Complexity and Repeated Implementation: Supplementary Material

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Abstract

This supplement presents some additional results and proofs that are omitted from the main paper.

1 Dictatorships with Multiple Payoffs

Our main analysis assumes that, for any individual i, the dictatorship $d(i)$ (over the range of the SCF) yields a unique (one-period) expected utility profile v^i if i acts rationally. Let us show that our results do not in fact depend on this assumption. For each $i \in I$, let $\mathcal{V}^i$ denote the set of payoff profiles that can be generated in $d(i)$ under rational dictator.[1]

We want to extend Lemma 2 (for the case of $I = 2$) and Lemma 6 (for the case of of $I \geq 3$) of the main text as follows: there are history-independent serial dictatorships/constant outcomes such that the corresponding payoffs satisfy the desired properties under *any* possible equilibrium play when the dictatorships generate multiple payoffs consistent with the dictators' rational choices. Once this extension is done, the regime constructions and the characterization arguments remain identical.

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[1] In Sections 2 and 3, we return to assuming that these are singleton sets.

We consider the case of two players only. In the general case with three or more players, our construction is built around providing correct intertemporal incentives just for two arbitrary, but fixed, players. The critical parts of Lemmas 2 and 6 of the main text therefore share the same constructions; it is straightforward to verify that the remainder of Lemma 6 also holds with the possibility of multiple dictatorial payoffs.

Suppose that $I = 2$, and modify condition ϕ as follows.

Condition ϕ^{**}. (i) For all i, there exists $\tilde{a}^i \in f(\Theta)$ such that $v_i(\tilde{a}^i) \leq v_i(f)$.

(ii) For all i, $v^i \in \mathcal{V}^i$ and $\gamma \in [0,1]$, $v(f) \neq \gamma v^i + (1-\gamma)v(\tilde{a}^i)$.

The following lemma extends Lemma 2 of the main text. For any regime R, let $W_i(R)$ denote the set of Nash equilibrium discounted average payoffs to player i.

Lemma 1.1 *Suppose that $I = 2$, and fix an SCF f that satisfies efficiency in the range and condition ϕ^{**}. Suppose also that $\delta \in \left(\frac{3}{4}, 1\right)$. Then, we obtain the following:*

(a) For each i, there exists a history-independent regime, referred to as S^i, such that, for any $w_i \in W_i(S^i)$, $w_i = v_i(f)$ and, for any $w_j \in W_j(S^i)$, $w_j < v_j(f)$, $j \neq i$.

(b) There exist history-independent regimes $\{X(t)\}_{t=1,2,\dots}$ and Y such that, for any t:

$$\max W_1(S^2) < \min W_1(Y) \leq \max W_1(Y) < \min W_1(X(t)) \leq \max W_1(X(t)) < v_1(f)$$
$$\max W_2(S^1) < \min W_2(X(t)) \leq \max W_2(X(t)) < \min W_2(Y_2) \leq \max W_2(Y_2) < v_2(f).$$

Proof. (a) Fix any i. By Lemma 1 of the main text, and since $\delta > \frac{1}{2}$, there exists an infinite sequence of two payoff profiles v_i^i and $v_i(\tilde{a}^i)$ such that the corresponding average discount payoff to i is exactly equal to $v_i(f)$. Consider a history-independent regime that alternates $d(i)$ and $\phi(\tilde{a}^i)$ according to the same sequence. If player i behaves rationally, his one-period payoff from $d(i)$ is unique and exactly equal to v_i^i; thus, the average payoff from the regime equals $v_i(f)$. Since f is efficient in the range, and given part (ii) of condition ϕ^{**}, $w_j < v_j(f)$ for any $w_j \in W_j(S^i)$.

