One-prime power hypothesis for conjugacy class sizes

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1 Introduction

To determine structural information about a finite group G given the set of conjugacy class sizes of G is an ongoing line of research, see [CC11] for an overview. How the arithmetic data given by the set of conjugacy class sizes is encoded varies, but one representation is via the bipartite graph $B(X)$. Let X be a set of positive integers and let $X^* = X \setminus 1$ (X may or may not contain the element 1). If $x \in X$ we denote the set of prime divisors of x by $\pi(x)$ and let $\rho(X) = \bigcup_{x \in X} \pi(x)$.

Definition. [IP10] The vertex set of $B(X)$ is given by the disjoint union of X^* and $\rho(X)$. There is an edge between $p \in \rho(X)$ and $x \in X^*$ if p divides x, i.e. if $p \in \pi(x)$.

In our context we let X be the set of conjugacy class sizes of a finite group G, and in this case we denote $B(X)$ by $B(G)$. In [Tae10] Taeri investigates the case when $B(G)$ is a cycle, or contains no cycle of length 4. In particular, he proves the following.

Theorem. [Tae10] Let G be a finite group and $Z(G)$ the centre of G. Suppose $G/Z(G)$ is simple, then $B(G)$ has no cycle of length 4 if and only if $G \cong A \times S$, where A is abelian, and $S \cong PSL_2(q)$ for $q \in \{4, 8\}$.

Taeri goes on to conjecture that the same conclusion holds if the assumption is just that G is finite and insoluble. In this paper we confirm Taeri's conjecture.

Main Theorem. If G is a finite insoluble group, then $B(G)$ has no cycle of length 4 if and only if $G = A \times S$, where A is abelian and $S \cong PSL_2(q)$ for $q \in \{4, 8\}$.

As Taeri comments, $B(G)$ having no cycle of length 4 is equivalent to G satisfying the *one-prime power hypothesis*, that is, if m and n are non-trivial conjugacy class sizes of G then either m and n are coprime or their greatest common divisor is a prime power. This is similar to the one-prime hypothesis introduced by Lewis to study character degrees [Lew95]. We use this terminology.

Throughout the paper G will be assumed to be a finite group. Most of the notation used will be standard. In particular, $Z(G)$ is the centre of G, the maximal normal soluble subgroup of G is denoted by $S(G)$, the maximal normal p-subgroup of G is denoted $O_p(G)$ and the Fitting and second Fitting subgroups are denoted by $F(G)$ and $F_2(G)$ respectively. The conjugacy class size of an element $x \in G$ will be denoted by $|x^G|$ and shall be called the *index* of $x \in G$. We say an element has *mixed index* if its index is not a prime power. The greatest common divisor of two numbers m and n shall be denoted by (m, n) and p will always be prime.

2 Preliminary Remarks

We begin by making some preliminary remarks.

Lemma 1. *Suppose N is a normal subgroup of a group G.*

(i) Let $x \in N$, then $|x^N|$ divides $|x^G|$.

(ii) Let $\bar{x} \in G/N = \bar{G}$, then $|\bar{x}^{\bar{G}}|$ divides $|x^G|$.

Let $C_G(x)$ be the centraliser of an element x in G. Then $C_G(x)$ is said to be minimal if $C_G(y) \leq C_G(x)$ for some $y \in G$ implies $C_G(y) = C_G(x)$. The following lemma is well-known.

Lemma 2. *Suppose x is a p-element with minimal centraliser. Then $C_G(x) = P_0 \times A$, where P_0 is a p-group and A is abelian.*

We have the following lemma.

Lemma 3. *Assume G satisfies the one-prime power hypothesis and there exists $x, y \in G$ with $C_G(x) < C_G(y)$. Then $|y^G|$ is a prime power.*

Proof. Let $|x^G| = m$ and $|y^G| = n$, then $(m, n) = n$ and hence n is a prime power, i.e. any non-minimal centraliser has prime power index.$\square$

The following result will prove useful.

Proposition 4. *[CC98, Theorem 1] All elements of prime power index in G lie in $F_2(G)$.*

Recall, G is called an F-group if whenever x and y are non-central elements of G satisfying $C_G(x) \leq C_G(y)$, then $C_G(x) = C_G(y)$. Rebmann has classified F-groups [Reb71].

Lemma 5. *(i) Suppose G satisfies the one-prime power hypothesis and $F(G)$, the Fitting subgroup of G, is central. Then G is an F-group.*

(ii)[Tae10] Suppose G is an insoluble F-group that satisfies the one-prime power hypothesis. Then $G \cong S \times A$ where $S \cong PSL_2(q)$ for $q \in \{4, 8\}$ and A is abelian.

