TORSION, TORSION LENGTH AND FINITELY PRESENTED GROUPS

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Abstract. We show that a construction by Aanderaa and Cohen used in their proof of the Higman Embedding Theorem preserves torsion length. We give a new construction showing that every finitely presented group is the quotient of some $C'(1/6)$ finitely presented group by the subgroup generated by its torsion elements. We use these results to show there is a finitely presented group with infinite torsion length which is $C'(1/6)$, and thus word-hyperbolic and virtually torsion-free.

1. Introduction

It is well known that the set of torsion elements in a group G, $\operatorname{Tor}(G)$, is not necessarily a subgroup. One can, of course, consider the subgroup $\operatorname{Tor}_1(G)$ *generated* by the set of torsion elements in G: this subgroup is always normal in G.

The subgroup $\operatorname{Tor}_1(G)$ has been studied in the literature, with a particular focus on its structure in the context of 1-relator groups. For example, suppose G is presented by a 1-relator presentation P with cyclically reduced relator R^k where R is not a proper power, and let r denote the image of R in G. Karrass, Magnus, and Solitar proved ([17, Theorem 3]) that r has order k and that every torsion element in G is a conjugate of some power of r; a more general statement can be found in [22, Theorem 6]. As immediate corollaries, we see that $\operatorname{Tor}_1(G)$ is precisely the normal closure of r, and that $G/\operatorname{Tor}_1(G)$ is torsion-free.

More generally, the manner in which $\operatorname{Tor}_1(G)$ is impacted by the *deficiency* $\operatorname{def}(G)$ of a finitely presented group G has also been investigated. The deficiency of G is the maximum value of $m-n$, where m and n are the number of generators and relators respectively as we range over all finite presentations of G. In [4, Corollary 3.6], Berrick and Hillman proved that if G is a finitely presentable group with $\operatorname{def}(G) > 0$, and $\operatorname{Tor}_1(G)$ is either finitely generated or locally finite,

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then $\mathrm{Tor}_1(G)$ is actually finite; again, in this situation $G/\mathrm{Tor}_1(G)$ is torsion-free. They claim that the question of whether $\mathrm{Tor}_1(G)$ is necessarily *trivial* under these hypotheses is open; using a result of Karras and Solitar [18, Main Theorem] one immediately sees that this triviality is indeed the case when G is presented by a 1-relator presentation.

In both cases described above, the quotient $G/\mathrm{Tor}_1(G)$ is torsion-free. Unfortunately, this is not always the case. Consider, for example, the group C presented by the following presentation: $\langle x, y, z \mid x^3 = e, y^3 = e, xy = z^3 \rangle$; it can be shown that C is a finitely presented word-hyperbolic group ([10, Proposition 3.5]), but that $C/\mathrm{Tor}_1(C) \cong \mathbb{Z}/3\mathbb{Z}$ ([10, Proposition 3.1]).

We can, however, iterate the process used to construct $\mathrm{Tor}_1(G)$ to produce an ascending chain of normal subgroups $\mathrm{Tor}_1(G) \leq \mathrm{Tor}_2(G) \leq \dots$ of G. For finite $n \in \mathbb{N}$, we define $\mathrm{Tor}_{n+1}(G)$ via $\mathrm{Tor}_{n+1}(G)/\mathrm{Tor}_n(G) = \mathrm{Tor}_1(G/\mathrm{Tor}_n(G))$; we define $\mathrm{Tor}_\omega(G) := \bigcup_{n\in\mathbb{N}} \mathrm{Tor}_n(G)$. The ordinal for which this chain stabilises is called the *torsion length* of G and denoted by $\mathrm{TorLen}(G)$. It turns out that $G/\mathrm{Tor}_\omega(G)$ is the universal torsion-free quotient of G: it is torsion-free, and all other torsion-free quotients uniquely factor through it (see [9, Corollary 3.4]). Thus $\mathrm{TorLen}(G)$ is always bounded above by ω; this bound is attained when the chain mentioned above does not stabilise at any finite stage. Intuitively, $\mathrm{TorLen}(G)$ is the minimal number of times we need to 'kill off' torsion to get a torsion-free quotient of G.

The notion of torsion length first appeared, independently, in both [11] and our earlier work [10]. In [11], Cirio *et al.* defined the *torsion degree* of a quantum group (here, quantum groups are C*-algebras equipped with a suitable comultiplication). The definition of torsion length aligns with torsion degree when a group is viewed as a quantum group via its associated C*-algebra. Further, they defined the notion of the "connected component at the identity" Q° of a quantum group Q and remarked that for an ordinary group G (again viewed as a quantum group via its associated C*-algebra) this object corresponds to $G/\mathrm{Tor}_\omega(G)$ ([11, Example 3.17]). They also constructed a descending ordinal indexed family of quantum subgroups G_α "converging" to G°; again, in the classical situation these objects correspond to the quotients $G/\mathrm{Tor}_\alpha(G)$.

The quotient $G/\mathrm{Tor}_\omega(G)$ was first studied in [6], where Brodsky and Howie investigated this object (they use the notation $\hat{G}$) for various families of groups. A group is *locally indicable* if every non-trivial finitely generated subgroup admits a surjection onto $\mathbb{Z}$: Brodsky and Howie showed that if a group has deficiency $\mathrm{def}(G) > 0$, then $G/\mathrm{Tor}_\omega(G)$ is locally indicable [6, Theorem 3.7]. They also showed that $G/\mathrm{Tor}_\omega(G)$ is locally indicable when G is 1-relator, or 2-relator with one relator having length ≤ 4, or 2-relator with with one relator having length 5 and the other has length ≤ 8, or at most 5-relator with each relator having length ≤ 3. These results appear as [6, Theorems 1.1–1.4].

In [10] we began a preliminary investigation of torsion length. One of the main results of that work, which we generalise here in Theorem 6.13, was the following theorem:

Theorem. [10, Theorem 3.3] There is a family of finitely presented groups $\{P_n\}_{n\in\mathbb{N}}$ such that:

1. $P_{n+1}/\operatorname{Tor}_1(P_{n+1}) \cong P_n$,
2. $\operatorname{TorLen}(P_n) = n$.

We then showed that a construction used to prove a classic embedding theorem of Higman, Neumann and Neumann (every countable group embeds into a 2 generator group) preserved torsion length. This fact, used with the theorem mentioned above, allowed us to arrive at the following result:

Theorem. [10, Theorem 3.10] There exists a 2-generator recursively presented group Q for which $\operatorname{TorLen}(Q) = \omega$.

This paper aims to extend [10, Theorem 3.10]. In Theorem 5.7, we prove the following:

Theorem. *There exists a finitely presented group F with* $\operatorname{TorLen}(F) = \omega$.

We do this by showing that a particular construction used in a proof of the Higman Embedding Theorem preserves this invariant.

Let us be more precise. The Higman Embedding Theorem [16] states that a finitely generated, recursively presented group embeds into a finitely presented group. There are many proofs of this result, but these arguments share a common theme: they are all constructive. One must begin with a finite generating set for the group, and an algorithm that computes its relations, and then explicitly build a finitely presented group from this data. In this paper we pick a particular construction, due to Aanderaa and Cohen [1, 2] and presented in [12], examine it in detail, and conclude that the torsion length of the finitely presented group so constructed is the same as that of the recursively presented group that we started with.

The existence of a finitely presented group with infinite torsion length is then an immediate consequence of [10, Theorem 3.10]: take the recursively presented group constructed in *loc. cit.* and apply the Aanderaa-Cohen construction.

Section 6 of this paper is concerned with improving Theorem 5.7. The following result appears as Theorem 6.10.

Theorem. *There exists a finitely presented word-hyperbolic virtually special group W with* $\operatorname{TorLen}(W) = \omega$. *In particular, W is virtually torsion-free.*

This is done using small cancellation theory. Of particular importance is the following theorem, whose content is contained in Proposition 6.4 and Theorem 6.7; this result is also of independent interest. Combined with Theorem 5.7, it proves Theorem 6.10.