(b) Fix any t. To show the claim, given part (a), it suffices to construct two history-independent regimes $X'(t)$ and Y' that are convex combinations of S^1 and S^2 with respective coefficients $\lambda(t)$ and μ on the former such that the following payoff properties hold:

$$\begin{aligned}\lambda(t)v_1(f)+(1-\lambda(t))\min W_1(S^2) &> \mu v_1(f)+(1-\mu)\max W_1(S^2)\\ \lambda(t)\max W_2(S^1)+(1-\lambda(t))v_2(f) &< \mu\min W_2(S^1)+(1-\mu)v_2(f).\end{aligned}$$

Let $\overline{w}_i := \max W_i(S^j)$ and $\underline{w}_i := \min W_i(S^j)$ for each i and j, $i \neq j$. If $\overline{w}_i = \underline{w}_i$ for any i, it is straightforward to see that there exist some $\lambda(t) > \mu$ that satisfy the above two inequalities. Consider otherwise. Then, the two inequalities above can be re-written, respectively, as

$$\begin{aligned}\lambda(t) &> \frac{v_1(f)-\overline{w}_1}{v_1(f)-\underline{w}_1}\mu+\frac{\overline{w}_1-\underline{w}_1}{v_1(f)-\underline{w}_1}\\ \lambda(t) &> \frac{v_2(f)-\underline{w}_2}{v_2(f)-\overline{w}_2}\mu \quad (>\mu).\end{aligned}$$

Since $v_1(f) > \overline{w}_1 > \underline{w}_1$, it is easily seen that, for $\mu < 1$,

$$\frac{v_1(f)-\overline{w}_1}{v_1(f)-\underline{w}_1}\mu+\frac{\overline{w}_1-\underline{w}_1}{v_1(f)-\underline{w}_1} \in (0,1).$$

Thus, one can fix μ sufficiently close to 0, and choose

$$\lambda(t) \in \left(\frac{v_1(f)-\overline{w}_1}{v_1(f)-\underline{w}_1}\mu+\frac{\overline{w}_1-\underline{w}_1}{v_1(f)-\underline{w}_1}, 1\right).$$

We modify $X'(t)$ and Y' to obtain $X(t)$ and Y as in the proof Lemma 2 of the main text. ■

2 Alternative Regime Construction with Strict Nash Equilibria

Here, we offer an alternative construction for the two-agent case in which truth-telling equilibria are such that any deviation from truth-telling leads to a *strict* payoff decrease. The purpose of this analysis is to obtain repeated implementation results that do not invoke self-selection in the range, as well as to incorporate the alternative complexity measure of Definition 8 in the main text. The results however require sufficiently large δ. The construction and its equilibrium properties below can be similarly extended to the case of $I \geq 3$.

Let $g'(1)$ denote an extensive-form mechanism such that:

- Stage 1 - Each agent $i = 1, 2$ announces a state, θ_i, from Θ.
- Stage 2 - Each agent announces an integer, z_i, from the set $\mathcal{Z} \equiv \{0, 1, 2\}$.

The outcome function is such that a constant outcome, $f\left(\tilde{\theta}\right)$ for some arbitrary but fixed $\tilde{\theta} \in \Theta$, is always implemented.

Let g' be an extensive-form mechanism such that:

- Stage 1 - Each agent $i = 1, 2$ announces a state, θ_i, from Θ.
- Stage 2 - Each agent announces an integer, z_i, from the set $\mathcal{Z}$.

The outcome function is given below:

(i) If $\theta_1 = \theta_2 = \theta$, $f(\theta)$ is implemented.

(ii) Otherwise, $f\left(\tilde{\theta}\right)$ for some arbitrary but fixed $\tilde{\theta} \in \Theta$ is implemented.

Note that this mechanism differs from g^e in the main text in that it does not invoke the self-selection condition when the agents announce different states.

Next, we define regime R' inductively with mechanism $g'(1)$ enforced in period 1 and the transition rules below.

Period 1:

Let (z_1, z_2) be the integers announced. The transition rules in period 1 are as follows.

- Rule C.1: If $z_1 = z_2 = 0$, the mechanism next period is g'.
- Rule C.2: If $z_1 > 0$ and $z_2 = 0$ ($z_1 = 0$ and $z_2 > 0$), the continuation regime is S^1 (S^2).
- Rule C.3: Suppose that $z_1, z_2 > 0$. Then, we have the following:
  - Rule C.3(i): If $z_1 = z_2 = 1$, the continuation regime is $X \equiv X(\tilde{t})$ for some arbitrary $\tilde{t}$, with the payoffs henceforth denoted by x.
  - Rule C.3(ii): If $z_1 = z_2 = 2$, the continuation regime is $X(1)$.
  - Rule C.3(iii): If $z_1 \neq z_2$, the continuation regime is Y.