Proof. (i) As $F(G)$ is central so is $F_2(G)$ and thus G has no elements of prime power index by Proposition 4. Applying Lemma 3 gives that G is an F-group.
(ii) This is a combination of [Tae10, Lemma 4] and [Tae10, Theorem 1].$\square$

Consider the following property. Let G be a finite non-abelian group with proper normal subgroup N and suppose all the conjugacy class sizes outside of N have equal sizes. Isaacs proved that in this situation then either G/N is cyclic, or else every non-identity element of G/N has prime order [Isa70]. We combine this result with Proposition 4 and a result of Qian to give the following lemma.

Lemma 6. *Suppose G is a finite group with at most one conjugacy class size that is not a prime power. Then either G is soluble or $G/F_2(G) \cong PSL_2(4)$.*

Proof. By Proposition 4 all elements outside of $F_2(G)$ have the same conjugacy class size. Applying [Isa70] gives that $G/F_2(G)$ is a non-soluble group with all elements of prime order. The result follows from [Qia05].$\square$

This lemma leads us to ask the following question. Suppose G is a finite group with at most one conjugacy class that is not a prime power, does it follow that G is soluble?

Groups in which all elements have prime power order are well studied and all called CP-groups. Delgado and Wu have given a full description of locally finite CP-groups, the following considers the special case when the Fitting subgroup is trivial.

Theorem 7. *[DW02] Let G be a finite CP-group with trivial Fitting subgroup. Then either G is simple and isomorphic to one of $PSL_2(q)$ where $q \in \{4, 7, 8, 9, 17\}$, $PSL_3(4)$, $Sz(8)$, $Sz(32)$ or G is isomorphic to M_{10}.*

The following observation is useful.

Lemma 8. *Suppose G satisfies the one-prime power hypothesis and that N is a normal subgroup of G. If $\bar{x} \in \bar{G} = G/N$ has mixed index in $\bar{G}$, then $|x^G| = |(xn)^G|$ for all $n \in N$.*

Proof. Note that $|\bar{x}^{\bar{G}}|$ divides both $|x^G|$ and $|(xn)^G|$. So, by the one-prime power hypothesis, the result follows. $\square$

3 Main Result

The property of satisfying the one-prime power hypothesis does not (clearly) restrict to normal subgroups (however we know of no examples where this is not the case). We do have the following.

Lemma 9. *Suppose G satisfies the one-prime power hypothesis and r is a prime dividing $|G|$. If N is a normal r-complement in G then N also satisfies the one-prime power hypothesis.*

Proof. Suppose not, then there exist $x, y \in N$ with $|x^N| \neq |y^N|$ and distinct primes p and q with pq dividing both $|x^N|$ and $|y^N|$. As G satisfies the one-prime power hypothesis this forces $|x^G| = |y^G|$. However note that $\frac{|x^G|}{|x^N|}$ divides $|G/N|$ and is thus a power of r, and similarly for y, so $|x^G| \neq |y^G|$, a contradiction.□

We first consider the case where there is only one mixed index.

Proposition 10. *Suppose G satisfies the one-prime power hypothesis and all elements of mixed index have index m. Then G is soluble.*

Proof. By Lemma 6 we can assume $G/F_2(G)$ is isomorphic to $PSL_2(4)$. Furthermore, if there exists a prime power index, say r^a with r not dividing m then G is quasi-Frobenius and hence soluble by [Kaz81]. So we can assume otherwise.

Let $\bar{G} = G/F_2(G)$. Since $\bar{G}$ has elements of index 12, 15 and 20 we see that m is divisible by 60. Let $x \in G$ with $\bar{x}$ of order 2. Then $|\bar{x}^{\bar{G}}| = 15$. But in G the index of x has to be m, so we see that $F_2(G)$ has to have a non-central 2-subgroup. We can argue similarly to show $F_2(G)$ has to have non-central 3 and 5 subgroups.

Suppose $x, y \in F_2(G)$, that x and y commute and have coprime orders. Suppose further that $|x^G| = p^a$ and $|y^G| = q^b$. If $p \neq q$ then $|xy|^G$ is divisible by just two different primes and so cannot equal m, a contradiction.

So assume $x, y \in F_2(G)$ with $|x^G| = p^a$, $|y^G| = q^b$ and $p \neq q$. Given that the indices of x and y are prime powers we can assume that each of x and y have prime power orders. Assume first that the orders of x and y are coprime. $C_G(x)$ contains a Sylow r-subgroup of G for each prime $r \neq p$. If y is not a p-element it, or some conjugate of it, is in $C_G(x)$ which contradicts the above assertion. So y is a p-element and x is q-element. Let r be a prime distinct from p and q and dividing the order $G/F_2(G)$.

