# Much ado about nothing

## The superconformal index and Hilbert series of three dimensional $\mathscr{N} = 4$ vacua

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This dissertation is submitted for the degree of
*Doctor of Philosophy*

Queens' College September 2018

In dedication to Lydia Prieg

# Declaration

This dissertation is based on research carried out while a graduate student at the Department of Applied Mathematics and Theoretical Physics from October 2014 to June 2018. The material in section 4 is based on the work [19] done in collaboration with Prof N. Dorey, N. Lohitsiri, Prof D. Tong and Dr C. Turner. Sections 5 and 6 are wholly the work of the author.

No part of this work has been submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

Alexander E. Barns-Graham
September 2018

# Acknowledgements

First and foremost, I thank my supervisor Nick Dorey, for teaching and advising me during the PhD. His insight and mastery of the subject remains an inspiration. I thank my collaborators Alex Arvanitakis, Nakarin Lohitsiri, David Tong, Paul Townsend and Carl Turner. Working with all of them has been a pleasure and a privilege.

I have had the great fortune of meeting and talking to many knowledgeable people over the past four years. I thank Alex Abbott, Louise Anderson, Jack Bartley, Philip Boyle-Smith, Santiago Cabrera, Felipe Contatto, Laurence Cooper, Joe Davighi, Cohl Furey, Julia Goedecke, Amihay Hanany, Joe Kamnitzer, Chris King, Charlotte Kirchhoff-Lukat, Wicher Malten, Nick Manton, Kai Roehrig, David Skinner, Joe Waldron, Claude Warnick, Ben Webster, Jack Williams, Kenny Wong, Antoni Woss, Yegor Zenkevich and Tomáš Zemen for providing many illuminating conversations on physics and mathematics. I especially thank Andrew Singleton and Sam Crew, who provided great help much of the work.

Thank you to the PhD group in DAMTP for providing such lively coffee and lunch times conversations, the quality of the conversation has likely added months to this PhD through lost work time! Thank you to all my graduate friends from Queens' for providing a welcome break from physics.

Thank you to Manda Stagg for making this whole department run as smoothly as it does, and for all your great recommendations of coffee shops and music.

I would like to thank my family for raising me and for their love and support.

Finally, I thank my wife who not only provided a lot of help with the grammar and proofreading; but has also been a great moral support for the past four years. This thesis is dedicated to you.

# Abstract

We study a quantum mechanical $\sigma$-model whose target space is a hyperKähler cone. As shown by Singleton, [184], such a theory has superconformal invariance under the algebra $\mathfrak{osp}(4^*|4)$. One can formally define a superconformal index that counts the short representations of the algebra. When the hyperKähler cone has a projective symplectic resolution, we define a regularised superconformal index. The index is defined as the equivariant Hirzebruch index of the Dolbeault cohomology of the resolution, hereafter referred to as the index. In many cases, the index can be explicitly calculated via localisation theorems. By limiting to zero the fugacities in the index corresponding to an isometry, one forms the index of the submanifold of the target space invariant under that isometry. There is a limit of the fugacities that gives the Hilbert series of the target space, and often there is another limit of the parameters that produces the Poincaré polynomial for $\mathbb{C}^\times$-equivariant Borel-Moore homology of the space.

A natural class of hyperKähler cones are Nakajima quiver varieties. We compute the index of the $A$-type quiver varieties by making use of the fact that they are submanifolds of instanton moduli space invariant under an isometry.

Every Nakajima quiver variety arises as the Higgs branch of a three dimensional $\mathcal{N} = 4$ quiver gauge theory, or equivalently the Coulomb branch of the mirror dual theory. We show the equivalence between the descriptions of the Hilbert series of a line bundle on the ADHM quiver variety via localisation, and via Hanany's monopole formula.

Finally, we study the action of the Poisson algebra of the coordinate ring on the Hilbert series of line bundles. We restrict to the case of looking at the Coulomb branch of balanced $ADE$-type quivers in a certain infinite rank limit. In this limit, the Poisson algebra is a semiclassical limit of the Yangian of $ADE$-type. The space of global sections of the line bundle is a graded representation of the Poisson algebra. We find that, as a representation, it is a tensor product of the space of holomorphic functions with a finite dimensional representation. This finite dimensional representation is a tensor product of two irreducible representations of the Yangian, defined by the choice of line bundle. We find a striking duality between the characters of these finite dimensional representations and the generating function for Poincaré polynomials.

# Table of contents

# List of figures

# List of tables

# Chapter 1

# Introduction

Quantum field theory is one of the greatest achievements of the scientific community. It is a theory that has had unprecedented experimental success[1]. Yet despite its accomplishments, there still remain basic questions that we are unable to answer. This is because the success has been through the use of perturbation series in the coupling via summing Feynman diagrams. However, as soon as the coupling becomes large, we are unable to use this technique. We can only trust at most a handful of terms in the asymptotic expansion, and they do not give us much information.

In order to answer some of these questions, we turn to toy models. The heuristic is to make use of symmetries of the theory in order to constrain physical quantities to the point where we can compute them analytically. However, there are theoretical restrictions on what symmetries are possible.

In 1967, Coleman and Mandula showed in [57] that, under reasonable assumptions, the most general Lie algebra of symmetries of the S-matrix is the Poincaré group direct sum with some compact Lie algebra of flavours. However, there are ways around this no-go theorem. If one relaxes the assumption of massive particles, then the theory can be conformal. Two other ways around the Coleman-Mandula theorem take advantage of the fact that it restricts the possibility of Lie algebras, but not more general algebras. In less than three dimensions, one can have a quantum group symmetry, and in any number of dimensions one could have supersymmetry (with supersymmetry a very similar theorem to Coleman-Mandula holds, proven in 1974 by Haag, Sohnius and Lopuszanski in [97]). An example of a quantum field theory that has all three of these symmetries is planar four dimensional $\mathscr{N}=4$ super

[1] The current experimental measures of the magnetic moment of the electron, [99], are as precise as measuring the distance from the Earth to the moon to within the breadth of a human hair

Yang-Mills[2], which is a superconformal symmetry with Yangian symmetry, see chapter I of [23] and the references therein. All three of these symmetries have been used to calculate many observables exaxtly, and much has been learned about QFT from this theory, because of this it is colloquially known as the "harmonic oscillator of the 21$^{\text{st}}$ century".

The particular toy model that we will be looking at is superconformal quantum mechanics, with $\mathfrak{osp}(4^*|4)$ symmetry. In order to have this much symmetry, the target space must be a hyperKähler cone. The Poisson algebra of holomorphic functions on the target space can be the semiclassical limit of a quantum group, and so in these theories we find supersymmetry, conformal symmetry and sometimes quantum group symmetry.

This model has a plethora of applications to physics. In many supersymmetric quantum field theories, the space of BPS states of a given charge are described by the cohomology of a quiver, see [70, 6, 7] and references therein, which is readily computed in this model. The AdS/CFT correspondence, [144], is poorly understood in the case of $\text{AdS}_2/\text{CFT}_1$, [181, 47]. This particular case is important as the near horizon geometries of extremal black holes tends to be of the form $\text{AdS}_2 \times K$ for $K$ some compact manifold, [31, 187]. The superconformal index of the quantum mechanics that we consider is related, in a certain limit, to the black hole entropy of the dual extremal black hole.

A procedure known as discrete light cone quantisation (DLCQ) compactifies a null direction of a quantum theory on a circle, [168, 167]. One finds a non-negative tower of Kaluza-Klein modes, with a finite number of degrees of freedom for each KK mode. This suggests the tantalising idea that we could use finite dimensional quantum mechanics to solve quantum field theories. However, the theory generally has issues with strong coupling to zero modes that make it untenable, [104]. Nonetheless, for maximally supersymmetric theories these problems are avoided. Thus, DLCQ provides a viable route to understanding mysterious theories such as the six dimensional $A_N$ (2,0) theory. The DLCQ of this theory is given by an $\mathfrak{osp}(4^*|4)$ quantum mechanics, whose target space is instanton moduli space, [3, 4].

The superconformal quantum mechanical model has a purely geometrical interpretation. The Hilbert space is the space of normalisable differential forms, and all the generators of $\mathfrak{osp}(4^*|4)$ have geometrical interpretations, [202, 184, 185]. We study the superconformal index of the superconformal quantum mechanics, and find that it is the Hirzebruch index of the target space.

A large class of hyperKähler cones is given by the vacua of three dimensional $\mathcal{N} = 4$ quiver gauge theories. We use these as a class of examples of $\mathfrak{osp}(4^*|4)$ quantum mechanics

[2]Of course this theory is four dimensional and four is not less than three, but via $AdS/CFT$-correspondence it is dual to a string worldsheet description, which is two dimensional.

that we can study. The vacua are the states in the quantum theory with lowest energy, which for a supersymmetric theory is zero.

In general, the question of what are the vacua of a quantum field theory is hard. The low energy of a theory can be strongly coupled, for example, in three dimensions where the classical dimension of the gauge coupling is one[3], we expect large coupling as we go to low energy. Nonetheless, for eight supercharge theories, there has been a large amount of progress in computing the vacua. In the seminal work [178], Seiberg and Witten computed the vacua of four-dimensional pure $SU(2)$ gauge theory, and through this work showed the first ever derivation of confinement in a four dimensional quantum field theory. They extended it to the full vacua of all four dimensional $SU(2)$ UV-complete gauge theories in [179].

In four dimensions, one has to be careful about whether the theory exists in the UV, as the gauge coupling is marginal. In three dimensions, there are no such worries, as the theory is asymptotically free. The problem of understanding the vacua of three dimensional eight supercharge theories was begun in [180]. The vacua splits into two hyperKähler cones, known as the Higgs and Coulomb branch that meet at the origin, with singular submanifolds along which mixed branches lie.

Three dimensional $\mathcal{N} = 4$ quantum field theories possess a quantum duality known as three dimensional mirror symmetry, [114]. This duality states the Higgs branch of one theory is isomorphic to the Coulomb branch of the mirror dual theory and vice versa. The advantage of this duality, for our purposes, is that it generally gives two different ways of computing the same observable. One often finds that a limit one is interested in is difficult to take in Higgs branch language, but quite simple to take in Coulomb branch language or vice versa.

In chapter 2 of this thesis, we review aspects of three dimensional quiver gauge theory. We start with the field content, the Lagrangian, the vacua and the relevent non-renormalisation theorems. We then give a detailed review of how Nakajima quiver varieties correspond to the Higgs branch, and how the chiral ring of the Coulomb branch can give us the Coulomb branch. We then review three dimensional mirror symmetry, three dimensional Seiberg “duality" (with the quotation marks explained in the chapter) and how these dualities can be understood from branes. We finally discuss deformation quantisation of the chiral ring and how the Ω-deformation can produce such a quantisation.

In chapter 3, we begin our study of the superconformal quantum mechanics. We take the target space to be the vacua of three dimensional quiver gauge theory. This model was first developed by Singleton in 2014-2016 in [184, 185]. It is a generalisation of the ideas first laid out by Witten in 1982 in [202]. It gives us a model where we can understand the

[3] In conventions where the gauge covariant derivate is $D = \partial + A$.

Hilbert space geometrically in terms of differential forms. After defining the theory and the superconformal index, we are left with the task of computing the index. To do this, it is necessary to regularise the theory. We do this by taking the projective symplectic resolution, and see that the superconformal index is invariant under the choice of resolution. Finally, we show how equivariant K-theory localisation theorems can be used to explicitly compute the graded partition function of the theory in many cases.

Chapter 4 is a description of the work in [19]. In this work the Hilbert series of instanton moduli space in the presence of a five dimensional Chern-Simon term is shown to be equal to the Hilbert series of the Coulomb branch of the mirror dual quiver. This derivation reflects the work of Nakajima [159] that describes the Coulomb branch as a Cherkis bow variety. We briefly review this. We show how this derivation also works for a certain linear quiver.

In chapter 5, we compute the supeconformal index of chapter 3 for linear quivers (and affine $A$-type quivers). We show how to compute the fixed points, using an observation of [154] that describes the linear quiver variety as a connected $\mathbb{C}^{\times}$-invariant subspace of instanton moduli space, and the fact, proven in this thesis, that the superconformal index is stable under restrictions to subspsaces. We use this to compute the superconformal index of the superconformal quantum mechanics whose target space is the Higgs branch of an $A$- or $\hat{A}$-type quiver, for any linear quiver.

We give a rigorous proof of a result from the string theory literature [85] about how to derive different linear quiver Hilbert series from "nicer" linear quivers. Using this result, we are able to write the Hilbert series and the superconformal index in terms of special symmetric functions.

In chapter 6, we investigate the action of the Poisson algebra of holomorphic functions on sections of line bundles of hyperKähler cones. We consider the models whose Poisson algebra is the classical limit of the Yangian. This is derived from a certain infinite rank limit of the Coulomb branch of balanced $ADE$-type quivers. We explicitly decompose the sections of line bundles into representations of the Poisson algebra. We further find a striking duality, where the character of sections of line bundles of a particular charge contains the generating function of Poincaré polynomials of finite rank quiver varieties.

In chapter 7, we conclude with a summary of the main results and a discussion of avenues for further work.

In appendix A, we summarise the Jeffrey-Kirwan pole procedure. This is the method used to check if the superconformal index is indeed invariant under wall crossing. We give a brief description of the theory, a simple calculation in the case of counting line bundles on $\mathbb{CP}^1$, and then a more convoluted calculation of a graded count of line bundles on a simple linear quiver - an example where wall crossing does happen.

# Chapter 2

# Three dimensional $\mathscr{N} = 4$ quiver gauge theories

In this chapter we review aspects of three dimensional $\mathscr{N} = 4$ quantum field theory and mathematics relevant to this thesis.

We start in section 2.1 by giving the field content and the UV-Lagrangian of such quantum field theories. This will lead us to writing the scalar potential, and a definition of quiver field theory. We then, in section 2.2, look at the classical and quantum vacua, reviewing the relevant non-normalisation theorems. We discuss the two types of vacua for quiver field theories, the Higgs and Coulomb branch, in sections 2.2.1 and 2.2.2 respectively. The Higgs branch section reviews two constructions of the Higgs branch: the hyperKähler quotient and the geometric invariant theory construction. We define the branch algebraically through the coordinate ring, which is the chiral ring of the field theory. We then, in section 2.2.3, define the Hilbert series of the chiral ring of the Higgs branch via a Molien integral. In section 2.3, we review some three dimensional dualities and the brane construction for $A$-type and $\hat{A}$-type quiver gauge theories. In section 2.3.1, we review three dimensional mirror symmetry. In section 2.3.2, we review the Hanany-Witten brane construction. In section 2.3.3, we discuss the non-duality that is three dimensional Seiberg duality. In section 2.4, we discuss the Poisson algebra of both the Higgs and Coulomb branch. In section 2.4.1, we discuss the deformation quantisation of a Poisson algebra generally, and in section 2.4.2, we describe a way within three dimensional gauge theory to realise such a quantisation of the chiral ring.

## 2.1 Field content and UV Lagrangian

We start by describing the main subject of study, supersymmetric quantum field theories with 8 supercharges. We will be primarily interested in three dimensional 8 supercharge theories, i.e. $\mathcal{N} = 4$ 2 + 1-dimensional gauge theories (we will often say three-dimensional, but we are always working in Minkowski signature). The reference for our convention is [192, 200].

We use gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1}$$

The reality of the gamma-matrices in this basis shows that in $2+1$ dimensions one can have Majorana spinors. Note that in $3+0$ dimensions, it is not possible to write a real representation of the gamma-matrices, and hence the minimal spinor is the Dirac spinor.

We use supercharges $Q_{ia\alpha}$ for $i, a, \alpha = 1, 2$; $i$ for $SU(2)_{\text{Lorentz}}$, $a$ for $SU(2)_H$ and $\alpha$ for $SU(2)_C$. $SU(2)_H \times SU(2)_C$ is the largest group of symmetries of the following anticommutator that commutes with Lorentz group and is called the R-symmetry.

$$\{Q_{ia\alpha}, Q_{jb\beta}\} = -2\varepsilon_{ab}\varepsilon_{\alpha\beta}\sigma^{\mu}_{ij}P_{\mu} + 2\varepsilon_{ij}(\varepsilon_{ab}Z^C_{\alpha\beta} + \varepsilon_{\alpha\beta}Z^H_{ab}). \tag{1.2}$$

Here $\sigma^{\mu}_{ij}$ are the Pauli matrices, and the central charges are

$$Z^H_{11} = (Z^H_{22})^\dagger \sim \zeta_{\mathbb{C}}, \quad Z^H_{12} \sim i\zeta_{\mathbb{R}}, \quad Z^C_{11} = (Z^C_{22})^\dagger \sim \varphi + m_{\mathbb{C}}, \quad Z^C_{12} \sim i(\phi_3 + m_{\mathbb{R}}), \tag{1.3}$$

where the variables on the right hand side can be found in table 2.1.

The presence of so much supersymmetry in the theory allows us to say quite a lot. First of all there are only two types of field multiplets for such a theory (not withstanding half hypermultiplets) and the UV Lagrangian has only a few deformation parameters. The vacua are constrained by non-renormalisation theorems to the extent where we can explicitly describe these objects as varieties in many cases.

Perhaps the simplest way to construct the UV Lagrangian of our theory is dimensional reduction from the $\mathcal{N} = 1$ six dimensional theory. There are two different types of fields, known as the vector multiplet and the hypermultiplet, and the theory has an $SU(2)_H$ R-symmetry.

The vector multiplet in six dimensions has field content: a gauge field $A$, and a Weyl fermion $\psi$ in the adjoint of the gauge group. The $SU(2)_H$ R-symmetry acts trivially on $A$ and acts as a doublet on $\psi$.

The hypermultiplet is commonly referred to as the "matter" of the theory. It can be in any quaternionic representation of the gauge group, we shall consider ones of the form $R \oplus \overline{R}$, for $R$ some complex representation of the gauge group. The field content of this multiplet is given by two complex scalars $(q, \tilde{q})$ in the representation $R \times \overline{R}$ of the gauge group and one Weyl spinor $\lambda$. The $SU(2)_H$ symmetry acts on the scalars as a doublet, but does not rotate the hypermultiplet spinors.

Through dimensional reduction we take the fields to be independent of the directions $x^3, x^4, x^5$. The Weyl fermion in six dimensions becomes four Majorana fermions in three dimensions, the components $A_3, A_4$ and $A_5$ of the gauge field becomes a triplet of scalars, $\phi$, and the scalars remain unchanged. The $SU(2)_C$ subalgebra of the six dimensional Lorentz algebra, rotating $x^3, x^4, x^5$, becomes an R-symmetry of the three dimensional theory.

We shall write the two different 3-dimensional $\mathscr{N} = 4$ supermultiplets in 4-dimensional $\mathscr{N} = 1$ superspace notation.

The 4-dimensional $\mathscr{N} = 2$ vector multiplet has content $(\Phi, \Phi^\dagger, V)$, where $\Phi$ is a 4-dimensional $\mathscr{N} = 1$ adjoint chiral multiplet and $V$ is a 4-dimensional $\mathscr{N} = 1$ adjoint vector multiplet. In the Wess-Zumino gauge:

$$\begin{aligned}
\Phi &= \varphi + i\theta\sigma^\mu\overline{\theta}\partial_\mu\varphi + \frac{1}{4}\theta^4\partial^2\varphi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\overline{\theta} + \theta^2 F\,, \\
\Phi^\dagger &= \varphi^\dagger - i\theta\sigma^\mu\overline{\theta}\partial_\mu\varphi^\dagger + \frac{1}{4}\theta^4\partial^2\varphi^\dagger + \sqrt{2}\overline{\theta}\overline{\psi} + \frac{i}{\sqrt{2}}\overline{\theta}^2\partial_\mu\theta\sigma^\mu\overline{\psi} + \overline{\theta}^2 F^\dagger\,, \\
V &= -\theta\sigma^\mu\overline{\theta}A_\mu + i\theta^2\overline{\theta}\overline{\lambda} - i\overline{\theta}^2\theta\lambda + \frac{1}{2}\theta^4 D\,.
\end{aligned} \tag{1.4}$$

Upon dimensional reduction to three dimensions, we will have that $A_3$ becomes a scalar. We write $\varphi = \phi_1 + i\phi_2$ and $A_3 = \phi_3$. $\phi$ is a triplet under $SU(2)_C$. See table 2.1 for a summary of all fields and their symmetries.

We define the 4-dimensional $\mathscr{N} = 1$ superfields $W_\alpha := -\frac{1}{4}\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}}D_\alpha V$ and $\overline{W}_{\dot{\alpha}} := -\frac{1}{4}D^\alpha D_\alpha \overline{D}_{\dot{\alpha}} V$. Where $D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu$ and $\overline{D}_{\dot{\alpha}} := -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$.

The 4-dimensional $\mathscr{N} = 2$ hypermultiplet is a 4-dimensional $\mathscr{N} = 1$ chiral multiplet $\mathscr{Q}$ in representation $R$ of the gauge group and a 4-dimensional $\mathscr{N} = 1$ anti-chiral multiplet $\tilde{\mathscr{Q}}$ in representation $\overline{R}$ of the gauge group. We write them as:

$$
\begin{aligned}
\mathscr{Q} &= q + i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}q + \frac{1}{4}\theta^4\partial^2 q + \sqrt{2}\theta\omega - \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\omega\sigma^{\mu}\overline{\theta} + \theta^2 H\,,\\
\mathscr{Q}^{\dagger} &= q^{\dagger} - i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}q^{\dagger} + \frac{1}{4}\theta^4\partial^2 q^{\dagger} + \sqrt{2}\overline{\theta}\overline{\omega} + \frac{i}{\sqrt{2}}\overline{\theta}^2\theta\sigma^{\mu}\partial_{\mu}\overline{\omega} + \overline{\theta}^2 H^{\dagger}\,,\\
\tilde{\mathscr{Q}} &= \tilde{q} + i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\tilde{q} + \frac{1}{4}\theta^4\partial^2 \tilde{q} + \sqrt{2}\theta\tilde{\omega} - \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\tilde{\omega}\sigma^{\mu}\overline{\theta} + \theta^2 \tilde{H}\,,\\
\tilde{\mathscr{Q}}^{\dagger} &= \tilde{q}^{\dagger} - i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\tilde{q}^{\dagger} + \frac{1}{4}\theta^4\partial^2 \tilde{q}^{\dagger} + \sqrt{2}\overline{\theta}\overline{\tilde{\omega}} + \frac{i}{\sqrt{2}}\overline{\theta}^2\theta\sigma^{\mu}\partial_{\mu}\overline{\tilde{\omega}} + \overline{\theta}^2 \tilde{H}^{\dagger}\,.
\end{aligned}
\tag{1.5}
$$

We summarise the three dimensional field content with the following table[1].

| | $G$ | $G_H$ | $SL(2;\mathbb{R})$ | $SU(2)_H$ | $SU(2)_C$ |
|---|---|---|---|---|---|
| $q$ | $R\otimes\mathbb{C}$ | $\overline{\text{fund}}$ | $\underline{1}$ | $\underline{2}$ | $\underline{1}$ |
| $\tilde{q}$ | $\overline{R}\otimes\mathbb{C}$ | fund | $\underline{1}$ | $\underline{2}$ | $\underline{1}$ |
| $\omega$ | $R\otimes\mathbb{C}$ | $\overline{\text{fund}}$ | $\underline{2}$ | $\underline{1}$ | $\underline{2}$ |
| $\tilde{\omega}$ | $\overline{R}\otimes\mathbb{C}$ | fund | $\underline{2}$ | $\underline{1}$ | $\underline{2}$ |
| $\phi$ | adj | $\underline{1}$ | $\underline{1}$ | $\underline{1}$ | $\underline{3}$ |
| $\psi$ | adj | $\underline{1}$ | $\underline{2}$ | $\underline{2}$ | $\underline{2}$ |
| $A_{\mu}$ | connection | $\underline{1}$ | $\underline{3}$ | $\underline{1}$ | $\underline{1}$ |
| $(D,F)$ | adj | $\underline{1}$ | $\underline{1}$ | $\underline{3}$ | $\underline{1}$ |
| $m$ | $\underline{1}$ | adj | $\underline{1}$ | $\underline{1}$ | $\underline{3}$ |
| $\zeta$ | $\pi_1(G)^{\vee}$ | $\underline{1}$ | $\underline{1}$ | $\underline{3}$ | $\underline{1}$ |
| $Q$ | $\underline{1}$ | $\underline{1}$ | $\underline{2}$ | $\underline{2}$ | $\underline{2}$ |

Table 2.1 Summary of the field content, the masses, Fayet-Iliopoulos terms and supercharges. See section 2.1.1 for the definition of $m,\zeta$ and $G_H$. “fund" is the fundamental representation and “adj" is the adjoint representation. $Q$ represents the supercharges of the theory. $SL(2;\mathbb{R})\cong Spin(2,1)$ is the double cover of the Lorentz group, $SO(2,1)$.

We can think of $\zeta$ as the scalar component of a frozen twisted vector multiplet and we can think of $m$ as the scalar component of a vector multiplet for the frozen gauge group $G_H$.

[1] We have ignored the possibility of half hypermultiplets as well as twisted multiplets as they will be of little relevance to our work. The half hypermultiplets come about when $R$ (or a simple part of it) is pseudoreal and we can essentially eliminate $\tilde{q}$. The twisted multiplets come about by acting with the idempotent outer automorphism that swaps $SU(2)_H$ and $SU(2)_C$.

### 2.1.1 Lagrangian

We first write down the Lagrangian for the four dimensional theory. We have gauge coupling $\tau = \frac{4\pi i}{e^2} + \frac{\theta}{2\pi}$ generally, but we will take $\theta = 0$ (upon dimensional reduction the $\theta$-term becomes trivial anyway). The trace is over the gauge indices.

The fields combine to form the Lagrangian, written in $\mathscr{N} = 1$ notation:

$$\begin{aligned}\mathscr{L}_{4d} =&\operatorname{tr}\left[\int d^4\theta \frac{1}{e^2}\Phi^\dagger e^{2[V,-]}\Phi + \left(\frac{-i\tau}{8\pi}\int d^2\theta W^\alpha W_\alpha + \text{h.c.}\right)\right.\\ &\left. + \int d^4\theta \left(\mathscr{Q}^\dagger e^V \mathscr{Q} + \tilde{\mathscr{Q}} e^{-V}\tilde{\mathscr{Q}}^\dagger\right) + \sqrt{2}\left(\int d^2\theta \tilde{\mathscr{Q}}\Phi\mathscr{Q} + \text{h.c.}\right)\right].\end{aligned} \tag{1.6}$$

The $\sqrt{2}$ arises by following the convention of [200]. The flavour group is defined by the exact sequence

$$G \to U(R) \to G_H \to 1\,. \tag{1.7}$$

We can add deformations in the form of Fayet-Iliopoulos terms and mass terms [123],

$$\begin{aligned}\mathscr{L}_{FI} &= -\int d^4\theta \operatorname{tr}\zeta_{\mathbb{R}} V + \left(\int d^2\theta \operatorname{tr}\zeta_{\mathbb{C}}\Phi + \text{h.c.}\right),\\ \mathscr{L}_M &= \sqrt{2}\left(\int d^2\theta \tilde{\mathscr{Q}} m_{\mathbb{C}}\mathscr{Q} + \text{h.c.}\right) + \int d^4\theta \left(\mathscr{Q}^\dagger(e^{-2m_{\mathbb{R}}\theta\overline{\theta}} - 1)\mathscr{Q} + \tilde{\mathscr{Q}}(e^{2m_{\mathbb{R}}\theta\overline{\theta}} - 1)\tilde{\mathscr{Q}}^\dagger\right).\end{aligned} \tag{1.8}$$

We pause to note that these Lagrangians are three dimensional. This is crucial to define the real mass term $m_{\mathbb{R}}$, as in four dimensions the quantity $\theta\overline{\theta}$ is not a Lorentz scalar, but in three dimensions it is. This is because the charge conjugation matrix in 2 + 1 dimensions is $C = \varepsilon$, [136].

In three dimensions one can add a Chern-Simons term [87]

$$\mathscr{L}_{\text{CS}} := \frac{k}{4\pi}\int d^4\theta \int_0^1 dt \operatorname{tr} V\overline{D}^\alpha \left(e^{-tV} D_\alpha e^{tV}\right) - \left(\frac{k}{4\pi}\int d^2\theta \operatorname{tr}\Phi^2 + \text{h.c.}\right), \tag{1.9}$$

with $k$ integer-quantised so that the action is invariant under large gauge transformations. However, this term will break the supersymmetry down to $\mathscr{N} = 3$ supersymmetry [122][2]. See the discussion in chapter V of [123] for good reasons for why one might not expect it to be possible to have an $\mathscr{N} = 4$ supersymmetric topologically massive super Yang-Mills theory in three dimensions. This breaking of supersymmetry makes the description of quantum vacua more difficult. For this reason, we will no longer consider it in this thesis.

[2]This is because of the presence of the Yang-Mills term. If there is no Yang-Mills term, then one can have $\mathscr{N} > 3$ supersymmetry. For example, the ABJM theory has $\mathscr{N} = 6$, [2].

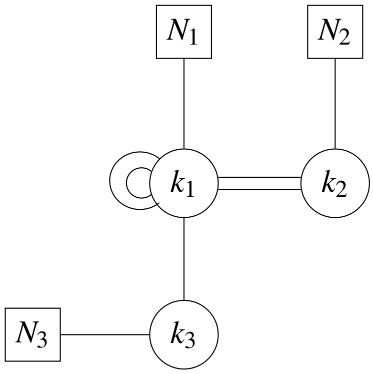


Fig. 2.1 An example of a quiver. Note that we can have multiple edges from one gauge node to another, and edges from a gauge node to itself.

The Lagrangian for 3d $\mathcal{N}=4$ we will look at is

$$\mathscr{L}_{3d} = \mathscr{L}_{4d\to 3d} + \mathscr{L}_{FI} + \mathscr{L}_{M}\,, \tag{1.10}$$

$\mathscr{L}_{4d\to 3d}$ is defined by taking the Lagrangian $\mathscr{L}_{4d}$, making all fields independent of $x^3$, writing $A_3$ as $\phi_3$, a scalar field, and noting that all Weyl fermions descend to two Majorana fermions.

We see via simple dimension counting that $[e]=\frac{1}{2}$, and so the theory is asymptotically free. This means that, unlike in four dimensions where we have to be careful about the matter multiplets so that we have a well-defined UV theory, we always have a well-defined theory. In the IR, the theory is strongly coupled.

The scalar potential is

$$\begin{aligned} U = {} & 2\left(|\tilde{q}(\varphi+m_{\mathbb{C}})|^2 + |(\varphi+m_{\mathbb{C}})q|^2\right) + m_{\mathbb{R}}^2\left(|q|^2+|\tilde{q}|^2\right) + \frac{1}{4e^2}|[\phi_3,\varphi]|^2 \\ & + \frac{1}{4}|\phi_3 q|^2 + \frac{1}{4}|\tilde{q}\phi_3|^2 + \frac{1}{2}\left|\frac{1}{e^2}[\varphi^\dagger,\varphi] + e^2(\zeta_{\mathbb{R}} - q^\dagger q + \tilde{q}\tilde{q}^\dagger)\right|^2 + e^2|\zeta_{\mathbb{C}} - \sqrt{2}\tilde{q}q|^2\,. \end{aligned} \tag{1.11}$$

## 2.1.2 Quiver field theory

A specific type of 3d $\mathcal{N}=4$ field theories that we will study are those that can be defined by quivers. A quiver is a directed graph, $\Gamma=(V,E)$, where $V \ni i$ is the set of nodes and $E \ni (i,j)$ the set of directed arrows (we allow multiple edges as well as loops). In order to define an 8 supercharge theory, we need $\Gamma$ to be a symmetric directed graph. This means for every $(i,j)\in E$ there is a corresponding $(j,i)\in E$. Such a quiver defines an 8 supercharge theory (otherwise it defines a 4 supercharge theory). We shall call such a quiver an *eight supercharge quiver*.

We assign a set of strictly positive integers $k \in \mathbb{N}^V$ and a set of non-negative integers $N \in \mathbb{N}_0^V$ to every node. The gauge group of our theory is defined to be

$$G = \prod_{i \in V} U(k_i)\,. \tag{1.12}$$

Pick a subset $\Omega \subset E$ such that $\Omega \cup \overline{\Omega} = E$ and $\Omega \cap \overline{\Omega} = \varnothing$, where $\overline{*} : E \to E$ is a bijection such that $\overline{(i,j)} = (j,i)$. For every $h = (i,j) \in \Omega$ we have a hypermultiplet in the $(\underline{k_i}, \overline{\underline{k_j}})$ of $G$, and for every $i \in V$ we have $N_i$ hypermultiplets in the $\underline{k_i}$ of $G$.

The Fayet-Iliopoulos terms live in $\mathbb{R}^V \otimes \mathbb{R}^3$ and the mass terms live in $\mathbb{R}^{\sum_i N_i} \otimes \mathbb{R}^3$, the Cartan subalgebra of the flavour group $\prod_i U(N_i)$ tensor product with $\mathbb{R}^3$.

We draw our quivers by writing every gauge node as a circle and each flavour node as a box, both with their respective ranks written inside. We draw all the arrows in $\Omega$ as undirected lines between the corresponding nodes. Figure 2.1 is an example of a (rather exotic) quiver gauge theory.

Assumming that there are no edge loops (edges from a node to itself), the quiver $\Gamma = (V,E)$ defines a Lie algebra as follows (see [117, 118] for details):

The $n \times n$ matrix, where $n := |V|$ and we have labelled the gauge nodes $1,\ldots,n$, defined as, for $a,b = 1,\ldots,n$,

$$C_{ab} := \begin{cases} 2\,, \text{ if } a = b, \\ -\#\text{ of edges } a \to b\,, \text{ if } a \neq b, \end{cases} \tag{1.13}$$

is a generalised Cartan matrix.

Let $\alpha_a := (\delta_{1a},\ldots,\delta_{na}) \in \mathbb{Z}^n$ for $a = 1,\ldots,n$, be the standard basis of $\mathbb{Z}^n$. We introduce a bilinear form on $\mathbb{Z}^n$ via

$$(\alpha_a, \alpha_b) = \frac{1}{2} C_{ab}\,. \tag{1.14}$$

We call $\alpha_a$ a fundamental root for each $a = 1,\ldots,n$. We define $\Pi$ as the set of fundamental roots. For any fundamental root $\alpha_a$, we define the fundamental reflection $r_{\alpha_a} \in \mathrm{Aut}\,\mathbb{Z}^n$ via

$$r_{\alpha_a}(\lambda) := \lambda - 2(\lambda, \alpha_a)\alpha_a\,, \text{ for } \lambda \in \mathbb{Z}^n\,. \tag{1.15}$$

The group $W(\Gamma)$ generated by all the fundamental reflections is the Weyl group of the graph. $R := \bigoplus_{a=1}^n \mathbb{Z}\alpha_a$ is the set of roots, and $R_+ := \bigoplus_{a=1}^n \mathbb{N}_0\alpha_a$ is the set of positive roots.

The Kac-Moody algebra, $\mathfrak{g}(\Gamma)$, generated by $\Gamma$ is the complex Lie algebra with $3n$ generators $e_a, f_a, h_a$ for $a=1,\dots,n$, and the following relations, for $a,b=1,\dots,n$,

$$\begin{aligned}
[h_a,h_b] &= 0\,,\\
[e_a,f_a] &= h_a\,,\\
[e_a,f_b] &= 0\,,\ \text{if } a\neq b\,,\\
[h_a,e_b] &= C_{ab}e_b\,,\\
[h_a,f_b] &= -C_{ab}f_b\,,\\
(\mathrm{ad}\,e_a)^{1-C_{ab}}e_b &= 0\,,\ \text{if } a\neq b\,,\\
(\mathrm{ad}\,f_a)^{1-C_{ab}}f_j &= 0\,,\ \text{if } a\neq b\,.
\end{aligned} \tag{1.16}$$

If the quiver is of *ADE*-type, then $\mathfrak{g}(\Gamma)$ is the simple Lie algebra whose Dynkin diagram is $\Gamma$.

## 2.2 The classical and quantum vacua

In a quantum field theory with two or more non-compact spatial directions, the non-normalisable zero modes of the theory are non dynamical, and must be fixed in order to define the theory. This is related to the phenomena that there are no Goldstone bosons in two dimensions [56]. This choice is a choice of superselection sector, [5], and is given by a choice of vacuum for the fields to tend to at spatial infinity. If there is one or fewer spatial dimensions, the wavefunction spreads over the moduli space of vacua instead.

Since we are working on $\mathbb{R}^{2,1}$, we need to pick a vacuum state. The classical Hamiltonian of the field theory can be written as the sum of positive semidefinite terms

$$H = ||D\text{fields}||^2 + U\,. \tag{2.17}$$

Thus, the vacua are given by the zero frequency Fourier modes, such that the potential is minimised and, since the origin is such a solution, they are all gauge equivalence classes of scalar values that are roots of the potential.

We look at the minima of the function $U$ in equation (1.11). First we see that $m_{\mathbb{R}}\neq 0$ necessarily requires that $q=\tilde{q}=0$. Let us set $m_{\mathbb{R}}=m_{\mathbb{C}}=0$, then the potential contains the terms (ignoring the various coefficients that arose from our scaling convention) $|\tilde{q}\varphi|^2+|\varphi q|^2+|\phi_3 q|^2+|\tilde{q}\phi_3|^2$. These mean that if $(q,\tilde{q})$ has a non-zero vev, then $\phi$ will be restricted into having some components zero; and if $\phi$ has a non-zero vev, then $(q,\tilde{q})$ will be restricted into having some components zero. In the case where some components of the vector multiplet scalars are non-zero and some components of the hypermultiplet scalars are non-zero, we get what is known as a mixed branch.

If the vector scalars are zero, then we have the space of gauge orbits of $(q,\tilde{q})$ vacuum expectation values that annihilate

$$U(\phi = m = 0) = \frac{1}{2}e^2\left|(\zeta_{\mathbb{R}} - q^\dagger q + \tilde{q}\tilde{q}^\dagger)\right|^2 + e^2|\zeta_{\mathbb{C}} - \sqrt{2}\tilde{q}q|^2 . \tag{2.18}$$

This space is known as the *Higgs branch* and a generic point of it generically (of course this depends on the choice of $R$, but any faithful representation will do) completely breaks the gauge group. To have any solutions at all for $\zeta$ generic, one needs $R$ to be "large enough" relative to the size of $G$. The space has a natural action of the flavour group $G_H \geqslant S\left(\prod_i U(\mathbb{C}^{N_i})\right)$ inherited from the action on the hypermultiplets. It is a hyperKähler space.

If the hypermultiplet scalars are zero, then we necessarily must turn off $\zeta_{\mathbb{R}}$ and $\zeta_{\mathbb{C}}$, but are free to turn on $m_{\mathbb{R}}$ and $m_{\mathbb{C}}$. Classically, the mass has no effect.

$$U(\zeta = q = \tilde{q} = 0) = \frac{1}{4e^2}|[\phi_3,\varphi]|^2 + \frac{1}{2}\left|\frac{1}{e^2}[\varphi^\dagger,\varphi]\right|^2 . \tag{2.19}$$

We can give vacuum expectation values to $\phi$ such that each component commutes with the other. The mutually commuting matrices can be diagonalised by gauge transformations, and thus, we find that our vacuum expectation values live in the Cartan subalgebra of $G$, $\mathfrak{h}_G$, generically breaking the gauge group to the maximal torus times the Weyl group of $G$, $W$. This is not the end of the story though, any three dimensional abelian gauge field can be dualised to a periodic scalar[3]. We relax our path integral to integrate over field strengths $F$ that do not obey the Bianchi identity, but enforce it with a Lagrange multiplier. Thus, our partition function is

$$Z[\chi] = \int [D\text{fields}]e^{iS_{3d} + i\int d^3x\chi DF} . \tag{2.20}$$

The crucial point to note is that if $\chi' - \chi = a \in \mathbb{R}$, then we would have a difference in the action of the form

$$\begin{aligned} \Delta S &= i\int d^3x aDF \\ &= ia\int_{S^2_\infty} F . \end{aligned} \tag{2.21}$$

The quantity $\int_{S^2_\infty} F$ is quantised by the values of $\pi_2(S^2) \cong \mathbb{Z}$. This means that the path integral is unchanged for certain discrete shifts of $\chi$, and thus this scalar is periodic. Integrating out

[3]Here we are really working in the quantum theory at the tree level, instead of purely classically.

$F$ allows one to swap the photon for the periodic scalar $\chi$, which we can give a vacuum expectation value. Thus, we have moduli space $(\mathfrak{h}_G \otimes \mathbb{R}^3 \times S_1^{\mathrm{rk}(G)})/W = (\mathbb{C} \times \mathbb{C}^\times)^{\mathrm{rk}(G)}/W$, where $\mathrm{rk}(G)$ is the rank of $G$. This space is known as the *Coulomb branch*. Note that this manifold is hyperKähler, and the topological symmetries $U(1)^V$, given by rotating $\chi$ and leaving $\phi$ invariant, act natually on it.

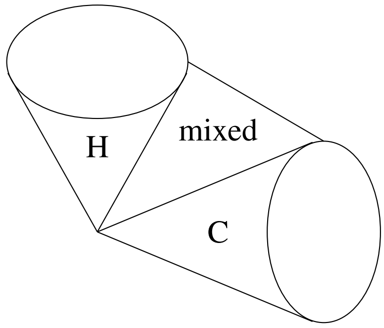


Fig. 2.2 A schematic picture of the vacua. The Higgs branch and the Coulomb branch are hyperKähler cones that intersect at the origin, and mixed branches lie between them.

We see that if $\zeta = 0$ and $m = 0$, then we have two hyperKähler manifolds that intersect at a single point, with singular submanifolds along both corresponding to mixed branches that stretch between the two. Generic values of $\zeta$ will lift the Coulomb branch and the mixed branches, while generic values of $m$ will lift the Higgs branch and the mixed branches.

The description so far has been purely classical. Upon quantisation, one needs to worry about quantum corrections to the potential. For a generic quantum field theory, we expect corrections to lift most of the vacua. However, there are theories where vacua are protected. For example, if the vacua are an orbit of a non-anomalous global symmetry, then they will be protected, with the choice of a particular point of the vacuum manifold breaking this symmetry. This is spontaneous symmetry breaking. Another way that the vacua can be protected are non-renormalisation theorems due to the presence of supersymmetry. The non-renormalisation theorems that are relevant to us are:

1. In the paper [11], it is proven for 4d $\mathcal{N} = 2$ theories that if the Coulomb branch metric depends non-trivially on some parameter, then the Higgs branch metric is independent of that parameter, and vice versa. The argument is elegant, and simply states that any such term in a Lagrangian will lead to non-Lorentz invariant terms in the action. Since we can promote the gauge coupling to a background vector multiplet, we see that the Higgs branch metric must be independent of the coupling, and hence, has no quantum corrections. This argument holds equally well in three dimensions.

2. The Coulomb branch, on the other hand, does receive quantum corrections. However, it is not lifted. Indeed, for it to be lifted we would need to generate a mass term for the vector multiplet scalars. However, our supersymmetry outlaws the possibility of such a superpotential. The only other ways one could generate a mass would be with a Chern-Simons term, which we have decided not to consider and cannot be induced due to the lack of a parity anomaly in our theory, and by turning on $\zeta_{\mathbb{R}}$. This argument can be found in [180].

3. In the low energy theory on a generic point of the Coulomb branch, we expect a three dimensional $\mathcal{N} = 4$ $\sigma$-model with target space the quantum Coulomb branch, which is a $4\text{rk}(G)$ real-dimensional manifold (up to codimension 1 singularities, using the word manifold for such spaces is a common abuse of notation in the physics literature). This much supersymmetry necessarily dictates that the manifold be hyperKähler [8, 80].

So we may conclude that our quantum Higgs branch is the classical one, which is hyperKähler, and the quantum Coulomb branch is a $4\text{rk}(G)$ real-dimensional hyperKähler manifold. We have a picture of the vacua in figure 2.2.

We shall now say some words on how one actually goes about computing what these manifolds are for quiver gauge theories.

### 2.2.1 Higgs branch of quiver gauge theory - Nakajima quiver varieties and Kempf-Ness theorem

For a three dimensional $\mathcal{N} = 4$ quiver gauge theory, the Higgs branch coincides with an object in the mathematical literature known as a Nakajima quiver variety. These objects, as well as being hyperKähler manifolds, are also quasi-projective varieties, and are in fact affine varieties for $\zeta_{\mathbb{R}}$ vanishing. As a purely mathematical object, these objects have many interesting properties lying on the intersection of representation theory and geometry.

Our starting data for a Nakajima quiver variety is identical to that of the quiver field theory. We have

1. An 8 supercharge quiver $\Gamma = (V, E)$;

2. $k \in \mathbb{N}^V$, $N \in \mathbb{N}_0^V$; and

3. $\zeta \in \mathbb{R}^{3V}$.

We have set $m = 0$.

There are two constructions that one can now follow. One is the hyperKähler quotient, this was first done in [157]; the second is via geometric invariant theory, this was first done in [158]. We shall describe both.

First, we shall define a hyperKähler manifold. To define a hyperKähler manifold, we first define a Kähler manifold

**Definition 1.** *A* Kähler manifold *is a manifold $\mathscr{X}$ with a complex structure $I$, a metric $g$ and a symplectic form $\omega$, such that for all $u,v \in T_x\mathscr{X}$ and for all $x \in \mathscr{X}$, we have that*

$$g(u,v) = \omega(u,Iv)\,. \tag{2.22}$$

An example of a Kähler manifold is $\mathbb{C}$. In complex coordinates $z,\bar{z}$, the geometrical structures are

$$\begin{aligned} g &= dz d\bar{z}\,, \\ \omega &= dz \wedge d\bar{z}\,, \\ I &= \begin{pmatrix} i & \\ & -i \end{pmatrix}\,. \end{aligned} \tag{2.23}$$

**Definition 2.** *A* hyperKähler manifold *is a manifold (though we will allow singularities) with three covariantly constant orthogonal endomorphisms of the tangent bundle, $(I,J,K)$, satisfying the quaternion algebra identities*[4]

$$I^2 = J^2 = K^2 = IJK = -1\,. \tag{2.24}$$

For $N$ a hyperKähler manifold, the action of $I,J$ and $K$ will give the tangent space, $TN$, the structure of a quaternionic vector space, so the real dimension of a hyperKähler manifold is divisible by 4. Furthermore, $I,J$ and $K$ are all complex structures for $N$. There are three associated symplectic forms, $\omega_a$ for $a=1,2,3$, where $\omega_a = -(gI, gJ, gK)$ so that $(N,I,\omega_1,g)$ is a Kähler manifold and similarly for $J$ and $K$. Given a choice of complex structure, say $I$, then $\omega_2 + i\omega_3$ is a holomorphic symplectic form on $N$. We shall often write $\omega_{\mathbb{R}} \equiv \omega_1$, and $\omega_{\mathbb{C}} \equiv \omega_2 + i\omega_3$. Notice the similarity with how we wrote our 8 supercharge Lagrangian; the choice of 4 supercharge subalgebra that we choose in order to write our Lagrangian is completely analogous to the choice of a complex structure.

[4]First written down when Hamilton carved them into Brougham Bridge Dublin on a walk with his wife. HyperKähler manifolds were first defined by Calabi [45]

An example of a hyperKähler manifold is $\mathbb{C}^2$. In complex coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$, the geometrical structures are

$$\begin{aligned} g &= dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2\,, \\ \omega_1 &= i dz_1 \wedge d\bar{z}_1 + i dz_2 \wedge d\bar{z}_2\,, \\ \omega_2 &= dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2\,, \\ \omega_3 &= -i dz_1 \wedge dz_2 + i d\bar{z}_1 \wedge d\bar{z}_2\,, \end{aligned} \tag{2.25}$$

and

$$I_1 = \begin{pmatrix} i & & & \\ & -i & & \\ & & i & \\ & & & -i \end{pmatrix}, \quad I_2 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad I_3 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}. \tag{2.26}$$

The hyperKähler quotient was first described in [106]. In its most general form we have a Lie group $G$ with a smooth, Hamiltonian, isometric, triholomorphic action on a hyperKähler manifold, $N$. To be clear on notation, this means that $\mathscr{L}_X \omega_a = 0$ for $\mathscr{L}_X$ the Lie derivative representing the action of the element $X$ of the Lie algebra associated to $G$, $\mathfrak{g}$, and there are three independent moment maps,

$$\begin{aligned} \mu_a &: N \to \mathfrak{g}^*\,, \\ \mu_a(g \cdot x) &= \mathrm{Ad}^\#(g)\mu_a(x)\,, \quad \forall g \in G, x \in N\,, \\ d\mu_a(X) &= i_X \omega_a\,, \quad \forall X \in \mathfrak{g}\,. \end{aligned} \tag{2.27}$$

Here $\mathrm{Ad}^\#$ is the coadjoint action of $G$ on $\mathfrak{g}^*$, defined by $\langle \mathrm{Ad}^\#(g)\ell, \mathrm{Ad}(g)X \rangle = \langle \ell, X \rangle$ for all $\ell \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. So the second line is a $G$-equivariance condition.

