4ITr(\Gamma^2) = 0 \]  

(3.102)

The constraint equation in this notation becomes,

\[ \frac{1}{4} Tr(\mathbf{K}^2) - \frac{1}{4} (Tr \mathbf{K})^2 + Tr(\Gamma^2) - \frac{1}{2} (Tr(\Gamma))^2 = 0 \]  

(3.103)

In terms of the scale factors and the angular velocities, the constraint equation becomes,

\[ \frac{1}{4} \left[ \left( \frac{d \ln \Gamma_1}{d \tau} \right)^2 + \left( \frac{d \ln \Gamma_2}{d \tau} \right)^2 + \left( \frac{d \ln \Gamma_3}{d \tau} \right)^2 \right] - \frac{1}{4} \left[ \frac{d \ln \Gamma_1}{d \tau} + \frac{d \ln \Gamma_2}{d \tau} + \frac{d \ln \Gamma_3}{d \tau} \right]^2 \]  

(3.104)

\[ + \frac{1}{2} I_1 \Omega_1^2 + \frac{1}{2} I_2 \Omega_2^2 + \frac{1}{2} I_3 \Omega_3^2 + \frac{1}{2} \left[ \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 - 2 \Gamma_1 \Gamma_2 - 2 \Gamma_1 \Gamma_3 - 2 \Gamma_2 \Gamma_3 \right] = 0 \]

Under a certain redefinition, these become the Einstein field equations for the metric functions for diagonal Bianchi VIII and IX models. This redefinition shall be discussed in the following section. We shall be using these evolution equations extensively in the remainder of the thesis to perform various analyses in the Bianchi IX universe.

### 3.6.1 Models with Fixed Kasner axes

In the vacuum case,

\[ \mathbf{K} - \mathbf{\tilde{K}} = \mathbf{U}^{-1} \left( \frac{d}{d \tau} \mathbf{U} \right) - \left( \frac{d}{d \tau} \mathbf{U} \right) \mathbf{U}^{-1} = 0 \]  

(3.105)
This means that $\Omega$ vanishes for unequal $\Gamma_i$ which denotes the general case. Thus $O$ is a constant, and we set it to unity. Thus there is no rotation of the principal axes of the model and they coincide with the fixed triad basis vectors $e^\alpha_a(x)$ all the way up to the singularity. The equations of motion thus simplify as a result of this. We denote

$$\Gamma_1 = e^{-2\beta^1}, \quad \Gamma_2 = e^{-2\beta^2}, \quad \Gamma_3 = e^{-2\beta^3}$$

and

$$d\tau = e^{-\beta^1+\beta^2+\beta^3} dt$$

In these time coordinates, this gives rise to the diagonal Bianchi VIII and IX equations in empty space,

$$\frac{2}{d\tau^2} \frac{d^2 \beta^1}{d\tau^2} = e^{-4\beta^1} - \left(e^{-2\beta^2} - e^{-2\beta^3}\right)^2 \tag{3.108}$$

$$\frac{2}{d\tau^2} \frac{d^2 \beta^2}{d\tau^2} = e^{-4\beta^2} - \left(e^{-2\beta^3} - e^{-2\beta^1}\right)^2 \tag{3.109}$$

$$\frac{2}{d\tau^2} \frac{d^2 \beta^3}{d\tau^2} = e^{-4\beta^3} - \left(e^{-2\beta^1} - e^{-2\beta^3}\right)^2 \tag{3.110}$$

and

$$\frac{\left(\frac{d\beta^1}{d\tau}\right)^2 + \left(\frac{d\beta^2}{d\tau}\right)^2 + \left(\frac{d\beta^3}{d\tau}\right)^2}{\left(\frac{d\beta^1}{d\tau} + \frac{d\beta^2}{d\tau} + \frac{d\beta^3}{d\tau}\right)^2} = -\frac{1}{2} \left(e^{-4\beta^1} + e^{-4\beta^2} + e^{-4\beta^3} - 2e^{-2\beta^1-2\beta^2} - 2e^{-2\beta^1-2\beta^3} - 2e^{-2\beta^2-2\beta^3}\right) \tag{3.111}$$

The difference between Bianchi VIII and IX arises in the sign of the structure constants. For Bianchi IX we have,

$$C^\alpha_\beta = \text{diag}(1, 1, 1) \tag{3.112}$$

and the metric in the triad as,

$$\eta = \text{diag}(\Gamma_1, \Gamma_2, \Gamma_3) \tag{3.113}$$

For Bianchi VIII, we have,

$$C^\alpha_\beta = \text{diag}(1, 1, -1) \tag{3.114}$$

and the metric in the triad as,

$$\eta = \text{diag}(\Gamma_1, \Gamma_2, -\Gamma_3) \tag{3.115}$$

Thus the difference between the two models arises only in terms like $e^{-2\beta^1-2\beta^2} - e^{-2\beta^1-2\beta^3}$ where they appear with an opposite sign for Type VIII and Type IX. On approach to the
singularity, however, if we use the standard assumption of maximum anisotropy where \( \Gamma_1 \gg \Gamma_2 \gg \Gamma_3 \), then the terms with mixed exponentials, like, \( e^{-2\beta_1} - e^{-2\beta_2} \ll e^{-4\beta_2} \) if \( \Gamma_1 \gg \Gamma_2 \) and so on. Thus near the singularity, these terms drop out and we are left with the equations of motion, for both Type VIII and Type IX,

\[
\begin{align*}
2 \frac{d^2 \beta_1}{d\tau^2} &= e^{-4\beta_1} - \left( e^{-2\beta_2} - e^{-2\beta_3} \right) \\
2 \frac{d^2 \beta_2}{d\tau^2} &= e^{-4\beta_2} - \left( e^{-2\beta_3} - e^{-2\beta_1} \right) \\
2 \frac{d^2 \beta_3}{d\tau^2} &= e^{-4\beta_3} - \left( e^{-2\beta_1} - e^{-2\beta_2} \right)
\end{align*}
\]

and

\[
\begin{align*}
\left( \frac{d \beta_1}{d\tau} \right)^2 + \left( \frac{d \beta_2}{d\tau} \right)^2 + \left( \frac{d \beta_3}{d\tau} \right)^2 - \left( \frac{d \beta_1}{d\tau} + \frac{d \beta_2}{d\tau} + \frac{d \beta_3}{d\tau} \right)^2 = -\frac{1}{2} \left( e^{-4\beta_1} + e^{-4\beta_2} + e^{-4\beta_3} \right)
\end{align*}
\]

**3.6.2 Oscillatory approach to the singularity**

The description of the model above, in terms of the principle values of the matrix \( \Gamma \), is equivalent to that obtained in terms of the scale factors \( a, b \) and \( c \), which are given by the spacetime metric written in the relevant triad as follows,

\[
g_{\alpha\beta} = \eta_{ab} e_a^\alpha e_b^\beta, \quad \eta_{ab} = \text{diag}(a^2, b^2, c^2)
\]

The Einstein’s equations on approach to the singularity in terms of these scale factors are given by,

\[
\begin{align*}
\frac{dabc}{abc} &= -\frac{1}{a^2b^2c^2} \left( a^4\lambda^2 - b^4\mu^2 - c^4v^2 \right) \\
\frac{abc}{abc} &= -\frac{1}{a^2b^2c^2} \left( -a^4\lambda^2 + b^4\mu^2 - c^4v^2 \right) \\
\frac{ab\dot{c}}{abc} &= -\frac{1}{a^2b^2c^2} \left( -a^4\lambda^2 - b^4\mu^2 + c^4v^2 \right)
\end{align*}
\]
Here the quantities $\lambda$, $\mu$ and $\nu$ are defined in terms of the triad basis vectors as,

$$\lambda = (e^1_\alpha - e^1_\beta) e^\alpha_2 e^\beta_3$$  \hspace{1cm} (3.125)$$

$$\mu = (e^2_\alpha - e^2_\beta) e^\alpha_1 e^\beta_3$$  \hspace{1cm} (3.126)$$

$$\nu = (e^3_\alpha - e^3_\beta) e^\alpha_1 e^\beta_2$$  \hspace{1cm} (3.127)$$

Substituting in,

$$|\lambda| a^2 = e^{-2\beta_1}, \quad |\mu| b^2 = e^{-2\beta_2}, \quad |\nu| c^2 = e^{-2\beta_3}$$  \hspace{1cm} (3.129)$$

into the Einstein equations in terms of the scale factors, we regain the equations obtained in the previous section in terms of $\beta_i$. We also assume that the scale factors close to the singularity can be written in terms of their Kasner asymptotes,

$$(a^2, b^2, c^2) = (a_0^2 t^{2p_1}, b_0^2 t^{2p_2}, c_0^2 t^{2p_3})$$  \hspace{1cm} (3.130)$$

with some factors $a_0, b_0$ and $c_0$ dependent on the coordinates $x^\alpha$. Furthermore,

$$p_1 + p_2 + p_3 = 1$$  \hspace{1cm} (3.131)$$

The quantities $p_i$ are called Kasner indices. To obtain the simplification of the equations of motion on approach to the singularity, when $a \gg b \gg c$ or $\Gamma_1 \gg \Gamma_2 \gg \Gamma_3$, we have effectively disregarded the 3-space components of the Ricci tensor in the vacuum Einstein equations. The condition for this assumption to remain sensible is

$$a \sqrt{ks^{-1}} \ll 1, \quad b \sqrt{ks^{-1}} \ll 1, \quad c \sqrt{ks^{-1}} \ll 1$$  \hspace{1cm} (3.132)$$

where $s = a_0 b_0 c_0$. As $t$ decreases however, this condition is violated at some critical time instant $t = t_c$. For example if the negative Kasner index corresponds to $a$, then,

$$a(t_c) \sqrt{ks^{-1}} \sim 1$$  \hspace{1cm} (3.133)$$

Adding and rewriting the Einstein’s equations in this regime, we get,

$$\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2 - \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)^2 + \frac{1}{2a^2 b^2 c^2} (a^4 \lambda^2 + b^4 \mu^2 + c^4 \nu^2) = 0$$  \hspace{1cm} (3.134)$$
From this equation we see that the onset of the critical time is signalled by the condition that \[ [(a^4 \lambda^2 + b^4 \mu^2 + c^4 \nu^2)(abc)^{-2}] \sim t^{-2} \] and hence of the order of the other terms in the equation. Thus if at the onset of each epoch at \( t + t_0 \), the scale factors are given by,

\[
(a^2, b^2, c^2) = (a_0^2 t^{2p_1}, b_0^2 t^{2p_2}, c_0^2 t^{2p_3})
\]

(3.135)

then at the critical time at the end of the epoch, they are given by,

\[
(a^2, b^2, c^2) = (\tilde{a}_0^2 t^{2\tilde{p}_1}, \tilde{b}_0^2 t^{2\tilde{p}_2}, \tilde{c}_0^2 t^{2\tilde{p}_3})
\]

(3.136)

The new Kasner indices \( \tilde{p}_i \forall i = 1, 2, 3 \) must follow the Kasner relations of the indices and their squares summing up to 1. The relation between the new indices and the ones at the onset of the epoch are given by,

\[
\tilde{p}_1 = -\frac{p_1}{1 + 2p_1}, \quad \tilde{p}_2 = \frac{p_2 + 2p_1}{1 + 2p_1}, \quad \tilde{p}_3 = \frac{p_3 + 2p_1}{1 + 2p_1}
\]

(3.137)

Thus, for example, in the first epoch if \( p_1 > p_2 > p_3 \) and \( c^2 \) was increasing or that \( p_3 < 0 \), then by these conversion relations, in the following epoch \( \tilde{p}_3 > 0 \) and \( \tilde{p}_2 < 0 \). In general, the negative Kasner index jumps to the scale factor which had the smallest positive Kasner index in the preceding epoch. In order to satisfy the Kasner relations, the Kasner indices can be parametrised in terms of a parameter \( u \geq 1 \) as follows,

\[
p_1 = \frac{u}{1 + u + u^2}, \quad p_2 = \frac{1 - u}{1 + u + u^2}, \quad p_3 = \frac{u(1 + u)}{1 + u + u^2}
\]

(3.138)

Thus if, in a particular epoch, the scale factors are described by,

\[
(a^2, b^2, c^2) \sim \left(t^{2p_1(u)}, t^{2p_2(u)}, t^{2p_3(u)}\right)
\]

(3.139)

then in the next epoch, the scale factors are described by,

\[
(a^2, b^2, c^2) \sim \left(t^{2\tilde{p}_2(u-1)}, t^{2\tilde{p}_1(u-1)}, t^{2\tilde{p}_3(u-1)}\right)
\]

(3.140)

From the equations of motion, we see that this process repeats and generates the sequence \( u^{(1)}, u^{(1)} - 1, u^{(1)} - 2 \) and so on. As \( u \geq 1 \), this evolution can be followed till \( u^{(1)} - \lfloor u^{(1)} \rfloor + 1 \) where \( \lfloor u^{(1)} \rfloor \) denotes the integer value of \( u^{(1)} \). Thus this system follows a regime of infinite oscillations with the Kasner indices switching in accordance to the rule described, on approach to the singularity.
3.7 Cosmic no-hair theorems for collapsing universes

In this section, we shall consider the equations of motion of Bianchi Class A, excluding Type IX in the first instance, and Type IX separately. We shall consider the effect of an ultra-stiff ideal fluid, with equation of state $p > \rho$, where $p$ and $\rho$ are the pressures and energy densities respectively. This fluid has a similar equation of state as the ekpyrotic field in terms of its effective equation of state [87]. However, it is to be noted, that the idealised equation of state is an approximation to the ekpyrotic field as the dynamics of the field is ignored in this case.

3.7.1 Bianchi I-VIII

We begin with the continuity equation (3.50) for the expansion-normalised density $\Omega$ in the time coordinate given by (3.43). In this case $\gamma > 2$ as the field is ultra-stiff, we find that for any initially contracting Bianchi Class A type I-VIII,

$$\Omega' \leq 0$$ (3.141)

Equality occurs for $K = \Sigma^2 = 0$ for any non-vacuum orbit of the $G_3$ isometry group of the Bianchi cosmologies ($\Omega > 0$). Thus $\Omega$ is a monotonically decreasing function of $\tau$, while bounded from below by the equality. Thus, $\lim_{\tau \to -\infty} \Omega' = 0$. This implies that $\lim_{\tau \to -\infty} K = \lim_{\tau \to -\infty} \Sigma = 0$. Recalling the fact that $\Sigma_+^2 + \Sigma_-^2 = \Sigma^2$, the latter limit means that $\lim_{\tau \to -\infty} \Sigma_+ = \lim_{\tau \to -\infty} \Sigma_- = 0$. By the Friedmann constraint (3.48), this also implies that $\lim_{\tau \to -\infty} \Omega = 1$. Let us consider the equations for the expansion normalised curvature variables, $N_i$.

$$N'_1 = (q - 4\Sigma_+)N_1$$ (3.142)

$$N'_2 = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2$$ (3.143)

$$N'_3 = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3$$ (3.144)

$$N'_4 = (q - 4\Sigma_-)N_4$$ (3.145)

We can assume that for each $N_i$, there exists a parameter $\epsilon > 0$ such that $N'_i/N_i > \epsilon$ for sufficiently negative $\tau$. This implies that $\lim_{\tau \to -\infty} N_i = 0$. Thus ‘the spatially flat and isotropic FRW universe is the global sink for all ultra-stiff, orthogonal, initially contracting Bianchi models of type I-VIII(class A) [87].
3.7.2 Bianchi Type IX

As $n_i > 0$ for Bianchi Type IX, the curvature parameter $K$ is not semi-positive definite and hence $H$ is no longer a monotonic function of time. Thus the appropriate normalisation to recast the Einstein equations in their expansion-normalised form is given by,

$$D \equiv \sqrt{H^2 + \frac{1}{4}(n_1n_2n_3)^{2/3}}$$  \hspace{1cm} (3.146)

The expansion normalised variables then become,

$$(\bar{H}, \bar{\Sigma}_\pm, \bar{N}_i, \bar{\Omega}) \equiv \left(\frac{H}{D}, \frac{\sigma_\pm}{D}, \frac{n_i}{3D} \rho \right)$$  \hspace{1cm} (3.147)

The definitions of $D$ and $\bar{H}$ imply the following constraint equation,

$$\bar{H}^2 + \frac{1}{4}(\bar{N}_1\bar{N}_2\bar{N}_3)^{2/3} = 1$$  \hspace{1cm} (3.148)

This equation also implies that $\bar{H}$ is bounded as $-1 \leq \bar{H} \leq 1$. A corresponding time variable can also be defined as,

$$\frac{dt}{d\bar{\tau}} = \frac{1}{D}$$  \hspace{1cm} (3.149)

All time variables taken with respect to this time variable will be denoted by $\star$. The evolution equation for $D$ is given by,

$$D^\star = -(1 + \bar{q})\bar{H}D$$  \hspace{1cm} (3.150)

The other Einstein equations in these variables become,

$$\bar{H}^\star = -(1 - \bar{H}^2)\bar{q}$$  \hspace{1cm} (3.151)

$$\bar{\Sigma}_\pm^\star = -(2 - \bar{q})\bar{H}\bar{\Sigma}_\pm - \bar{S}_\pm$$  \hspace{1cm} (3.152)

$$\bar{N}_1^\star = (\bar{H}\bar{q} - 4\bar{\Sigma}_\pm)\bar{N}_1$$  \hspace{1cm} (3.153)

$$\bar{N}_2^\star = (\bar{H}\bar{q} + 2\bar{\Sigma}_\pm + 2\sqrt{3}\bar{\Sigma}_-)\bar{N}_2$$  \hspace{1cm} (3.154)

$$\bar{N}_3^\star = (\bar{H}\bar{q} + 2\bar{\Sigma}_\pm - 2\sqrt{3}\bar{\Sigma}_-)\bar{N}_3$$  \hspace{1cm} (3.155)

where

$$\bar{q} = \frac{1}{2}(3\gamma - 2)(1 - \bar{V}) + \frac{3}{2}(2 - \gamma)\bar{\Sigma}^2$$  \hspace{1cm} (3.157)

$$\bar{\Sigma}^2 = \bar{\Sigma}_\pm^2 + \bar{\Sigma}_-^2$$  \hspace{1cm} (3.158)
\[ \bar{V} \equiv \frac{1}{12} \left[ \bar{\mathcal{N}}_1^2 + \bar{\mathcal{N}}_2^2 + \bar{\mathcal{N}}_3^2 - 2\bar{\mathcal{N}}_1\bar{\mathcal{N}}_2 - 2\bar{\mathcal{N}}_2\bar{\mathcal{N}}_3 - 2\bar{\mathcal{N}}_1\bar{\mathcal{N}}_3 + 3(\bar{\mathcal{N}}_1\bar{\mathcal{N}}_2\bar{\mathcal{N}}_3)^{2/3} \right] \] (3.159)

From the above definition, we see that \( \bar{V} \geq 0 \). The relation between \( \bar{V} \) and \( K \) is given by

\[ \bar{V} = \bar{H}^2 \left[ K + (N_1N_2N_3)^{2/3}/4 \right]. \]

The Friedmann constraint in these variables becomes,

\[ \Sigma^2 + \bar{V} + \bar{\Omega} = 1 \] (3.160)

And the equation for \( \bar{q} \) becomes,

\[ \bar{q} = 2\Sigma^2 + \frac{1}{2}(3\gamma - 2)\bar{\Omega} \] (3.161)

The corresponding continuity equation in these variables is given by,

\[ \bar{\Omega}^* = \bar{\Omega}\bar{H} \left[ -(3\gamma - 2)\bar{V} + 3(2 - \gamma)\Sigma^2 \right] \] (3.162)

Thus when \( \gamma > 2 \), \( \bar{q} \geq 2 \) with equality iff \( \bar{\Omega} = \Sigma^2 = 0 \). Using the fact that \( -1 \leq \bar{H} \leq 1 \), and evolution equation for \( \bar{H} \), we find that \( \bar{H}^* \leq 0 \). This means that \( \bar{H} \) is a monotonically decreasing function of \( \bar{\tau} \). \( \bar{q} > 0 \) for \( \bar{\Omega} > 0 \) and hence this implies that

\[ \lim_{\bar{\tau} \to -\infty} \bar{H} = 1 \quad \text{and} \quad \lim_{\bar{\tau} \to -\infty} \bar{H} = -1 \] (3.163)

Thus an initially expanding model will go through \( \bar{H} = 0 \) and hence undergo a recollapse. Once this occurs, (3.162) implies that \( \bar{\Omega}^* \geq 0 \) with equality iff \( \bar{V} = \Sigma^2 = 0 \). As before, \( \bar{\Omega} \leq 1 \), meaning that \( \bar{\Omega} \) is bounded, this implies that \( \lim_{\bar{\tau} \to +\infty} \bar{\Omega} = 0 \). This further implies,

\[ \lim_{\bar{\tau} \to +\infty} \bar{\Omega} = 1, \quad \lim_{\bar{\tau} \to +\infty} \bar{V} = 0, \quad \lim_{\bar{\tau} \to +\infty} \Sigma^2 = 0 \] (3.164)

Further from (3.161), we have,

\[ \lim_{\bar{\tau} \to +\infty} q = \frac{1}{2}(3\gamma - 2) \] (3.165)

From the evolution equations for \( \bar{N}_i \) we once again have a parameter \( \epsilon > 0 \) such that \( \bar{N}_i^* < -\epsilon \) and thus, \( \lim_{\bar{\tau} \to +\infty} \bar{N}_i = 0 \). Thus a Bianchi IX type cosmology also isotropises in the presence of the ultra-stiff ideal field on approach to the singularity at \( \bar{\tau} \to \infty \). It appears that this field also mitigates the chaotic oscillatory behaviour on approach to the singularity that is usually seen in vacuum Bianchi IX models or those sourced with ordinary sources of matter (those with equations of state that are less than stiff).
The cosmic no-hair theorem given in [87] is thus “All initially contracting spatially homogeneous, orthogonal Bianchi type I-VIII cosmologies and all Bianchi type IX universes that are sourced by an ultra-stiff fluid with an equation of state such that (\(\gamma - 2\)) is positive-definite collapse into an isotropic singularity, where the sink is the spatially flat and isotropic FRW universe”

In the following chapter we shall see that this conclusion on the successful isotropisation of Bianchi Class A models is seriously affected by the addition of ultra-stiff anisotropic pressures.
Chapter 4

Effect of anisotropic pressures on isotropising a bouncing model

4.1 Introduction

One of the first questions to ask when considering alternatives (or additions) to standard inflation is whether that alternative solves the problems that inflation claims to solve. For example, one can ask whether the present-day isotropy and homogeneity of the universe can be achieved through a cosmology which underwent contraction and bounce at some time (or times) in the past. It has been claimed that models implementing a phase of ekpyrosis, or a phase of scalar field-driven fast contraction can indeed solve this problem [75], [79], [36]. In effect, this model claims to solve the anisotropy problem by introducing a scalar field with negative potential energy, which behaves as an ideal fluid with ultra-stiff equation of state $p \gg \rho$. Its isotropic density therefore grows faster than the anisotropies in a contracting universe because the latter diverge no faster than an effective $p = \rho$ fluid. However, this simple analysis assumes that the matter pressure distribution is isotropic. A full analysis needs to include the effects of matter sources with anisotropic pressure distributions on approach to the singularity. Since the isotropic pressure is assumed to exceed the energy density, it should be permitted for the average pressure to exceed the energy density of the anisotropic fluid as well. The need to include anisotropic pressures on approach to the singularity can be understood by considering the greater relative abundance of free-streaming particles in an anisotropic universe, compared to an isotropic one. This is because interactions all become collisionless at a higher temperature in the case of anisotropic expansion, than in the isotropic case. Their interaction rates can be written as $\Gamma = \sigma n v \sim g \alpha^2 T$, where $\sigma$ is the interaction cross section, $n$ is the number density of particles, $v$ is the average velocity of the particles,
\( \alpha \) is the generalised structure constant associated with any interaction mediated by some gauge boson, \( T \) is the temperature of the universe and \( g \) is the effective number of relativistic degrees of freedom of particles at the temperature \( T \). This cosmological expansion rate will exceed the interaction rate, \( H \simeq g^{1/2}T^2m_p^{-1} \), whenever \( T > g^{1/2}\alpha^2m_p \sim 10^{16}\text{GeV} \) in simple unified models. In the preceding line \( m_p \) is the Planck mass. If the expansion is anisotropic down to a temperature \( T_F \), the expansion rate is faster, by a factor of \( T/T_F \). As \( \Gamma/H = (T_F/T^2)g^{1/2}\alpha^2m_p \) at \( T > T_F \), collisional equilibrium is even harder to maintain when \( T > T_F \). Thus particles free stream sooner in an anisotropic universe. However, graviton production near \( T = m_p \) also produces a population of collisionless particles whose free streaming will reduce significant anisotropic pressures if the expansion dynamics are anisotropic [91, 16].

In this chapter, I investigate the effects of anisotropic pressures in the Bianchi Class A homogeneous, anisotropic cosmologies, generalising the study of these effects in the simple Bianchi type I cosmological model by Barrow and Yamamoto [26]. I carry out a generalized phase-plane analysis for all the cosmologies of this type but then focus on the closed Bianchi type IX cosmologies and carry out numerical calculations to study their behaviour near any initial singularity or expansion minimum when this kind of anisotropic matter content is present in addition to an ultra-stiff isotropic fluid. I will show that in these most general homogeneous and anisotropic cosmologies it is essential to include the effects of anisotropic pressures as well as shear anisotropy. When the anisotropic pressures are stiffer on average than the isotropic pressures then they determine the nature of any singularity (or bounce) and it will be dominated by anisotropy, contrary to the situation expected in the standard ekpyrotic picture which ignores anisotropic pressures.

This chapter is organised as follows. I begin by presenting the generalised Einstein field equations in an expansion-normalised dynamical system for the non-tilted Bianchi Class A models containing isotropic ultra-stiff \((\rho > \rho)\) matter content as well as a second ultra-stiff matter source with positive density and anisotropic pressures. I first perform a stability analysis on this system for an initially contracting universe to see if a phase of ekpyrosis is really successful in suppressing the anisotropies in the presence of a dominant anisotropic pressure fluid. I also seek solutions to these equations in the limit of small anisotropy and give a new Bianchi I exact solution. In the next section, I study explicitly the evolution of a contracting, anisotropic but spatially homogeneous universe near the initial singularity in the presence of the matter content prescribed, then specializing to the Bianchi type IX universe. I then show the results of our numerical calculations in this universe and compare our results to the results of the stability analysis of the previous section. In the last section the conclusions are drawn.
For the numerical analyses in the remainder of this thesis, Wolfram Mathematica is used. As the systems studied here are the Bianchi IX universe, the behaviour is highly oscillatory. To capture these rapid oscillations, an adaptive step size method of integration has been used. Higher accuracy and precision goals for the integrator result in a trade-off with computation time. The qualitative features of the solutions are independent of the choice of initial conditions, however, the length of integration time needed for these features to show up depend on the initial conditions. Thus the initial conditions satisfying the Friedmann constraint are chosen so that they cause the relevant features to show up within a finite amount of integration time to prevent the build up of round-up error or unnecessarily long computation times.

4.2 Ekpyrotic models

This model has been briefly reviewed in Chapter 2. As mentioned there, this model also claims to solve the problem of growing shear by incorporating the ekpyrotic phase \[75, 36\]. This ekpyrotic phase has also been used in other cosmological bouncing models as a way to deal with the problem of growing anisotropies in a contracting universe \[37\], and so merits closer investigation. For simplicity, we shall focus on the single-field ekpyrotic model and first describe the effects of ekpyrosis in a Bianchi I universe with the ekpyrotic field and an ultra-stiff energy source with anisotropic pressure. The ekpyrotic field is a scalar field, \( \phi \) rolling down rapidly on a steep negative potential. This can be viewed as driving the contraction of the universe. To see how it might suppress the anisotropies, we write down its effective equation of state \[75\].

\[
p = (\gamma - 1)\rho, \quad \gamma \gg 2.
\]  

(4.1)

The anisotropy energy density scales as \( 1/l^6 \), and behaves like a source with \( \gamma = 2 \), \( l \) being the time dependent mean scale factor of the universe. Thus, the ekpyrotic phase simply introduces a source which scales with scale factor faster than the energy density in the anisotropy because \( \gamma > 2 \), see \[47\]. As the universe contracts, this term dominates over the anisotropy in the Friedmann equation, apparently solving the problem of isotropising the universe before it enters the hot big bang phase – or at least preventing the new expanding phase beginning with highly anisotropic dynamics. This should also result in significant dissipation and particle production which would reduce anisotropy and generate entropy \[91, 22\]. We ignore these complicated effects here.
The simplest form of an anisotropic but spatially homogeneous universe is the Bianchi I (or Kasner) universe [78, 123]. The metric is

\[ ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \]

(4.2)

The Einstein equations for this model gives [128]

\[ 3H^2 = \sigma^2 + \rho_{\text{matter}}, \]

(4.3)

\[ \sigma_{\alpha\beta} + 3H\sigma_{\alpha\beta} = \mu P_{\alpha\beta} \]

(4.4)

where \( \sigma_{\alpha\beta} \) is the shear tensor which follows the relation

\[ \sigma^2 = \frac{1}{2} \sigma_{\alpha\beta} \sigma_{\alpha\beta} \]

(4.5)

and \( \mu \) is the anisotropic pressure fluid density and the definition of \( P_{\alpha\beta} \) is shown in (4.11). Also \( \rho_{\text{matter}} \) refers to the total energy density of the matter components of the system, i.e., the matter with isotropic pressures as well as the matter with anisotropic pressure. If we have only a fluid with isotropic pressure, then the right-hand side of (4.4) vanishes and we can write the shear energy density, \( \sigma^2 \) in the Friedmann constraint, (4.3) as \( \Sigma^2/l^6 \), where \( \Sigma^2 \) is constant. Hence, an ekpyrotic field with equation of state, \( p_\phi \gg \rho_\phi \) would dominate over the anisotropy when \( l \to 0 \) and the singularity is approached. We can give a new exact Bianchi type I solution of Einstein’s equations in a form which illustrates this in the particular representative case with \( P_{\alpha\beta} = 0 \) and \( p = 3\rho \) where the metric is exactly integrable ((4.2)):

\[ a(t) = \left( (t^2 + C_2t)^{1/2}(\sqrt{t} + \sqrt{(C_2 + t)})^{2(3q_1 - 1)} \right)^{1/3} \]

(4.6)

\[ b(t) = \left( (t^2 + C_2t)^{1/2}(\sqrt{t} + \sqrt{(C_2 + t)})^{2(3q_2 - 1)} \right)^{1/3} \]

(4.7)

\[ c(t) = \left( (t^2 + C_2t)^{1/2}(\sqrt{t} + \sqrt{(C_2 + t)})^{2(3q_3 - 1)} \right)^{1/3} \]

(4.8)

\[ \sum_i q_i = 1 = \sum_i q_i^2 \]

(4.9)

Thus, we see that at early times this solution tends to the flat Friedmann solution for \( p = 3\rho \) 'matter': \( a \sim t^{1/6}, b \sim t^{1/6} \) and \( c \sim t^{1/6} \) as \( t \to 0 \), and at late times approaches the Kasner solution \( a \sim t^{q_1}, b \sim t^{q_2} \) and \( c \sim t^{q_3} \) with condition (4.9) as \( t \to \infty \); fuller details can be found in the Appendix. Thus, this solution provides a simple description of the transition from an
isotropic initial state to a Kasner-like anisotropic future. This is the opposite trend to the evolution of a $p < \rho$ perfect-fluid model.

