THE SIZE-RAMSEY NUMBER OF POWERS OF PATHS

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Abstract. Given graphs G and H and a positive integer q, say that G *is q-Ramsey for H*, denoted $G \to (H)_q$, if every q-colouring of the edges of G contains a monochromatic copy of H. The *size-Ramsey number* $\hat{r}(H)$ of a graph H is defined to be $\hat{r}(H) = \min\{|E(G)| \colon G \to (H)_2\}$. Answering a question of Conlon, we prove that, for every fixed k, we have $\hat{r}(P_n^k) = O(n)$, where P_n^k is the kth power of the n-vertex path P_n (i.e., the graph with vertex set $V(P_n)$ and all edges $\{u, v\}$ such that the distance between u and v in P_n is at most k). Our proof is probabilistic, but can also be made constructive.

§1. Introduction

Given graphs G and H and a positive integer q, say that G *is q-Ramsey for H*, denoted $G \to (H)_q$, if every q-colouring of the edges of G contains a monochromatic copy of H. When $q = 2$, we simply write $G \to H$. In its simplest form, the classical theorem of Ramsey [24] states that for any H there exists an integer N such that $K_N \to H$. The *Ramsey number* $r(H)$ of a graph H is defined to be the smallest such N. Ramsey problems have been well studied and many beautiful techniques have been developed to estimate Ramsey numbers. For a detailed summary of developments in Ramsey theory, see the excellent survey of Conlon, Fox and Sudakov [7].

A number of variants of the classical Ramsey problem are also under active study. In particular, Erdős, Faudree, Rousseau and Schelp [12] proposed the problem of determining the smallest number of edges in a graph G such that $G \to H$. Define the *size-Ramsey number* $\hat{r}(H)$ of a graph H to be

$$\hat{r}(H) := \min\{|E(G)| \colon G \to H\}.$$

In this paper, we are concerned with finding bounds on $\hat{r}(H)$ in some specific cases.

For any graph H it is not difficult to see that $\hat{r}(H) \leqslant \binom{r(H)}{2}$. A result due to Chvátal (see, e.g., [12]) shows that in fact this bound is tight for complete graphs. For the n-vertex path P_n, Erdős [11] asked the following question.

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Question 1.1. *Is it true that*

$$\lim_{n\to\infty} \frac{\hat{r}(P_n)}{n} = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\hat{r}(P_n)}{n^2} = 0?$$

Answering Erdős' question, Beck [3] proved that the size-Ramsey number of paths is linear, i.e., $\hat{r}(P_n) = O(n)$, by means of a probabilistic construction. Alon and Chung [2] provided an explicit construction of a graph G with $O(n)$ edges such that $G \to P_n$. Recently, Dudek and Prałat [10] gave a simple alternative proof for this result (see also [21]). More generally, Friedman and Pippenger [14] proved that the size-Ramsey number of bounded-degree trees is linear (see also [8, 15, 17]) and it is shown in [16] that cycles also have linear size-Ramsey numbers.

A question posed by Beck [4] asked whether $\hat{r}(G)$ is linear for all graphs G with bounded maximum degree. This was negatively answered by Rödl and Szemerédi, who showed that there exists an n-vertex graph H and maximum degree 3 such that $\hat{r}(H) = \Omega(n \log^{1/60} n)$. The current best upper bound for bounded-degree graphs is proved in [19], where it is shown that for every Δ there is a constant c such that for any graph H with n vertices and maximum degree Δ:

$$\hat{r}(H) \leqslant cn^{2-1/\Delta} \log^{1/\Delta} n.$$

For further results on size-Ramsey numbers the reader is referred to [5, 18, 25].

Given an n-vertex graph H and an integer $k \geqslant 2$, the kth power H^k of H is the graph with vertex set $V(H)$ and all edges $\{u, v\}$ such that the distance between u and v in H is at most k. Answering a question of Conlon [6] we prove that all powers of paths have linear size-Ramsey numbers. The following theorem is our main result.

Theorem 1.2. *For any integer $k \geqslant 2$,*

$$\hat{r}(P_n^k) = O(n). \tag{1.3}$$

Since $C_n^k \subseteq P_n^{2k}$, the next corollary follows directly from Theorem 1.2.