The hyperKähler quotient is an extension of the Kähler quotient, which itself is an extension of the symplectic quotient, also known as Marsden-Weinstein quotient. We pick an element $\zeta \in \mathfrak{Z} \otimes \mathbb{R}$, where $\mathfrak{Z} \subseteq \mathfrak{g}^*$ is the space of $G$-invariants, and take the quotient

$$\mathfrak{N}_\zeta := \mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)/G \tag{2.28}$$

If $N$ is smooth and $\zeta$ is a regular value for $\mu$ so that $d\mu$ has full rank, then $\mathfrak{N}_\zeta$ is a hyperKähler manifold. We will deal with a slightly more general case where $N$ is smooth, but $\zeta$ is not regular and instead have a hyperKähler manifold with singulaties.

We return to our hyperKähler quotient definition of the Nakajima quiver variety. We define the affine space of complex matrices

$$\begin{aligned} M \equiv M(k,N) := & \bigoplus_{(i,j)\in\Omega} \mathrm{Hom}\left(\mathbb{C}^{k_i},\mathbb{C}^{k_j}\right) \oplus \mathrm{Hom}\left(\mathbb{C}^{k_j},\mathbb{C}^{k_i}\right) \\ & \oplus \bigoplus_{i\in V} \mathrm{Hom}\left(\mathbb{C}^{N_i},\mathbb{C}^{k_i}\right) \oplus \mathrm{Hom}\left(\mathbb{C}^{k_i},\mathbb{C}^{N_i}\right) \\ & \cong \left(\mathbb{C}^2\right)^{\sum_{(i,j)\in\Omega} k_i k_j + \sum_{i\in V} k_i N_i} . \end{aligned} \tag{2.29}$$

This space is hyperKähler because $\mathbb{C}^2 \cong \mathbb{H}$ is hyperKähler and hyperKählerity is preserved under Cartesian product. Elements $(X,\tilde{X},q,\tilde{q}) \in M$ transform under $g \in G \equiv G_k = \prod_i GL(\mathbb{C}^{k_i})$ as

$$(X,\tilde{X},q,\tilde{q}) \mapsto (gXg^{-1}, g\tilde{X}g^{-1}, gq, \tilde{q}g^{-1}) . \tag{2.30}$$

This action is smooth (except for the zeroes), Hamiltonian, isometric and triholomorphic. So, we have three moment maps

$$\begin{aligned} \mu_{\mathbb{R}} &:= [X,X^\dagger] + [\tilde{X},\tilde{X}^\dagger] + qq^\dagger - \tilde{q}^\dagger\tilde{q} \in \prod_{a\in V} \mathfrak{u}(k_a) , \\ \mu_{\mathbb{C}} &:= [X,\tilde{X}] + q\tilde{q} \in \prod_{a\in V} \mathfrak{gl}(k_a) . \end{aligned} \tag{2.31}$$

We define the *Nakajima quiver variety* as

$$\mathfrak{M}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}} \equiv \mathfrak{M}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}(k,N) := \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/G . \tag{2.32}$$

The moment maps enforce exactly the same condition as equation (2.18), and the quotient by $G$ is nothing more than identifying gauge orbits. This hyperKähler manifold is exactly the Higgs branch of the quiver gauge theory with $m = 0$.

If $\zeta = 0$, then since the origin has a non-trivial stabiliser under $G$, there will be a singularity at the corresponding point in the quotient. We define the smooth subset of $\mathfrak{M}_\zeta$,

$$\mathfrak{M}^{\mathrm{reg}}_\zeta := \{x \in \mu^{-1}(\zeta) | \text{stabiliser of } x \text{ is trivial}\} . \tag{2.33}$$

Assuming $\Gamma$ is of *ADE*-type, then for $\theta \in R_+$, where, $R_+$ is the set of positive roots defined by $\Gamma$, we define

$$D_\theta := \left\{ x = (x_k) \in \mathbb{R}^V \middle| \sum_k x_k \theta_k = 0 \right\} . \tag{2.34}$$

Then we have

**Theorem 1.** *(Theorem 2.8 of [157]) Suppose*

$$\zeta \in \mathbb{R}^3 \otimes \mathbb{R}^V \setminus \bigcup_{\theta \in R_+} \mathbb{R}^3 \otimes D_\theta \,, \tag{2.35}$$

*then the regular locus $\mathfrak{M}_\zeta^{reg}$ coincides with $\mathfrak{M}_\zeta$. So, $\mathfrak{M}_\zeta$ is nonsingular, and, moreover, the hyperKähler metric is complete.*

Physically, if $\zeta \in D_\theta$ for $\theta \in R_+$, then there will be a singular submanifold where one can give non-zero vevs to $\varphi$ such that $[\varphi^\dagger, \varphi]$ lies in the subalgebra defined by $\theta$, and so lies at the intersection of the Higgs branch and a mixed branch.

We call $\zeta$ *generic* if it satisfies the condition (2.35) for $\Gamma$ of $ADE$-type, and more generally we call $\zeta$ generic if $\mathfrak{M}_\zeta$ is nonsingular. The question of when such $\zeta$ exist is addressed later in this section.

In order to compare with the geometric invariant theory quotient construction of the Nakajima quiver variety, we make a brief general point about hyperKähler quotients. For $N$ a hyperKähler manifold, $\mu_\mathbb{C}$ is a holomorphic map, in complex structure $I$, on $N$, since for all $X \in \mathfrak{g}$, with associated vector field $v_X$, and vector fields $Y$,

$$\begin{aligned} d\langle \mu_\mathbb{C}, X\rangle(IY) &= g(JIY, v_X) + ig(KIY, v_x) \\ &= id\langle \mu_\mathbb{C}, X\rangle(Y) \,. \end{aligned} \tag{2.36}$$

Thus, $\mu_\mathbb{C}^{-1}(\zeta_\mathbb{C}) \subset N$ is a complex submanifold (in complex structure $I$), and inherits the Kähler structure. So, we can view the hyperKähler quotient as the Kähler quotient by the action of $G$ with moment map $\mu_\mathbb{R}$.

$$\mathfrak{M}_\zeta = \mu^{-1}(\zeta)/G = \mu_\mathbb{C}^{-1}(\zeta_\mathbb{C}) /\!/_{\zeta_\mathbb{R}} G \,, \tag{2.37}$$

where $\mu_\mathbb{C}^{-1}(\zeta_\mathbb{C}) /\!/_{\zeta_\mathbb{R}} G$ is the symplectic quotient of $\mu_\mathbb{C}^{-1}(\zeta_\mathbb{C})$ by $G$.

If $\zeta_\mathbb{R} \in \mathbb{Z}$, then we can describe the symplectic map as shifted by some $\rho : G \to U(1)$, so that $d\rho : \mathfrak{g} \to 2\pi i \mathbb{R}$. We take moment map

$$\begin{aligned} &\mu^\rho : \mu_\mathbb{C}^{-1} \to \mathfrak{g}^* \,, \\ &\mu^\rho(x)(X) = \mu(x)(X) - \frac{1}{2\pi i} d\rho(X) \,\forall x \in \mu_\mathbb{C}^{-1}(\zeta_\mathbb{C}), X \in \mathfrak{g} \,. \end{aligned} \tag{2.38}$$

Then we have that

$$\mu_\mathbb{C}^{-1}(\zeta_\mathbb{C}) /\!/_{\zeta_\mathbb{R}} G = (\mu^\rho)^{-1}(0)/G \,. \tag{2.39}$$

One can see that we have swapped $\zeta_\mathbb{R}$ for $\frac{1}{2\pi i} d\rho$.

We now review the geometric invariant theory quotient, and how it is equivalent to the symplectic quotient via the Kempf-Ness theorem. Geometric invariant theory was started by Mumford in 1965 with his book, whose most recent edition is [152]. Since the components of $\mu_{\mathbb{C}}$ are irreducible as polynomials, $\mu_{\mathbb{C}}^{-1}(0)$ is an affine variety, and so we are interested in the case of affine varieties; a good reference for which is [110].

We shall look at two different quotients. The first is known as the affine quotient. It gives us the Nakajima quiver variety when $\zeta_{\mathbb{R}}=0$. The second is more general. It gives us the Nakajima quiver variety when $\zeta_{\mathbb{R}}\in\mathbb{Z}$. We note that in both cases the construction gives us the spaces as varieties, so purely analytic information such as the metric is inaccessible with this construction, but as we shall see later, we will have all the information required to compute the physical quantities that we are interested in.

We let $G_{\mathbb{C}}$ be a complex reductive group (equivalently, the complexification of some compact Lie group), $W$ an affine variety and $\mathbb{C}[W]$ the coordinate ring of $W$.

We first state some definitions

**Definition 3.** *Let $Z$ be a variety with a $G_{\mathbb{C}}$ action.*

- *$Z/G_{\mathbb{C}}$ is the quotient of $Z$ by the equivalence relation: $z\sim z'$ iff $\overline{G_{\mathbb{C}}\cdot z}\cap\overline{G_{\mathbb{C}}\cdot z'}\neq\varnothing$. It has the quotient topology.*
- *$Z/G_{\mathbb{C}}$ is a good quotient if the quotient map $f:Z\to Z/G_{\mathbb{C}}$ is such that $f^*:\mathscr{O}(U)\to\mathscr{O}(f^{-1}(U))^G$ is an isomorphism for all $U\subseteq Z/G_{\mathbb{C}}$ open.*
- *A geometric quotient of $Z$ is a good quotient such that the fibres are $G_{\mathbb{C}}$-orbits, i.e for all $\tilde{z}\in f(Z)$ there is a $z\in Z$ such that $f^{-1}(\tilde{z})=G_{\mathbb{C}}\cdot z$.*

The affine geometric invariant theory quotient is given simply by the map $\mathbb{C}[W]^{G_{\mathbb{C}}}\hookrightarrow\mathbb{C}[W]$, which induces a morphism $W\to W/\!/G_{\mathbb{C}}:=\mathrm{Spec}\ \mathbb{C}[W]^{G_{\mathbb{C}}}$.

The Kempf-Ness theorem then tells us that for $W=\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ and $G_{\mathbb{C}}$ the complexification of $G$,

$$\mathfrak{M}_{0,\zeta_{\mathbb{C}}}\cong_{\text{homeo}} W/\!/G_{\mathbb{C}}\,. \tag{2.40}$$

In physics terminology, we would call $\mathbb{C}[\mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})]^{G_{\mathbb{C}}}$ the space of all gauge invariant Higgs branch operators modulo the $F$-term equations. We shall state the Kempf-Ness theorem in a slightly more general form later in this section.

Now we look at the more general situation that corresponds to $\zeta_{\mathbb{R}}\in\mathbb{Z}$. By equation (2.39), we see that we will need to twist our construction by a character, $\rho:G_{\mathbb{C}}\to\mathbb{C}^\times$. This approach of using a character to twist the quotient was first used by King in [127]. We define a character $\rho:G_{\mathbb{C}}\to\mathbb{C}^\times$ and a trivial line bundle on $W$, $L:=W\times\mathbb{C}$. We use $\rho$ to lift the action of $G_{\mathbb{C}}$ on $W$ to $L$ via

$$g \cdot (v,c) = (g \cdot v, \rho(g)c)\,, \tag{2.41}$$

for $g \in G_{\mathbb{C}}$ and $(v,c) \in L$. Denote the information of the line bundle with the $G_{\mathbb{C}}$ action by $L_\rho$. Note that $L_\rho^{\otimes n} = L_{\rho^n}$ for all $n \in \mathbb{Z}$. There is an induced action on the sections given by, for $g \in G_{\mathbb{C}}$ and $\sigma \in H^0(W, L_\rho^{\otimes n})$,

$$g \cdot \sigma(v) = \rho^n(g)\sigma(g^{-1}v)\,. \tag{2.42}$$

Note that we can think of invariant sections of this line bundle as semi-invariant functions on $W$ with weight $\rho^n$,

$$H^0(W, L_\rho^{\otimes n})^{G_{\mathbb{C}}} \cong \mathbb{C}[W]^{G_{\mathbb{C}}}_{\rho^n} := \{f \in \mathbb{C}[W] | f(g\cdot v) = \rho^n(g) f(v) \,\forall\, v \in W, g \in G_{\mathbb{C}}\}\,, \tag{2.43}$$

where for $\mathscr{F}$ a vector bundle/sheaf on some space $X$, $H^0(X,\mathscr{F})$ is the space of global sections of $\mathscr{F}$ on $X$.

Consider the graded algebra

$$R := \bigoplus_{n \geqslant 0} H^0(W, L_\rho^{\otimes n})\,, \tag{2.44}$$

and its invariant graded subalgebra $R^{G_{\mathbb{C}}} = \bigoplus_n R_n^{G_{\mathbb{C}}}$, where $R_n^{G_{\mathbb{C}}} = \mathbb{C}[W]^{G_{\mathbb{C}}}_{\rho^n}$.

We define our quotient via the Proj construction, which we now review (see [100] II 2 for more details):

Let $S$ be an $\mathbb{N}_0$-graded ring. Define $S_+ := \bigoplus_{i>0} S_i$.

We define the set Proj $S$ to be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of $S_+$. We define a topology on Proj $S$ by taking all closed sets to be of the form $V(\mathfrak{a}) := \{\mathfrak{p} \in \text{Proj } S | p \supseteq \mathfrak{a}\}$, for $\mathfrak{a}$ some homogeneous ideal of $S$.

The sheaf of rings $\mathscr{O}$ is defined on Proj $S$ as follows. For each $\mathfrak{p} \in$Proj $S$, we define the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where $T$ is the multiplicative system consisting of all homogeneous elements of $S$ which are not in $\mathfrak{p}$. For any open subset $U \subseteq$Proj $S$, we define $\mathscr{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that $s$ is locally a quotient of elements of $S$: for each $\mathfrak{p} \in U$, there exists a neighborhood $V$ of $\mathfrak{p}$ in $U$, and homogeneous elements $a, f \in S$, of the same degree, such that for all $\mathfrak{q} \in V, f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f \in S_{(\mathfrak{q})}$.

$\mathscr{O}$ is a sheaf. Moreover, (Proj $S, \mathscr{O}$) is a scheme.

The inclusion $R^{G_{\mathbb{C}}} \hookrightarrow R$ induces a rational map

$$W \to W//_{\rho}G_{\mathbb{C}} := \text{Proj } R^{G_{\mathbb{C}}} . \quad (2.45)$$

This map is undefined on the null cone

$$N := \{v \in W | f(v) = 0 \forall f \in \bigoplus_{n>0} R_n^{G_{\mathbb{C}}}\} . \quad (2.46)$$

Following Mumford

**Definition 4.** *If $v \in W$, then*

- *$v$ is $\rho$-semistable, if there is an invariant section $\sigma \in H^0(W, L_\rho^{\otimes n})^{G_{\mathbb{C}}} = \mathbb{C}[W]_{\rho^n}^{G_{\mathbb{C}}}$, for some $n > 0$, such that $\sigma(v) \neq 0$.*
- *$v$ is $\rho$-stable, if $\dim(G_{\mathbb{C}})_x = 0$ and there is an invariant section $\sigma \in H^0(W, L_\rho^{\otimes n})^{G_{\mathbb{C}}} = \mathbb{C}[W]_{\rho^n}^{G_{\mathbb{C}}}$, for some $n > 0$, such that $\sigma(v) \neq 0$ and the action of $G_{\mathbb{C}}$ on the open affine subset $W_\sigma := \{u \in W | \sigma(u) \neq 0\}$ is closed (that is, all $G_{\mathbb{C}}$-orbits in $W_\sigma$ are closed).*

*The open subsets of $\rho$-stable and $\rho$-semistable points will be denoted $W^{\rho\text{-}s}$ and $W^{\rho\text{-}ss}$ respectively.*

In [127], King showed the following for quiver varieties, our exact statement is taken from [153]:

**Lemma 1.** *Let $x = (X, \tilde{X}, q, \tilde{q}) \in \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})$ and $\rho : G_{\mathbb{C}} \to \mathbb{C}^\times$, $\rho(g) := \prod_{i \in V} \det g_i$.*

- *$x$ is $\rho$-semistable if and only if there is no non-zero $V$-graded subspace $A$ of $\bigoplus \mathbb{C}^{k_i}$ such that $X(A), \tilde{X}(A) \subseteq A$ and $A \subseteq \text{Ker } \tilde{q}$.*
- *The notion of $\rho$-semistable and $\rho$-stable are equivalent, i.e. all $\rho$-semistable points are $\rho$-stable.*

By definition, $W^{\rho\text{-ss}}$ is the complement of the null cone $N$. The morphism $W^{\rho\text{-ss}} \to W//_{\rho}G_{\mathbb{C}}$ is the GIT quotient with respect to $\rho$.

**Theorem 2.** *(Mumford) The GIT quotient $\varphi : W^{\rho\text{-}ss} \to W//_{\rho}G_{\mathbb{C}}$ is a good quotient for the action of $G_{\mathbb{C}}$ on $W^{\rho\text{-}ss}$. Moreover, there is an open subset $W^{\rho\text{-}s}/G_{\mathbb{C}} \subset W//_{\rho}G_{\mathbb{C}}$ whose preimage under $\varphi$ is $W^{\rho\text{-}s}$ and the restriction $\varphi : W^{\rho\text{-}s} \to W^{\rho\text{-}s}/G_{\mathbb{C}}$ is a geometric quotient (which in particular is an orbit space).*

In general, the GIT quotient $W/\!/_\rho G_\mathbb{C}$ =Proj $R^{G_\mathbb{C}}$ with respect to $\rho$ is quasi-projective. We note that if $\rho$ is the trivial character, then in this construction we recover the affine GIT quotient $W \to W/\!/G_\mathbb{C}$.

The key result is the affine Kempf-Ness theorem, which states

**Theorem 3.** *(Affine Kempf-Ness theorem) Let $G_\mathbb{C}$ be the complexification of a compact Lie group $G$, acting linearly on an affine variety $W$, and suppose $G$ acts unitarily with respect to a metric on $W$. Given a character $\rho : G_\mathbb{C} \to \mathbb{C}^\times$, let $\mu^\rho : W \to \mathfrak{g}^*$ denote the moment map for this action. Then the inclusion $(\mu^\rho)^{-1}(0) \subset W^{\rho-ss}$ induces a homeomorphism*

$$\mu^{-1}(0)/G \cong_{\text{homeo}} W/\!/_\rho G_\mathbb{C}\,. \tag{2.47}$$

So notably, if we take $W = \mu_\mathbb{C}^{-1}(\zeta_\mathbb{C})$, we get that for $\zeta_\mathbb{R} \in \mathbb{Z}$ with an associated character $\rho$,

$$\mathfrak{M}_{\zeta_\mathbb{R},\zeta_\mathbb{C}} \cong_{\text{homeo}} \mu_\mathbb{C}^{-1}(\zeta_\mathbb{C})/\!/_\rho G_\mathbb{C}\,. \tag{2.48}$$

This implies equation (2.40) when $\rho = 1$.

As we have seen the space $\mathfrak{M}_0$ is generally singular, while the space $\mathfrak{M}_\zeta$ can often be smooth. We are interested in how the Poisson algebra of the smooth space is related to the non-smooth space. For this we must review the theory of symplectic resolutions:

For $\mathscr{X}$ a Kähler space, a *resolution* of $\mathscr{X}$ is a proper surjective morphism, $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$, such that $\tilde{\mathscr{X}}$ is smooth, and $\pi^{-1}(\mathscr{X}^{\text{reg}}) \to \mathscr{X}^{\text{reg}}$ is an isomorphism. If $\pi$ is a projective morphism, then this a *projective resolution*.

This map is holomorphic, so

$$\bar{\partial}\pi^* = \pi^*\bar{\partial}\,. \tag{2.49}$$

A symplectic resolution is one where $\pi^*\omega$, the pullback of the symplectic form on $\mathscr{X}^{\text{reg}}$, can be extended to a symplectic form on all of $\tilde{\mathscr{X}}$.

Since $\mathscr{X} := \mathfrak{M}_0$ is a symplectic normal affine variety, see [59] for this result, we know that if $\mathscr{X}$ has a projective resolution, then it has the following properties[5]:

- Since the $\mathbb{C}^\times$-action on $\mathscr{X}$ contracts $\mathscr{X}$ to a unique fixed point, $o \in \mathscr{X}$; imbues the coordinate ring with a grading, $\mathbb{C}[\mathscr{X}] = \bigoplus_{a\in\mathbb{N}_0} \mathbb{C}^a[\mathscr{X}]$, such that $\mathbb{C}^0[\mathscr{X}] = \mathbb{C}$, and

$$\left\{\mathbb{C}^a[\mathscr{X}], \mathbb{C}^b[\mathscr{X}]\right\} \subseteq \mathbb{C}^{a+b-2}[\mathscr{X}]\,, \quad \forall a,b \geqslant 0\,, \tag{2.50}$$

  the $\mathbb{C}^\times$-action has a canonical lift to an algebraic $\mathbb{C}^\times$-action on $\tilde{\mathscr{X}}$, giving the Poisson algebra $\Gamma(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}})$ a grading.

[5] See [89] for a more detailed list of the properties as well as proofs.

- We have that

$$\pi^* : \mathbb{C}[\mathscr{X}] \to \Gamma(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}}) \tag{2.51}$$

  is an isomorphism of graded Poisson algebras, see [119].

- For all $q > 0$,

$$H^q(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}}) = 0\,. \tag{2.52}$$

- Any Hamiltonian holomorphic vector field action on $\mathscr{X}$ corresponds to a function in $\mathbb{C}[\mathscr{X}]$. These functions then lift to $\Gamma(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}})$, where they define Hamiltonian vector fields. Indeed, note that if

$$\mathscr{L}_{X_f}\omega = 0\,, \tag{2.53}$$

  then we have that

$$\begin{aligned} \mathscr{L}_{X_{\pi^* f}}\tilde{\omega} &= d i_{X_{\pi^* f}}\tilde{\omega} \\ &= d\pi^* i_{X_f}\omega \\ &= \pi^* d i_{X_f}\omega = \pi^* \mathscr{L}_{X_f}\omega = 0\,. \end{aligned} \tag{2.54}$$

So, if a projective symplectic resolution of $\mathscr{X}$ exists, it is equivariant with respect to $\mathbb{C}^\times \times G_H$.

If $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{X}}'$ are two equivariant projective symplectic resolutions of $\mathscr{X} = \mathfrak{M}_0$, then they are birational. This means the Picard groups are isomorphic, [36].

When the underlying quiver has no edge loops, it is known that certain values of $\zeta_{\mathbb{R}}$, known as generic values, provide an equivariant symplectic resolution of the manifold. We discussed this earlier in this section, see proposition 3.22 of [158] for another statement of this. More generally, a question we may ask is: when are there projective symplectic resolutions of Nakajima quiver varieties? This was answered recently by Bellamy and Schedler in [24], but we shall not go into its details. We shall merely content ourselves with the fact that in all examples considered in this thesis, a projective symplectic resolution does exist.

Moreover, we have that

**Theorem 4.** *(A simple generalisation of 3.4 in [172]) If $\vec{\zeta}$ and $\vec{\zeta}'$ are both generic, then $\mathfrak{M}_{\vec{\zeta}}$ and $\mathfrak{M}_{\vec{\zeta}'}$ are $G_H$-equivariant diffeomorphic.*

This result is crucial, and restricts wall crossing phenomena for the observables we define in chapter 3. The theorem is essentially because the hyperKähler structure (specifically the existence of $\zeta_{\mathbb{C}}$ as well as $\zeta_{\mathbb{R}}$) means that the non-generic values (when generic values exists) form a codimension 3 subset of the space of Fayet-Iliopoulos parameters, and hence one can always go around a "wall" via a homotopically unique path.

We now make a brief point about the possibility of turning on a complex mass parameter $m_{\mathbb{C}} \in \mathfrak{h}_H$ (the Cartan subalgebra of the Lie algebra of $G_H$). We need to consider the term in $U$ (equation (1.11)). In order to make manifest the action we add in the previously suppressed identity terms,

$$U \supset 2|\tilde{q}(\varphi \otimes 1_{G_H} + 1_G \otimes m_{\mathbb{C}})|^2 + 2|(\varphi \otimes 1_{G_H} + 1_G \otimes m_{\mathbb{C}})q|^2 . \tag{2.55}$$

The vacua are now points of our hyperKähler construction that also obey the equation (2.55). Since the action of $\varphi$ is by an action that we quotient out, we are asking for points on $\mathfrak{M}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}$ that are invariant under the $U(1)$ generated by the complex mass $m_{\mathbb{C}} \in \mathfrak{h}_H$. They need not exist in general, and they need not be isolated. However, in all the examples we consider there will be finitely many isolated vacua. In these cases we have access to localisation theorems that allow us to compute indices.

Finally, it is clear from inspecting equation (1.11) that turning on $m_{\mathbb{R}}$ will completely lift the Higgs branch.

### 2.2.2 The Coulomb branch of a quiver gauge theory - Chiral rings and Hilbert series

As we have already stated in section 2.2, the Coulomb branch, unlike the Higgs branch, receives quantum corrections. This has meant that a mathematical construction of the Coulomb branch, analogous to that of the Nakajima quiver variety, was much harder to find. Despite this difficulty, there has been much progress over the past few years and definitions of the Coulomb branch of varying degrees of rigour were given in [155, 38, 44, 43], while a finite dimensional hyperKähler quotient construction specific for the Coulomb branch of $A$ and $\hat{A}$-type quiver gauge theories was given in [159]. Due to the complexity of these constructions and their lack of direct relevance to our work, we will spend very little time on them. However, the construction of [159] will be relevant to the calculations in chapter 4. We postpone discussion of that paper until that chapter.

These results above in a large part were inspired by the works of [64, 60, 63, 61, 62], to name but a few. In these works a recipe for computing an invariant of the Coulomb branch of certain quiver varieties is given. This invariant determines much of the complex structure of the Coulomb branch. Its computation relies on two fundamental assumptions:

1. The Coulomb branch is an affine variety;
2. The chiral ring (defined below) is equal to the coordinate ring of the Coulomb branch.

As far as the author is aware there are no physical reasons (apart from three dimensional mirror symmetry, see section 2.3) for these two assumptions to be true in general, nonetheless, they are strongly suspected to be true. In fact, the definition of the Coulomb branch in [155, 38] is taken to be the spectrum of a commutative ring, whose definition is heavily inspired by the chiral ring, and so agrees with these assumptions.

Call the quantum Coulomb branch vacuum $\mathcal{M}_C$. For a given choice of Coulomb branch vacuum $u \in \mathcal{M}_C$, we have a quantum field theory, which is composed of local operators $\mathcal{O}(x)$ for $x \in \mathbb{R}^{2,1}$. These operators have an operator product expansion, [201], but this is too complicated to compute in general (just knowing the spectrum itself is beyond current techniques). However, one can make progress by looking at a simpler subring of the space of all operators, notably the chiral ring, [138], whose elements are the chiral operators. A chiral operator is a (product of) local operator(s), such that it is annihilated by two supercharges, $Q_1, Q_2$, lying within an $\mathcal{N} = 2$ subalgebra of the $\mathcal{N} = 4$ Poincaré superalgebra[6],

$$[Q_i, \mathcal{O}(x)] = 0\,, \quad i = 1,2\,. \tag{2.56}$$

By Leibniz rule, if $\mathcal{O}_1(x)$ is chiral and $\mathcal{O}_2(y)$ is chiral, then the product $\mathcal{O}_1(x)\mathcal{O}_2(y)$ is chiral.

We further have chosen the $Q_i$ such that $\frac{\partial}{\partial x} = \{Q_i, Q\}$ for some $i = 1,2$ and $Q$, and so we have that

$$\frac{\partial}{\partial x}\langle \mathcal{O}(x)\rangle_u = \langle \{Q_i, [Q, \mathcal{O}(x)]\}\rangle_u = 0\,, \tag{2.57}$$

where in the first equality we used the super Jacobi identity and the chirality of $\mathcal{O}(x)$, and in the second equality we used that $Q_i$ is a symmetry of the theory. The subscript $u$ is to signify that we are taking the vacuum expectation value in the vacuum $u$. Thus, we have that the expectation value of any chiral operator is independent of its position. By the cluster decomposition principle, for a proof of which see [9], for chiral operators $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$\begin{aligned} \langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle_u &= \langle \mathcal{O}_1(x)\rangle_u \langle \mathcal{O}_2(y)\rangle_u \\ &=: \langle \mathcal{O}_1\rangle_u \langle \mathcal{O}_2\rangle_u\,. \end{aligned} \tag{2.58}$$

Since $\langle \mathcal{O}\rangle_u$ is a number, we see that the chiral ring is a commutative ring. In general we may have Grassmannian odd chiral ring operators, in which case we do not quite have a commutative ring. However, since we assume the chiral ring gives the coordinate ring of the Coulomb branch, we assume that all chiral operators are bosonic. It is not known to the author whether there is any good reason for this to be true except for three dimensional mirror symmetry, see section 2.3.

[6]This choice of $\mathcal{N} = 2$ subalgebra corresponds to a choice of complex structure. It is this complex structure that we are referring to when we say that functions on $\mathcal{M}_C$ are holomorphic.

By the operator product expansion, we have that for $\mathscr{O}_i$ and $\mathscr{O}_j$ chiral operators[7],

$$\langle \mathscr{O}_i \mathscr{O}_j \rangle_u = \sum_k c_{ij}{}^k \langle \mathscr{O}_k \rangle_u \,, \tag{2.59}$$

where $c_{ij}{}^k \in \mathbb{C}$.

The chiral ring is thus fully specified by the spectrum of all chiral operators $\{\mathscr{O}_i\}$, and their commutative multiplication determined by the structure constants $c_{ij}{}^k = c_{ji}{}^k$.

The final statement that we need is, for $\mathscr{O}$ an element of the chiral ring, $\langle \mathscr{O} \rangle_u$ depends holomorphically on $u$. The argument, lifted from [109], works by varying the non-holomorphic information, namely the $F^\dagger$-term and the $D$-term and showing that this lead to $Q_i$-exact changes to the path integral.

So, for any element $\mathscr{O}$ of the chiral ring, we define a holomorphic function

$$\begin{aligned} f_{\mathscr{O}} : \mathscr{M}_C &\to \mathbb{C}\,, \\ u &\mapsto \langle \mathscr{O} \rangle_u \,. \end{aligned} \tag{2.60}$$

It is clear that this mapping from operator to function is injective and that we must have

$$f_{\mathscr{O}_1} f_{\mathscr{O}_2} = \sum_k c_{ij}{}^k f_{\mathscr{O}_k} \,. \tag{2.61}$$

It is then claimed in the literature that the space of holomorphic functions that one gets by this correspondence is precisely the set of regular functions. Thus, the chiral ring is the coordinate ring of $\mathscr{M}_C$. So all we need in order to calculate $\mathscr{M}_C$ is the spectrum and the structure constants. This argument applies equally to the Higgs branch, and since we can compute the coordinate ring by other methods, we can see that on the Higgs branch, the chiral ring and the coordinate ring are indeed the same ring.

The object of $\mathscr{M}_C$ that we will calculate is an object known as the Hilbert series. This gives us the spectrum of the theory, as well as strong restrictions on the structure constants. Indeed, in some cases the Hilbert series is sufficient to determine the structure constants.

**Definition 5.** *For an affine variety $Z$ with group action $H$, the* Hilbert series *is the character of the coordinate ring under the induced action of $H$.*

So all we need to do is to compute the spectrum and its action under symmetries of the theory. We shall do so by thinking about what operators we know to be in the theory.

[7] Note that the terms in the OPE that depend on position would be $Q_i$ exact and so do not appear in the correlation function.

The tree level theory Coulomb branch has a factor $T^{\text{rk}(G)}$ from the dual photon, $\chi$. This dual photon in Maxwell's equation is the connection for the magnetic field. Indeed, in classical electrodynamics we have from Maxwell's equations that

$$\partial_\nu(\star F)^\nu = \partial^2\chi = k\,, \tag{2.62}$$

where $k$ is the magnetic source. Thus, in the quantum theory we should expect 't Hooft monopole operators, [108]. These are disorder operators, i.e. operators that create or destroy topological quantum numbers. The particular topological quantum numbers that they create or destroy is the vortex number. The vortex number is the charge under the topological[8] $U(1)$ with current tr$(\star F)$. The torus of all such $U(1)$'s is given by the Pontryagin dual of the gauge group:

$$T_C := \pi_1(G)^\vee := \text{Hom}(\pi_1(G), \mathbb{C}^\times)\,. \tag{2.63}$$

The vortex quantum number, $\mathfrak{n}$, lives in $\mathfrak{n} \in \pi_1(G)$. Note that this Pontryagin dual is exactly the space where the character $\rho$, of section 2.2.1, that resolves the Higgs branch, lives.

The magnetic charge of an operator in QED is quantised by Dirac quantisation, [74]. More generally, in non-abelian gauge theory the magnetic charge lives in the Weyl chamber of the coweight lattice of the gauge group, i.e. the weight lattice of the Langlands dual gauge group[9] , [90].

Following the direction of attack of [86], instead of defining the monopole operators in some CFT in the IR as done in [34, 33, 32], we shall define the monopole operators in the UV. An element of the coweight lattice of $G$, $\vec{m}$, is equivalent to a homomorphism $\mathfrak{m} : \mathfrak{u}(1) \to \mathfrak{g}$. A monopole operator of charge $\vec{m}$, $\mathcal{O}_{\vec{m}}(y)$, is defined by the boundary conditions in the path integral required when it is present. We require a Dirac monopole singularity[10] :

$$F(x) = \frac{\mathfrak{m}(1)}{2} \star d\frac{1}{|x-y|}\,. \tag{2.64}$$

Since we want an element of the chiral ring, supersymmetry forces one of the triplet of scalars in the vector multiplet, say $\phi_3$ (this choice being equivalent to the choice of $Q_1, Q_2$), to obey the Bogomoln'yi equation $d\phi_3 = \star F$. Letting the other two vector multiplet scalar fields have trivial boundary conditions, we have defined the bare monopole operator $\mathcal{O}^{\text{bare}}_{\vec{m}}$.

[8]This symmetry can be understood as coming from "large" gauge transformations, ones that are non-zero at infinity. That we can only see the $U(1)$ factors and not the full group is because instantons in the theory, in this case monopole operators, break the group, [170].

[9]This can be seen from noting that that electric charge lives in the weight lattice, and Dirac quantisation requires magnetic charge to "eat" electric charge to give an integer.

[10]The trivial Weyl group action on the space of magnetic charges is due to gauge invariance under the Weyl group at the boundary

We then allow ourselves to “dress" the bare monopole operators with the holomorphic gauge invariants of the complex scalar $\phi_1 + i\phi_2$. These are given by the Casimirs of the gauge group broken by $\vec{m}$.

The claim is that this fully describes the spectrum of the chiral ring. We now want to see how our operators are graded under the symmetries that act on them. There are two symmetries that we have available: the $U(1)_C \leqslant SU(2)_C$, and the $T_C$ defined in equation (2.63).

Before describing the grading of these operators, we shall describe a deformation that we are allowed to perform on the Coulomb branch, analogous to the deformation $\rho$, of the Higgs branch that, via geometric invariant theory, led to a resolved manifold. This deformation is the introduction of background magnetic charge, that is charge brought about by gauging the flavour symmetry of the theory, introducing some boundary conditions for a specific charge in the coweight lattice of the flavour symmetry, and then freezing this gauge symmetry. It was first done for the Hilbert series in [61]. We shall call the background magnetic charge $\vec{m}_F$. It lives in the Weyl chamber of the coweight lattice of $G_H$. Just as the chiral ring for $\vec{m}_F = 0$ is the coordinate ring, the chiral ring in the presence of background magnetic charge is given by the global sections of a line bundle on $\mathscr{M}_C$ defined by $\vec{m}_F$.

In the paper [86], the $U(1)_C$ of a bare monopole operator for a quiver gauge theory, defined on the 8 supercharge quiver $\Gamma = (V,E)$, is given as[11]

$$\begin{aligned}\Delta(\vec{m},\vec{m}_F) = &\sum_{a\in V}\sum_{i=1}^{N_a}\sum_{\ell=1}^{k_a}|m_{a,\ell} - m_{F\,a,i}| - \sum_{a\in V}\sum_{\ell,\ell'=1}^{k_a}|m_{a,\ell} - m_{a,\ell'}| \\ &+ \sum_{(a,b)\in E}\sum_{\ell=1}^{k_a}\sum_{\ell'=1}^{k_b}|m_{a,\ell} - m_{b,\ell'}|\,.\end{aligned} \tag{2.65}$$

The value of this charge when $\vec{m}_F = 0$ is incredibly important. Indeed, it splits the theory into three separate categories

1. If $\Delta(\vec{m},0) \geqslant 1$ for all $\vec{m} \neq 0$, then we call the theory *good*. Most notably it means that the chiral ring's $\mathbb{Z}$-grading by $U(1)_C$ is in fact a $\mathbb{N}_0$-grading, with the only operator of charge 0 being the constant operator. An affine variety with such a coordinate ring has a $\mathbb{C}^\times$-action whose maximal compact subgroup is $U(1)_C$, see for example [156], and any hyperKähler manifold with a $\mathbb{C}^\times$-action has a $\mathbb{H}^\times \geqslant SU(2)$ action rotating the $\mathbb{CP}^1$ of complex structures, [188]. This conical action on $\mathscr{M}_C$ is the geometric action of the dilatation operator in the low energy gauge theory, and so the UV $U(1)_C$-charge is useful information in the IR.

[11] We will take the definition of their $\Delta$ times 2.

One can see that the inequality holds iff for all $a \in V$

$$N_a + \sum_{b \in V} C_{ab} k_b \geqslant 0\,, \tag{2.66}$$

where $C$ is the generalised Cartan matrix associated to the quiver $\Gamma$. The theory is called *balanced* and has monopole operators of charge 2, if the inequality above is saturated at every node. It is shown in section 5.4 of [86] that if a theory is balanced, then it is necessarily a Dynkin diagram of $ADE$-type with flavour nodes or of $\hat{A}\hat{D}\hat{E}$-type, but with no flavour nodes.

We expect for good quivers a non-empty Higgs branch and an interacting superconformal field theory at the intersection of the Higgs branch and the Coulomb branch.

2. If $\Delta(\vec{m}, 0) \leqslant 0$ for some $\vec{m} \neq 0$, then we call the theory *bad*. The UV $U(1)_C$ charge has no interpretation in the IR. They generically have empty Higgs branch. Very little is known about the IR of bad quivers, but recent progress has been made using Seiberg-like dualities in [204, 20, 14, 71]. We won't discuss them anymore in this thesis, but we will remark in passing that if any node $a \in V$ has

$$N_a + \sum_{b \in V} C_{ab} k_b \leqslant -2\,, \tag{2.67}$$

then the theory is bad.

3. If $\Delta(\vec{m}, 0) \geqslant 1$ for all $\vec{m} \neq 0$, and the bound is saturated for some non-zero $\vec{m}$, then we call the theory *ugly*. The same argument for good quivers still holds, in that we still have that $\mathscr{M}_C$ is a hyperKähler cone and the conical action contains $U(1)_C$. However, it necessarily has free hypermultiplets given by the monopole operator with $\Delta = 1$.

We must have that

$$N_a + \sum_{b \in V} C_{ab} k_b \geqslant -1\,, \tag{2.68}$$

with at least one node saturating the bound. This is a necessary, but not sufficient condition for the quiver to be ugly, because if every node obeys this inequality, the theory may be bad.

It is very simple to compute the effect of dressing by Casimirs on the quantum numbers of the monopole operator. Any degree $d$ product of Casimirs has $U(1)_C$ charge $d$, and the Casimirs are freely generated.

The topological quantum numbers corresponding to $T_C = \prod_{a \in V} U(1)$ of the bare monopole operator, with magnetic charge $\vec{m}$, are given by $\left(\sum_{\ell=1}^{k_i} m_{a,\ell}\right)_a$. The dressing by Casimirs has no effect on these topological quantum numbers.

We are now ready to compute the Coulomb branch Hilbert series for any good or ugly quiver. We introduce fugacities $\tau$ for the $U(1)_C$ R-symmetry, and $z_a$ for the topological symmetries. We compute

$$\mathrm{HS}(\mathscr{M}_C) = \mathrm{tr}_{\mathbb{C}[\mathscr{M}_C]} \left( \tau^{\mathscr{D}} \prod_{a \in V} z_a^{\mathscr{J}_a} \right) , \tag{2.69}$$

where we say $\mathscr{J}_a$ are the generators of $T_C$ and $\mathscr{D}$ is the generator of $U(1)_C$.

If the quiver is balanced, then it is expected that the topological symmetry is enhanced in the quantum theory to some group $G_C$ whose maximal torus is $T_C$, [114, 86, 21]. For such an enhancement we would require operators of $U(1)_C$ charge 2, as that is the charge of a current, and the balance conditions (the saturation of the inequality (2.66)) are exactly the conditions needed to satisfy this requirement. For a balanced $ADE$-type quiver, $G_C$ is a simple Lie algebra and its Dynkin diagram is the quiver. The Hilbert series of a quiver is invariant under the action of the Weyl group of $G_C$.

A simple example of this symmetry enhancement is the quiver $\Gamma = ((1), \varnothing)$, i.e. a single point, with $N_1 = 2k_1$=2. For such a theory, one finds that the Hilbert series of the theory has the symmetry $(z_1, z_2) \mapsto (z_2, z_1)$.

Given the Hilbert series, as we have discussed, one knows the spectrum of the chiral ring and, restrictions on the structure constants. This is discussed in [28]. A very simple example of the ambigiuity in the structure constants that arises from the Hilbert series is if we had two local operators $\mathscr{O}_1$ and $\mathscr{O}_2$ whose only grading was $\tau$ and $\tau^2$ respectively, then we could not tell the difference in the Hilbert series between the relations $\mathscr{O}_2 = 0$ and $\mathscr{O}_1^2 = 0$.

### 2.2.3 The chiral ring of the Higgs branch

We have given a method for computing the Hilbert series of the chiral ring of the Coulomb branch. We now describe how to compute the Hilbert series of the chiral ring of the Higgs branch. We specifically talk about the construction on the Hilbert series of the Higgs branch when $\zeta_{\mathbb{R}} = \zeta_{\mathbb{C}} = 0$.

We have by geometric invariant theory that

$$\mathfrak{M}_{0,0} \cong \mathrm{Spec}\ \mathbb{C}[\mu_{\mathbb{C}}]^G , \tag{2.70}$$

which we can think of as all polynomial functions on affine space, modulo the ideal generated by $\mu_{\mathbb{C}}$, that are gauge invariant. Counting this is given quite simply as

$$\mathrm{HS} = \left\langle \mathrm{PE}\left[\mathrm{ch}(q\mathbb{C}\oplus\tilde{q}\mathbb{C}) - \mathrm{ch}(\mu_{\mathbb{C}}\mathbb{C})\right], 1\right\rangle_G , \tag{2.71}$$

where $\langle\chi,\psi\rangle_G$ is the inner product of $G$-characters defined for all compact Lie groups; 1 is the trivial character; ch is the character with respect to the gauge group $G$, the flavour group $G_H$ and the scaling symmetry $\mathbb{C}^\times$; and PE is the plethystic exponential

$$\mathrm{PE}[f(t_1,\ldots,t_n)] := \exp\left(\sum_{r=1}^{\infty}\frac{f(t_1^r,\ldots,t_n^r)}{r}\right) . \tag{2.72}$$

The plethystic exponential gives the character of a finitely generated ring with finitely many relations in terms of the character of the generators,

$$\mathrm{PE}\left[\sum_i t_i - \sum_j s_j\right] = \frac{\prod_j(1-s_j)}{\prod_i(1-t_i)} . \tag{2.73}$$

The Weyl integration formula, see for example chapter IV 1 of [40], tells us how to write the character dot product as an integral over the maximal torus of $G$, $T(G)\cong U(1)^{\sum_{a\in V}k_a}$. We can write

$$\mathrm{HS} = \prod_{a\in V}\left(\prod_{\ell=1}^{k_a}\oint\frac{dw_{a,\ell}}{2\pi i}\prod_{\substack{\ell\neq\ell'\\1}}^{k_a}\left(1-\frac{w_{a,\ell}}{w_{a,\ell'}}\right)\right)\mathrm{PE}\left[\mathrm{ch}(q\mathbb{C}\oplus\tilde{q}\mathbb{C}) - \mathrm{ch}(\mu_{\mathbb{C}}\mathbb{C})\right] , \tag{2.74}$$

where $w_{a,\ell}$ are the fugacities for the gauge symmetries, $G$, and are integrated out in the Hilbert series.

## 2.3 Three dimensional dualities and Hanany-Witten branes

As we have seen the two manifolds, the Higgs branch and the Coulomb branch, are very similar. They are both hyperKähler cones, one is resolved by $\zeta$, the other by $m$. This may lead us to wonder if there is a form of duality between these two branches? The answer as it turns out is yes. In this section, we shall describe this duality, known in the literature as “3d mirror symmetry"; and how it can be understood from string theory embeddings. We shall also have a brief definition of the three dimensional form of Seiberg duality.

### 2.3.1 3d mirror symmetry

The duality of 3d mirror symmetry is an IR duality of two different 3d $\mathcal{N}=4$ gauge theories, say theory $A$ and theory $B$. The duality states that if $A$ and $B$ are mirror dual theories, then they have the following identification of objects given in table 2.2.

| Theory $A$ | Theory $B$ |
|---|---|
| $\mathcal{M}_H^A$ | $\mathcal{M}_C^B$ |
| $\mathcal{M}_C^A$ | $\mathcal{M}_H^B$ |
| $G_H^A$ | $G_C^B$ |
| $G_C^A$ | $G_H^B$ |
| $\zeta_{\mathbb{R}}^A$ | $m_{\mathbb{R}}^B$ |
| $\zeta_{\mathbb{C}}^A$ | $m_{\mathbb{C}}^B$ |
| $m_{\mathbb{R}}^A$ | $\zeta_{\mathbb{R}}^B$ |
| $m_{\mathbb{C}}^A$ | $\zeta_{\mathbb{C}}^B$ |

Table 2.2 The dictionary for 3d $\mathcal{N}=4$ mirror symmetry.

This allows us to extend our complete description of the effects of the mass and Fayet-Iliopoulos deformations of the Higgs branch to such a complete description of the Coulomb branch. All we need to do is swap the role of $\zeta$ and $m$. Just as $\zeta_{\mathbb{R}}$ resolves the Higgs branch manifold, $m_{\mathbb{R}}$ resolves the Coulomb branch. Furthermore, there equivalence means that the Picard group of line bundles on $\mathcal{M}_H^A$ is isomorphic to the Picard group of line bundles on $\mathcal{M}_C^B$.

Mirror symmetry was first discovered in the paper [114], specifically for the quivers for $ADE$-instantons and their mirror duals. Four months later it was found that the duality could be understood in terms of $S$-duality in string theory for certain theories, [98]. The derivation of duality from string theory lead to papers such as [66] being able to derive further mirror pairs they found via string theory [67, 68], while [33, 32] provided field theoretic proofs for mirror symmetry in certain abelian theories in the large flavour limit. The string theory construction of Hanany and Witten in [98] is illuminating for what we shall look at in this thesis. Hence, we review it in the next subsection.

The mirror dual of a quiver gauge theory is difficult to calculate in general, in fact no recipe is known for how to calculate it for a general quiver. Sometimes the mirror dual theory to a quiver is not Lagrangian, and so not a quiver theory. For example, in [114] it was found that the mirror dual to a $\hat{E}_{6,7,8}$ quiver gauge theory is non-Lagrangian and, more generally in [27], a large family of non-Lagrangian theories was found from the M-theory construction of [12, 13, 84] that are mirror dual to Lagrangian theories.

In the case that the mirror pair is a quiver, the mapping between the global groups $G_C$ and $G_H$ does contain some information that constrains the mirror quiver. $G_H$ tells us about the number of nodes (equal to the rank of $G_H$), and $G_C$ tells us what the multi-set $\{N_a\}_a$ will be.

Certain families of quivers are closed under mirror symmetry, a simple example of which is the linear quiver, whose action under 3d mirror symmetry was given in [85, 63].

Since we have $\mathcal{M}_C^A = \mathcal{M}_H^B$ we would expect the Hilbert series of both manifolds to match. A derivation of this matching is given in chapter 4, where we derive the effect of mirror symmetry for specific theories .

### 2.3.2 The Hanany-Witten construction

In chapter 5, we will talk extensively about the $A$- and $\hat{A}$-type quivers. All of these quivers are realisable in the field theory limit of Hanany-Witten brane set-ups of [98]. We briefly discuss this brane set-up and the field theory limit. The notation is the same as [98].

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| NS5 | x | x | x | x | x | x | | | | |
| D5 | x | x | x | | | | | x | x | x |
| D3 | x | x | x | | | | x | | | |

Table 2.3 The brane configurations for a Hanany-Witten brane set-up. The x denotes a direction in which the brane spans and a blank is where it has a definite value

We work in Type IIB superstring theory on $\mathbb{R}^{9,1}$ (or $\mathbb{R}^{8,1} \times S^1$ for $\hat{A}$-type quivers). Such a theory has 10 dimensional $\mathcal{N} = 2$ supersymmety, this has 32 real supercharges. Write $Q_L$ and $Q_R$ for the two 16-component Majorana-Weyl supercharges, generated by left and right moving worldsheet degrees of freedom respectively. We consider a brane set-up as described by table 2.3.

The D5-brane is invariant under all supercharges $\varepsilon^L Q_L + \varepsilon^R Q_R$ such that

$$\varepsilon^L = \Gamma_0\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\varepsilon^R\,. \tag{3.75}$$

The NS5-brane is invariant under all supercharges $\varepsilon^L Q_L + \varepsilon^R Q_R$ such that

$$\varepsilon^L = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\varepsilon^L\,, \quad \varepsilon^R = -\Gamma_0\Gamma_1\Gamma_2\Gamma_2\Gamma_4\Gamma_5\varepsilon^R\,. \tag{3.76}$$

One can explicitly calculate and find that there are exactly 8 real degrees of freedom for $\varepsilon_L$ and $\varepsilon_R$ together. Furthermore, any such $(\varepsilon_L, \varepsilon_R)$ necessarily obeys

$$\varepsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_6\varepsilon_R\,. \tag{3.77}$$

Thus we find the D3-brane can be present without breaking any further supersymmetries.