However, if we relax the assumption of having energy sources with only isotropic pressure, we can no longer write down the form of the anisotropy energy density in (4.3) the simple form, $\Sigma^2/l^6$, since the right-hand side of (4.4) no longer vanishes. In fact, the anisotropy may diverge faster than the ekpyrotic fluid in powers of $l^{-1}$ in any particular direction as $t \to 0$, depending on the pressure component of the matter source in that direction. Hence, we can no longer be sure that adding a matter component with $w \gg 1$ solves the problem of isotropising the universe on approach to the singularity. This will be investigated in more detail and for more general forms of anisotropic spatially homogeneous universes in 4.3.1.

### 4.3 Bianchi Class A models of types I-VIII

In this section, we investigate the assumption that an ultra-stiff energy source suppresses the anisotropies near a singularity in an initially contracting universe. We do this for the Bianchi Class A models [58], which generalise the Bianchi type I models because they allow the presence of anisotropic spatial curvature. However, now we add an ultra-stiff anisotropic pressure source comoving with the isotropic fluid source to see if ekpyrosis still manages to suppress the anisotropies. The investigation of whether the anisotropies are suppressed by the ekpyrotic phase has been done in the case of an empty anisotropic spatially homogeneous geometry, the Kasner universe [59], but without the anisotropic pressure fluid. Studies regarding the inclusion of an anisotropic fluid in the Bianchi universes have also been made in [39]. However, here we follow the approach similar to the one used in [26, 87] and present a more general analysis for all the Bianchi models included in Class A with the aim of finding the conditions under which the Friedmann-Lemaître (FL) fixed point is an attractor for a contracting universe on approach to the collapse.

The ultra-stiff isotropic matter considered in this section is a null-energy-condition-violating fluid with negative energy density. This negative energy density is introduced to induce a bounce at early times instead of a singularity. This is because many bouncing cosmologies consider fields that effectively behave as a ghost field to facilitate the bounce and which also behave as a stiff or ultra-stiff matter source [25, 36, 43].

The energy-momentum tensor can be written as follows:

$$T^\text{total}_{ab} = T^I_{ab} + T^A_{ab}, \quad (4.10)$$
where the superscripts, \( I \) and \( A \) denote "isotropic" and "anisotropic" respectively. The anisotropic fluid energy-momentum tensor can be written explicitly as

\[
T_{ab}^A = \mu \{ u_a u_b + (\gamma - 1)(g_{ab} + u_a u_b) + \mathcal{P}_{ab} \}. \tag{4.11}
\]

In the rest of this work, the isotropic and anisotropic fluid energy densities will be referred to as \( \rho \) and \( \mu \) respectively and the isotropic fluid will have equation of state \( p = (\gamma - 1)\rho \) while the anisotropic pressure tensor \( \mathcal{P}_{ab} \) has diagonal elements \( (\gamma_i - \gamma) \), for all \( i = 1, 2, 3 \), respectively, with average value \( \gamma = (\gamma_1 + \gamma_2 + \gamma_3)/3 \). The isotropic fluid energy density, \( \rho \), is that of a ghost field and so \( \rho < 0 \).

The Einstein equations for this class of cosmological models can be written with the specified matter content as follows [128]:

\[
\dot{H} = -H^2 - \frac{2}{3} \sigma^2 - \frac{1}{6}(\rho + 3p) - \frac{1}{6}(\mu + p_1 + p_2 + p_3), \tag{4.12}
\]

where the anisotropic pressures are defined to be \( p_i = (\gamma - 1)\mu \) for \( i = 1, 2, 3 \).

\[
\dot{\sigma}_{\alpha\beta} = -3H\sigma_{\alpha\beta} + 2\varepsilon^{\mu\nu}_{(\alpha} \sigma_{\beta)\mu} \Omega_\nu - (3)S_{\alpha\beta} + \mathcal{P}_{\alpha\beta} \mu, \tag{4.13}
\]

\[
\dot{n}_{\alpha\beta} = -Hn_{\alpha\beta} + 2\sigma^{\mu}_{(\alpha} n_{\beta)\mu} + 2\varepsilon^{\mu\nu}_{(\alpha} n_{\beta)\mu} \Omega_\nu \tag{4.14}
\]

\[
\dot{\rho} = -3\gamma H \rho, \tag{4.15}
\]

\[
\dot{\mu} = -3\gamma H \mu - \sigma_{\alpha\beta} \mathcal{P}^{\beta\alpha} \mu. \tag{4.16}
\]

The Friedmann constraint is given by

\[
H^2 = -\frac{\rho}{3} + \frac{\mu}{3} + \frac{\sigma^2}{3} - \frac{(3)R}{6}. \tag{4.17}
\]

The following linear combinations of the independent components of the shear tensor are introduced,

\[
\sigma_+ \equiv \frac{1}{2}(\sigma_{22} + \sigma_{33}),
\]

\[
\sigma_- \equiv \frac{1}{2\sqrt{3}}(\sigma_{22} - \sigma_{33}).
\]

Diagonalising the stress tensor it is seen that all other components of the stress evolution equation are not dynamical. Similarly, the trace-free spatial Ricci tensor \( (3)S_{\alpha\beta} \) and the constant tensor \( \mathcal{P}_{\alpha\beta} \) are diagonal and their components \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) and \( S_+ \) and \( S_- \) are given by analogous expressions. Explicitly, the expansion-normalized combinations of the
4.3 Bianchi Class A models of types I-VIII

The dynamical components of the Ricci tensor are given by

\[ J_+ = \frac{1}{6H^2}[(n_2 - n_3)^2 - n_1(2n_1 - n_2 - n_3)], \]
\[ J_- = \frac{1}{2\sqrt{3}H^2}(n_3 - n_2)(n_1 - n_2 - n_3). \]  

The scalar curvature is given by,

\[ (3)R = -\frac{1}{2}[n_1^2 + n_2^2 + n_3^2 - 2(n_1n_2 + n_2n_3 + n_3n_1)]. \]  

We want to set up the phase space so that we can study the evolution of the quantities with respect to the expansion of the universe, i.e., with respect to \( H \equiv \dot{l}/l \) where \( l(t) \) is a generalised mean scale factor. We begin by introducing expansion-normalised variables as follows,

\[ \Omega = -\frac{\rho}{3H^2}, \quad Z = \frac{\mu}{3H^2}, \quad \Sigma^2 = \frac{\sigma^2}{3H^2}, \quad K = -\frac{(3)R}{6H^2}. \]

The Bianchi Class A universe will now be determined completely if we solve the Einstein’s field equations in these new variables for the state vector \( \{H, \Sigma_+, \Sigma_-, N_1, N_2, N_3, \Omega, Z\} \). We find that the Friedmann constraint(4.17) becomes,

\[ \Omega + Z + \Sigma^2 + K = 1 \]

In terms of the expansion-normalised variables, the Einstein field equations become:

\[ \Sigma'_\pm = -(2-q)\Sigma_\pm - J_\pm + 3P_\pm Z \]  
\[ N'_1 = (q - 4\Sigma_+)N_1 \]  
\[ N'_2 = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2 \]  
\[ N'_3 = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3 \]  
\[ \Omega' = [-(3\gamma - 2)K + 3(2 - \gamma)\Sigma^2 + 3(\gamma_\ast - \gamma)Z]\Omega \]  
\[ Z' = [3(2 - \gamma_\ast)\Sigma^2 - 3(\gamma - \gamma_\ast)\Omega - (3\gamma_\ast - 2)K - 6(P_+, \Sigma_+ + P_\Sigma_-)]Z \]

Here, all time derivatives ‘ are taken with respect to a new time coordinate \( \tau \) which is defined by the relation,

\[ \frac{dt}{d\tau} = \frac{1}{H} = \frac{l}{\dot{l}}, \]

\[ ^{1}\text{The minus signs in the definition of } \Omega \text{ and } K \text{ ensure that they are positive when } \rho < 0 \text{ and } (3)R < 0 \text{ in our system.} \]
Table 4.1 Equilibrium points of the Bianchi I-VIII dynamical systems in the presence of the anisotropic stress fluid

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_+$</th>
<th>$\Sigma_-$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$\Omega$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FL$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$P_1^+(II)$</td>
<td>$\Sigma_1^{(1)}$</td>
<td>0</td>
<td>$n_{p1}^{(1)}$</td>
<td>0</td>
<td>0</td>
<td>$\Omega_{p1}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$P_2^+(II)$</td>
<td>$\Sigma_2^{(1)}$</td>
<td>0</td>
<td>0</td>
<td>$n_{p2}^{(1)}$</td>
<td>0</td>
<td>$\Omega_{p2}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$P_3^+(II)$</td>
<td>$\Sigma_3^{(1)}$</td>
<td>0</td>
<td>0</td>
<td>$n_{p3}^{(1)}$</td>
<td>0</td>
<td>$\Omega_{p3}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$P_1^+(VI_0)$</td>
<td>$\Sigma_1^{V_1}$</td>
<td>0</td>
<td>0</td>
<td>$n_{V1}^{(1)}$</td>
<td>0</td>
<td>$\Omega_{V1}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$P_2^+(VI_0)$</td>
<td>$\Sigma_2^{V_1}$</td>
<td>0</td>
<td>0</td>
<td>$n_{V2}^{(1)}$</td>
<td>0</td>
<td>$\Omega_{V2}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>$P_3^+(VI_0)$</td>
<td>$\Sigma_3^{V_1}$</td>
<td>0</td>
<td>0</td>
<td>$n_{V3}^{(1)}$</td>
<td>0</td>
<td>$\Omega_{V3}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td>Kasner</td>
<td>$\mathcal{A}_1$</td>
<td>$\alpha_2^{-(1/2)}$</td>
<td>$\alpha_2^{-(1/2)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{A}_2$</td>
<td>$\alpha_2^{-(1/2)}$</td>
<td>$\alpha_2^{-(1/2)}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The deceleration parameter, $q = -\dot{H}/\dot{l}^2$ is given by

$$\dot{H} = -(1 + q)H^2$$

In the expansion-normalized variables, using the Friedmann constraint, we find that $q$ is given by

$$q = 2\Sigma^2 - \frac{1}{2}(3\gamma - 2)\Omega + \frac{1}{2}(3\gamma - 2)Z.$$  

4.3.1 Stability analysis

I consider the evolution equations in the expansion-normalised variables and perform a phase-plane analysis for them. The equilibrium points are first identified. In 4.1, the explicit forms of the quantities referred to in the Table are given in the Appendix. On examination of these expressions, we find that for cases of ultra-stiff matter, as well as for cases when the anisotropic fluid is stiffer than the isotropic fluid, all these points become unphysical except for the $FL$, Kasner, $\mathcal{A}_1$ and $\mathcal{A}_2$ points.

We are interested in understanding the behaviour of this system of equations with respect to the FL fixed point ($FL$) in the asymptotic past. Thus, we linearise this system of equations
around this fixed point to obtain the following equations:

\[
\Sigma'_\pm = \frac{3}{2}(\gamma - 2)\Sigma_\pm + 3\mathcal{P}_\pm Z, \quad (4.32)
\]

\[
N'_i = -\frac{1}{2}(3\gamma - 2)N_i, \quad \forall i = 1, 2, 3 \quad (4.33)
\]

\[
\Omega' = -3(\gamma - \gamma_s)Z, \quad (4.34)
\]

\[
Z' = -3(\gamma - \gamma_s)Z. \quad (4.35)
\]

We see from the linearised equation for \( \Omega \) that there exists a zero eigenvalue and so this is a critical variable. Thus we use a second-order stability analysis. We perform the following transformations on the system,

\[(\Omega, Z) \rightarrow (\omega, \tilde{Z}),\]

where \( \omega = \Omega - Z, \tilde{Z} = \Omega + Z \). All the other variables remain the same in this linear transformation and thus have not been shown explicitly. The \( \Omega \) equation then becomes controlled by

\[
\omega' = \frac{3}{4}(\gamma_s - \gamma)(\omega^2 - \tilde{Z}^2) + F(\Sigma_+, \Sigma_-, N_1, N_2, N_3). \quad (4.36)
\]

Using the Lyapunov procedure for the critical case with a single zero eigenvalue when no quadratic cross-terms multiplying the critical and non-critical eigenvectors are present, we set the stable variables to zero (see [11]). We find that the power of the leading term in the critical variable \( \bar{u} \) is then 2 and so the \( \tilde{Z} = 1 \) point and hence the \( \Omega = 1 \) point is past unstable.

We now attempt to find solutions to these equations in the limit of small anisotropy. We define the ratio of the anisotropic fluid densities and the isotropic fluid densities as,

\[
Q = \frac{\mu}{\rho}. \quad (4.37)
\]

The evolution equation for \( Q \) is

\[
\dot{Q} = [-3(\gamma_s - \gamma)H - 6(\mathcal{P}_+ \sigma_+ + \mathcal{P}_- \sigma_-)]Q. \quad (4.38)
\]

In the case of small anisotropy, we approximate the values of the Hubble expansion rate and the isotropic energy density by their values in the flat Friedmann universe,

\[
H = \frac{2}{3\gamma t}, \quad \rho = \frac{4}{3\gamma^2 t^2}. \quad (4.39)
\]
From the diagonalised equation for the shear, (4.13), we get the approximate solution,

\[
\sigma_\pm = -\frac{\gamma \mathcal{T}_\pm}{2} + \frac{1}{2} \mathcal{P}_\pm \mathcal{Q} \gamma t + \kappa_1 t^{-2/\gamma}.
\]  

(4.40)

where \( \kappa_1 \) is a constant. Substituting this value for the anisotropy in equation (4.38), we get to linear order

\[
\dot{Q} = 3(\gamma_* - \gamma)HQ,
\]  

(4.41)

which has the solution,

\[
Q \sim t^{-2(\gamma_* - \gamma)/\gamma}.
\]  

(4.42)

The linear analysis suffices as long as \( \gamma \neq \gamma_* \). We see that if the anisotropic fluid is stiffer than the isotropic fluid, so that \( \gamma_* > \gamma \), then the anisotropic fluid density does not tend to its FL value, \( \mu = 0 \), at early times.

The analysis of the Kasner equilibrium point follows the same pattern as the analysis for the FL point, in the sense that a zero eigenvalue appears in the computation. On applying the Lyapunov procedure [11], we find an even power of the eigenvector with the zero eigenvalue occurs to leading order and so this is also not an attractor in the asymptotic past.

4.4 Bianchi IX universe with isotropic ghost field and fluid with anisotropic pressures

We now consider the specific example of an anisotropic but spatially homogeneous closed universe of Bianchi type IX. This is the most interesting case because it contains the closed isotropic FL universe as a special case. It also displays the most general chaotic dynamics on approach to the singularity [98], [31] in the absence of stiff or super-stiff matter fields. It is assumed in this work that the matter and radiation sources could be neglected near the singularity and the vacuum dynamics is asymptotically approached for any perfect fluid source with \( 0 \leq p < \rho \). As is well known and has been reviewed in Chapter 3, the chaotic type IX evolution is well-approximated by an infinite succession of Kasner epochs, which occur in any finite open interval of proper time around \( t = 0 \). At any instant two of the scale factors oscillate with approximate Kasner initial conditions at the beginning of each epoch while the third decreases monotonically with time as \( t \to 0 \) [81]. The sequence of oscillatory Kasner configurations appears to be chaotic in nature and the discrete dynamics can be solved exactly to find the smooth invariant measure [45, 6]. It is a non-separable measure of the sort that characterises a double-sided continued fraction map. However, the inclusion of a stiff matter fluid with equation of state \( p = \rho \) [30, 97] results in an inevitable termination of the
chaotic oscillations on approach to the singularity, after which all three scale factors evolve monotonically (but not in general isotropically) to zero as $t \to 0$ because the Kasner solution for $p = \rho$ matter permits all the Kasner indices to be simultaneously positive and the initial state is quasi-istropic [13]. The chaotic oscillatory sequence towards the singularity ends: no further oscillations occur.

Thus, the inclusion of a stiff matter fluid in the Bianchi IX system ultimately suppresses the chaotic behaviour of the scale factors near the singularity. In the Misner’s Hamiltonian picture this corresponds to the universe point eventually having too low a perpendicular velocity component relative to the potential wall it is approaching as the walls expand on approach to the singularity. It never reaches the wall and remains moving as if there are no potential walls (ie as in a Bianchi type I universe). No further transpositions of Kasner behaviours occur. All the known ways in which chaotic behaviour can be avoided in type IX universes exploit this feature directly or indirectly and are linked to the dimension of space in an interesting way [52]. Clearly, in the ekpyrotic scenario [59], a phase of ekpyrotic evolution, which is equivalent to domination by an ultra-stiff fluid with $p > \rho$, will have a more pronounced effect of suppressing the anisotropy energy domination and driving the dynamics towards isotropy. We investigate if this conclusion is sustained in the presence of anisotropic pressures.

The type IX universe is also interesting because it reduces to the closed FL universe in the isotropic limit, and this has been shown to possess very simple cyclic behaviour in the presence of ghost stiff matter content ($p = \mu < 0$) and radiation [25]. This model therefore seems to be a suitable candidate to test the results of our stability analysis of the previous section and also to learn more about the explicit behaviour of the scale factors, their Hubble rates, and the shear anisotropy tensor.

The matter considered in the following analysis is, as before, an ultra-stiff ghost field plus a stiffer anisotropic pressure field. The ghost field is included because, if it dominates at small times, it will create a bounce at a non-zero expansion volume minimum. The dynamics will be driven towards isotropy if a bounce occurs. By contrast, if the ultra-stiff anisotropic pressure field dominates over the isotropic ghost field then it should drive the dynamics towards an anisotropic Weyl curvature singularity.

### 4.4.1 Field equations

In this section we analyse a diagonal Bianchi type IX universe containing an isotropic ultra-stiff ghost field (with negative density) and another fluid with positive density and anisotropic pressures. We will test the possibility that on approach to a singularity the ultra-stiff ghost field will dominate over the anisotropic pressures and so cause the universe to isotropise
and bounce at a finite expansion minimum. However, if the average anisotropic pressure becomes larger than that of the ghost field then we expect the singularity to be restored because the anisotropic pressures will dominate the dynamics at the singularity. The ghost field is included here simply as a device to bring about a simple bounce at finite radius; however, some editions of the ekpyrotic scenario do include an effective ghost by allowing the sign of the gravitational coupling to change because it is determined by a time-dependent scalar field [41, 27].

In the following, we consider a Bianchi IX universe with scale factors \(a(t), b(t)\) and \(c(t)\), containing an isotropic, ultra-stiff ghost field (negative energy density, \(p > \rho\)) as well as an anisotropic ultra-stiff field, with average stiffness exceeding that of the isotropic ghost field. The energy density of the isotropic ghost field is given by \(\rho\) and it has pressure \(p\) with equation of state,

\[
p = (\gamma - 1)\rho,
\]

and the ultra-stiff condition requires \(\gamma > 2\). The energy density of the anisotropic pressure "fluid" is denoted by \(\mu\), as before. The equation of state in the \(i\)th direction, where \(i = 1, 2, 3\) and denotes the 3 spatial directions, is given by,

\[
p_i = (\gamma_i - 1)\mu,
\]

and the ultra-stiff condition requires that some of the \(\gamma_i\), or their mean value, exceed 2. As is evident from the above equations, the \(p_i\)’s will be equal. The field equations for such a type IX universe are:

\[
\begin{align*}
\frac{\ddot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{a}}{ab} + \frac{a^2}{4b^2c^2} + \frac{b^2}{4a^2c^2} - \frac{3c^2}{4a^2b^2} + \frac{1}{2a^2} + \frac{1}{2b^2} - \frac{1}{2c^2} &= -(p + p_3), \\
\frac{\ddot{b}}{b} + \frac{\dot{c}}{c} + \frac{\dot{b}}{bc} + \frac{b^2}{4a^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3a^2}{4b^2c^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{1}{2a^2} &= -(p + p_1), \\
\frac{\ddot{c}}{c} + \frac{\dot{a}}{a} + \frac{\dot{c}}{ca} + \frac{a^2}{4b^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3b^2}{4a^2c^2} + \frac{1}{2c^2} + \frac{1}{2a^2} - \frac{1}{2b^2} &= -(p + p_2), \\
\frac{\dot{a}b}{ab} + \frac{\dot{b}c}{bc} + \frac{\dot{c}a}{ca} + \frac{1}{2a^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{a^2}{4b^2c^2} - \frac{b^2}{4a^2c^2} - \frac{c^2}{4a^2b^2} &= \rho + \mu.
\end{align*}
\]
We note that these equations can be solved exactly in the special case of the axisymmetric Bianchi IX universe, with $b = c$ and also $p_1 = p_2 = p_3 = \rho$, and for other equivalent cyclic permutations. The solutions have the form,

\begin{align*}
a(\tau)^2 &= A \text{sech}(A \tau) \\
b(\tau)^2 &= \frac{B^2}{4A} \cosh(A \tau) \text{sech}^2 \left( \frac{B}{2} \tau \right)
\end{align*}

subject to the constraints,

\begin{align*}
\rho &= \frac{M^2}{4a^2b^2} \\
A^2 + M^2 &= B^2
\end{align*}

where $dt = ab^2d\tau$ [13].

For the general ekpyrotic case with $p_1 \neq p_2$, an exact solution is unobtainable. Thus, we resort to finding a numerical solution for the full Bianchi IX evolution, including both ultra-stiff isotropic and anisotropic pressures. To facilitate the numerical integration, the scale factors in the three directions are rewritten in terms of their logarithms as \[ a(t) \equiv e^{\alpha(t)}, b(t) \equiv e^{\beta(t)}, c(t) \equiv e^{\delta(t)} \tag{4.46} \]

The field equations can be rewritten as a first-order system by an appropriate choice of variables (see Appendix for details). Three new quantities are introduced to achieve this:

\begin{align*}
x &\equiv \alpha'(t) - \beta'(t), \tag{4.47} \\
y &\equiv \alpha'(t) - \delta'(t), \tag{4.48} \\
H &= \frac{1}{3} \left( \alpha'(t) + \beta'(t) + \delta'(t) \right). \tag{4.49}
\end{align*}

In all the calculations that follow, the equation of state parameters for the isotropic and anisotropic fluids are set to be $\gamma = 5$, $\gamma_1 = 12$, $\gamma_2 = 18$, $\gamma_3 = 21$. These are representative values that capture the essential behaviour that occurs whenever the anisotropic ekpyrotic fluid is stiffer than the isotropic one ($\gamma_i > \gamma$). Also, the results of the numerical integrations performed by using different sets of initial conditions (Kasner and those that satisfy the Friedmann constraint) have been plotted in the evolution of the scale factors, $a(t), b(t)$ and $c(t)$, using the definitions given in (4.46).

As noted above, the Mixmaster behaviour seemed to occur in the form of epochs [31, 30] with the memory of the ‘initial’ data being erased in successive Kasner epochs on approach
to the singularity. Thus, we choose Kasner-like ‘initial’ conditions for the variables we are integrating over. We then examine the effect of the ultra-stiff fluids on their evolution. The initial values are as follows:

\[
x(\tau_0) \quad y(\tau_0) \quad H(\tau_0) \quad \alpha(\tau_0) \quad \beta(\tau_0) \quad \delta(\tau_0) \quad \rho(\tau_0) \quad \mu(\tau_0)
\]

\[
m_{k1} - m_{k2} \quad m_{k1} - m_{k3} \quad \frac{(m_{k1} + m_{k2} + m_{k3})}{3} \quad m_{k1} \quad m_{k2} \quad m_{k3} \quad s \quad v
\]

where the three Kasner indices are expressed in terms of the parameter \(u\), as usual, by:

\[
m_{k1} = -u/(u^2 + u + 1), \quad m_{k2} = (u + 1)/(u^2 + u + 1) \quad \text{and} \quad m_{k3} = u(u + 1)/(u^2 + u + 1)
\]

with \(s = 0.269943, \quad v = 0.20\). For the purposes of this computation, \(u = -6\pi\) and \(\tau_0 = -0.002\). The equations are evolved from \(t = \tau_0\) to \(t = \tau_f\), where \(\tau_f = -255\). The singularity (if it occurs) is taken to be at \(t = 0\) and indeed this is where all the quantities blow up in our computation.\(^2\)

The values of the indices \(m_i, \forall i = 1, 2, 3\), have been chosen according to the familiar Kasner vacuum parametrisation, described for example in [31], and so satisfy \(m_1 + m_2 + m_3 = 1 = m_1^2 + m_2^2 + m_3^2\).

The initial hypersurface for the numerical computation has a geometry that describes a flat, empty, anisotropic spacetime. This choice of initial conditions do not exactly satisfy the Friedmann constraint exactly (4.45). The integration could only be carried out for an ordinary (NEC-satisfying) ultra-stiff isotropic ghost field. The numerical calculations show oscillations of the scale factors before they are replaced by a nearly monotonic evolution towards the volume minimum or singularity, as described in [30]. This evolution is depicted in 4.1 for the case including only the isotropic ghost field. For the case including both the isotropic ultra-stiff field and an anisotropic pressure field (stiffer than the isotropic field) the evolution is shown in 4.2.

On examining 4.1 and 4.2, we observe the following features. The initial conditions used do not satisfy the Friedmann constraint and therefore later on in this section, the equations are solved again with initial conditions that do satisfy this constraint. There is more than one branch of solutions but at least one of the scale factors approaches the singularity in a slightly oscillatory manner as predicted in [30], while one of them stays nearly constant. On inclusion of the anisotropic energy density, the solutions seem to tend towards a contraction to a collapse, a fact that will be further verified in the case which takes into account the Friedmann constraint while picking initial conditions. In order to study the evolution of the shear and the near-singularity behaviours of the scale factors and the Hubble rates, we try to

---

\(^2\)The integration is carried out in negative time as we are interested in a contracting universe approaching the singularity and so we integrate backwards in time. The sign of the time coordinate is not relevant as it can be made positive by introducing a constant shift which would not affect our results.
4.4 Bianchi IX universe with isotropic ghost field and fluid with anisotropic pressures

Fig. 4.1 Evolution of the scale factors for Bianchi type IX from Kasner-like initial conditions towards the singularity. The universe contains isotropic ($\rho > 0$) ultra-stiff fluid only and no anisotropic pressure field included.

Fig. 4.2 Evolution of the scale factors for a Bianchi type IX universe from Kasner-like initial conditions towards the singularity. The universe contains an ultra-stiff isotropic fluid ($\rho > 0$) and a fluid with anisotropic pressure.
find initial conditions that satisfy the Friedmann constraint (4.45) and take into consideration the curvature and the matter content of the spacetime. Accordingly, we choose,

\[
\begin{align*}
\begin{array}{cccccccccc}
x(\tau_0) & y(\tau_0) & H(\tau_0) & \alpha(\tau_0) & \beta(\tau_0) & \delta(\tau_0) & \rho(\tau_0) & \mu(\tau_0) \\
m_1' - m_2' & m_1' - m_3' & (m_1' + m_2' + m_3')/3 & m_1 & m_2 & m_3 & -s & v \\
\end{array}
\end{align*}
\]

where \(m_1' = 0.594778, m_2' = 0.167825, m_3' = 0.276172, m_1 = 1.19144, m_2 = 2.24155, m_3 = 1.22871, s = 0.2175397,\) and \(v = 0.20;\) \(\tau_0\) is the initial instant of time. For the present case, we have chosen \(\tau_0 = 1.6\). The equations are evolved from \(t = \tau_f\) to \(\tau_0 = 1.6\), where \(\tau_f = -25\). These initial conditions satisfy the Friedmann constraint (4.45) with an error of only \(\epsilon \sim O(10^{-8})\).

From the results of the numerical computation, we find the following evolutionary features:

**Scale-factor evolution**

In the figures shown, that is in 4.3 and 4.4, the logarithms of the scale factors (i.e., \(\alpha, \beta, \delta\)) have been plotted. The 4.3 shows the evolution of the scale factors with the inclusion of only an isotropic ultra-stiff ghost field and 4.4 shows the evolution of the scale factors with the inclusion of both the isotropic, ultra-stiff ghost field and the anisotropic pressure, which is also ultra-stiff field and with greater average stiffness than the ghost field. In the absence of the anisotropic pressure field, we see that the scale factors undergo periodic bounces, with a phase of expansion, contraction and a turnaround. On inclusion of the anisotropic pressure field, the periodic bouncing behaviour is destroyed and the scale factor evolution seems to undergo gentle oscillations towards ultimately a collapse. One of the scale factors in the \(c(t)\) direction remains almost constant throughout the evolution. We study the near-singularity behaviour in more detail by focusing on the evolution in a small time interval near to \(t = 0\). The Figure 4.5 shows the evolution of the scale factors with the inclusion of only an isotropic ultra-stiff ghost field very close to the singularity and the Figure 4.6 shows the evolution of the scale factors with the inclusion of both the isotropic, ultra-stiff ghost field as well as the anisotropic pressure, ultra-stiff (with greater stiffness than the ghost field) field very close to the singularity.

Near the singularity the solutions show the following behaviours. The scale factors for the case including only the isotropic ghost fluid do not in fact collapse to a singularity. They undergo a non-singular bounce, as expected from our experience of the isotropic closed universe and that of the Kasner universe with ghost field and radiation. However, the bounces in the three directions seem to occur almost simultaneously. It is also interesting to note that
Fig. 4.3 Scale factor evolution in Bianchi type IX with initial conditions satisfying the Friedmann constraint with isotropic ultra-stiff ghost field ($\rho < 0$) and no ultra stiff anisotropic pressure field included.

