

THE KÄHLER-RICCI FLOW AND OPTIMAL DEGENERATIONS

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ABSTRACT. We prove that on Fano manifolds, the Kähler-Ricci flow produces a “most destabilising” degeneration, with respect to a new stability notion related to the H -functional. This answers questions of Chen-Sun-Wang and He.

We give two applications of this result. Firstly, we give a purely algebro-geometric formula for the supremum of Perelman’s μ -functional on Fano manifolds, resolving a conjecture of Tian-Zhang-Zhang-Zhu as a special case. Secondly, we use this to prove that if a Fano manifold admits a Kähler-Ricci soliton, then the Kähler-Ricci flow converges to it modulo the action of automorphisms, with any initial metric. This extends work of Tian-Zhu and Tian-Zhang-Zhang-Zhu, where either the manifold was assumed to admit a Kähler-Einstein metric, or the initial metric of the flow was assumed to be invariant under a maximal compact group of automorphism.

1. INTRODUCTION

A basic question in Kähler geometry is which Fano manifolds admit Kähler-Einstein metrics. The Yau-Tian-Donaldson conjecture [40, 19, 35], resolved by Chen-Donaldson-Sun [11], relates this to K-stability of the manifold.

Theorem 1.1. [11] *A Fano manifold admits a Kähler-Einstein metric if and only if it is K-stable.*

In this paper our main focus is the situation when a Fano manifold X does not admit a Kähler-Einstein metric. In analogy with the Harder-Narasimhan filtration of unstable vector bundles, and more generally with optimal destabilising one-parameter subgroups in geometric invariant theory [24, 8], one expects that in this case X has an optimal destabilising degeneration.

One precise conjecture in this direction is due to Donaldson [20], predicting that the infimum of the Calabi functional on X is given by the supremum of the Donaldson-Futaki invariants $\text{DF}(\mathcal{X})$ over all test-configurations \mathcal{X} for X . While this conjecture remains open in general, we show that an analogous result holds in the Fano case, if we replace the Calabi functional by the H -functional, and the Donaldson-Futaki invariant with the H -invariant, which we define by analogy with an invariant introduced by Tian-Zhang-Zhang-Zhu [36] for holomorphic vector fields.

Theorem 1.2. *Let X be a Fano manifold. We have*

$$\inf_{\omega \in c_1(X)} H(\omega) = \sup_{\mathcal{X}} H(\mathcal{X}), \quad (1.1)$$

where $H(\omega)$ is the H -functional, and the supremum is taken over the H -invariants of all test-configurations \mathcal{X} for X .

Our proof builds closely on the previous works [9, 10, 21]. The H-functional is defined by

$$H(\omega) = \int_X h e^h \omega^n,$$

where h is the Ricci potential of ω normalised so that e^h has average 1. As far as we can tell this functional first appears in the literature as the difference of the Mabuchi and Ding functionals in Ding-Tian [17, p. 69], where it is called E , and it is observed that $H(\omega) \geq 0$ with equality only if ω is Kähler-Einstein. The functional was shown to be monotonic along the Kähler-Ricci flow in Pali [26] and Phong-Song-Sturm-Weinkove [30]. He [21] studied the functional on the space of Kähler metrics in more detail, viewing it as analogous to the Calabi functional and giving lower bounds for it in terms of Tian-Zhang-Zhang-Zhu’s invariant [36] for vector fields. A “moment map” interpretation of the H-functional has been described by Donaldson [18]. For the precise definitions of a test-configuration and the H-invariant, see Definitions 2.1 and 2.2 below.

In view of Theorem 1.2 it is natural to say that X is H-stable, if $H(\mathcal{X}) < 0$ for all non-trivial special degenerations (which are simply test-configurations with \mathbb{Q} -Fano central fibre). It follows that then the infimum of $H(\omega)$ is zero, or equivalently that X is “almost Kähler-Einstein” in the sense of [2] and in particular X is K-semistable. Conversely in Lemma 2.5 we show that K-semistability implies H-stability, hence H-stability is not enough to detect the existence of a Kähler-Einstein metric in general.

Given Theorem 1.2 it is natural to ask whether the supremum on the right hand side of (1.1) is achieved by a test-configuration. Our next result shows that this is the case, as long as we allow a slight generalisation of the notion of test-configurations to what we call \mathbb{R} -degenerations (see Definition 2.7). In fact such an optimal \mathbb{R} -degeneration is given by the filtration constructed by Chen-Sun-Wang [9] using the Kähler-Ricci flow.

Corollary 1.3. *On a Fano manifold that does not admit a Kähler-Einstein metric, the Kähler-Ricci flow produces an optimal \mathbb{R} -degeneration, with maximal H-invariant.*

As we mentioned above, such an optimal degeneration should be thought of as analogous to the Harder-Narasimhan filtration for an unstable vector bundle. The corollary answers questions of Chen-Sun-Wang [9, Question 3.8] and He [22, Question 3]. It is natural to conjecture that optimal degenerations in this sense are unique. By the above result, this is related to a conjecture of Chen-Sun-Wang regarding uniqueness of the degenerations induced by the Kähler-Ricci flow [9, Conjecture 3.7].