Theorem. *Let $P = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a finite presentation with all relators freely reduced, cyclically reduced, and distinct. For any $k \in \mathbb{N}$, define the finite presentation $P_t^k := \langle x_1, \dots, x_m, t \mid (r_1 t)^k, \dots, (r_n t)^k, t^k \rangle$. Then P_t^k presents a $C'(2/k)$ small cancellation group. Moreover, for $k \geq 12$, we have $P_t^k/\operatorname{Tor}_1(P_t^k) \cong P$ and so* $\operatorname{TorLen}(P_t^k) = \operatorname{TorLen}(P) + 1$.

A part of the above theorem appeared in the work [7] of Bumagin and Wise; see Remark 6.11.

In Section 7 we finish with a discussion of some open problems relating to torsion length and torsion subgroups.

1.1. **Notation.** A presentation $P = \langle X|R\rangle$ is said to be a *recursive presentation* if X is a finite set and R is a recursive enumeration of relations; it is said to be a *finite presentation* if both X and R are finite. A group G is said to be *finitely* (respectively, *recursively*) *presentable* if it can be presented by a finite (respectively, recursive) presentation. If P, Q are group presentations denote their free product presentation by $P * Q$: this is given by taking the disjoint union of their generators and relations. If $g_1, \dots, g_n$ are elements of a group G, we write $\langle g_1, \dots, g_n\rangle$ for the subgroup in G generated by these elements and $\langle\langle g_1, \dots, g_n\rangle\rangle^G$ for the normal closure of these elements in G. Let ω denote the smallest infinite ordinal. Let $|X|$ denote the cardinality of a set X. If X is a set, let X^{-1} be a set of the same cardinality as and disjoint from X along with a fixed bijection $*^{-1} : X \to X^{-1}$. Write X^* for the set of finite reduced words on $X \cup X^{-1}$.

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2. $\mathrm{Tor}_n(G)$, HNN extensions and Britton's lemma

Definition 2.1. [9, Definition 3.1] Given a group G, inductively define $\mathrm{Tor}_n(G)$ as follows:

$$\mathrm{Tor}_0(G) := \{e\},$$
$$\mathrm{Tor}_{n+1}(G) := \langle\langle\ \{g \in G \mid g\,\mathrm{Tor}_n(G) \in \mathrm{Tor}\left(G/\mathrm{Tor}_n(G)\right)\}\ \rangle\rangle^G,$$
$$\mathrm{Tor}_\omega(G) := \bigcup_{n\in\mathbb{N}} \mathrm{Tor}_n(G).$$

Observe that $\mathrm{Tor}_{i+1}(G)/\mathrm{Tor}_i(G) = \mathrm{Tor}_1\left(G/\mathrm{Tor}_i(G)\right)$.

Lemma 2.2. [9, Corollary 3.4] *$G/\mathrm{Tor}_\omega(G)$ is torsion-free. Moreover, if $f : G \to H$ is a group homomorphism from G to a torsion-free group H, then $\mathrm{Tor}_\omega(G) \le \ker(f)$, and so f factors through $G/\mathrm{Tor}_\omega(G)$.*

Definition 2.3. [10, Definition 2.5] The *Torsion Length* of G, $\mathrm{TorLen}(G)$, is the smallest ordinal n such that $\mathrm{Tor}_n(G) = \mathrm{Tor}_\omega(G)$.

HNN extensions play a critical role in this paper; we briefly introduce them here.

Definition 2.4. Let G be a group, and suppose there are isomorphisms $\varphi_i : A_i \to B_i$ for $1 \le i \le n$, where A_i and B_i are subgroups of G. Define the *HNN extension* $G*_{\varphi_1,\dots,\varphi_n}$ with stable letters $t_1, \dots, t_n$ by

$$G*_{\varphi_1,\dots,\varphi_n} := (G * F_n)/\langle\langle\{t_i^{-1} a t_i \varphi_i(a^{-1}) \mid a \in A_i, 1 \le i \le n\}\rangle\rangle^{G*F_n},$$

where $\{t_1, \dots, t_n\}$ is a free generating set of F_n.

If $\varphi_i = \mathrm{id}_{A_i}$ for all $1 \le i \le n$, we write $G*_{A_1,\dots,A_n}$ for $G *_{\varphi_1,\dots,\varphi_n}$.

Definition 2.5. Let $G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension with stable letters $t_1,\dots,t_n$. Then a t_i-*pinch* is a word of the form $t_i^{-1}gt_i$ where $g\in A_i$ or $t_igt_i^{-1}$ where $g\in B_i$. A word w is said to be *reduced* if no subword of w is a t_i-pinch for any i.

Theorem 2.6 (Britton's lemma, [21, Theorem 11.81])**.** *Let $H = G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension with stable letters $t_1,\dots,t_n$, and let w be a word in H. If $w = e$ in H, then w contains a t_i-pinch as a subword, for some i.*

Corollary 2.7. *Let $G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension. Then G embeds into $G*_{\varphi_1,\dots,\varphi_n}$.*

Given a group G we write $\langle G;X|R\rangle$ to denote $(G*F_X)/\langle\langle R\rangle\rangle^{G*F_X}$, where R is any subset of $G*F_X$.

3. Good subgroups of HNN extensions

The notion of a good subgroup was introduced in [12, Proposition 1.34], and named so in [23, Definition 2].

Definition 3.1. Let $H = G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension. A *good subgroup* of G with respect to the HNN extension H is a subgroup $K\le G\le H$ such that $\varphi_i(K\cap A_i) = K\cap B_i$ for all $1\le i\le n$.

Lemma 3.2. [12, Proposition 1.34] *Let $H := G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension of G with stable letters $t_1,\dots,t_n$. Suppose $K\le G$ is a good subgroup of G with respect to the HNN extension H, and let $\psi_i : K\cap A_i\to K\cap B_i$ be the restriction of φ_i to $K\cap A_i$. Let K' be the subgroup of H generated by K and $t_1,\dots,t_n$. Then, the natural map*

$$\nu_K : K*_{\psi_1,\dots,\psi_n}\to K'$$

is an isomorphism. Moreover, $K'\cap G = K$.

We now study good subgroups which are normal.

Definition 3.3. Let $H := G*_{\varphi_1,\dots,\varphi_n}$ be an HNN extension of G with stable letters $t_1,\dots,t_n$. Let $K\trianglelefteq G$ be a good subgroup of G with respect to the HNN extension H. Let $\overline{\varphi}_i : A_i/(K\cap A_i)\to B_i/(K\cap B_i)$ be the induced isomorphism for each $1\le i\le n$. Define the following HNN extension with stable letters $\overline{t}_1,\dots,\overline{t}_n$:

$$H_K := (G/K)*_{\overline{\varphi}_1,\dots,\overline{\varphi}_n}$$

There is a surjective homomorphism

$$\phi_K : H\twoheadrightarrow H_K$$

which sends $g\mapsto gK$ for all $g\in G$, and $t_i\mapsto \overline{t}_i$ for all $1\le i\le n$.

Lemma 3.4. *Let G be a group, and $H := G*_{\varphi_1,\dots,\varphi_n}$ an HNN extension of G with stable letters $t_1,\dots,t_n$. Let $K\trianglelefteq G$. Then K is a good subgroup of G with respect to the HNN extension H if and only if $\langle\langle K\rangle\rangle^H\cap G = K$ in H.*

Proof.
$\Leftarrow$: Assume that $\langle\langle K\rangle\rangle^H\cap G = K$ in H. Take $1\le i\le n$ and suppose $x\in A_i\cap K$. We know that $\varphi_i(x)\in B_i$: we need to verify that $\varphi_i(x)\in K$. However, it is immediate that $\varphi_i(x) = t_i^{-1}xt_i\in\langle\langle K\rangle\rangle^H$, and thus that $\varphi_i(x)\in\langle\langle K\rangle\rangle^H\cap B_i =$

$\langle\!\langle K \rangle\!\rangle^H \cap G \cap B_i = K \cap B_i$; it follows that $\varphi_i(A_i \cap K) \subseteq B_i \cap K$. The inclusion $\varphi_i(A_i \cap K) \supseteq B_i \cap K$ can be proved in a similar fashion.
$\Rightarrow$: Suppose K is a good subgroup of G with respect to the HNN extension H, and take ϕ_K as in Definition 3.3. Then it is clear that $K \leq \langle\!\langle K \rangle\!\rangle^H \cap G \leq \ker(\phi_K) \cap G \leq K$; the last inequality here is a consequence of Theorem 2.6. □

Lemma 3.5. *Let H, K and ϕ_K be as in Definition* 3.3. *Then* $\ker(\phi_K) = \langle\!\langle K \rangle\!\rangle^H$.

