Supplementary Material for

‘Intermittency between avalanche regimes on grain piles’

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I. DATA PROCESSING FOR INTERVAL EXTRACTION

i. Experiments with continuous inflow and a fixed laser profile scanner

We distinguish avalanches by the value of the mean squared rate of change of height in the field of view of the laser profile scanner. The scanner, fixed at a distance X downslope, measures surface heights $h_{j,k}$ at $n = 512$ positions x_j in the channel's centerline, at times t_k separated by an interval $\delta t = 0.05$ s, and so this value may be expressed as

$$\langle \dot{h}^2 \rangle = \frac{1}{n} \sum_j \frac{(h_{j,k+1} - h_{j,k})^2}{\delta t^2}. \tag{1}$$

When no avalanching grains are in the field of view of the scanner, we expect that $h_{j,k}$ will be subject to independent, random measurement errors, approximately normally distributed, with standard deviation the measurement error of the scanner, $\epsilon = 0.09$ mm. Therefore, we expect that $n\delta t^2 \langle \dot{h}^2 \rangle / 2\epsilon^2 \sim \chi^2_n$. While an avalanche is in the field of view of the scanner, the time for each grain to move past x_j is approximately $0.005 \text{ s} \ll \delta t$, while the time for the surface height to vary substantially, after averaging over individual grains, is of the order of $1 \text{ s} \gg \delta t$. Therefore, we expect the standard deviation of $h_{j,k}$ over timescales of δt to be on the order of half the diameter of the largest grains, $d_{\max} = 0.71$ mm. Supposing $h_{j,k+1} - h_{j,k}$ to be normally distributed, we have that $2n\delta t^2 \langle \dot{h}^2 \rangle / d^2_{\max} \sim \chi^2_n$.

We register grains to be moving (respectively, static) when, after smoothing, $\langle \dot{h}^2 \rangle$ has a value more than 10 standard deviations away from the value expected for static (respectively, moving) grains. Specifically, we smooth $\langle \dot{h}^2 \rangle$ with a locally-weighted least-squares quadratic regression, with time span 1 s, and register an avalanche to have started (respectively, stopped) when, after a period in which grains in the scanner's field of view were static (respectively, moving), the smoothed value first exceeds (respectively, falls below) a critical value C_{move} (respectively, C_{stop}. Critical values are given by

$$C_{\text{move}} = \frac{2\epsilon^2}{\delta t^2} \left(1 + \sqrt{\frac{2}{n}} \right), \qquad C_{\text{stop}} = \frac{d^2_{\max}}{2\delta t^2} \left(1 - \sqrt{\frac{2}{n}} \right). \tag{2}$$

We define the time interval between avalanches as the time between one avalanche being registered to have stopped, and the subsequent avalanche being registered to have started. To remove the effect of individual outlying measurements, arising from misidentification of the surface by the scanner at a given time t_k, we exclude time intervals with a duration of less than 2 s.

ii. Experiments with interrupted inflow and full-length height profiles

We detect the position of a stopped avalanche front on a stationary pile by considering the changes in surface height between each pair of consecutive full-length height profiles. After each avalanche has come to a stop, the laser profile scanner is moved by an overhead traverse between 28 fixed positions at distances X_i downslope, 75 mm apart, and at each position the scanner measures surface heights 20 times along a 120 mm section of the channel's centerline, at distances downslope $x_{i,j}$, constant to within 1 mm between consecutive height profiles. We calculate the mean heights over these 20 measurements to obtain a full-length height profile, and subtract the mean heights from the previous full-length height profile to obtain the changes in surface height, $\Delta h_{i,j}$. Due to error in $x_{i,j}$, $\Delta h_{i,j}$ has random variation from 0 of the order of the grain size even in the absence of a physical change in surface height, but only where an avalanche has passed is there systematic variation of $\Delta h_{i,j}$.

We register a stopped avalanche front where the change in local gradient exceeds a critical value. We smooth $\Delta h_{i,j}$ as a function of $x_{i,j}$ with a robust locally-weighted least-squares linear regression of span approximately 40 mm, assigning lower weight to outliers and neglecting data points outside 6 mean absolute deviations, and we interpolate with a cubic spline the change in surface height Δh at 1 mm intervals along the entire profile. We calculate the local gradient $\Delta\theta$ of Δh and identify as the location of a new stopped avalanche front any peak of $\Delta\theta$ with a magnitude greater than the difference between the grains' angle of repose θ_r and their maximum angle of stability, θ_m. For the construction sand we use this is 4°, while for our glass beads this is 2°. If no such peak exists we conclude that the avalanche propagated the entire length of the channel, leaving no stopped avalanche front.