Our approach enables us to prove new results about the Kähler-Ricci flow on Fano manifolds. First we give an algebro-geometric interpretation of the supremum of Perelman’s μ -functional [27] on Fano manifolds that do not admit Kähler-Einstein metrics. This builds again on work of He [21], and as a special case answers Conjecture 3.4 in Tian-Zhang-Zhang-Zhu [36].

Theorem 1.4. *Let $\mu(\omega)$ denote Perelman’s μ -functional. We have*

$$\sup_{\omega \in c_1(X)} \mu(\omega) = nV - \sup_{\mathcal{X}} H(\mathcal{X}).$$

In addition the μ -functional tends to its supremum along the Kähler-Ricci flow starting from any initial metric.

As shown by Berman [4], the Donaldson-Futaki invariant of special degenerations can also yield upper bounds for the μ -functional, however it is not known whether the supremum can be characterised in that way.

Finally, using a result of Tian-Zhang-Zhang-Zhu [36], we obtain a general convergence result for the Kähler-Ricci flow, assuming the existence of a Kähler-Ricci soliton.

Corollary 1.5. *Suppose X is a Fano manifold admitting a Kähler-Ricci soliton ω_{KRS} , and let $\omega \in c_1(X)$ be an arbitrary Kähler metric. The Kähler-Ricci flow starting from ω converges to ω_{KRS} , up to the action of the automorphism group of X .*

When ω_{KRS} is actually Kähler-Einstein, this was proven by Tian-Zhu [37, 38] using Perelman's estimates [27, 34] (see also [13]). When ω_{KRS} is a general Kähler-Ricci soliton, then Tian-Zhang-Zhang-Zhu [36] proved the convergence result under the assumption that the initial metric ω is invariant under a maximal compact group of automorphisms of X .

We emphasise that the techniques used in the present note are not really new. Instead the novelty in our work is the observation that by using a slightly different notion of stability, the older techniques yield new stronger results.

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Notation and conventions: We normalise dd^c so that $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. A \mathbb{Q} -Fano variety is a normal variety X such that $-K_X$ is an ample \mathbb{Q} -line bundle and such that X has log terminal singularities. The volume of X is denoted by $V = (-K_X)^n = \int_X c_1(X)^n$. When X is smooth, given a metric $\omega \in c_1(X)$, we define its Ricci potential h to be such that $\text{Ric } \omega - \omega = dd^c h$ and $\int_X e^h \omega^n = V$. When X is a \mathbb{Q} -Fano variety, given a positive metric p on $-K_X$ with curvature ω , we set the Ricci potential to be $h = \log\left(\frac{\omega^n}{p}\right) + C$, with the constant C chosen such that $\int_X e^h \omega^n = V$. By a smooth Kähler metric on X we mean a $(1,1)$ -form on the regular part of X obtained locally by restricting a smooth Kähler metric under a local embedding into \mathbb{C}^N , see for example [14].

2. H-STABILITY

2.1. Analytic aspects. Let X be a \mathbb{Q} -Fano variety. We first focus on the analytic aspects of H-stability, postponing the algebraic description to Section 2.2. Here we will restrict ourselves to considering special degenerations in the sense of Tian [35]. In Section 2.2 we will define \mathbb{R} -degenerations.

Definition 2.1. [35] A *special degeneration* of X is a normal \mathbb{Q} -Fano family $\pi : \mathcal{X} \rightarrow \mathbb{C}$, together with a holomorphic vector field v on \mathcal{X} , a real multiple of which

generates a \mathbb{C}^* -action on \mathcal{X} covering the natural action on \mathbb{C} . In addition the fibre \mathcal{X}_t over t is required to be isomorphic to X for one, and hence all, $t \in \mathbb{C}^*$. We call \mathcal{X} a product special degeneration if $\mathcal{X} \cong X \times \mathbb{C}$, while \mathcal{X} is trivial if in addition the vector field is trivial on the X factor.

We now define the H-invariant to be a certain integral over \mathcal{X}_0 , which was first considered by Tian-Zhang-Zhang-Zhu [36] in the case of product special degenerations with X smooth. We remark that this H-invariant is not linear in the vector field v , and so it is important to allow “scalings” of \mathbb{C}^* -actions in the above definition.

Definition 2.2. Choose a smooth Kähler metric $\omega_0 \in c_1(\mathcal{X}_0)$, and let h_0 be the corresponding Ricci potential. Let θ_0 be a Hamiltonian for the induced holomorphic vector field v on \mathcal{X}_0 . Then we define the *H-invariant* $H(\mathcal{X})$ to be

$$H(\mathcal{X}) = \int_{\mathcal{X}_0} \theta_0 e^{h_0} \omega_0^n - V \log \left(\frac{1}{V} \int_{\mathcal{X}_0} e^{\theta_0} \omega_0^n \right),$$

where the integrals can be performed on the regular part of \mathcal{X}_0 .