Proof. The containment $\langle\!\langle K \rangle\!\rangle^H \subseteq \ker(\phi_K)$ is immediate.

Let $x \in \ker(\phi_K)$. We induct on the total number of occurrences of t_i or t_i^{-1} over all the i's, where $1 \leq i \leq n$, in the normal form of x in H: if x has none, then $x \in K$.

Assume that for some i, either t_i or t_i^{-1} appears at least once in x. By Britton's lemma $\phi_K(x)$ has a subword of the form $\bar{t}_i^{-1} a \bar{t}_i$ where $a \in A_i/(A_i \cap K)$ or $\bar{t}_i b \bar{t}_i^{-1}$ where $b \in B_i/(B \cap K)$. Thus x has a subword of the form $t_i^{-1} a' t_i$ where $a' \in A_i K$ or $t_i b' t^{-1}$ where $b \in B_i K$. Without loss of generality, we assume the former.

This subword $t_i^{-1} a' t$ is of the form $t_i^{-1} a k t_i$, where $a \in A_i$ and $k \in K$. But $t_i^{-1} a t_i = b \in B$, for some $b \in B$. We can therefore write x as $\lambda_1 t_i^{-1} a k t_i \lambda_2 = \lambda_1 b t_i^{-1} k t_i \lambda_2$. Observe that $t_i^{-1} k t_i \in \langle\!\langle K \rangle\!\rangle^H$, and thus that $t_i^{-1} k t_i \lambda_2 = \lambda_2 y$ where $y \in \langle\!\langle K \rangle\!\rangle^H$. We can therefore rewrite $x = \lambda_1 b \lambda_2 y$; from this we see that $\lambda_1 b \lambda_2 \in \ker(\phi_K)$. By induction, we have that $\lambda_1 b \lambda_2 \in \langle\!\langle K \rangle\!\rangle^H$. This tells us that $x \in \langle\!\langle K \rangle\!\rangle^H$. □

Corollary 3.6. *Let H, K and H_K be as in Definition* 3.3. *Then ϕ_K induces an isomorphism*

$$\overline{\phi}_K \;:\; H/\langle\!\langle K \rangle\!\rangle^H \xrightarrow{\cong} H_K.$$

4. The Higman embedding construction

The Higman Embedding Theorem states that a finitely generated, recursively presented group can be embedded in a finitely presented group. In this section we provide an overview of a proof of this result, introducing notation and constructions that will be used later in this paper.

4.1. **Modular machines and their connection to Turing machines.**

Modular machines are an alternative way of formalising the notion of mechanical computation: they simulate Turing machines in a very natural way using integers rather than tapes. This can often be useful in group theoretic applications; for example, there is a a proof of the Higman embedding theorem using modular machines (due to Aanderaa and Cohen and described in detail in Section 4.3) which is particularly transparent for the purposes of this paper.

Definition 4.1. A *modular machine* $\mathcal{M}$ consists of an integer $m > 1$ and a finite set of quadruples each of the form (a, b, c, R) or (a, b, c, L), where $m > a \geq 0$ and $m > b \geq 0$ and $m^2 > c \geq 0$. We require that, for each such pair (a, b), there is at most one quadruple of $\mathcal{M}$ of the form $(a, b, *, *)$.
A *modular machine configuration* is an ordered pair $(\alpha, \beta) \in \mathbb{N}^2$. We write $(\alpha, \beta) \underset{\mathcal{M}}{\longrightarrow} (\alpha_1, \beta_1)$, called a *computational step* of $\mathcal{M}$, if $\alpha = um + a$ and $\beta = vm + b$

(with $0 \leq a, b < m$) and there exists c such that either:
1. $(a, b, c, R) \in \mathcal{M}$ and $\alpha_1 = um^2 + c$ and $\beta_1 = v$, or
2. $(a, b, c, L) \in \mathcal{M}$ and $\alpha_1 = u$ and $\beta_1 = vm^2 + c$.

Note that the action of $\mathcal{M}$ on (α, β) depends only on the class of (α, β) modulo m. This is why we call $\mathcal{M}$ a *modular* machine.
We write $(\alpha, \beta) \xrightarrow[\mathcal{M}]{*} (\alpha', \beta')$ if there exists a finite sequence

$$(\alpha, \beta) = (\alpha_1, \beta_1) \xrightarrow[\mathcal{M}]{} (\alpha_2, \beta_2) \xrightarrow[\mathcal{M}]{} \dots \xrightarrow[\mathcal{M}]{} (\alpha_n, \beta_n) = (\alpha', \beta')$$

Such a sequence is called a *computation* of $\mathcal{M}$.
If, for $\alpha = um + a$, $\beta = vm + b$ $(0 \leq a, b < m)$, no quadruple of $\mathcal{M}$ begins with (a, b), then we say (α, β) is *terminal*. If $(0, 0)$ is terminal in $\mathcal{M}$, then we define the *halting set of* $\mathcal{M}$, denoted $H_0(\mathcal{M})$, by

$$H_0(\mathcal{M}) := \{(\alpha, \beta) \mid (\alpha, \beta) \xrightarrow[\mathcal{M}]{*} (0, 0)\}$$

The following result by Aanderaa and Cohen [1] (paraphrased), along with an analysis of its proof, shows that for each Turing machine T there is a modular machine $\mathcal{M}(T)$ which simulates the action of T and conversely, that any modular machine can be simulated by a Turing machine. Thus these two notions of computation are equivalent. One can find a more detailed discussion of this material in [1].

Theorem 4.2 ([1, Theorem 2])**.** *Let T be a Turing machine. Then, from T, we can construct a modular machine $\mathcal{M}(T)$ whose halting set $H_0(\mathcal{M}(T))$ is computationally equivalent to the halting set $\Omega(T)$ of T. Stated formally: $\Omega(T) \equiv_m H_0(\mathcal{M}(T))$.*

4.2. **Simulating a modular machine by a finitely presented group.**

We begin by describing how a modular machine can be simulated by a finitely presented group. This construction is then used in a proof of the Higman Embedding Theorem.

The idea is to follow the construction in [12, pp.266–268]. This was derived from [1], where a detailed exposition of modular machines can be found. We felt, however, that the exposition in [23] was slightly clearer, so we replicate here the argument presented there (the differences are only slight).