This brane configuration breaks the Lorentz group from $SO(1,9)$ to $SO(1,2)\times SO(3)_C\times SO(3)_H$, where $SO(1,2)$ acts on $(x_0,x_1,x_2)$, $SO(3)_C$ acts on $\mathbf{m}=(x_3,x_4,x_5)$ and $SO(3)_H$ on $\mathbf{w}=(x_7,x_8,x_9)$. $SU(2)_C$ is the double cover of $SO(3)_C$ and $SU(2)_H$ is the double cover of $SO(3)_H$. Thus, the super Poincaré group of this brane set-up is isomorphic to the three dimensional $\mathscr{N}=4$ super Poincaré group.

The fivebranes are infinite in extent. We denote the $i^{\text{th}}$ D5-brane's $(x_6,x_7,x_8,x_9)$-value as $(t_i,\mathbf{w}_i)$. We denote the $j^{\text{th}}$ NS5-brane's $(x_6,x_3,x_4,x_5)$-value as $(z_j,\mathbf{m}_j)$. The D3-branes are finite in the $x_6$-direction. Their ends will terminate on either type of fivebrane (that D-branes can end on other branes was found out in [186, 194]), giving us three different types of D3-brane. A D3-brane can connect two NS5-branes $i$ and $i'$ iff $\mathbf{w}_i=\mathbf{w}_{i'}$. The D3-brane has an arbitrary value of $(x_3,x_4,x_5)$, call this $\mathbf{x}$. A D3-brane can connect two D5 branes $j$ and $j'$ iff $\mathbf{m}_j=\mathbf{m}_{j'}$ having an arbitrary value of $(x_7,x_8,x_9)$, call this $\mathbf{y}$. Finally, a D3-brane can connect an NS5 to a D5, however the D3-brane's $(x_3,x_4,x_5)$-position is fixed by the D5 and its $(x_7,x_8,x_9)$-position is fixed by the NS5. Hence, it has no moduli in its position.

Our mirror symmetry is then enforced by $RS$, where $S$ is the $SL_2\mathbb{Z}$ transfomation

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{3.78}$$

which exchanges the two kinds of fivebranes, leaving the D3-brane invariant; and $R$ is a rotation, mapping $x_j\mapsto x_{j+4}$ and $x_{j+4}\mapsto -x_j$ for $j=3,4,5$.

We must pick boundary conditions for the D3-branes that set to zero half of the massless fields on the D3-brane world-volume. Supersymmetry allows boundary conditions in which either the vector multiplet or the hypermultiplet vanishes on the boundary, known as Dirichlet boundary conditions for the vector and hypermultiplet respectively; and Neumann boundary conditions, where the normal derivatives/components vanish. We find that the vector multiplet scalars are given by $\mathbf{x}$ and three of the four hypermultiplet scalars are given by $\mathbf{y}$. This forces the choice of boundary condition for the D3-branes. A D3-brane ending on an NS5 has Dirichlet boundary conditions for the vector multiplet and a D3-brane ending on a D5-brane has Dirichlet boundary conditions for the hypermultiplet.

The fields in our theory come from open string excitations where the ends are both on some brane, with one brane being a D3. The possibilities directly relevant for us are:

1. Stack of $k$ D3-branes between two NS5s. Strings from D3-brane to another D3-brane in this stack provide a vector multiplet with scalars $\mathbf{x}$ and vector $a_\mu$. The gauge group's maximal torus is $U(1)^k$, and is enhanced by Chan-Paton factors to $U(k)$ when the D3-branes are coincident. We call this gauge group the electric gauge group. The coupling constant, $\frac{1}{e^2}$, is proportional to the distance between the two NS5-branes.

2. Strings from stack of $k$ D3-branes between two NS5-branes to stack of $k'$ D3-branes between two NS5-branes, with one shared NS5-brane, provide a hypermultiplet of mass $\mathbf{x}_L - \mathbf{x}_R$, where $\mathbf{x}_L$ and $\mathbf{x}_R$ are the $\mathbf{x}$-values of the "left" and "right" D3-branes. The hypermultiplet is in the $(\underline{k}, \overline{\underline{k'}})$ of $U(k) \times U(k')$.

3. Stack of $k$ D3-branes between two NS5-branes. Strings from D3-branes to a D5-brane in-between (in the $x_6$-direction) the two NS5-branes provide a hypermultiplet in the fundamental of $U(k)$ with mass parameters $\mathbf{x} - \mathbf{m}$.

From this, one can see how to derive the quiver. For any two consecutive NS5-branes we have a gauge node, whose rank is determined by the number of D3-branes spanning the NS5-branes. The flavour rank for this node is determined by the number of D5-branes within the two NS5-branes. The hypermultiplets of point 2 provide bifundamental matter between neighbouring gauge nodes. The quiver we derive in this way will be linear if the spacetimes is Minkowski and circular if we compactify the $x_6$-direction.

For several D5-branes in-between two NS5-branes, one has field theory observables $\mathbf{m}_i - \mathbf{m}_j$. These are the mass parameters of the field theory. Likewise the positions of the NS5 branes, $\mathbf{w}_i$, provide the Fayet-Iliopoulos terms $\mathbf{w}_i - \mathbf{w}_j$. This matches what we would expect from our field theory discussion in section 2.1.

If the D3-branes intersect no D5-branes and left and right D3-branes do not meet on the NS5-branes, then all the hypermultiplets are massive and the moduli of the D3-branes describes the Coulomb branch. When all the D3-branes connect all neighbouring (in the $x_6$-direction) fivebranes, the hypermultiplets become massless and the moduli of the D3-branes describes the Higgs branch. Anything in between describes mixed branches.

To derive what the mirror dual is, we act with our element $RS$. The papers [195, 92] show that $S$-duality acts as electromagnetic duality, swapping $b$ with $a_\mu$. We then get a different brane picture, which is not of our described form, and so, we do not know how to write down the corresponding quiver. The solution to this problem is provided by a phenomenon called the Hanany-Witten transition. What we do is move NS5-branes through D5-branes until we are in a position where we can read off the quiver. The interesting phenomenon that Hanany and Witten found was that when the 5-branes pass through each other, D3-branes can be created or destroyed. To determine what happens one needs a certain charge for the NS5-brane and the D5-brane to be conserved. The charge for a specific NS5-brane is

$$L_{\mathrm{NS}} = \frac{1}{2}(r-l) + (L-R)\,, \tag{3.79}$$

where $r$ is the number of D5-branes to the right of the NS5-brane; $l$ is the number of D5-branes to the left; $R$ is the number of D3-branes which end to the right of the NS5; and $L$ the number of D3-branes that end to the left of the NS5. The same equation holds for D5, but now with $r$ and $l$ counting NS5-branes to the left and right.

### 2.3.3 Three dimensional Seiberg "duality"

The first instance of Seiberg duality was found in four dimensions in [177]. It was found to be an $S$-duality in the sense that it swaps electric gauge groups for magnetic gauge groups and strong coupling for weak coupling. The three dimensional form of it was first found in the papers [124, 1]. The three dimensional version is not an $S$-duality. It says the Higgs branch of a theory is isomorphic to the Higgs branch of the Seiberg dual theory.

We should be careful about our exact statement. It was shown in [14] that Seiberg duality is not a true IR duality. However, it is true that the resolved Higgs branch of one quiver gauge theory is equal to the resolved Higgs branch of the Seiberg "dual" theory. One can understand this from the brane picture of [98], described briefly in section 2.3.2. One can pass five-branes through each other. This creates or destroys three-branes in a way that can be calculated by the conservation of magnetic charge. However, if the five-branes are of the same type, then we expect them to swap magnetic charge as they pass through each other. Thus, the quiver gauge theory is not changed at all. The way out of this, for NS5-branes, is to turn on a real Fayet-Iliopoulos parameter (and of course for D5-branes we turn on a real mass parameter). This means that they never actually meet, and so we get a Hanany-Witten transition that changes the quiver gauge theory. Turning on the FI parameter resolves the Higgs branch, and this is why we have the equality of resolved Higgs branches of Seiberg dual theories.

Two quiver gauge theories, with quivers of $ADE$-type, are Seiberg dual if, in the classification given by equations (3.30) of chapter 6,they have equal $\lambda$, and the $\mu$ value of one is in a Weyl orbit of the $\mu$ value of the other. A theory is good if $\mu$ is dominant, and thus since every coweight has a unique dominant coweight that it is in the Weyl group orbit of, every $A$-type quiver with non-empty Higgs branch is Seiberg dual to some $T^{\sigma}_{\rho}$ theory, which will have the same Hilbert series and superconformal index. Note that any Seiberg dual quiver can be found by doing the one described for $A_1$ quivers in section 5.1, by treating the neighbouring gauge nodes of the quiver as flavour symmetries and then gauging these symmetries to glue the quiver back together. In [37], the unresolved Higgs branch is calculated for $\mu$ not dominant (they in fact calculate the mirror dual Coulomb branch for an $ADE$-type quiver). They find that the unresolved Higgs branch is *not* the same in general.

## 2.4 Quantisation of the chiral ring

Since both the Coulomb branch and the Higgs branch are hyperKähler manifolds, they have a holomorphic symplectic form, $\omega_{\mathbb{C}}$. This provides a Poisson bracket structure on the chiral ring in the usual way, via

$$\{f,g\} := \omega_{\mathbb{C}}(X_f, X_g)\,, \tag{4.80}$$

where $X_f$ is the unique vector field such that $i_{X_f}\omega_{\mathbb{C}} = df$. One can extend this to arbitrary holomorphic line bundles with connection $D$, via defining a line bundle valued vector field as $i_{X_f}\omega_{\mathbb{C}} = Df$.

This means that the chiral ring has the structure of a Poisson algebra. Pointwise multiplication forms an associative $\mathbb{C}$-algebra, the Poisson bracket has Lie algebra structure, and we have that the Poisson bracket acts as a derivation for pointwise multiplication.

Given any holomorphic vector bundle, we have that the Poisson algebra acts on the global sections via the Lie derivative. Hence, the global sections of any holomorphic vector bundles forms a representation of the Poisson algebra of the coordinate ring.

### 2.4.1 Deformation quantisation

In [73] in 1925, Paul Dirac wrote down what he thought the correct rule for quantisation of a classical system should be in the Heisenberg picture as

$$xy - yx = \frac{ih}{2\pi}\{x,y\}\,. \tag{4.81}$$

Deformation quantisation is a mathematical generalisation of this idea. We take some some Poisson manifold, that is a manifold whose space of functions form a Poisson $\mathbb{C}$-algebra. A deformation quantisation of $A$ is some algebra $(A_h, *)$, that as a vector space

$$A_h \cong_{\text{vector-space}} A \otimes_{\mathbb{C}} \mathbb{C}[[h]]\,. \tag{4.82}$$

$*$ is an associative product such that for $a,b \in A_h$, we have a $c \in A_h$ such that

$$a * b =: c = \sum_{k=0}^{\infty} h^k c_k(x)\,, \tag{4.83}$$

and

(i) $c_k$ are polynomials in $a_k$, $b_k$ and their derivatives;

(ii) $c_0(x) = a_0(x) b_0(x)$;

(iii) $[a,b] \cong a * b - b * a = -ih\{a_0, b_0\} + \mathcal{O}(h^2)\,.$

(i) means locality of $*$-product, (ii) means $A_h$ is a deformation of the commutative algebra of functions and (iii) is known as the correspondence principle (a terminology going back to the 1920's). See for example [52].

A simple example: the universal enveloping algebra of Lie algebra $\mathfrak{g}$, with product given by tensor product and grading by the degree, is the deformation quantisation of functions on $\mathfrak{g}^*$, with canonical Kirillov-Souriau Poisson bracket defined by the Lie bracket.

The question of whether such deformations exist more generally was solved for $C^\infty$-functions on symplectic manifolds in [69, 78]. An equivariant generalisation for symplectic manifolds was found in [147]. The question of general Poisson manifolds in [134]. The case of quantisation of holomorphic functions with a holomorphic symplectic form was solved in [166] and [29] deals with the case of a smooth symplectic algebraic variety, whose Poisson algebra is given by the regular functions.

The conclusion of this is that quantisations do exist for the chiral ring of $\mathfrak{M}_{\zeta_\mathbb{R}, \zeta_\mathbb{C}}$ for $\zeta_\mathbb{R}$ generic, and as we will show in section 3.1, this holds also for $\zeta_\mathbb{R} = 0$. Whenever we say such statements for the Higgs branch, the same statements hold for Coulomb branches with a mirror dual Higgs branch.

There are two important points to make. First of all, we have a map from $A_h$ to $A \cong A_h/hA$, known as the semiclassical limit, see for example [91]. We map $a \in A_h$ to $\bar{a} := a + hA_h$. By property (ii), we see that the product becomes commutative, but we induce a Poisson bracket via

$$\{\bar{a}, \bar{b}\} = \overline{\frac{1}{h}[a,b]}\,. \tag{4.84}$$

By the correspondence principle, this is exactly the original Poisson algebra of $A$. The second point is that this semiclassical limit means that we can pullback our representation of the Poisson algebra on global sections of holomorphic vector bundles to a representation of the quantised chiral ring. We will have an action of $A$ on global sections of some smooth bundle with connection via Lie derivative, which obeys (for a symplectic manifold)

$$\begin{aligned}[][\mathscr{L}_{\bar{a}}, \mathscr{L}_{\bar{b}}] &= \mathscr{L}_{\{\bar{a},\bar{b}\}} \\ &= \mathscr{L}_{\overline{\frac{1}{\hbar}[a,b]}}\,. \end{aligned} \tag{4.85}$$

The conclusion of this is that, if the quantised chiral ring is a quantum group with known representations, then we would expect to be able to decompose our global sections into these representations. We do exactly this for infinite rank $ADE$-type quiver Coulomb branch line bundles in section 6.3.

### 2.4.2 The Ω-deformation

The Ω-deformation provides us a physical way to realise a quantisation of the chiral ring in three dimensional field theory.

Supercharges

$$\begin{aligned} Q^H_\alpha &:= Q_{12\alpha} - Q_{21\alpha}\,, \\ Q^C_a &:= Q_{1a2} - Q_{2a1}\,, \end{aligned} \tag{4.86}$$

are invariant under $SU(2)_{EH} \leqslant SU(2)_H \times SU(2)_{\text{Lorentz}}$ diagonal and $SU(2)_{EC} \leqslant SU(2)_C \times SU(2)_{\text{Lorentz}}$ diagonal respectively. A choice of complex structure for both the Higgs and Coulomb branch fixes a choice of $Q^H$ and $Q^C$. $(Q^H)^2 = 2Z^C_{11} \sim \varphi + m_{\mathbb{C}}$, so $Q^H$ is nilpotent on the Higgs branch, and $(Q^C)^2 = 2Z^H_{11} \sim \zeta_{\mathbb{C}}$, so $Q^C$ is nilpotent on the Coulomb branch. One considers the twisted theory brought about by considering our observables to lie only in the cohomology of either $Q^H$ or $Q^C$. The resulting theory is Rozansky-Witten theory [174].

The Ω-deformation is a deformation of the supersymmetry algebra and Lagrangian[12] such that

$$(Q^{H \text{ or } C})^2 = \varepsilon \mathscr{L}_V\,, \tag{4.87}$$

where $V$ is the vector field generating spatial rotations on $\mathbb{R}^2$.

The resulting theory's observables are restricted to the origin of the spatial component. This orders the operators and thus provides a quantisation of the chiral ring [22, 43, 205].

[12] see [183] for the details

# Chapter 3

# Indices of Nakajima quiver varieties

In this chapter, we define an $\mathscr{N} = (4,4)$ quantum mechanical model, whose partition function we call the superconformal index. We write it as $\mathscr{Z}$. We also define an index that we call the Hilbert series to be the equivariant Euler character of equivariant sheaves. We show that for Nakajima quiver varieties it coincides with the notion of Hilbert series that we discussed in the previous chapter. Both indices have problems with their definitions caused by the singularities of the target space. We resolve this issue by working on the projective symplectic resolution of the target space.

In section 3.1, we define the quantum mechanical model, and talk about its $\mathfrak{osp}(4^*|4)$-symmetry. In section 3.2, we discuss the representation theory of $\mathfrak{osp}(4^*|4)$, culminating in us defining the superconformal index of any unitary $\mathfrak{osp}(4^*|4)$ representation. In section 3.3, we define the superconformal index of the quantum mechanics and write it as a sum of equivariant Euler characters. In section 3.3.1, we define the Hilbert series to be the equivariant Euler character, and discuss dependence on the choice of resolution. In section 3.3.2, we review how one computes an equivariant Euler character using localisation theorems in equivariant K-theory. In section 3.3.3, we compute the superconformal index of a simple theory using Chen-Ruan cohomology. In section 3.4, we relate the superconformal index and Hilbert series to various objects in higher dimensional quantum theories as a form of brief motivation.

## 3.1 The quantum mechanics

The construction of the index we use is developed in the thesis [185] and the paper [75], details of the quantum mechanics can also be found in [184]. We summarise the details here for ease of reference. The starting point is a hyperKähler cone, $\mathscr{X}$, which admits a triholomorphic closed homothety. This will be the unresolved Higgs (or Coulomb) branch of a three dimensional $\mathcal{N}=4$ quiver gauge theory, $\mathscr{X} := \mathfrak{M}_0$. Let $G_H$ be the group of isometries[1] on $\mathscr{X}$ generated by Hamiltonian vector fields.

The following discussion in this section is formal. The only non-singular choice of $\mathscr{X}$ that contain the amount of symmetry we want that we know of is flat space, $\mathbb{R}^{4n}$. We will in all other cases choose a regularisation given by the projective symplectic resolution of $\mathscr{X}$, $\tilde{\mathscr{X}}$.

We will look at supersymmetric quantum mechanics with target space $\mathscr{X}$. The action is given by

$$S = \int dt \left( \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + i g_{\mu\nu} \psi^{\dagger\mu} \frac{D}{Dt} \psi^\nu - \frac{1}{4} R_{\mu\nu\rho\sigma} \psi^{\dagger\mu} \psi^{\dagger\nu} \psi^\rho \psi^\sigma \right), \tag{1.1}$$

where $X^\mu$ are the coordinates on $\mathscr{X}$, $\psi^\mu$ and $\psi^{\dagger\mu}$ are complex conjugate fermionic variables, these are Grassmann-odd sections of the cotangent bundle of $\mathscr{X}$, and

$$\frac{D}{Dt} \psi^\mu := \dot{\psi}^\mu + \dot{X}^\nu \Gamma^\mu_{\nu\rho} \psi^\rho . \tag{1.2}$$

We define

$$\Pi_\mu := g_{\mu\nu} \dot{X}^\nu \tag{1.3}$$

to be the covariant momentum.

The commutation relations, found by quantising the Poisson brackets with respect to the Faddeev-Jackiw prescription (see chapter 2 of [185]) gives non-zero commutation relations

$$\begin{gathered} [X^\mu, \Pi_\nu] = i\delta^\mu_\nu, \quad [\Pi_\mu, \Pi_\nu] = -R_{\rho\sigma\mu\nu} \psi^{\dagger\rho} \psi^\sigma , \\ [\Pi_\mu, \psi^\nu] = i\Gamma^\nu_{\mu\rho} \psi^\rho , \quad [\Pi_\mu, \psi^{\dagger\nu}] = i\gamma^\nu_{\mu\rho} \psi^{\dagger\rho} , \quad \{\psi^\mu, \psi^{\dagger\mu}\} = g^{\mu\nu} . \end{gathered} \tag{1.4}$$

This $\sigma$-model has $\mathcal{N} = (1,1)$ supersymmetry generated by supercharges

$$\begin{aligned} Q &:= g_{\mu\nu} \dot{X}^\nu \psi^{\dagger\nu} , \\ Q^\dagger &:= g_{\mu\nu} \dot{X}^\mu \psi^\nu . \end{aligned} \tag{1.5}$$

As noted by Witten in [202], the Hilbert space for this quantum mechanics is $\Omega^*(\mathscr{X};\mathbb{C})$, the space of global forms with $L^2$-inner product

$$(\alpha,\beta) = \int_{\mathscr{X}} \alpha \wedge \star\overline{\beta} , \text{ for } \alpha,\beta \in \Omega^*(\mathscr{X};\mathbb{C}). \tag{1.6}$$

[1] We call this group $G_H$ as we will often take $\mathscr{X}$ to be the Higgs branch of a quiver gauge theory. However, we could have also taken $\mathscr{X}$ to the Coulomb branch, for which, in our convention, we would label the group of isometries as $G_C$.

In this geometric interpretation, $\psi^{\dagger\mu}$ acts as the raising operator $dX^\mu\wedge$, and $\psi^\mu$ acts as the lowering operator $g^{\mu\nu}\iota_{\partial_\nu}$. So, $Q$ acts as the de Rham derivative $d$, and $Q^\dagger$ as $\delta$, the coderivative. The Hamiltonian, $H = \frac{1}{2}\{Q, Q^\dagger\} = \Delta$, is the Laplacian.

Since the manifold is hyperKähler, the space of forms admits an $SO(5)$-action, [197]. The supersymmetries $Q$ and $Q^\dagger$ both transform in a **4** under this $SO(5)$, and so we can extend the $\sigma$-model to $\mathscr{N} = (4,4)$ supersymmetry. The Cartan of $SO(5)$ provides the Dolbeault bigrading of the space of forms. The new supercharges come from the complex structures of the manifold. They are

$$\begin{aligned} Q^a &= -i\psi^{\dagger\mu} I^a{}_\mu{}^\nu \Pi_\nu\,, \\ Q^{a\dagger} &= i\Pi^\dagger_\nu I^a{}_\mu{}^\nu \psi^\mu\,, \end{aligned} \tag{1.7}$$

where $a = 1,2,3$ is the $SU(2)_H$ index from the hyperKähler structure rotating the complex structures, and $I^a$ are the complex structures[2]. $Q^a$ acts as $i(\bar\partial - \partial)$, where $\partial$ is with respect to the complex structure $I^a$. $Q^{a\dagger}$ acts as $i(\partial^\dagger - \bar\partial^\dagger)$.

The target space is a hyperKähler cone. The conical action is given by a triholomorphic closed homothety. This provides the space with the action of the conformal algebra[3] $\mathfrak{so}(2,1)$. This conformal algebra, $\mathfrak{so}(5)$ and the supercharges $Q$ and $Q^\dagger$ close onto the superconformal algebra $\mathfrak{osp}(4^*|4)$. This is an action on the Hilbert space of forms, not the underlying manifold (this subtlety will be important in section 3.3, when we define the superconformal index and try to compute it with localisation theorems).

The bosonic subalgebra of the superconformal algebra is

$$\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(4)\,. \tag{1.8}$$

The conformal algebra, $\mathfrak{so}(2,1) \leqslant \mathfrak{osp}(4^*|4)$, is generated by the dilatation, $D$; the Hamiltonian, $H$; and the special conformal operator, $K$. If $\mathscr{X} = \mathbb{C}^{2n}$ with coordinates

[2] In our convention this would be denoted as $SU(2)_C$ if $\mathscr{X}$ were the Coulomb branch of a quiver instead of the the Higgs branch.

[3] Section 5 of [185] provides necessary conditions for when this happens, and proves that instanton moduli space, i.e. the Higgs branch of the Jordan quiver, is such an example. The proof there easily lifts to the Higgs branch of any quiver gauge theory with $\zeta_\mathbb{R} = \zeta_\mathbb{C} = 0$.

$Z^i = X^i + iY^i$ for $i = 1, \dots, 2n$, then these are given by

$$\begin{aligned} H &= -2\sum_{i=1}^{2n} \frac{\partial}{\partial Z_i}\frac{\partial}{\partial \overline{Z}_i}, \\ D &= -in - i\sum_{i=1}^{2n}\left(Z_i\frac{\partial}{\partial Z_i} + \overline{Z}_i\frac{\partial}{\partial \overline{Z}_i}\right), \\ K &= \frac{1}{2}\sum_{i=1}^{2n} Z_i\overline{Z}_i\,. \end{aligned} \tag{1.9}$$

We are interested in computing the spectrum of $D$. However, in general, this is too difficult a task. A simpler object to compute is one whose existence follows from the Hilbert space being a representation of the superconformal algebra $\mathfrak{osp}(4^*|4)$. We review the representation theory of this superalgebra, as well as the invariant of the theory that is easier to compute than the full spectrum of $D$, the superconformal index. The reference for this review is [185].

## 3.2 Representation theory and superconformal index of $\mathfrak{osp}(4^*|4)$

We make the following isospectral tranformation on the algebra to give a presentation that is easier to work with:

$$X \mapsto e^{-\mu K} e^{\frac{1}{2}\mu^{-1}H} X e^{-\frac{1}{2}\mu^{-1}H} e^{\mu K}, \quad \forall X \in \mathfrak{osp}(4^*|4)\,, \tag{2.10}$$

for some $\mu \in (0,\infty)$. One finds

$$\begin{aligned} iD &\mapsto L_0 := \mu^{-1}(H + \mu^2 K)\,, \\ H &\mapsto 2\mu L_+ := H - \mu^2 K - i\mu D\,, \\ K &\mapsto -\frac{1}{2\mu}L_- := -\frac{1}{4}H + \frac{\mu^2}{4}K - \frac{i}{4\mu}D\,. \end{aligned} \tag{2.11}$$

$L_0$ is represented by a second-order differential operator on the Hilbert space, the Laplacian with a regulating potential given by $K$. Hence, we expect it to have a discrete spectrum. Moreover, since $H$ has a positive spectrum and $K = e^{\frac{i\pi}{2}(H+K)} H e^{-\frac{i\pi}{2}(H+K)}$, we have that $L_0$ has a non-negative spectrum.

The Cartan subalgebra of $\mathfrak{g}_B$ is generated by $L_0, J_3, M$ and $N$, with $J_3$ the Cartan generator of $\mathfrak{su}(2)$, and $M$ and $N$ the Cartan generators of $\mathfrak{usp}(4)$. The weight lattice is $\varepsilon_1\mathbb{Z}\oplus\varepsilon_2\mathbb{Z}\oplus\delta_1\mathbb{Z}\oplus\delta_2\mathbb{Z}$, defined such that if $v_\lambda$ has eigenvalues $(\Delta, -2j, -m, -n)$ under $(L_0, 2J_3, M, N)$, then $v_\lambda$ has weight

$$\lambda = \frac{\Delta}{2}(\varepsilon_1 + \varepsilon_2) - j(\varepsilon_1 - \varepsilon_2) - m\delta_1 - n\delta_2 \,. \tag{2.12}$$

In order for $v_\lambda$ to be a lowest weight vector for a unitary irreducible representation of $\mathfrak{osp}(4^*|4)$, it is necessary that $\Delta \geqslant 0$, $(2j,m,n) \in \mathbb{N}_0^3$ and $m \geqslant n$.

As is standard in representation theory, the unitary irreducible lowest weight representations are classified by the values of the lowest weight. There are BPS bounds on $\Delta$. If $\Delta$ disobeys the bound, then the irreducible lowest weight representation is not unitary, and if $\Delta$ saturates the bound, then the lowest weight vector is annihilated by more elements than if it is not saturated. In the work [185], Singleton found a full classification of unitary irreducible lowest weight representations. Writing $Q^{\pm}_{i\alpha}$ for the supercharge corresponding to the root $\varepsilon_i \pm \delta_\alpha$ for $i, \alpha = 1,2$ we have:

**Theorem 5.** *(From chapter 7 of [185]) Unitary irreducible lowest weight representations of* $\mathfrak{osp}(4^*|4)$ *with lowest weight vector* $v_\lambda$ *and lowest weight* $\lambda$ *split into five possible cases:*

1. *Generic/long representations* $L(\Delta, j, m, m)$*, with* $\Delta > 2(j+m+1)$*;*

2. *Semishort representations* $SS(j,m,n)$*, with* $\Delta = 2(j+m+1)$*;*

3. *1/4-BPS representations* $S(m,n)$*, with* $\Delta = 2m$*,* $j = 0$ *and* $m \neq n$*,*

$$Q^-_{11} v_\lambda = Q^-_{21} v_\lambda = 0; \tag{2.13}$$

4. *1/2-BPS representations* $S(m)$*, with* $\Delta = 2m = 2n$*,* $j = 0$*, and*

$$Q^-_{11} v_\lambda = Q^-_{21} v_\lambda = Q^-_{21} v_\lambda = Q^-_{22} v_\lambda = 0; \tag{2.14}$$

   *and*

5. *The trivial representation* **1***, with* $\Delta = m = n = j = 0$*.*

*We call the 1/4-BPS, 1/2-BPS and trivial representations short representations.*

We now construct the superconformal index of $\mathfrak{osp}(4^*|4)$. It is a function that sends representations of $\mathfrak{osp}(4^*|4)$ to polynomials in fugacities, but is more “stable” than a character. It is linear with respect to direct sums. Superconformal indices were first introduced in [128] for four dimensional superconformal field theory, and extended to three, five and six dimensions in [30]. The idea is that since short and semishort representations have fewer states than long representations, their existence in the theory should be insensitive to continuous changes in the parameters of the theory. The exception to this is when the dimension of a long representation is lowered to the unitarity bound; upon hitting the bound, it splits into a sum of (semi)short representations. So, if one makes a linear function that evaluates to zero on both long representations and sums of (semi)short representations that a long representation can decompose into, then this function will be invariant under continuous deformations of our theory. This is what the superconformal index does, and is the sense in which the superconformal index is more stable than a $\mathfrak{osp}(4^*|4)$ character. The price for this stability is that it contains less information (for one, any information that does depend on the continuous parameters in the theory).

Singleton in [185] found that the long representations decompose on unitarity bounds as follows (writing $\Delta_{SS} := 2(j+m+1)$)

$$\begin{aligned} &L(\Delta_{SS}+d,j,m,n) \to SS(j,m,n) \oplus SS(j-\frac{1}{2},m+1,n)\,, \text{ as } d\to 0 \text{ for } j>0\,, \\ &L(\Delta_{SS}+d,0,m,n) \to SS(0,m,n) \oplus S(m+2,n)\,, \text{ as } d \to 0\,. \end{aligned} \tag{2.15}$$

This means that if $\mathscr{I}$ is our superconformal index, then we must have that

$$\mathscr{I}\big(SS(j,m,n)\oplus SS(j-\frac{1}{2},m+1,n)\big) = \mathscr{I}\big(SS(0,m,n)\oplus S(m+2,n)\big) = 0\,. \tag{2.16}$$

In order to define our index, we first pick the supercharge $q := Q^-_{11}$, with $s := q^\dagger$, then

$$\{q,s\} = E := \frac{1}{2}(L_0 + 2J_3 + 2M)\,. \tag{2.17}$$

The coweight $E$ has the property that for $\lambda$ the lowest weight of an irreducible unitary lowest weight representation,

$$E(\lambda) = \frac{1}{2}(\Delta - 2j - 2m) = \begin{cases} 0 & \text{short}\,, \\ 1 & \text{semishort}\,, \\ >1 & \text{long}\,. \end{cases} \tag{2.18}$$

Given a completely reducible representation, $R$, of $\mathfrak{osp}(4^*|4)$, we shall calculate the submodule, $R_0$, of states for which $E=0$. On $R_0$ we can then grade by the weights of symmetries that lie in the little group of $E$, $\mathrm{Stab}(E)$. However, since we are going to look at states annihilated by $E$ (and hence $q$ and $s$), we quotient the little group by its ideal $I = \langle E,q,s\rangle$. One finds

$$\mathrm{Stab}(E)/I \cong \mathfrak{u}(2|1)\,. \tag{2.19}$$

This algebra has rank 3. So that the algebra commutes with $q$ and $s$, as well as $E$, we ignore the supertrace $\mathfrak{u}(1)$, leaving us with $\mathfrak{su}(2|1)$. Using the notation of appendix D.1 in [185], we have that

$$\mathfrak{su}(2|1) = \langle L_0 - 2J_3, N, v_{\pm 2\delta}, Q^{\pm}_{22}, S^{\pm}_{22} \rangle \,. \tag{2.20}$$

Concretely, if $e_{AB}$ is the $3\times 3$ supermatrix with 1 in the $A^{\text{th}}$ row and $B^{\text{th}}$ column and zeroes elsewhere, then we can represent $\mathfrak{su}(2|1)$ as

$$\begin{aligned} N = e_{11} - e_{22}, \quad v_{2\delta_2} = e_{12}, \quad v_{-2\delta_2} = e_{21}, \quad L_0 - 2J_3 = e_{11} + e_{22} + 2e_{33}, \\ Q^{+}_{22} = \sqrt{2}e_{32}, \quad Q^{-}_{31} = \sqrt{2}e_{31}, \quad S^{+}_{22} = \sqrt{2}e_{23}, \quad S^{-}_{22} = \sqrt{2}e_{13} \,. \end{aligned} \tag{2.21}$$

We grade by the fugacities:

- $y$ for the $\mathfrak{su}(2)$ factor, with generator $N$, the other Cartan of $\mathfrak{usp}(4)$, for which a $(p,q)$ form has eigenvalue $p-d$, for $d$ the quaternionic dimension of $\mathscr{X}$ ($M$ acts with eigenvalue $q-d$ - this is seen explicitly in the quantum mechanics by writing $M$ and $N$ in the coordinates of the quantum mechanics). Since this is a fugacity for $\mathfrak{su}(2)$, we will have $y \mapsto 1/y$ symmetry; and

- $\tau$ for the other $\mathfrak{u}(1)$ factor, given by $\frac{1}{2}L_0 - J_3$. We call the power of $\tau$ the scaling dimension, because of its relations to the state's eigenvalue under $L_0$, which is the similarity transform of $D$ (equation (2.11)). Since we that the dimension is bounded by $2j+m$ (this is a unitarity bound for $\mathfrak{osp}(4^*|4)$), we have that the exponent of $\tau$ must always be positive.

Due to equation (2.18) and the fact that the only raising operator that lowers a state's $E$ value is the nilpotent $Q^{-}_{21}$, we can fully classify the $R_0$ content. It follows from the following theorem

**Theorem 6.** *(From chapter 7 of [185]) For $V_\lambda$ a unitary irreducible lowest weight representation, the $E=0$ content, $V_0$, is given as follows:*

- *If $V_\lambda$ is short, then $V_0$ is the irreducible lowest weight representation of $\mathfrak{u}(2|1)$ with lowest weight vector $v_\lambda$.*

- *If $V_\lambda$ is semishort, then $V_0$ is the lowest weight irreducible representation the irreducible lowest weight representation of $\mathfrak{u}(2|1)$ with lowest weight vector $Q^{-}_{21}v_\lambda$.*

- *If $V_\lambda$ is long, then $V_0 = \{0\}$.*

One then finds the following decomposition of the $E = 0$ content into representations of $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \subseteq \mathfrak{su}(2|1)$:

$$\begin{aligned} S(m,m)_0 &= \mathbf{m+1}_m \oplus \mathbf{m}_{m+1}\,, \\ S(m,0)_0 &= \mathbf{1}_m \oplus \mathbf{1}_{m+1} \oplus \mathbf{0}_{m+2}\,, \\ S(m,n)_0 &= \mathbf{n}_m \oplus \mathbf{n+1}_{m+1} \oplus \mathbf{n-1}_{m+1} \oplus \mathbf{n}_{m+2}\,, m > n > 0\,, \\ SS(j,m,0)_0 &= \mathbf{0}_{m+2j+2} \oplus \mathbf{1}_{m+2j+3} \oplus \mathbf{0}_{m+2j+4}\,, \\ SS(j,m,n)_0 &= \mathbf{n}_{m+2j+2} \oplus \mathbf{n+1}_{m+2j+3} \oplus \mathbf{n-1}_{m+2j+3} \oplus \mathbf{n}_{m+2j+4}\,. \end{aligned} \tag{2.22}$$

Here $\mathbf{m}_n$ means the $m$-dimensional irreducible representation of $SU(2)$ with charge $n$ under $U(1)$.

We now define our index as[4]

$$\begin{aligned} \mathscr{X}(R) :&= \mathrm{tr}_R \left( (-)^{M+N} e^{-\beta E} \tau^{\frac{1}{2}L_0 - J_3} y^N \right) \\ &= \mathrm{tr}_{R_0} \left( (-)^{M+N} \tau^{-M-2J_3} y^N \right)\,. \end{aligned} \tag{2.23}$$

The last line has used that $\frac{1}{2}L_0 - J_3 - E = -2J_3 - M$.

Writing, for $n \in \mathbb{N}$, $\chi(n) := y^{n-1} + y^{n-3} + \cdots + y^{1-n}$ and $\chi(0) = 0$, we have that

$$\begin{aligned} \mathscr{X}(S(m,m)) &= \tau^m(\chi(m+1)) - \tau\chi(m))\,, \\ \mathscr{X}(S(m,n)) &= (-)^{m+n}\tau^m((1+\tau^2)\chi(n+1)) - \tau(\chi(n) + \chi(n+2)))\,, m \neq n\,, \\ \mathscr{X}(SS(j,m,n)) &= (-)^{m+n}\tau^{m+2j+1}((1+\tau^2)\chi(n+1)) - \tau(\chi(n) + \chi(n+2)))\,. \end{aligned} \tag{2.24}$$

We can see that this obeys equation (2.16).

Of course, one can always further grade by any symmetries that commute with $\mathfrak{osp}(4^*|4)$.

## 3.3 The superconformal index and Hilbert series of the quantum mechanics

It is argued in [185] that despite the non-compactness of $\mathscr{X}$, Hodge theoretic arguments still hold. Namely

$$\{\text{states such that } E = 0\} \cong \{q\text{-cohomology}\} \cong \{s\text{-cohomology}\}\,. \tag{3.25}$$

[4] There is a subtlety here in our choice of $(-)^F$, see chapter 7.4 of [185] for more on this.

After the similarity tranformation (2.10), the Hilbert space changes to the space of forms on $\mathscr{X}$ that have finite norm under the following inner product

$$(\alpha,\beta)=\int_{\mathscr{X}} d^n x\sqrt{g}\alpha\wedge\overline{\beta}e^{-\mu K}\,. \tag{3.26}$$

From [185] chapter 7, we know that if $\beta$ is any form on $\mathscr{X}$ and $\alpha=\beta e^{-\mu K}$, then

$$s\alpha=\frac{1}{\sqrt{\mu}}\overline{\partial}\beta\, e^{-\mu K}\,. \tag{3.27}$$

Hence $s$ acts as a Dolbeault operator, up to the overall exponential factor.

Having finite norm under (3.26) restricts the functions quite dramatically. The space of zero forms in the Hilbert space that have $E=0$ is the coordinate ring of $\mathscr{X}$.

In section 7.3 of [185], Singleton shows that on affine space $\mathbb{C}^{2n}$ with complex coordinates $(q^i,\tilde{q}_i)$, $i=1,\dots,n$,

$$\{\text{states with } E=0\}\cong\mathbb{C}[q,\tilde{q},dq,d\tilde{q}]\,, \tag{3.28}$$

where the right hand side is the space of polynomials in the Grassmann even $q$ and $\tilde{q}$ and the Grassmann odd $dq$ and $d\tilde{q}$. The dot product is given by equation (3.26), recall $K=\sum_i(|q^i|^2+|\tilde{q}_i|^2)$.

Thinking of $\mathbb{C}^{2n}$ as an affine variety, we can identify the $s$-cohomology with the sheaf cohomology in the Zariski topology, with sheaves given by holomorphic $p$-forms. The analytic and Zariski cohomology are famously the same for projective varieties, [182]. However our space is not projective, notably it is non-compact. The functions that this inner product throws away are ones that have essential singularities at infinity, for example $e^{q^2}$ - these are the functions that stop analytic and Zariski cohomology being the same[5].

In the examples that we deal with, there are two important subtleties. These are:

1. $\mathscr{X}$ often contains singularities;
2. $\mathscr{X}$ is never compact, and hence the spectrum is infinite in size. This causes the index to be divergent.

In the case that $\mathscr{X}$ is instanton moduli space, these problems respectively correspond to instantons shrinking to point-like particle, and instantons being able to fly away to infinity (UV and IR singularities respectively).

We want a suitable regularisation to deal with the most general case of singular hyper-Kähler cones. We cannot define the sheaf of holomorphic $p$-forms on $\mathscr{X}$, as the cotangent

[5]Functions like $e^q$ are included in the Hilbert space, but they are in the completion, with respect to the inner product (3.26), of the space of polynomials.

space is not defined at the singular points (there is no such problem for the holomorphic functions). So we instead work on the projective symplectic resolution[6] of $\mathscr{X}$, $\tilde{\mathscr{X}}$. This breaks the $\mathfrak{osp}(4^*|4)$ supersymmetry, but preserves the $\mathfrak{su}(2|1)$ supersymmetry.

The space of $E=0$ states on $\mathbb{C}^{2n}$ is the sheaf cohomology of $\mathbb{C}^{2n}$; $s$ acts as a Dolbeault operator on the regular part of forms in the Hilbert space; and the space of functions is the coordinate ring for $\tilde{\mathscr{X}}$ (recall from section 2.2.1 that $\Gamma(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}}) \cong \mathbb{C}[\mathscr{X}]$). Because of this, we define the $s$-cohomology on the Hilbert space to be the Dolbeault cohomology of regular forms on $\tilde{\mathscr{X}}$. This is the sheaf cohomology of $\tilde{\mathscr{X}}$. Due to the importance of this definition for this chapter, we repeat it:

$$\{s\text{-cohomology}\} \cong \{\text{regular Dolbeault cohomology on } \tilde{\mathscr{X}}\}. \tag{3.29}$$

This index on the resolved space reproduces known results for what the superconformal index should be on instanton moduli space, as calculated in [4]. Namely, the 1/2-BPS states. In the paper [4], the authors compute that the 1/2-BPS states are given by the Poincaré polynomial for Borel-Moore homology of instanton moduli space. In section 6.2, we find that the superconformal index on the resolved space contains this information. Furthermore, we find in section 3.3.3 that the superconformal index on the unresolved space $\mathbb{C}^2/\mathbb{Z}_2$ (defined in this case by taking $\mathbb{Z}_2$-invariant states), does not contain the 1/2-BPS states (most easily found by setting $\tau = y$, because, as can be seen from equation (2.24), only the 1/2-BPS representations survive.). There is a further discussion of this in the yet to be published work [75].

For an $\mathscr{X}$ with a $\mathbb{C}^\times$-action such as ours ($\mathbb{C}[\mathscr{X}]$ graded by $\mathbb{N}_0$, with $\mathbb{C}^0[\mathscr{X}] = \mathbb{C}$ and the symplectic form homogeneous with respect to this grading), Namikawa in [162] showed that there are only finitely many non-isomorphic projective symplectic resolutions of $\mathscr{X} = \mathfrak{M}_0$. We may ask, given two non-isomorphic equivariant projective symplectic resolutions of $\mathscr{X}$, $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{X}}'$, do they have the same Dolbeault cohomology? If this is the case, then the cohomology of $\tilde{\mathscr{X}}$ is really an invariant of $\mathscr{X}$, and our regularisation by working on the resolution makes sense. Theorem 4 of chapter 2 tells us that they are indeed diffeomorphic, and hence we should expect the same cohomology. Moreover, the diffeomorphism is $G_H$-equivariant. This means that a grading under $G_H$ is preserved under changing the choice of projective symplectic resolution. We conjecture that the $\mathbb{C}^\times$-grading of the superconformal index is invariant under different choices of resolution.

One can use Molien-like integrals as found in section 2.2.3 to compute the equivariant Euler character (defined in section 3.3.1) for a generic vector bundle. One then has to take

[6] See the discussion near the end of 2.2.1 for the definition of the projective symplectic resolution.

a specific pole procedure, called the Jeffrey-Kirwan procedure, to recreate the appropriate pole structure. The procedure depends upon a specific choice of parameter known as the Jeffrey-Kirwan parameter, which is conjectured in the literature to be identified with the Fayet-Ilioupolous parameter $\zeta_{\mathbb{R}}$. This allows one to actually compute the Hilbert series for different resolutions, and compare the answer. We discuss this, as well as explaining how to do the procedure in appendix A.

When $\mathscr{X} = \mathfrak{M}_0$ is a hyperKähler orbifold[7] the cohomological hyperKähler resolution conjecture states that if $\tilde{\mathscr{X}}$ is also hyperKähler, then the cohomology on $\tilde{\mathscr{X}}$ is the orbifold cohomology on $\mathscr{X}$. See conjecture 6.3 of [175] for the first statement of this conjecture, and [176] for a slightly more sophisticated wording of it. The orbifold cohomology was first defined in [53], the key point is that it depends solely on $\mathscr{X}$, and so is independent of the choice of resolution. The conjecture was proven for the cases of Kummer surfaces and Hilbert scheme of K3 surfaces in [82, 81], but remains open in general. Due to how the orbifold cohomology is constructed, when $\mathscr{X}$ is a hyperKähler orbifold, the cohomology of $\tilde{\mathscr{X}}$ contains, as a subring, the cohomology of $\mathscr{X}$.

We still have the problem that our naive superconformal index will be divergent, due to the infinite dimensional Hilbert space. The resolution to this issue is to grade the space by the action of $\langle -M-2J_3\rangle \oplus \mathfrak{g}_H$. This solves the problem, because each homogeneous subspace of fixed weight under $\langle -M-2J_3\rangle \oplus \mathfrak{g}_H$ has finite dimension (this is proved later in this section).

We write $\mathscr{J}_i$ for the generators of the Cartan subalgebra of $\mathfrak{g}_H$ (the Lie algebra of $G_H$).

We will now define our *superconformal index*, $\mathscr{Z}$:

$$\begin{aligned}\mathscr{Z}(\tilde{\mathscr{X}};\tau,y,Z) &:= \mathrm{tr}_{\Omega^*(\tilde{\mathscr{X}})}\left((-)^{M+N}e^{-\beta E}y^N\tau^{-M-2J_3}\prod_i z_i^{\mathscr{J}_i}\right)\\ &= \sum_{q=0}^{2d}\sum_{p=0}^{2d}(-)^{p+q}\mathrm{tr}_{H^{p,q}(\tilde{\mathscr{X}})}\left(y^N\tau^{-M-2J_3}\prod_i z_i^{\mathscr{J}_i}\right).\end{aligned} \tag{3.30}$$

Assuming (3.29), this is the same index as we talked about in section 3.2. $y$ is the fugacity for an $\mathfrak{su}(2)$ symmetry. This means that $\mathscr{Z}$ has a $y \mapsto 1/y$ symmetry.

To compute this object, we will use localisation theorems for equivariant K-theory. To this end, we wish to write $\mathscr{Z}$ as a sum of equivariant Euler characters of equivariant sheaves.

[7]There are orbifold singularities when, for every point in the level set of the moment map, the stabiliser of the gauge group at that point is finite. In general, we expect worse singularities. See [133] for a partial resolution of symplectic quotients to a space with at worst orbifold singularities and [115] for this procedure applied to hyperKähler quotients.

We know by Dolbeault's theorem (for example, see [93]) that

$$H^q(\tilde{\mathscr{X}};A^p(\tilde{\mathscr{X}}))=H^{p,q}(\tilde{\mathscr{X}})\,, \tag{3.31}$$

where the right hand side is the $\bar{\partial}$-Dolbeault cohomology, and the left hand side is the sheaf cohomology of $A^p(\tilde{\mathscr{X}})$, the sheaf of holomorphic $p$-forms on $\tilde{\mathscr{X}}$.

This means that we can rewrite (3.30) as

$$\mathscr{Z}(\tilde{\mathscr{X}};\tau,y,Z):=\sum_{q=0}^{2d}\sum_{p=0}^{2d}(-)^{p+q}y^{d-p}\mathrm{tr}_{H^q(\tilde{\mathscr{X}};A^p(\tilde{\mathscr{X}}))}\left(\tau^{-M-2J_3}\prod_i z_i^{\mathscr{J}_i}\right)\,. \tag{3.32}$$

The right hand side of (3.32) is not manifestly a sum of equivariant Euler characters. Specifically, $-M-2J_3$ is not an action on the base manifold, but instead an action on the forms. Equivariance means the action on the fibre is equal to the action induced by the one on the base manifold. If $-M-2J_3=\mathscr{J}+aE+bN$ for some $a,b\in\mathbb{C}$, with $\mathscr{J}$ an action by Lie derivatives with respect to some vector field, then we can take out the factor of $t^{p-d}$, giving us the sum of equivariant Euler characters of the sheaf of holomorphic $p$-forms.