Fig. 4.4 Scale factor evolution in Bianchi type IX with initial conditions satisfying the Friedmann constraint with isotropic ultra-stiff ghost field ($\rho < 0$) and anisotropic pressure ultra-stiff field included, on approach to the singularity.
if the stiffness of the anisotropic fluid is increased, so that on the average it is stiffer than
the isotropic fluid, but is not stiff in one or two directions, then the scale factors in those
directions remain nearly constant. If the stiffness is less than that of the isotropic fluid, or
the initial conditions are such that the ultra-stiff anisotropic fluid is negligible compared to
the isotropic fluid density, they show similar behaviour to the isotropic case and undergo a
bounce after which the scale factors all begin to re-expand. In all other cases, they contract
until they are very near the singularity. This means that near the expected singularity, the
isotropic fluid scale factors re-expand, but the scale factors in the anisotropic fluid case all
seem to contract towards a singularity. In all cases, the shear for the case containing the
isotropic fluid alone is lower than when the anisotropic fluid is present.

The behaviour just described can be better understood if we try to find the Kasner-like
asymptotes for the system. We assume power-law time-evolution of the scale factors in the
simpler case of a Bianchi I universe with only an anisotropic pressure fluid present. This
amounts to neglecting the anisotropic spatial 3-curvature terms in Einstein’s equations on
approach to a singularity and this will be justified below. We take the scale factors in this
asymptotic limit to be

\[ a(t) \sim t^{m_1}, \quad b(t) \sim t^{m_2}, \quad c(t) \sim t^{m_3}, \]  \hspace{1cm} (4.50)

where the \( m_i \)'s for \( i = 1, 2, 3 \) are constants. Substituting them into the Bianchi I field equations
we find, from the evolution equation for the volume, \( V = abc \), that

\[ (m_1 + m_2 + m_3)(m_1 + m_2 + m_3 - 1) = 3(\gamma - 2)\mu_0, \]  \hspace{1cm} (4.51)

Solving for \( m_1 + m_2 + m_3 \), we get

\[ m_1 + m_2 + m_3 = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 12(\gamma - 2)\mu_0} \equiv B. \]  \hspace{1cm} (4.52)

Substituting the power-law forms of the scale factors specified above in

\[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}b}{ab} = -(\gamma_3 - 1)\mu, \]  \hspace{1cm} (4.53)

\[ \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}c}{bc} = -(\gamma_1 - 1)\mu, \]  \hspace{1cm} (4.54)

\[ \frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}a}{ca} = -(\gamma_2 - 1)\mu, \]  \hspace{1cm} (4.55)

and adding, we get

\[ m_1^2 + m_2^2 + m_3^2 = \frac{4}{3}B - \frac{1}{3}B^2 - 2(\gamma - 1)\mu_0 \equiv A \]  \hspace{1cm} (4.56)
We solve the continuity equation for the anisotropic pressure field $\mu$ which is given as follows,

$$\dot{\mu} + \gamma_1 \frac{\dot{a}}{a} \mu + \gamma_2 \frac{\dot{b}}{b} \mu + \gamma_3 \frac{\dot{c}}{c} \mu = 0$$

and substituting in the power law expressions for the scale factors $a, b$ and $c$ which have been assumed for the purpose of this computation, we get,

$$\mu \propto t^{-(\gamma m_1 + \gamma m_2 + \gamma m_3)}$$

Equating the powers of $t$ in any of the Bianchi I field equations, we get,

$$\gamma m_1 + \gamma m_2 + \gamma m_3 = 2$$

The explicit forms of the $m_i$'s that result are given below. Before giving them, we define the following new quantities:

$$C = 8 \gamma_1 - 4 \gamma_2 - 4 \gamma_3 + 2B \gamma_1^2 + 2B \gamma_2^2 - 2B \gamma_1 \gamma_2 - 2B \gamma_1 \gamma_3$$

$$D = 2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1 \gamma_2 - \gamma_2 \gamma_3 - \gamma_3 \gamma_1).$$

$$E = 8 - 4B \gamma_2 - A \gamma_2^2 - A \gamma_3^2 + B^2 \gamma_2^2 + B^2 \gamma_3^2 - 4B \gamma_2^2 + 2A \gamma_2 \gamma_3$$

$$F = -4 \gamma_1^2 + (B - 2) \gamma_2^2 + B \gamma_1 \gamma_2^2 - B \gamma_1 \gamma_3^2 + B \gamma_2 \gamma_3$$

$$G = -4 \gamma_1^2 + (B - 2) \gamma_3^2 + B \gamma_1 \gamma_3^2 - B \gamma_1 \gamma_2^2 + B \gamma_3 \gamma_2$$

Therefore, the explicit forms of the $m_i$'s become:

$$m_1 = \frac{C \pm \sqrt{C^2 - 4DE}}{2D}$$

$$m_2 = \frac{2 - B \gamma_3}{\gamma_2 - \gamma_3} + \frac{F}{D(\gamma_2 - \gamma_3)} + \frac{(\gamma_1 \pm \gamma_3)\sqrt{C^2 - 4DE}}{2D}$$

$$m_3 = \frac{2 - B \gamma_2}{\gamma_3 - \gamma_2} + \frac{G}{D(\gamma_3 - \gamma_2)} + \frac{(\gamma_1 \pm \gamma_2)\sqrt{C^2 - 4DE}}{2D}.$$
Effect of anisotropic pressures on isotropising a bouncing model

In order for these expressions to be real, and hence to have real Kasner exponents, we must have \( C^2 > 4DE \). It is easy to see that this is true in the limit \( \gamma_1 > \gamma_2, \gamma_3 \), where we retain terms in the evaluation of \( C^2 > 4DE \) in the first two leading orders of \( \gamma_1 \).

**Shear tensor**

We had initially set out to investigate the effect of the anisotropic pressure fluid on the evolution of the anisotropies. Thus, we next look at the behaviour of the shear tensor on approach to the singularity (or bounce). The shear tensor in the Bianchi type IX spacetime is given as,

\[
\sigma^2 = \frac{1}{6} \left\{ (H_\alpha - H_\beta)^2 + (H_\beta - H_\delta)^2 + (H_\delta - H_\alpha)^2 \right\},
\]

(4.68)
4.4 Bianchi IX universe with isotropic ghost field and fluid with anisotropic pressures

Fig. 4.7 Evolution of the shear in a Bianchi type IX universe with initial conditions satisfying the Friedmann constraint near the singularity.

where \( H_\alpha = \dot{\alpha}, H_\beta = \dot{\beta} \) and \( H_\delta = \dot{\delta} \). We focus on the near-singularity behaviour of the shear tensor. This evolution is shown in 4.7. On examining the figure, we find that with only the isotropic fluid present, the shear remains at a very small nearly constant small positive value. However, when we include the anisotropic pressure fluid, the shear rises and keeps rising to increasingly positive values until the singularity is reached. This is true as long as the anisotropic pressure in at least one direction is less stiff than the pressure of the isotropic ghost fluid. This is equivalent to the requirement that one third of the equation of state parameter \( \gamma_i \) be less than the overall equation of state parameter of the isotropic ultra-stiff ghost fluid. Thus, although the anisotropic fluid may be ultra-stiff and stiffer than the isotropic fluid, it may not be stiffer in a particular direction. This causes the assumption that an energy source that behaves like ultra-stiff matter suppresses the anisotropies near the singularity in a contracting universe to break down. If the anisotropic pressures are stiffer than the isotropic pressure in all three directions (\( \gamma_i / 3 > \gamma \) for all \( i = 1, 2, 3 \)) then the anisotropic stress is more greatly suppressed when the anisotropic pressure fluid is included compared to when only the ghost isotropic fluid is present. This is expected because it is simply the standard ekpyrotic model with a stiffer fluid present in all three scale factor directions to suppress the anisotropic stress. However, when anisotropic pressure fields have equations of state in each direction ensuring that the anisotropic stress is not suppressed, then the universe fails to undergo a bounce and re-expansion beyond the contracting phase. Instead, the contraction accelerates towards a collapse singularity in the Weyl curvature.

In addition to observing these trends, we also note the following general features. When the stiffness of the anisotropic fluid is less than the stiffness of the isotropic ghost fluid, the three scale factors all undergo bounces. The stiffness of the anisotropic fluid determines when this bounce occurs. If it becomes stiff on average (\( \gamma_\star = 2 \)), the scale factors begin to oscillate on approach to the singularity. When the anisotropic fluid is ultra-stiff on average
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(stiffer than the isotropic ghost fluid), but its initial conditions are such that its density is negligible or very small (less than half of the initial isotropic energy density) the scale factors begin to show a turnaround at an expansion minimum. The point of bounce is pushed towards the value at which the turnaround occurs for the isotropic case, as the anisotropic energy density is decreased.

4.5 Conclusions

In this chapter, we have focussed on the effects of pressure anisotropies in simple ekpyrotic [79, 59] cyclic universe scenarios that are more general and complicated than those first studied by Barrow and Yamamoto [26]. Pressure anisotropies have been ignored in all other studies of ekpyrotic and cyclic universes. It is important to include them in the discussion because collisionless particles will be abundant near the Planck scale where graviton production is rapid and asymptotically-free interactions will not be in equilibrium. In addition, we find that if the average anisotropic pressure is allowed to exceed the energy density, just as the isotropic pressure does in the ekpyrotic scenario, then an isotropic singularity (or bounce) will be unstable unless the isotropic density is overwhelmingly larger than the anisotropic density. The anisotropic ultra-stiff fluid will drive a contracting universe to an anisotropic singularity. Evolution from cycle to cycle will accumulate anisotropic distortions to the dynamics.

More formally, we find that the anisotropy, even in the simplest case of a Bianchi I universe with anisotropic pressures present, cannot be expressed simply as a power-law evolution of the mean scale factor. Using a phase-space analysis for the general field equations for the Bianchi Class A group of cosmologies, we find that the presence of an ultra-stiff fluid with anisotropic pressures prevents the isotropic Friedmann-Lemaître universe from being an attractor for the initially contracting universe. More specifically, we analysed the field equations in the case of the Bianchi IX universe. We solved these equations numerically containing ultra-stiff fluids with both anisotropic and isotropic (ghost) pressures. The anisotropies grow when an anisotropic pressure fluid with dominant stiffness is included: the universe contracts and hits a singularity. This contrasts with the case containing only the isotropic ghost fluid, where the universe undergoes a non-singular bounce. Our results confirm that the inclusion of anisotropic pressures is essential in any general analysis of cyclic cosmologies and their behaviour in the presence of deviations from perfect expansion isotropy. They will be an important factor to consider in all future iterations of the cyclic universe scenario in its several forms.
Chapter 5

Cyclic mixmaster universe

5.1 Introduction

In 1922 Alexander Friedmann [62] first noted the existence of ‘periodic worlds’ in his solutions of Einstein’s equations for isotropic and homogeneous universes with positive spatial curvature. But the physical study of cyclic universes in general relativistic cosmology begins with the work of Tolman [125], who first considered the simple situation of a closed Friedmann universe with zero cosmological constant, $\Lambda$, and non-negative pressure. The evolution of Tolman’s cyclic universes can be continued periodically through the big crunch singularity if it is assumed that no new physics arises there and the evolution can be extended smoothly through it $^1$. As a next step, Tolman incorporated the general consequences of an increase of entropy from cycle to cycle, in accordance with the second law of thermodynamics. This produced a monotonic increase in the maximum size and length of successive cycles, which continues forever. If this entropy increase is modelled as an increase in the dimensionless entropy per baryon in a mixture of radiation and baryons (ignoring baryon non-conserving interactions) then the increase is a reflection of the asymmetry in the pressure, $p$, from cycle to cycle, as energy is transferred from the baryonic ($p = 0$) to the radiation ($p \neq 0$) gas. Tolman’s work attracted periodic interest by other astronomers like Zanstra [132] in the 1950s, before Zeldovich and Novikov turned Tolman’s result into a theorem for rather restrictive equations of state of matter obeying Friedmann’s equations and the laws of thermodynamics. They also assumed that the cycles could not be continued indefinitely into the past because they would become smaller than the smallest finite-sized elementary particles (not assumed

$^1$Tolman’s analysis, which assumed no equation of state linking the pressure, $p \geq 0$, to the density, $\rho$, required an additional assumption. To avoid a finite-time (‘sudden’) singularity occurring where $p \to \infty$ with finite $\rho$ before the expansion maximum is reached one must stipulate some control over $p$, for example $p < C\rho$ for constant $C > 0$ [8].
pointlike in those days). As the cycles continue to increase in size, an oscillating universe appears increasingly ‘flat’, although it is closed with positive spatial curvature [82]. This might even provide an explanation for the proximity of the expansion dynamics to flatness today that differs in detail from that of the standard one-cycle inflationary universe model (although the latter generates proximity to flatness by a large entropy increase in one cycle through ‘reheating’ rather than by a progressive build up over many cycles by all processes). However, in what follows we will show that it does not share other features of an inflationary universe at late times.

Generalisations of this simple oscillatory Tolman universe produced some interesting new features. Barrow and Dabrowski [17] show that if there is a positive cosmological constant ($\Lambda > 0$) then the sequence of growing oscillations always comes to an end after a finite proper time and the dynamics evolve towards an ever-expanding de Sitter asymptote as $t \to \infty$ [17]. This end to the oscillations occurs no matter how small the positive value of the cosmological constant is: the cycles grow until they inevitably produce one that is large enough for the $\Lambda$ term to eventually dominate the dynamics at large size and its effect is to stop a further contraction from occurring. Notice that the final state is always one which is close to flatness and only just (depending on the exact size of the inter-cycle entropy jump) dominated by the $\Lambda$ energy density – rather like our universe, in fact.

5.2 Simple isotropic bounces

The assumption of a ‘bounce’ occurring at zero expansion scale and infinite density is computationally (and physically) awkward. However, it can be improved upon by introducing a simple ‘ghost’ field, with negative density, $\rho < 0$, that will cause the expansion to go through a smooth minimum at the beginning and at the end of each cycle instead of through a singularity where $\rho = \infty$. Ghost fields have often been used in bouncing cosmology scenarios to effect a non singular bounce such as in [38]. As illustrations, we can find two simple exact solutions which are of use in more complicated situations. Suppose that we have a closed Friedmann universe with scale factor, $a(t)$, containing two ‘fluids’ having densities $\rho > 0$ and $\rho_g < 0$. The second ‘ghost’ fluid with negative density, $\rho_g$, acts as a model stress to dominate at small $a$ and effect a bounce at $a = a_{\text{min}}$, while the conventional fluid with positive density, $\rho$, dominates at larger $a$. This situation continues to exist until the spatial curvature creates an expansion maximum at $a = a_{\text{max}}$. We give two solutions which are useful models of this type of behaviour for more detailed analyses and illustrate the effects of the two fields:

*Ghost fluid with $p_g = \rho_g \propto a^{-6} < 0$ and conventional radiation fluid with $p = \rho / 3 \propto a^{-4} > 0$*
The Friedmann equation (setting $8\pi G = c = 1$) is

$$\frac{\dot{a}^2}{a^2} = -\frac{\Sigma}{a^6} + \frac{\Gamma}{a^4} - \frac{1}{a^2},$$

with $\Sigma \geq 0$ and $\Gamma \geq 0$ being constants. The exact solution for the scale factor, when $\Gamma^2 \geq 4\Sigma$, can be written simply in terms of the expansion maximum and minimum radii in conformal time, defined by $dt = ad\eta$, as [25]

$$a^2(\eta) = \frac{1}{2} \left[ a_{\text{max}}^2 + a_{\text{min}}^2 + (a_{\text{max}}^2 - a_{\text{min}}^2) \sin 2(\eta + \eta_0) \right],$$

where the integration constant $\eta_0$ can be set to zero without loss of generality and

$$a_{\text{min}}^2 \equiv \frac{\Gamma - \sqrt{\Gamma^2 - 4\Sigma}}{2},$$

$$a_{\text{max}}^2 \equiv \frac{\Gamma + \sqrt{\Gamma^2 - 4\Sigma}}{2}.$$

Oscillatory solutions occur when $\Gamma^2 > 4\Sigma$ and $\Gamma^2 = 4\Sigma$ gives a static universe. In the high radiation entropy ($\propto \rho^{3/4} \propto \Gamma^{3/4}$) limit $\Gamma^2 \gg 4\Sigma$, we have $a_{\text{max}} \to \Gamma$ and $a_{\text{min}} \to \Sigma/\Gamma$ and we see that the maxima grow and the minima decrease in size if we let the radiation entropy grow from cycle to cycle.

Ghost fluid with $p_g = \rho_g/3 \propto a^{-4} < 0$ and conventional dust fluid with $p = 0, \rho \propto a^{-3} > 0$

The Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = -\frac{\Gamma}{a^4} + \frac{M}{a^3} - \frac{1}{a^2},$$

and, if $M^2 \geq 4\Gamma$, a new exact solution can be written simply in conformal time $dt = ad\eta$ in terms of the expansion maximum and minimum radii as [25]

$$a(\eta) = \frac{1}{4} \left[ a_{\text{max}} + a_{\text{min}} + (a_{\text{max}} - a_{\text{min}}) \sin(\eta + \eta_0) \right],$$

where

$$a_{\text{min}} = \frac{1}{2} \left[ M - \sqrt{M^2 - 4\Gamma} \right],$$

$$a_{\text{max}} = \frac{1}{2} \left[ M + \sqrt{M^2 - 4\Gamma} \right].$$
In the limit that $M^2 \gg 4\Gamma$, we have $a_{\text{min}} \to \Gamma/M$ and $a_{\text{max}} \to M$, so if we introduce an increase in matter entropy ($\propto M$) from cycle to cycle then we will have successively increasing maxima and decreasing minima.

In what follows, we shall add a ghost field to an oscillating anisotropic, spatially homogeneous universe in order to produce a smooth bounce at finite values of the scale factor where the densities are non-singular. It would be possible to effect a smooth bounce with a scalar field with quadratic potential which has been investigated in Mixmaster universes [22], however the probability of this bounce occurring is small, $O(a_{\text{min}}/a_{\text{max}})$, in any universe with $a_{\text{min}} \ll a_{\text{max}}$.

All of the above discussion has focussed upon simple isotropic closed universes with $S^3$ spatial topology. The situation in simple anisotropic universes of Kantowski-Sachs type was studied in detail by Barrow and Dąbrowski [17] and produces a more complicated scenario for cycle to cycle evolution. The Kantowski-Sachs universes have special $S^2 \times S^1$ spatial topology and are far from generic even amongst homogeneous anisotropic universes, although they have inhomogeneous generalisations with no symmetries found by Szekeres [122]. Only closed (compact space sections) universes with $S^2 \times S^1$ or $S^3$ topologies possess maximal hypersurfaces and so can recollapse and bounce when gravitationally attractive matter is present [12]. Whether or not they will do so depends on the matter content of the universe.

In this chapter we will study the dynamics of a cyclic Bianchi type IX ‘Mixmaster’ universe with $S^3$ spatial topology. This is the most general spatially homogeneous anisotropic closed universe and it contains the closed Friedmann universe as an isotropic special case. Exact solutions are only known for the axisymmetric special cases in vacuum, containing stiff matter ($p = \rho$) or electromagnetic fields, or a combination of both. [113, 13]. We are particularly interested in the behaviour of these anisotropic universes on approach to an expansion maximum of the volume and the behaviour of the three expansion scale factors there. Do the cycles grow in maximum size and do they become increasingly anisotropic from cycle to cycle? We will confine our attention to the case where the fluid in the universe is comoving, although in a subsequent study we will generalise this to fluids with non-comoving velocities. The type IX universe behaves quite differently to the simple Bianchi type I anisotropic universe because it has both expansion anisotropy (shear) and 3-curvature anisotropy. The 3-curvature anisotropy has no Newtonian analogue. The 3-curvature dynamics are complicated and the sign of the 3-curvature varies in time and is only positive when the expansion dynamics are sufficiently close to isotropy. An expansion maximum will only occur when the 3-curvature, $(^3 R)$, becomes positive (as it is all the time in the closed Friedmann universes). We want to discover if increasing entropy increases the size
of successive expansion maxima, as in the cyclic Friedmann models, but also to determine what happens to the expansion anisotropy over successive cycles. We can also incorporate a positive or negative cosmological constant to see if it can terminate a sequence of oscillations in a cyclic type IX universe in the same way that it does in an isotropic closed universe.

In order to follow the Bianchi type IX evolution smoothly from cycle to cycle, we introduce a stiff ghost field to create an expansion minimum at non-zero volume in every cycle, as discussed above. This field has no significant effect on the expansion maxima or the behaviour of the dynamics in its vicinity. It is well known that the Bianchi IX model displays formal chaotic behaviour in all its degrees of freedom as the volume tends to zero [14]. However, there is only an unbounded number of chaotic oscillations of the scale factor on an open interval $0 < t < T$ around the time origin for finite $T$: an infinite number of scale factor oscillations occur on any such interval no matter how small the value of $T$. In any finite interval $T_1 < t < T$, not including $t = 0$, the number of oscillations is finite and not technically chaotic. For realistic choices of $T_1 \approx 10^{-43}\text{s}$ for the start of classical cosmology, there will be less than about 12 Mixmaster oscillations even if they continued all the way from $T_1$ up to the present day [134]. This is because the overall expansion scale changes rapidly with the number of scale factor oscillations, which occur in logarithmic time. Thus, if there is a bounce at finite volume the issue of chaotic Mixmaster oscillations [31] is irrelevant to a discussion of the long-term dynamics.

Some bouncing cosmologies deal with the situation at bounce by incorporating a so-called phase of ekpyrosis where there is effectively an ultra stiff isotropic fluid with a $p \gg \rho$ field added to the matter content of the universe [121]. If there are no anisotropic pressures, this drives the dynamics towards isotropy as a singularity is approached. However, we should expect anisotropic pressures to be larger than the energy density (as is assumed for the isotropic pressure), due to the dominance of collisionless particles when $T > 10^{15}\text{GeV}$. The presence of these anisotropic pressures during the phase of ekpyrosis can reinstate the distorting effects of anisotropies and they diverge on approach to the bounce [26, 18].

In this chapter, I shall consider the effect of expansion and curvature anisotropies on a model of a bouncing type IX universe. This model incorporates several bounces, and we increase the entropy of the constituent matter content via an injection at each bounce. The increased entropy in each bounce leads to a higher maxima and longer cycles, as found in the original analysis of Tolman. The claim is that with increasing maxima, simple isotropic bouncing models, such as the isotropic Friedmann universe approaches flatness and begins to resemble the present-day universe in this respect. This claim shall be investigated in more general circumstances. We confirm the increasing volume maxima in successive bounces and the eventual cessation of bounces in the presence of a cosmological constant but with a more
complicated transitional evolution for the isotropisation of the three scale factors. However, we find that in the absence of a cosmological constant successive cycles become increasingly anisotropic despite the increase in size and approach to flatness. This is quite different to the long-term evolution predicted by inflation.

We also investigate the effects of a negative cosmological constant. This always produces collapse to a future singularity [124]. Our aim in doing this is to construct an anisotropic version of the simple Friedmann universes which can all be transformed into simple harmonic oscillators in conformal time after rescaling the expansion scale factor [7]. These models are studied as a further simple example of a bouncing type IX universe. They include ‘domain-wall’ matter \( p = -2/3 \rho \), as well as a negative cosmological constant in a closed Friedmann universe. We can also include an ultra-stiff matter field with isotropic pressures \( p = 5 \rho \) to subdue the anisotropies on approach to the bounce. In the absence of pressure anisotropies, this should work and allow our model to propagate further without hitting a singularity. In a later study we will include both non-comoving velocities and associated pressure anisotropies into the analysis of cyclic type IX universes.

### 5.2.1 Entropy

Our next task is to follow the consequences of a growth of entropy in the constituents of the system. The definition of a cosmological entropy is still debated. Here, we shall use only the thermodynamic entropy of the radiation or matter content and ignore any contribution from a ‘gravitational entropy’ that might be associated with Weyl curvature, gravitational clustering, or the area of the particle horizon [46]. We will use a very simple toy model of entropy injection in our model of a bouncing universe. We are, in this analysis, not interested in the physical origin of the production of entropy by non-equilibrium processes like quantum particle production or viscous anisotropy damping, which are dramatic entropy producers as \( t \to 0 \), [22]. Rather, we will consider sudden entropy increase at each expansion minimum and also use the dependence of the size of the expansion maximum on the entropy to determine the effect of increasing the entropy in a cycle. To circumvent issues regarding the non-conservation of baryon number, and hence making the definition of entropy per baryon ambiguous, we shall consider the effects of an increase of entropy of radiation. To model this, we consider first the definition of the entropy of radiation, \( S \), which is given by,

\[
S \propto T^3 V,
\]

where \( T \) is the temperature at that instant and \( V \) is the volume of the universe. We assume that the entropy per unit volume, once injected at the minima, remains constant throughout the
duration of the cycle until the next minimum (we ignore all particle-antiparticle annihilations and massive particles that become non-relativistic). The radiation energy density varies as $\rho_r = C_r V^{-4/3}$. Hence, during each cycle, the quantity $T^3V$ is a constant, implying that $\rho_r \propto T^4$. So, we can write the entropy as

$$S \propto T^3 \left( \frac{C_r}{\rho_r} \right)^{3/4} \propto C_r^{3/4}. \quad (5.2)$$

Thus, we can assess the effect of increasing the entropy of radiation by an increase in the constant $C_r$. It has been shown previously, in works such as [125], that an increase in the entropy of radiation leads to an increase in the expansion maxima in closed Friedmann universes. We expect this to occur also for a Bianchi IX universe. However, we also wish to study how the shape of the anisotropy behaves as the cycles get bigger with entropy injection. For this we need to specify the form of the type IX metric and analyze the field equations.

This is a good approximation to the more physical model of allowing energy exchange between some of the constituent fields of the model. For example, between the ghost fluid and the radiation fluid. This can be seen for example in [21], where explicit energy transfer terms between the scalar field and the radiation term are included, and the same growth in expansion maxima from cycle to cycle can be seen as shown here in this chapter. Another example is in [73], where in the bottom right panel in Figure 1 we can see the same increase in expansion maxima from cycle to cycle. In this case the bounce occurs in a flat universe with negative $\Lambda$ to cause the bounce. In general bulk viscosity entropy production is proportional to $H^2$ and is hence zero anyway at the bounce.

### 5.3 Closed isotropic bouncing universes with entropy increase

In this section, we review the results of [17] in the effect of entropy increase on closed, isotropic universes. We consider the Friedmann equation with ordinary dust, radiation and cosmological constant.

$$\frac{a^2}{a^2} = \frac{1}{3} (\rho_r + \rho_m) + \frac{\Lambda}{3} - \frac{1}{a^2} \quad (5.3)$$
The conservation laws for radiation and dust are given from the continuity equation respectively as,

\[ \rho_r a^4 = 3C_r = \text{const.} \]  \hspace{1cm} (5.4)

\[ \rho_d a^3 = 3C_r = \text{const.} \]  \hspace{1cm} (5.5)

Thus the Friedmann equation becomes in terms of the integration constants \(C_r\) and \(C_m\),

\[ \frac{\dot{a}^2}{a^2} = \frac{C_r}{a^4} + \frac{C_m}{a^3} + \frac{\Lambda}{3} - \frac{1}{a^2} \]  \hspace{1cm} (5.6)

The entropy is increased by increasing \(C_r\) as explained in section 5.2.1.

**Matter dominated universes**  The parametric solution for matter dominated universes, \(C_r = \Lambda = 0\), is given by [130]

\[ a(\tau) = \frac{C_m}{2} \left[ 1 - \cos(\tau - \tau_0) \right] \]  \hspace{1cm} (5.7)

\[ t(\tau) - t_0 = \frac{C_m}{2} \left[ \tau - \sin(\tau - \tau_0) \right] \]  \hspace{1cm} (5.8)

where \(t_0\) and \(\tau_0\) are constants. Let \(n\) be an integer marking the number of a subsequent cycle in an oscillating universe and \(C_{mn}\) be the value of the constant \(C_m\) in the \(n\)th cycle,

\[ a_n(\tau) = \frac{C_{mn}}{2} \left[ 1 - \cos \tau_n \right] \]  \hspace{1cm} (5.9)

and,

\[ t_n(\tau) - t_{0n} = \frac{C_{mn}}{2} \left[ \tau_n - \sin \tau_n \right] = \frac{\alpha^{n-1}}{2} C_{m1} \left[ \tau_n - \sin \tau_n \right] \]  \hspace{1cm} (5.10)

where \(t_{0n}\) is the value of the constant \(t_0\) in the \(n\)th cycle. where the recursion relation for the constants \(C_{mn}\) is given by

\[ C_{mn} = \alpha C_{mn-1} \hspace{1cm} (\alpha = \text{const.} > 1) \]  \hspace{1cm} (5.11)

The parameter \(\tau_n\) must be \(2\pi(n-1) \leq \tau_n \leq 2\pi n\), and the cosmological time at the end of the \(n\)th cycle is given by,

\[ t_n = \left( \sum_{i=1}^{n} \alpha^{i-1} \right) \pi C_{m1} \]  \hspace{1cm} (5.12)
The maximum height of the \( n \text{th cycle}, \)

\[
R_{n,\text{max}} = \alpha^{n-1} C_{m1}
\]  

(5.13)

The cosmological time instant at which this occurs,

\[
t_{n,\text{max}} = \left[ \left( \sum^{n-1}_{i=1} \alpha^{i-1} \right) + \frac{\alpha^{n-1}}{2} \right] \pi C_{m1}
\]  

(5.14)

The constant \( t_{0n} \) is given by,

\[
t_{0n} = \left[ \left( \sum^{n-1}_{i=1} \alpha^{i-1} \right) + (1-n)\alpha^{n-1} \right] \pi C_{m1}
\]  

(5.15)

As the cycles grow larger, we take the \( n \to \infty \) limit to find that the expansion maxima height and the time period of each cycle tends to infinity.