Both $C_G(x)$ and $C_G(y)$ can be assumed to contain a Sylow r-subgroup of G. Let u be an r-element of mixed index, there is one because r divides the order of $G/F_2(G)$. Taking conjugates we can assume $x, y \in C_G(u)$. By Lemma 2, $C_G(u) = R_0 \times A$ where A is an abelian r'-subgroup which must contain both x and y, a contradiction as x and y do not commute. So if $x, y \in F_2(G)$ with $|x^G| = p^a$ and $|y^G| = q^b$ with $p \neq q$ then x and y are both l-elements for some prime l. If there is an l'-element of prime power index then we can apply the previous argument. So every l'-element has mixed index. So G satisfies the hypothesis that every l'-element of G has the same index, using [Cam74], we get G is soluble. We end this paragraph by noting that if the proposition is not true then there is a prime p so that every element, x, of prime power index has $|x^G| = p^a$ for some a.

Note that if M is the subgroup generated by all the elements of prime power index then $M \subseteq F_2(G)$ and every element not in M has index m. As G/M is not soluble it is isomorphic to $PSL_2(4)$ and so $M = F_2(G)$.

Let t be a prime such that $t \neq p$. Any element of prime power index contains a Sylow t-subgroup of G in its centraliser and so centralises $O_t(G)$. Now $O_t(G) \subseteq Z(F_2(G))$. As $F_2(G)$ is metanilpotent if P is the Sylow p-subgroup of $F_2(G)$ then PF is normal in $F_2(G)$. But $PF = PU$ where U is the product of O_t for all $t \neq p$. So U is central in $F_2(G)$ and hence $PF = P \times U$ and P is normal in G.

There exist p-elements of mixed index otherwise all p-elements of G have p-power index and $G = P \times H$ for H some p'-subgroup of G, by [CC98], but such a group cannot satisfy the conditions of the proposition. Assume that there exists a p-element x of mixed index in $F_2(G)$ so $x \in P$. Then $C_G(x) = P_0 \times A_0$ where P_0 is a p-group and A_0 is an abelian p'-group. Let $m = p^e m_0$ where $(m_0, p) = 1$, then $[G : A_0] = p^f m_0$ for some f. Also A_0 cannot be central in G otherwise there would be no p'-elements of mixed index which is false. Then $A_0 \subseteq C_G(P)$, by an application of Thompson's Lemma [Gor68, 5.3.4]. As $x \in P$, A_0 is the Hall p'-subgroup of $C_G(P) = Z(P) \times A_0$. So A_0 is a normal abelian p'-subgroup of G. Furthermore, A_0 is central in $F_2(G)$ as it commutes with all elements that generate F_2 and since it is not central it follows that $m = 60$ and thus p is a divisor of 60. So, there exists a p-element, say y, of mixed index not in $F_2(G)$. Then $C_G(y) = P_1 \times A_1$ and, again by [Gor68, 5.3.4], A_1 centralises P but $|A_1| = |A_0|$ as x and y have the same index. This implies that $C_G(A_0) > F_2(G)$ so A_0 is central in G, a contradiction.

The last case to consider is that there are no elements of mixed index in P. That means that all the p-elements of $F_2(G)$ have index a power of p. By [CC98] it follows that $F_2(G) = P \times A$ where A has order prime to p and A is normal in G and central in $F_2(G)$. As A is not central we see that $p = 5$. Let y be a p-element of mixed index not in $F_2(G)$. Then $C_G(y) = P_1 \times A_1$ and A_1 centralises P by [Gor68, 5.3.4]. As A_1 is a subgroup of A it centralises P and y but P and y generate the Sylow p-subgroup of G and hence A_1 is in the centre of G. Then no p' element can have mixed index which is false as there are both 2 and 3 elements of mixed index.□

We are now ready to prove the main theorem.

Theorem 11. *Suppose G is insoluble and satisfies the one-prime power hypothesis. Then $G \cong PSL_2(q) \times A$ for $q \in \{4, 8\}$ where A is abelian.*

Proof. We suppose the result is not true and take G to be a counterexample of minimal order.

(i) Case 1: Suppose $\bar{G} = G/F_2(G)$ has elements of mixed order.