We consider the problem of free quantum mechanics on $\mathbb{C}^{2n}$, with complex coordinates $(q_i,\tilde{q}_i)_{i=1}^n$. Consider the action of $-M-2J_3+N$ on the forms

$$\alpha=\prod_{i=1}^n q_i^{a_i}\bar{q}_i^{\bar{a}_i}\tilde{q}_i^{b_i}\bar{\tilde{q}}_i^{\bar{b}_i}dq_i^{\delta_i}\wedge d\bar{q}_i^{\bar{\delta}_i}\wedge d\tilde{q}_i^{\varepsilon_i}\wedge d\bar{\tilde{q}}_i^{\bar{\varepsilon}_i}\,, \tag{3.33}$$

with $a,\bar{a},b,\bar{b}\in\mathbb{N}_0^n$ and $\delta,\bar{\delta},\varepsilon,\bar{\varepsilon}\in\{0,1\}^n$, then we have that[8]

$$\begin{aligned}2J_3\alpha&=\sum_{i=1}^n(a_i+b_i-\bar{a}_i-\bar{b}_i)\alpha\,,\\ M\alpha&=\sum_{i=1}^n(\bar{\delta}_i+\bar{\varepsilon}_i-n)\alpha\,,\\ N\alpha&=\sum_{i=1}^n(\delta_i+\varepsilon_i-n)\alpha\,.\end{aligned} \tag{3.34}$$

So, $-M-2J_3+N$ acts as the $\mathbb{C}^\times$-scaling[9], which we call $\mathscr{D}$:

$$\mathscr{D}\alpha=(-M-2J_3+N)\alpha=\sum_{i=1}^n(a_i+b_i-\bar{a}_i-\bar{b}_i+\delta_i+\varepsilon_i-\bar{\delta}_i-\bar{\varepsilon}_i)\alpha\,. \tag{3.35}$$

[8]Explicit expressions for the $\mathfrak{osp}(4^*|4)$ generators on flat space can be found in appendix E.5 of [185].

[9]On flat space, this is the Lie derivative with respect to the Hamiltonian vector field $\sum_{i=1}^n\left(q_i\frac{\partial}{\partial q_i}+\tilde{q}_i\frac{\partial}{\partial\tilde{q}_i}-\bar{q}_i\frac{\partial}{\partial\bar{q}_i}-\bar{\tilde{q}}_i\frac{\partial}{\partial\bar{\tilde{q}}_i}\right)$.

So the conclusion is that on $\mathbb{C}^{2n}$ we can write the superconformal index as

$$\begin{aligned}\mathscr{I}(\mathbb{C}^{2n};\tau,y,Z) &= \mathrm{tr}_{\Omega^*(\mathbb{C}^{2n})}\left((-)^{M+N}\left(\frac{y}{\tau}\right)^N \tau^{\mathscr{D}}\prod_i z_i^{\mathscr{J}_i}\right) \\ &= \sum_{q=0}^{2n}\sum_{p=0}^{2n}(-)^{p+q}\left(\frac{y}{\tau}\right)^{p-n}\mathrm{tr}_{H^q(\mathbb{C}^{2n};A^p(\mathbb{C}^{2n}))}\left(\tau^{\mathscr{D}}\prod_i z_i^{\mathscr{J}_i}\right).\end{aligned} \tag{3.36}$$

This is a sum of equivariant Euler characters[10].

We would like to make the conclusions for our superconformal index on $\tilde{\mathscr{X}}$. Since the action of $-M-2J_3+N$ is $\mathscr{D}$ on $\mathbb{C}^{2n}$, the induced action on the hyperKähler quotient $\mathscr{X}$ is the same. Thus the unique lift[11] of $\mathscr{D}$ to $\tilde{\mathscr{X}}$ is exactly the action of $-M-2J_3+N$ on $\tilde{\mathscr{X}}$ by definition.

So we may conclude

$$\mathscr{I}(\tilde{\mathscr{X}};\tau,y,Z) := \sum_{q=0}^{2d}\sum_{p=0}^{2d}(-)^{p+q}\left(\frac{y}{\tau}\right)^{p-d}\mathrm{tr}_{H^q(\tilde{\mathscr{X}};A^p(\tilde{\mathscr{X}}))}\left(\tau^{\mathscr{D}}\prod_i z_i^{\mathscr{J}_i}\right). \tag{3.37}$$

Equation (3.37) is a sum of equivariant Euler characters, and so we are ready to compute this object by equivariant localisation.

### 3.3.1 The Hilbert series AKA the equivariant Euler character

In order to compute (3.32) by equivariant localisation, we need to introduce the equivariant Euler character. To do this we shall introduce the Hilbert series, discussed in chapter 2. At first glance, our definition of the Hilbert series will look different to the one introduced in chapter 2, but we shall show they are the same. We will also show that the grading we use makes the Hilbert series a well-defined quantity, resolving the issue of non-compactness.

We first define the Hilbert series, HS, to be the equivariant Euler character of the structure sheaf:

$$\begin{aligned}\mathrm{HS}\left(\mathscr{O}_{\tilde{\mathscr{X}}};\tau,Z\right) &:= \sum_{q=0}^{2d}(-)^q\mathrm{tr}_{H^q(\tilde{\mathscr{X}};\mathscr{O}_{\tilde{\mathscr{X}}})}\left(\tau^{\mathscr{D}}\prod_i z_i^{\mathscr{J}_i}\right) \\ &= \mathrm{ch}_T(H^0(\tilde{\mathscr{X}};\mathscr{O}_{\tilde{\mathscr{X}}})) \\ &= \mathrm{ch}_T(\mathbb{C}[\mathscr{X}]),\end{aligned} \tag{3.38}$$

where $\mathrm{ch}_T$ is the character function for $T$-modules; $T$ is the maximal torus of $G_H\times\mathbb{C}^\times$; and $H^q$ is taking the Dolbeault cohomology, which we know is the $s$-cohomology. The

[10]We promote the sheaf $A^p(\tilde{\mathscr{X}})$ to an equivariant sheaf by acting with Lie derivatives.

[11]This result is reviewed near the end of section 2.2, and is from [89].

penultimate line follows from equation (2.52) from chapter 2, and the last line from equation (2.51) from chapter 2.

This index is well-defined for quiver varieties, in the sense that every homogeneous component corresponding to a fixed $\mathbb{C}^\times \times G_H$ weight is finite-dimensional. Indeed, the coordinate ring is generated by all gauge invariant operators that one can form (modulo the $F$-terms, of course), which amounts to taking various paths in the quiver that either start and end on two (possible the same) flavour nodes, or finish on the same gauge node and we take a trace. The power of $\tau$ for this operator is the length of the path. For any fixed integer $n$, there are clearly only a finite number of paths of length $n$ on a finite quiver, and so with $\tau^{L_0}$ alone, this is a well-defined quantity.

In the case that $\mathscr{X}$ is the Higgs branch, this is the same formal power series as the Molien integral of section 2.2.3, and in the case that $\mathscr{X}$ is the Coulomb branch, this gives the same formal power series as the monopole formula of section 2.2.2.

When the resolved space exists, and has isolated fixed points under $\mathbb{C}^\times \times G_H$, we can use localisation to compute the index. We do not have a definitive list of which quivers have a resolution with isolated fixed points, but we do know that the Higgs branch of the $A$- and $\hat{A}$-type quivers, and the Coulomb branch of the $DE$-type quiver for certain choices of flavour groups are all examples where this is the case. They will be discussed later in this thesis. Knowing all such quivers would be interesting, as it would give the entire space of quiver gauge theories where these techniques can be applied.

We shall also be interested in vector bundle valued sheaf cohomology on the space $\tilde{\mathscr{X}}$. Including line bundle valued sheaf cohomology, corresponding to baryonic charges on the Higgs branch or background magnetic charges on the Coulomb branch. We want to grade by the action of $\mathbb{C}^\times \times G_H$, and a necessary requirement of this is that the vector bundle is equivariant.

We would now like to make the same conclusions as we did for the Hilbert series above (about the well-definedness of the quantity), but for finite rank holomorphic vector bundle valued cohomology. Our first problem is the one of identifying the correct Hilbert space. Any finite rank vector bundle $V \to \tilde{\mathscr{X}}$ is locally free, i.e. there exists an open covering $\mathscr{U} = \{U_\alpha\}$ such that $\Gamma(V, U_\alpha) \cong \Gamma(\tilde{\mathscr{X}}, \mathscr{O}_{\tilde{\mathscr{X}}})^{\oplus r} \cong \mathscr{O}_{\mathscr{X}}|_{U_\alpha}^{\oplus r}$. Thus, we can use our inner product on the patches $U_\alpha$, and use a partition of unity to obtain an inner product on global sections[12]. On each patch, we have exactly the desired behaviour of restricting to regular sections. On top of this, since the vector bundle is holomorphic, we can choose our trivialisation such that the transitions functions are holomorphic. So, the action of $s$ as $\bar{\partial}$ on the open sets

[12]This is not a canonical inner product, but any such inner product will have to discard any sections with essential singularities at infinity, and this is all we need.

lifts to the whole space. We conclude that the finite rank holomorphic vector bundle valued $s$-cohomology on $\tilde{\mathscr{X}}$ is exactly the locally free coherent sheaf cohomology on $\tilde{\mathscr{X}}$.

We generalise the Hilbert series of equation (3.38) to arbitrary holomorphic equivariant finite rank vector bundles[13].

$$\mathrm{HS}(V;\tau,Z) := \sum_{q=0}^{2d} (-)^q \mathrm{ch}_T(H^q(\tilde{\mathscr{X}};V)) . \tag{3.39}$$

The question of whether this index is defined, i.e. is each homogeneous component finite-dimensional, is equivalent to the question of whether it is defined for the trivial bundle, $\underline{\mathbb{C}}$, whose associate sheaf is the structure sheaf, $\mathscr{O}_{\tilde{\mathscr{X}}}$, i.e. the Hilbert series we defined in (3.38). This is because of a localisation formula that we will later prove, and that $V$ being a finite rank vector bundle means that the associated sheaf $\mathscr{V}$ is coherent and locally free, and so locally a finite direct sum of copies of $\mathscr{O}_{\tilde{\mathscr{X}}}$.

### 3.3.2 Localisation

We have showed that the Hilbert series for the trivial bundle is well-defined. It remains to show that the more general Hilbert series of equation (3.39) is well-defined and how compute it. In order to do so, we shall use K-theoretic localisation theorems. Note that our Hilbert series is defined for equivariant holomorphic vector bundles, where it coincides with algebro-geometric sheaf cohomology. The sheaf cohomology can then be extended to arbitrary equivariant coherent sheaves, and it is necessary to do so in order to use the theorems of K-theory. We shall do this, but under the understanding that when we want to compute something that corresponds to our quantum mechanics, we must restrict ourselves to only the equivariant coherent sheaves that are locally free.

We shall conclude this section by computing the superconformal index.

We shall work in a slightly more general set-up: We assume that we have a symplectic $T$-variety $\mathscr{X}$, with $T$ some torus (this is the maximal torus of $G_H \times \mathbb{C}^\times$), and $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ an equivariant symplectic resolution, such that there is a unique fixed point of $\mathscr{X}$ under $T$, and the $T$-fixed points of $\tilde{\mathscr{X}}$ are finite and isolated.

Atiyah and Bott's localisation theorem [15, 16] would imply that equation (3.39) is given by a sum over fixed points. However, their formulae apply for compact manifolds, and of

[13]For $\tilde{\mathscr{X}}$ the cotangent bundle of a flag variety over some complex reductive group $G$, which is the smooth resolution for the quiver gauge theory $T^{\rho}_{\sigma}(G)$[14], Broer in [41] showed that the line bundle-valued higher cohomologies on $\tilde{\mathscr{X}}$ vanish when the line bundle is defined by a dominant weight. This means in that case that the Hilbert series of the line bundle is counting only the global sections of the line bundle, just as it is does for the structure sheaf in all cases.

course $\tilde{\mathscr{X}}$ is non-compact. The solution to this problem is related to the solution in well defining the object in the first place. The argument we use is lifted from [161], which was originally for the Jordan quiver.

We shall take $K^T(\tilde{\mathscr{X}})$ to be the Grothendieck group of $T$-equivariant locally free sheaves on $\tilde{\mathscr{X}}$. This is the set of elements that we eat to give our Hilbert series.

Because $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ is an equivariant proper morphism between $T$-varieties, we have an induced "transfer" homomorphism $\pi_!$ between the Grothendieck groups given by the alternating sum of higher direct image sheaves $\sum_i (-)^i R^i \pi_*$, see chapter 3 lemma 6.2.6 of [199]. In particular

$$\pi_! : K^T(\tilde{\mathscr{X}}) \to K^T(\mathscr{X}) . \tag{3.40}$$

Let $\mathscr{R} := \mathbb{Q}(\tau, Z)$ be the fraction field of the representation ring $R(T)$.

The localisation theorem for equivariant K-theory is due to [193], though a prelimary version is in [18]. This theorem tells us that the natural inclusion map, $\iota : \tilde{\mathscr{X}}^T \to \tilde{\mathscr{X}}$, of the $T$-fixed points of $\tilde{\mathscr{X}}$ into $\tilde{\mathscr{X}}$ induces a homomorphism, $\iota_*$, that is an isomorphism after localisation.

$$\iota_* : K^T(\tilde{\mathscr{X}}^T) \otimes_{R(T)} \mathscr{R} \xrightarrow{\cong} K^T(\tilde{\mathscr{X}}) \otimes_{R(T)} \mathscr{R} . \tag{3.41}$$

$\tilde{\mathscr{X}}^{\tilde{T}}$ is a discrete set of points, and hence

$$K^T(\tilde{\mathscr{X}}^T) \otimes_{R(T)} \mathscr{R} \cong \bigoplus_{\substack{x \\ \text{fixed points}}} \mathscr{R} . \tag{3.42}$$

Since $o \in \mathscr{X}$ is the unique fixed point under $T$ of $\mathscr{X}$, we have by the same localisation theorem that, for $\iota_0 : \mathscr{X}^T \hookrightarrow \mathscr{X}$ the inclusion of the fixed points,

$$\iota_{0*}^{-1} : K^T(\mathscr{X}) \otimes_{R(T)} \mathscr{R} \xrightarrow{\cong} K^T(\mathscr{X}^T) \otimes_{R(T)} \mathscr{R} \cong \mathscr{R} . \tag{3.43}$$

The inverse of $\iota_*$ is given by the following formula

$$\iota_*^{-1}(-) = \bigoplus_{\substack{x \\ \text{fixed points}}} \iota_x^*(-) \mathrm{PE}[\mathrm{ch}_T(T_x^* \tilde{\mathscr{X}}; \tau, Z)] , \tag{3.44}$$

where PE is the plethystic exponential defined in equation (2.72) in section 2.2.3; and $\iota_x^*$ is the pull-back homomorphism with respect to the inclusion $\iota_x : \{x\} \hookrightarrow \tilde{\mathscr{X}}$. The pullback homomorphism is defined via the isomorphism of $K^T(\tilde{\mathscr{X}})$ and the Grothendieck group of $\tilde{T}$-equivariant locally free sheaves.

**Proposition 1.** *(from Nakajima and Yoshioka) Let $\mathscr{V}$ be a $T$-equivariant locally free sheaf on $\tilde{\mathscr{X}}$. Then we have that*

$$\sum_{i=0}^{\dim_{\mathbb{C}} \tilde{\mathscr{X}}} (-)^i \operatorname{ch}_T H^i(\tilde{\mathscr{X}}, \mathscr{V}) = \sum_{\substack{x \\ \text{fixed points}}} \iota_x^*(\mathscr{V}) \mathrm{PE}[\operatorname{ch}_T(T_x^* \tilde{\mathscr{X}}; \tau, Z)] . \tag{3.45}$$

The result follows from noting that the following square commutes (this result is due to [193])

$$\begin{array}{ccc} K^T(\tilde{\mathscr{X}}) \otimes_{R(T)} \mathscr{R} & \xrightarrow[(\iota_*)^{-1}]{\cong} & \bigoplus_x \mathscr{R} \\ \pi_! \Big\downarrow & & \Big\downarrow \Sigma_x \\ K^T(\mathscr{X}) \otimes_{R(T)} \mathscr{R} & \xrightarrow[(\iota_{0*})^{-1}]{\cong} & \mathscr{R} \end{array}$$

and the observation that $(\iota_{0*})^{-1} = \operatorname{ch}_T$, which is a consequence of the identity $(\iota_{0*})\operatorname{ch}_T = 1$ (this follows immediately from equation (3.43)).

Note that the bottom route of the square is exactly our Hilbert series, this is because

$$\begin{aligned} \mathrm{HS}(V;\tau,Z) &= \sum_i (-)^i \operatorname{ch}_T(H^i(\tilde{\mathscr{X}};\mathscr{V})) \\ &= \sum_i (-)^i \operatorname{ch}_T(R^i \pi_*(\mathscr{V})) \\ &= (\iota_{0*})^{-1} \pi_!(\mathscr{V}) . \end{aligned} \tag{3.46}$$

The second line of the above equation relies on proposition 8.5 in III 8 of [100].

Equating both routes of the commuting square gives

$$\mathrm{HS}(V;\tau,Z) = \sum_{\substack{x \\ T\text{-fixed point}}} \operatorname{ch}_T(V_x;\tau,Z) \mathrm{PE}[\operatorname{ch}_T(T_x^* \tilde{\mathscr{X}};\tau,Z)] . \tag{3.47}$$

It is clear that the right hand side is well-defined (finite coefficient of each monomial in $\tau$ and $Z$), because near a single point we can trivialise the bundle $V$ to a finite sum of copies of an open set restriction of $\mathscr{O}_{\tilde{\mathscr{X}}}$, which we know has a well-defined character. This means that we have well-defined the Hilbert series in general, as we set out to show.

We can now compute our superconformal index using localisation. Recall from equation (3.37) that

$$\mathscr{Z}(\tilde{\mathscr{X}};\tau,y,Z) := \sum_{q=0}^{2d}\sum_{p=0}^{2d}(-)^{p+q}\left(\frac{y}{\tau}\right)^{p-d}\mathrm{tr}_{H^q(\tilde{\mathscr{X}};A^p(\tilde{\mathscr{X}}))}\left(\tau^{\mathscr{D}}\prod_i z_i^{\mathscr{J}_i}\right) \tag{3.48}$$

So we may write

$$\begin{aligned}\mathscr{Z}(\tilde{\mathscr{X}}) &= \sum_{p=0}^{2d}(-)^p\left(\frac{y}{\tau}\right)^{p-d}\mathrm{HS}(A^p(\tilde{\mathscr{X}});\tau,Z)\\ &= \sum_{p=0}^{2d}(-)^p\left(\frac{y}{\tau}\right)^{p-d}\sum_{\substack{x\\ T\text{-fixed point}}}\mathrm{ch}_T(\Lambda^p(T_x^*\tilde{\mathscr{X}});\tau,Z)\mathrm{PE}[\mathrm{ch}_T(T_x^*\tilde{\mathscr{X}};\tau,Z)]\\ &= \left(\frac{\tau}{y}\right)^d\sum_{\substack{x\\ T\text{-fixed point}}}\mathrm{PE}\left[\left(1-\frac{y}{\tau}\right)\mathrm{ch}_T(T_x^*\tilde{\mathscr{X}};\tau,Z)\right].\end{aligned} \tag{3.49}$$

For a compact manifold, $\mathscr{Z}$ would be the Hirzebruch $\chi_y$-index. See, for example, [105]. For a non-equivariant construction of a quantum mechanics on a compact manifold that gives the Hirzebruch $\chi_y$-index as the partition function, see [107]. Since this quantity coincides with the Hirzebruch $\chi_y$-index of $\tilde{\mathscr{X}}$, whenever we talk about $\mathscr{Z}$ in the rest of this thesis, we will define it as this topological quantity.

There are two important points that we should emphasize

- $\mathscr{Z}(\tilde{\mathscr{X}};\tau,y,Z)$ has $y\mapsto 1/y$ symmetry[15], this is because it is the Cartan of the $SU(2)$ Lefschetz action, whose raising operator is wedging with $\omega_{\mathbb{C}}$;
- The Hilbert series of the manifold is given by the coefficient of the lowest power of $y$ (equivalently the highest power of $y$) divided by $\tau^d$.

In equation (7.34) in [185], Singleton writes a Lagrangian for a $\sigma$-model, such that if we took the target to be $\tilde{\mathscr{X}}$, then we would have a $\mathfrak{su}(2|1)$ supersymmetric quantum mechanics whose Witten index is $\mathscr{Z}$. One could then turn on chemical potentials for our gradings and perform supersymmetric localisation. This should give precisely the same answer as found in equation (3.49) using equivariant K-theory localisation. Note that at each fixed point of $\tilde{\mathscr{X}}$, the contribution to $\mathscr{Z}$ is that of a free $\mathfrak{su}(2|1)$ quantum mechanics.

[15] If we taken $\tau$ to be the fugacity for the scaling symmetry, $\mathscr{D}$, this would be equivalent to using a fugacity $\tilde{y} := y\tau$ to count the $p$-grading of a $(p,q)$-form. We would have had less trouble in using equivariant localisation theorems, but at the cost of losing the manifest $y\mapsto 1/y$ symmetry.

In all examples we compute in chapter 5, the superconformal index can written in the form

$$\mathscr{Z} = \sum_{\substack{x \\ T\text{-fixed point}}} \prod_{n,m \in F_x} \mathrm{PE}\left[(-)^n Z^m \mathscr{Z}(S(n,0))\right], \tag{3.50}$$

where $\mathscr{Z}(S(n,0))$ is defined in equation (2.24), and $F_x$ is some set of weights for $\mathbb{C}^\times \times G_H$ at each fixed point. This has manifest $\mathfrak{su}(2|1)$ symmetry, being composed of plethystic exponential of $\mathfrak{su}(2|1)$ representations.

Note that in order for the superconformal index to have this property, it is necessary that there is some finite Laurent polynomials, $p_x(\tau, Z) \in \mathbb{N}_0[\tau^{\pm 1}, Z^{\pm 1}]$, labelled by the fixed points such that

$$\mathrm{ch}_T(T^*_x(\tilde{\mathscr{X}}); \tau, Z) = p_x(\tau^{-1}, Z^{-1}) + \tau^2 p_x(\tau, Z), \tag{3.51}$$

so that

$$\mathscr{Z} = \sum_{\substack{x \\ T\text{-fixed point}}} \mathrm{PE}\left[p_x(\tau, Z)(1 - \tau y)(1 - \tau / y)\right]. \tag{3.52}$$

### 3.3.3 A simple example

For the sake of clarity, we work through one example where it is known that the orbifold cohomology of [53] is the same as the cohomology of the symplectic resolution.

Using the orbifold cohomology, we shall calculate the superconformal index, and compare it to the localisation formulae. The quiver gauge theory we look at is known as $T(SU(2))$. It can be found in figure 3.1. It is self-mirror, which means its Coulomb branch is identical to its Higgs branch. The Higgs branch is $\mathbb{C}^2/\mathbb{Z}_2$, which has an orbifold singularity at the origin. It is identical to the moduli space of one centred $SU(2)$ instanton, and its projective symplectic resolution is the space $T^*\mathbb{CP}^1$.

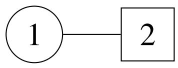


Fig. 3.1 The $T(SU(2))$ theory.

Chen Ruan cohomology involves taking the cohomology of the smooth part of the manifold, as well as the addition of twisted sectors, which live at the orbifold singularities. In the case of $\mathbb{C}^2/\mathbb{Z}_2$, there is one twisted sector. It is the point set, $\{*\}$. So we have

$$H^{p,q}_{\mathrm{orb}}(\mathbb{C}^2/\mathbb{Z}_2) = H^{p,q}(\mathbb{C}^2/\mathbb{Z}_2) \oplus H^{p-1,q-1}(\{*\}). \tag{3.53}$$

The ordinary cohomology bigraded-ring, $H^{p,q}(\mathbb{C}^2/\mathbb{Z}_2)$, is simply given by taking $\mathbb{Z}_2$-invariant holomorphic forms. $H^{0,0}(\{*\}) = 1$ and $H^{p,q}(\{*\}) = 0$ for $(p,q) \neq (0,0)$.

The only terms that survive in the index for $\mathbb{C}^2$, upon restricting to $\mathbb{Z}_2$-invariant forms, would be the terms with even powers of $z$. Hence we have that

$$Z(\mathbb{C}^2/\mathbb{Z}_2) = \frac{1}{2}\left(\frac{\tau}{y}\frac{(1-yz)(1-y/z)}{(1-\tau z)(1-\tau/z)} + \frac{\tau}{y}\frac{(1+yz)(1+y/z)}{(1+\tau z)(1+\tau/z)}\right) + 1\,. \tag{3.54}$$

One can then test this against localisation formulae. Using the analysis of section 5, we know there are two fixed points, and their contribution to the index is

$$Z(\mathbb{C}^2/\mathbb{Z}_2) = \frac{\left(1-\frac{y}{\tau z^2}\right)\left(1-\frac{1}{\tau y z^2}\right)}{\left(1-\frac{1}{z^2}\right)\left(1-\frac{1}{\tau^2 z^2}\right)} + \frac{\left(1-\frac{yz^2}{\tau}\right)\left(1-\frac{z^2}{\tau y}\right)}{(1-z^2)\left(1-\frac{z^2}{\tau^2}\right)}\,. \tag{3.55}$$

One can check that equations (3.54) and (3.55) are indeed the same.

## 3.4 Relation to higher dimensional indices

We shall briefly summarise some relations between our indices and quantum field theories in higher dimensions. The discussion is brief, and in no way an exhaustive list. It serves only to show some motivation.

The Hilbert series that we have computed for our quantum mechanics is identical to the Hilbert series of sections 2.2.2 and 2.2.3. This is because, as mentioned before, the Higgs and Coulomb branch of quiver gauge theories are affine varieties and the holomorphic functions over the vacua form a coherent sheaf, and hence the higher cohomology vanishes by Cartan's theorem B, and so we are counting the holomorphic functions graded by homothety and isometries. One in fact finds on the Higgs branch that the poles of the Molien integral are in a one-to-one correspondence with the fixed points of the manifold (see appendix A for why this is the case).

Perhaps the most famous correspondence is the one where the quiver is the Jordan/ADHM quiver. This quiver with dimensions $(k,N)$ has Higgs branch isomorphic as a complex variety[16] to the moduli space of $k$ $SU(N)$-instantons on $\mathbb{C}^2$, as first calculated in [17]. In this case, we find that both the Hilbert series and the superconformal index agree with the K-theoretic Nekrasov partition function, [164, 165].

Consider five-dimensional $\mathcal{N}=1$ gauge theory (eight supercharges) with gauge group $SU(N)$ and no matter. The partition function receives contributions from solitonic excitations. These are instantons, and for instanton number $k$ the contribution is exactly the Hilbert series for the Jordan quiver with dimensions $(k,N)$. While if we include a massive adjoint

[16]It is isometric to the moduli space of calorons, that is $k$ $SU(N)$-instantons on $S^1\times\mathbb{R}^3$.

hypermultiplet, of mass $m$, with $y = e^m$ , into our five dimensional theory (this giving what is called $\mathcal{N} = 1^*$), then the effect is to dress the theory with the Dirac zero modes in the adjoint representation. This is equivalent to dressing with Grassmann-valued cotangent vectors, and hence gives the superconformal index - this is most clearly seen by looking at the penultimate line of equation (3.49).

Section 2 of [19] provides a derivation for the Hilbert series of the Jordan quiver from a 4+1 dimensional $U(N)$ Yang-Mills field theory with a single real adjoint scalar and no supersymmetry. We discuss this work in chapter 4.

In the paper [173], the authors compute the partition functions of three dimensional $\mathcal{N} = 4$ quiver gauge theories on $S^2 \times S^1$ and $S^3$. Since these spacetimes are compact, we expect the theory in the IR to be a $\sigma$-model over the vacuum manifolds. Indeed, one finds that there are two limits of these partition functions, called the Higgs limit and the Coulomb limit, giving the Hilbert series of the Higgs branch and the Coulomb branch respectively.

# Chapter 4

# The ADHM quiver and Chern-Simons terms

This chapter is based on the work [19], done in collaboration with David Tong, Nick Dorey, Carl Turner and Nakarin Lohitsiri. The paper itself discussed five dimensional instantons in the presence of a large magnetic field, forcing the particles to lie in the lowest energy state. In analogy with the quantum Hall Effect in 2 + 1 dimensions, where a large magnetic field forces the particles to lie in the lowest Landau levels. Thus, one may expect the theory to tell us about the four dimensional Quantum Hall Effect. The author's contribution was a collaboration with Nick Dorey, deriving mirror symmetry for the ADHM quiver.

In section 2.2.2, we wrote the formula for the Hilbert series of the Coulomb branch of a three dimensional $\mathscr{N} = 4$ quiver gauge theory. By mirror symmetry, we would expect this to be equal to the Hilbert series of the Higgs branch of the mirror dual theory. In this section, we show that this is the case for the ADHM quiver. Our starting point is the Molien integral, see section 2.2.3, for the ADHM quiver. From this, we derive the Coulomb branch expression for its mirror dual, the $\hat{A}$-type quiver. We do this for an arbitrary line bundle on the Higgs branch, and through this see how the background magnetic charge for cycle excitations is enforced on the Coulomb branch.

In order to show the equivalence, we shall need to use some techniques involving symmetric functions. We summarise in section 4.1 all the formulae and definitions needed first. Our reference for this material is [142]. In section 4.2, we write the expression for the Hilbert series on the Higgs branch and mirror dual Coulomb branch with charge. In section 4.3, we derive the Coulomb branch expression. In sections 4.4 and 4.5, we derive some expressions for contour integrals relevant to our proof. In section 4.6, we talk about how are derivation is related to the Cherkis bow construction of the Coulomb branch, as found in [159]. Finally, in section 4.7, we derive the equality of the Hilbert series for a pair of mirror dual linear quiver, using this as an opportunity to talk about a novel gluing procedure on the Higgs branch, mirror dual to a well known procedure on the Coulomb branch, first described in [61].

## 4.1 Some symmetric function notation

To start, we define the set of variables $W = \{w_1, \ldots, w_n\}$. Associated to each partition $\lambda$, with the length $\ell(\lambda) \leqslant n$, there is a non-zero symmetric polynomial in $W$ over the field $\mathbb{Q}(t)$ (finite Laurent polynomials in $t$ with rational coefficients) called the Hall-Littlewood polynomial,

$$P_\lambda(W;t) := \frac{1}{N_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left( \prod_l w_l^{\lambda_l} \prod_{l<m} \frac{w_l - t w_m}{w_l - w_m} \right), \tag{1.1}$$

where $S_n$ is the symmetry group acting on the indices of the $W$ variables by permutations. Here $N_\lambda$ is a normalisation factor defined by

$$N_\lambda(t) := \frac{\varphi_{n-\ell(\lambda)}(t) \prod_{j \geqslant 1} \varphi_{m_j(\lambda)}(t)}{(1-t)^n}, \tag{1.2}$$

where

$$\varphi_a(t) = \prod_{j=1}^{a} (1-t^j) \quad \text{and} \quad m_j(\lambda) = |\{i \geqslant 1 : \lambda_i = j\}|. \tag{1.3}$$

The Hall-Littlewood polynomials with variables $W$ and $t$ form a basis for all symmetric functions in $W$ over the field $\mathbb{Q}(t)$. There is an inner product on symmetric polynomials. It is defined by the integral

$$\begin{aligned} \langle P_\lambda, P_\mu \rangle_t &:= \oint_C d\mu[W;t]\; P_\lambda(W;t) P_\mu(W^{-1};t) \\ &:= \frac{1}{n!} \left( \prod_{l=1}^{n} \frac{1}{2\pi i} \oint_C \frac{dw_l}{w_l} \right) \prod_{l \neq m} \frac{1 - w_l/w_m}{1 - t w_l/w_m}\; P_\lambda(W;t) P_\mu(W^{-1};t) \\ &= \frac{1}{N_\lambda(t)} \delta_{\lambda\mu}. \end{aligned} \tag{1.4}$$

We can further generalise our polynomials to be a basis for all finite symmetric Laurent polynomials in $W$ over $\mathbb{Q}(t)$. To do this we first define the set of $n$ ordered integers

$$\mathfrak{A} := \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{Z}^n | \zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_n\} \cong \mathbb{Z}^n / S_n. \tag{1.5}$$

For any $\zeta \in \mathfrak{A}$, we can define a partition $\lambda$ via $\lambda_l := \zeta_l - \zeta_n$ for $l = 1, \ldots, n-1$ and $\lambda_n = 0$. We define

$$N_\zeta(t) := N_\lambda(t)\,. \tag{1.6}$$

We define the shifted Hall-Littlewood polynomials by

$$\Psi_\zeta(W;q) := \prod_l w_l^{\zeta_n} P_\lambda(W;q)\,. \tag{1.7}$$

Extending the inner product in equation (1.4) to finite symmetric Laurent polynomials in the obvious way, one can quite easily show that

$$\left\langle \Psi_\zeta, \Psi_\eta \right\rangle_t = \frac{1}{N_\zeta(t)} \delta_{\zeta\eta}\,. \tag{1.8}$$

A simple trick, that we will use extensively, is the following: for $\zeta, \eta \in \mathfrak{A}$, we can write a product of two of these polynomials shifted by an arbitrary constant $a \in \mathbb{Z}$, namely

$$\Psi_\zeta(W)\Psi_\eta(W^{-1}) = \Psi_{\zeta+(a^n)}(W)\Psi_{\eta+(a^n)}(W^{-1}) \tag{1.9}$$

The Cauchy identity for Hall-Littlewood polynomials states that

$$\prod_{l,m} \frac{1 - t z_l y_m}{1 - z_l y_m} = \sum_\lambda P_\lambda(Z;t) Q_\lambda(Y;t)\,, \tag{1.10}$$

where

$$\begin{aligned} Q_\lambda(X;t) &:= b_\lambda(t) P_\lambda(X;t)\,, \\ b_\lambda(t) &:= \prod_{i \geqslant 1} \varphi_{m_i(\lambda)}(t)\,. \end{aligned} \tag{1.11}$$

The Hall-Littlewood polynomials form an algebra under multiplication, known as the Hall algebra. The structure constants, ${f^\lambda}_{\mu\nu}(t) \in \mathbb{Z}[t]$, are defined by

$$P_\mu(X;t) P_\nu(X;t) = \sum_\lambda {f^\lambda}_{\mu\nu}(t) P_\lambda(X;t)\,. \tag{1.12}$$

With these structure constants, one can define Hall-Littlewood polynomials for skew Young tableaux

$$\begin{aligned} Q_{\lambda/\mu}(X;t) &:= \sum_\nu {f^\lambda}_{\mu\nu}(t) Q_\nu(X;t)\,, \\ P_{\lambda/\mu}(X;t) &:= \frac{b_\mu(t)}{b_\lambda(t)} Q_{\lambda/\mu}(X;t)\,. \end{aligned} \tag{1.13}$$

The Schur polynomial, $s_\lambda(X)$, can be defined as a Hall-Littlewood polynomial with $t = 0$.

We define the Kostka polynomials relating the Schur polynomials and the Hall-Littlewood polynomials by

$$s_\alpha(X) = \sum_\mu K_{\alpha\mu}(t) P_\mu(X;t)\,. \tag{1.14}$$

It is non-trivially true that $K_{\alpha\mu}(t) \in \mathbb{N}_0[t]$, this was shown in [137].

We define

$$P_\lambda\left(\frac{X}{1-q};t\right) := P(X, qX, q^2X, \ldots;t)\,. \tag{1.15}$$

We define the Milne polynomial as

$$Q'_\lambda(X;t) := \sum_\mu K_{\mu\lambda}(t) s_\mu(X)\,. \tag{1.16}$$

A result from [130] tells us that

$$Q'_\lambda(X;t) = Q_\lambda\left(\frac{X}{1-t};t\right)\,. \tag{1.17}$$

Another Cauchy identity states

$$\begin{aligned}\prod_{l,m} \frac{1}{1-x_l y_m} &= \sum_\lambda s_\lambda(X) s_\lambda(Y) \\ &= \sum_\mu Q'_\mu(X;t) P_\mu(Y;t)\,.\end{aligned} \tag{1.18}$$

## 4.2 The Hilbert series of the Higgs and Coulomb branch

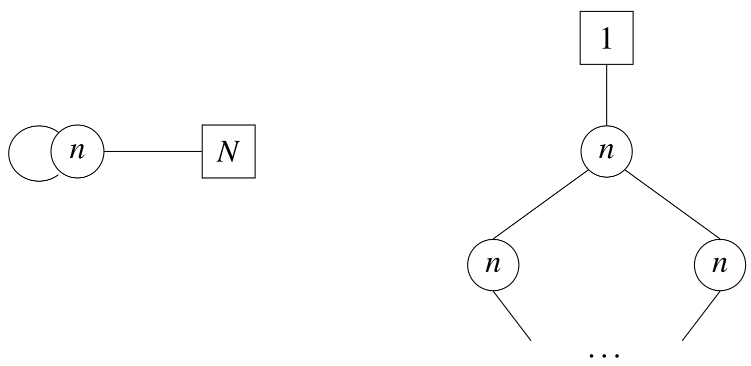


Fig. 4.1 On the left we have the ADHM quiver and on the right we have its mirror dual, the $\hat{A}_N$ quiver.

The Hilbert series of the ADHM quiver with background baryonic charge can be understood as the Nekrasov partition function, [164], for five dimensional $\mathcal{N} = 1$ $SU(N)$ super Yang-Mills compactified on $\mathbb{R}^4 \times S^1$ in an $\Omega$-background, [165]. The baryonic charge $k$ can be identified with a five dimensional Chern-Simons charge, as discussed in [191]. The fugacities $x$ and $\tau$ are identified with the parameters of the $\Omega$-background, and the fugacities $z_i$ are identitied with the Coulomb branch parameters of the five dimensional SUSY theory.

From section 2.2.3, we see that the Hilbert series for the Higgs branch of the ADHM quiver is given by the Molien integral,

$$\begin{aligned} \mathrm{HS}(k) :=& \frac{1}{n!}\prod_{l=1}^{n}\left(\oint_C \frac{dw_l}{2\pi i w_l^{k+1}}\prod_{i=1}^{N}\frac{1}{(1-\tau w_l z_i)(1-\tau/w_l z_i)}\right) \\ &\times \prod_{l\neq m}(1-w_l/w_m)\prod_{l,m=1}^{n}\frac{1-\tau^2 w_l/w_m}{(1-\tau w_l/x w_m)(1-\tau x w_l/w_m)}\,. \end{aligned} \tag{2.19}$$

Here, the contour $C$ is defined to be the unit $n$-torus.

In section 2.2.2, we gave the Hilbert series for the Coulomb branch of a generic three dimensional $\mathcal{N} = 4$ quiver gauge theory. Here we restate the result, but this time with a quiver with a cycle and a background magnetic charge from this excitation.

In order to understand how this excitation acts, it is best to think of the brane picture that we briefly summarised in section 2.3.2. In that picture, there are $k$ units of magnetic flux associated for the global symmetry associated to the periodic boundary condition around the $x^6$-circle. This shift in charge corresponds, schematically, to a shift in the branes relative position. In the Hilbert series, this manifests by having one of the charges shifted by a constant $(k^n)$. We shall see that it does not matter where this shift is. Indeed, one can share the shift amongst several nodes.

The bare dimension of a monopole operator with such a magnetic charge $k$ is

$$\Delta[\{\zeta^i\};k] = \sum_{i,j=1}^{N}\sum_{a,b=1}^{n}\left(-\delta_{ij}|\zeta^i_a-\zeta^j_b|+\delta_{i,j+1}|\zeta^i_a-\zeta^j_b+k\delta_{j,N}|\right)+||\zeta^N||\,, \tag{2.20}$$

and the Hilbert series is given by

$$\mathrm{HS}_C(k) = \sum_{\zeta\in\mathfrak{A}^N}\tau^{\Delta[\{\zeta^i\};k]}x^{|\zeta^N|}\prod_{i=1}^{N}z_i^{|\zeta^i-\zeta^{i-1}+k\delta_{i,1}|}\left[\prod_{l\in\mathbf{Z}}\prod_{a=1}^{m_l(\zeta^i)}\frac{1}{1-\tau^{2a}}\right]. \tag{2.21}$$

## 4.3 From Higgs to Coulomb

The machinery above can be brought to bear on our partition function.

**Theorem 7.** *The Hilbert series of the Higgs branch of the ADHM quiver with baryonic charge k is equal to the Hilbert series of the Coulomb branch of the mirror dual quiver with magnetic charge k. In equations,*

$$HS_C(k) = \mathrm{HS}(k)\,. \tag{3.22}$$

*Proof.* To start, we write the second line of equation (2.19) as

$$\begin{aligned}&\prod_{l\neq m}(1-w_l/w_m)\prod_{l,m=1}^{n}\frac{1-\tau^2 w_l/w_m}{(1-\frac{\tau}{x}w_l/w_m)(1-\tau x w_l/w_m)}\\ =&\frac{1}{(1-\tau^2)^n}\prod_{l\neq m}\frac{1-w_l/w_m}{1-\tau^2 w_l/w_m}\prod_{l,m}\frac{(1-\tau^2 w_l/w_m)(1-\tau^2 w_l/w_m)}{(1-\tau x w_l/w_m)(1-\frac{\tau}{x}w_l/w_m)}\,,\end{aligned} \tag{3.23}$$

and define functions

$$Q[W,\tilde{W};x,\tau] := \prod_{l,m}\frac{(1-\tau^2 w_l/w_m)(1-\tau^2\tilde{w}_l/\tilde{w}_m)}{(1-\tau x w_l/\tilde{w}_m)(1-\frac{\tau}{x}\tilde{w}_l/w_m)} \tag{3.24}$$

and

$$\pi_f(W;z) := \prod_{l=1}^{N}\frac{1}{1-\tau w_l z},\quad \pi_{\bar{f}}(W;z) := \prod_{l=1}^{N}\frac{1}{1-\tau/w_l z}\,. \tag{3.25}$$

The partition function can then be written as

$$\mathrm{HS}(k) = \frac{1}{(1-\tau^2)^n}\oint_C d\mu[W;\tau^2]\prod_{l=1}^{n}w_l^{-k}\left(\prod_{i=1}^{N}\pi_f(W;z_i,\tau)\pi_{\bar{f}}(W;z_i,\tau)\right)Q[W,W;x,\tau]\,. \tag{3.26}$$

To convert our Higgs branch expression into a Coulomb branch expression for the mirror dual theory, we will make use of “Dirac delta functions" for symmetric polynomials. These allow us to rewrite the Hilbert series as an integral over the maximal torus of the gauge group of the affine quiver, $T(U(n)^N) \cong T^{nN} \cong C^N$. To do this, we first define the function

$$K[W,\tilde{W}] := \sum_{\zeta\in\mathfrak{A}} N_\zeta(\tau^2)\Psi_\zeta(W;\tau^2)\Psi_\zeta(\tilde{W}^{-1};\tau^2)\,, \tag{3.27}$$

where $\mathfrak{A}$ is the set of $n$ ordered integers defined in equation (1.5), and the shifted Hall-Littlewood polynomials $\Psi$ are defined in equation (1.7). Then, using the measure defined in equation (1.4), we have for any symmetric Laurent polynomial $f$ in $W$ over $\mathbb{Q}(\tau^2)$

$$f(W) = \oint_C d\mu[\tilde{W};\tau^2]\, K[W,\tilde{W}]\, f(\tilde{W})\,. \tag{3.28}$$

This follows from the fact that the functions $\Psi_\zeta$ form a linear $\mathbb{Q}(\tau^2)$-basis for symmetric finite Laurent polynomials, and the orthogonality property outlined in equation (1.8). There is a notational subtlety here; the sum should not be inside the integral, but rather

$$f(W) = \sum_{\zeta\in\mathfrak{A}} \oint_C d\mu[\tilde{W};\tau^2]\, N_\zeta(\tau^2)\, \Psi_\zeta(W;\tau^2)\, \Psi_\zeta(\tilde{W}^{-1};\tau^2)\, f(\tilde{W})\,. \tag{3.29}$$

It is understood that whenever we write $K$ we mean that the sum sits outside the integral. This is similiar to how the Dirac delta function is not well-defined as a function unless inside an integral.

Defining $W^{(0)} \equiv W$ and $W^{(N+1)} \equiv \tilde{W}$, we insert a complete set of states to rewrite the integral as

$$\begin{aligned} \mathrm{HS}(k) = \frac{1}{(1-\tau^2)^n} \oint_C d\mu[W;\tau^2] \oint_C d\mu[\tilde{W};\tau^2] \prod_{i=1}^{N} \oint_C d\mu[W^{(i)};\tau^2] \\ Q[W,\tilde{W}] \prod_{i=1}^{N} \pi_f(W^{(i)},z_i)\pi_{\bar{f}}(W^{(i)},z_i) \prod_{i=1}^{N+1} K[W^{(i-1)-1},W^{(i)-1}] \end{aligned} \tag{3.30}$$

This motivates us to define, for $\zeta,\eta\in\mathfrak{A}$,

$$M_{\zeta\eta}(z) := N_\zeta(\tau^2) \oint_C d\mu[W;\tau^2]\, \Psi_\zeta(W;\tau^2)\, \pi_f(W;z,\tau)\, \pi_{\bar{f}}(W;z,\tau)\, \Psi_\eta(W^{-1};\tau^2)\,, \tag{3.31}$$

and

$$\mathscr{O}_{\zeta\eta}(k) := \frac{N_\zeta(\tau^2)}{(1-\tau^2)^n} \oint_C d\mu[W;\tau^2] \oint_C d\mu[\tilde{W};\tau^2]\, \Psi_{\zeta-(k^n)}(W)\, \Psi_\eta(\tilde{W}^{-1})\, Q[\tilde{W},W]\,, \tag{3.32}$$

The partition function can then be written in the simple form

$$\mathrm{HS}(k) = \mathrm{tr}_{\mathscr{H}} \left[\mathscr{O}(k) M(z_1) M(z_2) \dots M(z_N)\right], \tag{3.33}$$

where the trace is over the vector space of finite symmetric Laurent polynomials, which is spanned by $\{\Psi_\zeta(W;\tau^2)|\zeta\in\mathfrak{A}\}$. Once again, the sums over $\mathfrak{A}$ are outside the integrals.

The problem has been reduced to evaluating $M_{\zeta\eta}(z)$ and $\mathscr{O}_{\zeta\eta}(k)$. We start with $M_{\zeta\eta}(z)$, and use the trick in equation (1.9) to restrict to the case where all the elements of $\zeta$ and $\eta$ are negative, so that we can write

$$\begin{aligned} M_{\zeta\eta}(z) = N_\zeta(\tau^2) \oint_C d\mu[W;\tau^2]\, \Psi_\zeta(W;\tau^2)\, \pi_{\bar{f}}(W) \sum_{\chi\in\mathscr{P}_n} \oint_C d\mu[\tilde{W};\tau^2] \\ \frac{1}{(1-\tau^2)^n} P_\chi(W;\tau^2)\, Q_\chi(\tilde{W}^{-1};\tau^2)\, \Psi_\eta(\tilde{W}^{-1};\tau^2)\pi_f(\tilde{W})\,. \end{aligned} \tag{3.34}$$

The set $\mathscr{P}_n$ is the set of all partitions of length $\leqslant n$.

Next, we swap the sum with the integral, use the Cauchy identity (1.10) and expand $\Psi_\zeta$ and $\Psi_\eta$ as a sum over permutations, so that the integrand in the summand is a rational function. Using the symmetry of the integrand to get rid of the sum over permutations, we have

$$\begin{aligned} M_{\zeta\eta}(z) = & \frac{1}{N_\eta(\tau^2)(1-\tau^2)^n} \prod_{l=1}^n \left( \oint_C \frac{dw_l}{2\pi i w_l} \oint_{\tilde{C}} \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{w_l^{\zeta_l}}{1-\tau/w_l z} \frac{\tilde{w}_l^{-\eta_l}}{1-\tau\tilde{w}_l z} \right) \\ & \times \prod_{l,m=1}^n \frac{1-\tau^2 w_l/\tilde{w}_m}{1-w_l/\tilde{w}_m} \prod_{l<m} \left( \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l} \frac{1-w_l/w_m}{1-\tau^2 w_l/w_m} \right). \end{aligned} \tag{3.35}$$

An important subtlety is that the contour for $\tilde{W}$ has changed to $\tilde{C} := (1+\varepsilon)C$ for some sufficiently small $\varepsilon > 0$. This is so that it doesn't interfere with the rest of the pole structure. The denominator, $1 - w_l/\tilde{w}_m$, that arises from the Cauchy identity after swapping the sum of partitions with the integral over $\tilde{W}$ means that the form we integrate is not defined when the contour is such that there is a point where the value of $w_l$ and $\tilde{w}_m$ coincide. The justification for this choice of contour comes from noting that, if we evaluate the poles for $\tilde{w}_l$ in the order $\tilde{w}_n, \tilde{w}_{n-1}, \ldots, \tilde{w}_1$, and choose poles within the contour, then we find that the only poles are parameterised by a permutation, $\sigma \in S_n$, with $\tilde{w}_l = w_{\sigma(l)}$. (Recall that $\eta_l < 0$, so there are no poles at 0.) Upon evaluating the residue we then find exactly the original expression for $M_{\zeta\eta}(z)$.

Considering the relative ordering of the decreasing sequences $\zeta, \eta \in \mathfrak{A}$, we redefine the labels as follows

$$\begin{aligned} \zeta_l &\equiv \zeta_{i,\alpha_i} \quad \text{for } i = 1, \ldots, i_{\max},\ \alpha_i = 1, \ldots, n_i\,, \\ \eta_l &\equiv \eta_{i,\tilde{\alpha}_i} \quad \text{for } i = 1, \ldots, i_{\max},\ \tilde{\alpha}_i = 1, \ldots, \tilde{n}_i\,, \end{aligned} \tag{3.36}$$

such that $(n_1, \ldots, n_{i_{\max}})$ and $(\tilde{n}_1, \ldots, \tilde{n}_{i_{\max}})$ are compositions of $n$ (with $n_1$ and $\tilde{n}_{i_{\max}}$ possibly zero, but all other values strictly positive integers) and $\zeta_{i,\alpha_i} \leqslant \zeta_{i,\beta_i}\ \forall\ \alpha_i \geqslant \beta_i$, and $\eta_{i,\tilde{\alpha}_i} \leqslant \eta_{i,\tilde{\beta}_i}$ $\forall\ \tilde{\alpha}_i \geqslant \tilde{\beta}_i$ and, finally, $\zeta_{i,\alpha} \geqslant \eta_{i,\tilde{\alpha}} \geqslant \zeta_{i+1,\alpha'}\ \forall \alpha, \tilde{\alpha}, \alpha'$. With these definitions, we can now evaluate the matrix elements.