**Radiation dominated universe**  The solution for radiation dominated universe with \( C_{m} = \Lambda = 0 \) is given by,

\[
Z_n(t) = \alpha_n^2(t) = -t^2 + 2t_{0n}t + (C_r - t_{0n}^2)
\]  

(5.16)

where \( t_{0n} \) is the value of the constant \( t_{0} \) in the \( n \text{th cycle} \). The recursion relation is then,

\[
C_{rn} = \beta C_{rn-1} \quad (\beta = \text{const.} > 1)
\]  

(5.17)

where \( C_{nn} \) be the value of the constant \( C_{m} \) in the \( n \text{th cycle} \). The cosmological time at the end of the \( n \text{th cycle} \) is,

\[
t_n = 2\sqrt{C_{r1}} \sum^{n}_{i=1} \beta^{i-\frac{1}{2}}
\]  

(5.18)

The expansion maxima height in the \( n \text{th cycle} \),

\[
a_{n,\text{max}} = \beta^{\frac{n-1}{2}} \sqrt{C_{r1}}
\]  

(5.19)

The cosmological time at which this occurs,

\[
t_{n,\text{max}} = 2\sqrt{C_{r1}} \left[ \left( \sum^{n-1}_{i=1} \beta^{i-\frac{1}{2}} + \frac{1}{2} \beta^{\frac{n-1}{2}} \right) \right]
\]  

(5.20)

In the limit of \( n \to \infty \), the time period of the cycle as well as the height of the cycle goes to infinity.
Matter and radiation universe  In this case, we have only $\Lambda = 0$, while all the other parameters ($C_r$ and $C_m$) are non-zero. The solution for the scale factor is given by,

$$a(\tau) = \frac{C_m}{2} \left[ \sqrt{1 + \frac{4C_r}{C_m^2} (1 - \cos(\tau - \tau_0))} + \left(1 - \sqrt{1 + \frac{4C_r}{C_m^2}}\right) \right]$$  (5.21)

and for the cosmological time is,

$$t(\tau) - t_0 = \frac{C_m}{2} \left[ (\tau - \tau_0 - \tau(\alpha)) - \sqrt{1 + \frac{4C_r}{C_m^2} (\sin(\tau - \tau_0) - \sin \tilde{\tau}(\alpha))} \right]$$  (5.22)

where

$$\alpha = \frac{4C_r}{9C_m^2}$$  (5.23)

and

$$\tilde{\tau}(\alpha) = \frac{3}{2} \frac{\alpha}{3\alpha + \frac{1}{9}(1 + \sqrt{1 + 9\alpha})}$$  (5.24)

The maximum value of the scale factor in the $n$th cycle is given by,

$$a_{n,\text{max}} = \frac{1}{2} \left( C_{mn} + \sqrt{C_{mn}^2 + 4C_{mm}} \right)$$  (5.25)

and this occurs at the instant of cosmological time given by,

$$t_{\text{max},n} = (2n - 1) \sqrt{C_r} + \frac{C_{mn}}{2} \left[ \frac{\pi}{2} + (2n - 1) \arcsin \left( \frac{C_{mn}}{\sqrt{C_{mn}^2 + 4C_r}} \right) \right]$$  (5.26)

Thus we see that the size of the universe, that is the expansion maxima height grows with an increase of $C_r$.

Positive $\Lambda$-term plus radiation universe  The solution for the scale factor is given by,

$$Z(t) = a(t)^2 = \frac{3}{2\Lambda} \left\{ 1 - \cosh \left[ 2(t - t_0) \sqrt{\frac{\Lambda}{3}} \right] + \lambda \sinh \left[ 2(t - t_0) \sqrt{\frac{\Lambda}{3}} \right] \right\}$$  (5.27)

Given that we define $\lambda_n = \sqrt{\frac{4}{3} C_r \Lambda}$ for the $n$th cycle, we see that $\lambda$ increases due to an increase of entropy through $C_r$ according to the recursion relation,

$$\lambda_{n+1} = \sqrt{\beta} \lambda_n$$  (5.28)
5.4 The model of the bouncing Bianchi type IX universe

Thus the general solution for the square of the scale factor in the $n$th cycle is given by,

$$Z_n(t) = \frac{3}{2\Lambda} \left[ 1 - \frac{1}{2} (1 - \lambda_n)^n \exp \left( 2t \sqrt{\frac{\Lambda}{3}} \right) - \frac{1}{2} (1 + \lambda_n)^n \exp \left( -2t \sqrt{\frac{\Lambda}{3}} \right) \right]$$

(5.29)

The maxima is given by,

$$a_{n,\text{max}} = \frac{3}{2\Lambda} \left[ 1 - \sqrt{1 - \lambda_n^2} \right]$$

(5.30)

$$t_{n,\text{max}} = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \left( n - \frac{1}{2} \right) \ln \left( \frac{1 + \lambda_n}{1 - \lambda_n} \right)$$

(5.31)

The lifetime of the universe is calculated from the expression for the time at the beginning of each cycle,

$$t_n = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} (n - 1) \ln \left( \frac{1 + \lambda_n}{1 - \lambda_n} \right)$$

(5.32)

The lifetime of the $n$th cycle is given by,

$$t_{n+1} - t_n = \frac{1}{2} \sqrt{\frac{3}{\Lambda}} \left[ n \ln \frac{1 + \beta \lambda_n}{1 - \beta \lambda_n} + (n - 1) \ln \left( \frac{1 - \lambda_n}{1 + \lambda_n} \right) \right]$$

(5.33)

The size of the universe increases with the increase in radiation entropy, but the cycles cease and the universe expands forever when the condition $\lambda_n > 1$ is reached. The lifetime of each cycle also grows as $C_r$ increases. The influence of the cosmological constant is only felt when the universe has grown to sufficient size, before which, it behaves like a radiation dominated universe. The cycles cease in a de-Sitter asymptote and the universe enters exponential expansion. These behaviours will be studied and in certain cases replicated for the closed, anisotropic universe, bouncing model.

5.4 The model of the bouncing Bianchi type IX universe

5.4.1 Einstein equations for diagonal type IX universes

We aim to study the effects of entropy growth in anisotropic oscillating models of Bianchi type IX. We divide the problem into several sub-cases. We are interested in discovering whether the present-day universe would isotropise and approach flatness in a bouncing universe model after many cycles of entropy growth. In all cases, we take our cosmological model to be the spatially homogeneous Bianchi Type IX spacetime containing radiation with pressure $p = \rho/3$, and a ‘dust’ matter field with equation of state $p = 0$. We use the radiation field to assess the effects of increasing the entropy of radiation from cycle to
cycle in a bouncing universe. If only these two fluids are present, the universe will evolve towards a strong curvature singularity and experience only one cycle unless we assume a periodic continuation through the singularity. If we add the ultra-stiff ghost field with $\rho_g < 0$ and $p_g/\rho_g \gg 1$ then we create a smooth non-singular bounce and can follow the evolution through several cycles. We can then introduce a growth of entropy in the radiation field in the vicinity of the bounce in order to study the long-term effects of the shear anisotropy and the 3-curvature on the expansion maximum. In the last section of the chapter, we introduce a positive cosmological constant and also a negative cosmological constant, to see how the evolution is changed by their presence. In the latter case, we will add a domain-wall fluid ($p = -2\rho/3$) and an ultra-stiff fluid ($p = 5\rho$) to facilitate smooth non-singular bounces.

Note that when a ghost field is added to create a non-singular bounce it means that the dynamics will be dominated by this isotropic matter field at the expansion minima and so it will have a small isorotising effect over small time intervals around the minima. However, it is outweighed by the lengthening of the evolution time produced by the growing size of successive maxima. It would be possible to effect non-singular bounces with an anisotropic ghost field but this complication has been avoided here.

We study the solution of the Einstein equations for a diagonal spatially homogeneous Bianchi type IX universe with metric

$$\begin{align*}
 ds^2 &= dt^2 - \gamma_{ab}(t) e_\mu^a e_\nu^b dx^\mu dx^\nu, \\
 \gamma_{ab}(t) &= \text{diag}[a^2(t), b^2(t), c^2(t)], \\
 e_\mu^a &= \begin{bmatrix}
 \cos z & \sin z \sin x & 0 \\
 -\sin z & \cos z \sin x & 0 \\
 0 & \cos x & 1
 \end{bmatrix},
\end{align*}$$

containing non-interacting perfect fluids, each with a perfect fluid equation of state $p = (\gamma - 1)\rho$. The equations of state parameters for the radiation, dust and ghost fields are given by $\gamma_r = 4/3$, $\gamma_m = 1$ and $\gamma_g = 2$, respectively. The orthogonal expansion scale factors $a(t)$, $b(t)$ and $c(t)$ for the Bianchi type IX universe with the specified matter content satisfy the field equations:

$$\begin{align*}
 \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{a^2}{4b^2c^2} + \frac{\dot{b}^2}{4a^2c^2} - \frac{3c^2}{4a^2b^2} + \frac{1}{2a^2} + \frac{1}{2b^2} - \frac{1}{2c^2} &= - \sum_{i=r,m,g} (\gamma_i - 1)\rho, \tag{5.35}
\end{align*}$$
The model of the bouncing Bianchi type IX universe

\[
\dot{b} + \frac{\dot{c}}{c} + \frac{\dot{b}}{bc} + \frac{b^2}{4a^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3a^2}{4b^2c^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{1}{2a^2} = - \sum_{i=r,m,g} (\gamma_i - 1) \rho, \tag{5.36}
\]

\[
\dot{c} + \frac{\dot{a}}{a} + \frac{\dot{c}}{ca} + \frac{a^2}{4b^2c^2} + \frac{c^2}{4a^2b^2} - \frac{3b^2}{4a^2c^2} + \frac{1}{2a^2} + \frac{1}{2c^2} - \frac{1}{2b^2} = - \sum_{i=r,m,g} (\gamma_i - 1) \rho. \tag{5.37}
\]

The constraint equation reduces to,

\[
\frac{\dot{a}b}{ab} + \frac{\dot{b}}{bc} + \frac{\dot{c}}{ca} + \frac{1}{2a^2} + \frac{1}{2b^2} + \frac{1}{2c^2} - \frac{a^2}{4b^2c^2} - \frac{b^2}{4a^2c^2} - \frac{c^2}{4a^2b^2} = \sum_{i=r,m,g} \rho \tag{5.38}
\]

From the fluid continuity equations, we have,

\[
\rho_r(t) \propto (abc)^{-4/3} \\
\rho_s(t) \propto (abc)^{-2} \\
\rho_m(t) \propto (abc)^{-1}
\]

If we introduce a new time coordinate, \(\tau\), by defining

\[
d\tau = dt/abc \tag{5.39}
\]

then the field equations become (' denotes \(d/d\tau\)):

\[
2(\ln a)^\prime \prime + a^4 - (b^2 - c^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \tag{5.40}
\]

\[
2(\ln b)^\prime \prime + b^4 - (c^2 - a^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \tag{5.41}
\]

\[
2(\ln c)^\prime \prime + c^4 - (a^2 - b^2)^2 = a^2b^2c^2 \sum_{i=r,m,g} (\rho_i - p_i), \tag{5.42}
\]
and the constraint equation simplifies to,

\[
4[(\ln a)'(\ln b') + (\ln b)'(\ln c') + (\ln c)'(\ln a')] = a^4 + b^4 + c^4 - 2c^2(a^2 + b^2) - 2a^2b^2 + 4a^2b^2c^2 \sum_{i=r,m,g} \rho_i.
\]

Before we attempt to study the full numerical evolution of the type IX equations of motion with an increase in entropy of the radiation field in the presence of the ultra-stiff ghost field and a dust field, we shall try to construct an approximate parametric solution for the type IX evolution containing only the radiation field. A detailed study of the behaviour of the most general Bianchi type universes at intermediate times has been conducted in refs. [54, 92]. What they reveal is that at a very early time there is a reduction in anisotropy by quantum effects which is significant. To the future of such a time, the evolution enters a long quasi-axisymmetric phase. Two scale factors are larger than the third and differences between the first two are insignificant compared to their size relative to the other. This situation is familiar from the evolutionary pattern during isotropisation of Kasner metrics containing collisionless particles [91] where the anisotropic pressures created by the particles mimic the 3-curvature anisotropies in type IX. It is as if the dynamics has entered a time-reverse of one of the long cycles that a Bianchi type IX universe encounters on approach to small times. Following Doroshkevich et al [54], we use the approximation \( a = b \gg c \), which reduces the equations (5.40)-(5.42) to,

\[
\begin{align*}
(\ln a)'' + a^2c^2 &= \frac{1}{3}\rho_r a^4 c^2, \\
(\ln c)'' &= \frac{1}{3}\rho_r a^4 c^2, \\
2(\ln a)'(\ln c)' + (\ln a)^2 &= -a^2c^2 + \rho_r a^4 c^2.
\end{align*}
\]

We find the parametric solution quoted in [54], as follows. Defining \( \omega \), the ratio of the two terms on the right-hand side of (5.46) by

\[
\omega^2 \equiv \frac{a^2c^2}{3\rho_r a^4 c^2},
\]

we can express the radiation density in terms of the scale factors using \( \rho_r = C_r(a^2c)^{-4/3} \), and find \( \omega \) in terms of the scale factors as \( 3\omega^2 = a^{2/3}c^{4/3}/C_r \). Hence, using (5.44)-(5.46),
we can write

\[ 2(\ln \omega)'' = \frac{2}{3} (\rho_r a^4 c^2 - a^2 c^2) = \frac{2}{3} \rho_r a^4 c^2 (1 - 3\omega^2). \] (5.48)

Inspecting the above equation we see that the \( \rho_r a^4 c^2 \) term must be a function of the form \( f(\omega, \omega', \omega'') \). Our next task is to determine the form of this function, so that once we substitute this form in, we can get a self consistent solution for the evolution equation for the parameter \( \omega \), both by following the route of using the equation (5.47) to obtain \( \omega \) in terms of the scale factors and then using the equations of motion of the scale factors; and also by using the functional form in terms of the parameter \( \omega \) that we choose for \( \rho_r a^4 c^2 \) in its own evolution equation. After a few simple trials, it luckily turns out that the ansatz \( \rho_r a^4 c^2 = \alpha \omega' \) gives us the self consistent solution we desire, by following both of the routes we described. We shall demonstrate that this is in fact the case. We choose the ansatz \( \rho_r a^4 c^2 = \alpha \omega' \) and absorb the constant of integration \( C_r \) into the constant \( \alpha \). Then we have,

\[ (\ln \omega)' = \frac{1}{3} \alpha (\omega_0 + \omega - \omega^3). \] (5.49)

where \( \omega_0 \) is a constant of integration. We now examine the equation we get for \( \rho_r a^4 c^2 \) by substituting in the equations of motion This is,

\[ (\ln \rho_r a^4 c^2)'' = \frac{2}{3} \rho_r a^4 c^2 (1 - 6\omega^2). \] (5.50)

Using our ansatz, \( \rho_r a^4 c^2 = \alpha \omega' \), we also find that

\[ (\ln \alpha \omega')'' = \frac{2}{3} \alpha \omega' (1 - 6\omega^2). \] (5.51)

We can write the left hand side of this equation as \( (\ln \alpha + \ln \omega')'' = (\ln \omega')'' \) as \( \ln \alpha \) is a constant. We can integrate the above equation once as,

\[ (\ln \omega')' = \frac{2}{3} (\omega_0 - 2\omega^3 + \omega) \] (5.52)

We can write out the left hand side of this equation as follows,

\[ (\ln \omega')' = \omega' \frac{d}{d\omega} (\ln(\omega'/\omega) + \ln\omega) \] (5.53)
Differentiating and cancelling factors of \( \omega' \) from the first term in the brackets in the above equation, and then substituting in the right hand side of equation (5.48) for \( (\ln \omega)' \), we get,

\[
(\ln \omega')' = (\ln \omega)' + \frac{1}{3} \alpha \omega(1 - 3\omega^2)
\]  

(5.54)

Using the expression we found previously for \( (\ln \omega)' \), in equation (5.48), we recover the same right hand side as equation (5.52) up to some additive integration constants, confirming that our ansatz gives us a self-consistent solution. Hence, we can write the evolution equation for the parameter \( \omega \) as,

\[
(\ln \omega)' = Q^{1/2}(\omega_0 + \omega - \omega^3),
\]  

(5.55)

where we have redefined our constant \( \alpha \) to be the constant \( 3Q^{1/2} \) for notational consistency with ref. [92]. In conclusion, we have the following radiation-era solution in terms of the parameter \( \omega \) as in [92]:

\[
a(\tau) = 3^{1/2}Q(\omega_0 + \omega - \omega^3),
\]  

(5.56)

and

\[
c(\tau) = \frac{3^{1/2}\omega^{3/2}}{Q^{1/2}(\omega_0 + \omega - \omega^3)^{1/2}}.
\]  

(5.57)

By inspecting the solution, from the evolution equation of the parameter \( \omega \), that is (5.48), we find that, for \( \omega = -1, 0, 1 \), the equation yields a simple form,

\[
(\ln \omega)' = Q^{1/2}\omega_0.
\]  

(5.58)

However, for the special choices of \( \omega = -1, 0 \), the scale factor \( c(\tau) \) becomes imaginary and zero, respectively. On computing the volume maxima in terms of \( \omega \) for all positive values of \( \omega \) numerically, we find that the value of \( \omega \) at which the volume maximum occurs is very close to 1. Thus, we conclude that \( \omega \sim 1 \) is the point marking the volume maximum as well as the endpoint of the validity of this parametric solution.  

We now use the axisymmetric solution for the Type IX universe to see how the anisotropy behaves at the maximum of each cycle as we increase the radiation entropy, which is proportional to \( C_3^{3/4} \). Keeping in mind that the limit \( \omega \to 1 \) corresponds to the instant of maximum volume of the expansion, we can see how the ratios of the scale factors in the two

\[\text{Well before the volume maximum is approached the effects of the anisotropic 3-curvature are similar to the addition of a trace-free anisotropic pressure stress (long-wavelength homogeneous gravitational wave modes) on a background of simple Bianchi I form [16] or Friedmann form [69]. In the presence of isotropic black body radiation the two scale factors evolve as } a(t) \sim t^{1/2}(\ln t)^{2n_1} \text{ and } c(t) \sim t^{1/2}(\ln t)^{n_2}, \text{ with } 2n_1 + n_2 = 0, \text{ so the volume } a^2b \sim t^{3/2}, \text{ evolves as in Friedmann, to leading order [54, 16] but the shear falls more slowly than when the 3-curvature is isotropic.}\]
5.4 The model of the bouncing Bianchi type IX universe

directions behave in this limit. In the limit of $\omega \rightarrow 1$, the ratio $a/c$ reduces to,

$$\frac{a^2}{c^2} = Q^3 \omega_0^3.$$  \hspace{1cm} (5.59)

We have seen that the quantity $Q$ is related to the constant $\alpha$, and hence to the constant $C_r$ (which has been absorbed into the constant $\alpha$ as stated above). Therefore an increase in the entropy of radiation (note that equation (5.39) ensures that $dS/dt > 0$ if and only if $dS/d\tau > 0$) causes a corresponding increase in the ratio of the scale factors, which indicates that the universe is becoming more anisotropic as the heights of the successive expansion maxima increase.

5.4.2 Adding a positive cosmological constant

We can use a similar approximation, but this time in comoving proper time, $t$, instead of the conformal time coordinate, $\tau$. Using the approximation that $c \ll a = b$, we can reduce the Einstein equations to the following form:

$$2\ddot{a} = \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \Lambda,$$  \hspace{1cm} (5.60)

$$\ddot{c} = \frac{\dot{c}^2}{c^2} + \frac{\dot{a} \dot{c}}{c} = \Lambda,$$

where $\Lambda$ is the cosmological constant. If we rewrite the first equation in the form

$$a \frac{d}{da} \left( \dot{a}^2 \right) + \dot{a}^2 = \Lambda a^2 - 1,$$ \hspace{1cm} (5.61)

it integrates to,

$$\dot{a}^2 = \frac{\Lambda a^2}{3} - 1 + \frac{C_1}{a},$$ \hspace{1cm} (5.62)

where $C_1$ is an integration constant. In order to determine the value of the constant $C_1$, we can choose Kasner initial conditions where the scale factor $a(t) \sim t^{2/3}$ as $t \rightarrow 0$. Substituting this into equation (27), we get $C_1 = 4/9$ as $t \rightarrow 0$. The above equation resembles the equation for ordinary dust and a cosmological constant in an isotropic closed Friedmann universe. In the case of cosmological constant domination, the solution for $a(t)$ tends to the de Sitter solution, that is,

$$a(t) \sim \exp \left( \frac{\sqrt{\Lambda} t}{3} \right)$$ \hspace{1cm} (5.63)
Under the approximation $a \sim b \gg c$, the Friedmann constraint becomes,

$$\frac{\dot{a}^2}{a^2} + 2\frac{\dot{a} \dot{c}}{ac} + \frac{1}{a^2} = \Lambda \quad (5.64)$$

Substituting in (27), we get,

$$2\frac{\dot{a} \dot{c}}{ac} + C_1 = \frac{2\Lambda}{3} \quad (5.65)$$

Substituting in the solution for the cosmological constant dominated $a(t)$, and taking the limit of very late times (which is the limit in which we expect cosmological constant domination), we get

$$c(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \quad (5.66)$$

Thus we see that for cosmological constant domination isotropisation is achieved as $t \to \infty$, with the scale factors evolving towards,

$$a(t) = b(t) \sim c(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \quad (5.67)$$

It is worth noting that this result is not just a case of the standard cosmic no-hair theorem for spatially homogeneous universes due to Wald [129] because the three-curvature can be positive in type IX universes and all cosmological no-hair theorems assume that $3\mathcal{R} \leq 0$, [74], [20], [15]. Ostensibly, this is to ensure the universe does not suffer collapse to a future singularity before the $\Lambda$ term can dominate. However, the Kantowski-Sachs spatially homogeneous universe has $3\mathcal{R} > 0$ and need not approach the de Sitter metric at large $t$ when $\Lambda > 0$, [77]. In fact, the conditions necessary (and sufficient) for type IX models with $\Lambda = 0$ to recollapse are extremely subtle [12, 8] and examples have been found where type IX universes expand forever even though the sum of the density and the three principal pressures is positive [40]. Typically, the 3-curvature is negative as long as the dynamics are significantly anisotropic ($3\mathcal{R} = 2/c^2 - b^2/2c^4$ for axisymmetric 5.34 when $a = b \gg c$). This causes the expansion to continue until there is sufficient isotropisation for the three-curvature to become positive and only then does an expansion maximum of the volume become possible. This occurs unless a positive cosmological constant comes to dominate before it is reached, as in equation (32).

Hence, we see that the results of Barrow and Dabrowski [17] showing the inevitable termination of oscillations in a closed oscillation universe with $\Lambda > 0$ continue to hold in Bianchi type IX universes. Eventually, cycles will grow large enough for the $\Lambda$ term to dominate the dynamics. When it does so, it quickly stops the scale factors from reaching
expansion maxima. They all continue to expand and the dynamics are increasingly dominated
by the $\Lambda$ term and approach the de Sitter metric.

## 5.5 Numerical solutions of the type IX equations

We move now to consider the numerical integration of the full (non-axisymmetric) type IX
equations; first, in the presence only of radiation, $\rho_r$ and then with radiation and a ghost field,
$\rho_g$, that will create a smooth bounce at a finite volume minimum. In each case we will be
interested in the behaviour with ($\dot{S} > 0$) and without ($\dot{S} = 0$) entropy increase and with and
without a cosmological constant. These computations will enable us to confirm the general
picture found from the analytic approximations of the previous section.

### 5.5.1 Radiation universe with no entropy increase: $\dot{S} = 0$

**No ghost field: $\rho_g = 0$**

In the case of the Bianchi IX universe containing just the radiation field we have followed
the initial assumption of [54] that two of the scale factors approach the same value in the
expanding half of the cycle. Thus on approach to the maximum of the scale factor, the
universe resembles an axisymmetric type IX universe. The scale factors that are similar
to each other approach their maxima first and then, after reaching their maxima, they start
contracting and then oscillate around an almost constant value. The third scale factor
approaches its maximum at a later time and goes past the peaks of the other two scale factors
before reversing into contraction. We can now follow the evolution of the quantities in this
scenario, without needing to assume axial symmetry, by computing the behaviour of the
scale factors from various sets of initial conditions.

We start with initial conditions that are similar to the ones chosen in [54], that is,

$$[a(t), b(t), c(t)] \propto [t^{1/2}, t^{1/2}, t^{5/8}]. \quad (5.68)$$

In this case, in the absence of the ghost field, we find that the universe is unable to re-expand
after collapsing. The addition of the radiation or dust field does not cause a qualitative change
in the behaviour of the scale factors, but causes the system to reach stiffness faster. Thus we
choose values for the initial conditions for the density of radiation to be $\rho_r(t_i) = 1$ where
$t_i$ refers to the initial instant of time at which we start the integration. The scale factors in
two directions oscillate about each other as they follow the evolution trend of the volume
scale factor. The third scale factor is then smaller than the other two scale factors and does
not display the oscillatory behaviour undergone by the other two scale factors. The shear and the 3-curvature show oscillatory profiles before blowing up on approach to the strong curvature singularity at the ‘big crunch’. The presence of the singularity at the end of the collapse phase is inferred by the fact that the density of the matter and radiation components diverge there.

Running the simulation with arbitrarily selected initial conditions or even Kasner-like initial conditions, makes the collapse occur closer to the starting instant, in comparison to the case done with the initial conditions in (33) and little information can be extracted from the results. For the Kasner initial conditions, however, before the collapse occurs, the individual scale factors show some oscillatory behaviour.

![Figure 5.1](image)

Fig. 5.1 Time evolution of the volume (left) and three orthogonal scale factors (right) of a type IX universe with only radiation and no ghost field in comoving proper time $t$ during a single cycle. The blue dashed, green dotted and yellow solid lines correspond to scale factors $a(t)$, $b(t)$ and $c(t)$ respectively.

We see a single oscillation of the volume scale factor in Figure 5.1a. The behaviours of the individual scale factors are seen in Figure 5.1b, where the blue dashed line corresponds to the scale factor $a(t)$, the green dotted and the yellow solid lines to the scale factors $b(t)$ and $c(t)$, respectively. As we have noted before, the scale factors $a(t)$ and $b(t)$ show small oscillations around each other, while the scale factor $c(t)$ has a much smaller amplitude and does not show such oscillations. This is reminiscent of a long era in the evolution of type IX on approach to an initial singularity seen in [31]. On looking at the shear and the spatial
3-curvature, we see that they display oscillatory behaviour as well, before blowing up on approach to the singularity. The shear evolution is shown in Figure 5.2a and the 3-curvature evolution is shown in Figure 5.2b.

(a) 
(b) 

Fig. 5.2 From left to right: Evolution of shear and 3-curvature scalars during a single cycle of a type IX universe containing radiation and no ghost field, with comoving proper time, \(t\).

**Ghost field present: \(\rho_g \neq 0\)**

We create a simple bouncing cosmological model by adding a ghost field to create a non-singular bounce. The other fields in the system are the radiation fields and the dust fields. As before, we again choose initial values for the radiation, the dust and the ‘ghost’ fields to be of order 1, as \(\rho_r(t_i) = 8\), \(\rho_m(t_i) = 5\) and \(\rho_g(t_i) = -5\), respectively. Again, as long as the initial conditions for the densities are of the same order, their exact numerical value does not much affect the results of the computation qualitatively. Changing these numbers significantly changes the number of bounces the system undergoes in the same time frame of integration but qualitatively the features do not alter. Of course, as one might expect, changing the initial conditions for the density of the ‘ghost’ field to be orders of magnitude smaller than the radiation and dust fields causes the system to collapse and not undergo the non-singular bounce. On evolving this system through several successions of bounces, we find that the scale factors oscillate rapidly from cycle to cycle as do the energy densities of the radiation, matter and ghost field. The square of the shear tensor and the 3-curvature also show similar oscillatory behaviour.
Fig. 5.3 Evolution of volume (left) and individual scale factors (right) with radiation, ordinary dust and the ghost field included, with time. The blue dashed, green dotted and yellow solid lines correspond to $a(t)$, $b(t)$ and $c(t)$ respectively.

The evolution of the volume scale factor from cycle to cycle is oscillatory, as can be seen in Figure 5.4a, as are the behaviors of the scale factors in the three orthogonal directions. As before, in Figure 5.4b, the blue dashed, solid yellow, and dotted green lines trace the $a(t)$, $b(t)$ and $c(t)$ scale factors. The shear and the 3-curvature also display oscillatory behaviour and are shown in Figures 5.6a and 5.6b respectively.
5.5 Numerical solutions of the type IX equations

Fig. 5.5 From left to right: Evolution of shear and curvature with radiation, ordinary dust and the ghost field included, with time

Therefore, we see that adding the ghost field is essential to avoid collapse to a singularity after just one cycle, and to allow it to actually propagate smoothly through successive bounces. We will include the ghost field in the model in the rest of the chapter when we examine the effects of entropy injection, or the effects of adding a cosmological constant, so that we can evolve the model through a series of cycles without encountering singularities.

5.5.2 Radiation universe with entropy increase: \( \dot{S} > 0 \)

Ghost field present: \( \rho_g \neq 0 \)

We now consider our bouncing anisotropic cosmological model with dust, radiation and a ghost field to prompt and to allow it to propagate through several cycles when there is entropy increase from cycle to cycle.

We start with Kasner initial conditions, and with the same initial scale-factor evolution equation (33) and the respective energy densities as before (\( \rho_r(t_i) = 8, \rho_m(t_i) = 5 \) and \( \rho_g(t_i) = -5 \)). This time, we increase the value of the constant of radiation \( (C_r) \) by a factor of 2, to model the effects of entropy increase on the dynamics. We find that the volume scale factor shows an increase in cycle-to-cycle expansion maxima as expected. The individual scale
factors proceed through several chaotic oscillations in each cycle, and the three directions seem to oscillate increasingly out of phase as the volume maxima get larger.

Fig. 5.7 Evolution of the volume (left) and individual scale factors (right) of a type IX universe with entropy increase from cycle to cycle \((dS/dt > 0)\), with time. The blue dashed, green dotted, and solid yellow lines correspond to the orthogonal scale factors \(a(t)\), \(b(t)\) and \(c(t)\) respectively. The model includes radiation, dust matter and the ghost field. The ghost field ensures that the cycles bounce smoothly at finite values of the volume.

Figure 6.2a represents the evolution of the volume scale factor and Figure 6.2b represents the evolution of the individual scale factors.

To see if greater expansion-volume maxima lead to an increase in the anisotropy, we plot the square of the shear tensor (Figure 6.7b), denoted by \(\sigma^2\), and we see that the shear tensor indeed shoots up to larger and larger values at each successive minimum as the corresponding radiation maximum is increased. We can see that a similar increase occurs when we track the difference in the expansion rates of the scale factors in the three directions. A significant increase in the differences of the expansion rates in the \(a\) and the \(b\) directions, and in the
difference of the expansion rates in the $b$ and $c$ directions is seen as the expansion maxima get bigger. There is not such a large increase in the difference in the expansion rates in the $a$ and $c$ directions. We can also look at the 3-curvature (Figure 6.7a), and we find that it oscillates to a greater extent initially but as time increases, the amplitude and frequency of these oscillations decrease and the universe seems to approach flatness, albeit with strong expansion and 3-curvature anisotropy.

![Graphs showing the evolution of shear and 3-curvature scalars](image)

(a) (b)

Fig. 5.9 Evolution of shear (left) and the 3-curvature (right) scalars with time in a type IX universe with entropy increase. The model includes radiation, dust and the ghost field to create smooth non-singular bounces.