Period $t \geq 2$:

Consider any date $t \geq 2$. Let (θ_1, θ_2) and (z_1, z_2) be the states and integers announced in period t. The transitions rules are as follows.

- Rule D.1: If $\theta_1 \neq \theta_2$, the continuation regime is X.
- Rule D.2: If $\theta_1 = \theta_2$ and $z_1 = z_2 = 0$, the mechanism next period is g'.
- Rule D.3: If $\theta_1 = \theta_2$, $z_1 > 0$ and $z_2 = 0$ ($z_1 = 0$ and $z_2 > 0$), the continuation regime is S^1 (S^2).
- Rule D.4: Suppose that $\theta_1 = \theta_2$ and $z_1, z_2 > 0$. Then, we have the following:
  - Rule D.4(i): If $z_1 = z_2 = 1$, the continuation regime is X.
  - Rule D.4(ii): If $z_1 = z_2 = 2$, the continuation regime is $X(t)$.
  - Rule D.4(iii): If $z_1 \neq z_2$, the continuation regime is Y.

This regime modifies R^e in the main text in the following way. In the first period, the planner enforces a constant outcome but the integer play generates essentially identical transition rules as in R^e. In any period after the first, the agents play a sequential revelation mechanism with integers g', but the transition rules when playing g' is identical to the corresponding features of R^e only if the two agents announce the same state in Stage 1; otherwise, the continuation regime is X, which generate continuation payoffs strictly dominated by $v(f)$. With some abuse of notation, let us define the set of histories, partial histories, strategies and payoffs as in the main text.

To examine the set of equilibria of regime R', note first that the statements of Lemma 4 of the main text can be extended here as follows: conditional on any history beyond the first period at which mechanism g' is played *and the same state announced in Stage 1*, the two players must either report 0 for sure and obtain $v(f)$, or uniquely mix between 1 and 2 and obtain strictly less than $v(f)$; for $t = 1$, this also holds by similar arguments. Second, as in Lemma 5 of the main text, in any WPEC, the agents must play 0 for sure on- or off-the-equilibrium. (Note that, following any partial history involving disagreement in Stage 1, the integer play does not affect the continuation game because of Rule D.1.)

In the next lemma, we establish that the players must always report the same state *after the first period*. The basic intuition is that, otherwise, the continuation payoff of each agent i falls short of $v_i(f)$ and hence a deviation would occur in the previous period's integer stage. Note that this argument could not work if disagreement occurred in the first period; in order to avoid such coordination failure, we implement a constant outcome in period 1.

Lemma 2.1 *Fix any WPEC of regime R'. Also, fix any $t \geq 2$ and $\mathbf{h}^t \in \mathbf{H}^t$. Then, the agents always report the same state for sure.*

Proof. Let $r(\theta, \underset{\sim}{\theta})$ denote the probability with which partial history $(\theta, \underset{\sim}{\theta}) \in D_z$ occurs at $\mathbf{h}^t$ under the given WPEC, and let $a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}}$ represent the corresponding outcome. Also, let $\underset{\sim}{\Theta}' = \{(\theta^1, \theta^2) \in \Theta^2 \mid \theta^1 = \theta^2\}$ denote the set of state profiles in which the players agree and $\underset{\sim}{\Theta}'' = \Theta^2 \backslash \underset{\sim}{\Theta}'$ denote the set of state profiles in which they disagree.

By definition, at $\mathbf{h}^t$ mechanism g' is played. Therefore, in the previous period $t-1$ of history $\mathbf{h}^t$, one of the following must be true: (i) the same mechanism was in force, and moreover, by the transition rules of R', the players announced the same state in Stage 1 and integer 0 in Stage 2, or (ii) the players were in period 1 and announced 0 for sure. Then, by previous arguments, it must be that $\pi_i^{\mathbf{h}^t} = v_i(f)$ for all i.