Corollary 1.4. *For any integer $k \geqslant 2$,*

$$\hat{r}(C_n^k) = O(n). \tag{1.5}$$

Throughout the paper we use big O notation with respect to $n \to \infty$, where the implicit constants may depend on other parameters. For a path P, we write $|P|$ for the number of vertices in P. For simplicity, we omit floor and ceiling signs when they are not essential.

The paper is structured as follows. In Section 2 we introduce some preliminary definitions and give an outline of the proof. The proof of Theorem 1.2 is given in Section 3. In Section 4, we mention some related open problems.

§2. Outline of the proof

To prove Theorem 1.2, we will show that there exists a graph G with $O(n)$ edges such that $G \to P_n^k$.

To construct G we begin by taking a pseudo-random graph H with bounded degree. The existence of such an H will be proved in Lemma 3.1. Given H^k, we then take a *complete blow-up*, defined as follows.

Definition 2.1. Given a graph H and a positive integer t, the *complete-t-blow-up of H*, denoted H_t is the graph obtained by replacing each vertex v of H by a complete graph with $r(K_t)$ vertices, the *cluster* $C(v)$, and by adding, for every $\{u, v\} \in E(H)$, every edge between $C(u)$ and $C(v)$.

Note that we replace each vertex with a clique on $r(K_t)$ vertices rather than t vertices as might have been expected.

The following immediate fact states that complete blow-ups of powers of bounded-degree graphs have a linear number of edges. This makes them valid candidates for showing $\hat{r}(P_n^k) = O(n)$.

Fact 2.2. *Let k, t, a and b be positive constants. If H is a graph with $|V(H)| = an$ and $\Delta(H) \leqslant b$, then $|E(H_t^k)| = O(n)$.*

The heart of the proof is to show that, given any 2-colouring of the edges of H_t^k, we can find a monochromatic copy of P_n. To do this we will use the fact that H satisfies a particular property (Lemma 3.2). We shall also make use of the following result.

Theorem 2.3 (Pokrovskiy [23, Theorem 1.7])**.** *Let $k \geqslant 1$. Suppose that the edges of K_n are coloured with red and blue. Then K_n can be covered by k vertex-disjoint blue paths and a vertex-disjoint red balanced complete $(k+1)$-partite graph.*

We remark that we do not need the full strength of this result, in the sense that we do not need the complete $(k+1)$-partite graph to be balanced; it suffices for us to know that the vertex classes are of comparable cardinality. Such a result can be derived easily by iterating Lemma 1.5 in [23], for which Pokrovskiy gives a short and elegant proof (see also [22, Lemma 1.10]).

We shall also use the classical Kővári–T. Sós–Turán theorem [20], in the following simple form.

Theorem 2.4. *Let G be a balanced bipartite graph with t vertices in each vertex class. If G contains no $K_{s,s}$, then G has at most $4t^{2-1/s}$ edges.*

Let us now give a brief outline of how we find our monochromatic copy of P_n^k in a 2-edge coloured H_t^k. Suppose the edges of H_t^k have been coloured red and blue by an arbitrary colouring χ. Recall that H_t^k is obtained by blowing up H^k; in particular, the vertices v of H^k become large complete graphs $C(v)$ in H_t^k. By the choice of parameters, Ramsey's theorem tells us that each such $C(v)$ contains a monochromatic copy $B(v)$ of K_t. We may assume without loss of generality that at least half of the $B(v)$ are blue.

Let F be the subgraph of H induced by the vertices v such that $B(v)$ is blue. We shall define an auxiliary edge-colouring χ' of F^k. By using Theorem 2.3 we shall be able to find either (i) a blue P_n in F^k under χ' or (ii) a P_n in F (not in F^k) with certain additional properties. The path in (ii) will be found applying Lemma 3.2 with the sets A_i being the vertex classes of a red complete $(k+1)$-partite subgraph of F^k. This red complete $(k+1)$-partite subgraph of F^k will be found using Theorem 2.3, applied to a suitable red/blue coloured complete graph (we complete F^k with its auxiliary colouring χ' to a red/blue coloured complete graph by considering non-edges of F^k red).

In case (i), where we find a blue P_n in F^k under the colouring χ', we shall be able to find a blue P_n^k in H_t^k. In case (ii), the properties of the path P_n found in F will ensure the existence of a red P_n^k in F^k. It will then be easy to find a red P_n^k in $F_t^k \subseteq H_t^k$. The idea of defining an auxiliary graph on monochromatic cliques as above was used in [1].