We can understand this intuitively in another way. The shear is at its highest near the minimum of each cycle. As the expanding phase of the cycle begins, the shear is diluted rather slowly as $\sigma \sim (t \ln t)^{-1}$. Until the shear is diluted sufficiently, the universe cannot recollapse. When the shear enters the contracting phase, shear anisotropy accumulates. The longer the model evolves before the bounce, the greater amount of shear anisotropy is accumulated. Thus, by injecting radiation entropy, and effectively increasing the expansion maximum, we give the shear anisotropy more time to increase in the contracting phase, and thus, more time to dilute in the expanding phase. Hence, despite the increase in the shear anisotropy, we still see the universe recollapse and bounce.

We can also compare the pattern of entropy growth in the anisotropic Bianchi IX case with its isotropic subcase, the closed Friedmann universe.
We see in Figure 5.11a for the case of the Bianchi IX universe and in Figure 5.11b for the case of the isotropic closed Friedmann universe, that the variation of successive volume scale factor maxima in the Friedmann and the type IX universes with increasing radiation entropy show very similar, almost linear behaviour (this would be different if we injected entropy according to a different rule). We also show what happens to the range of the bounce with entropy injection in the Bianchi IX case.
5.6 Adding a cosmological constant

Fig. 5.12 Range of bounces versus entropy of radiation in the presence of ordinary dust and radiation in oscillating Bianchi IX universes.

It should come as no surprise that the range is also increasing fairly linearly with the injection of entropy, as can be seen in Figure 5.12. The increase in volume maxima simply means that the model takes a longer time to recollapse.

5.6 Adding a cosmological constant

We have seen in previous work that the addition of cosmological constant to the closed Friedmann model which incorporates increasing volume maxima with the injection of radiation entropy results in the model ceasing to oscillate before expanding exponentially towards the de Sitter metric [17]. Now we study the effects of adding either a positive or a negative cosmological constant in type IX universes. The motivation for doing this, for the case of the positive cosmological constant, is to see if a similar exponential expansion to the isotropic case takes place. It is also interesting to investigate the effect the expansion prompted by the cosmological constant has on the anisotropy and the spatial 3-curvature. The negative cosmological constant models create an interesting anisotropically recollapsing counterpart to the closed Friedmann models. These models are among the simplest versions of a closed isotropic bouncing universe as they incorporate a negative cosmological constant as well as a curvature ‘field’ which behaves as a fluid with equation of state \( p = -\frac{1}{3}\rho \). They admit a simple periodic solution in the isotropic case. It is expected that when anisotropy is
included, in the absence of an ultra stiff matter field, the universe will quickly approach an anisotropic singularity since the negative $\Lambda$ term is only influential at large volumes to effect a collapse but has negligible effect as the future singularity is reached.

5.6.1 Positive cosmological constant

We study the effect of adding a positive cosmological constant to a type IX model containing radiation with entropy injection, a dust field, and the ultra-stiff ($p = 5\rho$) ghost field to ensure a smooth non-singular bounce. The results depend on the value of the cosmological constant relative to the initial energy densities of the other fields. If the cosmological constant is set to have a value of the same order of magnitude as the other components in the system, then the model displays a very sudden increase in all the scale factors in each of the three directions without any of the oscillating behaviour that we have seen in the other cases: it quickly asymptotes to de Sitter behaviour. On the other hand, if the cosmological constant is too small then its effects cannot be seen in the time interval that we have set for integration. Thus, for an intermediate range of values of the cosmological constant, we find that as soon as the cosmological constant starts to dominate, the volume scale factor which was hitherto showing an oscillatory behaviour with increasing maxima from cycle to cycle does not recollapse beyond the point of maximum. Instead, it enters a final phase of exponential expansion. The individual scale factors all undergo exponential expansion, asymptoting to de Sitter behaviour. However, they have different rates of expansion in this phase, with two of the scale factors having nearly the same rate, oscillating around each other in this phase (this is reminiscent of the axisymmetric behaviour that motivated our analytic approximation in section III above). The third scale factor expands much less than the other two. As soon as the cosmological constant starts to dominate, the shear and the 3-curvature, which were oscillating before $\Lambda$-domination as in the previous case with $\Lambda = 0$, now start oscillating with smaller and smaller amplitudes as time progresses. For the purposes of our computation, we set the cosmological constant to be a value which is approximately $\Lambda \approx 3H^2$ which can be taken to mark the onset of cosmological constant domination.
Fig. 5.13 Evolution of (left) the volume, and (right) the individual scale factors of a type IX universe with positive $\Lambda$, with time, $t$. The blue dashed, green dotted and solid yellow lines correspond to scale factors $a(t)$, $b(t)$ and $c(t)$, respectively. The model includes radiation, dust and a ghost field to create non-singular bounces. Note that the oscillations cease after a finite time when $\Lambda$ term dominates the dynamics at large volume. All scale factors then asymptote to the de Sitter expansion after a few transitionary oscillations.

In Figure 5.14a we show the evolution of the volume scale factor and, in Figure 5.14b, the evolution of the individual scale factors, where the blue dashed, yellow solid, and green dotted lines represent the scale factors $a(t)$, $b(t)$ and $c(t)$, respectively. We can see the domination of the $\Lambda$ term leading to de Sitter like expansion in all of the scale factor directions by looking at the Hubble rates.
Fig. 5.15 Evolution of the Hubble rates in the presence of a positive cosmological constant with time. The blue dashed, green dotted and solid yellow lines trace the Hubble rates $\dot{a}(t)/a(t)$, $\dot{b}(t)/b(t)$ and $\dot{c}(t)/c(t)$. Oscillations cease when $\Lambda$ dominates. The Hubble rates then undergo an anisotropic transition phase before eventually approaching isotropic de Sitter-like expansion where the individual Hubble rates approach the same constant value.

We see in Figure 5.15, that the Hubble rates undergo several oscillations and enter into a transition phase when the cosmological constant starts to dominate. However, with positive cosmological constant domination the oscillations cease, and as the scale factors undergo exponential expansion, their Hubble rates tend to a constant value.

Figures 5.17a and 5.17b show the behaviour of the shear tensor squared $\sigma^2$ and of the 3-curvature respectively. They show a decrease with time, with the shear showing oscillations, as before, but with decreasing amplitude. The curvature also shows oscillations as before but falls to very small values as the cosmological constant starts to dominate.
5.6 Adding a cosmological constant

Fig. 5.16 (Left) Evolution with time, $t$, of the shear and (right) 3-curvature scalars in type IX universes with positive $\Lambda$. The model includes radiation, dust and a ghost field to create non-singular bounces.

5.6.2 Negative cosmological constant

We can also try to look at the case of the negative cosmological constant to find the role of anisotropies in the simple isotropic universe model. This is the closed Friedmann universe with curvature parameter $k = +1$, consisting of ‘domain wall matter’ which is just matter with an equation of state $p = -2\rho/3 \propto a^{-1}$ and a negative cosmological constant. The Friedmann equation is now, ($\Sigma > 0$, constant):

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \left( \Lambda + \frac{\Sigma}{a} \right) - \frac{1}{a^2}.
\]  

(5.69)

This model has been studied as a simple model of a bouncing universe admitting a simple solution (see Figure 12).
Consider an extension of this model to the anisotropic type IX case. In the absence of a stiff field to smooth out the anisotropies on approach to the bounce, it is expected that the universe on collapse will not be able to re-expand from a singularity. To prevent this, we add an ultra stiff matter field, with positive energy density, as we expect the bounce to be produced by the negative cosmological constant. We find that the volume scale factor undergoes a bounce. The individual scale factors undergo several oscillations with different time periods and different amplitudes. The shear undergoes oscillations with amplitudes that decrease as the volume scale factor expands and starts increasing again as the contraction phase begins. The 3-curvature also shows oscillatory behaviour as the expansion phase is followed by the contraction.

On changing the sign of the cosmological constant in the model, with the same initial conditions as we have been using previously, we see that with the injection of entropy into the radiation field, to facilitate an increase in the expansion maximum, the volume scale factor undergoes irregular oscillations. These can be seen in Figure 5.20a.
5.6 Adding a cosmological constant

Fig. 5.19 Evolution of the (left) volume and (right) individual scale factors for a type IX universe with negative $\Lambda$, with time, $t$. The blue dashed, green dotted and yellow solid lines correspond to $a(t)$, $b(t)$ and $c(t)$, respectively. The model includes an ultra stiff matter field ($p = 5\rho$) with positive energy density in addition to the negative cosmological constant.

The individual scale factors are seen in Figure 5.20b and are given as before by the blue dashed, green dotted and solid yellow lines representing the scale factors $a(t)$, $b(t)$ and $c(t)$, respectively. They appear to undergo oscillations as well.
Fig. 5.21 (Left) Evolution of the shear and (right) the 3-curvature, with negative $\Lambda$, versus time, $t$. The model includes an ultra stiff field ($\rho = 5\rho$) with positive energy density in addition to a negative cosmological constant.

Looking at Figures 5.22a and 5.22b, the shear and the curvature also show several rapid and irregular oscillations as they approach minima. On collapse of the model, they show no sign of approaching isotropy, blowing up instead. To demonstrate that the long-term evolution is chaotic, we find that on slightly changing the initial conditions by $\pm 0.001$ (where the initial conditions we have chosen for $[x, y, H]$ as well as the densities of the ghost and radiation fields are of order 1), we find that the number of oscillations in the shear and the curvature, as well as their amplitude and shape, drastically change (especially for the initial few cycles of the model). The behaviour for the shear for the second set of initial conditions are shown in Figure 5.24a.
Fig. 5.23 Evolution of the shear (a) and curvature (b) in the type IX universe with negative $\Lambda$ and ghost field, with time, with initial conditions differing by $\pm 0.001$, to illustrate the chaotic sensitivity of the dynamics to small changes in initial data over many cycles of time evolution.

Looking next at the 3-curvature we find a similar situation, where the shape of the oscillations as well as their frequency and amplitude vary greatly with the same slight change in the initial conditions. We see this in Figure, 5.24b.

It is interesting to note that this chaotic behaviour from cycle to cycle occurs in $t$ time, rather than in its logarithms as is the case in the chaotic behaviour seen on approach to the singularity if bounces do not occur. [31, 14].

## 5.7 Conclusions

We have investigated the fate of cyclic universes in the most general spatially homogeneous closed universes of Bianchi type IX. These models are considerably more general than those previously used to study oscillating universes. We include radiation, matter and a stiff ‘ghost’
field with negative density to produce a smooth, non-singular bounce at finite volume at the end of each cycle. A bounce at any finite volume, no matter how small, avoids the issue of chaotic oscillations [31] of the scale factors on approach to the expansion minima. We investigate the effects of an increase in radiation entropy from cycle to cycle, in accord with the Second Law of thermodynamics, and also the effects of adding a positive or negative cosmological constant to the Einstein equations. We find the following:

**Flatness and shape evolution:** When $\Lambda = 0$ we studied the evolution of type IX universes by analytic approximations and by numerical evolution (starting with Kasner initial conditions). The evolution follows an approximately axisymmetric form in which different scale factors attain their maxima at different times before turning around and collapsing towards their next minima. We found that as the size of the volume maximum increases, the model approaches ‘flatness’ in the same way that isotropic closed universes do. However, the dynamics do not approach isotropy as the volume maximum is approached or as the ensuing minimum is approached. The successive expansion maxima grow increasingly out of phase. The long-term dynamics are therefore anisotropic and differ significantly from those predicted for inflationary universes.

**General relativistic effects:** The late-time evolution of type IX universe is dominated by intrinsically general-relativistic effects associated with its 3-curvature anisotropy (for which there is no Newtonian analogue). The sign of the 3-curvature scalar, $3R$, can change with time. When the type IX universe is significantly anisotropic, $3R$ is negative and the dynamics cannot have an expansion maximum. The universe therefore keeps on expanding and eventually becomes close enough to isotropy for $3R$ to become positive and then it is able to experience a volume maximum and recollapse.

**The effects of entropy increase:** We injected radiation entropy at each finite expansion minimum to model the effect of increasing entropy. We found that the entropy increase leads to steady increase in the size of the volume maxima of successive cycles and to their temporal duration but these maxima are anisotropic.

**The effect of $\Lambda > 0$:** The addition of a cosmological constant is always found to bring the oscillations in the volume of the universe to an end. This occurs no matter how small the value of $\Lambda$ is. Oscillations of the universe occur, and grow anisotropically as in the case of $\Lambda = 0$ until the size of the maximum grows large enough for $\Lambda$ to become dynamically important there. Subsequently, after a few scale factor transitory changes it will dominate before any
5.7 Conclusions

expansion maximum can occur and accelerate the expansion towards an increasingly de Sitter-like metric evolution. This behaviour is in accord with cosmic no-hair theorems even though, technically, they do not apply to the type IX metric because it permits positive three-curvature, which is excluded by the theorems.

The effect of $\Lambda < 0$: The addition of a negative cosmological constant causes any cosmological model to recollapse, regardless of the sign of the 3-curvature. We use the addition of $\Lambda < 0$ to produce a simple bouncing model that experiences a finite non-singular minimum at the end of each cycle because of the presence of a ghost field. We follow the chaotic evolution of the scale factors, the shear, and the 3-curvature from cycle to cycle. We showed that there is sensitive dependence on initial conditions.

Our analysis introduced some simplifying assumptions. We consider only the diagonal Bianchi IX metric with fluids that possess comoving velocity fields and isotropic pressures. In a separate study, we will relax these assumptions and show that a similar analysis is possible which leads to similar conclusions. Thus we have shown that in the most general spatially homogeneous anisotropic cyclic universes in general relativity with $\Lambda = 0$ the growth of entropy leads to never-ending cycles of increasing size and duration, as Tolman first showed for isotropic models. However, although these cycles approach flatness they do not approach isotropy and do not resemble our observed universe. If we add $\Lambda > 0$ then, no matter how small the magnitude of $\Lambda$, the growing oscillations always come to an end and subsequently the dynamics pass through a quasi-isotropic phase before asymptoting towards the isotropic dynamics of a de Sitter metric. These analyses can also be readily extended to other cyclic universe scenarios that are based upon extensions of general relativity [121, 35, 71, 44, 24, 111, 101].
Chapter 6

Tilted Bianchi models and the effect of non-comoving velocities

6.1 Introduction

There have been studies of non-comoving matter in other Bianchi types, such as Bianchi type VII\(_0\), such as in [89, 90, 93, 94], but with a focus on the fate of primordial turbulence and the effects of collisionless particles on vortices. There has also been a study of the problem in a radiation-filled universe in ref. [68]. Here we extend these analyses, and those of cyclic universes, to include a Bianchi IX (‘mixmaster’) universe with non-comoving radiation. We also include a comoving null energy condition (NEC) violating, or ‘ghost’, field with \(p > \rho\) and \(\rho < 0\) [25, 21]. Its inclusion is simply a device to produce a bounce at a finite minimum of each cycle. It avoids the evolution falling into an open interval around \(t = 0\) which will produce chaotic mixmaster oscillations [98] and thus avoids the inclusion of infinite mixmaster oscillations on approach to the expansion minima. These oscillations are not likely to be physically relevant in classical cosmological evolution: if the universe bounces at the Planck time \((t_{pl} \sim 10^{-43}s)\) then only a few mixmaster oscillations are permitted up to the present \((t_0 \sim 10^{60}t_{pl})\) because they occur in log-log of the comoving proper time. This makes our problem more tractable from a numerical perspective. Our aim is study the effect of non-comoving radiation in the case of a cyclic mixmaster universe with thermal entropy growth, in the presence of both zero, positive and negative cosmological constant.

In section 2, we set up the Einstein equations for the problem, with a brief background to the tetrad formalism in Bianchi IX models, and give the energy-momentum tensor of the non-comoving radiation field that we are introducing. We derive the equations of motion and the evolution equation for the velocities that are normalised with the appropriate power of the
energy density of radiation. In section 3, we discuss the qualitative effects of entropy increase on the cycle to cycle evolution of closed isotropic and anisotropic universes and identify a new effect of entropy increase introduced by the presence of non-comoving velocities. In section 4, we provide some analytic analysis of the type IX equations with velocities before presenting our computational solutions of the Einstein equations with and without a cosmological constant of either sign in section 5, and give our conclusions in section 6.

6.2 The setup

6.2.1 The Einstein equations

For the purposes of studying the effect of non-comoving velocities in anisotropic closed cyclic universes, we choose the diagonal Bianchi IX universe with metric,

\[ ds^2 = dt^2 - \gamma_{ab} \epsilon^a_\mu \epsilon^b_\nu dx^\mu dx^\nu, \]

(6.1)

where

\[ \gamma_{ab} = \text{diag}[a(t)^2, b(t)^2, c(t)^2]. \]

(6.2)

In the matter sector, we introduce the energy-momentum tensor for a perfect field:

\[ T_{ab} = (\rho + p)u_a u_b - p \gamma_{ab}. \]

(6.3)

The 4-velocity of the perfect fluid with respect to our chosen tetrad frame is

\[ u_a = (u_0, u_1, u_2, u_3). \]

(6.4)

The relations of the components of the velocities with respect to the universe frame are given by,

\[ u_a = \epsilon^{\mu}_a \bar{u}_\mu, \]

(6.5)

where the \( \bar{u}_\mu \) are the components of the 4-velocity of the fluid with respect to the universe frame. We shall be working with the 4-velocity in the tetrad frame for consistency with the Ricci tensor, which is also written in the tetrad frame for the purposes of this computation. The components of this 4-velocity obey the normalisation,

\[ u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1. \]

(6.6)
Referring to [96], we find the following conditions on the energy-momentum tensor. The fluid vorticity $\omega_{ab}$ is zero if and only if the spatial velocity components $u_i$ are zero. Thus, for the general case of non-comoving fields, we do indeed have vorticity in our system. Thus, with reference to the orthonormal frame formalism, we have non-zero shear $\sigma_{ab}$, the curvature variables $n_{ab}$ as well as vorticity $\omega_{ab}$.

For the purposes of our computation, we consider non-interacting perfect fluids, with ideal equation of state,

$$p = (\gamma - 1)\rho, \quad (6.7)$$

and we can add their energy-momentum tensors together in the usual way. In our system, we include radiation with $\gamma = \gamma_r = 4/3$ and an ultra-stiff comoving ‘ghost’ field with equation of state $\gamma = \gamma_g = 5$, a value chosen simply for convenience in effecting a bounce. The densities and the pressures of the radiation and the ‘ghost’ field are given by $\rho_r$, $p_r$ and $\rho_g$ and $p_g$. The radiation field has velocities which are not comoving in the tetrad frame of reference. We normalise the 4-velocity of the radiation field so that the normalised velocity components are related to the velocity vector by $v_a = (\rho_r + p_r)^{1/\gamma - 1/2}u_a$, and denote the normalised velocity vector by

$$v = (v_0, v_1, v_2, v_3) \quad (6.8)$$

In our case, for black body radiation, $\gamma_r = 4/3$ and the normalised velocity components are therefore given by $v_a = (\rho_r + p_r)^{1/4}u_a$. Considering energy-momentum conservation in the tetrad frame, we get the conservation of particle current,

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g}s u^i) = 0, \quad (6.9)$$

where $s$ is the entropy density. For radiation, $s \propto \rho^{3/4}$, this yields the conservation law,

$$v_o^2 a^2 b^2 c^2 (\rho_r + p_r) = const \equiv L^3, \quad (6.10)$$

where we have labelled the constant $L^3$ for consistency with reference [68].

The second constraint equation for the components of velocity of the radiation field is

$$v_1^2 + v_2^2 + v_3^2 = L\delta \quad (6.11)$$

The constant $L$ has the dimensions of length and the constant $\delta$ is dimensionless. Close to isotropy, when the spatial components of the velocity 4-vector are negligible, we have $\delta \ll 1$. For the case of small velocities in a near-Friedmann radiation-dominated universe, we see that their spatial components are constant. For the dust-dominated universe, the spatial
components of the velocities fall as $1/a$ where $a(t)$ is the scale factor of the Friedmann universe and $t$ is the comoving proper time.

We have a further hydrodynamic equation of motion ($\nabla^a T_{ab} = 0$), and 4-velocity normalisation [81] to employ in what follows:

$$ (p + \rho) u^k \left( \frac{\partial u_i}{\partial x^k} - \frac{1}{2} u^l \frac{\partial g_{kl}}{\partial x^i} \right) = -\frac{\partial p}{\partial x^i} - u_k u^k \frac{\partial p}{\partial x^i} $$  \hspace{1cm} (6.12)

$$ u_i u^i = 1 $$  \hspace{1cm} (6.13)

### 6.2.2 Non-comoving velocities in a Bianchi type IX universe

We now ask what happens in anisotropic, spatially homogeneous universes, with scale factors $a, b, c$, when there are non-comoving velocities and vorticities. Suppose we first take the background expansion of the scale factors to have the same form as in the type IX universe without non-comoving velocities that we studied in ref. [19]. In our earlier study without velocities we found a long period of evolution during the radiation era (before the curvature creates a slow-down of the expansion near the volume maximum) where far from the expansion maximum, the scale factors evolve to a good approximation in a quasi-axisymmetric manner, as

$$ a(t) = a_0 t^{1/2} \ln(t)^{-1/2}, b(t) = b_0 t^{1/2} \ln(t)^{-1/2}, c(t) = c_0 t^{1/2} \ln(t). $$  \hspace{1cm} (6.14)

Note that the volume, $abc \propto t^{3/2}$, evolves like the Friedmann model [68]. The logarithmic corrections are familiar in the study of anisotropic universes with anisotropic 3-curvatures, trace-free radiation stresses in the presence of isotropic radiation, or long-wavelength gravitational waves [16]. They reflect the presence of a zero eigenvalue when we perturb around the shear variables around the isotropic model whereas the volume has a negative real eigenvalue. More generally, the effects of the velocities in the type IX radiation universe can be treated as test motions on an expanding radiation background governed by equations (6.9) and (6.10):

$$ a(t) = a_0 t^{1/2} \ln^\lambda(t), b(t) = b_0 t^{1/2} \ln^\mu(t), c(t) = c_0 t^{1/2} \ln^\nu(t), $$  \hspace{1cm} (6.15a)

$$ \lambda + \mu + \nu = 0, \text{ and } \lambda, \mu, \nu \text{ constants}, $$  \hspace{1cm} (6.15b)

$$ abc \propto t^{3/2}, $$  \hspace{1cm} (6.15c)

where (6.15a) reduces to the particular case (6.14) when $\lambda = \mu = -1/2$ and $\nu = 1$. 

Ignoring spatial gradients with respect to time variations, and taking the diagonal scale factors to be \(a(t), b(t),\) and \(c(t),\) for a radiation-dominated universe \((p = \rho / 3),\) these equations specialise to [81]

\[
abcu_0 \rho^{3/4} = t^{3/2} u_0 \rho^{3/4} = \text{constant}, \quad (6.16)
\]
\[
u_\alpha \rho^{1/4} = \text{constant}, \alpha = 1, 2, 3. \quad (6.17)
\]

If we solve them as \(t \to \infty\) with \(\lambda < \mu < \nu,\) then the dominant component of \(u^\alpha\) is \(u^1 = u_1 / a^2,\) which gives \(u_0^1 \approx u_1 u^1 = (u_1)^2 t^{-1} \ln^{-2\lambda}(t),\) and we get the dominant late-time behaviours from (6.16)-(6.17):

\[
\rho \propto \frac{\ln^{2\lambda}(t)}{t^2}, u_1 u^1 \propto \frac{1}{\ln^{3\lambda}(t)}, \quad (6.18)
\]
\[
T_1^1 \approx \rho u_1 u^1 \propto \frac{1}{t^2 \ln^{\lambda}(t)} \propto T_0^0 \quad (6.19)
\]
\[
T_2^2 \approx \rho u_2 u^2 \propto \frac{\ln^{\lambda - 2\mu}(t)}{t^2} \quad (6.20)
\]
\[
T_3^3 \approx \rho u_3 u^3 \propto \frac{\ln^{\lambda - 2\nu}(t)}{t^2} \quad (6.21)
\]

The corrections to the case with comoving velocities and zero vorticity are therefore only logarithmic in time during the radiation era. The scalar 3-velocity, has dominant asymptotic form

\[
V \equiv \sqrt{u_\alpha u^\alpha} \approx \ln^{-3\lambda}(t). \quad (6.22)
\]

For the quasi-axisymmetric radiation-dominated phase of the type IX evolution, we take \(\lambda = \mu = -1 / 2\) and \(\nu = 1\) and we see that the stresses induced by the velocities grow logarithmically in time compared to the other terms in the field equations (of order \(O(1/t^2)\)) present when the velocities are comoving. We see that the diagonal stress-tensor components \(T_0^0 \approx T_1^1 \approx T_2^2 \propto t^{-2} \ln^{1/2}(t)\) fall off slower than \(t^{-2}\) as \(t \to \infty,\) while \(T_3^3 \propto t^{-2} \ln^{-3/2}(t)\) falls off faster than \(t^{-2}.\) We can see explicitly that the 3-velocity, \(V,\) is expected to grow as \(\ln^{3/2}(t)\) in our approximation, which holds so long as the velocities are small enough for the perturbations not to disrupt the assumed (velocity-free) metric evolution (6.15a) and we are far from the expansion maximum. If there is an expansion maximum, then these asymptotic forms will be cut off when the approximate solution (6.14) breaks down and we need a numerical analysis to determine the detailed evolution in this regime, and from cycle
to cycle. However, we expect the presence of non-comoving velocities to introduce changes to the analysis that was made for type IX cyclic universes in our work [19].

If we repeat this analysis in an isotropic de Sitter background with late-time scale factor evolution approaching \(a = b = c = e^{H_0 t}\) before the volume maximum, then the asymptotic behaviour of radiation is \(u_\alpha u^\alpha = \text{const.}, u_0 = \text{const.}, \rho_r \propto e^{-4H_0 t}\), and the new terms induced in the field equations by the non-comoving velocities do not grow at late times. However, we note that the velocities produce a constant tilt relative to the normals to the surfaces of homogeneity and the asymptotic form at late times approaches de Sitter with a constant velocity field tilt (as also is seen in ref. [118]). In general, when the cosmological constant, \(\Lambda \equiv 3H_0^2\), is positive it will end the sequence of increasing oscillations in a cyclic closed universe, no matter how small its value, because the size of the universe will eventually become large enough for \(\Lambda\) to dominate before a maximum is reached in some future cycle [17].

6.2.3 Equations of motion

In the type IX universe, the evolution equations for the velocities are as follows (where overdot is \(d/dt\)):

\[
\dot{v}_1 + \frac{v_3 v_2}{v_0} \left( \frac{1}{c^2} - \frac{1}{b^2} \right) \left( 1 + 2L^3 w^{-1/2} \frac{a^4 - b^2 c^2}{(a^2 - b^2)(c^2 - a^2)} \right) = 0, \tag{6.23}
\]

\[
\dot{v}_2 + \frac{v_1 v_3}{v_0} \left( \frac{1}{a^2} - \frac{1}{c^2} \right) \left( 1 + 2L^3 w^{-1/2} \frac{b^4 - c^2 a^2}{(b^2 - c^2)(a^2 - b^2)} \right) = 0, \tag{6.24}
\]

\[
\dot{v}_3 + \frac{v_2 v_1}{v_0} \left( \frac{1}{b^2} - \frac{1}{a^2} \right) \left( 1 + 2L^3 w^{-1/2} \frac{c^4 - a^2 b^2}{(c^2 - a^2)(b^2 - c^2)} \right) = 0, \tag{6.25}
\]

where \(w \equiv (\rho_r + p_r)\). The evolution equations for the scale factors in cosmological time then become,

\[
(\ln a)^{-} + 3H(\ln a)^{+} + \frac{1}{2} \left( \frac{a^2}{b^2 c^2} - \frac{b^2}{c^2 a^2} - \frac{c^2}{a^2 b^2} \right) + \frac{1}{a^2} + 2L^3 \left( \frac{a^2 + c^2}{b^2 (c^2 - a^2)^3} v_1^2 - \frac{a^2 + b^2}{c^2 (a^2 - b^2)^3} v_2^2 \right) - \frac{2}{a^2} \left( \frac{w^{1/2} v_3^2}{a^2} + \frac{\rho_r - p_r}{2} - \frac{\rho_g + p_g}{2} \right)
\]

\[
= 2 \left( \frac{w^{1/2} v_3^2}{a^2} + \frac{\rho_r - p_r}{2} - \frac{\rho_g + p_g}{2} \right) \tag{6.26}
\]
6.3 Approximate analysis of the radiation era

\[
(\ln b)' + 3H(\ln b) = \frac{1}{2} \left( \frac{b^2}{c^2 a^2} - \frac{a^2}{b^2 c^2} - \frac{c^2}{a^2 b^2} \right) + \frac{1}{b^2} + 2L^3 \left( \frac{b^2 + a^2}{c^2 (b^2 - a^2)^3} v_1^2 - \frac{b^2 + c^2}{a^2 (b^2 - c^2)^3} v_1^2 \right)
\]

\[
(\ln c)' + 3H(\ln c) = \frac{1}{2} \left( \frac{c^2}{a^2 b^2} - \frac{a^2}{b^2 c^2} - \frac{b^2}{a^2 c^2} \right) + \frac{1}{c^2} + 2L^3 \left( \frac{c^2 + b^2}{a^2 (b^2 - c^2)^3} v_1^2 - \frac{c^2 + a^2}{b^2 (c^2 - a^2)^3} v_1^2 \right)
\]

These equations include a comoving ghost field (\(\rho_g\)) and the non-comoving radiation field (\(\rho_r\)). We give an approximate analysis of the solutions to these equations during the radiation era, far from expansion minima and maxima in the Appendix. We show there that the velocity components are constant up to logarithmic oscillatory factors during the era when the expansion dynamics are well approximated by 6.15a.

6.3 Approximate analysis of the radiation era

We seek an approximate solution of the velocity evolution equations in the type IX model during the radiation era. In our earlier study [19] without velocities we found a long period of evolution during the radiation era (before the curvature creates slow-down of the expansion near the volume maximum) with the scale factors evolving to a good approximation in a quasi-axisymmetric manner during the radiation era, as

\[
a(t) = a_0 t^{1/2} [\ln(t)]^{-1/2}, b(t) = b_0 t^{1/2} [\ln(t)]^{-1/2}, c(t) = c_0 t^{1/2} \ln(t). \tag{6.29}
\]

When the effects of the velocities in the Bianchi type IX radiation universe are small they can be treated as test motions on an expanding radiation background governed by equations (6.9) and (6.10). We examine a typical case where we choose

\[
v_3 = \text{constant}.
\]

This is consistent with the velocity evolution equation for \(v_3\) with \(1/a^2 = 1/b^2\). In the approximation \(a \gg b \gg c\) and \(b^4 > a^2 c^2\) for large \(t\) from ((6.2.3) and (6.2.3)), the evolution equations for \(v_1\) and \(v_2\) reduce to:
\[ \dot{v}_1 + \frac{v_2v_3}{v_0c^2} \left( 1 - \frac{2L^3}{w^{1/2}} \right) = 0, \]

\[ \dot{v}_2 - \frac{v_1v_3}{v_0c^2} \left( 1 + \frac{2L^3b^2}{a^2w^{1/2}} \right) = 0. \]

We assume non-relativistic velocities, so take \( v_0 = 1 \), and note that \( w = \rho_r + p_r = 4\rho_r/3 \).