Next, at $\mathbf{h}^t$, if the agents report the same state in Stage 1, by applying the arguments of Lemma 4 of the main text, the agents report zero for sure in Stage 2. It then follows that the continuation payoff profile after any $d = (\theta, \underset{\sim}{\theta})$ is $v(f)$ if $\underset{\sim}{\theta} \in \underset{\sim}{\Theta}'$ and, by Rule D.1, x if $\underset{\sim}{\theta} \in \underset{\sim}{\Theta}''$.

Therefore, we can write each i's continuation payoff at $\mathbf{h}^t$ as

$$\begin{aligned}
\pi_i^{\mathbf{h}^t} &= \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}'} r(\theta,\underset{\sim}{\theta}) \left[(1-\delta)u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta v_i(f)\right] + \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \left[(1-\delta)u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta x_i\right] \\
&= (1-\delta) \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\Theta^2} r(\theta,\underset{\sim}{\theta}) u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) + \delta \left[v_i(f) \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}'} r(\theta,\underset{\sim}{\theta}) + x_i \sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \right].
\end{aligned}$$

Since $\pi_i^{\mathbf{h}^t} = v_i(f)$ and $x_i < v_i(f)$ for all i, if $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) \neq 0$ then it must be that $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\Theta^2} r(\theta,\underset{\sim}{\theta})\ u_i(a^{\mathbf{h}^t,\theta,\underset{\sim}{\theta}},\theta) > v_i(f)$ for all i. But this is not feasible with f being efficient in the range. It therefore follows that $\sum_{\theta\in\Theta,\underset{\sim}{\theta}\in\underset{\sim}{\Theta}''} r(\theta,\underset{\sim}{\theta}) = 0$. ■

Next, define

$$\bar{\delta} = \max_{i\in I} \left\{ \frac{\max_{a,a'\in f(\Theta),\theta\in\Theta} \{u_i(a,\theta) - u_i(a',\theta)\}}{\max_{a,a'\in f(\Theta),\theta\in\Theta} \{u_i(a,\theta) - u_i(a',\theta)\} + (v_i(f) - x_i)} \right\} \in (0,1).$$

We obtain existence below.

Lemma 2.2 *If $\delta > \bar{\delta}$, regime R' admits a WPEC.*

Proof. Consider the following repeated game strategy profile. In period 1, each player announces 0 for sure. From period 2, each player always reports the true state followed by integer 0.

To see that this profile constitutes an SPE, by Rules C.2 and D.3, neither player wants to deviate at any integer-reporting stage; by Rule D.1, and since $\delta > \bar{\delta}$, deviation from the prescribed state-reporting strategy is not profitable. It is also clear that this SPE is a WPEC itself. ■

3 Construction with Simultaneous Mechanisms

Suppose that $I = 2$. Define g^* as a one-shot mechanism in which, for each $i = 1, 2$, $M_i = \Theta \times \mathcal{Z}$ and the outcome function is such that:

(i) If m_1 and m_2 are such that $\theta_1 = \theta_2 = \theta$ and $z_i = 0$ for some i, $f(\theta)$ is implemented.

(ii) If m_1 and m_2 are such that $\theta_1 \neq \theta_2$ and $z_i = 0$ for some i, an outcome from the set $L_1(\theta_2) \cap L_2(\theta_1)$, as defined by self-selection in the range, is implemented.

(iii) If m_1 and m_2 are such that $z_1 > 0$ and $z_2 > 0$, a constant outcome, $f\left(\tilde{\theta}\right)$ for some arbitrary but fixed $\tilde{\theta} \in \Theta$, is always implemented.

Next, we define regime R^* as follows. First, construct the continuation regimes $\{S^i\}_{i \in I}$, $\{X(t)\}_{t \in \mathbb{Z}_+}$ and Y such that the resulting payoffs satisfy the inequalities imposed in the main text. Then, there exists some fixed $\kappa > 0$ such that the payoff difference to any agent from any pair of such regimes is at least κ: $|w_i^j - x_i(t)| > \kappa$, $|w_i^j - y_j| > \kappa$ and $|x_i(t) - y_i| > \kappa$ for any t and any $i, j \in I$. Second, the transitions are such that mechanism g^* is played in $t = 1$, and if, at date $t \geq 1$, g^* is the mechanism played with $m_i = (\theta_i, z_i)$ being the message announced by $i = 1, 2$, the continuation mechanism or regime at the next period is given by the same rules as in regime R^e in the main text, that is:

- Rule E.1: If $z_1 = z_2 = 0$, then the mechanism next period is g^*.