§3. Proof of Theorem 1.2

Our first lemma guarantees the existence of bounded-degree graphs with the pseudo-randomness property we require.

Lemma 3.1. *For every positive constants ε and a, there is a constant b such that, for any large enough n, there is a graph H with $v(H) = an$ such that:*

(1) For every pair of disjoint sets S, $T \subseteq V(H)$ with $|S|, |T| \geqslant \varepsilon n$, we have $|E_H(S,T)| > 0$.
(2) $\Delta(H) \leqslant b$.

Proof. Fix positive constants ε and a. Let $c = 4a/\varepsilon^2$ and $b = 4ac$ and consider a sufficiently large n. Let $G = G(2an, p)$ be the binomial random graph with $p = c/n$. By Chernoff's inequality, with high probability we have $|E(G)| < (4a^2c)n$. Moreover, with high probability G satisfies (1) (with $H = G$) by the following reason: Let X_G be the number of pairs of disjoint subsets of $V(G)$ of size εn with no edges between them. Then, from the choice of c and using Markov's inequality, we have

$$\mathbb{P}[X_G \geqslant 1] \leqslant \mathbb{E}[X_G] \leqslant \binom{2an}{\varepsilon n}^2 \left(1 - \frac{c}{n}\right)^{(\varepsilon n)^2} < 2^{4an} \cdot e^{-c\varepsilon^2 n} = o(1).$$

Thus, there is a graph G with $|E(G)| < (4a^2c)n$ and $X_G = 0$.

Now let H be a subgraph of G obtained by iteratively removing a vertex of maximum degree until exactly an vertices remain. Then $\Delta(H) \leqslant b$, as otherwise, from the choice of b we would have deleted more than $b \cdot an > |E(G)|$ edges from G during the iteration, which contradicts property (1). Moreover, as H is an induced subgraph of G, (1) is maintained. This completes the proof of the lemma. □

We now show that any graph satisfying the hypothesis of Lemma 3.1 and property (1) also satisfies an additional property.

Lemma 3.2. *For every integer $k \geqslant 1$ and every $\varepsilon > 0$ there exists $a_0 > 0$ such that the following holds for any $a \geqslant a_0$. Let H be a graph with an vertices such that for every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geqslant \varepsilon n$ we have $|E_H(S,T)| > 0$. Then, for every family $A_1, \ldots, A_{k+1} \subseteq V(H)$ of pairwise disjoint sets each of size at least εan, there is a path $P_n = (x_1, \ldots, x_n)$ in H with $x_i \in A_j$ for all $1 \leqslant i \leqslant n$, where $j \equiv i \pmod{k+1}$.*

Algorithm 1:

Input : a graph H with $v(H) = an$ satisfying (1) and sets $A_i \subseteq V(H)$ $(1 \leqslant i \leqslant k+1)$ with $A_i \cap A_j = \varnothing$ for all $i \neq j$ and $|A_i| \geqslant \varepsilon an$ for all i.

Output : a path $P_n = (x_1, \dots, x_n)$ in H with $x_i \in A_j$ for all i, where $j \equiv i \pmod{k+1}$.

foreach $1 \leqslant i \leqslant k+1$ **do**
  $U_i \leftarrow A_i$; $D_i \leftarrow \varnothing$
while $|D_i| \leqslant |A_i|/2$ *for all* i **do**
  pick $x_1 \in U_1$ and let $P = (x_1)$; $r \leftarrow 1$; $U_1 \leftarrow U_1 \setminus \{x_1\}$
  while $1 \leqslant |P| < n$ **do**
    `//` $P = (x_1, \dots, x_r)$ `with` $r \geqslant 1$
    if $\exists u \in U_{r+1}$ *with* $\{x_r, u\} \in E(H)$ **then**
      $x_{r+1} \leftarrow u$; $U_{r+1} \leftarrow U_{r+1} \setminus \{u\}$
      $P \leftarrow (x_1, \dots, x_r, x_{r+1})$; $r \leftarrow r+1$
    else
      $D_r \leftarrow D_r \cup \{x_r\}$
      $P \leftarrow (x_1, \dots, x_{r-1})$; $r \leftarrow r-1$
  if $|P| = n$ **then**
    return P `// path has been found`
STOP with failure `// this will not happen`

To prove Lemma 3.2, we analyse a depth first search algorithm, adapting a proof idea in [5, Lemma 4.4]. More specifically, we run an algorithm (stated formally as Algorithm 1). Our algorithm receives as input a graph H with $v(H) = an$ satisfying property (1), and a family of pairwise disjoint sets $A_1, \dots, A_{k+1} \subseteq V(H)$ with $|A_i| \geqslant \varepsilon an$ for all i. The output of $\mathcal{A}$ is a path $P_n = (x_1, \dots, x_n)$ in H with $x_i \in A_j$ for all i, where $j \equiv i \pmod{k+1}$.