**Lemma 2.** *The value of $M_{\zeta\eta}(z)$ is saturated by the pole located at*

$$w_l = \frac{\tau^{2s_l}}{z} \quad \text{and} \quad \tilde{w}_l = \frac{\tau^{2\tilde{s}_l}}{z}\,, \tag{3.37}$$

*where*

$$\begin{aligned} s_l = s_{i,\alpha} &= \sum_{j=1}^{i-1}(n_j - \tilde{n}_j) + \alpha - \frac{1}{2}\,, \\ \tilde{s}_l = \tilde{s}_{i,\tilde{\alpha}} &= \sum_{j=1}^{i} n_j - \sum_{j=1}^{i-1}\tilde{n}_j + \frac{1}{2} - \tilde{\alpha}\,. \end{aligned} \tag{3.38}$$

*So if $\zeta_1 \geqslant \eta_1$, $s_1 = \frac{1}{2}$ and if $\eta_1 > \zeta_1$, $\tilde{s}_1 = -\frac{1}{2}$. Furthermore, the residue of this pole is such that we have*

$$M_{\zeta\eta}(z) = \frac{1}{\prod_{a\in\mathbb{Z}} \varphi_{m_a(\eta)}(\tau^2)} z^{|\eta|-|\zeta|}\tau^{\Delta_1[\zeta,\eta]}\,, \tag{3.39}$$

*where $\Delta_1[\zeta,\eta]$ is defined in equation (4.49).*

This is the topic of section 4.4. We are unable to prove this lemma, due to the complexity of the multi-dimensional contour integral. We are still able to have a rigorous proof of our theorem by noting that we need only rigorously evaluate $\mathscr{O}_{\zeta\eta}(k)$, and then rely on the fact that the rest of the theorem relies on showing non-abelian mirror symmetry for the linear quiver in section 4.7. The Mirković-Vyrbornov isomorphism of [148] rigorously proves mirror symmetry for linear quivers, see the discussion in section 6.3.1 for more on this.

This expression for $M_{\zeta\eta}(z)$ is invariant under shifts of the form $(\zeta,\eta) \mapsto (\zeta + (c^n), \eta + (c^n))$ for all $c \in \mathbb{Z}$. This is what we would expect, as we could see this symmetry already when we used equation (1.9).

For the other term $\mathscr{O}_{\zeta\eta}(k)$, we have that

**Lemma 3.**

$$\mathscr{O}_{\zeta\eta}(k) = \delta_{\zeta-(k^n)\,\eta} z^{|\eta|}\tau^{||\eta||} \tag{3.40}$$

*where $||\eta|| := \sum_{l=1}^{n} |\eta_l|$ and $|\eta| := \sum_{l=1}^{n} \eta_l$.*

We prove this in section 4.5.

Given these lemmas, we can write the Hilbert series as

$$\mathrm{HS}(k) = \sum_{\vec{\zeta}\in\mathfrak{A}^N} x^{|\zeta^{(1)}|}\tau^{||\zeta^{(1)}||} M_{\zeta^{(1)}\zeta^{(2)}}(z_1) M_{\zeta^{(2)}\zeta^{(3)}}(z_2)\dots M_{\zeta^{(N)}\zeta^{(1)}+(k^n)}(z_N)\,. \tag{3.41}$$

To see that this does indeed agree with (2.21), we shall expand out the $M's$. First we consider the bare dimension, written in equation (2.20). Our expression for the bare dimension is (defining $\zeta^{(N+1)} \equiv \zeta^{(1)}$)

$$
\begin{aligned}
\Delta[\{\zeta\};k] &= \sum_{i=1}^{N} \Delta_1[\zeta^{(i)}, \zeta^{(i+1)} + \delta_{iN}(k^n)] + ||\zeta^{(1)}|| \\
&= \sum_{i=1}^{N} \sum_{l,m=1}^{n} \left( |\zeta_l^{(i)} - \zeta_m^{(i+1)} + \delta_{iN} k| - \frac{1}{2}|\zeta_l^{(i)} - \zeta_m^{(i)}| - \frac{1}{2}|\zeta_l^{(i+1)} - \zeta_m^{(i+1)}| \right) + ||\zeta^{(1)}|| \\
&= \sum_{i=1}^{N} \sum_{l,m=1}^{n} \left( |\zeta_l^{(i)} - \zeta_m^{(i+1)} + \delta_{iN} k| - |\zeta_l^{(i)} - \zeta_m^{(i)}| \right) + ||\zeta^{(1)}|| \,.
\end{aligned}
\tag{3.42}
$$

We then redefine our variables via $\zeta^{(i)} = \zeta^{N+1-i}$, and write

$$
\begin{aligned}
\Delta[\{\zeta\};k] &= \sum_{i=1}^{N} \sum_{l,m=1}^{n} \left( |\zeta_l^i - \zeta_m^{i-1} + \delta_{i1} k| - |\zeta_l^i - \zeta_m^i| \right) + ||\zeta^N|| \\
&= \sum_{i=1}^{N} \sum_{l,m=1}^{n} \left( |\zeta_l^i - \zeta_m^{i-1} + \delta_{i1} k| - |\zeta_l^i - \zeta_m^i| \right) + ||\zeta^N|| \\
&= \sum_{i,j=1}^{N} \sum_{l,m=1}^{n} \left( \delta_{i\,j+1} |\zeta_l^i - \zeta_m^j + \delta_{jN} k| - \delta_{i\,j} |\zeta_l^i - \zeta_m^j| \right) + ||\zeta^N||
\end{aligned}
\tag{3.43}
$$

This is the expression in equation (2.20). We have

$$
\mathrm{HS}(k) = \sum_{\vec{\zeta} \in \mathfrak{A}^N} x^{|\zeta^N|} \tau^{\Delta[\{\zeta\};k]} \prod_{i=1}^{N} z_i^{|\zeta^{N-i+1} - \zeta^{N-i} + k\delta_{iN}|} \prod_{i=1}^{N} \prod_{a \in \mathbb{Z}} \frac{1}{\varphi_{m_a(\zeta^i)}(\tau^2)} \,. \tag{3.44}
$$

Upon acting with an element of the Weyl group of $U(N)$ such that $z_i \mapsto z_{N-i+1}$, we find we exactly reproduce equation (2.21),

$$
\mathrm{HS}(k) = \sum_{\vec{\zeta} \in \mathfrak{A}^N} x^{|\zeta^N|} \tau^{\Delta[\{\zeta\};k]} \prod_{i=1}^{N} z_i^{|\zeta^i - \zeta^{i-1} + k\delta_{i1}|} \prod_{i=1}^{N} \prod_{a \in \mathbb{Z}} \frac{1}{\varphi_{m_a(\zeta^i)}(\tau^2)} \,. \tag{3.45}
$$

$\square$

## 4.4 Evaluating $M_{\zeta\eta}(z)$

The integral of a meromorphic $(m,0)$ form $\omega$ on an oriented $m$ real dimensional submanifold, $D$, of $\mathbb{C}^m$ is defined by the homology class of $D$. The manifold we compute the homology of is that of $\mathbb{C}^m$ minus complex codimension 1 hyperplanes defined by the poles of $\omega$. Thus, we have a well-defined integral with our choice of homology $C \times \tilde{C}$ .

For the practical evaluation of our integral, we can evaluate each variable in turn, choosing to sum either inside or outside the contour. This leads to many different ways of evaluating the integral, but the answer at the end of the calculation is always defined and independent of these choices, as our homology class is well-defined.

For the evaluation of this integral, we have been unable to prove that the residue has the value we claim it does for general $n, \zeta$ and $\eta$, but we have extensive numerical evidence for small values of $n$. Note that the Mirković-Vybornov isomorphism means that this is not a problem (see sections 4.7 and 6.3.1).

Numerical evidence suggests that if we change the homology class to a homology class defined by the interleaving of $\zeta$ and $\eta$, then there is a unique pole contributing to the value of the integral given by the values

$$w_l = \tau^{2s_l}/z, \quad \tilde{w}_l = \tau^{2\tilde{s}_l}/z. \tag{4.46}$$

This homology class is defined as follows: the interleaving of $\zeta$ and $\eta$ allows us to define an ordering on the variables $W, \tilde{W}$ via $w_{i\alpha} \leqslant w_{j\beta}$ and $\tilde{w}_{i\alpha} \leqslant \tilde{w}_{j\beta}$ if $i < j$ or $i = j$ and $\alpha < \beta$ and $w_{i\alpha} \leqslant \tilde{w}_{j\beta}$ if $i \leqslant j$ and $\tilde{w}_{i\alpha} \leqslant w_{j\beta}$ if $i < j$. The interleaving defines the sets $P \subseteq \{w_1, \dots, w_n\}$ and $Q \subseteq \{\tilde{w}_1, \dots, \tilde{w}_n\}$ as the subset of elements where the respective corresponding $s_l$ and $\tilde{s}_l$ are positive, and their respective complements $\bar{P}$ and $\bar{Q}$, whose elements are such that the respective corresponding $s_l$ and $\tilde{s}_l$ are negative. The contour is then defined by the elements of $P \cup Q$'s size ordered by the ordering defined above, with the smallest element having the smallest contour and the largest element having the largest contour, while for the elements of $\bar{P} \cup \bar{Q}$, we do the opposite ordering with the smallest element having the largest contour and so on. For evaluating the integral, we sum all the poles of $P$ and $Q$ on the inside of the contour, and outside the contour for $\bar{P}$ and $\bar{Q}$.

With this choice of contour, one finds that upon splitting the form, we get

$$M_{\zeta\eta}(z) = \frac{1}{N_\eta(\tau^2)(1-\tau^2)^n} \prod_l \tau^{2\zeta_l s_l - 2\eta_l \tilde{s}_l} z^{\eta_l - \zeta_l}, \times (\star). \tag{4.47}$$

The left over part represented by $\star$ contains all the relevant pole structure and evaluates to one on the pole, while the term to the left of $\star$ evaluates to exactly the answer required on the pole.

Indeed, we find

$$\sum_{l}(2\zeta_l s_l - 2\eta_l \tilde{s}_l) = \Delta_1[\zeta,\eta]\,, \tag{4.48}$$

where

$$\Delta_1[\zeta,\eta] := \Delta_H[\zeta,\eta] - \frac{1}{2}\Delta_V[\eta] - \frac{1}{2}\Delta_V[\eta]\,, \tag{4.49}$$

with

$$\begin{aligned} \Delta_H[\zeta,\eta] &:= \sum_{l,m=1}^{n} |\zeta_l - \eta_m|\,, \\ \Delta_V[\zeta] &:= \sum_{l,m=1}^{n} |\zeta_l - \zeta_m|\,. \end{aligned} \tag{4.50}$$

This is because

$$\begin{aligned} \Delta_1[\zeta,\eta] &= \frac{1}{2}\sum_{l,m=1}^{n}\Big(2|\zeta_l-\eta_m| - |\zeta_l-\zeta_m| - |\eta_l-\eta_m|\Big) \\ &= \sum_{i\leqslant j}\sum_{\alpha_i=1}^{n_i}\sum_{\tilde{\alpha}_j=1}^{\tilde{n}_j}(\zeta_{i,\alpha_i}-\eta_{j,\tilde{\alpha}_j}) + \sum_{j<i}\sum_{\alpha_i=1}^{n_i}\sum_{\tilde{\alpha}_j=1}^{\tilde{n}_j}(\eta_{j,\tilde{\alpha}_j}-\zeta_{i,\alpha_i}) \\ &\quad - \sum_{i<j}\sum_{\alpha_i=1}^{n_i}\sum_{\alpha_j=1}^{n_j}(\zeta_{i,\alpha_i}-\zeta_{j,\alpha_j}) - \sum_{i=1}^{i_{\max}}\sum_{\alpha_i<\alpha_i'}(\zeta_{i,\alpha_i}-\zeta_{i,\alpha_i'}) \\ &\quad - \sum_{i<j}\sum_{\tilde{\alpha}_i=1}^{\tilde{n}_i}\sum_{\tilde{\alpha}_j=1}^{\tilde{n}_j}(\eta_{i,\tilde{\alpha}_i}-\eta_{j,\tilde{\alpha}_j}) - \sum_{i=1}^{i_{\max}}\sum_{\tilde{\alpha}_i<\tilde{\alpha}_i'}(\eta_{i,\tilde{\alpha}_i}-\eta_{i,\tilde{\alpha}_i'}) \\ &= \sum_{i=1}^{i_{\max}}\sum_{\alpha_i=1}^{n_i} 2s_{i,\alpha_i}\zeta_{i,\alpha_i} - \sum_{i=1}^{i_{\max}}\sum_{\tilde{\alpha}_i=1}^{\tilde{n}_i} 2\tilde{s}_{i,\tilde{\alpha}_i}\eta_{i,\tilde{\alpha}_i}\,. \end{aligned} \tag{4.51}$$

However, for the original homology choice it is more complicated. For example, if we take $n=2$ and $(\zeta;\eta)=(-1,-3;-2,-4)$, but with the original contour, then there no longer is a non-zero residue at the point $(w_1,w_2;\tilde{w}_1,\tilde{w}_2)=(\tau/z,\tau/z,\tau/z,\tau/z)$, and instead one finds the value of the matrix element is given by the sum of the residue of the pole at $(\tau/z,\tau^3/z;\tau^3/z,\tau/z)$ and $(\tau/z,0;0,\tau/z)$. However, the final value of $M_{\zeta\eta}(z)$ remains the same.

A case where these two homology choices are the same is when $\zeta_l \geqslant \eta_m$ for all $l$ and $m$. We prove the value of $M_{\zeta\eta}(z)$ is indeed what we want in this simpler case. $M_{\zeta\eta}(z)$ is

$$\begin{aligned} M_{\zeta\eta}(z) = \prod_{l=1}^{n} \oint_C \frac{dw_l}{2\pi i} \oint_{\tilde{C}} \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{w_l^{\zeta_l}}{w_l - \tau/z} \frac{\tilde{w}_l^{-\eta_l - 1}}{1 - \tau z \tilde{w}_l} \\ \prod_{l,m} \frac{\tilde{w}_m - \tau^2 w_l}{\tilde{w}_m - w_l} \prod_{l<m} \left( \frac{1 - \tilde{w}_m/\tilde{w}_l}{1 - \tau^2 \tilde{w}_m/\tilde{w}_l} \frac{1 - w_l/w_m}{1 - \tau^2 w_l/w_m} \right). \end{aligned} \tag{4.52}$$

First we need to argue that taking the poles at zero for the $w_l$ will lead to zero residue, meaning we can ignore these poles. We see this by considering evaluating the poles of the $\tilde{w}_l$'s before the $w_l$'s. $\tilde{w}_n$ only has poles at $w_l$, which then leads to $\tilde{w}_{n-1}$ only having poles at $w_{l'}$ for $l' \neq l$. This gives $\tilde{w}_l = w_{\sigma(l)}$ for some $\sigma \in S_n$, undoing our splitting into two integrals and giving us our original integral as in equation (3.31). The important point to note is that now each $w_l$ has lowest power in its Taylor expansion $\zeta_l - \eta_{\sigma^{-1}(l)} \geqslant 0$ (the $-1$ exponent goes away because of the numerator $\tilde{w}_{\sigma^{-1}(l)} - \tau^2 w_l \mapsto (1 - \tau^2) w_l$) and so there can't be a pole with any coordinate at zero.

With this in mind, we can now proceed to evaluating our integral. $w_1$ has a pole at 0 and a pole at $\tau/z$. Since we are ignoring the pole at 0, we need only consider the pole at $\tau/z$, the residue of which is

$$\begin{aligned} M_{\zeta\eta}(z) = \tau^{\zeta_1} z^{-\zeta_1} \prod_{l=2}^{n} \oint \frac{dw_l}{2\pi i} \frac{w_l^{\zeta_l}}{w_l - \tau^3/z} \prod_l \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l - 1}}{1 - \tau z \tilde{w}_l} \\ \prod_{l=2}^{n} \prod_m \frac{\tilde{w}_m - \tau^2 w_l}{\tilde{w}_m - w_l} \prod_l \frac{\tilde{w}_l - \tau^3/z}{\tilde{w}_l - \tau/z} \prod_{l<m} \frac{1 - \tilde{w}_m/\tilde{w}_l}{1 - \tau^2 \tilde{w}_m/\tilde{w}_l} \prod_{l<m,2}^{n} \frac{1 - w_l/w_m}{1 - \tau^2 w_l/w_m}. \end{aligned} \tag{4.53}$$

Note that this is an equality, as we know the pole at zero will have no contribution. For $w_2$ there is a pole at 0 and at $\tau^3/z$. The pole at $\tau^3/z$ has residue

$$\begin{aligned} M_{\zeta\eta}(z) = \tau^{\zeta_1 + 3\zeta_2} z^{-\zeta_1 - \zeta_2} \prod_{l=3}^{n} \oint \frac{dw_l}{2\pi i} \frac{w_l^{\zeta_l}}{w_l - \tau^5/z} \prod_l \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l - 1}}{1 - \tau z \tilde{w}_l} \\ \prod_{l=3}^{n} \prod_m \frac{\tilde{w}_m - \tau^2 w_l}{\tilde{w}_m - w_l} \prod_l \frac{\tilde{w}_l - \tau^5/z}{\tilde{w}_l - \tau/z} \prod_{l<m} \frac{1 - \tilde{w}_m/\tilde{w}_l}{1 - \tau^2 \tilde{w}_m/\tilde{w}_l} \prod_{l<m,3}^{n} \frac{1 - w_l/w_m}{1 - \tau^2 w_l/w_m}. \end{aligned} \tag{4.54}$$

This continues with $w_r$ having a pole at $\tau^{2r-1}/z$ and at 0, with us choosing the non-zero pole giving residue

$$M_{\zeta\eta}(z) = \tau^{\zeta_1+\cdots+(2r-1)\zeta_r} z^{-\zeta_1-\cdots-\zeta_r} \prod_{l=r+1}^{n} \oint \frac{dw_l}{2\pi i} \frac{w_l^{\zeta_l}}{w_l - \tau^{2r+1}/z} \prod_l \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l-1}}{1-\tau z \tilde{w}_l}$$
$$\prod_{l=r+1}^{n} \prod_m \frac{\tilde{w}_m - \tau^2 w_l}{\tilde{w}_m - w_l} \prod_l \frac{\tilde{w}_l - \tau^{2r+1}/z}{\tilde{w}_l - \tau/z} \prod_{l<m} \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l} \prod_{l<m,r+1}^{n} \frac{1-w_l/w_m}{1-\tau^2 w_l/w_m}. \tag{4.55}$$

Finally we evaluate $w_n$ at $\tau^{2n-1}/z$, ignoring the pole at 0 to get

$$M_{\zeta\eta}(z) = \tau^{\zeta_1+\cdots+(2n-1)\zeta_n} z^{-\zeta_1-\cdots-\zeta_n} \prod_l \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l-1}}{1-\tau z \tilde{w}_l} \prod_l \frac{\tilde{w}_l - \tau^{2n+1}/z}{\tilde{w}_l - \tau/z} \prod_{l<m} \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l}. \tag{4.56}$$

The only pole for $\tilde{w}_n$ is then $\tau/z$. Ignoring the factors of $\tau$ and $z$ at the front from the previous line, this gives

$$M_{\zeta\eta}(z) \propto \tau^{-\eta_n-1} z^{\eta_n+1} \prod_{l=1}^{n-1} \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l-1}}{1-\tau z \tilde{w}_l} \frac{1}{1-\tau^2}$$
$$\prod_{l=1}^{n-1} \frac{\tilde{w}_l - \tau^{2n+1}/z}{\tilde{w}_l - \tau^3/z} (\tau/z - \tau^{2n+1}/z) \prod_{l<m,1}^{n-1} \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l}, \tag{4.57}$$

where $\propto$ is used instead of an equals sign, because we are ignoring the powers of $\tau$ and $z$ arising from the $w$ integrals. The only pole for $\tilde{w}_{n-1}$ is $\tau^3/z$ with residue

$$M_{\zeta\eta}(z) \propto \tau^{-\eta_n-3\eta_{n-1}-3} z^{\eta_n+\eta_{n-1}+1} \prod_{l=1}^{n-2} \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l-1}}{1-\tau z \tilde{w}_l} \frac{1}{1-\tau^2} \frac{1}{1-\tau^4} \prod_{l=1}^{n-2} \frac{\tilde{w}_l - \tau^{2n+1}/z}{\tilde{w}_l - \tau^5/z}$$
$$(\tau^3/z - \tau^{2n+1}/z)(1-\tau^{2n}) \prod_{l<m,1}^{n-2} \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l}. \tag{4.58}$$

This continues with the only contributing pole for $\tilde{w}_{n-r+1}$ being $\tau^{2r-1}/z$ with residue

$$M_{\zeta\eta}(z) \propto \tau^{-\sum_{l=n-r+1}^{n}(2n-2l+1)\eta_l} z^{\sum_{l=n-r+1}^{n}\eta_l} \prod_{l=1}^{n-r} \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \frac{\tilde{w}_l^{-\eta_l-1}}{1-\tau z \tilde{w}_l} \prod_{l=1}^{r} \frac{1}{1-\tau^{2l}}$$
$$\prod_{l=1}^{n-r} \frac{\tilde{w}_l - \tau^{2n+1}/z}{\tilde{w}_l - \tau^{2r+1}/z} \prod_{l=n-r+1}^{n} (1-\tau^{2l}) \prod_{l<m,1}^{n-r} \frac{1-\tilde{w}_m/\tilde{w}_l}{1-\tau^2\tilde{w}_m/\tilde{w}_l}. \tag{4.59}$$

Taking $\tilde{w}_1 = \tau^{2n-1}/z$ gives our expected result.

### 4.4.1 An identity from the matrix element

We have that

$$M_{\zeta\eta}(z) = N_\eta(t^2)\oint_C d\mu[W;\tau^2]\Psi_\eta(W;t^2)\Psi_\zeta(W^{-1};t^2)\pi_f[W;z,t]\pi_{\bar{f}}[W;z,t]\,. \tag{4.60}$$

As in equation (1.9), we shift $\zeta$ and $\eta$ by the same amount, this time such that they both have non-zero length. Writing $\nu := \eta - \min(\zeta_n, \eta_n)$ and $\mu := \zeta - \min(\zeta_n, \eta_n)$, we get

$$M_{\zeta\eta}(z) = \frac{b_\nu(\tau^2)}{(1-\tau^2)^k}\oint_C d\mu[W;\tau^2]P_\nu(W;\tau^2)P_\mu(W^{-1};\tau^2)\pi_f[W;z]\pi_{\bar{f}}[W;z]\,. \tag{4.61}$$

Using the identities in section 4.1, we can rewrite this as

$$M_{\zeta\eta}(z) = b_\nu(\tau^2)z^{|\nu|-|\mu|}\sum_\lambda \frac{1}{\varphi_{n-l(\lambda)}(\tau^2)b_\lambda(\tau^2)}Q_{\lambda/\mu}\left(\frac{\tau}{1-\tau^2};\tau^2\right)Q_{\lambda/\nu}\left(\frac{\tau}{1-\tau^2};\tau^2\right)\,. \tag{4.62}$$

Now note that $l(\nu) \leqslant n$, which implies that $l(\lambda) \leqslant n$, and hence we have

$$M_{\zeta\eta}(z) = z^{|\nu|-|\mu|}\sum_\lambda \frac{1}{\varphi_{n-l(\lambda)}(\tau^2)}Q_{\lambda/\mu}\left(\frac{\tau}{1-\tau^2};t^2\right)P_{\lambda/\nu}\left(\frac{\tau}{1-\tau^2};\tau^2\right)\,. \tag{4.63}$$

The result of this section thus tells us the following identity, which is a generalisation of the principal specialisation result for Hall-Littlewood polynomials to partitions of finite length,

$$\sum_{\substack{\lambda\\ l(\lambda)\leqslant n}} \frac{1}{\varphi_{n-l(\lambda)}(\tau^2)}Q_{\lambda/\mu}\left(\frac{\tau}{1-\tau^2};\tau^2\right)P_{\lambda/\nu}\left(\frac{\tau}{1-\tau^2};\tau^2\right) = \frac{1}{\varphi_{n-l(\nu)}(\tau^2)b_\nu(\tau^2)}\tau^{\Delta_1[\mu,\nu]}\,. \tag{4.64}$$

## 4.5 Proof of lemma 3: Evaluating $\mathscr{O}_{\zeta\eta}(k)$

*Proof.* We note that $\mathscr{O}_{\zeta\eta}(k) = \mathscr{O}_{\zeta-(k^n),\eta}(0)$, so it is sufficient to evaluate $\mathscr{O}_{\zeta\eta}(0) \equiv \mathscr{O}_{\zeta\eta}$. Using the permutation invariance of the integrand, we can rewrite $\mathscr{O}_{\zeta\eta}$ as

$$\begin{aligned}\mathscr{O}_{\zeta\eta} = \frac{(1-\tau^2)^n}{N_\eta(\tau^2)}\prod_{l=1}^n\left(\oint_C\frac{dw_l}{2\pi i}\oint_C\frac{d\tilde{w}_l}{2\pi i}\right)\prod_{l=1}^n w_l^{\zeta_l}\tilde{w}_l^{-\eta_l}\prod_{l<m}(w_m-w_l)(\tilde{w}_m-\tau^2\tilde{w}_l)\\ \prod_{l>m}(\tilde{w}_m-\tilde{w}_l)(w_m-\tau^2 w_l)\prod_{l,m}\frac{1}{(w_m-\tau x\tilde{w}_l)(\tilde{w}_m-\frac{\tau}{x}w_l)}\,.\end{aligned} \tag{5.65}$$

We define $n_+, \tilde{n}_+$ such that $\eta_{n_+}, \zeta_{\tilde{n}_+} \geqslant 0$ and $\eta_{n_++1}, \zeta_{\tilde{n}_++1} < 0$. $w_1, \ldots, w_{n_+}$ only have poles within the unit circle at $\tau x \tilde{w}_l$, while $\tilde{w}_{\tilde{n}_++1}, \ldots, \tilde{w}_n$ only have poles within the unit circle at $\frac{\tau}{x} w_l$. We do these integrals, defining $\sigma : \{1, \ldots, n_+\} \to \{1, \ldots, n\}$ and $\tau : \{\tilde{n}_+ + 1, \ldots, n\} \to \{1, \ldots, n\}$ so $w_l = wx\tilde{w}_{\sigma(l)}$ for $l = 1, \ldots, n_+$ and $\tilde{w}_l = \frac{\tau}{x} w_{\tau(l)}$ for $l = \tilde{n}_+ + 1, \ldots, n$. Note that the factor $\prod_{l<m}(w_m - w_l)(\tilde{w}_l - \tilde{w}_m)$ guarantees that $\sigma$ and $\tau$ are injective.

Now we show that

$$\begin{aligned} \sigma\{1, \ldots, n_+\} &\subseteq \{1, \ldots, \tilde{n}_+\}, \\ \tau\{\tilde{n}_+ + 1, \ldots, n\} &\subseteq \{n_+ + 1, \ldots, n\}, \end{aligned} \tag{5.66}$$

otherwise the residue is zero. We prove this for $\sigma$ and note that a very similar proof will work for $\tau$. Suppose $\exists l \in \{1, \ldots, n_+\}$ such that $\sigma(l) > \tilde{n}_+$, then $w_l = \tau x \tilde{w}_{\sigma(l)} = \tau^2 w_{\tau\sigma(l)}$. Either $\tau\sigma(l) > l$, so the factor in the integrand $(1 - \tau^2 w_{\tau\sigma(l)}/w_l)$ evaluates to zero, or $\tau\sigma(l) < l \leqslant n_+ \implies w_{\tau\sigma(l)} = \tau x \tilde{w}_{\sigma\tau\sigma(l)} \implies \tilde{w}_{\sigma(l)} = \tau^2 \tilde{w}_{\sigma\tau\sigma(l)}$. Now either $\sigma(l) > \sigma\tau\sigma(l)$, so the factor $(\tilde{w}_{\sigma(l)} - \tau^2 \tilde{w}_{\sigma\tau\sigma(l)})$ evaluates to zero, or $\tilde{w}_{\sigma\tau\sigma(l)} = \frac{\tau}{x} w_{\tau\sigma\tau\sigma(l)}$. This process carries on, either we hit a zero, or we hit a fixed point of $\tau\sigma$ (as it is a permutation of a finite set). However, hitting a fixed point of $\tau\sigma$ would lead to $w_m = \tau^2 w_m$ for some $m$ with $w_m \neq 0$, a clear contradiction.

Now we show that if $\tilde{n}_+ > n_+$, then we get zero. Suppose that this is true, we extend our definition of $\tau$ to $\{n_+ + 1, \ldots, n\}$ in order that $\tau\{n_+ + 1, \ldots, n\} = \{n_+ + 1, \ldots, n\}$, and then extend our definition of $\sigma$ such that $\{\sigma(1), \ldots, \sigma(n_+), \sigma\tau(n_+ + 1), \ldots, \sigma\tau(\tilde{n}_+)\} = \{1, \ldots, \tilde{n}_+\}$. After doing these integrals, we consider $w_{\tau(n_++1)}, \ldots, w_{\tau(\tilde{n}_+)}$. As $m_{\tau(l)} < 0$, we consider the poles outside the unit circle, there is only one and it is at $w_{\tau(l)} = \frac{x}{\tau} \tilde{w}_m$ for $m = 1, \ldots, \tilde{n}_+$. The factor $\prod_{l=1}^{n_+} \prod_{n_++1}^{n} (\tilde{w}_{\sigma(l)} - \frac{\tau}{x} w_m)$ guarantees that the pole is $w_{\tau(l)} = \frac{x}{\tau} \tilde{w}_{\sigma\tau(l)}$, for $l = n_+ + 1, \ldots, \tilde{n}_+$.

Upon doing this integral, one finds the integrand

$$\begin{aligned} \mathscr{O}_{\zeta\eta} = \frac{(-)^{\#} x^{\#} \tau^{\#}}{N_\eta(\tau^2)} \prod_{l=\tilde{n}_++1}^{n} \oint \frac{dw_{\tau(l)}}{2\pi i w_{\tau(l)}} \prod_{l=1}^{\tilde{n}_+} \oint \frac{d\tilde{w}_l}{2\pi i \tilde{w}_l} \\ \prod_{l=\tilde{n}_++1}^{n} w_{\tau(l)}^{\zeta_{\tau(l)} - \eta_l} \prod_{l=1}^{n_+} \tilde{w}_{\sigma(l)}^{\zeta_l - \eta_{\sigma(l)}} \prod_{l=n_++1}^{\tilde{n}_+} \tilde{w}_{\sigma\tau(l)}^{\zeta_{\tau(l)} - \eta_{\sigma\tau(l)}}. \end{aligned} \tag{5.67}$$

This means that $\tilde{w}_1, \ldots, \tilde{w}_{\tilde{n}_+}$ can only receive poles within the unit circle from the monomial at zero, and will have a non-zero residue only if it is a simple pole. However, we have that $\zeta_{\tau(l)} - \eta_{\sigma\tau(l)} - 1 \leqslant -2$ for $l = n_+ + 1, \ldots, \tilde{n}_+$ because $\zeta_{\tau(l)} < 0$ and $\eta_{\sigma\tau(l)} \geqslant 0$ for such $l$. Hence this cannot be a simple pole. We conclude that the integral will evaluate to zero, unless $\tilde{n}_+ \leqslant n_+$. We now do the integral for $w_1, \ldots, w_{n_+} = \tau x w_{\sigma(1)}, \ldots, \tau x w_{\sigma(n_+)}$ and $\tilde{w}_{n_++1}, \ldots, \tilde{w}_n = \frac{\tau}{x} w_{\tau(n_++1)}, \ldots, \frac{\tau}{x} w_{\tau(n)}$. By our earlier argument, we have that $\sigma\{1,\ldots,n_+\} = \{1,\ldots,n_+\}$ and $\tau\{n_+ + 1,\ldots,n\} = \{n_+ + 1,\ldots,n\}$, so we may combine $\sigma$ and $\tau$ together to form $\sigma \in S_{n_+} \times S_{n-n_+} \subseteq S_n$. Upon performing the integral, we obtain that all the cross terms cancel out. We define the dot product of the Hall-Littlewood polynomials for $r \leqslant n$ variables to be

$$\begin{aligned}\langle \Psi_\theta, \Psi_{\theta'} \rangle^{(r)} &:= \frac{1}{N_\theta N_{\theta'}} \prod_{i=1}^{r} \oint \frac{dw_i}{2\pi i w_i} \sum_{\sigma \in S_r} \prod_{i=1}^{r} w_i^{\theta_i - \theta'_{\sigma^{-1}(i)}} \prod_{i<j} (1 - \tau^2 w_j / w_i)(1 - w_{\sigma(j)}/w_{\sigma(i)}) \\ &\prod_{i>j} (1 - \tau^2 w_{\sigma(j)}/w_{\sigma(i)})(1 - w_j/w_i) \prod_{i \neq j} \frac{1}{(1 - \tau^2 w_j/w_i)(1 - w_j/w_i)} \\ &= \delta_{\theta\theta'} \frac{1}{N_\theta^{(r)}(\tau^2)} ,\end{aligned} \tag{5.68}$$

where $N^{(r)}$ has an $r$ to indicate it is for partitions of size $r$ and not $n$. We find that the integral becomes, defining $\zeta_{(n_+)} := (\zeta_1,\ldots,\zeta_{n_+})$ and $\zeta^{(n_+)} := (\zeta_{n_+ + 1},\ldots,\zeta_n)$ (and similarly for $\eta$),

$$\begin{aligned}\mathscr{O}_{\zeta\eta} &= N_\zeta \tau^{\Sigma_1^{n_+} \zeta_l - \Sigma_{n_+ + 1}^{n} \eta_l} x^{\Sigma_1^{n_+} \zeta_l + \Sigma_{n_+ + 1}^{n} \eta_l} \langle \Psi_{\zeta_{(n_+)}}, \Psi_{\eta_{(n_+)}} \rangle^{(n_+)} \langle \Psi_{\zeta^{(n_+)}}, \Psi_{\eta^{(n_+)}} \rangle^{(n-n_+)} \\ &= \delta_{\zeta\eta} x^{|\eta|} \tau^{||\eta||}\end{aligned} \tag{5.69}$$

as required.

□

## 4.6 Cherkis bow varieties

Nakajima and Takayama in [159] showed that the Coulomb branch of any $A$-type and $\hat{A}$-type quiver can be described as a certain type of quiver variety known as the Cherkis bow. A physical derivation of this fact was first given in [54].

Our derivation of the Coulomb branch Hilbert series through manipulations of the Higgs branch Molien integral gives exactly what one would expect for the Molien integral of the Cherkis bow. We shall briefly describe the Cherkis bow variety that coincides with the Coulomb branch of the $\hat{A}$-quiver mirror dual to the ADHM quiver.

The Cherkis bow is composed of two different parts. The triangle and the two way. These two components are in figure 4.2. We either align them in a line or in a circle. An example of this is the one that we will consider, the diagram on the left of figure 4.2 is the Cherkis bow for the $\hat{A}$ quiver in figure 4.1. Schematically, the flavour nodes become two ways and the edges between gauge nodes become triangles.

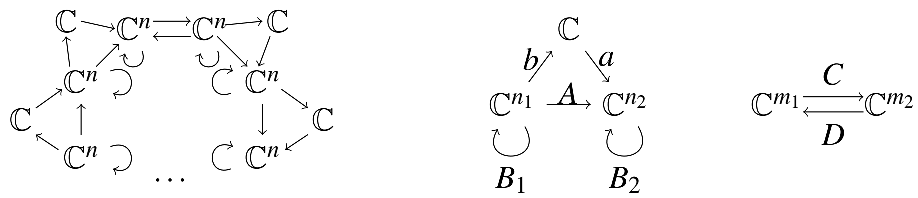

Fig. 4.2 On the left we have the Cherkis bow for the mirror dual of the ADHM quiver. The middle and the right diagrams show the constituents of a Cherkis bow, respectively the triangle and the two way.

The Cherkis bow variety is, up to stability conditions and deformation parameters, given by the solutions to some quadratic equations, modulo the group of invertible endormorphisms of the domains and codomains of $A, C$ and $D$, acting in the obvious way. We want to emphasise the quadratic equation for the triangle part, which is

$$B_2 A - A B_1 + ab = 0\,. \tag{6.70}$$

The Molien integral that we would expect for the triangle when $n_1 = n_2 = n$ is exactly our piece $M_{\zeta\eta}(z)$, where $z$ is the fugacity for the global symmetry from $\mathbb{C}^\times$-rotations of the $\mathbb{C}$ flavour.

## 4.7 Mirror symmetry for a linear quiver

The derivation of mirror symmetry for the ADHM quiver and its mirror dual $\hat{A}$ quiver leads to a derivation of mirror symmetry for a particular pair of linear quivers, shown in figure 4.3. In doing so, we find two interesting points: how to glue together two tails on the Higgs branch in order to produce an adjoint hypermultiplet; how to do so with a baryonic charge for the gauge node the adjoint hypermultiplet is non-trivially charged under.

In the notation of [63], the two quivers are $T^\sigma_\rho(SU(nN))$ (the diagram on the left in figure 4.3) and $T^\rho_\sigma(SU(nN))$ (the diagram on the right), where

$$\begin{aligned} \sigma &= (\underbrace{N-1,\dots,N-1}_{n},\underbrace{1,\dots,1}_{n})\,,\\ \rho &= (\underbrace{n,\dots,n}_{N})\,. \end{aligned} \tag{7.71}$$

Since $\rho$ is of the form $(a^b)$, for $a,b \in \mathbb{N}$, we have that $T^{\sigma}_{\rho}(SU(nN))$ is a balanced quiver. This means the Coulomb branch has full $SU(N)$ symmetry, as we would expect given the mirror dual.

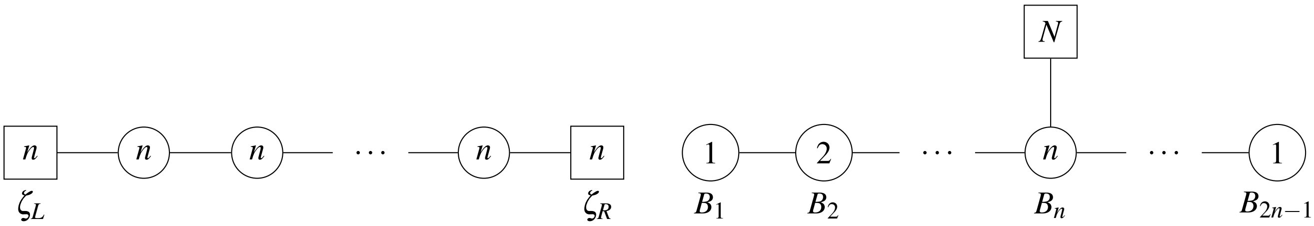


Fig. 4.3 On the left, we have the $A_{N-1}$ quiver with background magnetic charges $\zeta_L$ and $\zeta_R$, where $\zeta_L, \zeta_R \in \mathfrak{A}$. On the right, we have its mirror dual, the "fish-tailed quiver" with background baryonic charge $B_1, \ldots, B_{2k-1} \in \mathbb{Z}$.

We define $N_A$ as the value of the gauge rank of the $A^{\text{th}}$ node of the fish-tailed quiver (so $N_A = A$ for $A = 1, \ldots, n$). An expression for the Hilbert series of the Higgs branch of the fish-tailed quiver is given by the Molien integral

$$\begin{aligned} \mathrm{HS}_{\mathrm{lin}}(B_A) = & \frac{1}{1!^2 2!^2 3!^2 \ldots (n-1)!^2 n!} \prod_{A=1}^{2n-1} \prod_{a=1}^{N_A} \left( \oint \frac{dw_{A,a}}{2\pi i w_{A,a}} w_{A,a}^{-B_A} \right) \\ & \prod_{A=1}^{2n-1} \frac{\prod_{l \neq m} (1 - w_{A,l}/w_{A,m}) \prod_{l,m}^{N_A} (1 - \tau^2 w_{A,l}/w_{A,m})}{\prod_{l=1}^{N_{A+1}} \prod_{m=1}^{N_A} (1 - \tau w_{A+1,l}/w_{A,m})(1 - \tau w_{A,m}/w_{A+1,l})} \\ & \prod_{l=1}^{n} \prod_{i=1}^{N} \frac{1}{1 - \tau z_i/w_{n,l}} \frac{1}{1 - \tau w_{n,l}/z_i} . \end{aligned} \tag{7.72}$$

Note that we have an integral over the unit torus, $T^{N_A}$, for each node, with contributions from the baryonic charges, $B_A$; the bifundamental matter; constraints; Haar measure; and the fundamental matter at the $n^{\text{th}}$ node. We collapse this integral to an integral over the middle node (an integral of rank $n$), by noting that for the $w_{1,1}$ pole there are only two poles at $w_{1,1} = \tau w_{2,1}$ and $w_{1,1} = \tau w_{2,2}$. Upon integrating out $w_{1,1}$, one then integrates out $w_{2,l}$ finding poles at $w_{3,\sigma(l)}$ and so on, until we reach $w_{n,l}$. Doing the same thing from the right inwards gives the expression,

$$\mathrm{HS}_{\mathrm{lin}}(\zeta_L, \zeta_R) = \frac{1}{(1-\tau^2)^n} \oint_C d\mu[W;\tau^2] \Psi_{n_L}(W;\tau^2) \prod_{i=1}^{N} \left( \pi_f(W;z_i) \pi_{\bar{f}}(W;z_i) \right) \Psi_{n_R}(W^{-1};\tau^2) , \tag{7.73}$$

where the "fundamental" and "antifundamental" parts are defined in equation (3.25), $\Psi$ is defined in equation (1.7) and the baryonic charges $B$ correspond to the charges $\zeta_L$ and $\zeta_R$ via

$$\begin{aligned} \zeta_{L,1} &= B_1 + \cdots + B_k, & \zeta_{R,1} &= 0\,, \\ \zeta_{L,2} &= B_2 + \cdots + B_k, & \zeta_{R,2} &= -B_{k+1}\,, \\ &\dots & &\dots \\ \zeta_{L,k} &= B_k, & \zeta_{R,k} &= -(B_{k+1} + \cdots + B_{2k-1})\,. \end{aligned} \tag{7.74}$$

Note that we can shift $\zeta_L$ and $\zeta_R$ by an arbitrary constant, leaving the Hilbert series invariant by equation 1.9. So we can reach all possible background magnetic charges this way[1].

Just as in section 4.3, we convert our Higgs branch expression into a Coulomb branch expression via the use of "Dirac delta functions" for symmetric polynomials. These allow us to rewrite the Hilbert series as an integral over the maximal torus of the gauge group of the linear quiver, $U(n)^{N-1}$. Using the tricks of section 4.3, we can rewrite the Hilbert series of the Higgs branch as follows

$$\begin{aligned} \mathrm{HS}_{\mathrm{lin}}(\zeta_L, \zeta_R) =& \frac{1}{(1-\tau^2)^n} \oint_C d\mu[W;\tau^2]\Psi_{\zeta_L}(W) \prod_{i=1}^{N} \Bigg( \oint_C d\mu[W^{(i)};\tau^2] \\ & K[W^{(i-1)\,-1}, W^{(i)\,-1}]\pi_f(W^{(i)};z_i)\pi_{\bar{f}}(W^{(i)};z_i) \Bigg) \Psi_{\zeta_r}(W^{(N)\,-1}) \\ =& \frac{1}{(1-\tau^2)^n} \frac{1}{N_{\zeta_L}(\tau^2)} \sum_{\zeta\in\mathfrak{A}^N} M_{\zeta_L,\zeta^{(1)}}(z_1) M_{\zeta^{(1)},\zeta^{(2)}}(z_2) \dots M_{\zeta^{(N-1)},\zeta_R}(z_N)\,. \end{aligned} \tag{7.75}$$

Using the results of 4.3, one sees that this is exactly the expression for the Hilbert series of the Coulomb branch of the mirror dual theory.

The results of [61] tells us that if we set $\zeta_L = \zeta_R$ and sum over all values with appropriate factors, then we gauge and glue together the two flavour nodes of the $A_{N-1}$ quiver. The resulting quiver is the $\hat{A}_N$-quiver of figure 4.1. If we wanted to have a baryonic charge $k$, then we would shift $\zeta_R$ by $(k^n)$:

$$\sum_{\zeta^{(N)}\in\mathfrak{A}} (1-\tau^2)^n N_{\zeta^{(N)}}(\tau^2)\tau^{||\zeta^{(N)}||} x^{|\zeta^{(N)}|} \mathrm{HS}_{\mathrm{lin}}(\zeta^{(N)}, \zeta^{(N)} + (k^n)) = \mathrm{HS}(k)\,. \tag{7.76}$$

On the mirror side we can glue together the two tails of the fishtailed quiver by summing over the baryonic charges, as specified by $\zeta_L = \zeta_R$ and equations (7.74), leaving an adjoint hypermultiplet.

[1] This is because the flavour symmetry of a quiver is $S\left(\prod_a U(N_a)\right)$.

# Chapter 5

# *A*-type quivers

We defined in chapter 3 a superconformal index, and described how to compute it on spaces whose resolutions have isolated fixed points. In this chapter, we show how to apply this to $A$- and $\hat{A}$-type quivers by calculating their fixed points. The calculation relies on an observation by Nakajima in [154] that these quivers' Higgs branches can be thought of as $\mathbb{C}^{\times}$-fixed point submanifolds of instanton moduli space. We then make use of the fact that the superconformal index for instanton moduli space is already known, [164]. The culmination of this is theorem 9, where we write the Hilbert series and superconformal index of a general linear quiver with arbitrary background charge in closed form.

In section 5.1, we define the $A$-type quiver, distinguishing our class from the slightly less general $T^{\sigma}_{\rho}(SU(N))$ theories. In section 5.2, we do the fixed point construction on quantum mechanics, and see how it acts on the superconformal index, sending the index of the large manifold to the fixed point manifold in a certain limit of the fugacities. We then describe how the linear quivers sit inside the ADHM quiver, using the construction of [154]. This then leads to a description of the isolated fixed points of any linear quiver in terms of coloured Young tableaux. In section 5.3, we compute the superconformal index of a linear quiver, by taking the limit of the ADHM quiver and restricting to the appropriate fixed points. The scaling of the fugacities is crucial and is described in section 5.3.1, while the actual calculation on the index is in section 5.3.2. We work through an example in section 5.3.3, reproducing known results for the Hilbert series. In section 5.3.4, we introduce non-Abelian baryonic charges to the instanton moduli space, and then take the limit, in order to see which charges on the linear quiver can be described in this way. Finally, section 5.4 contains the main results of this chapter. Lemma 4 writes the Hilbert series or superconformal index with arbitrary background Abelian baryonic charge in terms of a $T_{\sigma}$-type theory. This leads to theorem 9, which allows us to write the Hilbert series or superconformal index in a closed form, with the sum over fixed points replaced with a sum over a Weyl orbit. This result for

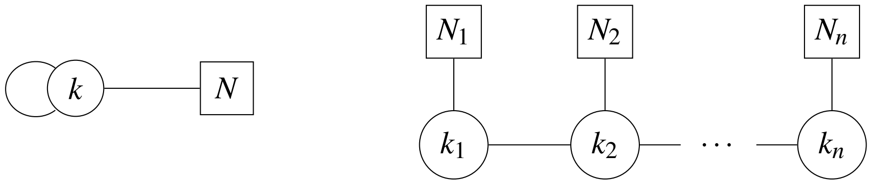


Fig. 5.1 The ADHM quiver on the left and a generic $A_n$ quiver on the right.

the Hilbert series is known and was done in [63]. Here we provide a proof using the fixed points, and so can extend the result to the superconformal index in a rigorous way. In section 5.5, we generalise these results to write a fixed point sum for $\hat{A}_n$-type quivers. In section 5.6, we prove lemma 4. In section 5.7, we prove theorem 9.

## 5.1 The linear quiver

The two quivers that we consider are shown in figure 5.1.

The linear quiver is a quiver gauge theory as defined in section 2.1.2, where the underlying graph is the Dynkin diagram of $A_n$. We shall be considering the Higgs branch of these theories. A large class of such theories was discussed in [63]. They called such theories $T^{\sigma}_{\rho}(SU(N))$. These theories are good theories, in the sense of section 2.2.2. However, we shall look at a slightly larger class of linear quivers. The ones where the Higgs branch exists.

For example, if we take the linear quiver for $A_1$, with dimensions $(k,N)$, then if $k \leqslant N/2$, we have $T^{(1^N)}_{(N-k,k)}(SU(N))$ theory. If $k > N/2$, then we no longer have a $T^{\sigma}_{\rho}(SU(N))$ theory, but we still have a well defined Higgs branch. In both cases the resolved Higgs branch is $T^*(\mathrm{Gr}(k,N))$, the cotangent bundle to the complex Grassmannian of $\mathbb{C}^k$ planes lying within $\mathbb{C}^N$. There is a duality, known as three-dimensional Seiberg duality, which relates the theory with $(k,N)$ to $(N-k,N)$. The equivalence of the Higgs branches follows from the fact that $\mathrm{Gr}(k,N) \cong \mathrm{Gr}(N-k,N)$.