(1) Define the group $K := \langle x, y, t \mid [x, y] = e \rangle$.
(2) For all $(r, s) \in \mathbb{Z}^2$, define the word $t(r, s) := y^{-s} x^{-r} t x^r y^s \in K$.
(3) Let $T := \langle \{t(r, s)\}_{(r,s) \in \mathbb{Z}^2} \rangle \leq K$.
(4) Observe that T is free with basis $\{t(r, s)\}_{(r,s) \in \mathbb{Z}^2}$.
(5) Observe that $T = \langle\langle t \rangle\rangle^K$.
(6) For $M > a \geq 0$, $N > b \geq 0$, define

$$K_{a,b}^{M,N} := \langle t(a, b), x^M, y^N \rangle \leq K,$$

$$T_{a,b}^{M,N} := \langle \{t(\alpha, \beta) \mid \alpha \equiv a \mod M,\ \beta \equiv b \mod N\} \rangle \leq T \leq K.$$

(7) Let $(i, j) \in \mathbb{Z}^2$, and $m, n \in \mathbb{Z}$. Observe that $T \cap \langle t(i, j), x^m, y^n \rangle$ is free with basis $\{t(r, s) \mid r \equiv i \mod m,\ s \equiv j \mod n\}$. In particular,

$$T \cap K_{a,b}^{M,N} = T_{a,b}^{M,N}.$$

(8) Observe that the correspondence $t \mapsto t(a,b)$, $x \mapsto x^M$, $y \mapsto y^N$ induces an isomorphism
$$K \to K_{a,b}^{M,N}.$$
This isomorphism sends $t(u,v)$ to $t(uM+a, vN+b)$ and thus induces an isomorphism
$$T \to T_{a,b}^{M,N}.$$
(9) Let $\mathcal{M} = \{(a_i, b_i, c_i, R) \mid i \in I\} \cup \{(a_j, b_j, c_j, L) \mid j \in J\}$ be a modular machine with modulus m.
(10) The maps in step (8) induce, for each $i \in I$ and $j \in J$, isomorphisms
$$\phi_i : K_{a_i,b_i}^{m,m} \to K_{c_i,0}^{m^2,1},$$
$$\varphi_j : K_{a_j,b_j}^{m,m} \to K_{0,c_j}^{1,m^2}.$$
(11) Form the HNN extension
$$K_{\mathcal{M}} := K *_{\{\phi_i\}_{i\in I}, \{\varphi_j\}_{j\in J}},$$
with stable letters $\{r_i\}_{i\in I}$ and $\{l_j\}_{j\in J}$. Note that $K_{\mathcal{M}}$ is finitely presented.
(12) Define the subgroup $T' := \langle T, \{r_i\}_{i\in I}, \{l_j\}_{j\in J}\rangle \leq K_{\mathcal{M}}$, where T is as in step (3).
(13) Define the set $H_0(\mathcal{M}) := \{(\alpha,\beta) \mid (\alpha,\beta) \xrightarrow[\mathcal{M}]{*} (0,0)\}$.
(14) Define $T_{\mathcal{M}} := \langle\{t(\alpha,\beta) \mid (\alpha,\beta) \in H_0(\mathcal{M})\}\rangle \leq K$.
(15) Define $T'_{\mathcal{M}} := \langle T_{\mathcal{M}}, \{r_i\}_{i\in I}, \{l_j\}_{j\in J}\rangle \leq K_{\mathcal{M}}$.
(16) Observe that $T'_{\mathcal{M}} = \langle t, \{r_i\}_{i\in I}, \{l_j\}_{j\in J}\rangle$.
(17) Observe that $t(\alpha,\beta) \in T'_{\mathcal{M}}$ iff $(\alpha,\beta) \in H_0(\mathcal{M})$.
(18) With the identity map $\theta : T'_{\mathcal{M}} \to T'_{\mathcal{M}}$, form the HNN extension
$$G_{\mathcal{M}} := K_{\mathcal{M}} *_\theta$$
with stable letter q.
(19) Observe that $q^{-1}t(\alpha,\beta)q = t(\alpha,\beta)$ in $G_{\mathcal{M}}$ iff $(\alpha,\beta) \in H_0(\mathcal{M})$.

Taking $\mathcal{M}'$ with nonrecursive halting set $H_0(\mathcal{M}')$ gives a finitely presented group $G_{\mathcal{M}'}$ with undecidable word problem.

For our purposes, a useful consequence of the above construction is that we can simulate any modular machine by a finitely generated group: see step (19) of Construction 4.2.

4.3. **The Higman Embedding Theorem.**

We now give an overview of the construction used in a particular proof of the Higman Embedding Theorem, taken directly from [12, pp.279–281]. We note that this proof originally comes from [2].

(1) Let $C = \langle c_1, \dots, c_n \mid S \rangle$ be a finitely generated recursively presented group, where S corresponds to the set $H_0(\mathcal{M})$ of a modular machine $\mathcal{M}$; see step (7). Denote the modulus of $\mathcal{M}$ by m. We assume that S covers *all* the trivial words in the group.
(2) Re-write every word in C as a word in the free monoid on $\{c_1, \dots, c_{2n}\}$ with c_i^{-1} replaced by c_{n+i}.

(3) To each word $w = c_{i_k}c_{i_{k-1}}\cdots c_{i_0}$ associate an m-ary representation $\alpha = \Sigma_{j=0}^{k} i_j m^j$.
(4) Define $I := \{\alpha \in \mathbb{N} \mid \alpha \text{ represents a word}\}$. That is, $\alpha = \Sigma_{j=0}^{k}\beta_j m^j$ where $1 \leq \beta_j \leq 2n$.
(5) For $\alpha \in I$, define $w_\alpha(c)$ to be the word formed from α.
(6) For $\alpha \in I$, write $w_\alpha(b)$, $w_\alpha(bc)$ for the words obtained from $w_\alpha(c)$ by replacing c_i with b_i and $b_i c_i$ respectively (where $\{b_i\}_{i=1}^{2n}$ are a new set of symbols).
(7) Observe that, for all $\alpha \in I$, we have that $w_\alpha(c) \in S$ iff $(\alpha, 0) \in H_0(M)$.
(8) Recall the group $K_{\mathcal{M}}$ from step (11) of 4.2. Define $U := \{t, \{r_i\}_{i\in I}, \{l_j\}_{j\in J}\}$; U is a subset of $K_{\mathcal{M}}$.
(9) Define $t_\alpha := t(\alpha, 0) \in K_{\mathcal{M}}$.
(10) Form the free product

$$H_1 := K_{\mathcal{M}} * (C \times \langle b_1, \dots, b_n | - \rangle) * \langle d | - \rangle,$$

and set $b_{n+i} := b_i^{-1}$ for $1 \leq i \leq n$.
(11) Observe that the sets $\{t_\alpha \mid \alpha \in I\}$ and $\{t_\alpha w_\alpha(b) d \mid \alpha \in I\}$ each form a free basis for the subgroups they respectively generate in H_1. The correspondence $t_\alpha \mapsto t_\alpha w_\alpha(b) d$ extends to an isomorphism ψ between these subgroups.
(12) Form the HNN extension

$$H_2 := H_1 *_\psi$$

with stable letter p.
(13) Define the subgroup

$$A := \langle t, x, d, b_1, \dots, b_n, p \rangle \leq H_2.$$

(14) For $1 \leq i \leq 2n$, define the subgroup

$$A_i := \langle t_i, x^m, b_i d, b_1, \dots, b_n, p \rangle \leq H_2.$$

(15) Observe that for all i, A is isomorphic to A_i via the map ψ_i induced by the correspondence sending $t \mapsto t_i$, $x \mapsto x^m$, $d \mapsto b_i d$, $b_j \mapsto b_j$ for all $1 \leq j \leq n$, and $p \mapsto p$.
(16) Observe that $\langle t, x, d, b_1, \dots, b_n \rangle$ and $\langle t_i, x^m, b_i d, b_1, \dots, b_n \rangle$ are both good in H_1 with respect to the HNN extension H_2. Therefore A, and the A_i for $1 \leq i \leq 2n$, are all HNN extensions.
(17) Define the subgroup

$$A_+ := \langle U, d, b_1, \dots, b_n, p \rangle \leq H_2.$$

(18) Define the subgroup

$$A_- := \langle U, d, b_1 c_1, \dots, b_n c_n, p \rangle \leq H_2.$$

(19) Observe that $\langle U, d, b_1, \dots, b_n \rangle$ is good in H_1 with respect to the HNN extension H_2. Therefore A_+ is an HNN extension.
(20) Observe that A_+ is isomorphic to A_- via the map $\psi_+ : A_+ \to A_-$ induced by the correspondence sending $u \mapsto u$ for all $u \in U$, $d \mapsto d$, $b_j \mapsto b_j c_j$ for all $1 \leq j \leq n$, and $p \mapsto p$.

(21) With the isomorphisms defined above, define the HNN extension

$$H_3 := H_2 *_{\psi_1, \dots, \psi_{2n}, \psi_+},$$

with stable letters $a_1, \dots, a_{2n}$ and k.

(22) Observe that H_3 is finitely presented, and $C \hookrightarrow H_3$.