The factors of V are included to ensure the integral equals zero if $\theta_0 = 0$. It is straight forward to check that $H(\mathcal{X})$ is independent of choice of Hamiltonian, i.e. it does not change if we replace θ by $\theta + C$ for $C \in \mathbb{R}$. We will later give an equivalent, algebro-geometric, definition of the H-invariant which will show that it is also independent of choice of $\omega_0 \in c_1(\mathcal{X}_0)$ (which follows from [36] with \mathcal{X}_0 smooth).

The definition of H-stability then simply requires control of the sign of the H-invariant of each special degeneration.

Definition 2.3. We say that X is *H-stable* if $H(\mathcal{X}) < 0$ for all non-trivial special degenerations of X .

By contrast recall the analytic definition of the Donaldson-Futaki invariant, which is the relevant weight for K-stability.

Definition 2.4. We define the *Donaldson-Futaki invariant* of \mathcal{X} to be

$$\text{DF}(\mathcal{X}) = \int_{\mathcal{X}_0} \theta_0 e^{h_0} \omega_0^n - \int_{\mathcal{X}_0} \theta_0 \omega_0^n.$$

We say that X is *K-semistable* if $\text{DF}(\mathcal{X}) \leq 0$ for all special degenerations \mathcal{X} , and X is *K-stable* if in addition $\text{DF}(\mathcal{X}) = 0$ only if $\mathcal{X}_0 \cong X$.

We remark that, in the literature, the opposite sign convention for the Donaldson-Futaki invariant is often used, but the present sign convention seems to be more natural in view of (1.1). The conventions also sometimes differ in the sign given to the Hamiltonian function corresponding to a \mathbb{C}^* -action. This makes little difference in the Donaldson-Futaki invariant, since it is linear, but the H-invariant is more sensitive to changes in such conventions.

The two invariants are related as follows:

Lemma 2.5. *We have $\text{DF}(\mathcal{X}) \geq H(\mathcal{X})$, with equality if and only if \mathcal{X} is trivial.*

Proof. This follows by Jensen’s inequality. Indeed, $V^{-1}\omega_0^n$ is a probability measure so by concavity of the logarithm

$$\log \int_{\mathcal{X}_0} e^{\theta_0} (V^{-1}\omega_0^n) \geq \int_{\mathcal{X}_0} \theta_0 (V^{-1}\omega_0^n),$$

with equality if and only if θ_0 is constant, i.e. \mathcal{X} is trivial. \square

Corollary 2.6. *If a Fano manifold is K-semistable, then it is H-stable. In particular, if a Fano manifold admits a Kähler-Einstein metric, then it is H-stable, but the converse is not necessarily true.*

Proof. The first part follows directly from Lemma 2.5. The second follows from the fact that the existence of a Kähler-Einstein metric implies K-semistability, but there are K-semistable manifolds that are not K-stable, and so do not admit a Kähler-Einstein metric (see Ding-Tian [16] and Tian [35]). \square

2.2. Algebraic aspects. We first define the algebro-geometric degenerations that we will consider. This will be a generalisation of the notion of a test-configuration [19], using the language of filtrations [39, 7]. Let X be an arbitrary projective variety, and for any ample line bundle $L \rightarrow X$ let us define the graded coordinate ring

$$R(X, L) = \bigoplus_{m \geq 0} H^0(X, mL).$$

For simplicity we write $R_m = H^0(X, mL)$. By a filtration of $R(X, L)$ we mean an \mathbb{R} -indexed filtration $\{F^\lambda R_m\}_{\lambda \in \mathbb{R}}$ for each m , satisfying

- (i) F is decreasing: $F^\lambda R_m \subset F^{\lambda'} R_m$ whenever $\lambda \geq \lambda'$,
- (ii) F is left-continuous: $F^\lambda R_m = \bigcap_{\lambda' < \lambda} F^{\lambda'} R_m$,
- (iii) For each m , $F^\lambda R_m = 0$ for $\lambda \gg 0$ and $F^\lambda R_m = R_m$ for $\lambda \ll 0$,
- (iv) F is multiplicative:

$$F^\lambda R_m \cdot F^{\lambda'} R_{m'} \subset F^{\lambda + \lambda'} R_{m+m'}.$$

The associated graded ring of such a filtration is defined to be

$$\text{gr} F^\lambda R(X, L) = \bigoplus_{m \geq 0} \bigoplus_i F^{\lambda_{m,i}} R_m / F^{\lambda_{m,i+1}} R_m,$$

where the $\lambda_{m,i}$ are the values of λ where the filtration of R_m is discontinuous.

Definition 2.7. An \mathbb{R} -degeneration for (X, L) is a filtration of $R(X, rL)$ for some integer $r > 0$, whose associated graded ring is finitely generated.

Let us recall here that as shown by Witt Nyström [39] (see also [7, Proposition 2.15]), test-configurations correspond to \mathbb{Z} -filtrations of $R(X, rL)$ whose Rees algebra is finitely generated. These are filtrations satisfying $F^\lambda R_m = F^{\lceil \lambda \rceil} R_m$ and that

$$\bigoplus_{m \geq 0} \left(\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda R_m \right)$$

is a finitely generated $\mathbb{C}[t]$ -algebra.