Since \( \rho_r \propto (abc)^{-4/3} \propto t^{-2} \), we write

\[ w^{1/2} = \frac{M}{t}, \]

where \( M \) is a positive constant. Therefore the radiation entropy, \( s \), depends on \( M \) via

\[ s \propto \rho^{3/4} \propto w^{3/4} \propto M^{3/2}. \]

Hence, we have approximately

\[ \dot{v}_1 + \frac{v_2v_3}{c^2t \ln^2(t)} \left( 1 - \frac{2L^3t}{M} \right) = 0, \quad (6.30) \]

\[ \dot{v}_2 - \frac{v_1v_3}{c^2t \ln^2(t)} \left( 1 - \frac{2L^3b^2t}{Ma^2_0} \right) = 0, \quad (6.31) \]

where \( v_3 \) is constant. At large times these equations are (and scaling \( a_0 = b_0 \))

\[ \dot{v}_1 = \frac{2L^3v_3v_2}{Mc_0^2 \ln^2(t)} \equiv \frac{Dv_2}{\ln^2(t)}, \quad (6.32) \]

\[ \dot{v}_2 = -\frac{2L^3v_3b_0^2v_1}{Mc_0^2a_0^2 \ln^2(t)} \equiv -\frac{Dv_2}{\ln^2(t)}, \quad (6.33) \]

where

\[ D = \frac{2L^3v_3}{Mc_0^2} \]

is a constant. Hence, we see immediately that

\[ v_1^2 + v_2^2 = E : E = \text{constant.} \quad (6.34) \]

Since \( v_1 = \sqrt{E - v_2^2} \), we have in (6.32)
\[ \dot{v}_1 = -v_2 \dot{v}_2 (E - v_2^2)^{-1/2} = \frac{Dv_2}{\ln^2(t)}, \]

hence

\[ \int \frac{dv_2}{\sqrt{E - v_2^2}} = -D \int \frac{dt}{\ln^2(t)} \]

Therefore,

\[ v_2 = \sqrt{E} \sin \left( -D \int \frac{dt}{\ln^2(t)} \right), \]

and so, by (6.34), we have

\[ v_1 = \sqrt{E} \cos \left( -D \int \frac{dt}{\ln^2(t)} \right). \]

The components \( v_1 \) and \( v_2 \) therefore undergo bounded oscillations while \( v_3 \) remains constant.

We can get a better approximation by keeping all the terms in equations (6.30) and (6.31). If we write them as

\[ \dot{v}_1 + \frac{Av_2}{t \ln^2(t)} (1 - Bt) = 0, \tag{6.35} \]

\[ \dot{v}_2 - \frac{Av_1}{t \ln^2(t)} (1 - Bt) = 0, \tag{6.36} \]

then \( v_1^2 + v_2^2 = E \), and hence we find a second order correction which confirms the oscillatory behaviour of the velocities with growing periods of oscillation:

\[ v_1 = E^{1/2} \cos \left( -\frac{1}{\ln(t)} - B \int \frac{dt}{\ln^2(t)} \right) \]

\[ v_2 = E^{1/2} \sin \left( -\frac{1}{\ln(t)} - B \int \frac{dt}{\ln^2(t)} \right) \]

### 6.4 Introducing entropy increase

We want to investigate the effect of the non-comoving velocities on a closed cyclic type IX universe when its radiation entropy increases from cycle to cycle, mirroring Tolman’s classic analysis [125]. The radiation entropy density is \( s \propto \rho^{3/4} \). As in the earlier analysis made in
ref. [19], we first consider a closed Bianchi IX universe containing radiation, dust, and a ghost field but no cosmological constant. The ghost field has negative density and is dominant when the singularity is approached but dynamically irrelevant far from the initial and final singularities in each large cycle. It is included only to create a bounce at finite volume. This avoids evolution into the open interval of time around a curvature singularity at $t = 0$ during which the dynamics will be chaotic [30, 98, 14, 45]. For realistic choices of $T_1 \approx 10^{-43}$ s as the start of classical cosmology, there will be less than about 12 Mixmaster oscillations even if they continued all the way from $T_1$ up to the present day [134, 55]. This is because the overall expansion scale changes rapidly with the number of scale factor oscillations, which occur in log-log time.

Fig. 6.1 Evolution of (a) the volume scale factors, and (b) the individual scale factors (left to right) with the increase in entropy with time $t$ in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, as well as a comoving dust field, and a comoving ghost field to facilitate the bounce. The blue starred, red dotted, and green lines correspond to the principal values of the 3-metric in the tetrad frame, scale factors $a(t), b(t)$ and $c(t)$ respectively.

### 6.4.1 Effects of entropy increase

The effects of an increase in entropy from cycle to cycle of an isotropic oscillating closed universe were first considered by Tolman [125]. He showed that there would be an increase in expansion volume maxima and cycle length from cycle to cycle as a consequence of the second law of thermodynamics. The total energy of the universe is zero in each cycle and successive oscillations drive the universe closer and closer to flatness.
If the dynamics are allowed to be anisotropic then we showed that, with \( \Lambda = 0 \), increasing entropy leads to the increase of volume maxima and cycle length in successive cycles but the anisotropy grows from cycle to cycle in a manner that displays sensitive dependence on ‘initial’ conditions. We investigated this development in the context of the Bianchi type IX universe with comoving fluid velocities - the most general closed spatially homogeneous universe containing an isotropic FLRW universe as a particular case [19]. The addition of \( \Lambda > 0 \) eventually terminates these oscillations, as in the isotropically expanding case.

In this chapter we add an extra generalisation – the addition of non-comoving velocities to the most general anisotropic closed universe evolution with entropy increase. According to (6.9), the entropy increase from cycle to cycle should lead to a new effect: the reduction of the velocity from cycle to cycle. However, it is important to keep in mind that the constant on the right hand side of the conservation equation resulting from (6.9), that is equation (6.11), does not remain constant from cycle to cycle. Close to isotropy, the energy density \( \rho \sim L^3(abc)^{-4/3} \), and the entropy density \( s \propto \rho^{3/4} \) for radiation. Increasing the entropy density from cycle to cycle means that \( L \) remains constant only per cycle but jumps to a higher value in the next cycle. Thus, the constraint equation (6.11) is valid in each cycle with the right hand side being equal to a new, larger constant in subsequent cycles if there is entropy increase. A way of modelling this problem is to ensure that the constraint is imposed simultaneously with the injection of entropy at each minima. Thus, if we increase the entropy, or in our case the energy density (as \( s \propto \rho^{3/4} \)) by a factor \( \Delta \), then the normalised velocities \( v_i = \rho^{1/4} u_i \) must be multiplied by a factor \( \Delta^{-1/4} \) to keep the constraint equation (6.11) unchanged. Thus we see that when the entropy increases, the velocities decrease as the evolution proceeds from cycle to cycle in accord with the second law of thermodynamics.

The sum of the square of the normalised velocities, \((\rho + p)^{1/4} u_\alpha\), oscillates initially but eventually settles down to a nearly constant value with small oscillations around this value even as oscillations proceed to higher and higher expansion maxima. In Figure 6.5, we show the constancy of this sum over one cycle. We have modelled the effects of radiation entropy, ‘\( s \)’, increase during a cycle of a closed universe by creating a sudden entropy increase at the start of each cycle\(^1\). This produces the increase in the expansion maximum of successive cycles, first discovered by Tolman [125].

We identify a new feature of isotropic, oscillating radiation universes: any non-comoving velocities and vorticities will diminish from cycle to cycle as the expansion maxima increase and flatness is approached in accord with the second law of thermodynamics. For the anisotropic case, the overall trend in velocity evolution is oscillatory and is made more

\(^1\) We assume that the additional radiation entropy is at rest relative to the comoving frame so that we are not adding angular momentum. The situation is analogous to the effect of quantum created particle at the Planck epoch on vortical motions, where the increase in inertia of created particles causes velocities to drop [9]
Fig. 6.3 Evolution of the squares of velocities of non-comoving radiation with the increase in entropy with time $t$ in a Bianchi IX universe containing non-comoving radiation, as well as comoving dust and the ghost fields, the latter to facilitate the bounce. The velocity constraint (6.11) has been imposed. An increase in entropy (energy density) causes a decrease in the velocities and vice versa. Where necessary in the last two figures, the figure has been magnified to capture the rapidly oscillating features of the plot. From left, clockwise, the entropy density ($s \propto \rho^{3/4}$), the square of the spatial components of the velocities, $u_1^2, u_3^2$ and $u_2^2$ are shown.
6.4 Introducing entropy increase

![Graph showing the velocity constraint equation](image)

**Fig. 6.5** The evolution of (6.11), the velocity constraint equation, over one cycle.

![Graph showing 3-curvature and shear](image)

**Fig. 6.6** Evolution of (a) the 3-curvature and (b) the shear with the increase of entropy with time $t$ in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, also containing a comoving dust field and a comoving ghost field to facilitate the bounce.
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complicated. This is because we have shown that flatness is approached with an increase in expansion maxima and the inclusion of non-comoving velocities changes the dependence of the energy density and hence of the entropy on the scale factors from the isotropic case (and the anisotropic case in the absence of these non-comoving velocities) [19]. Thus we can only observe a increase/decrease in the velocities with a corresponding decrease/increase in the entropy. Aside from this effect, the evolutionary impact of the non-comoving velocities on the evolution in a cyclic radiation universe found in case with comoving velocities is only asymptotically logarithmic in time [19].

6.4.2 Evolution with non-comoving velocities

To study the behaviour of this model under the influence of non-comoving matter we assume that only the radiation field possesses non-comoving velocities (i.e. the ghost field is comoving). In the case of Bianchi IX, we find that the scale factors do undergo a bouncing behaviour, (see Figure 6.2a), as in the case without the non-comoving velocities. The volume scale factor, $abc$, mimics the behaviour of cube of the scale factor in the isotropic Friedmann case and shows an increase in height of its expansion maxima as the entropy of the constituents is increased from cycle to cycle. The individual scale factors oscillate out of phase with each other and with different expansion maxima, similar to their behaviour without the non-comoving velocities, (see Figure 6.2b). However, the period of the volume oscillations is greater than in the comoving velocities case. Thus, the model takes longer to recollapse on average than in the comoving case, making each cycle last longer in comoving proper $t$ time.

The shear and the 3-curvature undergo oscillations which increase in amplitude and frequency near the minima and do not appear to fall to smaller and smaller values: see Figures 6.7a and 6.7b.

The velocity components themselves show oscillatory behaviour: see Figures 6.9a, 6.9b and 6.9c. However, the amplitude of their oscillations undergoes cyclic behaviour. The amplitudes of oscillations fall to their smallest values at the expansion minima of the scale factors. After the first oscillation, one of the velocity components starts undergoing very small oscillations around a nearly constant value. We give an approximate analytic analysis of this evolution in the Appendix.
Fig. 6.8 (a), (b), and (c): Evolution of the squares of the 3-velocity components of non-comoving radiation with the increase in entropy in time $t$ in a Bianchi IX universe consisting of non-comoving radiation, as well as comoving dust and the ghost fields, the latter to facilitate the bounce. Unlike in Figures 6.4b, 6.4c and 6.4d, the velocity constraint equation (6.11) has not been explicitly imposed. The evolution of $u_2(t)^2$ and $u_3(t)^2$ are highly oscillatory especially in the second cycle with very small time periods of oscillation, and to show this behaviour clearly, the plots are magnified and partly inset.
6.5 The effects of a cosmological constant

![Graphs showing evolution of shear and 3-curvature](image)

Fig. 6.10 Evolution of (a) the shear, and (b) the 3-curvature (left to right) and the individual scale factors with the increase in entropy with time $t$ in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, containing a comoving dust field and a comoving ghost field to facilitate the bounce, together with a positive cosmological constant.

6.5.1 Positive cosmological constant ($\Lambda > 0$)

Now we add a cosmological constant to the model. The effect of cosmological constant domination in the case of comoving velocities was to cause the model to change from a cyclical behaviour to asymptotically de Sitter like expansion [19] (note that the cosmic no hair theorems [129] do not cover the type IX case because the 3-curvature scalar can be positive).

As in the case of comoving velocities, the model is able to undergo cyclical behaviour until the maxima grow large enough for the cosmological constant to dominate at late times and then the dynamics approach a phase of quasi de Sitter expansion: see Figure 6.13a. The individual expansion rates oscillate while the model is still undergoing cyclical behaviour but approach a constant value $H_0 = \sqrt{\frac{\Lambda}{3}}$ signalling the onset of isotropic de Sitter behaviour, see Figure 6.13b.

In the de Sitter phase the shear and the curvature are diluted by expansion, as expected, and fall exponentially rapidly to very small values: see Figures 6.11a and 6.11b. The 3-curvature can be seen to change sign from negative values (when the dynamics are far from isotropy) to positive values (when the dynamics are close to isotropy). Positive 3-curvature is necessary for a volume maximum to occur.
6.5 The effects of a cosmological constant

Fig. 6.12 Evolution of (a) the volume scale factor, and (b) the individual directional Hubble rates (left to right) with the increase in entropy, and a positive cosmological constant, with time $t$ in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, also containing a comoving dust field and a comoving ghost field to facilitate the bounce. The blue starred, red dotted, and solid green lines correspond to derivatives of the principal values of the 3-metric in the tetrad frame, Hubble rates $\dot{a}/a$, $\dot{b}/b$ and $\dot{c}/c$ respectively. The model undergoes approach to de Sitter expansion when the cosmological constant eventually dominates the dynamics after cycles become large enough to ensure this.
Fig. 6.14 Evolution of the squares of velocity components of non-comoving radiation with the increase in entropy with time $t$ in a Bianchi IX universe consisting of non-comoving radiation, as well as a comoving dust field and a comoving ghost field to facilitate the bounce, and a positive cosmological constant. The graphs (a), (b) and (c) plot the squares of the spatial components of the 4-velocity in the tetrad frame, $u_1(t)^2$, $u_2(t)^2$, and $u_3(t)^2$, respectively.

The velocities themselves oscillate rapidly in each cycle, while the amplitudes of the oscillations rise or fall according to the growth or regression of the scale factors: see Figures 6.15a, 6.15b and 6.15c. The amplitudes of the oscillations of the velocities in two of the directions grow with the exponential expansion of the scale factors. As the scale factors expand further, the time period of the oscillations of the velocities also increases.
6.5.2 Negative cosmological constant ($\Lambda < 0$)

Fig. 6.16 Evolution of (a) the shear, and (b) the 3-curvature with the increase of entropy with time $t$ in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, as well as comoving dust and ghost field, the latter to facilitate the bounce, in the presence of a negative cosmological constant.

Adding a negative cosmological constant results in the universe always recollapsing [124], as this is just another null energy condition violating field. For the behaviour of the volume and individual scale factors, see Figures 6.19a and 6.19b.

The ghost field allows the model to undergo more cycles of oscillation. As we are not introducing an increase in entropy and all the cycles are of equal size, we shall focus on one cycle. The velocities all oscillate and increase with the volume of the universe. One of the velocities ($u_1(t)^2$) also oscillates but with smaller amplitude around a constant value. Only at the end of each cycle does this velocity component show an increase in the amplitude of oscillations: see Figures 6.21a, 6.21b and 6.21c.

The shear and the 3-curvature undergo oscillations, falling to their smallest values at the moments when the volume of the universe is at its highest: see Figures 6.17a and 6.17b. Again, we see the 3-curvature taking on negative values when the dynamics are significantly anisotropic and positive values when close to isotropy.
Tilted Bianchi models and the effect of non-comoving velocities

Fig. 6.18 Evolution of (a) the volume scale factor and (b) the individual scale factors with $t$ in the presence of a negative cosmological constant in a Bianchi IX universe where the radiation is not comoving with the tetrad frame, and containing a comoving dust field and a comoving ghost field to facilitate the bounce. The blue starred, red dotted and green solid lines correspond to the principal values of the 3-metric in the tetrad frame, scale factors $a(t)$, $b(t)$ and $c(t)$, respectively.

6.6 Conclusions

To complete the analysis of the shape of cyclic closed anisotropic universes, it is important to include the effects of non-comoving matter. In the current analysis, we have extended the results of [19] by including a radiation field that is not comoving with the reference tetrad frame. This tilted velocity field introduces vorticity, in addition to the shear and 3-curvature anisotropies, into the universe.

We found that, as in the comoving case, the expansion maxima increases with increasing entropy of the constituents from cycle to cycle, while the individual scale factors oscillate out of phase with each other. The overall dynamics approach flatness over many cycles but they become increasingly anisotropic. We find a new effect in oscillating universes with non-comoving velocities and vorticity. Over successive cycles of entropy increase the conservation of momentum and angular momentum ensures that there is a decrease in the magnitude of the velocities and vorticities in response to the increase of entropy. We modelled entropy increase per cycle by adding entropy at the start of each cycle of a closed universe. We also included a comoving ghost field with negative energy density in order to create a bounce at finite expansion minima and avoid the chaotic mixmaster regime as $t \to 0$ – it is not relevant in practice to post-Planck time evolution.

Our numerical study shows that the velocities oscillate many times and around an almost constant value per cycle, and the amplitude of the oscillations increases with the increase
Fig. 6.20 Evolution of the squares of the velocities of non-comoving radiation with time $t$ in a Bianchi IX universe consisting of non-comoving radiation, as well as comoving dust and ghost fields, the latter to facilitate the bounce, and a negative cosmological constant. Plots (a), (b) and (c) show the squares of the spatial components of the 4-velocity in the tetrad frame, $u_1(t)^2$, $u_2(t)^2$, and $u_3(t)^2$, respectively. The highly oscillatory behaviour of the velocity components with very short time period is captured by the magnified insets in each of the plots.
in expansion volume. The velocity in one of the directions tends to a constant value after initially undergoing several oscillations. On explicitly imposing the constraint equation arising out of particle number conservation, (see equation (6.11)), we find that an increase in entropy density( and hence energy density as for radiation $s \propto \rho^{3/4}$) produces a corresponding decrease in the components of the non-comoving velocity, and vice versa.

When we add a positive cosmological constant to a model containing radiation and a ghost field we confirm that the oscillations are sustained until the cosmological constant dominates the dynamics, after which the scale factors enter a period of quasi de Sitter expansion. The velocities oscillate with amplitude increasing with increasing scale factor as before, but after cosmological constant domination, the time period of oscillations starts increasing and they oscillate less rapidly around a constant value. The asymptotic state is de Sitter with a constant velocity field.

When we add a negative cosmological constant we find there is always collapse, as expected. Studying one cycle we see that the scale factors oscillate out of phase with each other. The velocities in two directions oscillate with increasing amplitude as the volume increases but decrease again with decreasing volume. The velocity in the third direction, however, oscillates with very small amplitude around a constant value, only increasing in oscillation amplitude at the end of each cycle when the volume is its smallest.

We conclude that the inclusion of non-comoving velocities has the effect of increasing the time period of the oscillations of the model. The velocities oscillate rapidly per cycle but with increasing amplitude as the volume of the universe increases, in at least two directions. In the third direction, the velocity oscillates around a constant value with very small amplitude, and hence remains nearly constant per cycle. It only increases in amplitude when the model collapses, before relapsing again to a nearly constant value during the next cycle. Our analysis has identified the principal ingredients of a general cyclic closed universe in the case of spatial homogeneity.
Chapter 7

Anisotropic models in Canonical Quantum Cosmology

7.1 Introduction

In the previous chapters, we have shown that a highly anisotropic initial state is unfavourable to sustaining a bouncing model. This analysis has been purely classical. While quantum effects, such as the effects of particle production on the process of isotropisation can be included, it is also worth investigating the effects of considering the problem from the perspective of quantising the gravitational sector. Thus we shall try to understand the effect of anisotropies on the probability of the universe to tunnel into existence. In order to do this, we shall employ the formalism of canonical quantum cosmology in general, and the tunnelling analogy [127] in particular. An alternative approach to this problem was made by the Hartle Hawking approach [72], which I will not discuss here. We shall consider 2 examples of anisotropic universes for this purpose: the flat Bianchi I universe containing the simplest form of expansion anisotropies; and the Bianchi IX universe, as it is the anisotropic analogue of the closed Friedmann universe, containing both anisotropic 3-curvature and expansion anisotropies.

7.2 The Hamiltonian formalism of general relativity

The formalism of quantum cosmology is largely dependent on the Hamiltonian formalism of general relativity. This is briefly reviewed in the subsequent section. For these purposes, we consider the metric of a 3-dimensional compact space $h_{ij}$ embedded in a 4-dimensional
manifold with metric $g_{\mu\nu}$. This is given by the $3+1$ metric,

$$\begin{align*}
\text{ds}^2 &= g_{\mu\nu}dx^\mu dx^\nu = -\left(N^2 - N_iN^i\right)dt^2 + 2N_idx^i dt + h_{ij}dx^i dx^j \\
&= -N^2 dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j 
\end{align*}$$

(7.1)

where $N$ and $N_i$ are the lapse and shift functions. The action is given by,

$$S = \frac{1}{2} \left[ \int_M d^4x \sqrt{-g} (R - 2\Lambda) + 2 \int_{\partial M} d^3x h^{1/2} K \right] + S_{\text{matter}}$$

(7.2)

where $K$ is the trace of the extrinsic curvature $K_{ij}$ at the boundary of the four-manifold $M$, given by $\partial M$. The extrinsic curvature is given by,

$$K_{ij} = \frac{1}{2N} \left[-\frac{\partial h_{ij}}{\partial t} + 2D_i(N_j)\right]$$

(7.3)

Here $D_i$ is the covariant derivative in the three-surface. Let us consider a scalar field (given by $\phi$) action for the matter sector.

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right]$$

(7.4)

The action in terms of the $3+1$ variables becomes,

$$S = \int d^3x dt dN h^{1/2} [K_{ij}K^{ij} - K^2 + 3R - 2\Lambda] + S_{\text{matter}}$$

(7.5)

From this, we can derive the Hamiltonian form of the action,

$$S = \int d^3x dt \frac{1}{2} \left[h_{ij} \pi^{ij} + \dot{\phi} \pi_\phi - N \mathcal{H} - N_i \mathcal{H}_i\right]$$

(7.6)

where $\pi^{ij}$ and $\pi_\phi$ are the momenta conjugate to $h_{ij}$ and $\phi$ respectively. From this we can see that the Hamiltonian is a sum of constraints and the Lagrange multipliers are the lapse $N$ and the shift $N_i$. The momentum constraint is given by,

$$\mathcal{H}_i = -2D_j \pi^{ij}_i + \mathcal{H}^\text{matter}_i = 0$$

(7.7)

The Hamiltonian constraint is given by,

$$\mathcal{H} = 2G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2} h^{1/2} (3R - 2\Lambda) + \mathcal{H}^\text{matter} = 0$$

(7.8)
where $G_{ijkl}$ is the DeWitt metric given by,

$$G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$$  \hspace{1cm} (7.9)

### 7.3 The superspace

The formalism of quantum cosmology has been to view the universe as a quantum object, all information regarding which is contained in a quantum wavefunction,

$$\psi(h_{ij}(x), \phi(x))$$  \hspace{1cm} (7.10)

This wavefunction is defined on the space of all 3 dimensional metrics and all matter field configurations $\phi(x)$, which is called the ‘superspace’. It is infinite dimensional, with a finite number of coordinates $(h_{ij}(x)), \phi(x)$ at every point $x$ on the 3-surface. The DeWitt metric along with some suitable metric for the matter sector provides the metric on the superspace. This wavefunction is also invariant under all 3-dimensional diffeomorphisms. The wavefunction in this case is a functional on the superspace. It does not have any explicit dependence on time, because general relativity is a parametrised theory and “time” is already contained in its dynamical variables $h_{ij}$ and $\phi$. This also shows in the fact that the relevant 3-surfaces are compact and do not change their relative position in the 4-manifold.

Following the Dirac quantisation procedure, the wavefunction is annihilated by the operator version of the classical constraints. Thus we make the following substitutions for the momenta,

$$\pi^{ij} \rightarrow -i \frac{\partial}{\partial h_{ij}}, \quad \pi_{\phi} \rightarrow -i \frac{\partial}{\partial \phi}$$  \hspace{1cm} (7.11)

The quantised version of the momentum constraint becomes,

$$H_i \Psi = 2i D_j \frac{\partial \Psi}{\partial h_{ij}} + H_{\text{matter}} \Psi = 0$$  \hspace{1cm} (7.12)

The momentum constraint is invariant under diffeomorphisms. We can see that by shifting the coordinates by $x^i \rightarrow x^i - \xi^i$, and writing the corresponding wavefunction as,

$$\Psi [h_{ij} + D_{(i}(\xi_{j)] = \Psi [h_{ij}] + \int d^3 x D_{(i}(\xi_{j)} \frac{\delta \Psi}{\delta h_{ij}}$$  \hspace{1cm} (7.13)
Integrating the last term by parts and dropping the boundary term as the manifold is assumed to be compact, the change in the wavefunction is given by,

$$\delta \Psi = - \int d^3x \xi_j D_i \left( \frac{\delta \Psi}{\delta h_{ij}} \right) = \frac{1}{2i} \int d^3x \xi_i \mathcal{H}^i \Psi$$

(7.14)

showing that the wavefunctions satisfying the momentum constraint remain unchanged. The Wheeler de Witt equation becomes,

$$\mathcal{H} \Psi = \left[ -G_{ijkl} \frac{\partial}{\partial h_{ij}} \frac{\partial}{\partial h_{kl}} - \frac{h^{1/2}}{2} \left( 3R - 2\Lambda \right) + \mathcal{H}_{\text{matter}} \right]$$

(7.15)

ignoring operator ordering ambiguities for the sake of simplicity. Interpreting the wavefunction is a widely debated subject. For example, [70] considers a peak in the wavefunction or any probability distribution constructed from it, as the marker of a physical prediction.

### 7.4 The minisuperspace formalism

Working in all generality in an infinite dimensional superspace is very hard. Thus canonical quantum cosmology tends to restrict the superspace to homogeneous and isotropic geometries and their perturbations. The justification for doing this is that these geometries are physically relevant to the universe we live in. This is of course problematic from a theoretical perspective as we effectively set several field modes and their momenta to zero simultaneously, violating the uncertainty principle. The restriction to minisuperspace can be thought of as a toy model, and it makes the problem more tractable, in that the configuration space now becomes finite dimensional.

In the minisuperspace approximation, the lapse function is taken to be homogenous $N = N(t)$, the shift is set to zero $N_i = 0$, and the metric then becomes,

$$ds^2 = -N^2(t)dt^2 + h_{ij}(x,t)dx^idx^j$$

(7.16)

The three metric is taken to be homogenous and is hence described by $t$. The configuration space is now finite-dimensional in the minisuperspace, and the configuration space is described by some finite number of other functions, for example $q^{\alpha}(t)$. These functions are typically some function of the metric functions $h_{ij}(x,t)$. For our purposes, $\alpha = 1, 2, 3$. Inserting this metric into the action, we see,

$$S = \int_0^1 dt N \left[ \frac{1}{2N^2} f_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - U(q) \right] = \int L dt$$

(7.17)
In the above, the lapse function can be scaled and $t$ shifted to have the time integration take place from 0 to 1. Here $f_{\alpha \beta}$ is the reduced version of the DeWitt metric (7.9) and has indefinite signature $(-, +, +, +, \ldots)$, with the $-$ corresponding to a gravitational variable. This form can be extended to the inclusion of matter, under some assumptions, so $f_{\alpha \beta}$ could also include matter variables. Varying with respect to $q^\alpha$ we obtain the field equations,

$$\frac{1}{N} \frac{d}{dt} \left( \frac{q^\alpha}{N} \right) + \frac{1}{N^2} \Gamma^\alpha_{\beta \gamma} q^\beta q^\gamma + f_{\alpha \beta} \frac{\partial U}{\partial q^\beta} = 0 \quad (7.18)$$

where $\Gamma^\alpha_{\beta \gamma}$ is the Christoffel symbol constructed from the metric $f_{\alpha \beta}$. The general solution of these equations has $(2n - 1)$ arbitrary parameters. The canonical momenta are defined as,

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = f_{\alpha \beta} \frac{\dot{q}^\beta}{N} \quad (7.19)$$

and the canonical Hamiltonian is,

$$H_c = p_\alpha q^\alpha - L = N \left[ \frac{1}{2} f_{\alpha \beta} p_\alpha p_\beta + U(q) \right] \equiv NH \quad (7.20)$$

The Hamiltonian form of the action is,

$$H(q^{\alpha}, p_\alpha) = \frac{1}{2} f_{\alpha \beta} p_\alpha p_\beta + U(q) = 0 \quad (7.21)$$

This also means that $N$ is the Lagrangian multiplier, enforcing the constraint that $H = 0$. On canonically quantising, we demand that the Hamiltonian, which is now promoted to an operator, annihilates the time dependent wavefunction. This equation is the Wheeler de Witt equation,

$$\hat{H}(q^{\alpha}, \hat{p}_\alpha) \Psi(q^{\alpha}) = 0 \quad (7.22)$$

Solving this equation will yield the wavefunction of the universe. The main problem with extracting information from the universe wavefunction is the absence of a unique way of defining boundary conditions. It thus seems that the boundary conditions must be imposed as an additional law to the system. One proposal is the no-boundary wavefunction by Hartle and Hawking. This says that the wavefunction should be given by an integral over all compact 4-geometries bounded by the 3-geometry $h_{ij}$,

$$\Psi(h_{ij}(x), \phi(x)) = \int [dg][d\phi] \exp[-S_E(g, \phi)] \quad (7.23)$$
This path integral, however, on performing the Euclidean rotation $t \rightarrow -i\tau$ as is often done in quantum field theory, is still badly divergent from below. There have been some attempts to fix this issue by analytic continuation.

Another proposal is the Lorentzian wavefunction, which involves integrating all Lorentzian histories which interpolate between a vanishing 3-geometry $\emptyset$ to $(h, \phi)$, and which lies to the past of $(h, \phi)$.

$$\psi(h_{ij}(x), \phi(x)) = \int_{\emptyset}^{\emptyset} [dg] [d\phi] \exp[i S(g, \phi)]$$  \hspace{1cm} (7.24)

In relation to this is the tunneling boundary condition which gives the same wavefunction as the Lorentzian path integral, and which says that only outgoing waves in $\psi(h_{ij}(x), \phi(x))$ should be allowed at the boundary. However, the definitions of ‘outgoing’ and ‘boundary’ in this case is still contested.