5. Properties of the embedding construction

In this section, the groups C, H_1, H_2 and H_3 will be as in Section 4.3.

Lemma 5.1. *Let X be a subset of C. Then*

(1) $\langle\langle X \rangle\rangle^{H_1}$ is good in H_1 with respect to the HNN extension H_2.
(2) $\langle\langle X \rangle\rangle^{H_2}$ is good in H_2 with respect to the HNN extension H_3.

Proof. We claim that the following is true:

$$\langle\langle X \rangle\rangle^{H_1} \cap \langle \{ t_\alpha \mid \alpha \in I \} \rangle = \{e\},$$
$$\langle\langle X \rangle\rangle^{H_1} \cap \langle \{ t_\alpha w_\alpha(b) d \mid \alpha \in I \} \rangle = \{e\}.$$

To see this, consider the map $\lambda : H_1 \to H_1$ induced by the identity maps on $K_{\mathcal{M}}$, $\langle b_1, \dots, b_n | - \rangle$, and $\langle d | - \rangle$, and the trivial map on C.

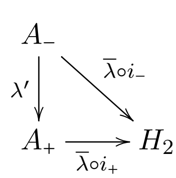
The map λ, restricted to $K_M * (\{e\} \times \langle b_1, \dots, b_n | - \rangle) * \langle d | - \rangle$, is injective, and thus injective on both $\langle \{ t_\alpha \mid \alpha \in I \} \rangle$ and $\langle \{ t_\alpha w_\alpha(b) d \mid \alpha \in I \} \rangle$. However, $\langle\langle X \rangle\rangle^{H_1}$ is contained in $\ker(\lambda)$.

This proves the first part of the lemma; we now move to the second.

Take the map λ defined above. It is clear that λ extends to a map $\overline{\lambda} : H_2 \to H_2$, sending $p \mapsto p$. Again, $\langle\langle X \rangle\rangle^{H_2} \le \ker(\overline{\lambda})$. As before, we see that the restriction of $\overline{\lambda}$ to $\langle K_M * (\{e\} \times \langle b_1, \dots, b_n | - \rangle) * \langle d | - \rangle, p \rangle$ is injective. It follows that $\langle\langle X \rangle\rangle^{H_2} \cap A = \langle\langle X \rangle\rangle^{H_2} \cap A_i = \{e\}$ for all $1 \le i \le 2n$.
Finally, consider the inclusions

$$\iota_- : A_- \to H_2$$
$$\iota_+ : A_+ \to H_2$$

Step (20) of 4.3 tells us that the restriction of λ to A_- is injective with image A_+, and thus induces an isomorphism $\lambda' : A_- \to A_+$; λ' is inverse to the map ψ_+ defined in (15) of 4.3. We see that $\overline{\lambda} \circ \iota_+ \circ \lambda' = \overline{\lambda} \circ \iota_- : A_- \to H_2$:



It is clear that $\overline{\lambda} \circ \iota_+$ is injective, and thus that $\overline{\lambda} \circ \iota_-$ is as well. Since $\langle\langle X \rangle\rangle^{H_2}$ is contained in $\ker(\overline{\lambda})$, we see that $\langle\langle X \rangle\rangle^{H_2} \cap A_- = \langle\langle X \rangle\rangle^{H_2} \cap A_+ = \{e\}$. This proves the last part of the lemma. □

Before we proceed, we need the following observation. It is proved in the same way that [10, Corollary 2.9] is, by using the torsion theorem for HNN extensions.

Lemma 5.2. *Let G be a group, and $\varphi : H \to K$ an isomorphism between subgroups $H, K \leq G$. Let $G*_\varphi$ be the associated HNN extension. Then*

$$\mathrm{Tor}_1(G*_\varphi) = \langle\langle \mathrm{Tor}_1(G) \rangle\rangle^{G*_\varphi} = \langle\langle \mathrm{Tor}(G) \rangle\rangle^{G*_\varphi}.$$

Lemma 5.3. *For all $m \geq 0$, the following hold:*

(1) $\mathrm{Tor}_m(H_1) = \langle\langle \mathrm{Tor}_m(C) \rangle\rangle^{H_1}$.
(2) $\mathrm{Tor}_m(H_1) \cap C = \mathrm{Tor}_m(C)$.

Proof.
By [10, Proposition 2.10], we know that

$$\mathrm{Tor}_m(H_1) = \langle\langle \mathrm{Tor}_m(K_{\mathcal{M}}) \cup \mathrm{Tor}_m(C \times \langle b_1, \dots, b_n | - \rangle) \cup \mathrm{Tor}_m(\langle d | - \rangle) \rangle\rangle^{H_1}.$$

However, $K_{\mathcal{M}}$, $\langle b_1, \dots, b_n | - \rangle$ and $\langle d | - \rangle$ are all torsion-free. It follows that

$$\mathrm{Tor}_m(H_1) = \langle\langle \mathrm{Tor}_m(C) \rangle\rangle^{H_1}$$

for all m. This proves part (1). For the second part, observe that there is a map $\mu : H_1 \to C$ induced by the trivial map on $K_{\mathcal{M}}$ and $\langle d | - \rangle$ and the standard projection to C on $C \times \langle b_1, \dots, b_n | - \rangle$. The map μ restricts to the identity on C and sends $\mathrm{Tor}_m(H_1)$ to $\mathrm{Tor}_m(C)$. The result follows. □

Lemma 5.4. *For $i = 1, 2$, and for all $m \geq 0$, the following hold:*

(1) $\mathrm{Tor}_m(H_{i+1}) = \langle\langle \mathrm{Tor}_m(H_i) \rangle\rangle^{H_{i+1}}$.
(2) $\mathrm{Tor}_m(H_i)$ *is good in H_i with respect to the HNN extension H_{i+1}.*

Proof. We prove this by induction on m. The result is obvious for $m = 0$.

We now come to the inductive step. Let $i \in \{1, 2\}$. Assume the statement is true for m. The induction hypothesis tells us that $\mathrm{Tor}_m(H_{i+1}) = \langle\langle \mathrm{Tor}_m(H_i) \rangle\rangle^{H_{i+1}}$ and that $\mathrm{Tor}_m(H_i)$ is good in H_i with respect to the HNN extension H_{i+1}. Thus, by Lemma 3.4,

$$\langle\langle \mathrm{Tor}_m(H_i) \rangle\rangle^{H_{i+1}} \cap H_i = \mathrm{Tor}_m(H_i).$$

Combining these facts, we see that $\mathrm{Tor}_m(H_{i+1}) \cap H_i = \mathrm{Tor}_m(H_i)$. As a consequence, the inclusion $H_i \to H_{i+1}$ induces an embedding

$$H_i / \mathrm{Tor}_m(H_i) \to H_{i+1} / \mathrm{Tor}_m(H_{i+1});$$

via this, we identify $H_i / \mathrm{Tor}_m(H_i)$ as a subgroup of $H_{i+1} / \mathrm{Tor}_m(H_{i+1})$.

Using Lemma 3.5 and Corollary 3.6, we see that

$$H_{i+1} / \mathrm{Tor}_m(H_{i+1}) = H_{i+1} / \langle\langle \mathrm{Tor}_m(H_i) \rangle\rangle^{H_{i+1}} \cong \langle H_i / \mathrm{Tor}_m(H_i); \mathrm{stable}_i \mid \mathrm{relations}_i \rangle$$

where stable_i, $\mathrm{relations}_i$ are, respectively, the stable letters and relations of the HNN construction of H_{i+1} from H_i. It then follows from Lemma 5.2 that

$$\mathrm{Tor}_1(H_{i+1} / \mathrm{Tor}_m(H_{i+1})) = \langle\langle \mathrm{Tor}_1(H_i / \mathrm{Tor}_m(H_i)) \rangle\rangle^{H_{i+1} / \mathrm{Tor}_m(H_{i+1})}.$$

The preimage of $\mathrm{Tor}_1(H_{i+1} / \mathrm{Tor}_m(H_{i+1}))$ in H_{i+1} is $\mathrm{Tor}_{m+1}(H_{i+1})$, and the preimage of $\langle\langle \mathrm{Tor}_1(H_i / \mathrm{Tor}_m(H_i)) \rangle\rangle^{H_{i+1} / \mathrm{Tor}_m(H_{i+1})}$ in H_{i+1} is $\langle\langle \mathrm{Tor}_{m+1}(H_i) \rangle\rangle^{H_{i+1}}$. Thus $\mathrm{Tor}_{m+1}(H_{i+1}) = \langle\langle \mathrm{Tor}_{m+1}(H_i) \rangle\rangle^{H_{i+1}}$, and (1) is proved for the case $m + 1$.