As it runs, the algorithm builds a path $P = (x_1, \dots, x_r)$ with $x_i \in A_j$ for all i and j with $j \equiv i \pmod{k+1}$. Furthermore, it maintains sets U_j and $D_j \subseteq A_j$ for all j, with the property that U_j, D_j, and $V(P) \cap A_j$ form a partition of A_j for every j. The cardinality of the sets U_j decrease as the algorithm runs, while the D_j increase. As the algorithm runs, we have $r = |P| < n$ and it searches for an edge $\{x_r, u\} \in E(H)$ where u belongs to the set U_{r+1} of *unused* vertices in A_{r+1}. If such a vertex $u \in U_{r+1}$ is found, then P is made one vertex longer by adding u to it. If there is no such vertex u, then x_r is declared a *dead end* and it is put into D_r. Moreover, the path P is shortened by one vertex; it becomes $P = (x_1, \dots, x_{r-1})$. Our algorithm iterates this procedure. If we find a path P with n vertices this way, then we are done.

We now analyse Algorithm 1.

Proof of Lemma 3.2. We will prove that Algorithm 1 returns a path P on line 13 as desired, instead of terminating with failure on line 14.

Fix an integer $k \geqslant 1$ and $\varepsilon > 0$. Let

$$a_0 = 2 + \frac{4}{\varepsilon(k+1)}, \tag{3.3}$$

fix $a \geqslant a_0$ and let n be sufficiently large. Let H be a graph with an vertices satisfying property (1), i.e., for every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geqslant \varepsilon n$ we have $|E_H(S,T)| > 0$. Let $A_1, \ldots, A_{k+1} \subseteq V(H)$ be a family of pairwise disjoint sets each of size at least εan.

First recall that U_i, D_i, and $V(P) \cap A_i$ form a partition of A_i for every i. Since the path P is always empty on line 4, at this point we have $|U_1| \geqslant |A_1| - |D_1| \geqslant |A_1|/2 > 0$. Then, line 4 is always executed successfully.

Suppose now that $\mathcal{A}$ stops with failure on line 14. Then, for some i, say $i = r$, the set $D_i = D_r$ became larger than $|A_r|/2 \geqslant \varepsilon an/2 \geqslant \varepsilon n$. Furthermore, we have $|P| < n$ and $|D_{r+1}| \leqslant |A_{r+1}|/2$ (indices modulo $k+1$) and hence,

$$|U_{r+1}| \geqslant |A_{r+1}| - |D_{r+1}| - |V(P) \cap A_{r+1}| \geqslant \frac{1}{2}|A_{r+1}| - \left\lceil \frac{n}{k+1} \right\rceil \geqslant \frac{1}{2}\varepsilon an - \frac{2n}{k+1} > \varepsilon n.$$

Note that this is the only place where the exact value of a_0 is used. Applying property (1) to the pair (D_r, U_{r+1}), we see that there is an edge $\{x, u\} \in E(H)$ with $x \in D_r$ and $u \in U_{r+1}$. Consider the moment in which x was put into D_r. This happened on line 10, when P had x as its foremost vertex and $\mathcal{A}$ was trying to extend P further into U_{r+1}. At this point, because of the edge $\{x, u\} \in E(H)$, we must have had $u \notin U_{r+1}$ (see line 6). Since the set U_{r+1} decreases as $\mathcal{A}$ runs, this is a contradiction and hence $\mathcal{A}$ does not terminate on line 14.