To calculate the effective time intervals between avalanches, we use the measured time intervals $T_j(0)$ between the resumption of inflow and the start of an avalanche, index j. For each X, we let $j_k(X)$ be the index of the kth avalanche that propagates a distance greater than X downslope, and we note that the kth effective time interval between avalanches passing X is

$$T_k(X) = \sum_{j=j_k(X)+1}^{j_{k+1}(X)} T_j(0). \tag{3}$$

II. DATA

The following tables list summary data for the experimentally-observed time intervals between avalanches. At distances X downslope, we observe a total of n intervals between avalanches, of which n_{QP} and n_{IR} are classified in the quasi-periodic and irregular regimes, respectively. In these regimes, the mean time intervals between avalanches are $\bar{T}_{QP}$ and $\bar{T}_{IR}$ respectively, and the standard deviations of the time intervals between avalanches, from these means, are ΔT_{QP} and ΔT_{IR}, respectively.

X / mm	n	n_{QP}	$\bar{T}_{QP}$ / s	ΔT_{QP} / s	n_{IR}	$\bar{T}_{IR}$ / s	ΔT_{IR} / s
186.0	304	253	4.58	0.98	36	4.96	2.41
306.0	273	102	6.49	0.61	130	8.45	4.81
394.0	272	105	7.71	0.98	131	11.44	7.94
514.0	232	49	8.61	1.37	144	14.43	10.81
634.0	179	0	-	-	130	18.63	13.79
754.0	155	55	9.12	1.56	82	17.93	13.98
844.0	143	66	12.21	2.47	40	25.03	19.97
964.0	163	60	10.45	1.93	64	30.50	21.06
1084.0	140	25	9.64	0.98	75	33.39	26.90
1204.0	134	25	14.12	2.79	65	36.14	30.88
1294.0	181	91	11.84	2.81	41	34.39	36.12
1414.0	156	70	9.88	1.48	50	40.21	42.09
1534.0	163	92	12.24	2.70	39	44.06	38.68
1664.0	144	75	11.24	2.01	17	51.15	41.67
1974.0	107	15	8.90	0.67	43	46.85	43.95
2054.0	94	0	-	-	44	52.14	34.80

TABLE I. Avalanche interval statistics for a constant inflow, flux $Q = 3.05$ cm^{3}s^{-1}, of angular construction sand, mean particle diameter $\bar{d} = 0.47$ mm.

X / mm	n	n_{QP}	$\bar{T}_{QP}$ / s	ΔT_{QP} / s	n_{IR}	$\bar{T}_{IR}$ / s	ΔT_{IR} / s
181.5	380	238	41.93	8.93	95	34.95	19.81
351.5	405	55	26.55	4.91	282	42.61	28.03
521.5	272	0	-	-	219	63.91	49.16
681.5	204	0	-	-	159	83.22	70.59
881.5	168	0	-	-	106	106.54	81.30
1031.5	162	37	31.51	3.96	83	132.03	90.67
1201.5	112	0	-	-	72	149.04	104.65
1371.5	96	0	-	-	50	176.61	129.32
1541.5	98	0	-	-	54	171.02	145.92
1711.5	83	0	-	-	37	219.54	164.61
1800.0	243	159	23.83	4.66	62	185.80	195.11
1881.5	122	22	23.47	3.27	53	195.67	174.83

TABLE II. Avalanche interval statistics for a constant inflow, flux $Q = 0.9$ cm^{3}s^{-1}, of angular construction sand, mean particle diameter $\bar{d} = 0.47$ mm.