- Rule E.2: If $z_1 > 0$ and $z_2 = 0$ ($z_1 = 0$ and $z_2 > 0$), then the continuation regime is S^1 (S^2).

- Rule E.3: If $z_1, z_2 > 0$, then we have the following:
  - Rule E.3(i): If $z_1 = z_2 = 1$, the continuation regime is $X \equiv X(\tilde{t})$ for some arbitrary but fixed $\tilde{t}$, with the payoffs henceforth denoted by x.
  - Rule E.3(ii): If $z_1 = z_2 = 2$, the continuation regime is $X(t)$.
  - Rule E.3(iii): If $z_1 \neq z_2$, the continuation regime is Y.

For this regime, with slight abuse of notation, let $\mathbf{H}^t$ continue to denote the set of histories at the beginning of period t when mechanism g^* is to be played. Then, we write player i's strategy as $\sigma_i : \mathbf{H}^\infty \times \Theta \to \triangle(\Theta \times \mathcal{Z})$. For any $\mathbf{h} \in \mathbf{H}^\infty$, $\pi_i^{\mathbf{h}}(R^*, \sigma)$ denotes i's continuation payoff in this regime under strategy profile σ.

Let $\overline{u} = \max_{i\in I, a\in A, \theta\in\Theta} u_i(a,\theta)$ and $\underline{u} = \min_{i\in I, a\in A, \theta\in\Theta} u_i(a,\theta)$, and define

$$\bar{\delta} = \frac{\overline{u} - \underline{u}}{\overline{u} - \underline{u} + \kappa}.$$

We next consider SPEs of the above regime.

Lemma 3.1 *Suppose that $\delta \in (\bar{\delta}, 1)$. Consider any SPE σ of regime R^*, and fix any t, $\theta^* \in \Theta$ and $\mathbf{h} \in \mathbf{H}^t$. For each $i = 1, 2$, let $r_i(\theta, z) = \sigma_i(\mathbf{h}, \theta^*)(\theta, z)$ be the equilibrium probability of i choosing (θ, z) at $(\mathbf{h}, \theta^*)$. Then, one of the following must hold:*

(a) For each i, $\sum_{\theta\in\Theta} r_i(\theta, 0) = 1$ and his continuation payoff at the next period is $v_i(f)$.

(b) For each i, $\sum_{\theta\in\Theta} r_i(\theta, 1) + \sum_{\theta\in\Theta} r_i(\theta, 2) = 1$ such that $\sum_{\theta\in\Theta} r_i(\theta, 1) > 0$ and $\sum_{\theta\in\Theta} r_i(\theta, 2) > 0$; $\sum_{\theta\in\Theta} r_i(\theta, 1)$ depends on $x(t)$; and i's continuation payoff at the next period is less than $v_i(f)$.

Proof. Fix any $\delta > \bar{\delta}$ and any SPE of R^*. Fix any t, $\theta^* \in \Theta$ and $\mathbf{h} \in \mathbf{H}^t$. Let z_i denote the integer that i ends up choosing at $(\mathbf{h}, \theta^*)$ and Π_i denote i's continuation payoff at the next period if both agents announce zero at the given history.

Also, for any i, whenever we mention a deviating strategy σ_i' that is identical to the equilibrium strategy σ_i everywhere except that at $(\mathbf{h}, \theta^*)$ it announces integer z', we mean the following: $\sum_{\theta\in\Theta} \sigma_i'(\mathbf{h}, \theta^*)(\theta, z') = 1$ and $\sigma_i'(\mathbf{h}, \theta^*)(\theta, z') = \sum_{z\in\mathcal{Z}} r_i(\theta, z)$, while $\sigma_i'(\mathbf{h}', \theta') = \sigma_i(\mathbf{h}', \theta')$ for any $(\mathbf{h}', \theta') \neq (\mathbf{h}, \theta^*)$.