Since $\sum_{1 \leqslant i \leqslant k+1} \big(|D_i| - |U_i|\big)$ increases as Algorithm 1 runs, we know the algorithm terminates. Therefore, we conclude that it returns a suitable path P as claimed. □

We are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $k \geqslant 1$ and let $\varepsilon = 1/3(k+1)$. Let a_0 be the constant given by an application of Lemma 3.2 with parameters k and ε. Set $a = \max\{6k, a_0\}$ and let b be given by Lemma 3.1 for this choice of a. Moreover, let H be a graph with $|V(H)| = an$ and $\Delta(H) \leqslant b$ be as in Lemma 3.1. Finally, put $t = (64k)^{2k}$ and $s = 2k$.

Let H_t^k be a complete-t-blow-up of H^k, as in Definition 2.1, and let $\chi\colon E(H_t^k) \to \{\text{red}, \text{blue}\}$ be an edge-colouring of H_t^k. We shall show that H_t^k contains a monochromatic copy of P_n^k under χ. By the definition of H_t^k, any cluster $C(v)$ contains a monochromatic copy $B(v)$ of K_t. Without loss of generality, the set $W := \{v \in V(H)\colon B(v) \text{ is blue}\}$ has cardinality at least $v(H)/2$. Let $F := H[W]$ be the subgraph of H induced by W, and let F' be the subgraph of $F_t^k \subseteq H_t^k$ induced by $\bigcup_{w \in W} V(B(w))$.

Given the above colouring χ, we define a colouring χ' of F^k as follows. An edge $\{u, v\} \in E(F^k)$ is coloured *blue* if the bipartite subgraph $F'[V(B(u)), V(B(v))]$ of F' naturally induced by the sets $V(B(u))$ and $V(B(v))$ contains a blue $K_{s,s}$. Otherwise $\{u, v\}$ is coloured *red*.

Claim 3.4. *Any 2-colouring of $E(F^k)$ has either a blue P_n or a red P_n^k.*

Proof. We apply Theorem 2.3 to F^k, where if an edge is not present in F^k, then we consider it to be in the red colour class. If F^k contains a blue copy of P_n, then we are done. Hence we may assume F^k contains a balanced, complete $(k+1)$-partite graph K with parts $A_1, \dots, A_{k+1}$ on at least $v(F^k) - kn \geqslant an/2 - kn$ vertices, with no blue edges between any two parts. As $a \geqslant 6k$, each one of these parts has size at least

$$\frac{1}{k+1}\left(\frac{1}{2}a - k\right)n \geqslant \varepsilon an. \tag{3.5}$$

By Lemma 3.2 applied to the collection of sets of vertices $A_1, \dots, A_{k+1}$ of $F \subseteq H$ (specifically F and not F^k), we see that $F[V(K)]$ contains a path with n vertices such that any consecutive $k+1$ vertices are in distinct parts of K. Therefore $F^k[V(K)]$ contains a copy of P_n^k in which every pair of adjacent vertices are in distinct parts of K. By the definition of K, such a copy is red. □

By Claim 3.4, F^k contains a blue copy of P_n or a red copy of P_n^k under the edge-colouring χ'. Thus, we can split our proof into these two cases.

Case 1. First suppose F^k contains a blue copy $(x_1, \dots, x_n)$ of P_n. Then, for every $1 \leqslant i \leqslant n-1$, the bipartite graph $F'[V(B(x_i)), V(B(x_{i+1}))]$ contains a blue copy of $K_{s,s}$, with, say, vertex classes $X_i \subseteq V(B(x_i))$ and $Y_{i+1} \subseteq V(B(x_{i+1}))$. As $|X_i| = |Y_i| = s = 2k$ for all $2 \leqslant i \leqslant n-1$, we can find sets $X_i' \subseteq X_i$ and $Y_i' \subseteq Y_i$ such that $|X_i'| = |Y_i'| = k$ and $X_i' \cap Y_i' = \varnothing$ for all $2 \leqslant j \leqslant n-1$. Let $X_1' = X_1$ and $Y_n' = Y_n$.

We now show that the set $U := \bigcup_{i=1}^{n-1} X_i' \cup \bigcup_{i=2}^{n} Y_i'$ provides us with a blue copy of P_{2kn}^k in $F' \subseteq H_t^k$. Note first that $|U| = 2k + 2k(n-2) + 2k = 2kn$. Let $u_1, \dots, u_{2kn}$ be an ordering of U such that, for each i, every vertex in X_i' comes before any vertex in Y_{i+1}' and after every vertex in Y_i'. By the definition of the sets X_i' and Y_i' and the construction of $F' \subseteq F_t^k \subseteq H_t^k$, each vertex u_j is adjacent in blue to $\{u_{j'} \in U \colon 1 \leqslant |j - j'| \leqslant k\}$. Thus, U contains a blue copy of P_{2nk}^k, as claimed.