All linear quiver gauge theories that are not a $T^{\sigma}_{\rho}(SU(N))$ theory, but have a well defined Higgs branch, are Seiberg dual to a $T^{\sigma}_{\rho}(SU(N))$ theory. We have a more nuanced discussion of this duality in section 2.3.3.

## 5.2 The linear quiver as a submanifold of instanton moduli space

The fundamental idea of this section and the next is to calculate the superconformal index of the Higgs branch of a linear quiver, by thinking of it as a $\mathbb{C}^{\times}$-fixed point submanifold of a larger manifold we know the superconformal index of, specifically, instanton moduli space. This $\mathbb{C}^\times$ action is a subgroup of the isometries, and notably does not include the $\mathbb{C}^\times$-scaling action generated by $\mathscr{D} - N$ (see section 3.3 for the definition of that $\mathbb{C}^\times$-action).

We shall first run through such a construction in the simplest example, free quantum mechanics on flat space. This example will provide intuition for the construction on the curved instanton moduli space, especially when one remembers that the superconformal index is given by the contribution of the quadratic fluctuations at each fixed point, which is free quantum mechanics.

We shall then describe how the Higgs branch of a linear quiver sits inside the Higgs branch of the ADHM quiver, and how we can describe the fixed points using the coloured Young tableaux fixed points of the ADHM quiver.

### 5.2.1 The construction for free quantum mechanics

Consider $\mathbb{C}^4$, with coordinate ring $\mathbb{C}[X_1, \tilde{X}_1, X_2, \tilde{X}_2]$. $X_i$ is charged[1] as $\tau s_i$ and $\tilde{X}_i$ is charged as $\tau/s_i$ for i=1,2. $\tau/s_i$ and $\tau s_i$ are fugacities for $\mathbb{C}^\times_{i,1} \times \mathbb{C}^\times_{i,2}$ rotating the target space $\mathbb{C}^2_i = \mathbb{C}_{i,1} \times \mathbb{C}_{i,2}$. The diagonal subgroup of $\mathbb{C}^\times_{1,1} \times \mathbb{C}^\times_{1,2}$ is counted with the same fugacity as the diagonal subgroup of $\mathbb{C}^\times_{2,1} \times \mathbb{C}^\times_{2,2}$, this is the scaling generated by $\mathscr{D} - N$. $dX_i$ and $d\tilde{X}_i$ are charged as $ys_i$ and $y/s_i$ respectively, where the fugacity $y$ is for a $\mathbb{C}^\times_y$ that rotates the cotangent fibres. The index of this SQM is

$$\mathscr{Z}(\mathbb{C}^4) = \left(\frac{\tau}{y}\right)^2 \prod_{i=1}^{2} \frac{(1-ys_i)(1-y/s_i)}{(1-\tau s_i)(1-\tau/s_i)} \,. \tag{2.1}$$

The Hilbert series is the coefficient of the highest power of $y$ divided by $\tau^2$,

$$\mathrm{HS}(\mathbb{C}^4) = \prod_{i=1}^{2} \frac{1}{(1-\tau s_i)(1-\tau/s_i)} \,. \tag{2.2}$$

We consider restricting to the hyperKähler submanifold invariant under the subgroup $\{(x, 1/x) | x \in \mathbb{C}^\times\} \subset \mathbb{C}^\times_{1,1} \times \mathbb{C}^\times_{1,2}$, namely $\mathbb{C}_{2,1} \times \mathbb{C}_{2,2}$. We do this by discarding all generators with non-zero power of $s_1$. To take this limit, we take the limit $s_1 \to 0$ in the index. This gives

$$\mathscr{Z}(\mathbb{C}^2) = \frac{\tau}{y} \frac{(1-ys_2)(1-y/s_2)}{(1-\tau s_2)(1-\tau/s_2)} \,. \tag{2.3}$$

[1] One should think of this as the charge of the operator given by multiplication by $X_i$, and similarly for the other variables.

We then get the Hilbert series by taking the highest power of $y$ divided by $\tau$,

$$\mathrm{HS}(\mathbb{C}^2) = \frac{1}{(1-\tau s_2)(1-\tau/s_2)}\,. \tag{2.4}$$

The conclusion of this is that this limit of the superconformal index of the original hyperKähler cone gives the superconformal index of the fixed point submanifold (which is also a hyperKähler cone).

**Theorem 8.** *Let $\mathscr{M}$ be a hyperKähler cone with isolated fixed points under $\mathbb{C}^\times \times T$, where $T$ is a torus action of isometries and $\mathbb{C}^\times$ the conical action. If $\mathscr{N} \subset \mathscr{M}$ is the fixed point submanifold (generally not connected) under a subgroup of isometries $T' \subset T$, then, if $(x_i)_{i\in I}$ are the fugacities corresponding to $T'$, we have that*

$$\lim_{x_i \to 0,\, i\in I} \mathscr{Z}(\mathscr{M}) = \mathscr{Z}(\mathscr{N})\,. \tag{2.5}$$

The proof of this follows trivially from the example of free quantum mechanics above and the localisation theorems of chapter 3.

### 5.2.2 The construction of the fixed point manifold

Here we describe how we construct the Higgs branch of the linear quiver from instanton moduli space. We give the construction, and then describe how the fixed points of instanton moduli space become the fixed points of the linear quivers.

#### The fixed point manifold

We explain the construction in [154]. This construction takes a certain $\mathbb{C}^\times$-subgroup of $T := \mathbb{C}^\times \times G_H$, and restricts $\mathfrak{M}_{\zeta_\mathbb{R},0}$ to the fixed point submanifold. This submanifold is a disjoint union of linear quivers.

We want the fixed points of the $\mathbb{C}^\times \ni t_1$ action on the set $\mu_\mathbb{C}^{-1}(0)$ given by

$$(X,\tilde{X},Q,\tilde{Q}) \mapsto (t_1 X, t_1^{-1}\tilde{X}, Q\rho_W(t_1)^{-1}, \rho_W(t_1)\tilde{Q})\,. \tag{2.6}$$

This corresponds to a choice of homomorphism $\rho_V : \mathbb{C}^\times \to GL(\mathbb{C}^k)$, such that

$$(t_1 X, t_1^{-1}\tilde{X}, Q\rho_W(t_1)^{-1}, \rho_W(t_1)\tilde{Q}) = (\rho_V(t_1)^{-1} X \rho_V(t_1), \rho_V(t_1)^{-1}\tilde{X}\rho_V(t_1), \rho_V(t_1)^{-1}Q, \tilde{Q}\rho_V(t_1))\,. \tag{2.7}$$

$\rho_V$ is a homomorphism, because the action of $GL(\mathbb{C}^k)$ is free on the space of stable maps. We choose the conjugacy class of $\rho_V$ and $\rho_W$. This fixes the linear quiver and we now show how.

The conjugacy class of $\rho_V$ is determined by an $\vec{n} \in \mathbb{Z}^k/S_k$, such that

$$t_1 \mapsto \begin{pmatrix} t_1^{n_1} & & & \\ & t_1^{n_2} & & \\ & & \ddots & \\ & & & t_1^{n_k} \end{pmatrix}. \tag{2.8}$$

Similarly, $\rho_W$'s conjugacy class is determined by $\vec{m} \in \mathbb{Z}^N/S_N$.

We order these integers from smallest to largest. Let $p := \min(m_1, n_1)$ and $q := \max(n_k, m_N)$. Define $n := q - p + 1$. For $a = 1, \dots, n$, we define the spaces

$$\begin{aligned} V_a &= \text{Eigenspace of } \mathbb{C}^k \text{ with eigenvalue } t_1^{q+1-a}, \\ W_a &= \text{Eigenspace of } \mathbb{C}^N \text{ with eigenvalue } t_1^{q+1-a}. \end{aligned} \tag{2.9}$$

Note that unless $m_1 \geqslant n_1$ and $m_N \leqslant n_k$, the fixed point set will be empty, so we may as well take $n = n_k - n_1 + 1$.

$X, \tilde{X} \in \mathrm{End}(\mathbb{C}^k)$, $Q \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ and $\tilde{Q} \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)$. However, we shall see that the fixed points respect the eigenspace structure of $\rho_V$ and $\rho_W$. We see from equation (2.7) that for $v \in V_i$,

$$\begin{aligned} t_1 X v &= \rho_V(t_1)^{-1} X \rho_V(t_1) v = t_1^{q+1-a} \rho_V(t_1)^{-1} X v, \\ \implies \rho_V(t_1) X v &= t_1^{q-a} X v. \end{aligned} \tag{2.10}$$

This implies that $X : V_a \to V_{a+1}$. Similarly, $t_1^{-1}\tilde{X} = \rho_V(t_1)^{-1}\tilde{X}\rho_V(t_1)$, $Q\rho_W(t_1)^{-1} = \rho_V(t_1)^{-1}Q$ and $\rho_W(t_1)\tilde{Q} = \tilde{Q}\rho_V(t_1)$ means

$$\begin{aligned} X &: V_a \to V_{a+1}, \\ \tilde{X} &: V_a \to V_{a-1}, \\ Q &: W_a \to V_a, \\ \tilde{Q} &: V_a \to W_a. \end{aligned} \tag{2.11}$$

So we exactly have the $A_n$ linear quiver. We define

$$k_a := \dim V_a, \quad N_a := \dim W_a. \tag{2.12}$$

The way to think of this in explicit calculations is that we pick a particular $\rho_W$, which - calling the Nakajima quiver variety associated to the linear quiver $M(\rho_V, \rho_W)$ - gives us a decomposition as

$$\coprod_{\rho_V} M(\rho_V, \rho_W) \subseteq \mathfrak{M}_{\zeta_{\mathbb{R}},0}\,. \tag{2.13}$$

**The fixed points of the fixed point manifold**

In the evaluation of the superconformal index of the linear quiver, the analysis of chapter 3 means that one need only consider the fixed points of the action of $T$. On instanton moduli space these fixed points famously correspond to $N$-coloured Young tableaux of total size $k$, [164]. For a particular choice of $\rho_W$, each fixed point will correspond to an individual $\rho_V$. We explain here how to work out which $\rho_V$, and hence which linear quiver, the fixed point is an element of. Note that since the fixed points are invariant under the whole of $(\mathbb{C}^\times)^{N+2}$, they are invariant under the particular $\mathbb{C}^\times$ we used to restrict to the linear quivers, and hence must lie in some linear quiver.

Moreover, the Higgs branch of a linear quiver is non-empty if and only if we can construct such a fixed point. The only if is trivial, as the fixed point is an element of the Higgs branch, while the other way is true because it must be closed under the action of $(\mathbb{C}^\times)^{N+2}$, lie within instanton moduli space, and every point on instanton moduli space flows under the action of $(\mathbb{C}^\times)^{N+2}$ to a fixed point, [160].

The fixed points are the stable maps $X, \tilde{X}, Q, \tilde{Q}$ obeying $\mu_{\mathbb{C}} = 0$, and

$$\begin{aligned} (w_l - w_m + \tau x) X_{lm} &= 0\,, \\ (w_l - w_m + \tau / x) \tilde{X}_{lm} &= 0\,, \\ (w_l - \tau - z_i) Q_{li} &= 0\,, \\ (w_l + \tau - z_i) \tilde{Q}_{il} &= 0\,, \end{aligned} \tag{2.14}$$

where $l, m = 1, \dots, k$ and $i = 1, \dots, N$. The coloured Young tableaux correspond to the values of gauge fugacities, $w \in \mathbb{C}^k$, with

$$w_{i,(\alpha,\beta)} = (\tau x)^{\alpha-1} (\tau/x)^{\beta-1} \tau / z_i\,. \tag{2.15}$$

Exactly $k$ components of the $2kN + 2k^2$ components of $(X, \tilde{X}, Q, \tilde{Q})$ are non-zero. They are, writing $l = 1, \dots, k$ as $(i, (\alpha, \beta))$ for $(\alpha, \beta) \in Y_i$ and $i = 1, \dots, N$,

$$X_{i(\alpha,\beta),i(\alpha+1,\beta)}, \tilde{X}_{i(\alpha,\beta),i(\alpha,\beta+1)}, \tilde{Q}_{i(1,1),i} \neq 0\,. \tag{2.16}$$

Suppose $e_{ais}$ is a basis for $\mathbb{C}^k$, for $a = 1, \dots, n$, $i = 1, \dots, N_a$ and $s \in Y_{ai}$, and $f_{ai}$ a basis for $\mathbb{C}^{N_a}$ for $a = 1, \dots, n$, $i = 1, \dots, N_a$. Then we have that $e_{ai(1,1)} \in V_a$, because $\tilde{Q}_{ai(1,1)\,ai} f_{ai} \propto$

$e_{ai(1,1)}$ and $\tilde{Q}_{ai(1,1)ai} \neq 0$. Now we see that if $(2,1) \in Y_{ai}$, then $Xe_{ai(2,1)} \propto e_{ai(1,1)}$, and so $e_{ai(2,1)} \in V_{a-1}$. Through this, we see that

$$e_{ai(\alpha,\beta)} \in V_{a-\alpha+\beta} \,. \tag{2.17}$$

This fully determines the value of the $k_a$'s. Note that there can be values of $a$ where $N_a = 0$ and $k_a \neq 0$.

The quiver $M(\rho_W, \rho_V)$ is equal to (using the notation of [63]) Higgs$\left[T^\rho_\sigma\left(SU(M)\right)\right]$, where $\sigma$ and $\rho$ are partitions of $M$ determined by $\rho_W$ and $\rho_V$. The number of fixed points on this manifold is given by

$$\#\text{ of fixed points} = \sum_{\rho \leqslant \nu \leqslant \sigma^\vee} K_{\nu\rho} K_{\nu^\vee \sigma} \,, \tag{2.18}$$

where $K_{\alpha\beta}$ are the Kostka numbers. The reason for this is explained in section 6.4.1, and comes from the expression for the Poincaré polynomial in [171].

We work through an example, to show how the Young tableaux are chosen. Take the linear quiver in figure 5.2

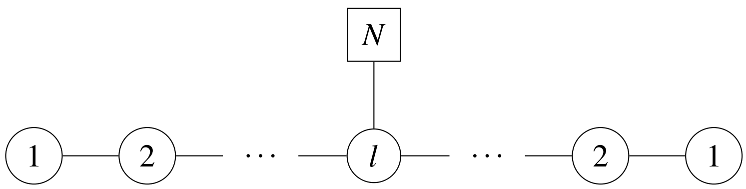


Fig. 5.2 An example of a linear quiver, $k = l^2$.

The coloured Young tableau that we restrict to are exactly the ones such that

$$\begin{aligned}
&\exists \text{ exactly } l\,, s \in Y_i \text{ s.t. } s = (b,b) \text{ for some } b \in \mathbb{N}\,,\\
&\exists \text{ exactly } l-1\,, s \in Y_i \text{ s.t. } s = (b-1,b) \text{ for some } b \in \mathbb{N}\,,\\
&\exists \text{ exactly } l-1\,, s \in Y_i \text{ s.t. } s = (b,b-1) \text{ for some } b \in \mathbb{N}\,,\\
&\exists \text{ exactly } l-2\,, s \in Y_i \text{ s.t. } s = (b-2,b) \text{ for some } b \in \mathbb{N}\,,\\
&\text{etc}\,.
\end{aligned} \tag{2.19}$$

So if $N = 1$, the only pole is given by a single square Young tableaux of height and width $l$.

## 5.3 The superconformal indices of the ADHM and linear quiver

We now use the analysis of the previous section to compute the superconformal index of the linear quiver. We need to first take the appropriate scaling of the fugacities, corresponding to the choice of $\mathbb{C}^\times$-subgroup of the isometries. We can then apply the limit of the fugacities to the superconformal index of the ADHM quiver.

We shall then work through some examples and look briefly at non-Abelian baryonic charge.

### 5.3.1 Scaling

Here we describe the scalings of the fugacities in our model, corresponding to the choice of $\mathbb{C}^\times$-subgroup of isometries. We take $\rho_W$ to be in the conjugacy class $(c_1,\ldots,c_N) \in \mathbb{Z}^N$ with $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N = 0$. There is an invariance corresponding to shifts $c_i \mapsto c_i + c$, for constant $c \in \mathbb{Z}$, as they are $SU(N)$ fugacities and not $U(N)$ fugacities. We take $\rho_V$ in the conjugacy class $(d_1,\ldots,d_k) \in \mathbb{Z}^k$, with $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k = 0$. $d$ also has a shift invariance. Hence, we can make both $c$ and $d$ start at 0. We defined $k_a := m_{a-1}(d)$, $N_a := m_{a-1}(c)$ and $n := d_1 + 1$, such that $\sum_{a=1}^n k_a = k$ and $\sum_{a=1}^n N_a = N$.

We split up the $z$ fugacities as

$$(z_1,\ldots,z_N) = (z_{1,1},\ldots,z_{1,N_1},z_{2,1},\ldots,z_{n,N_n}) . \tag{3.20}$$

This conjugacy class will correspond to rescaling the $z_i$ fugacities such that $z_{a,i} = x^a z_{\tilde{i}}$ is constant (where $\sum_{b=1}^{a-1} N_b + i = \tilde{i}$).

We split up the $w_l$ fugacities as

$$(w_1,\ldots,w_k) = (w_{1,1},\ldots,w_{1,k_1},w_{2,1},\ldots,w_{n,k_n}) . \tag{3.21}$$

This conjugacy class will correspond to rescaling the $w_l$ fugacities, such that $w_{a,l} = x^{-a} w_{\tilde{l}}$ is constant (where $\sum_{b=1}^{a-1} k_b + l = \tilde{l}$).

### 5.3.2 The superconformal indices

We first describe the Nekrasov partition function for the ADHM construction, making our conventions clear.

We follow the fugacity conventions of [126], where in equations (2.56) and (2.57) they have

$$a_i = \frac{\mu_i}{2}\,, \quad -\varepsilon_1 = i\frac{\gamma_1 - \gamma_R}{2}\,, \quad \varepsilon_2 = i\frac{\gamma_1 + \gamma_R}{2}\,, \quad m = i\frac{\gamma_2}{2}\,. \tag{3.22}$$

We then convert to our conventions via

$$z_i = e^{a_i}\,, \quad \tau = e^{\frac{\varepsilon_1+\varepsilon_2}{2}}\,, \quad x = e^{\frac{\varepsilon_1-\varepsilon_2}{2}}\,, \quad y = e^{m}\,. \tag{3.23}$$

The Nekrasov partition function for the ADHM quiver is given by

$$\mathscr{Z}_{k,N} := \sum_{\substack{\{Y_i\}\\ |\{Y_i\}|=k}} \prod_{i,j=1}^{N} \prod_{s\in Y_i} \frac{\sinh\frac{E_{ij}(s)-i(\gamma_2+\gamma_R)}{2}\sinh\frac{E_{ij}(s)+i(\gamma_2-\gamma_R)}{2}}{\sinh\frac{E_{ij}(s)}{2}\sinh\frac{E_{ij}(s)-2i\gamma_R}{2}}\,, \tag{3.24}$$

where $E_{ij}(s)$ is defined in equation (2.54) of [126].

We define the functions of a box, $s=(x,y)$, at row $x$ and column $y$ in the $i^{\text{th}}$ partition $Y_i$ of a coloured Young tableau $\vec{Y}$

$$f_{ij}(s) := -a_i(s) - l_j(s) - 1\,, \quad g_{ij}(s) := -a_i(s) + l_j(s)\,, \tag{3.25}$$

where $a_i(s) := Y_{ix} - y$ the arm length and $l_j(s) := (Y_j^{\vee})_y - x$ the leg length relative to $Y_j$. Given these functions, we can write the superconformal index of the ADHM quiver as

$$\mathscr{Z}_{k,N} = \sum_{\substack{\{Y_i\}\\ |\{Y_i\}|=k}} \prod_{i,j=1}^{N} \prod_{s\in Y_i} \mathrm{PE}\left[\tau^{g_{ij}(s)-1} x^{f_{ij}(s)} \frac{z_i}{z_j}\left(1+\tau^2-\tau(y+1/y)\right)\right]. \tag{3.26}$$

From the superconformal index we can always recover the Hilbert series by taking the coefficient of the highest power of $y$ (the coefficient of $y^{kN}$) in the expression divided by $\tau^{kN}$. This is

$$\mathrm{HS}_{k,N} = \sum_{\substack{\{Y_i\}\\ |\{Y_i\}|=k}} \prod_{i,j=1}^{N} \prod_{s\in Y_i} \frac{1}{\left(1-\frac{z_i}{z_j}\tau^{g_{ij}(s)+1}x^{f_{ij}(s)}\right)\left(1-\frac{z_j}{z_i}\tau^{1-g_{ij}(s)}x^{f_{ij}(s)}\right)}\,. \tag{3.27}$$

With our conventions now clear, we derive our expression for the linear quiver defined by the conjugacy classes of $\rho_V$ and $\rho_W$. We scale the fugacities in $\mathscr{Z}_{k,N}$ as described in subsection 5.3.1, and, using theorem 8, take the limit $x \to 0$. This gives

$$\mathscr{Z}_{\rho_V,\rho_W} = \sum_{\substack{\{Y_{a,i}\} \\ \rho_V,\rho_W}} \prod_{a,b=1}^{n} \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \prod_{\substack{s\in Y_{a,i} \\ f_{(a,i)(b,j)}(s)=a-b}} \mathrm{PE}\left[\frac{z_{a,i}}{z_{b,j}} \tau^{g_{(a,i)(b,j)}(s)-1}\left(1+\tau^2-\tau(y+1/y)\right)\right]. \tag{3.28}$$

In this expression, the $\substack{\{Y_{a,i}\} \\ \rho_V,\rho_W}$ means restricting the sum to all fixed points corresponding to the linear quiver $M(\rho_V,\rho_W)$. Note that unlike the instanton moduli space's superconformal index, a generic box from a coloured Young tableaux, associated to a fixed point within the manifold, need not contribute an individual term to the index. Indeed, if this were so then the highest power of $y$ in the index would be $kN$, which is strictly greater than the quaternionic dimension of $M(\rho_V,\rho_W)$.

Since the manifold is connected, and the contribution at each fixed point corresponds to the tangent space at that point, we would expect that the highest power of $y$ at each point would be the quaternionic dimension of the manifold $\dim_{\mathbb{H}} M(\rho_V,\rho_w) = \sum_a (k_a k_{a+1} + k_a N_a - k_a^2)$. This is a non-trivial combinatorial condition on the coloured Young tableaux that appears to be true.

The Hilbert series of the linear quiver is given by

$$\mathrm{HS}_{\rho_V,\rho_W} = \sum_{\substack{\{Y_{a,i}\} \\ \rho_V}} \prod_{a,b=1}^{n} \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \prod_{\substack{s\in Y_{a,i} \\ f_{(a,i)(b,j)}(s)=a-b}} \frac{1}{\left(1-\frac{z_{a,i}}{z_{b,j}}\tau^{g_{(a,i)(b,j)}(s)+1}\right)\left(1-\frac{z_{b,j}}{z_{a,i}}\tau^{1-g_{(a,i)(b,j)}(s)}\right)}. \tag{3.29}$$

From this we can conclude that

$$\lim_{\rho_W,x\to 0} \mathscr{Z}_{k,N} = \sum_{\rho_V} \mathscr{Z}_{\rho_V,\rho_W}. \tag{3.30}$$

This sum has multiplicity one for each $\rho_V$, but we might find that $\mathscr{Z}_{\rho_V,\rho_W} = 0$, and we may also have two equivalent linear quivers for different $\rho_V$'s, for example (1)-(1)-[1] and [1]-(1)-(1). Furthermore, we may not have a connected quiver for a specific $\rho_V$ and $\rho_W$.

### 5.3.3 An example: the $T_\sigma(SU(N))$-type theories

To highlight how this construction works, we look at the quiver in figure 5.3. We shall also find later in section 5.4 that the superconformal index of any linear quiver can be derived from the superconformal index of quivers of this form, and hence this derivation is crucial for our later derivation.

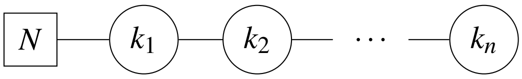


Fig. 5.3 A $T_\sigma(SU(N))$-type theory.

The quiver ranks obey $N \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n > 0$, otherwise the Higgs branch is empty. The calculation of the Hilbert series of this quiver was done via Lefschetz fixed point theorem directly in [63], we find that our analysis exactly reproduces their results for a choice of $k$ and $N$ such that it is a $T_\sigma(SU(N))$ theory.

Since there is only one flavour node, and it is on the far left node, the fixed points are coloured Young tableaux of length 1. They are given by $k_n$ lots of $(n)$, $k_{n-1}-k_n$ lots of $(n-1)$,..., $k_1-k_2$ lots of $(1)$ and $N-k_1$ lots of $\varnothing$. One then needs to sum over the Weyl group $S_N$ modulo the Weyl group of the Levi subgroup,

$$\tilde{W} := S_{k_n} \times S_{k_{n-1}-k_n} \times S_{k_1-k_2} \times S_{N-k_1}\,. \tag{3.31}$$

This is precisely the same parameterisation of the fixed points found in [63]. Define the composition of $N$

$$l_1 = k_n\,, \quad l_2 = k_{n-1}-k_n\,, \quad l_3 = k_{n-2}-k_{n-1}\,, \ldots, l_n = k_1-k_2\,, \quad l_{n+1} = N-k_1\,. \tag{3.32}$$

Thus, $\tilde{W} = \prod_{a=1}^{n+1} S_{l_a}$ and the complex dimension of the manifold is

$$d := \dim_{\mathbb{C}} \mathrm{Higgs}[T_\sigma(SU(N))] = 2\sum_{a>b} l_a l_b\,. \tag{3.33}$$

Suppose $Y_i = (a)$ and $Y_j = (b)$, then we have that $f_{ij}(s)$ is zero if and only if $b \leqslant a-1$ and $s = (1,a)$ (the last box). For this box we have that $g_{ij}(s) = -1$. Define the function on indices $h : \{1,\ldots,N\} \to \{0,1,\ldots,n\}$ via

$$i = l_1 + l_2 + \cdots + l_{h(i)} + j\,, \text{ for } j = 1,\ldots,l_{h(i)+1}\,. \tag{3.34}$$

Thus the superconformal index is

$$\mathscr{Z} = \sum_{w\in S_N/\tilde{W}} \prod_{h(i)>h(j)} w\left(\frac{\tau}{y}\frac{\left(1-\frac{z_i}{z_j}\frac{y}{\tau}\right)\left(1-\frac{z_j}{z_i}\tau y\right)}{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{z_j}{z_i}\tau^2\right)}\right)\,. \tag{3.35}$$

The Hilbert series, recovered by taking the highest power of $y$ divided by $\tau^{d/2}$, can be seen to be the same as the expression for the Hilbert series in [63]. It is a generalised Hall-Littlewood polynomial. We will discuss this is section 4.

### 5.3.4 Baryonic charges - Abelian and non-Abelian

One can compute the superconformal index not just of the theory in a vacuum, but also with a fixed background charge corresponding to some representation of the gauge group. Geometrically, this corresponds to taking sheaf cohomology valued over the locally free coherent sheaf defined by the vector bundle corresponding to this representation. We need only consider the sheafs corresponding to irreducible representations.

If we can compute our indices in the fixed point formalism for any background value of baryonic charge, then we can glue together our quivers in the sense of section 4.7. This means that we could calculate the superconformal index for many other theories. This idea on the Coulomb branch is used to compute the Hilbert series of the Sicilian theory in the paper [62]. However, at the moment we do not know how to deal with certain background charges. Specifically, the problem is that in order to evaluate the integral for general charges, one must use the Jeffrey-Kirwan procedure, this is summarised in appendix A, and this leads to different answers for different choices of Jeffrey-Kirwan parameter. There is presumably a correct choice of parameter for this procedure, but I have not yet investigated what it is. We will for now discuss the charges that one can obtain on the linear quiver by reducing from instanton moduli space.

For instanton moduli space, the gauge group is $U(k)$ and the irreducible representations are parameterised by a partition, $\lambda$, of length $< k$, and an integer, $c \in \mathbb{Z}$, corresponding to the $U(1)$ charge. If $\lambda$ is empty and $c \neq 0$, then the sheaf is invertible, i.e. a line bundle, and the index is just that of the 5 dimensional theory in the presence of a Chern-Simons term, see [19] and chapter 4. Note that no known correspondence under mirror symmetry exists for holomorphic vector bundles of rank larger than one on the Higgs branch. In fact, it is not known to the author how one would compute the Hilbert series on the Coulomb branch with non-Abelian background charge. The methods of chapter 4 certainly imply a way to approach this problem, but it is not clear whether this Hilbert series would have any physical interpretation.

We analyse the superconformal index of the instanton moduli space for arbitrary background charge via a Molien type integral (see section 2.2 of [46]),

$$\mathscr{Z}_{k,N}(c,\lambda) = \frac{\tau^{kN+k^2}}{y^{kN+k^2}} \frac{1}{k!} \prod_{l=1}^{k} \left( \oint_{\gamma_{J-K}} \frac{dw_l}{2\pi i w_l^{1+c}} \prod_{i=1}^{N} \frac{(1-yw_l z_i)(1-y/w_l z_i)}{(1-\tau w_l z_i)(1-\tau/w_l z_i)} \right) \prod_{l\neq m} (1-w_l/w_m)$$
$$s_\lambda(W^{-1}) \prod_{l,m=1}^{k} \left( \frac{(1-\tau^2 w_l/w_m)}{(1-\tau x w_l/w_m)(1-\frac{\tau}{x} w_l/w_m)} \frac{(1-yxw_l/w_m)(1-y\frac{x}{\tau^2} w_l/w_m)}{(1-\frac{\tau}{y} w_l/w_m)(1-y\tau w_l/w_m)} \right), \tag{3.36}$$

where $s_\lambda$ is a Schur polynomial.

The contour $\gamma_{J-K}$ describes the poles that one takes the residue at. If we have that $1+c+\lambda_1-N$ and $1+c+\lambda_{k-1}-N$ are negative, then the pole structure is just given by the usual bosonic poles described by coloured Young tableaux. If they are both positive, then one redefines the integration variable as $w_l \mapsto \xi_l = 1/w_l$ and then the pole structure for $\xi$ is given by coloured Young tableaux. For values where $1+c+\lambda_1-N$ is positive and $1+c+\lambda_{k-1}-N$ is negative, one must deal with the Jeffrey-Kirwan procedure[2] summarised in appendix A.

We rescale the variables by $x$, as described in section 5.3.1, and without the charges, we take the limit $x\to 0$ to obtain

$$\mathscr{Z}_{\rho_V,\rho_W} = \left(\frac{\tau}{y}\right)^{\sum_{a=1}^{n} k_a N_a + \sum_{a=1}^{n-1} k_a k_{a+1}} \left( \frac{(1-\tau^2)}{(1-\frac{\tau}{y})(1-\tau y)} \right)^{k} \prod_{a=1}^{n} \frac{1}{k_a!} \prod_{l=1}^{k_a} \left( \oint_{\gamma_{J-K}} \frac{dw_{a,l}}{2\pi i w_{a,l}} \right.$$
$$\left. \prod_{i=1}^{N_a} \frac{(1-yw_{a,l}z_{a,i})(1-y/w_{a,l}z_{a,i})}{(1-\tau w_{a,l}z_{a,i})(1-\tau/w_{a,l}z_{a,i})} \right) \prod_{a=1}^{n} \prod_{\substack{l\neq m \\ 1}}^{k_a} \frac{(1-w_{a,l}/w_{a,m})(1-\tau^2 w_{a,l}/w_{a,m})}{(1-\frac{\tau}{y}w_{a,l}/w_{a,m})(1-\tau y w_{a,l}/w_{a,m})}$$
$$\prod_{a=1}^{n-1} \prod_{l=1}^{k_a} \prod_{m=1}^{k_b} \frac{(1-yw_{a+1,m}/w_{a,l})(1-yw_{a,l}/w_{a+1,m})}{(1-\tau w_{a+1,m}/w_{a,l})(1-\tau w_{a,l}/w_{a+1,m})}. \tag{3.37}$$

Now we consider the effect of the charges on the limit. Note that

$$s_\lambda(W^{-1}) = \sum_{\sigma\in S_k/S_k^\lambda} \prod_{a,l} x^{\sigma(a)\lambda_{a,l}} w_{\sigma(a,l)}^{-\lambda_{a,l}} \prod_{(a,l)<(b,m)} \frac{1}{1-x^{\sigma(b)-\sigma(a)} w_{\sigma(a,l)}/w_{\sigma(b,m)}}. \tag{3.38}$$

[2] To be clear, one must also use the Jeffrey-Kirwan procedure in the other cases. It is just that in these cases, the contributing poles match the coloured Young tableaux structure.

This means that the power of $x$ is minimal if $\sigma(a,l)=(a,\sigma(l))$, i.e. that $\sigma \in \prod_a S_{k_a}$. The coefficient of this minimal power of $x$ is given by

$$s_\lambda(W^{-1}) \to \prod_{a=1}^{n} s_{\lambda_a}(W_a^{-1}), \tag{3.39}$$

where $\lambda_{a,l} \geqslant \lambda_{b,m}$ for all $a<b$, and $\lambda = \lambda_1 \cup \cdots \cup \lambda_n$. We conclude

$$\begin{aligned}\mathscr{Z}_{\rho_V,\rho_W}(c,\lambda) = &\left(\frac{\tau}{y}\right)^{\sum_{a=1}^{n} k_a N_a + \sum_{a=1}^{n-1} k_a k_{a+1}} \left(\frac{(1-\tau^2)}{(1-\frac{\tau}{y})(1-\tau y)}\right)^{k} \prod_{a=1}^{n} \frac{1}{k_a!} \prod_{l=1}^{k_a} \left( \oint_{\gamma_{J-K}} \frac{dw_{a,l}}{2\pi i w_{a,l}^{1+c}} \right.\\ &\left.\prod_{i=1}^{N_a} \frac{(1-yw_{a,l}z_{a,i})(1-y/w_{a,l}z_{a,i})}{(1-\tau w_{a,l}z_{a,i})(1-\tau/w_{a,l}z_{a,i})}\right) \prod_{a=1}^{n} \prod_{\substack{l\neq m\\1}}^{k_a} \frac{(1-w_{a,l}/w_{a,m})(1-\tau^2 w_{a,l}/w_{a,m})}{(1-\frac{\tau}{y}w_{a,l}/w_{a,m})(1-\tau y w_{a,l}/w_{a,m})}\\ &\prod_{a=1}^{n-1} \prod_{l=1}^{k_a} \prod_{m=1}^{k_b} \frac{(1-yw_{a+1,m}/w_{a,l})(1-yw_{a,l}/w_{a+1,m})}{(1-\tau w_{a+1,m}/w_{a,l})(1-\tau w_{a,l}/w_{a+1,m})} \prod_{a=1}^{n} s_{\lambda_a}(W_a^{-1}).\end{aligned} \tag{3.40}$$

## 5.4 Generalised Hall-Littlewood polynomials

While we have a fixed point sum for any *A*-type quiver, a generic *A*-type quiver has quite a complicated sum, corresponding to a sum of several Weyl group orbits (the Weyl group of the flavour group, possibly with some quotient by the Weyl group of a Levi subgroup of the flavour group). However, as we saw in section 5.3.3, if the *A*-type quiver has only one flavour group on a final node, then the fixed points lie in a single Weyl orbit, again with some possible quotient from a Levi subgroup. This makes these quivers much easier to work with, and their Hilbert series can be written in terms of a symmetric polynomial known as a generalised Hall-Littlewood polynomial, first defined in [94].

It turns out that we can reach all *A*-type quivers, by taking an *A*-type quiver with only one flavour group on a final node (written in this thesis as $T_\sigma(SU(N))$-type quivers), and taking certain limits of the flavour fugacities. Thus, we can write any fixed point sum as a sum over a single Weyl orbit with certain residues of fugacities (note that some of the elements of the orbit will have zero residue under some of the limits, so will not contribute). This then allows us to write the Hilbert series of an arbitrary *A*-type quiver in terms of a generalised Hall-Littlewood polynomial.

The paper [85] gave a string theory derivation of how one can reach any *A*-type quiver from a $T_\sigma(SU(N))$-type quiver, using the brane set-up described in section 2.3.2, and the papers [63, 61] showed it using the mirror dual Coulomb branch monopole formula. In this section, we show that this procedure corresponds to a procedure of gluing together the Young tableaux of the original fixed points to form a new fixed point, and provide a rigorous proof that it works. This proof allows us to rigorously state the results of [63], which write the Hilbert series of a $T_\rho^\sigma(SU(N))$ theory as a generalised Hall-Littlewood polynomial.

The procedure is more general than the work cited above, as it works for linear quivers that are not $T_\rho^\sigma(SU(N))$ theories, and also works on the level of the superconformal index, as it is a fixed point procedure.

For $n \in \mathbb{N}$, $k \in \mathbb{N}^n$, $N \in \mathbb{N}_0^n$ and $Z = (z_{ai} | a = 1, \ldots, n,\, i = 1, \ldots, N_a)$, let $\mathrm{HS}(k,N)(Z)$ be the Hilbert series of the $A_n$-type quiver with gauge ranks $k$, flavour ranks $N$ and flavour fugacities $Z$. We have the following lemma:

**Lemma 4.** *Let $k \in \mathbb{N}^n$ and $N \in \mathbb{N}_0^n$, with $N_1, N_H \neq 0$ and $N_a = 0$ for $a = 2, \ldots, H-1$. Let $Z := (z_{a,i} | i = 1, \ldots, N_a - 1$ for $a = 1, H$, $i = 1, \ldots, N_{H+1} + 1$ for $a = H+1$, $i = 1, \ldots, N_a$ for $a = H+2, \ldots n) =: (Z^{(1)}, Z^{(H)}, \ldots, Z^{(n)})$. Then*

$$\begin{aligned}
&\mathrm{HS}\big(k + (-1^H, 0^{n-H}), N + (-1^H, 1, 0^{n-H-1})\big)(Z) \\
=&\mathrm{Res}_{\omega=1} \mathrm{HS}(k,N)(Z^{(1)} \cup (\tau^{-H}\omega^{-H} z_{H+1,N_{H+1}+1}), Z^{(H)} \cup (\tau\omega z_{H+1,N_{H+1}+1}), \\
&Z^{(H+1)} \backslash (z_{H+1,N_{H+1}+1}), Z^{(H+2)}, \ldots, Z^{(n)}) \times \frac{1}{(H+1)(1-\tau^2)} \\
&\prod_{j=1}^{N_1-1} \frac{1}{1-\tau^H \frac{z_{1j}}{z_{H+1,N_{H+1}+1}}} \frac{1}{1-\tau^{2-H} \frac{z_{H+1,N_{H+1}+1}}{z_{1j}}} \prod_{j=1}^{N_H-1} \frac{1}{1-\tau \frac{z_{H,j}}{z_{H+1,N_{H+1}+1}}} \frac{1}{1-\tau \frac{z_{H+1,N_{H+1}+1}}{z_{H,j}}} \,.
\end{aligned} \tag{4.41}$$

*Similarly, for $k \in \mathbb{N}^n$ and $N \in \mathbb{N}_0^n$, with $N_1, N_2 \neq 0$, let $Z := (z_{a,i} | i = 1, \ldots, N_1 - 2$ for $a = 1$, $i = 1, \ldots, N_2 + 1$ for $a = 2$, $i = 1, \ldots, N_a$ for $a = 3, \ldots n) =: (Z^{(1)}, Z^{(2)}, \ldots, Z^{(n)})$. Then*

$$\begin{aligned}
&\mathrm{HS}\big(k + (-1, 0^{n-1}), N + (-2, 1, 0^{n-2})\big)(Z) \\
=&\mathrm{Res}_{\omega=1} \mathrm{HS}(k,N)(Z^{(1)} \cup (\tau^{-1}\omega^{-1} z_{2,N_2+1}, \tau\omega z_{2,N_2+1}), Z^{(2)} \backslash (z_{2,N_2+1}), Z^{(3)}, \ldots, Z^{(n)}) \\
&\times \frac{1}{(H+1)(1-\tau^2)} \prod_{j=1}^{N_1-1} \frac{1}{1-\tau \frac{z_{1j}}{z_{2,N_2+1}}} \frac{1}{1-\tau \frac{z_{2,N_2+1}}{z_{1j}}} \,.
\end{aligned} \tag{4.42}$$

The proof of this is in section 5.6.

We can use lemma 4 to write the Hilbert series of a linear quiver as a generalised Hall-Littlewood polynomial.

**Theorem 9.** *The Hilbert series of an A-type quiver with gauge nodes $k \in \mathbb{N}^n$, flavour nodes $N \in \mathbb{N}_0^n$ and background baryonic charge $B \in (-\mathbb{N}_0)^n$ is*

$$
\begin{aligned}
\mathrm{HS} =& \tau^{-\sum_{i=1}^{M_1}\sum_{a=1}^{n-h(i)+1} aB_a} Q_h^m\left(Z^{(1)} + (\tau+\tau^{-1})Z^{(2)} + \cdots + (\tau^{n-1}+\cdots+\tau^{1-n})Z^{(n)};\tau^2\right) \\
& \prod_{a=1}^{n} a!^{N_a} \prod_{\substack{a>b\\1}}^{n}\prod_{i=1}^{N_a}\prod_{j=1}^{N_b}\prod_{\ell=1}^{a} \frac{1}{\left(1-\tau^{2-a-b+2\ell}\frac{z_{a,i}}{z_{b,j}}\right)\left(1-\tau^{b-a+2\ell}\frac{z_{b,j}}{z_{a,i}}\right)} \\
& \prod_{a=1}^{n}\prod_{\ell=1}^{a}\prod_{\substack{i,j=1\\ i\neq j\,,\,\text{if } \ell=a-1}}^{N_a} \frac{1}{\left(1-\tau^{2-2a+2\ell}\frac{z_{a,i}}{z_{a,j}}\right)^{\ell}} \prod_{a=1}^{n}\prod_{\ell=1}^{a-1}\prod_{i,j=1}^{N_a} \frac{1}{\left(1-\tau^{2+2a-2\ell}\frac{z_{a,i}}{\_a,j}\right)^{\ell}},
\end{aligned}
\tag{4.43}
$$

*where $(M_0, M_1, \ldots, M_n) \in \mathbb{N}^{n+1}$ are the ranks of the $T_\sigma$-type theory that the linear quiver is a limit of (see equation (7.82) for the exact definition); h is defined in equation (3.34); $Q_h^m$ is the generalised Hall-Littlewood polynomial defined by*

$$
\begin{aligned}
Q_h^m(Z;\tau^2) :=& \sum_{w\in S_{M_0}/\tilde{W}} w\left(\prod_{i=1}^{M_1} z_i^{m_{h(i)}} \prod_{\substack{i>j\\ h(i)=h(j)}} \left(1-\frac{z_i}{z_j}\right)\left(1-\tau^2\frac{z_j}{z_i}\right)\prod_{i>j}\frac{1-\tau^2\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}}\right) \\
:=& \frac{1}{|\tilde{W}|}\sum_{w\in S_{M_0}} w\left(\prod_{i=1}^{M_1} z_i^{m_{h(i)}} \prod_{\substack{i>j\\ h(i)=h(j)}} \left(1-\frac{z_i}{z_j}\right)\left(1-\tau^2\frac{z_j}{z_i}\right)\prod_{i>j}\frac{1-\tau^2\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}}\right);
\end{aligned}
\tag{4.44}
$$

*and m is the mirror dual magnetic charge. It is defined by the baryonic charge B via*

$$
m_a := \sum_{b=1}^{n-a} B_b\,. \tag{4.45}
$$

*We can extend this to the superconformal index as follows*

$$
\begin{aligned}
\mathscr{Z}(B) =& \tilde{Q}_h^0\left(Z^{(1)} + (\tau+\tau^{-1})Z^{(2)} + \cdots + (\tau^{n-1}+\cdots+\tau^{1-n})Z^{(n)};\tau^2,y\right) \\
& \prod_{a=1}^{n} a!^{N_a} \prod_{\substack{a>b\\1}}^{n}\prod_{i=1}^{N_a}\prod_{j=1}^{N_b}\prod_{\ell=1}^{a} \frac{\left(1-y\tau^{1-a-b+2\ell}\frac{z_{a,i}}{z_{b,j}}\right)\left(1-y\tau^{-1+b-a+2\ell}\frac{z_{b,j}}{z_{a,i}}\right)}{\left(1-\tau^{2-a-b+2\ell}\frac{z_{a,i}}{z_{b,j}}\right)\left(1-\tau^{b-a+2\ell}\frac{z_{b,j}}{z_{a,i}}\right)} \\
& \prod_{a=1}^{n}\prod_{\ell=1}^{a}\prod_{\substack{i,j=1\\ i\neq j\,,\,\text{if } \ell=a-1}}^{N_a} \left(\frac{1-y\tau^{1-2a+2\ell}\frac{z_{a,i}}{z_{a,j}}}{1-\tau^{2-2a+2\ell}\frac{z_{a,i}}{z_{a,j}}}\right)^{\ell} \prod_{a=1}^{n}\prod_{\ell=1}^{a-1}\prod_{i,j=1}^{N_a} \left(\frac{1-y\tau^{1+2a-2\ell}\frac{z_{a,i}}{\_a,j}}{1-\tau^{2+2a-2\ell}\frac{z_{a,i}}{z_{a,j}}}\right)^{\ell},
\end{aligned}
\tag{4.46}
$$

*where d is the quaternionic dimension of the linear quiver and*

$$\tilde{Q}_h^m(Z;\tau^2,y) := \sum_{w\in S_{M_0}/\tilde{W}} w\left(\prod_{i=1}^{M_1} z_i^{m_{h(i)}} \prod_{i>j} \frac{\tau}{y}\frac{1-\tau^2\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}}\frac{1-\frac{y}{\tau}\frac{z_i}{z_j}}{1-\tau y\frac{z_i}{z_j}} \prod_{\substack{i>j\\ h(i)=h(j)}} \frac{y}{\tau}\frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\tau^2\frac{z_j}{z_i}\right)}{\left(1-\tau y\frac{z_j}{z_i}\right)\left(1-\frac{y}{\tau}\frac{z_i}{z_j}\right)}\right). \tag{4.47}$$

We prove this in section 5.7.

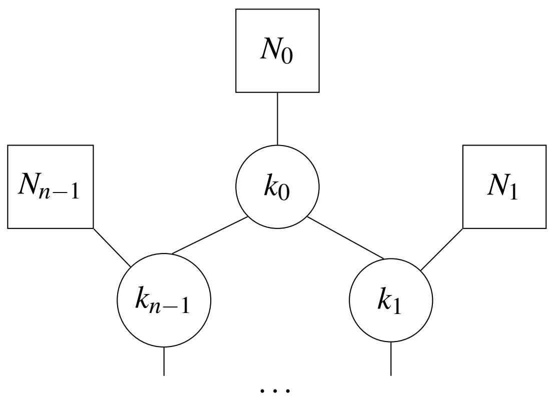

Fig. 5.4 A generic $\hat{A}_n$-type quiver.

## 5.5 $\hat{A}_n$-quivers

The construction of $A_n$-type quivers in section 5.2 can be easily adapted to give us the fixed points of generic $\hat{A}_n$-type quivers. The associated variety to a $\hat{A}_n$ quiver is the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_n$, [135].

We restrict $t_1$ in equation (2.7) to lie in a finite cyclic group, $t_1 \in \mathbb{Z}_n \subset \mathbb{C}^\times$. If we do this, then we have the same argument as for the linear quiver. $\mathbb{C}^k$ is split into $n$ pieces, $V_0,\ldots,V_{n-1}$, with $V_a$ having weight $t_1^a$, and similarly $\mathbb{C}^N$ splits into $n$ pieces $W_0,\ldots,W_{n-1}$. As before we define

$$k_a := \dim V_a\,, \quad N_a := \dim W_a\,. \tag{5.48}$$

The difference now is the periodicity, namely

$$\begin{aligned} X &: V_{n-1} \to V_0\,, \\ \tilde{X} &: V_0 \to V_{n-1}\,. \end{aligned} \tag{5.49}$$

This periodicity is important for identifying which $\hat{A}_n$-type quiver a particular fixed point lies in given the choice of $\rho_W$.

The superconformal index for $\hat{A}_n$ is

$$\mathscr{Z}_{\rho_V,\rho_W}(\hat{A}_n) = \sum_{\substack{\{Y_{a,i}\}\\ \rho_V,\rho_W}} \prod_{a,b=1}^{n} \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \prod_{\substack{s\in Y_{a,i}\\ f_{(a,i)(b,j)}(s)\equiv a-b\,(\mathrm{mod}\, n)}} \mathrm{PE}\left[\frac{z_{ai}}{z_{bj}} \tau^{g_{(a,i),(b,j)}(s)-1}(1+\tau^2-\tau(1/y+y))\right]. \tag{5.50}$$

In this expression, the $\substack{\{Y_{a,i}\}\\ \rho_V,\rho_W}$ means restricting the sum to all fixed points corresponding to the affine quiver fixed by $\rho_V$ and $\rho_W$.

## 5.6 Proof of lemma 4

*Proof.* We define $h := N_1$.

In the first case, we map the flavour fugacities $z_{1,h} \mapsto \tau^{-H}\omega^{-H} z_{H+1,N_{H+1}+1}$ and $z_{H,N_H} \mapsto \tau\omega z_{H+1,N_{H+1}+1}$. We want the residue of the Hilbert series of the original quiver at $\omega = 1$.