The notion of an \mathbb{R} -degeneration arises naturally from the work of Chen-Sun-Wang [9] on the Kähler-Ricci flow, and it corresponds to considering degenerations of a projective variety under certain real one-parameter subgroups of $GL_N(\mathbb{C})$. Moreover, as Chen-Sun-Wang do in [9, Section 3.1], we can approximate any \mathbb{R} -degeneration by a sequence of test-configurations. Indeed, given an \mathbb{R} -degeneration for (X, L) , let us denote by \bar{R} the associated graded ring of the filtration, and by $X_0 = \text{Proj } \bar{R}$ the corresponding projective variety. The filtration gives rise to a real one-parameter family of automorphisms of \bar{R} , by scaling the elements in

$$F^{\lambda_{m,i}} R_m / F^{\lambda_{m,i+1}} R_m$$

by $t^{\lambda_{m,i}}$. Without loss of generality we can suppose that \bar{R} is generated by the elements \bar{R}_1 of degree one (i.e. those coming from sections of L on X), and we set $N + 1 = \dim H^0(X, L)$. In this case we have an embedding $X_0 \subset \mathbb{P}^N$, and a real one-parameter group of projective automorphisms of X_0 given by $e^{t\Lambda}$, where Λ is a diagonal matrix with eigenvalues $\lambda_{1,i}$. In addition, as discussed in [9], we have an embedding $X \subset \mathbb{P}^N$, such that $\lim_{t \rightarrow \infty} e^{t\Lambda} \cdot X = X_0$ in the Hilbert scheme. Note that the matrices $e^{\sqrt{-1}t\Lambda}$ generate a compact torus T in $U(N)$, which must preserve X_0 , and therefore also acts on the coordinate ring \bar{R} . Perturbing the eigenvalues of Λ slightly to rational numbers, we can obtain a \mathbb{C}^* -subgroup ρ of the complexified torus $T^{\mathbb{C}}$ acting on \mathbb{P}^N , for which $\lim_{t \rightarrow 0} \rho(t) \cdot X = X_0$, i.e. we have a test-configuration for X with the central fiber X_0 , and the induced action on X_0 is a perturbation of $e^{t\Lambda}$.

In terms of the coordinate ring \bar{R} the action of $T^{\mathbb{C}}$ together with the m -grading defines the action of $\mathbb{C}^* \times T^{\mathbb{C}}$, and allows us to decompose

$$\bar{R} = \bigoplus_{m \geq 0} \bigoplus_{\alpha \in \mathfrak{t}^*} \bar{R}_{m,\alpha}$$

into weight spaces. The action of $e^{t\Lambda}$ corresponds to a choice of (possibly irrational) $\xi \in \mathfrak{t}$, so that the weight of its action on $\bar{R}_{m,\alpha}$ is $\langle \alpha, \xi \rangle$, and the approximating test-configurations obtained above give rise to a sequence of rational $\xi_k \in \mathfrak{t}$ such that $\xi_k \rightarrow \xi$. We will define the H-invariant of an \mathbb{R} -degeneration in such a way that it is continuous under this approximation procedure.

Definition 2.8. Let $\eta \in \mathfrak{t}$. For $t \in \mathbb{C}$, we define the *weight character* by

$$C(\eta, t) = \sum_{m \geq 0, \alpha \in \mathfrak{t}^*} e^{-tm} \alpha(\eta) \dim \bar{R}_{m,\alpha}.$$

By [12, Theorem 4], the weight character is a meromorphic function in a neighbourhood of $0 \in \mathbb{C}$, with the following Laurent series expansion:

$$C(\eta, t) = \frac{b_0(n+1)!}{t^{n+2}} + \frac{b_1(n+2)!}{t^{n+1}} + O(t^{-n}).$$

Moreover, b_0, b_1 are smooth functions of η . For the H-invariant, the key term will be the b_1 term of this expansion.

Remark 2.9. When the η is integral, the above construction reduces to a more well known equivariant Riemann-Roch construction, namely the constants b_0, b_1 are given by the asymptotics of the total weight

$$\sum_{\alpha \in \mathfrak{t}^*} \alpha(\eta) \cdot \dim \bar{R}_{m,\alpha} = b_0 m^{n+1} + b_1 m^n + O(m^{n-1}).$$

To define the H-invariant, we will also need the term

$$\frac{c_0}{n!} = \lim_{m \rightarrow \infty} \sum_{\alpha \in \mathfrak{t}^*} m^{-n} e^{-m^{-1}\alpha(\eta)} \dim \bar{R}_{m,\alpha}, \quad (2.1)$$

which is also a function of η .