Another approach to quantum cosmology is the “third quantisation” approach in which the universe wavefunction is promoted to an operator in quantum field theory and can be expressed in terms of the creation and annihilation operators. The problem of defining boundary conditions is now replaced by the problem of uniquely defining an ‘in’ state for the operator $\psi$. The radius of the universe plays the role of time in this case.

### 7.5 Tunneling: de Sitter minisuperspace

As a simple first example of the tunneling wavefunction, we shall consider the minisuperspace model corresponding to the closed de Sitter universe [127]. The gravitational action in this case is given by,

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2} - \rho_v \right)$$  \hspace{1cm} (7.25)

where $R$ is the Ricci scalar, and $\rho_v$ is the vacuum energy in the universe. The universe is assumed to homogeneous and isotropic. The metric is given by,

$$ds^2 = \sigma^2 [N(t)^2 dt^2 - a(t)^2 d\Omega_3^2]$$  \hspace{1cm} (7.26)

where $N$ is an arbitrary lapse function, $a(t)$ is the scale factor, $\sigma^2 = 1/12\pi^2$ is the normalisation constant and $d\Omega_3^2$ is the metric on the 3-sphere. Substituting the metric into the action, we get,

$$\mathcal{L} = \frac{1}{2} \left[ a \left( 1 - \frac{a^2}{N^2} \right) - \Lambda a^3 \right]$$  \hspace{1cm} (7.27)

and the momentum is given by,

$$p_a = -\frac{aa}{N}$$  \hspace{1cm} (7.28)
where $\Lambda = (1/6\pi)^2\rho_v$. The Lagrangian in (7.27) can be written in the canonical form as follows,
\[
\mathcal{L} = p_a \dot{a} - N \mathcal{H} \tag{7.29}
\]
where
\[
\mathcal{H} = -\frac{1}{2} \left( \frac{p_a^2}{a} + a - \Lambda a^3 \right) \tag{7.30}
\]
Variation of the above with respect to $N$, gives,
\[
\mathcal{H} = 0 \tag{7.31}
\]
For $N = 1$, we recover the equation of motion for $a$ as follows,
\[
a^2 + 1 - \Lambda a^2 = 0 \tag{7.32}
\]
The classical solution to this, which is effectively the Friedmann equation is,
\[
a(t) = H^{-1} \cosh(Ht) \tag{7.33}
\]
where $H = \Lambda^{1/2}$. We can quantise this model by replacing $p_a \rightarrow -i \partial / \partial a$ and hence obtaining the Wheeler deWitt equation from the quantised Hamiltonian constraint (7.31).
\[
\left( \frac{d^2}{da^2} + \frac{\gamma}{a} \frac{d}{da} - U(a) \right) \psi(a) = 0 \tag{7.34}
\]
where
\[
U(a) = a^2 (1 - H^2 a^2) \tag{7.35}
\]
and $\gamma$ is the operator ordering ambiguity between the operators $a$ and $p_a$. We consider the solutions of this equation in the semi-classical regime, that is, close to the turning points of the potential, where $d\psi/da \approx 0$. In these regions the $\gamma$ term does not affect the solution. The classically allowed solution ($U(a) > 0$) is given by $a \geq H^{-1}$. The WKB solutions in this region are given by,
\[
\psi_{\pm}(a) = [p(a)]^{-1/2} \exp[\pm i \int_{H^{-1}}^a p(a') da' \mp i\pi/4] \tag{7.36}
\]
where $p(a) = [-U(a)]^{1/2}$. The under-barrier solutions where $a < H^{-1}$ are given by,
\[
\tilde{\psi}_{\pm}(a) = [p(a)]^{-1/2} \exp[\pm \int_{H^{-1}}^a |p(a')| da'] \tag{7.37}
\]
For \( a \gg H^{-1} \),

\[
\hat{p}_a \psi_{\pm}(a) \approx \pm p(a) \psi_{\pm}(a) \tag{7.38}
\]

The ‘+’ and ‘−’ signs correspond to expanding and contracting universes respectively as given by the sign of the eigenvalue \( p(a) \) and hence \( \dot{a} \) (from the definition of \( p(a) \)). The way the tunneling scenario now imposes boundary conditions is by neglecting the solution where the universe started from an infinitely large size and contracted to a small size. Thus we have,

\[
\psi(a > H^{-1}) = \psi_-(a) \tag{7.39}
\]

The under-barrier wavefunction that we found from the WKB approximation is,

\[
\psi(a < H^{-1}) = \tilde{\psi}_+(a) - \frac{i}{2} \tilde{\psi}_-(a) \tag{7.40}
\]

Far away from the classical turning point \( a = H^{-1} \), the first term in the above equation dominates. The nucleation probability is then given as,

\[
\left| \frac{\psi(H^{-1})}{\psi(0)} \right| \sim \exp \left( -2 \int_0^{H^{-1}} |p(a')| \right) = \exp \left( -\frac{3}{8G^2 \rho_v} \right) \tag{7.41}
\]

Thus we see that a very large vacuum energy \( \rho_v \) reduces the probability of such a universe to come into existence exponentially. This was initially seen as a justification of sorts for having a small value of vacuum energy today.

### 7.6 Effect of anisotropies on tunnelling probability

There has been an ongoing effort for several years to attempt to find quantum cosmological solutions for anisotropic models [86, 85]. Studies [10] show that Bianchi I is the most general vacuum, compact model. The Bianchi IX universe has been studied in [85, 51], with exact solutions to the Wheeler de Witt equation being found in the axisymmetric case. In this section, we have solved the Wheeler de Witt equation in the case of small anisotropies in the Bianchi I and the Bianchi IX universes, and then used these solutions to find the tunnelling probabilities. In the end of the section, we comment on the case of large anisotropies.

#### 7.6.1 The Bianchi I universe

Let us consider the case of one of the simplest bouncing models, namely, the simple harmonic universe. This involves a universe with negative cosmological constant \( \Lambda < 0 \) and ‘domain-
wall’ matter where the equation of state is \( p = -\frac{2}{3}\rho \). This model has been studied in [66, 67] as a simple, stable bouncing solution in GR which avoids the singularity. The question of the stability of a scalar field model with a negative cosmological constant, to small perturbations when tunnelling from the oscillating state to zero size was explored in [99]. It was seen that, while there are both stable and unstable solutions, all solutions pertaining to tunnelling to zero size are unstable. In [100], the tunnelling decay rate of such a universe tunnelling to zero size, with the use of a massless scalar field acting as a ‘clock’ was calculated.

In this section, we shall extend the analysis, and consider the effect of anisotropies on the probability of a universe tunnelling into finite size from a potential barrier. In the case of the simple flat anisotropic universe, we shall see that the anisotropies behave essentially like massless scalar fields, allowing us to perform a semi-analogous analysis to [100].

In order to study the effect of anisotropies on the tunneling decay rate, we consider these fields in a Bianchi I universe, having a metric,

\[
ds^2 = dt^2 - a(t)^2 \left\{ e^{2\beta_+(t) + 2\sqrt{3}\beta_-(t)} dx^2 + e^{2\beta_+(t) - 2\sqrt{3}\beta_-(t)} dy^2 + e^{-2\beta_+(t) - 2\sqrt{3}\beta_-(t)} dz^2 \right\} \quad (7.42)
\]

The isotropic limit of this, is the flat Friedmann metric which is obtained when \( \beta_+ = \beta_- = 0 \). Also, for the rest of the chapter \( a = \exp(\alpha) \). The Friedmann equation then looks like,

\[
\frac{\dot{a}^2}{a^2} = \frac{1}{2}(\dot{\beta}_+^2 + \dot{\beta}_{-2}) + \frac{\Lambda}{3} + \frac{\sigma}{a} \quad (7.43)
\]

where \( \sigma \) is the initial condition for the density of the ‘domain-wall’ matter field that we have included. We are studying the tunnelling probability through a potential barrier from \( a = 0 \) to a finite volume. Thus we require a potential with at least 2 turning points, one of which is at \( a = 0 \) and the other at some finite \( a \). The potential barrier corresponds to positive values of the potential and the classically allowed regions correspond to negative values of the potential. In order to satisfy these conditions and to use as an illustrative example, the effect of anisotropies on tunnelling probability, we choose \( \sigma < 0 \) and \( \Lambda > 0 \). For the remaining part of this section we shall only be working with the absolute values of \( \sigma \) and \( \Lambda \) and use the appropriate sign. The Hamiltonian for such a model containing the cosmological constant and the domain-wall matter is,

\[
H = -\frac{1}{2a} \left( p_+^2 - p_-^2 - p_z^2 + \bar{\dot{U}}(a) \right) \quad (7.44)
\]
The potential $\bar{U}(a)$ is given by,

$$\bar{U}(a) = - (\Lambda a^4 - \sigma a^3)$$  \hspace{1cm} (7.45)

and the momentum terms are given by $p_a = -a \dot{a}$, $p_\pm = \beta_\pm a^3$. Quantising this model and replacing $p_a \to -\partial / \partial a$, $p_\pm \to -\partial / \partial \beta_\pm$, we get the Wheeler de Witt equation,

$$\left( -a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} + a^2 \bar{U}(a) \right) \Psi = 0$$  \hspace{1cm} (7.46)

We replace $\alpha = \ln(a)$. The universe wavefunction is $\Psi$ and can be written in terms of its separable solution given by $\Psi = A(\alpha) \bar{\beta}_+ (\beta_+) \bar{\beta}_- (\beta_-)$. The zeroes of the potential are thus at $\alpha \to -\infty$ and $\alpha = \ln(\sigma / \Lambda)$. The potential is plotted in Figure 7.1.

![Minisuperspace potential for the Bianchi I universe with $\Lambda = 2$ and $\sigma = -1$ and $b^2 = 0.01$ plotted against the scale factor $a$. The regions corresponding to classically allowed and disallowed regions are labelled as I, II and III, and will be used subsequently when writing solutions in these regions.](image)

The separable parts of the wavefunction corresponding to the anisotropies, $\bar{\beta}_\pm$ follow the equations,

$$\frac{d\bar{\beta}_+}{d\beta_+^2} = b_+^2 \bar{\beta}_+$$  \hspace{1cm} (7.47)

$$\frac{d\bar{\beta}_-}{d\beta_-^2} = b_-^2 \bar{\beta}_-$$  \hspace{1cm} (7.48)
Thus the equation for \( A(a) \) becomes,

\[
\frac{dA^2}{d\alpha^2} + (U(\alpha) - b_+^2 - b_-^2)A(a) = 0
\]

(7.49)

where \( U(\alpha) = e^{2\alpha} \hat{U}(\alpha) \). Denoting the modified potential as \( U_b \) and \( b^2 = b_+^2 + b_-^2 \) we have,

\[
U_b = U(\alpha) - b^2
\]

(7.50)

The modification of the potential leads to the creation of a new classically allowed region, region \( III \) in Figure 7.1. Furthermore, the modification also modifies the turning points. The change in the turning points can be found by solving for \( U_b = 0 \). Doing this perturbatively in \( b^2 \) we find that the change is given by,

\[
\delta \alpha = \frac{b^2}{|U'(\bar{\alpha})|}
\]

(7.51)

where \( \bar{\alpha} \) denotes the turning points \( \alpha \to -\infty \) and \( \alpha = \ln(\sigma/|\Lambda|) \). Thus we find the modified turning point \( \alpha_* \) to be,

\[
\alpha_* = \ln\left(\frac{\sigma}{\Lambda}\right) - \frac{b^2 \Lambda^3}{\sigma^4}
\]

(7.52)

The modification to the \( \alpha \to -\infty \) can be found by taking this limit on the modified potential \( U_b \) and solving for \( \lim_{\alpha \to \infty} U_b(\alpha_0) = 0 \). This gives,

\[
\alpha_0 = \frac{1}{3} \ln\left(\frac{b^2}{\sigma}\right)
\]

(7.53)

Far away from the turning points (\( \alpha = \alpha_0, \alpha_* \)), using the semiclassical approximation, the Wheeler de Witt equation has the following solution,

\[
A(\alpha) \approx \frac{C_1 e^{-i\pi/4}}{|-U_b(\alpha)|^{1/4}} e^{i\int_{\alpha_*}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'} + \frac{C_2 e^{i\pi/4}}{|-U_b(\alpha)|^{1/4}} e^{-i\int_{\alpha_*}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'}
\]

(7.54)

The solution underneath the potential barrier takes the form,

\[
A(\alpha) \approx \frac{D_1}{(U_b(\alpha))^{1/4}} e^{i\int_{\alpha_*}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'} + \frac{D_2}{(U_b(\alpha))^{1/4}} e^{-i\int_{\alpha_*}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'}
\]

(7.55)
Near the turning points, $\alpha = \alpha_0, \alpha_*$ this approximation breaks down, and then the potential is approximated as,

$$U(\alpha) \approx U(\alpha_*) + U'(\alpha_*)(\alpha - \alpha_*) = U'(\alpha_*)(\alpha - \alpha_*)$$  \hspace{1cm} (7.56)

as $U(\alpha_*) = 0$. Setting $U'(\alpha_*)(\alpha - \alpha_*) = z$, the WdW equation near a turning point,

$$\left(\frac{\partial^2}{\partial z^2} - z\right) \Psi(z) = 0$$  \hspace{1cm} (7.57)

The solutions are linear combinations of the asymptotic forms of the Airy functions, $Ai(z)$ and $Bi(z)$. The asymptotic forms of the Airy functions which will be used for the rest of the calculation are given below.

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3} z^{3/2}}$$  \hspace{1cm} (7.58)

$$Bi(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} e^{\frac{2}{3} z^{3/2}}$$  \hspace{1cm} (7.59)

$$Ai(-z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \sin \left[ \frac{2}{3} \frac{z^{3/2}}{4} \right]$$  \hspace{1cm} (7.60)

$$Bi(-z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \cos \left[ \frac{2}{3} \frac{z^{3/2}}{4} \right]$$  \hspace{1cm} (7.61)

Imposing boundary conditions to the right of the turning point $\alpha = \alpha_*$, we require that the solution be only outgoing. This is because we are interested in solutions that expand out after tunnelling through the barrier.

$$A_I(\alpha) = \frac{C_I e^{i\pi/4}}{|-U_b(\alpha)|^{1/4}} e^{-i \int_{\alpha}^{\alpha_*} \sqrt{-U_b(\alpha')} d\alpha'}$$  \hspace{1cm} (7.62)

To the left of $a = a_0$ we have a classically allowed region. We are not interested in trajectories that lead to $a < 0$ as these have no physical significance. Thus we impose boundary conditions which only involve trajectories that go to values of $a > a_0$.

$$A_{III}(\alpha) = \frac{C_{III} e^{i\pi/4}}{|-U_b(\alpha)|^{1/4}} e^{-i \int_{a_0}^{a} \sqrt{-U_b(\alpha')} d\alpha'}$$  \hspace{1cm} (7.63)
For the under-barrier solution, in region II, we have by matching to the solution by the WKB connection formula in region III,

\[
A_{\text{II}}(\alpha) = \frac{C_{\text{III}}}{|U_b(\alpha)|^{1/4}} \left[ -\frac{i}{2} e^{-\int_{\alpha_0}^{\alpha} \sqrt{U_b(\alpha')} d\alpha'} + e^{i\int_{\alpha_0}^{\alpha} \sqrt{U_b(\alpha')} d\alpha'} \right]
\] (7.64)

and by matching to the solution in region I, we have,

\[
A_{\text{II}}(\alpha) = \frac{1}{|U_b(\alpha)|^{1/4}} \left[ A' e^{-\int_{\alpha_*}^{\alpha} \sqrt{U_b(\alpha')} d\alpha'} + B' e^{i\int_{\alpha_*}^{\alpha} \sqrt{U_b(\alpha')} d\alpha'} \right]
\] (7.65)

As these are the same solution, the constants \(A'\) and \(B'\) can be written as,

\[
A' = C_{\text{III}} e^{K}
\] (7.66)

\[
B' = -\frac{iC_{\text{III}}}{2} e^{-K}
\] (7.67)

The constant \(C_I\) can also be written in terms of \(C_{\text{III}}\) as follows,

\[
C_I = \left( 2e^K - \frac{e^{-K}}{2} \right)
\] (7.68)

In all of the above definitions, \(K\) is defined to be,

\[
K = \int_{\alpha_0}^{\alpha_*} \sqrt{U_b(\alpha')} d\alpha'
\] (7.69)

The probability of the universe tunnelling through the potential barrier in region II to the classically allowed region in region I, is given by

\[
\Gamma = \frac{|A_I|}{|A_{\text{III}}|} = \left| \left( 2e^K - \frac{e^{-K}}{2} \right) \right|^2 e^{-2K}
\] (7.70)

**Evaluation of \(K\)**  Now, taking into account the shift in the turning points of the potential because of the anisotropy, we see that \(\alpha_0 = (1/3)\ln(b^2/\sigma)\) and \(\alpha_* = \sigma/\Lambda - b^2 \Lambda^3 / \sigma^4\). Thus \(K\) can be written as,

\[
K \approx \int_{-\alpha_0}^{\alpha_*} \sqrt{U(\alpha')} d\alpha'
\] (7.71)

The terms dependent on \(b^2\) in the potential are not considered as when integrated with the endpoints of the integral which are also dependent on \(b\), we get terms which are of order
higher than $b^2$. The zeroth order terms are given by,

$$K_0 = -rac{1}{4\Lambda^{3/2}} \frac{2\Lambda^2 - 3\Lambda \sigma + \sigma^2}{(\frac{\sigma}{\Lambda} - 1)^{1/2}} + \frac{\sigma^2}{4\Lambda^{3/2}} \arctan \left( \frac{\sigma}{\Lambda} - 1 \right)^{-1/2}$$  \quad (7.72)

Retaining terms upto order $b^2$, we find the perturbation to be,

$$\delta K = -\frac{2}{3} b + \frac{1}{5} \frac{\Lambda}{\sigma^{4/3}} b^{5/3} - b^2 \frac{\Lambda^{7/2}}{\sigma^4} \sqrt{\frac{\sigma}{\Lambda} - 1}$$  \quad (7.73)

The lowest order in the above is the first term and hence the tunnelling probability (7.70) is suppressed by a factor of $\sim \exp(-4b/3)$.

### 7.7 Bianchi IX models with vacuum energy

We shall now consider the anisotropic analogue to the closed de Sitter example, reviewed in Section 7.5. This is the Bianchi IX universe in the presence of a cosmological constant, studied in [51]. Let us consider such a model containing vacuum energy given by $\rho_v > 0$. It has a metric,

$$ds^2 = a(t)^2 \left[ dt^2 - (e^{2\beta})_{ij} \dot{e}^i \dot{e}^j \right]$$  \quad (7.74)

where $e^i$ for $i = 1, 2, 3$ are the triad one-forms, and $a(t)$ is the scale factor. The $\beta_{ij}$ are given by,

$$\beta_{ij} = \text{diag} \left( \beta_+ + \sqrt{3} \beta_-, \beta_+ - \sqrt{3} \beta_-, -2\beta_+ \right)$$  \quad (7.75)

where $\beta_\pm$ are anisotropy parameters. The action can be written as,

$$S = \int d^4x \sqrt{-g} \left( R - \rho_v \right)$$  \quad (7.76)

Canonically quantising the corresponding Hamiltonian, we obtain the Wheeler de Witt equation,

$$\left[ a^2 p \frac{\partial}{\partial a} \left( a^p \frac{\partial}{\partial a} \right) - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} - U(a, \beta_+, \beta_-) \right] \psi(a, \beta_+, \beta_-) = 0$$  \quad (7.77)

where $p$ represents the operator ordering ambiguity in the operators $a$ and $\partial/\partial a$. The super-potential $U$ is given by,

$$U(a, \beta_+, \beta_-) = a^4 \left[ 1 - a^2 H^2 - V(\beta_+, \beta_-) \right]$$  \quad (7.78)
where \( H^2 \sim p^2 \rho_v \), and the anisotropy potential \( V \) is given by,

\[
V(\beta_+, \beta_-) = 1 + \frac{1}{3} \left\{ e^{-8\beta_+} + 2e^{4\beta_+} \left[ \cosh(4\sqrt{3}\beta_-) - 1 \right] - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) \right\} \tag{7.79}
\]

This is bounded from below as at \( \beta_+ = \beta_- = 0, V = 0 \). The boundary of the classically forbidden region is given by the surface \( U(a, \beta_+, \beta_-) = 0 \),

\[
a^2 H^2 = 1 - V(\beta_+, \beta_-) \tag{7.80}
\]

\( V(\beta_+, \beta_-) \) is a growing function of \( \beta_\pm \) and hence the width of the barrier decreases with growing anisotropy, until \( V(\beta_+, \beta_-) > 1 \) at which point, the barrier disappears and the tunnelling analogy breaks down. As the model cannot be solved exactly, it shall be solved in the limiting cases of small and large anisotropy. Let us introduce,

\[
\beta^2 = \beta_+^2 + \beta_-^2 \tag{7.81}
\]

### 7.7.1 The case of small anisotropy

For small anisotropy, that is, \(|\beta_\pm| \ll 1\), the super-potential becomes,

\[
U(a, \beta_+, \beta_-) = a^4 (1 - H^2 a^2 - 8\beta^2) \tag{7.82}
\]

This is effectively a perturbation about the isotropic FRW model with a cosmological constant. The unperturbed model would then satisfy,

\[
\left[ a^2 - p \frac{\partial}{\partial a} \left( a^p \frac{\partial}{\partial a} \right) - U_0(a) \right] \psi^{(0)}(a) = 0 \tag{7.83}
\]

where,

\[
U_0(a) = a^4 (1 - H^2 a^2) \tag{7.84}
\]

The fixed points of this potential are at \( a = 0 \) and \( a = H^{-1} \). Let us denote these fixed points as \( a_0 \) and \( a_* \) respectively. The solution to this in the classically allowed region, \( a > H^{-1} \), subject to the tunnelling boundary condition, is given by

\[
\psi(a, \beta_\pm) = \psi^{(0)}(a, \beta_\pm) \tag{7.85}
\]
where

\[ \psi^{(0)}(a > H^{-1}) = ca^{-(p+1)/2/(H^2a^2 - 1)^{-1/4}} \exp\left(-\frac{i}{3H^2}(H^2a^2 - 1)^{3/2}\right) \]  

(7.86)

and

\[ \chi(a > H^{-1}, \beta \pm) = \frac{\sqrt{(H^2a^2 - 1)^2 + i^2}}{(\sqrt{H^2a^2 - 1} - i)(\sqrt{H^2a^2 - 1} + 3i)} \exp\left(-\frac{4ia^2b^2}{\sqrt{H^2a^2 - 1} + 3i}\right) \]  

(7.87)

\( C \) is a normalisation constant. The under-barrier wavefunction in the classically disallowed region is given by,

\[ \psi^{(0)}(a < H^{-1}) = C \exp\left(\frac{1}{4}i\pi\right) a^{-(p+1)/2/(1 - H^2a^2)^{-1/4}} \exp\left(\frac{(1 - H^2a^2)^{3/2}}{3H^2}\right) \]  

(7.88)

\[ \chi(a < H^{-1}, \beta \pm) = \frac{\sqrt{(1 - H^2a^2)^2 - 1}}{(\sqrt{1 - H^2a^2} - 1)} \exp\left(-\frac{4a^2\beta^2}{3 - \sqrt{1 - H^2a^2}}\right) \]  

(7.89)

The probability of this universe to tunnel through the potential barrier from the fixed point \( a_0 = 0 \) to the point \( a_\star \sim H^{-1} \) is evaluated by considering the value of the under-barrier wavefunction at the turning point \( a_\star \sim H^{-1} \) and is given by\(^1\),

\[ \left| \frac{\psi(H^{-1})}{\psi(0)} \right| \sim \exp\left(-\frac{4\beta^2}{3\rho_v}\right) \]  

(7.90)

Thus we see that the tunnelling probability is an exponentially decaying function of the anisotropy, here represented by \( \beta^2 \).

### 7.7.2 The case of large anisotropy

In the case of large anisotropy, the potential reads as,

\[ U(a, \beta) = a^4\left[1 - H^2a^2 - \frac{1}{3}\exp(-8\beta_+\right] \]  

(7.91)

In the limit of large anisotropy, the \((1 - H^2a^2)\) term can be neglected with respect to the anisotropy term. The potential above also is independent of \( \beta_- \) in this limit and it is assumed that the solution to the Wheeler de Witt equation is also independent of \( \beta_- \) in this limit. The

\(^1\)There is a modification to the actual value of this fixed point due to the anisotropy part of the potential given by \( \delta a_\star = \frac{\beta^2H^3}{2 + 3\rho_v} \).
solution then reads as,

\[ \psi(x) = x^\nu \left( C_1 K_\nu \left( \frac{1}{6} x \right) + C_2 I_\nu \left( \frac{1}{6} x \right) \right) \] (7.92)

where \( K_\nu \) and \( I_\nu \) are the modified Bessel functions, \( x = a^2 \exp (-4\beta) \) and \( \nu = \frac{1}{6} (p - 1) \).

Imposing the boundary condition that the universe must remain finite at large \( a \), this solution becomes,

\[ \psi(\beta_+ \to \infty) = C_1 x^\nu K_\nu \left( \frac{1}{6} x \right) \] (7.93)

This is appreciably different from zero at \( a \leq \exp(2\beta_+) \). Everywhere else, the wavefunction becomes,

\[ \psi \approx C_1 x^{(p-4)/6} \exp \left( -\frac{1}{6} x \right) \] (7.94)

Thus we see that overall, small anisotropies exponentially suppress the tunnelling probability for a universe to tunnel through a potential barrier. In the case of large anisotropies, the potential barrier vanishes and the tunnelling analogy breaks down. This fact can be an indication of the fact that at very high anisotropies, the sign of the curvature of the Bianchi IX universe changes and it starts resembling an open universe. In [133], a comment is made about the suppression of the probability of creation of non-compact topologies, and this idea is expanded upon in [88]. [88] as well as [49], however, point out that open and flat universes are not necessarily ruled out as probable starting points of the universe, if they are considered to be topologically non-trivial (for example by creating a 3-torus as in the flat case). The physical significance of the tunnelling analogy breaking down in the case of large anisotropy requires further investigation.

### 7.8 Conclusion

The minisuperspace formalism can be thought of as an approximation to a complete theory of quantum gravity and hence, quantum cosmology. However, it can still provide valuable insights about the very early universe and the conditions under which it came into existence. In this chapter, we have considered the effect of anisotropies on the tunnelling probability. We see that, both in the case of the simple Bianchi I case, and also in the case of the anisotropic closed universe, or Bianchi IX universe, while the tunnelling analogy is valid, the tunnelling probability is exponentially suppressed by the presence of anisotropies, thus making it less likely for a highly anisotropic universe to come into existence.
The next case of interest would be to construct an anisotropic analogue of the Simple Harmonic Universe using Bianchi IX. This model would be interesting in understanding questions like the effect of anisotropies on the probability that such a universe would remain oscillating. Following [100], if $Q \ll 1$ is the probability that a $\Lambda = 0$ universe would tunnel, then the probability that such a universe would keep oscillating after $N$ oscillations is given by,

$$P_N = (1 - Q)^N \approx e^{-QN} \quad (7.95)$$

Let us consider an oscillating universe that, like the example of the Simple Harmonic universe is oscillating between a minimum and a maximum radius, given by $a_-$ and $a_+$ respectively. Then the probability $Q$ can be calculated by the modulus of the ratio of the wave-functions in the classically allowed region to the right of $a_-$ and the classically disallowed region to the left of $a_-$. To compute these wave-functions we would need to solve the Wheeler de Witt equation. This can be done perturbatively for the case of small anisotropy. The case of the Bianchi IX model in which the anisotropies manifest as derivative operators as well as in the anisotropy potential, the situation is similar to that of a scalar field in a closed isotropic Friedmann universe in a potential. Thus the simple harmonic anisotropic universe case can be solved using the methods of [126]. This is work in progress.