Case 2. Now suppose F^k contains a red copy P of P_n^k. That is, F^k contains a set of vertices $\{x_1, \dots, x_n\}$ such that x_i is adjacent in red to all x_j with $1 \leqslant |j - i| \leqslant k$. We shall show that, for each $1 \leqslant i \leqslant n$, we can pick a vertex $y_i \in V(B(x_i))$ so that $y_1, \dots, y_n$ define a red copy of P_n^k in $F' \subseteq F_t^k \subseteq H_t^k$. We do this by applying the local lemma [13, p. 616] (a greedy strategy also works).

We have to show that it is possible to pick the y_i $(1 \leqslant i \leqslant n)$ in such a way that $\{y_i, y_j\}$ is a red edge in F' for every i and j with $1 \leqslant |i - j| \leqslant k$. Let us choose $y_i \in V(B(x_i))$ $(1 \leqslant i \leqslant n)$ uniformly and independently at random. Let $e = \{x_i, x_j\}$ be an edge in $P \subseteq F^k$. We know that e is red. Let A_e be the event that $\{y_i, y_j\}$ is a *blue* edge in F'. Since the edge e is red, we know that the bipartite graph $F'[V(B(x_i)), V(B(x_j))]$ contains no blue $K_{s,s}$. Theorem 2.4 then tells us that $\mathbb{P}[A_e] \leqslant 4t^{-1/s}$.

The events A_e are not independent, but we can define a dependency graph D for the collection of events A_e ($e \in E(P)$) by adding an edge between A_e and A_f if and only if $e \cap f \neq \varnothing$. Then $\Delta(D) \leqslant 4k$. Given that

$$4\Delta\mathbb{P}[A_e] \leqslant 64kt^{-1/s} = 1 \tag{3.6}$$

for all e, the Local Lemma tells us that $\mathbb{P}\big[\bigcap_{e\in E(P)} \bar{A}_e\big] > 0$, and hence a simultaneous choice of the y_i ($1 \leqslant i \leqslant n$) as required is possible. This completes the proof of Theorem 1.2. □

Throughout our proof we have used probabilistic methods to show the existence of G. We now briefly discuss how our proof could be made constructive. For instance, it suffices to take for H a suitable (n, d, λ)-graph as in Alon and Chung [2], namely, it is enough to have $\lambda = O(\sqrt{d})$ and d large enough with respect to k and $1/\varepsilon$.

§4. Open questions

We make no attempts to optimise the constant given by our argument, so the following question is of interest.

Question 4.1. *For any integer $k \geqslant 2$, what is $\limsup_{n\to\infty} \hat{r}(P_n^k)/n$?*

It is also interesting to consider what happens when more than two colours are at play. For $q \in \mathbb{N}$, let $\hat{r}_q(H)$ denote the *q-colour size-Ramsey number* of H, that is, the smallest number of edges in a graph that is q-Ramsey for H.

Conjecture 4.2. *For any $q, k \in \mathbb{N}$ we have $\hat{r}_q(P_n^k) = O(n)$.*

It is conceivable that in hypergraphs the size-Ramsey number (defined analogously as for graphs) of tight paths may be linear. Let $H_n^{(k)}$ denote the tight path of uniformity k on n vertices; that is, $V(H_n^{(k)}) = [n]$ and $E(H_n^{(k)}) = \big\{\{1, \ldots, k\}, \{2, \ldots, k+1\}, \ldots, \{n-k+1, \ldots, n\}\big\}$. The following question appears as Question 2.9 in [9].

Question 4.3. *For any $k \in \mathbb{N}$, do we have $\hat{r}(H_n^{(k)}) = O(n)$?*

Finally, we note that for fixed k, our main result implies the linearity of the size Ramsey number for the grid graphs $G_{k,n}$, the cartesian product of the paths P_k and P_n. Indeed our main result implies the linearity of the size Ramsey number for any sequence of graphs with bounded bandwidth. For the d-dimensional grid graph G_n^d, obtained by taking the cartesian product of d copies of P_n, we raise the following question.