X / mm	n	n_{QP}	$\bar{T}_{QP}$ / s	ΔT_{QP} / s	n_{IR}	$\bar{T}_{IR}$ / s	ΔT_{IR} / s
181.5	404	53	8.45	1.82	350	7.98	3.12
186.0	404	53	8.45	1.82	350	7.98	3.12
306.0	354	52	8.61	2.28	301	9.26	3.77
351.5	331	52	8.61	2.28	278	10.03	4.54
394.0	308	52	8.61	2.28	255	10.94	5.63
514.0	257	52	8.61	2.28	204	13.67	8.71
521.5	253	52	8.61	2.28	200	13.94	9.14
634.0	216	52	8.61	2.28	163	17.11	12.75
681.5	206	52	8.61	2.28	153	18.23	14.12
754.0	193	52	8.61	2.28	140	19.92	15.54
844.0	174	52	8.61	2.28	121	22.94	19.20
881.5	166	52	8.61	2.28	113	24.57	20.56
964.0	159	52	8.61	2.28	106	26.01	22.49
1031.5	152	52	8.61	2.28	99	27.85	24.12
1084.0	147	52	8.61	2.28	94	29.33	25.15
1201.5	136	52	8.61	2.28	83	33.22	29.57
1204.0	136	52	8.61	2.28	83	33.22	29.57
1294.0	128	51	8.78	2.34	76	36.28	35.80
1371.5	123	51	8.78	2.34	71	38.83	36.32
1414.0	117	51	8.78	2.34	65	42.42	39.52
1534.0	110	51	8.78	2.34	58	47.54	45.68
1541.5	109	51	8.78	2.34	57	48.37	45.65
1664.0	101	51	8.78	2.34	49	56.27	48.56
1711.5	99	51	8.78	2.34	47	58.66	49.67
1800.0	97	51	8.78	2.34	45	61.27	51.90
1881.5	95	51	8.78	2.34	43	64.12	52.99
1974.0	93	50	8.96	2.52	42	65.64	52.98
2054.0	93	50	8.96	2.52	42	65.64	52.98

TABLE III. Avalanche interval statistics for an interrupted inflow, flux $Q = 3.05$ cm^{3}s^{-1}, of angular construction sand, mean particle diameter $\bar{d} = 0.47$ mm.

X / mm	n	n_{QP}	$\bar{T}_{QP}$ / s	ΔT_{QP} / s	n_{IR}	$\bar{T}_{IR}$ / s	ΔT_{IR} / s
181.5	813	7	2.41	0.88	806	2.77	1.59
186.0	807	7	2.41	0.88	800	2.79	1.61
306.0	622	7	2.41	0.88	615	3.62	2.53
351.5	593	7	2.41	0.88	586	3.80	2.96
394.0	565	7	2.41	0.88	558	4.00	3.41
514.0	505	7	2.41	0.88	498	4.47	4.06
521.5	501	7	2.41	0.88	494	4.50	4.35
634.0	455	7	2.41	0.88	448	4.96	5.59
681.5	435	7	2.41	0.88	428	5.20	5.97
754.0	403	7	2.41	0.88	396	5.62	7.14
844.0	360	7	2.41	0.88	353	6.30	8.66
881.5	347	7	2.41	0.88	340	6.54	9.02
964.0	318	7	2.41	0.88	311	7.15	10.07
1031.5	295	7	2.41	0.88	288	7.72	11.21
1084.0	275	7	2.41	0.88	268	8.30	11.79
1201.5	244	7	2.41	0.88	237	9.38	13.05
1204.0	242	7	2.41	0.88	235	9.46	13.26
1294.0	222	7	2.41	0.88	215	10.34	14.54
1371.5	211	7	2.41	0.88	204	10.90	14.97
1414.0	205	7	2.41	0.88	198	11.23	15.34
1534.0	183	7	2.41	0.88	176	12.64	16.89
1541.5	183	7	2.41	0.88	176	12.64	16.89
1664.0	164	7	2.41	0.88	157	14.17	17.99
1711.5	162	7	2.41	0.88	155	14.35	18.10
1800.0	155	7	2.41	0.88	148	15.03	18.56
1881.5	149	7	2.41	0.88	142	15.66	19.05
1974.0	147	7	2.41	0.88	140	15.89	19.16
2054.0	145	7	2.41	0.88	138	16.12	19.21

TABLE IV. Avalanche interval statistics for an interrupted inflow, flux $Q = 1.22$ cm^{3}s^{-1}, of spherical glass beads, mean particle diameter $\bar{d} = 0.22$ mm.