Lemma 2.10. *The limit c_0 is well defined, and continuous in η .*

Proof. The existence of this limit can be seen as a consequence of the existence of the Duistermaat-Heckman measure

$$\text{DH} = \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{\alpha \in \mathfrak{t}^*} \dim \bar{R}_{m,\alpha} \delta_{m^{-1}\alpha(\eta)},$$

where $N_m = \dim \overline{R}_m$ and δ denotes the Dirac measure (see for instance Boucksom-Hisamoto-Jonsson [7, Section 5]). In our situation this measure has compact support, and

$$\frac{c_0}{V} = \int_{\mathbb{R}} e^{-\lambda} \mathrm{DH}(\lambda). \quad (2.2)$$

In order to show that c_0 in (2.1) is continuous in η , we argue by approximation. We can assume, as above, that \overline{R} is generated by the degree one elements \overline{R}_1 . This implies that if $\alpha \in \mathfrak{t}^*$ is such that $\overline{R}_{m,\alpha}$ is nonzero, then α is a sum of m weights appearing in the action on \overline{R}_1 , and so in particular $|\alpha| < Cm$ for a uniform C .

Suppose now that $\eta, \eta' \in \mathfrak{t}$, and $\overline{R}_{m,\alpha}$ is nontrivial. Then using the mean value theorem we obtain

$$\left| e^{-m^{-1}\alpha(\eta)} - e^{-m^{-1}\alpha(\eta')} \right| \leq Cm^{-1} |\alpha(\eta) - \alpha(\eta')| \leq C|\eta - \eta'|,$$

for a uniform C . Since $\dim \overline{R}_{m,\alpha} \leq \dim \overline{R}_m \leq dm^n$ for some $d > 0$, it follows that for each m we have

$$\left| \sum_{\alpha \in \mathfrak{t}^*} m^{-n} \left(e^{-m^{-1}\alpha(\eta)} - e^{-m^{-1}\alpha(\eta')} \right) \dim \overline{R}_{m,\alpha} \right| \leq Cd|\eta - \eta'|.$$

It now follows that if $\eta_k \rightarrow \eta$, and for each η_k the limit (2.1) defining $c_0(\eta_k)$ exists, then the limit defining $c_0(\eta)$ also exists and $c_0(\eta_k) \rightarrow c_0(\eta)$. \square

We now return to the case that X is a \mathbb{Q} -Fano variety.

Definition 2.11. Let \mathcal{X} be an \mathbb{R} -degeneration for $(X, -K_X)$. We define the H -invariant of \mathcal{X} to be

$$H(\mathcal{X}) = -V \log \left(\frac{c_0}{V} \right) - 2((n-1)!)b_1,$$

where $V = (-K_X)^n$ is the volume as before.

This definition agrees with the analytic one for special degenerations:

Proposition 2.12. *The analytic and algebraic definitions of the H -invariant agree for special degenerations.*

Proof. We can apply the formula (2.2) to the product degeneration $\mathcal{X}_0 \times \mathbb{C}$ induced by the action on \mathcal{X}_0 , together with [3, Proposition 4.1] to see that

$$c_0 = \int_X e^{\theta_0} \omega_0^n,$$

where ω_0 is any smooth S^1 -invariant metric on \mathcal{X}_0 and θ_0 is the Hamiltonian of the induced vector field.

What remains is to show that

$$b_1 = - \int_{\mathcal{X}_0} \theta_0 e^{\theta_0} \frac{\omega_0^n}{2(n-1)!},$$

but this is essentially standard when the vector field v on \mathcal{X} generates a \mathbb{C}^* -action [35, 4, 21], and both sides are linear under scaling v . \square

Remark 2.13. The algebraic definition of the Donaldson-Futaki invariant is

$$\mathrm{DF}(\mathcal{X}) = n! \left(b_0 - \frac{2}{n} b_1 \right).$$

Applying the finite Jensen's inequality one can prove $DF(\mathcal{X}) \geq H(\mathcal{X})$ for special degenerations, which is just the algebraic analogue of Lemma 2.5, however to characterise the equality case it seems advantageous to use the analytic representations of both quantities.

3. THE H-FUNCTIONAL

Let X be a \mathbb{Q} -Fano variety.

Definition 3.1. Let $\omega \in c_1(X)$ be a Kähler metric, with Ricci potential h , and define the *H-functional* to be

$$H(\omega) = \int_X h e^h \omega^n.$$

Recall here the normalisation of h so that e^h has average one. It will be useful to extend this definition to the case where ω is smooth merely on the regular locus X_{reg} , with continuous potential globally and also continuous Ricci potential. In this situation, we make the same definition, where ω^n means the Bedford-Taylor wedge product.

As we described in the Introduction, this functional goes back to Ding-Tian [17], is monotonic along the Kähler-Ricci flow (see Pali [26], Phong-Song-Sturm-Weinkove [30]) and was studied as a functional on the space of Kähler metrics in more detail by He [21].