We consider two cases in turn.

Case 1: No player randomizes over integers, i.e. $\sum_{\theta\in\Theta} r_i(\theta,0)=1$, $\sum_{\theta\in\Theta} r_i(\theta,1)=1$ or $\sum_{\theta\in\Theta} r_i(\theta,2)=1$ for all i.

In this case we show that each i must play 0 for sure, i.e. $\sum_{\theta\in\Theta} r_i(\theta,0)=1$. Suppose otherwise; then some i plays $z_i\neq 0$ for sure and the other announces z_j for sure. We derive contradiction by considering the following subcases.

Subcase 1A: $z_i>0$ and $z_j=0$.

The continuation regime at the next period is S^i (Rule E.2). Thus, j's equilibrium continuation payoff at $(\mathbf{h},\theta^*)$ is at most $(1-\delta)\overline{u}+\delta w_j^i$. Consider j deviating to another strategy identical to the equilibrium strategy except that it announces the positive integer other than z_i at this history. By (iii) of the outcome function of g^*, and by Rule E.3(iii), the corresponding continuation payoff is $(1-\delta)f(\tilde{\theta})+\delta y_j$. Since $y_j>w_j^i$ by construction, and since $\delta>\bar{\delta}$, the deviation is profitable. This is a contradiction.

Subcase 1B: $z_i>0$ and $z_j>0$.

The continuation regime is either X, $X(t)$ or Y (Rule E.3). Also, the current period's outcome is $f(\tilde{\theta})$. Suppose that the continuation regime is X or $X(t)$, and consider the same deviation as in Subcase 1A above by player 2. This deviation does not affect the distribution of current period's outcome but activates Y. Since $y_2>x_2(t)$ for any t, it follows that the deviation is profitable, a contradiction. If the continuation regime is Y, since $x_1>y_1$, player 1 can profitably deviate and we obtain a contradiction.

Thus, both players choose 0 for sure at this history, and g^* must be the mechanism at the next period. We next show that $\Pi_i=v_i(f)$ for all i. For this, suppose first that $\Pi_i<v_i(f)$ for some i. But then, consider i deviating to another strategy identical to the equilibrium strategy except that it announces a positive integer at this history. By (i) and (ii) of g^*, and by Rule E.2, such a deviation does not alter the current period's payoff but leads to a continuation payoff at the next period equal to $v_i(f)$, a contradiction. The rest follows as in the proof of Lemma 4 of the main text since f is efficient in the range.

Case 2: Some player randomizes over integers.

We proceed by first establishing the following two claims.

Claim 1: For each i, the continuation payoff from announcing 1 is greater than that from announcing 0, if $z_j>0$ for sure, $j\neq i$.

Proof of Claim 1. If i announces zero, by Rule A.2, his continuation payoff at the next period is w_i^j. If he announces 1, by Rules A.3(i) and A.3(iii), the continuation payoff at the next period is $x_i > w_i^j$ or $y_i > w_i^j$. Since $\delta > \bar{\delta}$, the current period's payoff does not matter.

Claim 2: Suppose that agent i announces 0 with positive probability, i.e. $\sum_{\theta \in \Theta} r_i(\theta, 0) > 0$. Then the other agent j must also announce 0 with positive probability and $\Pi_i \geq v_i(f)$. Furthermore, $\Pi_i > v_i(f)$ if j does not choose 0 for sure.

Proof of Claim 2. Let us show the first part of this claim by way of contradiction. So, suppose that $\sum_{\theta \in \Theta} r_j(\theta, 0) = 0$. Consider i deviating to a strategy identical to the equilibrium strategy except that it announces integer 1 for sure at this history. By Claim 1, the deviation is profitable, a contradiction against the assumption that $\sum_{\theta \in \Theta} r_i(\theta, 0) > 0$. The latter parts of the claim follow immediately as in the corresponding case of the proof of Lemma 4 of the main text.