III. STATISTICAL TESTS

i. R^2 value

Considering the quasi-periodic (respectively, irregular) regime, we conduct a linear regression of the non-dimensionalized mean intervals between avalanches $Q\bar{T}(X)/d^3$ on distances downslope X. We consider those values of $X \in [400, 2000]$ for which quasi-periodic (respectively, irregular) avalanches were observed in an experiment with a constant inflow of construction sand and with the laser profile scanner fixed at position X, whether with $Q = 3.05$ cm^{3}s^{-1} or with $Q = 0.9$ cm^{3}s^{-1}. For each X, we use the values of $Q\bar{T}/d^3$ arising from the experiments with a) a constant inflow of grains, b) an interrupted inflow of sand, and c) an interrupted inflow of glass beads. $\bar{T}$-values are listed in section II.

For the quasi-periodic regime, the 98% confidence interval for the best-fit gradient is $[-49, 28]$ mm^{-1} (under the assumption of independent, normally-distributed errors) and the coefficient of determination $R^2 = 0.011$, demonstrating that the data are consistent with $Q\bar{T}/d^3$ being constant within the considered range of X. For the irregular regime, the 98% confidence interval for the $Q\bar{T}/d^3$-intercept is $[-1.7 \times 10^5, 0.14 \times 10^5]$ and, restricting to an intercept of 0, the coefficient of determination $R^2 = 0.93$, demonstrating that the data are consistent with the model $Q\bar{T}/d^3 \propto X$ within the considered range of X.

ii. Hypothesis tests

We conduct likelihood ratio hypothesis tests to determine whether our observations of the non-dimensional mean intervals between avalanches, $Q\bar{T}/d^3$, are consistent with the hypotheses that the underlying probability density of avalanches stopping a distance X downslope is zero in the quasi-periodic regime and proportional to X^{-2} in the irregular regime. Under these hypotheses, $Q\bar{T}/d^3$ will be constant in the quasi-periodic regime and proportional to X in the irregular regime. As the probability density must be non-negative, the probability of a given avalanche stopping before X must monotonically increase with X and so, since all avalanches propagate from the top of the channel, the frequency with which avalanches pass X must monotonically decrease with X. Therefore, our alternative hypotheses are that the non-dimensional mean intervals between avalanches are monotonically increasing functions of X.

As in section III.i, we consider those values of $X \in [400, 2000]$ for which relevant data exists from an experiment with a constant inflow of grains, and consider all available data for each such X (see section II). We assume that, for a given X, the non-dimensional intervals observed in different experiments have equal means, as justified by Welch's tests on the data. Motivated by the central limit theorem, we further assume that the observed means are normally distributed, with variances given by $Q_\alpha^2 \mathrm{Var}_k(T_{\alpha,k}(X))/d_\alpha^6 n_\alpha(X)$ for indices α and k corresponding to different experiments and different intervals observed within an experiment, respectively.

For the quasi-periodic regime, we consider the hypotheses $H_0 : \bar{T}(X) = A$, for some $A \geq 0$, and $H_1 : \bar{T}(X)$ is a monotonically increasing function of X. Using the maximum likelihood framework, we find that the log-likelihood ratio (LLR) of H0 to H1 is equal to -0.99. Since the parameter set corresponding to H_0 lies on the boundary of that corresponding to H_1 and the difference in the dimensionality of the parameter spaces is 13, we have by Chernoff's extension of Wilk's theorem that under H_0, the approximate distribution of $\Lambda = -2LLR$ is given by a 50% chance that $\Lambda = 0$ and a 50% chance that $\Lambda \sim \chi^2_{13}$[1]. Under H_0, the probability of a more extreme value of Λ is therefore $p = 0.5$, and so our data are consistent with H_0.

For the irregular regime, we consider the hypotheses $H_0 : \bar{T}(X) = BX$, for some $B \geq 0$, and $H_1 : \bar{T}(X)$ is a monotonically increasing function of X. Using the maximum likelihood framework, we find that the log-likelihood ratio of H0 to H1 is equal to -9.5. Since the parameter set corresponding to H_0 lies in the interior of that corresponding to H_1 and the difference in the dimensionality of the parameter spaces is 21, we have by Wilk's theorem that, under H_0, we may take the approximation $\Lambda \sim \chi^2_{21}$[2]. Under H_0, the probability of a more extreme value of Λ is therefore $p = 0.59$, and so our data are consistent with H_0.

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[1] H. Chernoff, Ann. Math. Statist. **25**, 573 (1954).

[2] S. S. Wilks, Ann. Math. Statist. **9**, 60 (1938).