We now assume X is smooth. By the Jensen inequality we have $H(\omega) \geq 0$, with equality if and only if ω is a Kähler-Einstein metric. More generally if ω is a Kähler-Ricci soliton, with soliton vector field W , it is clear that $H(\omega) = H(W)$ as then the Hamiltonian, suitably normalised, equals the Ricci potential. Moreover, the critical points of $H(\omega)$ are precisely Kähler-Ricci solitons, and the gradient flow of the H-functional is simply the Kähler-Ricci flow (see [21, Proposition 2.2]). The H-functional thus plays the role for the Kähler-Einstein problem that the Calabi functional plays for the constant scalar curvature problem.

The goal of the present section is to prove that the infimum of the H-functional has an algebro-geometric interpretation.

Theorem 3.2. *We have*

$$\inf_{\omega \in c_1(X)} H(\omega) = \sup_{\mathcal{X}} H(\mathcal{X}),$$

where the supremum on the right hand side is taken over all \mathbb{R} -degenerations (or even just special degenerations). In addition the supremum is achieved by an \mathbb{R} -degeneration.

This will be a consequence of the work of He [21] to obtain the inequality $H(\omega) \geq H(\mathcal{X})$ for any metric ω and \mathbb{R} -degeneration \mathcal{X} , as well as the work of Chen-Sun-Wang [9], to obtain the equality.

Theorem 3.3. *Let \mathcal{X} be an \mathbb{R} -degeneration for X . Then for any metric $\omega \in c_1(X)$ we have $H(\omega) \geq H(\mathcal{X})$.*

Proof. First of all, by an approximation argument it is enough to show the result for test-configurations \mathcal{X} .

Next we can assume that \mathcal{X} is normal. The reason is that under normalisation the number c_0 does not change (see [7, Theorem 3.14]), while the number b_1 can

only decrease (see [7, Proposition 3.15] or [32, Proposition 5.1]). It follows that under normalisation the H-invariant can only increase.

Let φ_t be a geodesic ray in the space of Kähler potentials, induced by the normal test-configuration \mathcal{X} , with initial metric ω [31]. Let us denote the associated (weak) Kähler metrics by $\omega_t = \omega + dd^c \varphi_t$ and set

$$Y(t) = - \int_X \dot{\varphi}_t e^{h\varphi_t} \omega_t^n - V \log \left(V^{-1} \int_X e^{-\dot{\varphi}_t} \omega_t^n \right).$$

We now show

$$H(\omega) \geq \lim_{t \rightarrow \infty} Y(t)$$

This inequality is due to He [21]; for the readers' convenience we give the proof.

We can normalise the potentials φ_t so that

$$\int_X e^{-\dot{\varphi}_t} \omega_t^n = V, \tag{3.1}$$

since according to Berndtsson [5, Proposition 2.2] this integral is independent of t . Note also that, up to the addition of a constant, $-Y(t)$ is the derivative of the Ding functional along the ray of Kähler potentials φ_t . By Berndtsson [6], the Ding functional is convex along geodesics, and so $Y(t)$ is monotonically decreasing in t .

At the same time, by Jensen's inequality we have

$$V^{-1} \int_X (-h - \dot{\varphi}_0) e^h \omega^n \leq \log \left(V^{-1} \int_X e^{-h - \dot{\varphi}_0} e^h \omega^n \right) = 0,$$

and so

$$H(\omega) = \int_X h e^h \omega^n \geq - \int_X \dot{\varphi}_0 e^h \omega^n = Y(0).$$

By the monotonicity of $Y(t)$ it then follows that

$$H(\omega) \geq \lim_{t \rightarrow \infty} Y(t). \tag{3.2}$$

What remains is to relate this limit of $Y(t)$ to the H-invariant. For this note first that by Hisamoto [23, Theorem 1.1], the formula (2.2), and (3.1), we have

$$c_0 = \int_X e^{-\dot{\varphi}_t} \omega_t^n = V,$$

and so from Definition 2.11 we have

$$H(\mathcal{X}) = -2b_1(n-1)!.$$

Since $-Y(t)$ is, up to addition of a constant, the derivative of the Ding functional along the geodesic ray, the asymptotics of $Y(t)$ can be obtained from Berman [4, Theorem 3.11]. It follows that

$$H(\mathcal{X}) \leq \lim_{t \rightarrow \infty} Y(t),$$

and so with (3.2) the proof is complete. \square

The other main ingredient that we use is the following, due to Chen-Wang [10] and Chen-Sun-Wang [9] (see especially [9, p12, p16]).

Theorem 3.4. *Let $(X, \omega(t))$ be a solution of the Kähler-Ricci flow. The sequential Gromov-Hausdorff limit of $(X, \omega(t))$ as $t \rightarrow \infty$ is a \mathbb{Q} -Fano variety Y , independent of choice of subsequence, which admits a Kähler-Ricci soliton with soliton vector field W_Y . Assume that $W_Y \neq 0$. Then there exists a “two-step” \mathbb{R} -degeneration from X to Y , i.e. an \mathbb{R} -degeneration \mathcal{X}_a for X with \mathbb{Q} -Fano central fibre \bar{X} , and an \mathbb{R} -degeneration \mathcal{X}_b for \bar{X} with central fibre Y . The corresponding (real) one-parameter group of automorphisms on \mathcal{X}_b is induced by the soliton vector field on Y .*

Chen-Sun-Wang prove that the Donaldson-Futaki invariants of \mathcal{X}_a and \mathcal{X}_b are equal [9, Proposition 3.5], or equivalently the Futaki invariants of \bar{X} and Y are equal. We will require an analogous statement for the H-invariant.