The boundary of the classically forbidden region found by solving $U(a, \beta_+, \beta_-)$ gives us the turning points of the potential,

$$a_{\pm} = \frac{\sigma}{2H^2} \pm \frac{1}{H^2} \sqrt{\frac{\sigma^2 - 4H^2(1 - V(\beta_+, \beta_-))}{}} \quad (7.96)$$

In the case of small anisotropy, the potential becomes,

$$U(a, \beta_+, \beta_-) = a^4[1 - a^2H^2 + \sigma a + 8\beta^2] = U_0(a) + 8a^4b^2 \quad (7.97)$$

where $\beta^2 = \beta_+^2 + \beta_-^2$. The corresponding turning points,

$$a_{\pm} = \frac{\sigma}{2H^2} \pm \frac{1}{H^2} \sqrt{\frac{\sigma^2 - 4H^2(1 - 8b^2)}{}} \quad (7.98)$$

In order to simplify the subsequent calculations, we follow the conventions of [100] and write the potential in the following form,

$$U(\alpha) = \tilde{\beta}^{-2} e^{4\alpha} (1 - 8b^2 - 2e^{\alpha} + \gamma^{-2} e^{2\alpha}) \quad (7.99)$$
And the turning points in this notation becomes,

\[ \alpha_{\pm} = \ln \left[ \gamma^2 \pm \gamma^2 \sqrt{1 - \gamma^{-2}(1 - 8b^2)} \right] \]  
(7.100)

In the above, we have replaced \( a \) by \( \alpha = \ln \omega \gamma a \), and,

\[ \tilde{\beta} = \left( \frac{1}{12\pi^2} \right) = \frac{32\pi}{27} G^3 \sigma^2 \]  
(7.101)

Also,

\[ \omega = \sqrt{\frac{1}{3} |\Lambda|} \text{ and } \gamma = \sqrt{\frac{1}{12} |\Lambda|} \]  
(7.102)

The solution for the isotropic

7.8.1 Simple Harmonic Universe with anisotropies

In order to construct the anisotropic analogue of the Simple Harmonic universe, we simply add the ‘domain-wall’ matter term or the \( \sigma/a \) term to the action (7.76). The Hamiltonian then becomes,

\[ U(a, \beta_+, \beta_-) = a^4 \left[ 1 - a^2 H^2 + \sigma a - V(\beta_+, \beta_-) \right] \]  
(7.103)

The boundary of the classically forbidden region found by solving \( U(a, \beta_+, \beta_-) \) gives us the turning points of the potential,

\[ a_{\pm} = \frac{\sigma}{2H^2} \pm \frac{1}{H^2} \sqrt{\sigma^2 - 4H^2(1 - V(\beta_+, \beta_-))} \]  
(7.104)

In the case of small anisotropy, the potential becomes,

\[ \bar{U}(a, \beta_+, \beta_-) = a^4 \left[ 1 - a^2 H^2 + \sigma a + 8\beta^2 \right] = U_0(a) + 8a^4 b^2 \]  
(7.105)

where \( \beta^2 = \beta_+^2 + \beta_-^2 \). The corresponding turning points,

\[ a_{\pm} = \frac{\sigma}{2H^2} \pm \frac{1}{H^2} \sqrt{\sigma^2 - 4H^2(1 - 8b^2)} \]  
(7.106)

The interesting behaviour that we see for the potential, in the case of large anisotropy, on addition of the ‘domain-wall’ matter term is that although the barrier width decreases with increasing \( \beta \) as before, the vanishing of the barrier does not take place at \( V(\beta_+, \beta_-) > 1 \). Another difference between the isotropic case studied in [100], and the anisotropic case is that in the case of the small anisotropies, the perturbation to the potential results in the classically
allowed well to be shifted to the right of its position in the unperturbed isotropic case. However as in the Bianchi I case the $\partial^2 \Psi / \partial \beta^2_{\pm}$ act as massless scalar fields. The solution to the Wheeler de Witt equation is separable and can be written as $\Psi = A(a)\bar{\beta}_+(\beta_+)\bar{\beta}_-(\beta_-)$. Thus,

$$\frac{d\bar{\beta}_+}{d\beta^2_{\pm}} = b^2 \bar{\beta}_+$$ (7.107)

$$\frac{d\bar{\beta}_-}{d\beta^2_{\pm}} = b^2 \bar{\beta}_-$$ (7.108)

These eigenvalues modify the potential for $A(a)$ as,

$$\tilde{U}(a, \beta_+, \beta_-) = a^4 [1 - a^2 H^2 + \sigma a + 8 \beta_2^2] - b^2$$ (7.109)

where $b^2 = b^2_+ + b^2_-$. In order to simplify the subsequent calculations, we follow the conventions of [100] and write the potential in the following form,

$$U(\alpha) = \tilde{\beta}^{-2} e^{4\alpha} (1 - 8b^2 - 2e^\alpha + \gamma^{-2} e^{2\alpha})$$ (7.110)

And the turning points in this notation becomes,

$$\alpha_\pm = \ln \left[ \gamma^2 \pm \gamma^2 \sqrt{1 - \gamma^{-2}(1 - 8b^2)} \right]$$ (7.111)

In the above, we have replaced $a$ by $\alpha = \ln \omega \gamma a$, and,

$$\tilde{\beta} = \left( \frac{2G}{3\pi} \right) = \frac{32\pi}{27} G^3 \sigma^2$$ (7.112)

Also,

$$\omega = \sqrt{\frac{8\pi G}{3} |\Lambda|} \text{ and } \gamma = \sqrt{\frac{2\pi G \sigma^2}{3} |\Lambda|}$$ (7.113)

Thus, similar to the case of addition of a massless scalar field in [100], the anisotropies result in the formation of a new classically allowed region that was not present in the unperturbed case. In the case of small anisotropies, there is a modification to the turning points of the potential due to the $b^2$ term in the potential. This modification is given by,

$$\delta \alpha_\pm = \frac{b^2}{|U_b(\alpha_\pm)|} \approx \tilde{\beta}^2 b^2$$ (7.114)
Let the modified turning points then be \( \alpha_{1,2} = \alpha \pm \delta \alpha \). The solution to the Wheeler de Witt equation is given by the WKB approximation far from the turning points.

\[
A(\alpha) \approx \frac{C_1 e^{-i \pi/4}}{|-U_b(\alpha)|^{1/4}} e^{i \int_{\alpha}^{\alpha'} \sqrt{-U_b(\alpha')} d\alpha'} + \frac{C_2 e^{i \pi/4}}{|-U_b(\alpha)|^{1/4}} e^{-i \int_{\alpha}^{\alpha'} \sqrt{-U_b(\alpha')} d\alpha'}
\]

This is the solution far away from the turning points in the classically allowed region. The solution, far away from the turning points is given by,

\[
A(\alpha) \approx \frac{D_1}{(U_b(\alpha))^{1/4}} e^{i \int_{\alpha}^{\alpha'} \sqrt{-U_b(\alpha')} d\alpha'} + \frac{D_1}{(U_b(\alpha))^{1/4}} e^{-i \int_{\alpha}^{\alpha'} \sqrt{-U_b(\alpha')} d\alpha'}
\]

Around the turning points \( \alpha = \alpha_* \), the potential is linearised as before as,

\[
U(\alpha) \approx U(\alpha_*) + U'(\alpha_*)(\alpha - \alpha_*) = U'(\alpha_*)(a - a_*)
\]

as \( U(\alpha_*) = 0 \). Setting \( U'(\alpha_*)(\alpha - \alpha_*) = z \), the WdW equation near a turning point,

\[
\left( \frac{\partial^2}{\partial z^2} - z \right) \Psi(z) = 0
\]

The solutions are linear combinations of the asymptotic forms of the Airy functions, \( Ai(z) \) and \( Bi(z) \). We impose boundary conditions in each region, such that the outgoing modes to the right of the potential barrier in region IV is taken to vanish. Also, modes in the classically allowed region formed due to the addition of the derivative anisotropies (like the massless scalar field), are considered to be purely outgoing towards \( \alpha \rightarrow -\infty \) (and hence \( a = 0 \)). The reasoning behind this is that the singularity should be a ‘point of no return’ and hence trajectories should not be able to lead a universe back from a singularity. Using these boundary conditions, we have the solutions in regions IV and II to be,

\[
A^{IV}(\alpha) = \frac{A}{2(U_b(\alpha))^{1/4}} \exp \left( -\int_{\alpha_2}^{\alpha} \sqrt{U_b(\alpha')} d\alpha' \right)
\]

and

\[
A^{I}(\alpha) = \frac{Be^{-i \pi/4}}{(-U_b(\alpha))^{1/4}} \exp \left( -i \int_{\alpha}^{\alpha_0} \sqrt{-U_b(\alpha')} d\alpha' \right)
\]
Matching the WKB solutions to the asymptotic forms of the Airy functions in regions III and II, and reconciling solutions everywhere in terms of a single coefficient, we get,

\[ A^{III}(\alpha) = \frac{Be^{-i\pi/4}}{(-U_b(\alpha))^{1/4}} \left( \left( 2e^K + \frac{e^{-K}}{2} \right) e^{i\int_{\alpha}^{\alpha_2} \sqrt{-U_b(\alpha')} d\alpha'} + i \left( 2e^K - \frac{e^{-K}}{2} \right) e^{-i\int_{\alpha}^{\alpha_2} \sqrt{-U_b(\alpha')} d\alpha'} \right) \]  

(7.121)

and

\[ A^{II}(\alpha) = \frac{B}{2(U_b(\alpha))^{1/4}} \left( 2e^K e^{-\int_{\alpha_0}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'} - ie^{-K} e^{i\int_{\alpha_0}^{\alpha} \sqrt{-U_b(\alpha')} d\alpha'} \right) \]  

(7.122)

where,

\[ K = \int_{\alpha_0}^{\alpha_1} \sqrt{-U_b(\alpha')} d\alpha' \]  

(7.123)

The relations between \( A \) and \( B \) given by,

\[ A = iB \left( 2e^K - \frac{e^{-K}}{2} \right) e^{-iJ} \]  

(7.124)

\[ A = -iB \left( 2e^K + \frac{e^{-K}}{2} \right) e^{iJ} \]  

(7.125)

where,

\[ J = \int_{\alpha_1}^{\alpha_2} \sqrt{-U(\alpha, b)} d\alpha \]  

(7.126)

This also implies that,

\[ J = \left( n + \frac{1}{2} \right) \pi - i \frac{1}{2} \ln \left( \frac{1 - \frac{e^{-2K}}{4}}{1 + \frac{e^{-2K}}{4}} \right) \]  

(7.127)

where \( n \) is an integer. Following this paper, we must now evaluate \( J \) and \( K \) for our specific turning points and our potential.

**Evaluation of \( J \)** The integral to be evaluated is given by

\[ J = \int_{\alpha_1}^{\alpha_2} \sqrt{-U(\alpha, \beta) + b^2} d\alpha \approx J_0 + J_1 + J_2 \]  

(7.128)

where,

\[ J_0 = \int_{\alpha_-}^{\alpha_+} \sqrt{-U(\alpha, \beta)} d\alpha \]  

(7.129)
7.8 Conclusion

Evaluating this to second order in $\beta$ we get,

$$\frac{8\beta^2 \gamma^8 (3\gamma^4 + 2\gamma^2 - 4) (1 - \gamma^2)^{3/2}}{\beta^2} + \frac{2\gamma^8 (9\gamma^6 - 8\gamma^4 - 31\gamma^2 + 30) \sqrt{1 - \gamma^2}}{15\beta^2} \quad (7.130)$$

$J_1$ is given by,

$$J_1 = \frac{b^2}{2} \int_{\alpha_-}^{\alpha_+} \frac{1}{\sqrt{-U(\alpha, \beta)}} d\alpha = \frac{b^2 \beta^2 \pi (1 - 8b^2 - 3\gamma^2)}{4\gamma^2 (1 - 8b^2)^{5/2}} \quad (7.131)$$

and finally

$$J_2 \approx \int_{\alpha_1}^{\alpha_-} \sqrt{-U(\alpha, \beta)} d\alpha + \int_{\alpha_1}^{\alpha_+} \sqrt{-U(\alpha, \beta)} d\alpha \quad (7.132)$$

$$\approx \frac{2}{3} \left( \sqrt{-U'(\alpha, \beta)} \delta \alpha^{3/2} + \sqrt{-U'\alpha_+, \beta} \delta \alpha_+^{3/2} \right)$$

$$\approx \frac{2}{3} b^3 \left( \frac{1}{U'(\alpha, \beta)} + \frac{1}{U'(\alpha_+, \beta)} \right) \approx \beta^2 b^3$$

**Evaluation of $K$**  The integral to be evaluated is now given by

$$K = \int_{\alpha_-}^{\alpha_+} \sqrt{U_b(\alpha, \beta)} \quad (7.133)$$

Evaluating this again to second order in $b$ we get, for $K$ at first order

$$K_0 = \frac{\gamma^2 \left( -2\sqrt{1 - \gamma^2} \gamma^4 + 2\sqrt{1 - \gamma^2} + 3\gamma^2 + 3 (\gamma^2 - 1) \gamma \log(-1 - \gamma) \right)}{6\beta} - \frac{\gamma^2 \left( 3\gamma^2 \sqrt{1 - \gamma^2} - 3 (\gamma^2 - 1) \gamma \log \left( \gamma \left( \sqrt{1 - \gamma^2} - \gamma \sqrt{1 - \gamma^2} \right) \right) - 2 \right)}{6\beta} \quad (7.134)$$

and for $K$ at the next order,

$$b^2 \gamma^2 \left( 4\sqrt{1 - \gamma^2} \left( 2\gamma^4 + 3\sqrt{1 - \gamma^2} \gamma^2 - 2 \right) + 4\sqrt{1 - \gamma^2} \left( \gamma^4 \left( 4 - \frac{3}{\sqrt{1 - \gamma^2}} \right) - 4 \right) \right) - b^2 \gamma^2 \left( 3\gamma \left( 8\log \left( \gamma \left( \sqrt{1 - \gamma^2} - \gamma \sqrt{1 - \gamma^2} \right) \right) + \frac{(\gamma^2 - 1)}{\gamma} \left( \frac{4\gamma}{\sqrt{1 - \gamma^2}} - \frac{4\gamma^5}{\sqrt{1 - \gamma^2}} \frac{4\gamma^2}{\sqrt{1 - \gamma^2}} \right) \right) \right) \quad \frac{6\beta}{6\beta}$$

$$+ \frac{b^2 \gamma^2}{6\beta} 12(\gamma + 2\gamma \log((1 - \gamma)\gamma) + 2)$$
We have neglected $\delta K$ arising from the $b^2$ term in the potential as it is assumed that $\bar{b}^2 b^2 \ll 1$. Once the universe has tunneled from $a = 0$ to $a = a_-$, it is reasonable to assume that it will continue oscillating between $a_-$ and $a_+$. Thus the probability of the universe oscillating is given by the probability of the universe tunnelling into $a = a_-$. This is given by,

$$\Gamma \sim \exp(-2K)$$  \hspace{1cm} (7.135)

Thus we see that the anisotropy, given by $b^2$ exponentially suppresses this probability.
Chapter 8

Conclusions and Future Outlook

Inflation, despite its successes as a model for the early universe, has several shortcomings. Model-building efforts, while being addressed towards dealing with these shortcomings, should also be directed to exploring the possibility of alternative models. The hope for such an alternative to inflation is that they would be able to reproduce the standard predictions of Big Bang cosmology. In addition to this, these models could improve upon the existing problems with inflation. Finally, they should differ in their predictions from inflation, in order to be able to distinguish between the models. If, indeed, an alternative to inflation is found that is observationally indistinguishable from inflation, then this means that our confidence in inflation as a working theory of the early universe would diminish.

Standard Big Bang cosmology suffers from the problem of an initial singularity. Non-singular bouncing cosmologies show a way to alleviate this problem by hypothesising that the bounce or transition between the contracting and expanding phase occurred at finite volume. However, in most cases this requires an effective violation of the null energy condition. Furthermore, any cosmological model that incorporates a contracting phase is plagued by growing anisotropies with decreasing volume. A standard, well-verified prediction is the large scale isotropy and homogeneity in the present day universe. Inflation is able to explain this by diluting the anisotropies by an exponentially increasing volume. Bouncing cosmologies, on the other hand, seek to mitigate this problem by introducing an ultra-stiff ($p > \rho$) field to dominate over the anisotropies and smooth them out on approach to the bounce. However, there are several types of anisotropy that need to be investigated in order to ascertain the viability of cyclic cosmologies: simple expansion rate anisotropy, spatial curvature anisotropy, and pressure anisotropy. Simple expansion-rate anisotropies and 3-curvature anisotropies can always be dominated by an ultra-stiff perfect fluid with equation of state $p > \rho$. This is well appreciated and we confirm it for the Bianchi class A and type IX universes.
The inclusion of anisotropic pressures changes this story. Even if they are ultra-stiff on average and hence obeying the conditions of ekpyrosis, we see that isotropisation does not occur in the same way as in the absence of anisotropic pressures. We cannot prove a general cosmic no-hair theorem for the case of anisotropic pressures, as we could do in their absence, for the isotropisation of Bianchi Class A universes. On specialising to the Bianchi Class A, we see that they cause the model to collapse to an anisotropic singularity, and the shear to grow to higher and higher positive values. The anisotropic pressures source the shear and hence the shear anisotropies start growing faster than they would, in their absence, on approach to the bounce. In order for the bounce to occur, the field driving the mechanism of the bounce, whether it be curvature or a ghost field, must be greater than the anisotropies. As the anisotropy energy density is sourced by the anisotropic pressures, this condition is never fulfilled and the model’s bouncing behaviour is destroyed by growing anisotropies.

We then generalise the simple oscillatory closed universe solutions found by Barrow and Dabrowski [17], to include anisotropies. The anisotropic analogue of the closed Friedmann universe is the Bianchi type IX universe. On evolving the Bianchi IX model over several cycles with increasing entropy of constituent matter content (in our case, radiation), we note the following features of the shape of the anisotropic oscillating universe:

- An increase in entropy produces an increase in the volume of the expansion maxima, just as in the isotropic case.

- This does not, however, lead to a corresponding decrease in shear anisotropy or a dilution of the 3-curvature.

- Inclusion of the cosmological constant causes oscillations to cease at the point of domination of the cosmological constant. The model then asymptotes to de Sitter expansion, and this leads to expansion anisotropy and 3-curvature being diluted.

To complete this analysis of the cyclic, closed universe, we have then included fluids that possess velocities that are not comoving with the triad frame of reference of the diagonal Bianchi IX universe under consideration. This leads us to consider the effects of the introduction of a vorticity term in addition to the shear and the curvature terms that we had been considering so far. We find, that while the increase in expansion volume with an increase in entropy mimics the behaviour of the comoving case, a new effect can be seen. An increase in entropy results in a corresponding decrease in the magnitudes of the 3-velocities and vorticities of the non-comoving field. Furthermore, the velocities in one of the spatial directions tends to oscillate around a constant value. The addition of a cosmological constant in this case leads to oscillations ceasing and the model tending towards exponential expansion.
as before. The velocities all tend towards oscillation around a constant value at the moment of cosmological constant domination.

The story of the question of isotropisation does not end here. It is interesting to see whether a negative source for anisotropic pressures can provide an alternative mechanism for isotropisation to happen. Preliminary results from this investigation indicate that for Bianchi Class A, the local stability of the isotropic Friedmann-Lemaitre point can be proved in the asymptotic past, but not the global stability. The Bianchi IX model does not isotropise in the presence of negative anisotropic stress, unless an ultra-stiff equation of state is introduced. A possible motivation for introducing such exotic matter would be the inclusion of non-ideal equations of state, for example in [3]. This would introduce a region of parameter space for the parameters specifying this non-ideal equation of state for which isotropisation can occur. Note that the simplest example of the extension of the linear equation of state— the quadratic equation of state includes a $\rho^2$ term, similar to that found in loop quantum cosmological models [107]. Most ordinary sources of matter, like dust or radiation, would act like stiff or ultra-stiff fields due to this quadratic density term.

In the final chapter of the thesis, we have explored the quantum cosmological implications of the creation of an anisotropic universe. More specifically, we have considered simple cases of anisotropy such as the Bianchi I and the Bianchi IX models, and have shown that the tunnelling probability is exponentially suppressed by the anisotropies in the case of small anisotropies. In the case of large anisotropies, the potential barrier through which tunnelling occurs vanishes, and the tunnelling analogy breaks down. This could be, in the case of Bianchi IX, because the sign of the curvature changes from positive to negative in cases of large anisotropy and it is unclear how to apply the tunnelling analogy without creating some topologically non-trivial identifications to the open universe geometry.

It would be interesting to extend the analysis to more general models of anisotropic models, such as the Bianchi Class A models and Kantowski Sachs, to prove that anisotropy in general suppresses the probability of the universe tunnelling into existence. It is also worth investigating the regime of validity of the tunnelling analogy with respect to large anisotropy and understanding exactly the causes for the tunnelling analogy to break down in these cases. Furthermore, an interesting question to ask is what the probability is of an oscillating universe that has tunneled into existence to keep oscillating. Some preliminary calculations suggest that this probability too is suppressed by the anisotropies. This is still work in progress.
References


References


Appendix: Some exact solutions

A.1 New exact solution for $p = 3\rho$ fluid in Bianchi I spacetime

The field equations for the Bianchi I type spacetime are:

\[
\ddot{\alpha} + \dot{\alpha}^2 + \dot{\beta} + \dot{\beta}^2 + \alpha \dot{\beta} = -p, \tag{1}
\]

\[
\ddot{\beta} + \dot{\beta}^2 + \dot{\delta} + \dot{\delta}^2 + \beta \dot{\delta} = -p, \tag{2}
\]

\[
\ddot{\delta} + \dot{\delta}^2 + \ddot{\alpha} + \alpha^2 + \dot{\delta} \dot{\alpha} = -p, \tag{3}
\]

\[
\dot{\alpha} \dot{\beta} + \dot{\beta} \dot{\delta} + \dot{\delta} \dot{\alpha} = \rho, \tag{4}
\]

where the scale factors are expressed as $a(t) = \exp(\alpha(t))$, $b(t) = \exp(\beta(t))$, and $c(t) = \exp(\delta(t))$. Adding equations ((1))-((3)), we get

\[
2(\ddot{\alpha} + \ddot{\beta} + \ddot{\delta}) + 2(\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\delta}^2) + (\dot{\alpha} \dot{\beta} + \dot{\beta} \dot{\delta} + \dot{\delta} \dot{\alpha}) = -3p. \tag{5}
\]

Using the formula $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we get,

\[
2(\ddot{\alpha} + \ddot{\beta} + \ddot{\delta}) + 2(\dot{\alpha} + \dot{\beta} + \dot{\delta})^2 - 4(\dot{\alpha} \dot{\beta} + \dot{\beta} \dot{\delta} + \dot{\delta} \dot{\alpha}) + (\dot{\alpha} \dot{\beta} + \dot{\beta} \dot{\delta} + \dot{\delta} \dot{\alpha}) = -3p. \tag{6}
\]

Now substituting (4) we get,

\[
2(\ddot{\alpha} + \ddot{\beta} + \ddot{\delta}) + 2(\dot{\alpha} + \dot{\beta} + \dot{\delta})^2 + 3\rho = -3p. \tag{7}
\]

Defining the volume as $V \equiv \exp(A)$ where $A = (\alpha + \beta + \delta)$ we get,

\[
\dot{V} = 3\rho_0 V^{-3}. \tag{8}
\]
Solving this gives

\[ V^2 = C_1 t^2 + C_2 t + C_3. \]  

(9)

Subtracting equations ((1)) and ((2)) we get, for example,

\[ \ddot{\alpha} - \ddot{\beta} + 3\dot{A}(\dot{\alpha} - \dot{\beta}) = 0, \]  

(10)

and cyclic permutations. Thus we see that each of these combinations go as \( V^{-1} \). We can write then, by integrating the above,

\[ \alpha - \beta = 2l_1 \log[\sqrt{t} + C_2(C_2 + t)], \]  

(11)

and

\[ \alpha - \delta = 2l_2 \log[\sqrt{t} + C_2(C_2 + t)], \]  

(12)

where \( C_1 = 1, C_3 = 0 \). We already know that

\[ (\alpha + \beta + \delta) = \frac{1}{2} \log[t^2 + C_2 t]. \]  

(13)

By using the fact that \( 3\alpha = (\alpha + \beta + \delta) + (\alpha - \beta) + (\alpha - \delta) \), we obtain,

\[ 3\alpha = \log \left[ (t^2 + C_2 t)^{1/2}(\sqrt{t} + \sqrt{C_2(C_2 + t)})^{2(l_1 + l_2)} \right]. \]  

(14)

Thus,

\[ a(t) = \left( (t^2 + C_2 t)^{1/2}(\sqrt{t} + \sqrt{C_2(C_2 + t)})^{2(l_1 + l_2)} \right)^{1/3}, \]  

(15)

\[ b(t) = \left( (t^2 + C_2 t)^{1/2}(\sqrt{t} + \sqrt{C_2(C_2 + t)})^{2(l_2 - 2l_1)} \right)^{1/3}, \]  

(16)

\[ c(t) = \left( (t^2 + C_2 t)^{1/2}(\sqrt{t} + \sqrt{C_2(C_2 + t)})^{2(l_1 - 2l_2)} \right)^{1/3}. \]  

(17)

From the Friedmann constraint equation at late times (where \( \rho \to 0 \), we get the following constraint,

\[ l_1^2 + l_2^2 - l_1 l_2 = 1. \]  

(18)

We label the indices in the solutions for the scale factors as follows,

\[ 3q_1 = 1 + l_1 + l_2, \]  

(19)

\[ 3q_2 = 1 + l_2 - 2l_1, \]  

(20)

\[ 3q_3 = 1 + l_1 - 2l_2. \]  

(21)
Therefore, we have the full solution:

\[ a(t) = \left( (t^2 + C_2 t)^{1/2} (\sqrt{1 + \sqrt{C_2 (C_2 + t)^2}})^{2(3q_1 - 1)} \right)^{1/3}, \tag{22} \]

\[ b(t) = \left( (t^2 + C_2 t)^{1/2} (\sqrt{1 + \sqrt{C_2 (C_2 + t)^2}})^{2(3q_2 - 1)} \right)^{1/3}, \tag{23} \]

\[ c(t) = \left( (t^2 + C_2 t)^{1/2} (\sqrt{1 + \sqrt{C_2 (C_2 + t)^2}})^{2(3q_3 - 1)} \right)^{1/3}. \tag{24} \]

We see that at early times this solution tends to the flat Friedmann solution for \( p = 3 \rho \) fluid \( (a \sim t^{1/6}, b \sim t^{1/6}, c \sim t^{1/6}) \) as \( t \to 0 \), and at late times approaches the vacuum Kasner solution \( a \sim t^{q_1}, b \sim t^{q_2}, c \sim t^{q_3} \), with \( \sum_i q_i = 1 = \sum_i q_i^2 \), as \( t \to \infty \). Thus, this solution provides a simple exact description of the transition from an isotropic initial state to a Kasner-like anisotropic future in a particular case. It displays the opposite evolutionary trend to the evolution of a \( 0 \leq p < \rho \) perfect-fluid model.

### A.2 Fixed points

In order to perform the stability analysis on the Bianchi Class A system, we need to identify the fixed points of the system. They have been presented in a tabular form in 4.1. The explicit forms are given below.

On examination of the forms of the fixed points, we find that only the FL (Friedmann Lemaitre), Kasner and the \( A_1 \) and \( A_2 \) points are real for the case considered.

### A.3 Equations for the Bianchi IX numerical computation

In 4.4, a new system of variables was introduced to make the numerical computation of the system of Einstein’s equations simpler by reducing them to first-order differential equations. They are written explicitly as follows:

\[ x'(t) + 3H(t)x(t) = \frac{1}{2} (\gamma_1 - \gamma_2) \mu(t) + \exp(-2\alpha(t)) - \exp(-2\beta(t)) \]

\[ -(-\exp(-2\alpha(t) + 2\beta(t) - 2\delta(t)) + \exp(2\alpha(t) - 2\beta(t) - 2\delta(t)), \tag{25} \]
Appendix: Some exact solutions

\[\Sigma_{\Omega_{\Omega_1}} = \frac{1}{8} (3\gamma - 2)\]

\[n_{\Omega_{\Omega_1}} = \frac{3}{4} [(2 - \gamma) (3\gamma - 2)]^{1/2}\]

\[\Omega_{\Omega_{\Omega_1}} = \frac{3}{16} (\gamma - 6)\]

\[\Sigma_{\Omega_{\Omega_1}} = \frac{2\sqrt{3} \gamma (\gamma - 7) + 6 (2 - 3\gamma) P_{-}}{\sqrt{3} (2\gamma - 2\gamma_{-} - 9(\gamma - 2) P_{+} + 3(5\gamma - 2) P_{-})}\]

\[n_{\Omega_{\Omega_1}} = \sqrt{\frac{12(\gamma - 2) P_{-} \left(\sqrt{3} P_{-} - 3\right) \left(\sqrt{3} (\gamma - \gamma_{-}) + 3(3\gamma - 2) P_{+}\right)^2}{\left(\sqrt{3} (2\gamma - 2\gamma_{-} - 9(\gamma - 2) P_{+} + 3(5\gamma - 2) P_{-})\right)^2}} + \frac{2\sqrt{3} (\gamma - \gamma_{-}) + 6 (3\gamma - 2) P_{-}}{(3\gamma - 2) P_{-}}\]

\[\Omega_{\Omega_{\Omega_1}} = \frac{4 (6 - 6\gamma) - 3\gamma}{6\sqrt{3} P_{-}}\]

\[n_{\Omega_{\Omega_{\Omega_1}}} = \sqrt{\frac{6 (2 - \gamma) (3\gamma - 2)}{(2 - 3\gamma)^2}}\]

\[\Omega_{\Omega_{\Omega_{\Omega_1}}} = \frac{6 (3\gamma - 30) P_{-} + 2\sqrt{3} (2\gamma - 2\gamma_{+} + 3(\gamma - 2) P_{+})}{4\sqrt{3} (\gamma - \gamma_{-}) + (48 - 72\gamma) P_{-}}\]

\[\Sigma_{\Omega_{\Omega_{\Omega_{\Omega_1}}}} = \sqrt{\frac{6(3\gamma - 30) P_{-} + 2\sqrt{3} (3\gamma - 2) (\gamma - 2) P_{+}}{144(\gamma - 2) P_{-} \left(\sqrt{3} (\gamma - \gamma_{-}) + 6 (2 - 3\gamma) P_{-}\right)}}\]

\[n_{\Omega_{\Omega_{\Omega_{\Omega_1}}} = \sqrt{\frac{((63\gamma - 30) P_{-} + 2\sqrt{3} (3\gamma - 2) (\gamma - 2) P_{+})}{3(\gamma - 2) ((63\gamma - 30) P_{-} + 2\sqrt{3} (3\gamma - 2) (\gamma - 2) P_{+}) + 4(3\gamma - 2) (\gamma - \gamma_{-}) + (63\gamma - 30) P_{-} + 2(9\gamma - 3\gamma_{-} - 8))}}\]

\[\Omega_{\Omega_{\Omega_{\Omega_{\Omega_1}}} = \frac{1}{3\sqrt{3} P_{-}} n_{\Omega_{\Omega_{\Omega_{\Omega_1}}} = \sqrt{\frac{1}{3\sqrt{3} P_{-}}}}\]

\[Z_{\Omega_{\Omega_{\Omega_{\Omega_{\Omega_1}}}}} = \Sigma_{\Omega_{\Omega_{\Omega_{\Omega_{\Omega_1}}}}^{2}} + \Sigma_{\Omega_{\Omega_{\Omega_{\Omega_{\Omega_1}}}}^{2}} = 1\]
\[ y'(t) + 3H(t)y(t) = \frac{1}{2}(\gamma_1 - \gamma_3)\mu(t) + \exp(-2\alpha(t)) - \exp(-2\delta(t)) \]
\[ -(-\exp(-2\alpha(t) - 2\beta(t) + 2\delta(t)) + \exp(2\alpha(t) - 2\beta(t) - 2\delta(t))), \]

\[ 6(H')^2 = -\frac{\exp(-2\alpha(t) - 2\beta(t) + 2\delta(t))}{2} - \frac{\exp(-2\alpha(t) + 2\beta(t) - 2\delta(t))}{2} \]
\[ -\frac{\exp(2\alpha(t) - 2\beta(t) - 2\delta(t))}{2} + \exp(-2\alpha(t)) + \exp(-2\beta(t)) + \exp(-2\delta(t)) + 3\gamma_2\mu(t) + 3\gamma_3\rho(t), \]

\[ \rho'(t) = -3\gamma H(t)\rho(t), \]

\[ \mu'(t) = -3\mu(t)H(t) - (\gamma_1 - 1)\frac{(x(t) + y(t) + 3H(t))}{3}\mu(t) - (\gamma_2 - 1) \]
\[ \frac{(3H(t) - 2x(t) + y(t))}{3}\mu(t) - (\gamma_3 - 1)\frac{(x(t) - 2y(t) + 3H(t))}{3}\mu(t), \]

\[ \alpha'(t) = \frac{1}{3}(3H(t) + x(t) + y(t)), \]

\[ \beta'(t) = \frac{1}{3}(3H(t) - 2x(t) + y(t)), \]

\[ \delta'(t) = \frac{1}{3}(3H(t) + x(t) - 2y(t)). \]