Due to the other flavour fugacities having generic values, the only possible poles in $\omega - 1$ in the Hilbert series, when expressed as a sum over coloured Young tableaux $\{\vec{Y}\}$, are the ones from the interactions between $Y_{1,h}$ and $Y_{H,N_H}$. These are

$$\prod_{\substack{s\in Y_{1,h}\\ f_{1,hH,N_H}(s)=1-H}} \frac{1}{\left(1-\omega^{-H-1}\tau^{g_{1,hH,N_H}(s)-H}\right)\left(1-\omega^{H+1}\tau^{2+H-g_{1,hH,N_H}(s)}\right)} \prod_{\substack{s\in Y_{H,N_H}\\ f_{H,N_H 1,h}(s)=H-1}} \frac{1}{\left(1-\omega^{H+1}\tau^{H+2+g_{H,N_H 1,h}(s)}\right)\left(1-\omega^{-H-1}\tau^{-H-g_{1,hH,N_H}(s)}\right)}. \tag{6.51}$$

So we see that we get a non-zero residue at $\omega = 1$ if one of the four possibilities happen

1. $s \in Y_{1,h}$ such that $f_{1,hH,N_H}(s) = 1-H$ and $g_{1,hH,N_H}(s) = H$.
2. $s \in Y_{1,h}$ such that $f_{1,hH,N_H}(s) = 1-H$ and $g_{1,hH,N_H}(s) = H+2$.
3. $s \in Y_{H,N_H}$ such that $f_{H,N_H 1,h}(s) = H-1$ and $g_{H,N_H 1,h}(s) = -H-2$.
4. $s \in Y_{H,N_H}$ such that $f_{H,N_H 1,h}(s) = H-1$ and $g_{H,N_H 1,h}(s) = -H$.

Now because of the structure of the quiver we know that $Y_{1,h} = (m)$ for some $m \in \mathbb{N}_0$ and $Y^\vee_{H,N_h} = (H^i, \lambda^\vee)$ for $\lambda \in \mathscr{P}$ with $\ell(\lambda) < H$ and $i \in \mathbb{N}_0$. We will use this to show that only possible non-zero residue is the 4th case. For cases 1 and 2, note that

$$g_{1,hH,N_H}(s) = -a_{1,h}(s) + \ell_{H,N_H}(s) \leqslant \ell_{H,N_H}(s) \leqslant H-1 < H < H+2\,. \qquad (6.52)$$

So, these poles cannot be achieved. In case 3 and 4, we have

$$f_{H,N_H\,1,h}(s) = -1 - a_{H,N_H}(s) - \ell_{1,h}(s) \leqslant -1 - \ell_{1,h}(s) \leqslant H-1\,, \qquad (6.53)$$

with the left inequality saturated for $a_{H,N_H}(s) = 0$, and the right inequality saturated if $\ell_{1,h}(s) = -H$. This corresponds to taking $s$ to be the bottom right of $Y_{H,N_H}$ and having $i > m$. If this is the case, then we have

$$g_{H,N_H\,1,h}(s) = -a_{H,N_H}(s) + \ell_{1,h}(s) = -H\,. \qquad (6.54)$$

So, pole 4 is the only contribution. If we have a $\{\vec{Y}\}$ that does not satisfy condition 4, then it has zero residue and is not a fixed point of the final quiver.

In the second set up for the quiver ($H = 1 \iff N_2 \neq 0$), we take the scaling $z_{1,h} \mapsto \tau^{-1}\omega^{-1}z_{2,N_2+1}$ and $z_{1,1} \mapsto \tau\omega z_{2,N_2+1}$. The same pole structure holds, we just need to relabel $H, N_H$ as $1,1$.

The fixed points of the quiver we reach when we limit $\omega$ to one are given by $\{\vec{Y}\}\setminus\{Y_{1,h}, Y_{H,N_H}\}\cup\{Y_{H+1,N_{H+1}+1}(Y_{1,h}, Y_{H,N_H})\}$, with $Y_{H+1,N_{H+1}+1}(Y_{1,h}, Y_{H,N_H}) := \left(\left(Y_{H,N_H} - 1^H\right)^\vee \cup Y_{1,h}^\vee\right)^\vee$, where $\rho - (1^{\ell(\rho)}) := (\rho_1 - 1, \ldots, \rho_{\ell(\rho)} - 1)$. One can see that this tableau is in fact fixed by the requirement that we make the linear quiver we want to. Conversely, given a fixed point $\{Y'\}$ of the final quiver, we can reconstruct $Y_{1,h}$ and $Y_{H,N_H}$ from $Y_{H+1,N_{H+1}}$. For example, if $Y_{H+1,N_{H+1}} = \varnothing$, then $Y_{1,h} = \varnothing$ and $Y_{H,N_H} = (1^H)$. So, one sees that all fixed points of the final quiver can be constructed from a subset of the original quiver's fixed points.

What we now must do is actually compute the residue at $\omega = 1$ for the coloured Young tableaux, and see what we get.

The terms corresponding to the pair of tableaux $(Y_{a,i}, Y_{b,j})$ for $(a,i),(b,j) \neq (1,h),(H,N_H)$ are the same for both the initial and final quiver. The terms that will change are

1. $(Y_{1,h}, Y_{b,j})$,
2. $(Y_{b,j}, Y_{1,h})$,
3. $(Y_{H,N_H}, Y_{b,j})$,
4. $(Y_{b,j}, Y_{H,N_H})$,
5. $(Y_{H,N_H}, Y_{1,h})$, and

6. $(Y_{1,h}, Y_{H,N_H})$.

We will show that the terms for 1, 2, 3 and 4 are captured by

$$\prod_{\substack{s\in Y_{H+1,N_{H+1}+1}\\ f_{H+1,N_{H+1}+1\,b,j}(s)=H+1-b}} \frac{1}{1-\frac{z_{H+1,N_{H+1}+1}}{z_{b,j}}\tau^{1+g_{H+1,N_{H+1}+1\,b,j}(s)}}\frac{1}{1-\frac{z_{b,j}}{z_{H+1,N_{H+1}+1}}\tau^{1-g_{H+1,N_{H+1}+1\,b,j}(s)}}$$
$$\prod_{\substack{s\in Y_{b,j}\\ f_{b,jH+1,N_{H+1}+1}(s)=b-H-1}} \frac{1}{1-\frac{z_{b,j}}{z_{H+1,N_{H+1}+1}}\tau^{1+g_{b,jH+1,N_{H+1}+1}(s)}}\frac{1}{1-\frac{z_{H+1,N_{H+1}+1}}{z_{b,j}}\tau^{1-g_{b,jH+1,N_{H+1}+1}(s)}}, \tag{6.55}$$

except for some terms that factor out of the sum over fixed points. We will show the terms 5. and 6. factor out of the sum over fixed points. By construction, we have

$$\{s\in Y_{H+1,N_{H+1}+1}\}\cong\{s\in Y_{1,h}\}\sqcup\{s\in Y_{H,1}-(1^H)\}. \tag{6.56}$$

1. We have that for $s=(1,y)\in Y_{1,h}$,

$$\begin{aligned} 1-b=f_{1,hb,j}(s)&=y-m-(Y^\vee_{bj})_y\,,\\ g_{1,hb,j}(s)&=y-m+(Y^\vee_{bj})_j-1\,. \end{aligned} \tag{6.57}$$

2. $s=(x,y)\in Y_{bj}$. For $y\leqslant m$ have

$$\begin{aligned} f_{b,j\,1,h}(s)&=x+y-(Y_{bj})_x-2\\ &\leqslant b+y-(Y_{bj})_x-2\leqslant b-2<b-1\,. \end{aligned} \tag{6.58}$$

For $y>m$ have

$$\begin{aligned} f_{b,j\,1,h}(s)&=x+y-(Y_{bj})_x-1\\ &\leqslant b+y-(Y_{bj})_x-1\leqslant b-1\,, \end{aligned} \tag{6.59}$$

and so $x=b$ and $y=(Y_{bj})_b$. So, we only have a contribution if $(Y_{bj})_b>m$. It contributes

$$g_{b,j\,1,h}(s)=-b\,. \tag{6.60}$$

3. $s=(x,y)\in Y_{H,N_H}$

$$\begin{aligned} H-b=f_{H,N_H\,b,j}(s)&=x+y-i-\lambda_x-(Y^\vee_{bj})_y-1\,,\\ g_{H,N_H\,b,j}(s)&=1+y-i-\lambda_x+(Y^\vee_{bj})_y-x\,. \end{aligned} \tag{6.61}$$

4. Let $s=(x,y)\in Y_{b,j}$. For $y\leqslant i$ we have

$$\begin{aligned} f_{b,jH,N_H}(s) &= x+y-H-(Y_{bj})_x-1 \\ &\leqslant b+y-(Y_{bj})_x-H-1<b-H\,. \end{aligned} \tag{6.62}$$

   So, we have no contributions for $y\leqslant i$. For $y>i$,

$$\begin{aligned} b-H=f_{b,jH,N_H}(s) &= x+y-(Y_{bj})_x-\lambda^{\vee}_{y-i}\,, \\ g_{b,jH,N_H}(s) &= y-(Y_{bj})_x+\lambda^{\vee}_{y-i}-x\,. \end{aligned} \tag{6.63}$$

5. For $(Y_{1,h},Y_{H,N_H})$, we know $Y_{1,h}=(m)$ and $Y^{\vee}_{H,N_H}=(H^i)\cup\tau$ with $\tau_1<H$, so, we have that $s=1,\dots,m\in Y_{1,h}$ and that $f_{1,hH,N_H}(s)=-1+s-m-H+1=s-m-H\leqslant -H$. Thus, there are no contributions from this pairing.

6. The final pairing to consider is $(Y_{H,N_H},Y_{1,h})$. We have already worked out which box contributes and need only calculate

$$\mathrm{Res}_{\omega\to 1}\frac{1}{(1-\omega^{H+1}\tau^2)(1-\omega^{-H-1})}=\frac{1}{(H+1)(1-\tau^2)}\,. \tag{6.64}$$

Now we look at the terms arising from $(Y_{bj},Y_{H+1,N_{H+1}})$ and $(Y_{H+1,N_{H+1}},Y_{bj})$. They are of four types:

i $s=(x,y)\in Y_{H+1,N_{H+1}+1}$ with $x=H+1$, so $y\leqslant m$. This is the box that comes from $Y_{1,h}$.

$$\begin{aligned} H+1-b=f_{H+1,N_{H+1}+1\,b,j}(s) &= H+y-m-(Y^{\vee}_{bj})_y\,, \\ g_{H+1,N_{H+1}+1\,b,j}(s) &= y-m+(Y^{\vee}_{bj})_y-H-1\,. \end{aligned} \tag{6.65}$$

ii $s=(x,y)\in Y_{bj}$ with $y<i$.

$$f_{b,jH+1,N_{H+1}+1}(s)=\begin{cases} -2+x+y-(Y_{bj})_x-H\,, & y\leqslant m\,, \\ -1+x+y-(Y_{bj})_x-H\,, & m<y<i\,. \end{cases} \tag{6.66}$$

   Since $x+y-(Y_{bj})_x-H\leqslant b-H$, there can only be a solution if $m<y<i$, $x=b$ and $y=(Y_{bj})_b$. There is, in fact, at most one contribution. It happens precisely when $(Y_{bj})\in(m,i)$. It contributes

$$g_{b,jH+1,N_{H+1}+1}(b,(Y_{bj})_b)=H-b\,. \tag{6.67}$$

iii $s=(x,y)\in Y_{H+1,N_{H+1}+1}$ with $x\leqslant H$.

$$\begin{aligned} H+1-b=f_{H+1,N_{H+1}+1\,b,j}(s) &= -1+x+y-i-\lambda_x-(Y^\vee_{bj})_y\,, \\ g_{H+1,N_{H+1}+1\,b,j}(s) &= y-i-\lambda_x+(Y^\vee_{bj})_y-x\,. \end{aligned} \tag{6.68}$$

iv $s=(x,y)\in Y_{bj}$ with $y\geqslant i$.

$$\begin{aligned} b-H-1=f_{b,j\,H+1,N_{H+1}+1}(s) &= x+y-(Y_{bj})_x-\lambda^\vee_{y-i+1}-1\,, \\ g_{b,j\,H+1,N_{H+1}+1}(s) &= y-(Y_{bj})_x+\lambda^\vee_{y-i+1}-x\,. \end{aligned} \tag{6.69}$$

We now show that the contributions 1,2,3,4 are equal to i,ii,iii,iv, up to some universal factors. Bearing in mind the scalings $z_{1,h}=\tau^{-H}z_{H+1,N_{H+1}+1}$ and $z_{H,N_H}=\tau z_{H+1,N_{H+1}+1}$, 1 and i directly match. The rest are slightly trickier.

2 and ii match only if $(Y_{bj})_b\leqslant i$, otherwise 2 has an unmatched contribution.

3 and iii match, except for the terms $s=(x,y)\in Y_{H,N_H}$ such that $a_{HN_H}(s)=0$, which happens when $y=i+\lambda_x$, as we shaven these boxes off. The number of extra terms for 3 is given by the number of solutions to the equation

$$x-(Y^\vee_{bj})_{i+\lambda_x}=H+1-b\,,\ \text{for}\ x\in\{1,\dots,H\}\,. \tag{6.70}$$

For $(x,y)$ a contribution for 4, we have that $(x,y-1)$ is an equal contribution for iv (noting that for 4, $y>i$; for iv, $y\geqslant i$; and $i\geqslant 1$). So, we have a match, except for the terms $(x,y)\in Y_{bj}$ for iv with $y=(Y_{bj})_x$. The number of such terms is given by the number of solutions to the equation

$$x-\lambda^\vee_{(Y_{bj})_x-i+1}=b-H\,,\ \text{for}\ x\in\{1,\dots,(Y^\vee_{bj})_i\}\,. \tag{6.71}$$

All three types of extra terms contribute the same factor to the Hilbert series, namely

$$\text{extra term contribution}=\frac{1}{1-\frac{z_{bj}}{z_{H+1N_{H+1}+1}}\tau^{1+H-b}}\,\frac{1}{1-\frac{z_{H+1N_{H+1}+1}}{z_{bj}}\tau^{1-H+b}}\,. \tag{6.72}$$

To further simplify equations (6.70) and (6.72), we define the partition $\mu$ via its entries

$$\mu_a:=(Y_{bj})_a-i+1\,,\ \text{if}\ (Y_{bj})_a\geqslant i\,. \tag{6.73}$$

With this, the extra term from 2 and ii is

$$\mu_b \neq 0 \big( \iff \ell(\mu) = b \big) \, ; \tag{6.74}$$

(6.70) becomes

$$\mu^{\vee}(\lambda(x)+1) = x+b-H-1 \, , \text{ for } x \in \{1,\dots,H\} \, ; \tag{6.75}$$

and (6.72) becomes

$$\lambda^{\vee}(\mu(x)) = x-b+H \, , \text{ for } x \in \{1,\dots,\ell(\mu)\} \, . \tag{6.76}$$

We will show[3] that the number of solutions to (6.74) + the number of solutions to (6.75) is one greater than the number of solutions to (6.76) for $b=1$ or $H$ and exactly equal for $b>H$.

For $b=1$, $\mu=\varnothing$ or $\ell(\mu)=1$. It is clear in both cases.

For $b \geqslant H$, define

$$\begin{aligned} \chi_{\lambda,\mu} &:= |\{x|\lambda^{\vee}(\mu(x)) = x-b+H \, , x \in \{1,\dots,\ell(\mu)\}\}| \, , \\ \psi_{\lambda,\mu} &:= |\{x|\mu^{\vee}(\lambda(x)+1) = x+b-H-1 \, , x \in \{1,\dots,H\}\}| \, . \end{aligned} \tag{6.77}$$

We see that

$$\begin{aligned} \chi_{\lambda,\mu} &= |\{x|\lambda(x-b+H+1)+\frac{1}{2} < \mu(x) < \lambda(x-b+H)+\frac{1}{2} \, , x \in \{1,\dots,\ell(\mu)\}\}| \, , \\ \psi_{\lambda,\mu} &= |\{x|\mu(x) < \lambda(x-b+H)+\frac{1}{2} < \mu(x-1) \, , x \in \{1+b-H,\dots,b\}\}| \, . \end{aligned} \tag{6.78}$$

We define the merger of the parts of $\mu(i)$ and $\lambda(j)+\frac{1}{2}$ in descending order, where we define $\lambda(H-b+1),\dots,\lambda(0) := \infty$, as the list $\nu$. Define the statistic $\sigma$ via $\sigma(0) := 0$; $\sigma(k) := \sigma(k-1)-1$ if $\nu(k)$ comes from $\mu$; and $\sigma(k) := \sigma(k-1)+1$ if $\nu(k)$ comes from $\lambda$. Then we want

$$\begin{aligned} \chi_{\lambda,\mu} &= |\{k|\nu(k) \text{ comes from } \mu \, , \sigma(k) = 0\}| \, , \\ \psi_{\lambda,\mu} &= |\{k|\nu(k) \text{ comes from } \lambda \, , \sigma(k) = 1 \, , k > 1 \text{ if } b > H\}| \, . \end{aligned} \tag{6.79}$$

We need the condition in $\psi_{\lambda,\mu}$, $k>1$ for $b>H$, because we do not want to count the infinity term. Up to this subtlety, $\chi_{\lambda,\mu}$ is the number of steps in $\sigma$ from 1 to 0, and $\psi_{\lambda,\mu}$ is the number of steps in $\sigma$ from 0 to 1. For $b=H$, if $\sigma(H+\ell(\mu)) \leqslant 0$, then $\psi_{\lambda,\mu} = \chi_{\lambda,\mu}$, and if

[3] The author is grateful to [111] for providing the solution to this part of the problem

$\sigma(H+\ell(\mu))>0$, then $\psi_{\lambda,\mu}=\chi_{\lambda,\mu}+1$. For $b>H$, if $\sigma(b+\ell(\mu))\leqslant 0$, then $\psi_{\lambda,\mu}=\chi_{\lambda,\mu}-1$, and if $\sigma(b+\ell(\mu))>0$, then $\psi_{\lambda,\mu}=\chi_{\lambda,\mu}$.

Now we see that $\sigma(b+\ell(\mu))=b-\ell(\mu)$ and so the result follows for the cotangent spaces. □

## 5.7 Proof of theorem 9

*Proof.* Upon each use of lemma 4, we start from two different possible set ups. One is the quiver with $k=(k_1,k_2,\dots,k_H,k_{H+1},\dots)$ and $N=(h,0^{H-2},N_H,N_{H+1},\dots)$ with $H>2$, which we send to $k=(k_1-1,k_2-1,\dots,k_H-1,k_{H+1},\dots)$ and $N=(h-1,0^{H-2},N_H-1,N_{H+1}+1,\dots)$. The other set up is starting from $k=(k_1,k_2,\dots)$ and $N=(N_1,N_2,\dots)$, for $N_2\neq 0$, which is sent to $k=(k_1-1,k_2,\dots)$ and $N=(N_1-2,N_2+1,\dots)$.

We show this generates all possible linear quivers. Suppose we wanted the linear quiver $k=(k_1,\dots,k_n)$, $N=(N_1,\dots,N_n)$. We start with the quiver

$$k=(M_1,\dots,M_n)\,,\quad N=(M_0,0,\dots,0)\,, \tag{7.80}$$

with $M_0\geqslant M_1\geqslant\cdots\geqslant M_n$. We do

$$\begin{aligned}
&k=(M_1,\dots,M_n)\,,\quad N=(M_0,0,\dots,0)\\
&\mapsto k=(M_1-1,M_2,\dots,M_n)\,,\quad N=(M_0-2,1,\dots,0)\\
&\mapsto\cdots\mapsto k=(M_1-N_n,M_2,\dots,M_n)\,,\quad N=(M_0-2N_n,N_n,0,\dots,0)\\
&\mapsto k=(M_1-N_n-1,M_2-1,\dots,M_n)\,,\quad N=(M_0-2N_n-1,N_n-1,1,\dots,0)\\
&\mapsto\cdots\mapsto k=(M_1-2N_n,M_2-N_n,\dots,M_n)\,,\quad N=(M_0-3N_n,0,N_n,\dots,0)\\
&\mapsto\cdots\mapsto k=(M_1-(n-1)N_n,M_2-(n-2)N_n,\dots,M_n)\,,\quad N=(M_0-nN_n,0,\dots,N_n)\\
&\mapsto k=(M_1-(n-1)N_n-1,M_2-(n-2)N_n,\dots,M_n)\,,\quad N=(M_0-nN_n-2,1,\dots,N_n)\\
&\mapsto\cdots\mapsto k=(k_1,\dots,k_n)\,,\quad N=(N_1,\dots,N_n)\,.
\end{aligned} \tag{7.81}$$

So, we see that

$$\begin{aligned}
M_0&=\sum_{i=1}^{n} iN_i\,,\\
M_i&=k_i+\sum_{j=i+1}^{n}(j-i)N_j\,.
\end{aligned} \tag{7.82}$$

We show theorem 5.7 for the $T_\sigma(SU(N))$-type quiver with ranks $M \in \mathbb{N}^{n+1}$ defined in equation (7.82). From section 5.3.3, we know that the Hilbert series is given by

$$
\begin{aligned}
\mathrm{HS} &= \sum_{w\in S_{M_0}/\tilde{W}} w\left(\prod_{i=1}^{M_1} \tau^{-\sum_{a=1}^{n-h(i)+1} aB_a} z_i^{\sum_{a=1}^{n-h(i)+1} B_a} \prod_{h(i)>h(j)} \frac{1}{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{z_j}{z_i}\tau^2\right)}\right)\\
&= \prod_{i=1}^{M_1} \tau^{-\sum_{a=1}^{n-h(i)+1} aB_a} \prod_{i,j=1}^{M_0} \frac{1}{1-\tau^2\frac{z_i}{z_j}} \sum_{w\in S_{M_0}/\tilde{W}} w\left(\prod_{i=1}^{M_1} z_i^{\sum_{a=1}^{n-h(i)+1} B_a} \prod_{i>j} \frac{1-\tau^2\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}}\right)\\
&\quad \prod_{\substack{i>j\\ h(i)=h(j)}} \left(1-\frac{z_i}{z_j}\right)\left(1-\tau^2\frac{z_j}{z_i}\right)\\
&= Q_h^n(Z;\tau^2)\tau^{-\sum_{i=1}^{M_1}\sum_{a=1}^{n-h(i)+1} aB_a} \prod_{i,j=1}^{M_0} \frac{1}{1-\tau^2\frac{z_i}{z_j}}\,.
\end{aligned}
\tag{7.83}
$$

This proves the theorem in this case.

We now consider a general $A_n$-type quiver, with gauge and flavour ranks $(k,N)\in\mathbb{N}^n\times\mathbb{N}_0^n$ and $B\in-\mathbb{N}_0^n$.

All the poles that we take residues at when applying lemma 4 are contained in the product $\prod_{i,j=1}^{M_0}\frac{1}{1-\tau^2\frac{z_i}{z_j}}$. This means that all that happens to the generalised Hall-Littlewood polynomial is that the fugacities are scaled. The scalings of the fugacities from the quiver with ranks $M$ to reach the fugacities for the final quiver are

$$
\begin{aligned}
&z_{N_1+2N_2+\cdots+(r-1)N_r+aN_r+i} = \tau^{1-r+2a} z_{r,i}\,,\\
&r=1,\ldots,n\,,\ i=1,\ldots,N_r\,,\ a=0,1\ldots,r-1\,.
\end{aligned}
\tag{7.84}
$$

Defining $\alpha_i(\tau)=\tau^i+\tau^{i-2}+\cdots+\tau^{-i}$, we can write this as

$$
Z\mapsto (Z^{(1)},\alpha_1(\tau)Z^{(2)},\ldots,\alpha_{n-1}(\tau)Z^{(n)})\,. \tag{7.85}
$$

Now we take the residue at the product over roots. Writing Res for the iterated residues as defined by (7.81), we find

$$
\begin{aligned}
\mathrm{Res}\prod_{i,j=1}^{M_0}\frac{1}{1-\tau^2\frac{z_i}{z_j}} &= \prod_{a=1}^{n} a!^{N_a} \prod_{\substack{a>b\\1}}^{n}\prod_{i=1}^{N_a}\prod_{j=1}^{N_b}\prod_{\ell=1}^{a} \frac{1}{\left(1-\tau^{2-a-b+2\ell}\frac{z_{a,i}}{z_{b,j}}\right)\left(1-\tau^{b-a+2\ell}\frac{z_{b,j}}{z_{a,i}}\right)}\\
&\prod_{a=1}^{n}\prod_{\ell=1}^{a}\prod_{\substack{i,j=1\\ i\neq j,\ \text{if } \ell=a-1}}^{N_a} \frac{1}{\left(1-\tau^{2-2H+2\ell}\frac{z_{a,i}}{z_{a,j}}\right)^{\ell}} \prod_{a=1}^{n}\prod_{\ell=1}^{a-1}\prod_{i,j=1}^{N_a} \frac{1}{\left(1-\tau^{2+2a-2\ell}\frac{z_{a,i}}{\_a,j}\right)^{\ell}}\,.
\end{aligned}
\tag{7.86}
$$

Finally, we wish to extend these results to the superconformal index. We write the superconformal index for the $T_\sigma(SU(N))$-type quiver with ranks $M \in \mathbb{N}^{n+1}$ defined in equation (7.82):

$$\begin{aligned}
\mathscr{Z} &= \sum_{w \in S_N/\tilde{W}} \prod_{h(i)>h(j)} w\left( \frac{\tau}{y} \frac{1-\frac{y}{\tau}\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}} \frac{1-\tau y \frac{z_j}{z_i}}{1-\tau^2 \frac{z_j}{z_i}} \right) \\
&= \prod_{i,j=1}^{M_0} \frac{1-\tau y \frac{z_i}{z_j}}{1-\tau^2 \frac{z_i}{z_j}} \sum_{w \in S_{M_0}/\tilde{W}} w\left( \prod_{i>j} \frac{\tau}{y} \frac{1-\tau^2 \frac{z_i}{z_j}}{1-\frac{z_i}{z_j}} \frac{1-\frac{y}{\tau}\frac{z_i}{z_j}}{1-\tau y \frac{z_i}{z_j}} \right. \\
&\left. \prod_{\substack{i>j \\ h(i)=h(j)}} \frac{y}{\tau} \frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\tau^2 \frac{z_j}{z_i}\right)}{\left(1-\tau y \frac{z_j}{z_i}\right)\left(1-\frac{y}{\tau}\frac{z_i}{z_j}\right)} \right).
\end{aligned} \tag{7.87}$$

Since $y$ is generic and adds no poles, it is a simple matter of dressing each term with the appropriate fermion piece.

□

# Chapter 6

# Infinite rank limits

This chapter is based on the author's own currently unpublished work.

As discussed in section 2.4.1, we expect the Hilbert series at generic charge and the superconformal index to be characters of the quantisation of the Poisson algebra. In this chapter, we study the representation theory of this quantum group in a simple example. We work with quivers whose quantised Poisson algebra is a Yangian. This requires taking an infinite rank limit of the quiver theory. We find the Yangian character decomposition of the Hilbert series for generic background magnetic charge.

We shall define the well-known Poincaré polynomial of a resolved quiver variety. We find that it is related to both our superconformal index and Hilbert series. For a particular quiver, its Poincaré polynomial is a certain limit of the superconformal index, while the Hilbert series of the quivers with Yangian quantised symmetry contains the generating function for Poincaré polynomials.

The two main results of this section are:

1. The Hilbert series of an infinite rank limit of balanced quivers, for a certain background charge, gives the generating function of finite rank Poincaré polynomials;

2. The space of sections of line bundles of the infinite rank limit of balanced quivers for any background charge is the classical limit of a tensor product of Kirillov-Reshitikhin modules times the classical limit of the Yangian.

The second result is a generalisation of a result by Mozgovoy, giving the generating function of Poincaré polynomails as the partition function of a spin chain. It is a consequence of the Yangian symmetry of the infinite rank Coulomb branch Poisson algebra.

In section 6.1, we define some notation that we use throughout the chapter. In section 6.2, we define the Poincaré polynomial and the infinite rank limit of the Hilbert series of the Coulomb branch. We prove that this reproduces the generating function for Poincaré polynomials. In section 6.3, we show that the theory has Yangian symmetry. We define a fermionic form and show that this form corresponds to the graded chararacter of the classical limit of Yangian modules. We state result 2, about the infinite rank Coulomb branch Hilbert series for generic charge. In section 6.4, we restrict ourselves from balanced $ADE$-type quivers to balanced $A$-type quivers. In section 6.4.1, we compare our result for the Poincaré polynomial of $A$-type quivers to the results of [171], which contains another expression for the Poincaré polynomial. We prove that they are equivalent using manipulations of a spin chain. In section 6.4.2, we work through the infinite rank limit on the mirror dual side. We see that the chiral ring in this limit is a semiclassical limit of the $A$-type Yangian, as expected. Section 6.5 contains the proof of lemma 6, and section 6.6 contains the proof of theorem 12 (this is result 2).

## 6.1 Notation

We briefly define some notation used throughout this chapter.

Starting from an arbitrary quiver $\Gamma$, we write, for $e \in E$ an arrow in the quiver, $i(e)$ for the base of the arrow and $f(e)$ for the target. Both are in $V$, the set of vertices. $(-,-)$ is the standard non-degenerate bilinear form on $\mathfrak{h}^\vee$, the dual of the Cartan subalgebra of the Lie algebra, $\mathfrak{g}$, defined by the underlying quiver. $T$ is the $\mathbb{Z}$-valued quadratic form on the root lattice (the Tits form), given by $T(\alpha) = \frac{1}{2}(\alpha, \alpha)$. We define $Q$ to be the root lattice, $Q_+$ the positive root lattice, $P$ the weight lattice and $P_+$ the dominant weight lattice. See [118] for the standard definitions of these objects. Note that if $\Gamma$ is connected, then $T$ is positive definite iff $\Gamma$ is of $ADE$-type (proposition 4.9 of [118]).

Let $(\alpha_i)_{i\in V}$ be the simple roots of $\mathfrak{g}$. For any $\alpha \in Q$, we have $(\alpha^i)_{i\in V} \in \mathbb{Z}^V$. For $v \in P$, define $(v_i)_{i\in V}$ by $v_i := (v, \alpha_i)$. We have that $(v, \alpha) = \sum_{i\in V} v_i \alpha^i$, for $\alpha \in Q, v \in P$. For any $v \in P$ and $\alpha \in Q_+$, we define the $\tau^2$-binomial coefficients

$$[v, \alpha] := \prod_{i\in V} [v_i, \alpha^i], \quad [\infty, \alpha] := \prod_{i\in V} [\infty, \alpha^i], \tag{1.1}$$

as well as, for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$[n, m] := \frac{\prod_{k=1}^m (1 - \tau^{2n+2k})}{\prod_{k=1}^m (1 - \tau^{2k})}, \quad [\infty, m] := \frac{1}{\prod_{k=1}^m (1 - \tau^{2k})}. \tag{1.2}$$

For any expression $f$ depending on $\tau$, we define the conjugation $\overline{f(\tau)} := f(\tau^{-1})$, if it makes sense.

For $\zeta \in \mathscr{P}^V$, we define for $b \in \mathbb{N}$, $\zeta_b \in \mathbb{N}_0^V \cong Q_+$, via $\zeta_{ba} = \zeta_b^{(a)} \in \mathbb{N}_0$ for $a \in V$.
$\mathfrak{g}$ has the usual triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \,. \tag{1.3}$$

## 6.2 The Poincaré polynomial

We study the generating series for Borel-Moore homology. For $M$ a manifold (possibly non-compact), the Borel-Moore homology of $M$ is defined as the relative singular homology of the one point compactification of $M$, $\bar{M}$, with respect to the point at infinity, and so for compact manifolds the Borel-Moore homology is identical to the regular singular homology.

$$H_*^{BM}(M) := H_*(\bar{M}, \{\infty\}) \,. \tag{2.4}$$

We are interested in the $\mathbb{C}^\times$-equivariant Borel Moore homology of $\mathfrak{M}_\zeta$, $H_*^{\mathrm{BM}}(\mathfrak{M}_\zeta)$, for the definition of which see [77]. This $\mathbb{C}^\times$-action is defined by some

$$\lambda : \mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times T_H = T \cong (\mathbb{C}^\times)^{r+1} \,. \tag{2.5}$$

We assume that this is a *generic* action, this means that it has isolated fixed points. We have defined $r := \mathrm{rk}(G_H)$.

Theorem 3.7 (3) and (4) of [160] easily lift to any Nakajima quiver variety with isolated fixed points. This means that the cycle map from the Chow group to equivariant Borel-Moore homology is an isomorphism (see [83]), the odd components of the homology group vanish, and the homology group is freely generated. So we need only compute the Chow group. For this we can use the results of [55]. We find that each fixed point contributes a single generator, whose homology degree is given by the dimension of the (+)-attracting set at that point. That is, for $p \in \mathfrak{M}_\zeta$ a fixed point, the (+)-attracting set is

$$S_p = \{x \in \mathfrak{M}_\zeta \,|\, \lim_{t \to 0} \lambda(t) \cdot x = p\} \,. \tag{2.6}$$

We then define the Poincaré polynomial as the generating function of equivariant Borel-Moore homology:

$$\begin{aligned} P_{\mathfrak{M}_\zeta}(q) &:= \sum_{i=0}^{\dim_{\mathbb{C}} \mathfrak{M}_\zeta} \dim \left(H_{2i}^{BM}(\mathfrak{M}_\zeta)\right) q^i \\ &= \sum_{p \text{ fixed point}} q^{\dim_{\mathbb{C}} S_p} \,. \end{aligned} \tag{2.7}$$

The superconformal index of chapter 3 contains the Poincaré polynomial. We show how.

At each fixed point of $\mathfrak{M}_\zeta$, we want to know the dimension of the (+)-attracting set. A generic choice of $\lambda$ is given by

$$\lambda(t) = (t^m, t^{n_1}, \dots, t^{n_r}) \tag{2.8}$$

for some

$$n_1 > \cdots > n_r \gg m > 0\,. \tag{2.9}$$

We write our superconformal index as a function of the fugacities $\tilde{y}, \tau, Z$, where $\tilde{y} = y/\tau$. Then one maps the superconformal index under the fugacity mapping

$$\tau \mapsto s^m\,, \quad z_i \mapsto s^{n_i}\,, \tag{2.10}$$

for $s$ a generic fugacity. Finally, one takes the limit $s \to 0$. This produces a finite Laurent polynomial in $\tilde{y}$. We call this $Q_{\mathfrak{M}_\zeta}(\tilde{y})$.

For a particular fixed point one has a product of the form $\tilde{y}^{-d}\prod_a \frac{1-\tilde{y}s^{n_i}}{1-s^{n_i}}$, for $n_i$ some integers and $d$ the quaternionic dimension of the manifold. When $n_i$ is positive, the limit of the $i^{\text{th}}$ factor gives one. When $n_i$ is negative, it gives $\tilde{y}$. Furthermore, the sign of $n_i$ tells us about whether the tangent direction $i$ at the fixed point is an attracting or repelling one.

Thus, we have that the Poincaré polynomial is given by

$$P_{\mathfrak{M}_\zeta}(q) = q^{\dim_{\mathbb{H}} \mathfrak{M}_\zeta} Q_{\mathfrak{M}_\zeta}(q)\,. \tag{2.11}$$

### 6.2.1 The Poincaré polynomial and the Hilbert series

Consider a general eight supercharge quiver with flavour ranks $N \in \mathbb{N}_0^V$ specified and gauge ranks unspecified. In [102, 103], Hausel wrote the generating function for the Poincaré polynomial, summing over the possible values of $k$. For completeness we include his formula. It states that

$$\sum_{k\in\mathbb{N}^V}\sum_{i=0}^{d_{k,N}} b_{2i}(\mathfrak{M}(k,N))q^{d_{k,N}}X^k = \frac{\sum_{\vec{\pi}\in\mathscr{P}^I} q^{-\frac{1}{2}H[\vec{\pi},\vec{m}^{(+)}]}\prod_{i\in I}\frac{X_i^{|\pi_i|}}{\prod_{a\in\mathbb{N}}\varphi_{m_a(\pi_i)}(1/q)}}{\sum_{\vec{\lambda}\in\mathscr{P}^I} q^{-\frac{1}{2}H[\vec{\lambda},0]}\prod_{i\in I}\frac{X_i^{|\lambda_i|}}{\prod_{a\in\mathbb{N}}\varphi_{m_a(\lambda_i)}(1/q)}}\,, \tag{2.12}$$

where for $i \in \mathbb{N}$ we have defined

$$\begin{aligned} \varphi_i(t) &:= \prod_{j=1}^{i} (1-t^j)\,; \\ b_{2i}(M) &:= \dim_{\mathbb{R}}(H_{2i}^{BM}(M)\,; \end{aligned} \tag{2.13}$$

the definition of $H$ is found in equation (2.21); and $d_{k,N}$ is the complex dimension of the quiver variety with gauge ranks $k$ and flavour ranks $N$.

A striking feature of this equation is that the right hand side looks very similar to the monopole formula for the Hilbert series of the three dimensional theory defined by this quiver. We shall find in this section that, for $ADE$-type quivers, the right hand side does indeed come from the monopole formula of a certain infinite rank theory. With this equivalence, we are then able to generalise certain results of Mozgovoy, [150], to generic background charge. This makes manifest the representation theory of the Hilbert series under the Poisson algebra.

**Theorem 10.** *The generating function of the Poincaré polynomial of ADE-type quivers is equal to the Hilbert series of the infinite rank limit of the same ADE-type quiver, with magnetic charge at each node, $i \in V$, defined to be $m_i = (1^{N_i})$.*

In proving this theorem we start from a generic good/ugly quiver, and then in trying to take our infinite rank limit find that we are necessarily restricted to balanced $ADE$-type quivers.

*Proof.* As stated in section 2.2.2, provided a quiver gauge theory theory is good or ugly, the bare dimension of a monopole, with charge $\vec{n} \in \prod_{i\in V} \mathbb{Z}^{k_i}/S_{k_i}$ and background magnetic charge $\vec{m} \in \prod_{i\in V} \mathbb{Z}^{N_i}/S_{N_i}$, can be written as

$$\Delta[\vec{n},\vec{m}] = \sum_{i\in V}\left(-\sum_{\alpha,\beta}|n_{i\alpha}-n_{i\beta}| + \sum_{\alpha,\beta}|n_{i\alpha}-m_{i\beta}|\right) + \sum_{e\in E}\sum_{\alpha,\beta}|n_{i(e)\alpha}-n_{f(e)\beta}|\,. \tag{2.14}$$

The following argument is lifted from [86]. From the manipulations in section 2 of that paper, we can write

$$\begin{aligned} \Delta[\vec{n},\vec{m}] =& 2\sum_{i\in V}\left[\sum_{\alpha,\beta=1}^{k_i} B(n_{i\alpha},n_{i\beta}) - \sum_{\alpha=1}^{k_i}\sum_{\beta=1}^{N_i} B(n_{i\alpha},m_{i\beta})\right] - 2\sum_{e\in E}\sum_{\alpha=1}^{k_{i(e)}}\sum_{\beta=1}^{k_{f(e)}} B(n_{i(e)\alpha},n_{f(e)\beta}) \\ &+ \sum_{i\in V} e_i \sum_{\alpha=1}^{k_i}|n_{i\alpha}| + \sum_{i\in V} k_i \sum_{\beta=1}^{N_i}|m_{i\beta}|\,, \end{aligned} \tag{2.15}$$

where, just as in [86], we have defined the *excess*

$$e_i := -2k_i + N_i + \sum_{\substack{e\in E \\ i(e)=i}} k_{f(e)} + \sum_{\substack{e\in E \\ f(e)=i}} k_{i(e)}\,, \tag{2.16}$$

and the function

$$B(a,b) := \frac{1}{2}(|a|+|b|-|a-b|)\,. \tag{2.17}$$

This function has the property

$$B(a,b) = \begin{cases} \min(|a|,|b|)\,, & \text{if } a\cdot b > 0\,, \\ 0\,, & \text{o/w}\,. \end{cases} \tag{2.18}$$

Note that the excess, $e_i$, is exactly the term that must vanish for all $i \in V$ for the quiver to be balanced.

The Hilbert series is given by

$$\mathrm{HS}[\vec{m}] = \sum_{\vec{n}\in\prod_{i\in V}\mathbb{Z}^{k_i}/s_{k_i}} \tau^{\Delta[\vec{n},\vec{m}]} \prod_{i\in V} \frac{z_i^{|n_i|-|m_i|}}{\prod_{a\in\mathbb{Z}} \varphi_{m_a(n_i)}(\tau^2)}\,. \tag{2.19}$$

Now we note that because of equations (2.18) and (2.15), an individual summand factorises into the positive and negative parts of the vector charges $\vec{n}$. Writing $\mathscr{P}_\ell$ for all partitions of length $\leqslant \ell$, we have that

$$\begin{aligned} \mathrm{HS}[\vec{m}] = \sum_{\substack{\ell_{1i}+\ell_{2i}=k_i \\ \ell_1,\ell_2\in\mathbb{N}_0^V}} \sum_{\pi_i\in\mathscr{P}_{\ell_{1i}}} \sum_{\nu_i\in\mathscr{P}_{\ell_{2i}}} & \tau^{H(\vec{\pi},\vec{m}^{(+)})+H(\vec{\nu},\vec{m}^{(-)})+\sum_{i\in V}\left(e_i(|\pi_i|+|\nu_i|)+k_i\sum_{\beta=1}^{N_i}|m_{i\beta}|\right)} \\ \prod_{i\in V} & \frac{z_i^{|\pi_i|-|\nu_i|-|m_i^{(+)}|+|m_i^{(-)}|}}{\varphi_{k_i-\ell(\pi_i)-\ell(\nu_i)}(\tau^2)\prod_{a\in\mathbb{N}}\varphi_{m_a(\pi_i)}(\tau^2)\varphi_{m_a(\nu_i)}(\tau^2)}\,, \end{aligned} \tag{2.20}$$

where $\vec{m}^{(+)}$ is the partition of the positive parts of $\vec{m}$; $\vec{m}^{(-)}$ is the partition of the absolute value of the negative parts of $\vec{m}$; and $H$ is defined as

$$\begin{aligned} H(\vec{\lambda},\vec{\mu}) :=& 2\sum_{i\in V}\left[\sum_{\alpha,\beta=1}^{k_i} B(\lambda_{i\alpha},\lambda_{i\beta}) - \sum_{\alpha=1}^{k_i}\sum_{\beta=1}^{N_i} B(\lambda_{i\alpha},\mu_{i\beta})\right] \\ & -2\sum_{e\in E}\sum_{\alpha=1}^{k_{i(e)}}\sum_{\beta=1}^{k_{f(e)}} B(\lambda_{i(e)\alpha},\lambda_{f(e)\beta}) \\ &= 2T(\lambda^\vee) - 2(\lambda^\vee,\mu^\vee)\,. \end{aligned} \tag{2.21}$$

The last line of equation (2.21) follows from the following two identities identities: for $\lambda,\mu\in\mathscr{P}$

$$\begin{aligned} \langle\lambda,\mu\rangle :=& \sum_{i,j=1}^{\infty} B(\lambda_i,\mu_j) \\ =& \sum_{i,j=1}^{\infty}\min(i,j)m_i(\lambda)m_j(\mu) \\ =& \sum_{i=1}^{\infty}\sum_{j,k=1}^{\infty} m_j(\lambda)m_k(\mu) \\ =& \sum_{i=1}^{\infty}\lambda_i^\vee\mu_i^\vee\,, \end{aligned} \tag{2.22}$$

and for $\lambda,\mu\in\mathscr{P}^I$

$$\begin{aligned} (\lambda,\mu) =& \sum_{k=1}^{\infty}\sum_{i,j\in I}\lambda_{ik}\mu_{jk}C_{ij} \\ =& \sum_{k=1}^{\infty}\left(\sum_{i\in I}2\lambda_{ik}\mu_{ik} - \sum_{e\in E}\lambda_{i(e)k}\mu_{f(e)k}\right) \\ =& \sum_{i\in I}2\langle\lambda_i^\vee,\mu_i^\vee\rangle - \sum_{e\in E}\langle\lambda_{i(e)}^\vee,\mu_{f(e)}^\vee\rangle\,. \end{aligned} \tag{2.23}$$

We rescale our Hilbert series by $\tau^{-\sum_{i\in V}\frac{k_i}{2}\sum_{\beta=1}^{N_i}|m_{i\beta}|}$. We now want to take the limit $k_i\to\infty$ for each $i\in V$, such that the ratio between different flavour/gauge nodes is constant. We do this by scaling all dimension vectors by a common factor $c\in\mathbb{N}$, and send $c\to\infty$. In order for the scaling dimension to not diverge in this limit for some fixed non-zero $(\pi,\nu)$, it is necessary that $e_i=0$ for all $i\in V$. We assume that this is the case, this means that the quiver must be of $ADE$-type[1]. We then would like to know whether the Hilbert series has a $c\to\infty$ limit (i.e. is

[1] $\hat{A}\hat{D}\hat{E}$-type quivers can also be balanced, but the cost is no flavour symmetry, which is crucial for us in order to have background magnetic charge.

the balanced condition sufficient). For this we use Cauchy-Schwarz[2] to write

$$H(\vec{\lambda},\vec{\mu}) \geqslant \sqrt{(\lambda^\vee,\lambda^\vee)}\left(\sqrt{(\lambda^\vee,\lambda^\vee)} - 2\sqrt{(\mu^\vee,\mu^\vee)}\right), \tag{2.24}$$

and hence $H(\vec{\lambda},\vec{\mu}) \to \infty$ as $\max_i \ell(\lambda_i) \to \infty$. This guarantees that the grading provided by the $\mathbb{C}^\times$-action at any fixed homogeneous component is fixed for $c$ large enough, and hence the limit is well-defined.

Calling the rescaled Hilbert series in this limit $\overline{\mathrm{HS}}$, we get

$$\begin{aligned}\overline{\mathrm{HS}}[\vec{m}] = &\frac{1}{\varphi_\infty(\tau^2)^{|V|}} \sum_{\vec{\pi}\in\mathscr{P}^V} \tau^{H[\vec{\pi},\vec{m}^{(+)}]} \prod_{i\in V} \frac{z_i^{|\pi_i|-|m_i^{(+)}|}}{\prod_{a\in\mathbb{N}} \varphi_{m_a(\pi_i)}(\tau^2)} \\ &\sum_{\vec{\nu}\in\mathscr{P}^V} \tau^{H[\vec{\nu},m^{(-)}]} \prod_{i\in V} \frac{z_i^{|m_i^{(-)}|-|\nu_i|}}{\prod_{a\in\mathbb{N}} \varphi_{m_a(\nu_i)}(\tau^2)}\,.\end{aligned} \tag{2.25}$$

The Hilbert series in this limit has factorised into a positive charge piece, a negative charge piece and a plethystic piece. Equation (2.20)'s summand contains a factorisation into positive, negative and a plethystic piece, but it depends on the length of the positive and negative pieces, and so is not a factorisation of the whole Hilbert series.

If we define

$$m_i := (1^{N_i}),\ \text{for all } i\in V\,, \tag{2.26}$$

then we have that

$$\mathscr{G}[\vec{m}] = \frac{\overline{\mathrm{HS}}[\vec{m}]\prod_{i\in V} z_i^{N_i}}{\overline{\mathrm{HS}}[0]}\,, \tag{2.27}$$

where $\mathscr{G}[\vec{m}]$ is the generating function for the Poincaré polynomial of the Nakajima quiver variety the quiver defines, with the ranks of the gauge groups given by the topological charge in the Coulomb branch formula, and the ranks of the flavour groups given by background magnetic charge $\vec{m}$. This is because we have

$$\mathscr{G}[\vec{m}] = \frac{\sum_{\vec{\pi}\in\mathscr{P}^V} \tau^{H[\vec{\pi},\vec{m}^{(+)}]} \prod_{i\in V} \frac{z_i^{|\pi_i|}}{\prod_{a\in\mathbb{N}} \varphi_{m_a(\pi_i)}(\tau^2)}}{\sum_{\vec{\lambda}\in\mathscr{P}^V} \tau^{H[\vec{\lambda},0]} \prod_{i\in V} \frac{z_i^{|\lambda_i|}}{\prod_{a\in\mathbb{N}} \varphi_{m_a(\lambda_i)}(\tau^2)}}\,. \tag{2.28}$$

This is the formula in [102], upon $\tau \mapsto q^{-1/2}$ and $z_i \mapsto X_i$. □

[2] In order to use Cauchy-Schwarz, it is crucial that our quiver is of $ADE$-type, and that $\lambda$ and $\mu$ are real.

### 6.2.2 The Weyl group action

It was understood how the Weyl group of an ADE quiver acts on the corresponding Nakajima quiver variety's Poincaré polynomial in [141]. It is worth noting that this result is equivalent to the expected enhanced symmetry of the Coulomb branch of a balanced quiver. If we take a balanced quiver with some gauge and flavour ranks $\{\vec{k},\vec{N}\}$, then, for any $a \in \mathbb{N}$, we can consider the balanced quiver with the same underlying directed graph, and ranks $\{a\vec{k},a\vec{N}\}$. If the graph is of $ADE$-type, then it is expected [21] that the Coulomb branch is invariant under the Weyl group of this quiver. Notably, this will hold in the limit $a \to \infty$, and hence gives an alternative explanation for the Weyl symmetry of the Poincaré polynomial.