Lemma 3.5. *We have $H(\mathcal{X}_a) = H(\mathcal{X}_b)$.*

Proof. By [9, Proof of Lemma 3.4, Proposition 3.5], the weight decompositions of $H^0(\bar{X}, -rK_{\bar{X}})$ and $H^0(Y, -rK_Y)$ are isomorphic for all sufficiently large and divisible r , hence invariants created from these decompositions are equal. The result follows from the algebraic definition of the H-invariant, Definition 2.11. \square

We now proceed to the proof of Theorem 3.2.

Proof of Theorem 3.2. From Theorem 3.3 we already know that

$$\inf_{\omega \in c_1(X)} H(\omega) \geq \sup_{\mathcal{X}} H(\mathcal{X}),$$

taking the supremum over all \mathbb{R} -degenerations. To complete the proof we show that the \mathbb{R} -degeneration \mathcal{X}_a obtained from the Kähler-Ricci flow $\omega(t)$ in Theorem 3.4 satisfies

$$\lim_{t \rightarrow \infty} H(\omega(t)) = H(\mathcal{X}_a). \quad (3.3)$$

To see this, note that (Y, ω_Y) is the Gromov-Hausdorff limit of $(X, \omega(t))$, and so

$$\lim_{t \rightarrow \infty} H(\omega(t)) = H(\omega_Y), \quad (3.4)$$

where the former quantity is calculated on X and the latter is calculated on Y . Here to make sense of $H(\omega_Y)$ we recall how to define the Ricci potential in this situation. The regularity results for Kähler-Ricci solitons imply that ω_Y is smooth on the regular locus Y_{reg} , and has continuous potential on Y [3, Section 3.3]. Then as the Ricci potential is uniformly bounded in C^1 along the flow [34], the Ricci potential of ω_Y on the regular locus Y_{reg} extends to a continuous function on Y which we still call the Ricci potential, and one can define $H(\omega_Y)$ as usual. Moreover, this implies that to prove (3.3) one can work only on the smooth locus.

Denote by V the soliton vector field on Y . What we now show is that $H(V) = H(\omega_Y)$. This would be immediate if ω_Y were a smooth Kähler metric, however the regularity results described above do not imply this. Denote by h the Ricci potential of ω_Y , which is continuous by the above, and pick a smooth Kähler metric $\eta \in c_1(Y)$ with Ricci potential f and Hamiltonian θ . It follows from [4] that

$$\int_Y h e^h \omega_Y^n = \int_Y \theta e^f \eta^n,$$

where we have used that h is the Hamiltonian with respect to ω_Y of V . Indeed, both quantities are limit derivatives of a component of the Ding functional [4, 3].

For the remaining term, we use that h is continuous and so

$$\int_Y e^h \omega_Y^n = \int_Y e^\theta \eta^n,$$

as follows from the proof of [15, Theorem 3.14]. Thus $H(Y) = H(\mathcal{X}_b)$, and by Lemma 3.5 $H(\mathcal{X}_a) = H(\mathcal{X}_b)$, so (3.3) follows.

Note that the \mathbb{R} -degeneration \mathcal{X}_a has \mathbb{Q} -Fano central fiber, and so the approximating test-configurations are actually special degenerations. It follows that $\sup_{\mathcal{X}} H(\mathcal{X})$ can be computed by considering only special degenerations \mathcal{X} . \square

Remark 3.6. We should emphasise that the central fiber \overline{X} of the optimal degeneration does not necessarily agree with the Gromov-Hausdorff limit Y along the Kähler-Ricci flow. The first \mathbb{R} -degeneration with central fiber \overline{X} is analogous to the Harder-Narasimhan filtration of an unstable vector bundle, while the second \mathbb{R} -degeneration with central fiber Y is analogous to the Jordan-Hölder filtration of a semistable bundle.

4. APPLICATIONS TO THE KÄHLER-RICCI FLOW

We recall the following functionals introduced by Perelman [27]:

Definition 4.1. Denote by $S(\omega)$ the scalar curvature of ω . For a smooth function f on (X, ω) satisfying

$$\int_X e^{-f} \omega^n = V,$$

we define the W -functional to be

$$W(\omega, f) = \int_X (S(\omega) + |\nabla f|^2 + f) e^{-f} \omega^n.$$

The μ -functional is defined as

$$\mu(\omega) = \inf_{f \in C^\infty(X)} W(\omega, f).$$

The following extends [36, 21], who proved a special case of the following result, namely an upper bound over product special degenerations.

Theorem 4.2. *We have*

$$\sup_{\omega \in c_1(X)} \mu(\omega) = nV - \sup_{\mathcal{X}} H(\mathcal{X}).$$

In addition the supremum of μ is achieved in the limit along the Kähler-Ricci flow with any initial metric on X .