Let such an element be $\bar{u}$. Then we can assume $\bar{u}$ has order divisible by precisely two primes, p and q say, and further we can assume u similarly has order divisible by two primes p and q. We write $u = xy$ where x and y commute and x has p-power order and y has q-power order. As u is not an element of $F_2(G)$ it follows that u has mixed index, and as $\bar{u}$ has mixed order we also know that both x and y do not lie in $F_2(G)$ and thus also have mixed index. As $C_G(x)$ is minimal it follows from Lemma 2 that $C_G(x) = P_0 \times A$ where P_0 is a p-group and A is abelian. A similar statement holds for $C_G(y)$ and thus we obtain that $C_G(u) = C_G(x) = C_G(y)$ and is abelian. Now there exists z an element of mixed index different to $|u^G|$ otherwise all elements of $G/F_2(G)$ would be of prime power order [Isa70]. If $|z^G|$ is coprime to p then z centralises a Sylow p-subgroup and a conjugate of z lies in $C_G(x)$, but then the index of z divides the index of x, a contradiction. Thus both p and q divide $|z^G|$. So we have shown that there are only two mixed indices of elements of G and these are given by $|x^G|$ and $|z^G|$. Thus, by the one-prime power hypothesis there exist a pair of primes r and s say with r dividing $|x^G|$ and s dividing $|z^G|$ but the product rs does not divide any conjugacy class size in G. Thus, by [Itô53, Prop. 5.1], G has a normal r-complement (say), call this complement N. Then N satisfies the one-prime power hypothesis by Lemma 9. If N is soluble so is G, so we can assume N is insoluble. Thus, by induction, $N \cong S \times A$ where A is abelian and S is one of the simple groups $PSL_2(q)$ for q equal to 4 or 8. Note A must be central in G as otherwise G does not satisfy the one-prime power hypothesis. However, if A is central in G all r-elements have r-power index as the outer automorphism groups of these two simple groups have no elements of order r. Thus the Sylow r-subgroup is a direct factor of G by [CC98, Theorem A]. As G satisfies the one-prime power hypothesis, this forces the Sylow r-subgroup to be central. Thus, $G/Z(G) \cong S$, and all elements of the quotient are of prime power order, a contradiction.

ii) Case 2: Assume all elements of $G/F_2(G)$ have prime power order.

We can assume we have at least one mixed index by Proposition 4. If we have precisely one then G is soluble by Proposition 10. So we can assume there exist elements of mixed index which are not equal.

Let $\bar{G} = G/F_2(G)$. Let $\bar{x}$ be a p-element in $\bar{G}$. As $C_{\bar{G}}(\bar{x})$ is a p-group it follows that$|\bar{G}|/|\bar{G}|_p$ divides $|\bar{x}|^{\bar{G}}$ where $|\bar{G}|_p$ denotes the p-part of $|\bar{G}|$. A similar statement holds for all elements of $\bar{G}$.

If $|\bar{G}|$ were divisible by more that 3 primes this would force all elements outside of $F_2(G)$ to have the same conjugacy class size in G, a contradiction. Thus we can assume $|\bar{G}|$ is divisible by exactly 3 primes. Assume that p, q, r are the primes that divide the order of $G/F_2(G)$ and there is an element of index divisible by pqr. But every element not in $F_2(G)$ has index divisible by at least two of p, q or r so all elements would have the same index which we are assuming is not the case. So we must have that $|x|^G$ is coprime to p and likewise for other elements.

Now, consider $O_t(G) \neq 1$, there exists an element $x \in G \backslash F_2(G)$ such that $|x^G|$ and t are coprime. This follows from the argument above if t divides the order of $|\bar{G}|$. If not, note that the indices of any two elements $y, z \in G \backslash F_2(G)$ already have a prime in common that also divides $|\bar{G}|$. Thus $O_t(G) \leq C_G(x)$. Let $n \in F_2(G)$, then by Lemma 8, it follows that $O_t(G) \leq C_G(xn)$ and thus $O_t(G) \leq C_G(n)$. So, $C_G(O_t(G))$ is a normal subgroup of G containing $F_2(G)$. Since $F(G)$ is a direct product of $O_t(G)$ for all t, $F(G)$ is central in $F_2(G)$. It follows that $F(G) = F_2(G) = S(G)$.

As $\bar{G}$ has trivial Fitting subgroup it follows from Theorem 7 that $\bar{G}$ is a simple group which comes from a known list or is isomorphic to M_{10}. However M_{10} has order 720 and an element with index 90, see [ABL+], which contradicts the discussion above. Thus we can assume that $\bar{G}$ is simple. Note that $O_t(G)$, for any t, centralises some element not in $S(G)$ so $C_G(O_t(G))$ is a normal subgroup of G strictly containing $F_2(G)$. But as $\bar{G}$ is simple, $O_t(G)$ is central but then so is $F(G)$. But then, by Lemma 5, we have that $G \cong PSL_2(q) \times A$ for $q \in \{4, 8\}$ and A abelian, as required.$\square$

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