We have just proved that (1) is true for $m + 1$; combining this fact with Lemma 5.3 (1), we see that

$$\mathrm{Tor}_{m+1}(H_i) = \langle\langle \mathrm{Tor}_{m+1}(H_{i-1}) \rangle\rangle^{H_i} = \dots = \langle\langle \mathrm{Tor}_{m+1}(H_1) \rangle\rangle^{H_i} = \langle\langle \mathrm{Tor}_{m+1}(C) \rangle\rangle^{H_i}.$$

Lemma 5.1 then tells us that $\mathrm{Tor}_{m+1}(H_i)$ is good in H_i with respect to the HNN extension H_{i+1}. □

The next corollary now follows from Lemmas 5.3, 5.4 and 3.4:

Corollary 5.5. *For $i = 1, 2, 3$, and for all $m \geq 0$, the following hold:*

(1) $\mathrm{Tor}_m(H_i) = \langle\langle \mathrm{Tor}_m(C) \rangle\rangle^{H_i}$.
(2) $\mathrm{Tor}_m(H_i) \cap C = \mathrm{Tor}_m(C)$.

Theorem 5.6. *There is a uniform construction that, on input of a recursive presentation of a group C, outputs a finite presentation of a group H in which C embeds, with* $\mathrm{TorLen}(C) = \mathrm{TorLen}(H)$.

Proof. This is an immediate consequence of Corollary 5.5, taking $H = H_3$. As

$$\langle\langle \mathrm{Tor}_m(C) \rangle\rangle^{H_3} = \mathrm{Tor}_m(H_3)$$

for all m, $\mathrm{Tor}_m(C) = \mathrm{Tor}_{m+1}(C)$ implies that $\mathrm{Tor}_m(H_3) = \mathrm{Tor}_{m+1}(H_3)$.

Conversely, since

$$\mathrm{Tor}_m(H_3) \cap C = \mathrm{Tor}_m(C)$$

for all m, $\mathrm{Tor}_m(H_3) = \mathrm{Tor}_{m+1}(H_3)$ implies that $\mathrm{Tor}_m(C) = \mathrm{Tor}_{m+1}(C)$.

In conclusion, $\mathrm{Tor}_m(H_3) = \mathrm{Tor}_{m+1}(H_3)$ if and only if $\mathrm{Tor}_m(C) = \mathrm{Tor}_{m+1}(C)$, for any m. Thus the sequences $\mathrm{Tor}_j(H_3)$ and $\mathrm{Tor}_j(C)$ stabilise at precisely the same value of j (if at all), and so $\mathrm{TorLen}(H_3) = \mathrm{TorLen}(C)$. □

Theorem 5.7. *There exists a finitely presented group F with* $\mathrm{TorLen}(F) = \omega$.

Proof. In [10, Theorem 3.10], we proved that there is a 2-generator, recursively presented group with infinite torsion length. We now apply Theorem 5.6. □

An interesting exercise would be to construct an explicit finite presentation of such a group. Theoretically, this could be done by carefully following the constructions given above. The presentation that arises as the output of such a process, however, would undoubtedly be very complicated. A more straightforward presentation, perhaps giving a group that arises elsewhere in the literature, would be interesting.

6. A word-hyperbolic virtually special example

We now show various ways of constructing finitely presented virtually special groups with infinite torsion length. We thank Henry Wilton for initially suggesting that this is possible and pointing out an alternate way to prove it.

Definition 6.1. Let Γ be an undirected graph on finite vertex set labeled $1, \ldots, n$, and edge set E. The *right-angled Artin group* (RAAG), $A(\Gamma)$, associated to Γ is the group with presentation

$$\langle x_1, \ldots, x_n \mid [x_i, x_j] \ \forall \ \{i, j\} \in E \rangle.$$

A group G is said to be *special* if it is a subgroup of some RAAG. More generally, a group G is said to be *virtually special* if it contains a finite index subgroup which is special.

Every RAAG on n generators can be seen as an HNN extension of a RAAG on $n-1$ generators; it follows that every RAAG is torsion-free, and thus that virtually special groups are virtually torsion-free.

For the remainder of this section, if $P = \langle X|R\rangle$ is a group presentation we denote by $\overline{P}$ the group presented by P, and if $w \in X^*$ is a word in the generators of P then we denote by $\overline{w}$ the element of $\overline{P}$ represented by w.

Definition 6.2. Let $P = \langle X|R\rangle$ be a presentation where each $r \in R$ is freely reduced and cyclically reduced (as a word in X^*), and where R is *symmetrised* (i.e., closed under taking cyclic permutations and inverses).
A nonempty freely reduced word $w \in X^*$ is called a *piece* with respect to P if there exist two distinct elements $r_1, r_2 \in R$ that have w as maximal common initial segment.
Let $0 < \lambda < 1$. Then P is said to satisfy the $C'(\lambda)$ *small cancellation condition* if whenever w is a piece with respect to P and w is a subword of some $r \in R$, then $|u| < \lambda|r|$ as words.
A group is called a $C'(\lambda)$ group if it can be presented by a presentation satisfying the $C'(\lambda)$ small cancellation condition.

If $P = \langle X|R\rangle$ is a presentation of a group G where R is not symmetrised, we can take the symmetrised closure R_{sym} of R, where R_{sym} consists of all cyclic permutations of words in R and R^{-1} (with repetitions removed). Then R_{sym} is symmetrised and $P_{sym} = \langle X|R_{sym}\rangle$ is also a presentation of G. In a slight abuse of notation, we call the presentation $P = \langle X|R\rangle$ symmetrised if R is symmetrised. Observe that if R is finite, then so is R_{sym}.

The following theorem is a consequence of the substantial results of Agol [3] and Wise [24].

Theorem 6.3. *Let $P = \langle X|R\rangle$ be a finite presentation satisfying the $C'(1/6)$ small cancellation condition. Then $\overline{P}$ is both word-hyperbolic and virtually special.*

Proof. Finitely presented $C'(1/6)$ groups are known to be word-hyperbolic (see [13, 14]). One then uses [24, Theorem 1.2] and [3, Theorem 1.1] to show that $\overline{P}$ is virtually special. □

Proposition 6.4. *Let $P = \langle X \mid R = \{r_1, r_2, \dots\}\rangle$ be a presentation, with all words in R freely reduced, cyclically reduced, and distinct. For any $k \in \mathbb{N}$, define the presentation*

$$P_t^k := \langle X, t \mid t^k, (r_1 t)^k, (r_2 t)^k, \dots\rangle$$

Then $(P_t^k)_{sym}$ is symmetrised and satisfies the $C'(2/k)$ small cancellation condition. If P is finite, then $\overline{P_t^k}$ is word-hyperbolic and virtually special for all $k \geq 12$.

Proof. Let $S = \{t^k, (r_1 t)^k, (r_2 t)^k, \dots\}$. We first need to check that every $s \in S_{sym}$ is freely and cyclically reduced. But this follows from the fact that R is freely and cyclically reduced, along with the strategic placements of the t's. By definition, S_{sym} is symmetrised. We now show that S_{sym} satisfies the $C'(2/k)$ small cancellation condition.