Question 4.4. *For any integer $d \geqslant 2$, is $\hat{r}(G_n^d) = O(n^d)$?*

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References

[1] P. Allen, G. Brightwell, and J. Skokan, *Ramsey goodness and otherwise*, Combinatorica **33** (2013), no. 2, 125–160. ↑4

[2] N. Alon and F. R. K. Chung, *Explicit construction of linear sized tolerant networks*, Discrete Math. **72** (1988), no. 1-3, 15–19. MR975519 ↑2, 8

[3] J. Beck, *On size Ramsey number of paths, trees, and circuits. I*, J. Graph Theory **7** (1983), no. 1, 115–129. MR693028 ↑2

[4] ———, *On size Ramsey number of paths, trees and circuits. II*, Mathematics of Ramsey theory, 1990, pp. 34–45. MR1083592 ↑2

[5] I. Ben-Eliezer, M. Krivelevich, and B. Sudakov, *The size Ramsey number of a directed path*, J. Combin. Theory Ser. B **102** (2012), no. 3, 743–755. MR2900815 ↑2, 5

[6] D. Conlon, *Question suggested for the ATI–HIMR Focused Research Workshop: Large-scale structures in random graphs,* Alan Turing Institute, December 2016. ↑2, 9

[7] D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory*, Surveys in combinatorics 2015, 2015, pp. 49–118. MR3497267 ↑1

[8] D. Dellamonica Jr., *The size-Ramsey number of trees*, Random Structures Algorithms **40** (2012), no. 1, 49–73. MR2864652 ↑2

[9] A. Dudek, S. L. Fleur, D. Mubayi, and V. Rödl, *On the size-Ramsey number of hypergraphs*, J. Graph Theory **86** (2017), no. 1, 104–121. ↑8

[10] A. Dudek and P. Prałat, *An alternative proof of the linearity of the size-Ramsey number of paths*, Combin. Probab. Comput. **24** (2015), no. 3, 551–555. MR3326432 ↑2

[11] P. Erdős, *On the combinatorial problems which I would most like to see solved*, Combinatorica **1** (1981), no. 1, 25–42. MR602413 ↑1

[12] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161. MR479691 ↑1

[13] P. Erdős and L. Lovász, *Problems and results on* 3*-chromatic hypergraphs and some related questions*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, 1975, pp. 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. MR52#2938 ↑7

[14] J. Friedman and N. Pippenger, *Expanding graphs contain all small trees*, Combinatorica **7** (1987), no. 1, 71–76. MR905153 ↑2

[15] P. E. Haxell and Y. Kohayakawa, *The size-Ramsey number of trees*, Israel J. Math. **89** (1995), no. 1-3, 261–274. MR1324465 ↑2

[16] P. E. Haxell, Y. Kohayakawa, and T. Łuczak, *The induced size-Ramsey number of cycles*, Combin. Probab. Comput. **4** (1995), no. 3, 217–239. MR1356576 ↑2

[17] X. Ke, *The size Ramsey number of trees with bounded degree*, Random Structures Algorithms **4** (1993), no. 1, 85–97. MR1192528 ↑2

[18] Y. Kohayakawa, T. Retter, and V. Rödl, *The size-Ramsey number of short subdivisions of bounded degree graphs*, Random Structures Algorithms (2018). To appear, 36pp. ↑2

[19] Y. Kohayakawa, V. Rödl, M. Schacht, and E. Szemerédi, *Sparse partition universal graphs for graphs of bounded degree*, Adv. Math. **226** (2011), no. 6, 5041–5065. MR2775894 ↑2

[20] T. Kővári, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57. MR0065617 ↑3

[21] S. Letzter, *Path Ramsey number for random graphs*, Combin. Probab. Comput. **25** (2016), no. 4, 612–622. MR3506430 ↑2

[22] A. Pokrovskiy, *Partitioning edge-coloured complete graphs into monochromatic cycles and paths*, J. Combin. Theory Ser. B **106** (2014), 70–97. MR3194196 ↑3

[23] ———, *Calculating Ramsey numbers by partitioning colored graphs*, J. Graph Theory **84** (2017), no. 4, 477–500. ↑3

[24] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **S2-30** (1930), no. 1, 264. MR1576401 ↑1

[25] D. Reimer, *The Ramsey size number of dipaths*, Discrete Math. **257** (2002), no. 1, 173–175. MR1931501 ↑2

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