Now, similarly to the corresponding arguments in Lemma 4 of the main text, we can show that, in this Case 2, both players choose a positive integer for sure, i.e. for each i, $\sum_{\theta \in \Theta} r_i(\theta, 1) + \sum_{\theta \in \Theta} r_i(\theta, 2) = 1$. Clearly, the mixing probabilities must depend on $x(t)$. Furthermore, since for each i, $v_i(f)$ exceeds $x_i, x_i(t)$ or y, it follows that the continuation payoff at the next period must be less than $v_i(f)$. ■

We adopt the complexity notion as in Definition 6 of the main text. It is straightforward to identify that R^* has a WPEC in which each agent always announces the true state and integer 0 (for any δ). The next lemma characterizes WPECs of R^*.

Lemma 3.2 *Suppose that $\delta \in (\bar{\delta}, 1)$. Consider any WPEC of regime R^*, and fix any t, $\theta \in \Theta$ and $\mathbf{h} \in \mathbf{H}^t$ (on or off the equilibrium path). Then, each agent announces zero for sure at this history.*

Proof. Suppose not. Then, given the results of Lemma 3.1, there exists a WPEC, σ, such that, at some t, θ^* and $\mathbf{h}^t \in \mathbf{H}^t$, the two agents play integer 1 or 2 for sure with the mixing probabilities determined by $x(t)$. Furthermore, by construction, there exist t' and t'' such that $x(t') \neq x(t'')$ and, therefore, it follows that, for all i, we have $\sigma_i(\mathbf{h}^t, \theta^*) \neq \sigma_i(\tilde{\mathbf{h}}, \theta^*)$ for some $\tilde{\mathbf{h}}$. Define $\mathbf{H}' = \{\mathbf{h} \in \mathbf{H}^\infty | \sigma_i(\mathbf{h}, \theta^*) = \sigma_i(\mathbf{h}^t, \theta^*) \text{ or } \sigma_i(\mathbf{h}, \theta^*) = \sigma_i(\tilde{\mathbf{h}}, \theta^*)\}$.

Now, consider any $i = 1, 2$ deviating to another strategy σ_i' that is identical to the equilibrium strategy σ_i except that, for any $\mathbf{h} \in \mathbf{H}'$,

- $\sum_{\theta \in \Theta} \sigma_i'(\mathbf{h}, \theta^*)(\theta, 1) = 1$, i.e. the deviating strategy announces 1 for sure; and
- $\sigma_i'(\mathbf{h}, \theta^*)(\theta, 1) = \sum_{z \in \{0,1,2\}} \sigma_i(\tilde{\mathbf{h}}, \theta^*)(\theta, z)$, i.e. the deviating strategy mixes over the states in the same way that the equilibrium strategy plays after history $\tilde{\mathbf{h}}$ (and θ^*).

Thus, for any $\mathbf{h}, \mathbf{h}' \in \mathbf{H}'$, $\sigma_i'(\mathbf{h}, \theta^*) = \sigma_i'(\mathbf{h}', \theta^*)$. Since σ_i' is less complex than σ_i according to Definition 6 of the main text, we obtain a contradiction by showing that $\pi_i^{\mathbf{h},\theta^*}(\sigma_i', \sigma_{-i}, R^*) = \pi_i^{\mathbf{h},\theta^*}(\sigma, R^*)$ for any $\mathbf{h} \in \mathbf{H}'$.

Fix any $\mathbf{h} \in \mathbf{H}'$ and consider history $(\mathbf{h}, \theta^*)$. First, suppose that j plays 0 for sure. Then, by part (a) of Lemma 3.1, i also plays 0 for sure and obtains a continuation payoff equal to $v_i(f)$ at the next period in equilibrium. By (i) and (ii) of the outcome function of g^*, and by the second part of the deviating strategy specified above, the deviation does not affect the distribution of current period outcome and, by Rule E.2 and the first part of the deviating strategy, it also induces the same continuation payoff $v_i(f)$ at the next period. Second, suppose that j is mixing over integers at this history. Then, by part (b) of Lemma 3.1, j mixes between 1 and 2 in equilibrium. In this case, by (iii) of g^*, the current outcome is independent of announcements, while i is indifferent between choosing 1 and 2 in terms of the next period's continuation payoff. ■