1. Any cyclic permutation of $(r_i t)^k$ shares a piece with t^k of length at most one (and no piece with t^{-k}). Similarly, any cyclic permutation of $(r_i t)^{-k}$ shares a piece with t^{-k} of length at most one (and no piece with t^k). Such pieces have length at most $1/k$ of either word.
2. Consider shared pieces of cyclic permutations of pairs of words of the form $(r_i t)^k$ and $(r_j t)^k$. If $r_i = ab$ and $r_j = cd$, where a, b, c, d are words, then we are left with considering words of the form $bt(r_i t)^{k-1}a$ and $dt(r_j t)^{k-1}c$ respectively. As $r_i \neq r_j$, the initial overlap of these can be at most $bt \equiv dt$, followed by some overlap of r_i and r_j (of length at most $\min\{|r_i|, |r_j|\}$, as $r_i \neq r_j$ and t is acting as an end marker). So this initial overlap can have length at most $\min\{2|r_i|, 2|r_j|\} + 1$ which is less than $2/k$ of the length of either word.
3. By repeating the arguments as in step 2, we can show that cyclic permutations of pairs of words of the form $(r_i t)^{-k}$ and $(r_j t)^{-k}$ overlap at most $2/k$ of the length of either word.
4. We now consider shared pieces of cyclic permutations of pairs of words of the form $(r_i t)^k$ and $(r_j t)^{-k} = (t^{-1} r_j^{-1})^k$. If $r_i = ab$ and $r_j = cd$, where a, b, c, d are words, then the words we are considering must be of the form $bt(r_i t)^{k-1}a$ and $c^{-1}(t^{-1}r_j^{-1})^{(k-1)}t^{-1}d^{-1}$ respectively. An initial overlap cannot involve t or t^{-1}, and thus has length at most $\min\{|r_i|, |r_j|\}$; this is less than $1/k$ of the length of either word.

If follows that $(P_t^k)_{sym}$ satisfies the $C'(2/k)$ small cancellation condition; in the case where P is finite, we appeal to Theorem 6.3 to finish the proof. □

The following standard result was first proved in [13]; see [22, Theorem 6] for an explicit statement of the result.

Lemma 6.5 ([13, Theorem VIII]). *Let $P = \langle X | R \rangle$ satisfy the $C'(1/6)$ small cancellation condition. Then an element $g \in \overline{P}$ has order $n > 1$ if and only if there is a relator $r \in R$ of the form $r = s^n$ in X^*, with $s \in X^*$, such that g is conjugate to $\overline{s}$ in $\overline{P}$.*

Lemma 6.6. *Let $P = \langle X \mid R = \{r_1, r_2, \ldots\} \rangle$ be a presentation, with all words in R freely reduced, cyclically reduced, and distinct. Let P_t^k be as before, with $k \geq 12$. Then*

$$\mathrm{Tor}_1(\overline{P_t^k}) = \langle\!\langle \overline{t}, \overline{r_1 t}, \overline{r_2 t}, \ldots \rangle\!\rangle^{\overline{P_t^k}}$$

Proof. (In this proof, we take all normal closures to be in $\overline{P_t^k}$.)

Clearly $\{\overline{t}, \overline{r_1 t}, \overline{r_2 t}, \ldots\}$ are all torsion elements in $\overline{P_t^k}$, and so we have that $\langle\!\langle \overline{t}, \overline{r_1 t}, \overline{r_2 t}, \ldots \rangle\!\rangle \subseteq \mathrm{Tor}_1(\overline{P_t^k})$.

To show the converse, it suffices to show that $\mathrm{Tor}(\overline{P_t^k}) \subseteq \langle\!\langle \overline{t}, \overline{r_1 t}, \overline{r_2 t}, \ldots \rangle\!\rangle$. So take some torsion element $g \in \mathrm{Tor}(\overline{P_t^k})$ with $\mathrm{o}(g) = n$. By Proposition 6.4, $(P_t^k)_{sym}$ satisfies the $C'(1/6)$ small cancellation condition; thus, by Lemma 6.5 g is conjugate to some $\overline{s}$ with $s^n = r$ for some relator r of $(P_t^k)_{sym}$. If $r = t^k$ or t^{-k} then s is a power of t and so g is conjugated into $\langle\!\langle \overline{t}, \overline{r_1 t}, \overline{r_2 t}, \ldots \rangle\!\rangle$.
Otherwise, s^n is equal to some cyclic permutation of some $(r_i t)^k$ or $(r_i t)^{-k}$; it is enough to just consider the first case. Then, there is some cyclic permutation q of s such that $q^n = (r_i t)^k$ as words in X^*.

What word q can we have which satisfies $(r_it)^k = q^n$? If $|q| < |r_i|$, then q contains no t and thus q^n contains no t; a contradiction. If $|q| = |r_it|$ then $q = r_it$ and so g is conjugate to $\overline{q} = \overline{r_it}$ which clearly lies in $\langle\langle \overline{t}, \overline{r_1t}, \overline{r_2t}, \dots \rangle\rangle$. If $|q| > |r_it|$ then $q = (r_it)^z a$, where $r_i = ab$ is a decomposition of r_i. Thus $q^n = ((r_it)^z a)^n$, and this can only be equal to $(r_it)^k$ if $a = \varnothing$. In this case $q = (r_it)^z$ for some z, and so g is conjugate to $\overline{q} = \overline{(r_it)^z}$ which lies in $\langle\langle \overline{t}, \overline{r_1t}, \overline{r_2t}, \dots \rangle\rangle$. Thus $\mathrm{Tor}(\overline{P_t^k}) \subseteq \langle\langle \overline{t}, \overline{r_1t}, \overline{r_2t}, \dots \rangle\rangle$. □

Theorem 6.7. *Let $P = \langle X|R \rangle$ be a finite presentation with all words in R freely reduced, cyclically reduced, and distinct. Then, for any $k \geq 12$, $\overline{P_t^k}$ is word-hyperbolic, virtually special, and satisfies*

$$\overline{P_t^k} / \mathrm{Tor}_1(\overline{P_t^k}) \cong \overline{P}.$$

Thus, in this case, $\mathrm{TorLen}(\overline{P_t^k}) = \mathrm{TorLen}(\overline{P}) + 1$.

Proof. This follows immediately from Proposition 6.4 and Lemma 6.6. □

Remark 6.8. One may ask why the introduction of the extra generator t is necessary when constructing P_t^k. It is indeed true that, given a finite presentation $P = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$, the finite presentation $Q = \langle x_1, \dots, x_m \mid r_1^{k_1}, \dots, r_n^{k_n} \rangle$ (where $k_1, \dots, k_n \in \mathbb{N}_{\geq 1}$) presents a group $\overline{Q}$ with $\mathrm{TorLen}(\overline{Q}) - 1 \leq \mathrm{TorLen}(\overline{P}) \leq \mathrm{TorLen}(\overline{Q})$. The reader can easily verify this. However, it is not necessarily the case that $\overline{Q} / \mathrm{Tor}_1(\overline{Q}) \cong \overline{P}$. As an example, we can consider $P = \langle x, y, z | x, y^3, xy = z^3 \rangle$ and $Q = \langle x, y, z | x^3, y^3, xy = z^3 \rangle$. It is clear that P is just a presentation for the cyclic group with 9 elements, C_9. On the other hand, by [10, Proposition 3.1], $\overline{Q} / \mathrm{Tor}_1(\overline{Q}) \cong C_3$.

Remark 6.9. Every finitely generated $C'(1/6)$ group has been shown in [15, Corollary 1.4] to be *acylindrically hyperbolic* (this notion was first defined in [19]). Using this, the assumption that P is finite can be relaxed in Theorem 6.7 if we allow ourselves a slightly weaker conclusion. If we continue to assume that X is finite while no longer requiring R to be so, then [15, Corollary 1.4] implies - assuming the notation of Theorem 6.7 - that $\overline{P_t^k}$ is acylindrically hyperbolic. It still follows from 6.6 that $\overline{P_t^k} / \mathrm{Tor}_1(\overline{P_t^k}) \cong \overline{P}$. We thank the anonymous referee for bringing this to our attention.