Proof. It is shown in [21] that for all $\omega \in c_1(X)$ we have

$$\mu(\omega) \leq nV - H(\omega),$$

and hence

$$\sup_{\omega} \mu(\omega) \leq nV - \inf_{\omega} H(\omega).$$

Thus by Theorem 3.2 we have

$$\sup_{\omega} \mu(\omega) \leq nV - \sup_{\mathcal{X}} H(\mathcal{X}). \tag{4.1}$$

Let (Y, ω_Y) be the Gromov-Hausdorff limit of (X, ω) along the Kähler-Ricci flow as usual. Let h_Y be the Ricci potential of ω_Y . Then we have have, as in [9, p17] (but with different normalisations),

$$W(\omega_Y, h_Y) = nV - H(Y),$$

hence by Theorem 3.4 and Lemma 3.5 there is a special degeneration \mathcal{X}_a such that $nV - H(\mathcal{X}_a) = W(\omega_Y, h_Y)$.

What remains is to show that along the Kähler-Ricci flow ω_t we have

$$\lim_{t \rightarrow \infty} \mu(\omega_t) = W(\omega_Y, h_Y).$$

Let f_t denote the minimiser of the W functional on (X, ω_t) , i.e. $\mu(\omega_t) = W(\omega_t, f_t)$. It is well known [33] that f_t is smooth, and in addition the uniform control of the Sobolev constant [41, 42] and the scalar curvature [34] along the flow implies that we have a uniform bound $|\Phi_t| < C$, where $\Phi_t = e^{-f_t/2}$. In terms of Φ_t we have

$$W(\omega_t, f_t) = \int_X (S(\omega_t)\Phi_t^2 + 4|\nabla\Phi_t|^2 - 2\Phi_t^2 \ln \Phi_t) \omega_t^n. \quad (4.2)$$

From Bamler [1, p. 60] and Chen-Wang [10, Proposition 6.2] we know that on any compact subset $K \subset Y_{reg}$ of the regular part of Y , we have $f_t \rightarrow h_Y$ as $t \rightarrow \infty$, and elliptic regularity implies that this convergence holds for derivatives as well. It follows that

$$\lim_{t \rightarrow \infty} \int_K (S(\omega_t) + |\nabla f_t|^2 + f_t) e^{-f_t} \omega_t^n = \int_K (S(\omega_Y) + |\nabla h_Y|^2 + h_Y) e^{-h_Y} \omega_Y^n, \quad (4.3)$$

where we are viewing the ω_t as defining metrics on K for large t , using the smooth convergence of the metrics on the regular part of Y .

We know that $h_Y, \nabla h_Y$ are bounded, and so as we let K exhaust Y_{reg} on the right hand side of (4.3) we recover $W(\omega_Y, h_Y)$. On the left hand side we use the expression (4.2) in terms of Φ_t , and the fact that the singular set has codimension at least 4 [1], and so in particular we can choose K with the volume of $Y \setminus K$ being arbitrarily small. Although a priori the gradient $\nabla\Phi_t$ may concentrate on the singular set, it still follows that for any $\epsilon > 0$, we can choose K so that for sufficiently large t

$$\int_K (S(\omega_t) + |\nabla f_t|^2 + f_t) e^{-f_t} \omega_t^n \leq W(\omega_t, f_t) + \epsilon.$$

Exhausting Y_{reg} with compact sets K we obtain

$$\lim_{t \rightarrow \infty} W(\omega_t, f_t) \geq W(\omega_Y, h_Y).$$

Together with (4.1) this implies $\lim_{t \rightarrow \infty} \mu(\omega_t) = W(\omega_Y, h_Y)$, which is what we wanted to show. \square

Corollary 4.3. *Suppose X is a Fano manifold admitting a Kähler-Ricci soliton ω_{KRS} , and let $\omega \in c_1(X)$ be an arbitrary (not necessarily automorphism invariant) Kähler metric. The Kähler-Ricci flow starting from ω converges to ω_{KRS} , up to the action of the automorphism group of X .*

Proof. It is a result of Tian-Zhang-Zhang-Zhu [36] that if X admits a Kähler-Ricci soliton ω_{KRS} and $\omega(t)$ satisfies the Kähler-Ricci flow for arbitrary $\omega(0)$, then provided

$$\mu(\omega(t)) \rightarrow \sup_{\omega \in c_1(X)} \mu(\omega),$$

then the flow converges to ω_{KRS} modulo the action of automorphisms of X . Thus the claim follows from Theorem 4.2. \square

Remark 4.4. In some examples we expect that Theorem 4.2 can be used to identify the limit of the Kähler-Ricci flow on X , even if X does not admit a soliton, by finding degenerations that maximize the μ -functional. In their work on the Kähler-Ricci flow on S^2 with conical singularities, Phong-Song-Sturm-Wang [28, 29] use this approach to identify the limiting solitons along the flow.

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