Theorem 6.10. *There is a finitely presented word-hyperbolic virtually special group W with* $\mathrm{TorLen}(W) = \omega$. *In particular, W is virtually torsion-free.*

Proof. Let P be a finite presentation of a group with infinite torsion length; such things exist, by Theorem 5.7. Then, by Theorem 6.7, $\overline{P_t^{12}}$ is hyperbolic and virtually special, and $\mathrm{TorLen}(\overline{P_t^{12}}) = \mathrm{TorLen}(\overline{P}) + 1 = \omega$. Take $W = \overline{P_t^{12}}$. □

Remark 6.11. The main construction in [7] can be used to obtain a similar result to Theorem 6.7 above. Given a finite presentation $P = \langle A|R \rangle$, we see in equation (4) of [7, pp. 141] an explicit finite presentation of a $C'(1/6)$ group G and $N \triangleleft G$ such that $G/N \cong \overline{P}$, and moreover that $\overline{P}$ is isomorphic to $\mathrm{Out}(N)$ ([7, Theorem 11]). Further analysis shows that $N = \mathrm{Tor}_1(G)$, normally generated by 2 elements. However, both the finite presentation of G in [7] and its manner of construction seem to be substantially more complicated than the finite presentation P_t^{12} constructed above

Remark 6.12. In [10, Lemma 2.3] we showed that $\mathrm{Tor}_i(H) \leq \mathrm{Tor}_i(G)$ whenever $H \leq G$. However, this does not extend to bounding torsion length of subgroups, even for finitely presented groups. Using the fact that there are finitely presented groups of any torsion length ([10, Theorem 3.3]), including ω (Theorem 5.7), along with the Adian-Rabin construction ([8, Theorem 2.4]), one can show that given any finitely presented group H, and any ordinal $1 \leq n \leq \omega$, there is a finitely presented group of torsion length n into which H embeds.

We finish this section with an alternate construction for, and strengthening of, the result obtained as [10, Theorem 3.3].

Theorem 6.13. *Define the sequence of finite presentations* $P_0 := \langle - | - \rangle$, $P_1 := \langle t_1 \mid t_1^{12} \rangle$, $P_2 := \langle t_1, t_2 \mid (t_1^{12} t_2)^{12}, t_2^{12} \rangle$, *and, in general*

$$P_n := (P_{n-1})_{t_n}^{12} = \langle t_1, \dots, t_n \mid (\cdots (t_1^{12} t_2)^{12}) \cdots t_n)^{12}, \dots, (t_{n-1}^{12} t_n)^{12}, t_n^{12} \rangle$$

Then $\overline{P}_n$ *is a* $C'(1/6)$ *(and therefore word-hyperbolic and virtually special) group,* $\overline{P}_n / \mathrm{Tor}_1(\overline{P}_n) \cong \overline{P}_{n-1}$, *and* $\mathrm{TorLen}(\overline{P}_n) = n$.

Proof. This follows immediately from Theorem 6.7. □

7. Quotients

We are interested in the universal torsion-free quotients of finitely presented groups. We begin with the following observation.

Proposition 7.1. *Let G be a finitely presented group with infinite torsion length (see Theorem 6.10). Then* $G/\mathrm{Tor}_\omega(G)$ *is finitely generated and recursively presented, but* not *finitely presented.*

Proof. Assume $G/\mathrm{Tor}_\omega(G)$ is finitely presented. Then we have that $\mathrm{Tor}_\omega(G)$ must be the normal closure of finitely many elements of G; say $\mathrm{Tor}_\omega(G) = \langle\langle g_1, \dots, g_n \rangle\rangle^G$. But then each g_i lies in some $\mathrm{Tor}_{m_i}(G)$, and as the $\mathrm{Tor}_j(G)$ form a nested sequence we have that all the g_i lie in $\mathrm{Tor}_M(G)$ for $M = \max\{m_i\}$. Thus $\mathrm{Tor}_\omega(G) = \mathrm{Tor}_M(G)$, and so G has finite torsion length; a contradiction. □

With this in mind, we ask the following question:

Question 1. Is there a finitely presented group G for which $G/\mathrm{Tor}_1(G)$ is recursively presented but *not* finitely presented?

Note that if such a group were to exist, then using the Adian-Rabin construction ([8, Theorem 2.4]) one could construct a group G such that any sequence drawing from "finitely presented" and "not finitely presented" is realised by looking at the sequence $G/\mathrm{Tor}_1(G)$, $G/\mathrm{Tor}_2(G)$, ...

In the case of word-hyperbolic groups, however, it is always true that $G/\mathrm{Tor}_1(G)$ is finitely presented, as we now show; moreover, in this context a finite presentation for $G/\mathrm{Tor}_1(G)$ can be algorithmically constructed. We begin with a result of Papasoglu.

Theorem 7.2 ([20])**.** *There is a partial algorithm that, on input of a finite presentation P, halts if and only if $\overline{P}$ is a word-hyperbolic group. Moreover, when this algorithm does halt, it outputs a hyperbolicity constant δ for P.*

For a finitely generated group G with finite generating set X, we define the ball of radius r about the identity, $B_X(e,r)$, to be the set of elements

$$B_X(e,r) \coloneqq \{g \in G \mid \exists w \in X^* \text{ with } |w| \le r \text{ and } \overline{w} = g \text{ in } G\}.$$

The following standard lemma will be of use; the proof of [5, III.Γ Theorem 3.2] provides an argument to verify it:

Lemma 7.3. *Let G be a finitely presented word-hyperbolic group with hyperbolicity constant δ. Then any finite subgroup $H \le G$ is conjugate in G to some subgroup in the $(4\delta+2)$-ball around the origin. That is, there exists some $g \in G$ such that $g^{-1}Hg \subseteq B(e, 4\delta+2)$.*

Theorem 7.4. *Let $P = \langle X|R\rangle$ be a finite presentation of a word-hyperbolic group G with hyperbolicity constant δ. Let $S_{X,\delta}$ be the finite set*

$$S_{X,\delta} \coloneqq \{g \in \mathrm{Tor}(G) \mid \langle g\rangle \subseteq B_X(e, 4\delta+2)\}$$

Then $\langle\!\langle \mathrm{Tor}(G)\rangle\!\rangle^G = \langle\!\langle S_{X,\delta}\rangle\!\rangle^G$. Moreover, from P and δ we can algorithmically construct the set $S_{X,\delta}$.

Proof. Let g be a torsion element in G. Then, by Lemma 7.3, $\langle g\rangle$ is conjugate to a subgroup in the ball $B_X(e, 4\delta+2)$. Thus $\langle\!\langle \mathrm{Tor}(G)\rangle\!\rangle^G = \langle\!\langle S_{X,\delta}\rangle\!\rangle^G$, and so the first statement is proved.

Now, using the uniform solution to the word problem for hyperbolic groups (see [5, III.Γ Theorems 2.4–2.6]), we can identify a set of words (of length at most r) together representing all elements in $S_{X,\delta}$ as follows: enumerate all words of length at most r in X^*; call these $w_1, \dots, w_k$. For each w_i, compute minimal-length words for $w_i^2, w_i^3, \dots$ and so on until either some $\overline{w_i^m}$ lies outside $B_X(e, 4\delta+2)$ or is trivial. If, for w_i, the former occurs first, then discard w_i. If, for w_i, the latter occurs first, then add w_i to our set. At the end of this process, we will have formed the set $S_{X,\delta}$, algorithmically from P and δ. □

Using Theorems 6.7, 7.2 and 7.4, we immediately see the following:

Corollary 7.5. *Let G be a finitely presented word-hyperbolic group. Then $G/\mathrm{Tor}_1(G)$ is finitely presented. Moreover, given a finite presentation P for G, we can algorithmically construct from it a finite presentation for $G/\mathrm{Tor}_1(G)$. Finally,* any *finitely presented group Q can be obtained as $Q \cong G/\mathrm{Tor}_1(G)$ for some $C'(1/6)$ (and therefore word-hyperbolic) group G.*

Remark 7.6. Indeed, every finitely *generated* group H can be obtained as $H \cong G/\mathrm{Tor}_1(G)$ for some $C'(1/6)$ (and hence acylindrically hyperbolic) group G: see Remark 6.9. We thank the anonymous referee for pointing this out.

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