Note that this Weyl group action is exactly the Seiberg duality of section 2.3.3.

## 6.3 Yangian symmetry in the infinite rank limit

In the paper [150], Mozgovoy proves the fermionic Lusztig conjecture, conjectured by Lusztig [140]. It states that for an $ADE$-quiver the fermionic forms of [101] are related to the Poincaré polynomials of Nakajima's quiver varieties. The exact theorem that Mozgovoy proves states that, for a quiver of $ADE$-type,

$$\mathscr{G}(N) = \sum_{\lambda \in P} n(N,\lambda,\tau^{-2}) \mathrm{ch} M(\lambda)\,, \tag{3.29}$$

where $\mathscr{G}(N)$ is the generating function for the Poincaré polynomial, $n$ is a certain fermionic form defined in equation (3.42), $N$ is choice of flavour symmetries, and ch $M(\lambda)$ is the character of the Verma module over $\mathfrak{g}(\Gamma)$.

Since we have shown in the previous section that this generating function can be derived from an infinite rank Coulomb branch Hilbert series at a specific charge, an obvious question arises: If we generalise to generic background charge, do we get a similar result? We further have the question: What does this tell us about the representation theory?

We shall show that the result can be generalised to generic background charge, and that it shows the presence of Yangian symmetry.

### 6.3.1 The Yangian is the quantisation of the Coulomb branch

The affine Grassmannian was first introduced in 1988 by Kazhdan and Lusztig in [125]. It was shown in 2016 in [37] that the Coulomb branch of a good eight supercharge quiver of $ADE$-type is isomorphic as an affine variety to an object known as a Lusztig slice of the affine Grassmannian, $\mathrm{Gr}^{\bar{\lambda}}_{\mu}$. We define this object:

Let $G$ be the $ADE$-type group with Lie algebra $\mathfrak{g}$. $\lambda, \mu \in \mathfrak{h}$ dominant coweights. These give rise to elements $t^\lambda, t^\mu \in G[t, t^{-1}]$.

Let Gr:$= G((t))/G[[t]]$ be the *affine Grassmannian* of $G$. There is an action of $G[[t]]$ on Gr by left multiplication. Let $\mathrm{Gr}^\lambda := G[[t]]t^\lambda$ and $\mathrm{Gr}_\mu := G_1[t^{-1}]t^{w_0\mu}$, where $G_1[t^{-1}]$ is the kernel of the evaluation map $G[t^{-1}] \to G$ under $t^{-1} \mapsto 0$ and $w_0 \in W$ is the longest elements of the Weyl group of $G$.

$\mathrm{Gr}^{\bar{\lambda}}_\mu := \overline{\mathrm{Gr}^\lambda} \cap \mathrm{Gr}_\mu$ is the *Lusztig slice*. This is a transverse slice to $\mathrm{Gr}^\mu$ inside $\overline{\mathrm{Gr}^\lambda}$. It is the Coulomb branch of the $ADE$-type quiver. For $A$-type quivers we know that the Higgs branch is a nilpotent orbit intersect a Slodowy slice, this was proven in [143], and we know the mirror dual is also an $A$-type quiver. So we would expect an isomorphism between Lusztig slices of the $A$-type affine Grassmannian and Slodowy slices of $A$-type nilpotent orbits. This isomorphism is known as the Mirković-Vyrbornov isomorphism. It was proven in [148], while its generalisation to the quantisation was proven in [198].

The correspondence between the gauge and flavour dimensions, $k$ and $N$, of the $ADE$-type quiver and $\lambda$ and $\mu$ is as follows. We call the underlying $ADE$-Dynkin graph of the quiver $\Gamma = (V, E)$. Define $\alpha_i$ to be the simple positive roots of $\Gamma$, $\omega_i$ the fundamental weights and $C$ the Cartan matrix of $\Gamma$. The relation is given by

$$\begin{aligned} \alpha &:= \sum_{i \in V} k_i \alpha_i\,, \\ \lambda &= \sum_{i \in V} N_i \omega_i\,, \\ \mu &= \lambda - \alpha\,. \end{aligned} \tag{3.30}$$

Note that for $A$-type we have that $T^\rho_\sigma(SU(N))$ corresponds to $\lambda$ and $\mu$ via

$$\lambda_i = \sigma^\vee_i - \sigma^\vee_{i+1}\,, \quad \mu_i = \rho_i - \rho_{i+1}\,. \tag{3.31}$$

It was shown in the works [121, 120] that the quantisation of the coordinate ring, in the sense of section 2.4, of the Lusztig slice of the affine Grassmannian is an object called the truncated shifted Yangian.

$\mathrm{Gr}^{\bar{\lambda}}_\mu$ carries a natural Poisson structure. Moreover, there is a map of graded Poisson algebras $\mathrm{gr}(Y^\lambda_\mu(\mathbf{R})) \to \mathbb{C}[\mathrm{Gr}^{\bar{\lambda}}_\mu]$. This is an isomorphism modulo nilradical ideals, [121]. This means that we can think of the quantisation of the Coulomb branch, in the sense of section 2.4, as the truncated shifted Yangian.

We define the shifted Yangian briefly, and give an outline of how to define the truncated shifted Yangian. See [52] for an introduction to the Yangian, and [121] for more details on the truncated shifted Yangian. The *Yangian* is the $\mathbb{C}$-algebra with generators $E_a^{(r)}, F_a^{(r)}, H_a^{(r)}$ for $a \in V$ and $r \in \mathbb{N}$ with relations

1. $[H_a^{(r)}, H_b^{(s)}] = 0$;
2. $[E_a^{(r)}, F_b^{(s)}] = \delta_{ab} H_a^{(r+s-1)}$;
3. $[H_a^{(1)}, E_b^{(s)}] = C_{ab} E_b^{(s)}$;
4. $[H_a^{(r+1)}, E_b^{(s)}] - [H_a^{(r)}, E_b^{(s+1)}] = \frac{C_{ab}}{2}(H_a^{(r)} E_b^{(s)} + E_b^{(s)} H_a^{(r)})$;
5. $[H_a^{(1)}, F_b^{(s)}] = -C_{ab} F_b^{(s)}$;
6. $[H_a^{(r+1)}, F_b^{(s)}] - [H_a^{(r)}, F_b^{(s+1)}] = -\frac{C_{ab}}{2}(H_a^{(r)} F_b^{(s)} + F_b^{(s)} H_a^{(r)})$;
7. $[E_a^{(r+1)}, E_b^{(s)}] - [E_a^{(r)}, E_b^{(s+1)}] = \frac{C_{ab}}{2}(E_a^{(r)} E_b^{(s)} + E_b^{(s)} E_a^{(r)})$;
8. $[F_a^{(r+1)}, F_b^{(s)}] - [F_a^{(r)}, F_b^{(s+1)}] = -\frac{C_{ab}}{2}(F_a^{(r)} F_b^{(s)} + F_b^{(s)} F_a^{(r)})$;
9. $a \neq b, i = 1 - C_{ab} \implies \mathrm{sym}[E_a^{(r_1)}, [E_a^{(r_2)}, \ldots [E_a^{(r_i)}, E_b^{(s)}] \ldots]] = 0$;
10. $a \neq b, i = 1 - C_{ab} \implies \mathrm{sym}[F_a^{(r_1)}, [F_a^{(r_2)}, \ldots [F_a^{(r_i)}, F_b^{(s)}] \ldots]] = 0$,

where sym is symmetrising over the indices $r_1, \ldots, r_i$.

This is the filtered presentation of the Yangian. The filtration is given by $\deg X^{(r)} = r$ for $X = E_a, F_a, H_a$.

Denote $Y_\mu$, the *shifted Yangian*, to be the subalgebra with PBW basis given by ordered monomials in $E_a^{(r)}$, $H_a^{(r)}$, $F_a^{(s)}$ for $r > 0$ and $s > \langle \mu, \alpha_a \rangle$. This is proposition 3.11 of [121]. This object was first defined in [42] as the quantisation of the coordinate ring of Slodowy slices in $\mathfrak{gl}_n$.

The truncation was introduced in [121]. We take definition from chapter 3.2 of [120], we shall not list the details. The idea is to quotient by a two sided ideal generated by elements labelled by $a \in V$, whose degree is greater than $k_a$.

We now discuss the quantisation of our infinite rank limit. Firstly, at finite rank we take the quiver to be balanced, i.e. $\mu = 0$ (for $A$-type quivers this corresponds to $\rho = (b^{n+1})$ for some $b \in \mathbb{N}$). This means that there is no shifting at all, which is what we expect, as the shifting would break the expected enhanced topological symmetry. In the infinite rank limit we have $k_a \to \infty$ for all $a \in V$, removing the truncation.

This means that the quantised Coulomb branch chiral ring for the infinite rank limit of a balanced $ADE$-quiver is the Yangian of the $ADE$-group that quiver defines.

## 6.3.2 The fermionic form

We shall define Hatayama's fermionic form. We show that the presence of Hatayama's fermionic form indicates that the representation is a classical limit of the Yangian. To show this, we must discuss the representation theory of the current algebra $\mathfrak{g}[t]$, the quantised loop algebra $U_q(\mathbf{L}\mathfrak{g})$, and the relation between the quantised loop algebra and the Yangian.

We shall define a type of tensor product for the current algebra called the *fusion product*. This will be defined on a certain set of modules called the *classical Kirillov-Reshetikhin modules*. These are the classical limit of the *quantum Kirillov-Reshetikhin modules*, simple modules of the quantum loop algebra. The fusion product is the classical limit of the tensor product of quantum loop algebra modules (theorem 3.1 of [163]). We then use the results of [88] to show that this is a classical limit of the tensor product of Yangian representations.

### Current algebra

The current algebra $\mathfrak{g}[t]$ is defined to be $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$ as a vector space, with brackets induced by, for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{N}_0$

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} \,. \tag{3.32}$$

This is a $\mathbb{N}_0$-graded algebra, $\mathfrak{g}[t] = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{g} \otimes t^n$.

For $u \in \mathbb{C}$, we have an algebra morphism known as the evaluation morphism

$$\begin{aligned} \mathrm{ev}_u : \mathfrak{g}[t] &\to \mathfrak{g} \,, \\ x \otimes t^n &\mapsto u^n x. \end{aligned} \tag{3.33}$$

For $V$ a representation of $\mathfrak{g}$, we have that $\mathrm{ev}_u^*(V)$ is a representation of $\mathfrak{g}[t]$ of the same dimension as $V$.

A $\mathfrak{g}[t]$-module $V$ is called *cyclic*, if there exists a $v \in V$ such that $U(\mathfrak{g}[t])v = V$. Any cyclic module inherits a filtration from the grading of $\mathfrak{g}[t]$. This filtration $v \in V_0 \subset V_1 \subset \dots$ is defined as

$$V_n := U^{(\leqslant n)}(\mathfrak{g}[t])v \,, \tag{3.34}$$

where $U^{(\leqslant n)}(\mathfrak{g}[t])$ is the subspace of $U(\mathfrak{g}[t])$ with degree $\leqslant n$.

There is an associated graded $\mathfrak{g}[t]$-module

$$\mathrm{gr}\, V = \bigoplus_{n \in \mathbb{N}_0} V_n / V_{n-1} \,. \tag{3.35}$$

This grading is clearly $\mathfrak{g}$-equivariant.

We define a special class of $\mathfrak{g}[t]$-modules called the *classical Kirillov-Reshetikhin modules*. These modules take their name from the work of Kirillov and Reshitikhin [131], where they introduced finite dimensional simple modules of $Y(\mathfrak{g})$ with highest weight $m\omega_i$ for $i \in V$ and $m \in \mathbb{N}$. In the works [48, 49], a classical Kirillov-Reshetikhin module was defined as a $\mathfrak{g}[t]$-module. Their definition is (lifting the presentation from [10]):

**Definition 6.** *A* classical Kirillov-Reshetikhin module*, denoted* $\mathrm{KR}_{a,m}(u)$*, where* $a \in V$*,* $m \in \mathbb{N}$ *and* $u \in \mathbb{C}^{\times}$*, is the graded module of the filtered module generated by the action of* $U(\mathfrak{g}[t])$ *on some cyclic* $v \in \mathrm{KR}_{a,m}(u)$ *such that, for* $\mathfrak{g}[t] \ni x[n]_u := x \otimes (t-u)^n$ *for* $x \in \mathfrak{g}$ *and* $n \in \mathbb{N}_0$*,*

1. $x[n]_u v = 0$ *if* $x \in \mathfrak{n}_+$ *and* $n > 0$*;*
2. $h_b[n]_u v = m\delta_{n0}\delta_{ab} v$*;*
3. $f_b[n]_u v = 0$ *if* $n \geqslant \delta_{ab}$*;*
4. $f_a[0]_u^{m+1} v = 0$*.*

$\mathrm{KR}_{a,m}(u)$ is cyclic by definition.

The classical Kirillov-Reshetikhin modules $\mathfrak{g}$-modules as well as being $\mathfrak{g}[t]$-modules. Writing $L(\lambda)$ for the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$, we have that the classical Kirillov-Reshitikhin modules decompose as

$$\mathrm{KR}_{a,m}(u) = L(n\omega_a) \oplus \bigoplus_{\mu < n\omega_a} c_{\mu,n,a} L(\mu)\,, \tag{3.36}$$

for $c_{\mu,n,a} \in \mathbb{N}_0$. The work [48] gives an explicit decomposition for $A$- and $D$-type:

- For $A_n$, we have $\mathrm{KR}_{a,m}(\zeta)) \cong_{\mathfrak{sl}(n+1)} L(m\omega_a)$.
- For $D_n$, we have $\mathrm{KR}_{a,m}(\zeta) \cong_{\mathfrak{so}(2n)} \bigoplus_{\mu \in P(a,m)} L(\mu)$, where $P(a,m) \subset P_+$ is defined recursively:

$$\begin{aligned} P(a,m) &= P(a-1,m) + P(a,1) \text{ for } a = 1,\dots,n-2\,, \\ P(a,1) &= \{\omega_a, \omega_{a-2}, \omega_{a-4}, \dots\} \text{ for } a = 1,\dots,n-2\,, \\ P(a,m) &= m\omega_a \text{ for } a = n-1, n\,. \end{aligned} \tag{3.37}$$

For $E$-type it is only known for certain Dynkin nodes.

For $u_1,\dots,u_p \in \mathbb{C}$ pairwise distinct, consider the filtered $\mathfrak{g}[t]$-module,

$$W(\vec{v}) = \bigotimes_{i=1}^{p} \mathrm{KR}_{a_i,v_i}(u_i)\,. \tag{3.38}$$

By the work [79], $W(\vec{v})$ is a cyclic $\mathfrak{g}[t]$-module (if each component is generated by $v_i$, then $W(\vec{v})$ is generated by $v_1 \otimes \cdots \otimes v_p$). There is an associated graded space. We call the graded space the *fusion product*. We write it as

$$\mathrm{gr}\, W(\vec{v},u) = \mathrm{KR}_{a_1,v_1} * \cdots * \mathrm{KR}_{a_p,v_p}(u_1,\dots,u_p)\,. \tag{3.39}$$

Since the grading is $\mathfrak{g}$-equivariant, there is an expansion in characters of the form

$$\sum_{n\in\mathbb{N}_0} q^n \mathrm{ch}_{\mathfrak{g}}\big((\mathrm{gr}\, W(\vec{v},u))_n\big) = \sum_{\lambda\in P_+} M_{\vec{v},\lambda,u}(q)\mathrm{ch}_{\mathfrak{g}} L(\lambda)\,, \tag{3.40}$$

where $M_{\vec{v},\lambda,u}(q) \in \mathbb{N}_0[q]$.

The Feigin-Loktev conjecture states that $M_{\vec{v},\lambda,u}(q)$ does not depend on $u$. Moreover, for $\mathfrak{g}$ simply-laced, di Francesco and Kedem proved in [72] that in fact

$$M_{\vec{v},\lambda,u}(q) = m(\vec{v},\lambda,\tau^2)\,, \tag{3.41}$$

where the right hand side is Hatayama's fermionic form. We now give the definition of the fermionic forms $m$ and $n$.

$n(\vec{v},\lambda,\tau^2)$ is defined as

$$\begin{aligned} n(\vec{v},\lambda,\tau^2) := \sum_{\substack{\zeta\in\mathscr{P}^V\\ |\zeta|=|\vec{v}|-\lambda}} \prod_{i\in V} x_i^{|\zeta^{(i)}|} \tau^{2\sum_{i\in V}\sum_{a=1}^{\infty} v_a^{(i)}\zeta_a^{(i)} - 2\sum_{a=1}^{\infty} T(\zeta_a)} \\ \times \prod_{a\geqslant 1} \left[ \sum_{b=1}^{a} (v_b-\zeta_b), \zeta_a - \zeta_{a+1} \right]. \end{aligned} \tag{3.42}$$

The fermionic form $m(\vec{v},\lambda,\tau^2)$ is defined with the same summand as $n$, but the sum over $\zeta \in \mathscr{P}^V$ is restricted to $\zeta$ such that

1. $\sum_{\ell=1}^{k}(v_k - \zeta_k) \in P_+$ for all $k \geqslant 1$ (this is known as having non-negative vacancy numbers); and

2. $|\zeta| = |\vec{v}| - \lambda$.

The fermionic forms $n(\vec{\nu},\lambda,q)$ and $m(\vec{\nu},\lambda,q)$, where $\vec{\nu}\in\mathscr{P}^V$ and $\lambda\in P$, were first defined in [101]. In that paper they were conjectured to be equal for $\lambda\in P_+$. This was proven to be the case in [72].

### Quantum loop algebra

We now discuss the quantum loop algebra $U_q(\mathbf{L}\mathfrak{g})$. The loop algebra $\mathbf{L}\mathfrak{g}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ is defined in much the same way as the current algebra with $[x\otimes t^m,y\otimes t^n]=[x,y]\otimes t^{m+n}$, the only difference being $m,n\in\mathbb{Z}$. For $c\in\mathbb{C}$, $\varphi_c:\mathbf{L}\mathfrak{g}\to\mathbf{L}\mathfrak{g}$ is a Lie algebra automorphism defined via

$$\varphi_c(x\otimes f(t))\mapsto x\otimes f(t+c) \text{ for } x\in\mathfrak{g}, f(t)\in\mathbb{C}[t]. \tag{3.43}$$

Let $q$ be an indeterminate. For $\ell\in\mathbb{Z}$, $s,s'\in\mathbb{N}_0$ with $s\geqslant s'$ set

$$[\ell]_q:=\frac{q^\ell-q^{-\ell}}{q-q^{-1}},\quad [s]_q!:=[s]_q[s-1]_q\dots[1]_q,\quad \begin{bmatrix} s\\ s'\end{bmatrix}_q:=\frac{[s]_q!}{[s']_q![s-s']_q!}. \tag{3.44}$$

$U_q(\mathbf{L}\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra with generators $x_{a,r}$, $\kappa_a^{\pm1}$, $h_{a,m}$ for $a\in V$, $r\in\mathbb{Z}$, $m\in\mathbb{Z}\backslash\{0\}$, and the following relations for $a,b\in V$, $r,r'\in\mathbb{Z}$, $m,m'\in\mathbb{Z}\backslash\{0\}$:

1. $\kappa_a\kappa_a^{-1}=\kappa_a^{-1}\kappa_a=1$;
2. $[\kappa_a,\kappa_b]=[\kappa_a,h_{b,m}]=[h_{a,m},h_{b,m'}]=0$;
3. $\kappa_a x^\pm_{b,r}\kappa_a^{-1}=q^{\pm C_{ab}}x^\pm_{b,r}$;
4. $[h_{a,m},x^\pm_{b,r}]=\pm\frac{1}{m}[mC_{ab}]_q x^\pm_{b,r+m}$;
5. $[x^+_{a,r},x^-_{b,r'}]=\delta_{ab}\frac{\phi_{a,r+r'}-\phi^-_{a,r+r'}}{q-q^{-1}}$;
6. $x^\pm_{a,r+1}x^\pm_{b,r'}-q_i^{\pm C_{ab}}x^\pm_{b,r'}x^\pm_{a,r+1}=q^{\pm C_{ab}}x^\pm_{a,r}x^\pm_{b,r'+1}-x^\pm_{b,r'+1}x^\pm_{a,r}$; and
7. for $a\neq b$ and all sequences of integers $r_1,\dots,r_{1-C_{ab}}$

   $\sum_{\sigma\in S_{1-C_{ab}}}\sum_{k=0}^{1-C_{ab}}(-)^k\begin{bmatrix} s\\ k\end{bmatrix}_q x^\pm_{a,r_{\sigma(1)}}\dots x^\pm_{a,r_{\sigma(k)}}x^\pm_{b,r'}x^\pm_{a,r_{\sigma(k+1)}}\dots x^\pm_{a,r_{\sigma(1-C_{ab})}}=0.$

The $\phi^\pm_{a,r}$'s are determined by equating coefficients of powers of $u$ in the formula

$$\sum_{r=0}^{\infty}\phi^\pm_{a,\pm r}u^{\pm r}=\kappa_a^{\pm1}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty}h_{a,\pm m}u^{\pm m}\right), \tag{3.45}$$

and $\phi^{\pm}_{a,\mp r} = 0$ for $r > 0$.

Let $U_q(\mathbf{L}\mathfrak{n}_\pm)$ be the subalgebra of $U_q(\mathbf{L}\mathfrak{g})$ generated by $\{x^{\pm}_{a,r} | a \in V, r \in \mathbb{Z}\}$ and $U_q(\mathbf{L}\mathfrak{h})$ be the subalgebra generated by $\{k_a^{\pm 1}, h_{a,m} | a \in V, m \in \mathbb{Z}\backslash\{0\}\}$. One can show from the defining relations that a PBW-like exression holds,

$$U_q(\mathbf{L}\mathfrak{g}) = U_q(\mathbf{L}\mathfrak{n}_-)U_q(\mathbf{L}\mathfrak{h})U_q(\mathbf{L}\mathfrak{n}_+)\,. \tag{3.46}$$

### Quantum loop algebra representation theory

A $U_q(\mathbf{L}\mathfrak{g})$-module $M$ is *of type 1* if

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad M_\lambda = \{v \in M | k_a^{\pm 1} v = q^{\pm \langle h_a, \lambda \rangle} v\}\,, \tag{3.47}$$

where $h_a$ are the generators of $\mathfrak{h}$.

We only consider $U_q(\mathbf{L}\mathfrak{g})$-modules of type 1.

Let $P_q^+$ be the monoid under coordinate-wise multiplication of $V$-tuples of polynomials $\pi = (\pi_1(u), \ldots, \pi_n(u))$ such that each $\pi_i(u)$ is expressed as

$$\pi_i(u) = (1 - a_1 u)(1 - a_2 u) \ldots (1 - a_k u), \tag{3.48}$$

for some $k \in \mathbb{N}_0, a_j \in \mathbb{C}(q)^\times$. Define a map wt: $P_q^+ \to P^+$ via

$$\mathrm{wt}(\pi) = \sum_{a \in V} \deg \pi_a \omega_a. \tag{3.49}$$

Say a $U_q(\mathbf{L}\mathfrak{g})$-module $V$ is $\ell$-highest weight with $\ell$-highest weight vector $v$ if

$$x^{+}_{a,r} v = 0 \text{ for } a \in V, r \in \mathbb{Z}, \quad U_q(\mathbf{L}\mathfrak{h})v = \mathbb{C}(q)v, \quad V = U_q(\mathbf{L}\mathfrak{g})v. \tag{3.50}$$

In [50], Chari and Pressley showed that for every $\pi \in P_q^+$ there is a unique (up to isomorphism) simple finite-dimensional $\ell$-highest weight $U_q(\mathbf{L}\mathfrak{g})$-module, which we denote by $L_q(\pi)$, such that its $\ell$-highest weight vector $v_\pi$ (unique up to scalar multiplication) satisfies

$$\phi^{\pm}_{a,\pm r} v_\pm = \gamma^{\pm}_{a,\pm r} v_\pi \text{ for } a \in V, r \in \mathbb{N}_0, \tag{3.51}$$

where $\gamma^{\pm}_{a,r}$ are rational functions in $q$ determined by the formula

$$\sum_{r=0}^{\infty} \gamma^{+}_{a,r} u^r = q^{\deg \pi_a} \frac{\pi_a(q^{-1}u)}{\pi_a(qu)} = \sum_{r=0}^{\infty} \gamma^{-}_{a,-r} u^{-r}. \tag{3.52}$$

$\pi$ is called the *Drinfeld polynomial* of $L_q(\pi)$.

**Definition 7.** *Given $a \in V, \ell \in \mathbb{N}$ and $\xi \in \mathbb{C}(q)^\times$, define $\pi_{a,\ell,\xi} \in P_q^+$ by*

$$(\pi_{a,\ell,\xi})_b(u) = \begin{cases} \prod_{k=0}^{\ell-1}(1 - \xi q^{2k} u) & \text{if } b{=}a, \\ 1 & \text{o/w.} \end{cases} \tag{3.53}$$

*The simple $U_q(\mathbf{L}\mathfrak{g})$-module associated to this $L_q(\pi_{a,\ell,\xi})$ is called the* quantum Kirillov-Reshitikhin module. *Denote it $W_q^{a,\ell}(\xi)$.*

### Classical limit of quantum loop algebra

Let $\mathscr{A}$ be the local subring of $\mathbb{C}(q)$ defined by

$$\mathscr{A} = \{f/g | f, g \in \mathbb{C}[q], g(1) \neq 0\}. \tag{3.54}$$

An $\mathscr{A}$-lattice $L$ of a $\mathbb{C}(q)$-vector space $V$ is a free $\mathscr{A}$-submodule such that $V \cong \mathbb{C}(q) \otimes_{\mathscr{A}} L$. Let $U_{\mathscr{A}}(\mathbf{L}\mathfrak{g}) \subseteq U_q(\mathbf{L}\mathfrak{g})$ be the $\mathscr{A}$-subalgebra generated by $\kappa_a^{\pm 1}, x_{a,r}^{\pm}$ for $a \in V, r \in \mathbb{Z}$, similarly for triangular components. It is an $\mathscr{A}$-lattice and a sub-coalgebra of $U_q(\mathbf{L}\mathfrak{g})$ (note the quantised loop algebra is a Hopf algebra, and so has a coalgebra structure).

Let $P_{\mathscr{A}}^+$ be the submonoid of $P_q^+$ consisting of $\pi = (\pi_1(u), \dots, \pi_n(u))$ such that $\pi_a(u) \in \mathscr{A}^\times[u]$ for all $a \in V$. The following result is from [48, 51]:

**Lemma 5.** *Assume that $W$ is a finite-dimensional $\ell$-highest weight $U_q(\mathbf{L}\mathfrak{g})$-module with $\ell$-highest weight $\pi \in P_{\mathscr{A}}^+$, and let $v_\pi$ be the $\ell$-highest weight vector. Then the $U_{\mathscr{A}}(\mathbf{L}\mathfrak{g})$-submodule $L = U_{\mathscr{A}}(\mathbf{L}\mathfrak{g})v_\pi \subseteq W$ is an $\mathscr{A}$-lattice of $W$.*

Given an $\mathscr{A}$-module $M$, denote by $M_{\mathbb{C}} := \mathbb{C} \otimes_{\mathscr{A}} M$, where $\mathbb{C}$ is regarded as an $\mathscr{A}$-module through the isomorphism $\mathbb{C} \cong \mathscr{A}/(q-1)\mathscr{A}$. For an element $v \in M$, write $\bar{v} = 1 \otimes v \in M_{\mathbb{C}}$. There exists a surjective $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} U_{\mathscr{A}}(\mathbf{L}\mathfrak{g})_{\mathbb{C}} &\to U(\mathbf{L}\mathfrak{g}), \\ x_{a,r}^+ &\mapsto \overline{x_{a,r}^+} := e_a \otimes t^r, \\ x_{a,r}^- &\mapsto \overline{x_{a,r}^-} := f_a \otimes t^r, \\ \kappa_a^{\pm 1} &\mapsto \overline{\kappa_a^{\pm 1}} := 1, \\ h_{a,m} &\mapsto \overline{h_{a,m}} := h_a \otimes t^m, \end{aligned} \tag{3.55}$$

where $h_a, e_a, f_a$ for $a \in V$ are the Chevalley generators of $\mathfrak{g}$.

Assume that $V$ is a finite-dimensional $\ell$-highest weight $U_q(\mathbf{L}\mathfrak{g})$-module with $\ell$-highest weight $\pi \in P_{\mathscr{A}}^{+}$, and let $v_\pi$ be an $\ell$-highest weight vector. Set $L = U_{\mathscr{A}}(\mathbf{L}\mathfrak{g})v_\pi \subseteq V$. Through the $\mathbb{C}$-algebra homomorphism, $L_{\mathbb{C}}$ becomes an $\mathbf{L}\mathfrak{g}$-module, which is called the *classical limit* of $W$ and denote the associated graded $\mathbf{L}\mathfrak{g}$-module by $\overline{W}$. Note that $\dim_{\mathbb{C}(q)} W = \dim_{\mathbb{C}} \overline{W}$.

For sufficiently large $N$, $\mathfrak{g} \otimes (t-c)^N \mathbb{C}[t]$ acts trivially on $\mathrm{KR}_{a,\ell}(c)$. Hence, when $c \neq 0$ we have

$$\mathbf{L}\mathfrak{g} \twoheadrightarrow \mathfrak{g} \otimes (\mathbb{C}[[t-c]]/(t-c)^N \mathbb{C}[[t-c]]) \cong \mathfrak{g} \otimes (\mathbb{C}[t] \otimes (t-c)^N \mathbb{C}[t]) \qquad (3.56)$$

induced by Taylor expansion. So classical Kirillov-Reshitikhin modules are uniquely lifted to $\mathbf{L}\mathfrak{g}$-modules.

**Proposition 2.** *(Proposition 2.8 in [163]) Let $a \in V, \ell \in \mathbb{N}$ and $\xi \in \mathscr{A}^\times$, and set $c = \xi(1) \in \mathbb{C}^\times$. Then the classical limit $\overline{W_q^{a,\ell}(\xi)}$ is isomorphic to $\mathrm{KR}_{a,\ell}(c)$ as a $\mathbf{L}\mathfrak{g}$-module.*

The point of all of this is the following theorem:

**Theorem 11.** *(Theorem 3.1 in [163]) Let $a_1,\dots,a_p \in V, \ell_1,\dots,\ell_p \in \mathbb{N}$ and $\xi_1,\dots,\xi_p \in \mathscr{A}^\times$. Assume that the tensor product $W_q^{a_1,\ell_1}(\xi_1) \otimes \cdots \otimes W_q^{a_p,\ell_p}(\xi_p)$ is $\ell$-highest weight. If $\xi_1(1) = \cdots = \xi_p(1) = c \in \mathbb{C}^\times$, then the classical limit of the tensor product is isomorphic as a $\mathfrak{g}[t]$-module to the pullback with respect to $\varphi_c$ of the fusion product of $\mathrm{KR}_{a_k,\ell_k}$:*

$$\overline{W_q^{a_1,\ell_1}(\xi_1) \otimes \cdots \otimes W_q^{a_p,\ell_p}(\xi_p)} \cong \varphi_c^*(\mathrm{KR}_{a_1,\ell_1} * \cdots * \mathrm{KR}_{a_p,\ell_p}(0)). \qquad (3.57)$$

### Relation to the Yangian

The work [196] was the first to show that for $\mathfrak{g}$ of $ADE$-type, if $\mathfrak{C}$ (resp. $\mathfrak{D}$) is the abelian category of finite dimensional $U_q(\mathbf{L}\mathfrak{g})$-(resp. $Y_\hbar(\mathbf{L}\mathfrak{g})$-)modules such that the Drinfeld polynomials of the simple factors have roots in $q^{\mathbb{Z}}$ (resp. $\mathbb{Z}$), then the characters of simple modules in $\mathfrak{C}$ and $\mathfrak{D}$ are the same.

A more sophisticated form of this statement was in the work of [88]. Given a finite dimensional representation of $Y_\hbar(\mathfrak{g})$, they constructed an action of $U_q(\mathbf{L}\mathfrak{g})$. In this work they showed that if $\hbar \in \mathbb{C}\backslash\mathbb{Q}$ and $q = e^{\pi i \hbar}$, then there is an exact faithful functor

$$\Gamma : \mathrm{Rep}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \to \mathrm{Rep}_{\mathrm{fd}}(Y_q(\mathbf{L}\mathfrak{g})), \qquad (3.58)$$

where $\mathrm{Rep}_{\mathrm{fd}}(\mathfrak{a})$ is the abelian category of finite dimensional representations of the algebra $\mathfrak{a}$.

Theorem 6.3 of [88] gives that if $\hbar \in \Pi \subset \mathbb{C}$, where $\Pi$ is a subset of $\mathbb{C}$ stable under shifts by $\frac{1}{2}\mathbb{Z}\hbar$ and $x - y \notin \mathbb{Z}_{\neq 0}$ for all $x, y \in \Pi$, and $\Omega = e^{2\pi i \Pi}$, where $\Omega \subset \mathbb{C}^\times$ is stable under

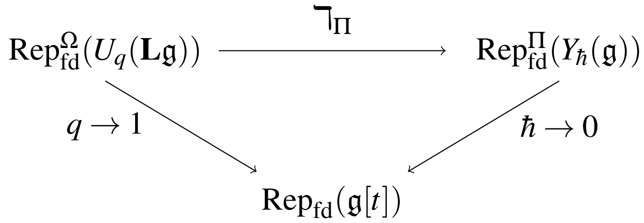


Fig. 6.1 The commuting diagram of categories.

multiplication by $q$, then the functor $\Gamma$ restricts to a functor

$$\Gamma_\Pi : \mathrm{Rep}^{\Pi}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g})) \to \mathrm{Rep}^{\Omega}_{\mathrm{fd}}(Y_q(\mathbf{L}\mathfrak{g})), \tag{3.59}$$

where $\mathrm{Rep}^{\Pi}_{\mathrm{fd}}(Y_\hbar(\mathfrak{g}))$ is the subcategory of representations whose roots of the associated Drinfeld polynomial lie in $\Pi$ (similarly for $U_q(\mathbf{L}\mathfrak{g})$ and $\Omega$).

The functor $\Gamma_\Pi$ admits an inverse functor:

$$\daleth_\Pi : \mathrm{Rep}^{\Omega}_{\mathrm{fd}}(U_q(\mathscr{L}\mathfrak{g})) \to \mathrm{Rep}^{\Pi}_{\mathrm{fd}}(Y_{\bar{h}}(\mathfrak{g}))\,. \tag{3.60}$$

If $\Pi$ is a fundamental domain for $u \mapsto u+1$, then $\Gamma_\Pi$ gives rise to an isomorphism of categories.

In order to show this, they constructed an action of $Y_\hbar(\mathfrak{g})$ on $U_q(\mathbf{L}\mathfrak{g})$-modules. This action commutes with the classical limit (see figure 6.1), and hence, we see that the fusion product of classical Kirillov-Reshitikhin modules is the classical limit of the tensor product of quantum Kirillov-Reshitikhin modules for the Yangian.

### 6.3.3 The result

We want to extend the rank of the flavour group at any particular node $N_a$ to a partition. We call this collection of partitions $\vec{\nu} \in \mathscr{P}^V$. For $\nu \in \mathbb{N}_0^V$, define $r(\nu,\tau) \in \mathbb{Q}(\tau)[\![x_i | i \in V]\!]$ as (chapter 1 equation (3) of [150])

$$r(\nu,\tau^{-2}) := \sum_{\zeta \in \mathscr{P}^V} \prod_{i \in V} x_i^{|\zeta^{(i)}|} \tau^{2(\nu,\zeta_1) - 2\sum_{a=1}^{\infty} T(\zeta_a)} \prod_{a \geqslant 1} [\infty, \zeta_a - \zeta_{a+1}]\,. \tag{3.61}$$

In order to prove the fermionic Lusztig conjecture Mozgovoy proves Theorem 1.2 in [150], which states

$$n(\nu,\tau^{-2}) = r(\nu,\tau^2) r(0,\tau^{-2})\,. \tag{3.62}$$

We want this formula, but for $\vec{v} \in \mathscr{P}^V$, instead of $v \in P_+$. We extend the definition of $r$ to

$$r(\vec{v}, \tau^{-2}) := \sum_{\zeta \in \mathscr{P}^V} \prod_{i \in V} x_i^{|\zeta^{(i)}|} \tau^{2\sum_{i\in V}\sum_{a=1}^{\infty} v_a^{(i)} \zeta_a^{(i)} - 2\sum_{a=1}^{\infty} T(\zeta_a)} \prod_{a \geqslant 1} [\infty, \zeta_a - \zeta_{a+1}] \,. \tag{3.63}$$

Define the generating function of $n(\vec{v}, \lambda, \tau^2)$ as

$$n(\vec{v}, \tau^2) := \sum_{\lambda \in P} n(\vec{v}, \lambda, \tau^2) x^{v - \lambda} \,. \tag{3.64}$$

**Lemma 6.** *We have that*

$$n(\vec{v}, \tau^{-2}) = r(\vec{v}, \tau^2) r(0, \tau^{-2}) \,. \tag{3.65}$$

We prove this in section 6.5.

Noting that $r(\vec{0}, \tau) = r(0, \tau)$, we find that the proof in chapter 8 of [150] exactly follows through. For an $ADE$-type quiver, the Coulomb branch formula with arbitrary non-negative background charge obeys

$$\mathscr{G}(\vec{v}) = \sum_{\lambda \in P} n(\vec{v}, \lambda, \tau^2) \prod_{i \in V} z_i^{\lambda - |\vec{v}|} \prod_{\alpha \in Q_+} \frac{1}{1 - z^{\alpha}} \,, \tag{3.66}$$

where $Q_+$ is the set of positive roots. We review this argument in section 6.6.

We have that under the action of any element $w$ of the Weyl group of $\Gamma$ that[3]

$$n(v, w \cdot \lambda, \tau^2) = (-)^{l(w)} n(v, \lambda, \tau^2) \,. \tag{3.67}$$

Hence, by the Weyl character formula we have that

$$\mathscr{G}(\vec{v}) = \sum_{\lambda \in P_+} n(\vec{v}, \lambda, \tau^2) \prod_{i \in V} z_i^{-|\vec{v}|} \mathrm{ch}\, L(\lambda) \,, \tag{3.68}$$

where $L(\lambda)$ is the irreducible representation of highest weight $\lambda$. Notably, for an $A$-type quiver ch $L(\lambda)$ is the Schur polynomial $s_\lambda$.

**Theorem 12.** *Let $\Gamma$ be a balanced ADE-type quiver, with $G$ the corresponding Lie group, and $\mathfrak{g}$ the corresponding Lie algebra. Choose ranks $(ck, cN)$, for $c \in \mathbb{N}, k \in \mathbb{N}^V$ and $N \in \mathbb{N}_0^V$, and let $\mathscr{X}_c$ be the Coulomb branch. For $m \in \mathbb{Z}^N$, we define the line bundle on $\mathscr{X}_c$ as the chiral operators with background magentic charge $m$, and denote this line bundle as $\mathscr{L}_{m,c}$.*

[3]This is conjecture 8.3 in [101], and still remains unproven in the literature. The lack of proof of this identity means that theorem 12 is not yet fully proven.

*Then as a representation of the Yangian $Y(\mathfrak{g})$*

$$\lim_{c\to\infty} Z^{|m|}\tau^{-c\sum_i ||m^{(i)}||k_i}\Gamma(\mathscr{X}_c, \mathscr{L}_{m,c}) \cong V\left(m^{(+)}\right) \otimes V\left(m^{(-)}\right)^{\dagger} \otimes \mathrm{gr}(Y(\mathfrak{g}))\,. \tag{3.69}$$

*where $V(\mu)$ is the classical limit of the tensor product of quantum Kirillov-Reshitikhin modules defined by $\mu \in \mathscr{P}^V$ and $\mathbb{C}^\times$-action corresponding to the grading, $V(\mu)^\dagger$ is the same but with the ADE-charge flipped ($z_i \mapsto 1/z_i$), $\mathrm{gr}(Y(\mathfrak{g}))$ is the semi-classical limit of the Yangian $Y(\mathfrak{g})$, which is $\mathbb{C}^\times$-graded.*

We prove this in section 6.6.

## 6.4 *A*-type quivers

Here we restrict our attention to *A*-type quivers. In section 6.4.1, we show the expression for the Poincaré polynomial of an *A*-type Nakajima quiver variety of Proudfoot and Schedler in [171] matches our expression, utilising an involution of the spin chain partition function. In section 6.4.2, we explicitly find for a specific linear quiver that the Poisson algebra in the infinite rank limit is the semiclassical limit of the Yangian of $\mathfrak{gl}_N$, as expected.

### 6.4.1 Matching the result

Note that the Poincaré polynomial generating function can have a further grading, given by the fact that the quiver varieties are all cones and thus admit a $\mathbb{C}^\times$-action that leaves the homological degree invariant. Indeed, Proudfoot and Schedler [171] showed that the Poincaré polynomial for $\mathfrak{M}_{\zeta,\sigma\rho} = \mathrm{Higgs}\left[T^{\sigma}_{\rho}(SU(N))\right]$ is

$$P_{\mathfrak{M}_{\zeta,\sigma\rho}}(y,\tau) = \tau^{2n[\rho]-2n[\sigma]} \sum_{\sigma\leqslant\nu\leqslant\rho^\vee} \tau^{2n[\nu]-2n[\nu^\vee]} K_{\nu\sigma}(y) K_{\nu^\vee\rho}(\tau^{-2})\,. \tag{4.70}$$

Here $y$ is counting twice the homological degree and $\tau$ is counting the weights under the $\mathbb{C}^\times$ action.

If we set $\tau$ to 1, we have that

$$P_{\mathfrak{M}_{\zeta,\sigma\rho}}(y,1) = \sum_{\sigma\leqslant\nu\leqslant\rho^\vee} K_{\nu\sigma}(y) K_{\nu^\vee\rho}\,. \tag{4.71}$$

We should be able to match this formula to the one given by Mozgovoy in [150]. In this section we show exactly that.

We define the involution $\omega$, a ring automorphism of the ring of symmetric functions defined by $\omega s_\lambda = s_{\lambda^\vee}$. Note that $\omega^2 = 1$.

The formula from Mozgovoy says that for an $A$-type quiver

$$\sum_{k\in\mathbb{N}^V}\sum_a q^{a-\dim_{\mathbb{H}}\mathfrak{M}_{\zeta,(k,N)}}\dim H^{\mathrm{BM}}_{2a}(\mathfrak{M}_{\zeta,(k,N)})Z^{\mu(k,N)} = \sum_{\lambda\in\mathscr{P}} m(N,\lambda,q^{-1})s_\lambda(Z)\,, \tag{4.72}$$

where $\mu_i(k,N) := N_i + k_{i+1} + k_{i-1} - 2k_i$, this is in the Weyl orbit of a partition, $\rho$, for generic $(k,N)$. We are slightly abusing notation by summing over the partition $\lambda$ associated to a dominant weight. Note that if $\psi$ is a dominant weight of $A_n$, then $\psi$ defines a partition via $\lambda(\psi)_i = \sum_{j=i}^n \psi_j$.

To compare equations (4.72) and (4.71), we need to restrict $k$ in equation (4.72) such that it defines a good quiver. We expand the right hand side of equation (4.72) in monomials,

$$\sum_{k\in\mathbb{N}^V}\sum_a q^{a-\dim_{\mathbb{H}}\mathfrak{M}_{\zeta,(k,N)}}\dim H^{\mathrm{BM}}_{2a}(\mathfrak{M}_{\zeta,(k,N)})Z^{\mu} = \sum_{\lambda,\rho\in\mathscr{P}} m(N,\lambda,q^{-1})K_{\lambda\rho}m_\rho(Z)\,. \tag{4.73}$$

Note that the monomial is a sum over the Weyl group, and any individual $\rho$ on the right hand side gives the Poincaré polynomial for a family of Seiberg dual quivers, which, from section 2.3.3, we know are all identical (note that we are working purely with resolved Higgs branches). Picking $k$ such that we have a good quiver (corresponding to the term $Z^\rho$ in the monomial symmetric polynomial), this then implies that

$$P(\mathfrak{M}_{\zeta,\rho\sigma},q) = q^{\dim_{\mathbb{H}}\mathfrak{M}_{\zeta,\rho\sigma}}\sum_\lambda m\big(((m_i(\sigma)^i))_{i\in V},\lambda,q^{-1}\big)K_{\lambda\rho}\,, \tag{4.74}$$

where $\tilde{\rho} = (\sum_{j=i}^n \rho_j)_i$, $N_i = m_i(\sigma)$ and $((m_i(\sigma)^i))_{i\in V}$ is defined in equation (4.80). Note that $|((m_i(\sigma)^i))_{i\in V}| = |\sigma| = |\rho| = |\lambda|$.

We look at the partition function for the $SU(n+1)$ spin chain, where we grade by the weights of the diagonal $SU(n+1)$ as well as the energy, which we count with $q$. If the sites are

$$\omega((N_i^{\,i}))_{i\in V} := \bigotimes_{i=1}^n [i,0,\ldots,0]^{\otimes m_i(\sigma)}\,, \tag{4.75}$$

then, we know from [129] that the partition function is

$$\mathfrak{X} = \sum_\lambda K_{\lambda\sigma}(q)s_\lambda(Z)\,. \tag{4.76}$$

Note that the Kostka polynomial is related to the fermionic form of [101] via

$$m\big(\omega((m_i(\sigma)^i))_{i\in V},\lambda,q\big)q^{c(\omega((m_i(\sigma)^i))_{i\in V})} = K_{\lambda\,\sigma}(q)\,, \tag{4.77}$$

where $c(R)$ is the absolute value of the minimal dimension in $m\big(R,\lambda,q\big)\in\mathbb{N}_0[q^{-1}]$. We have that

$$c(((m_i(\sigma)^i))_{i\in V}) = \dim_{\mathbb{H}}\mathfrak{M}_{\zeta,\rho\sigma}\,, \tag{4.78}$$

because we know that $H_0^{\mathrm{BM}}(\mathfrak{M}_{\zeta,\rho\sigma})\neq 0$ and equation (4.72).

We consider the action of the involution $\omega$ on the spin chain partition function. We act in two different ways on the spin chain, one way directly on the partition function and the other on the sites. The first way gives partition function

$$\omega\mathfrak{X} = \sum_{\lambda} K_{\lambda\sigma}(q)s_{\lambda^\vee}(Z)\,. \tag{4.79}$$

The other way changes the sites as

$$\omega^2((m_i(\sigma)^i))_{i\in V} = ((m_i(\sigma)^i))_{i\in V} := \bigotimes_{i=1}^{n}[0,\dots,0,1,0,\dots,0]^{\otimes m_i(\sigma)}\,, \tag{4.80}$$

giving partition function

$$\omega\mathfrak{X} = \sum_{\lambda} q^{c(((m_i(\sigma)^i))_{i\in V})} m\big(((m_i(\sigma)^i))_{i\in V},\lambda,q\big)s_\lambda(Z)\,. \tag{4.81}$$

We then compare our two different expressions for $\omega\mathfrak{X}$. Using that $m_\lambda$ is a basis for symmetric functions, we deduce

$$\sum_{\tilde{\rho}} q^{\dim_{\mathbb{H}}\mathfrak{M}_{\zeta,\rho\sigma}} m((N_i{}^i)_i,\rho,q)K_{\tilde{\rho}\,\lambda^\vee} = \sum_{\tilde{\rho}} K_{\tilde{\rho}\mu}(q)K_{\tilde{\rho}^\vee\lambda^\vee}\,. \tag{4.82}$$

The conclusion is that the right hand side of equation (4.74) is the right hand side of equation (4.71).

We finish by noting that the $\tau$-grading in equation (4.70) implies that there may be another grading of the infinite rank Coulomb branch. The author is unaware currently whether this is the case, and what the interpretation of such a grading would be.

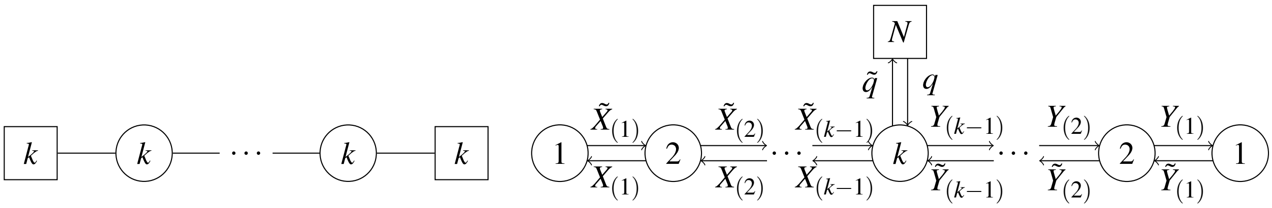


Fig. 6.2 The same two quivers found in figure 4.3 in chapter 4, but with the hypermultiplets on the right hand side explicitly labelled. The limit we consider is $k \to \infty$ on the Coulomb branch of the left hand side and the Higgs branch of the right hand side. Note that the length of the quiver on the right hand side goes to infinity.

### 6.4.2 The infinite rank limit on the Higgs branch - Poisson algebra

In this section, we shall investigate the mirror dual of the infinite rank limit of a balanced Coulomb branch quiver of section 6.2.1. We shall calculate the Poisson algebra given by the coordinate ring of the mirror dual Higgs branch in this limit. This motivates our result of section 6.3.1, that states the quantised algebra is $Y(\mathfrak{g})$. The particular quiver we consider is pictured with its mirror dual in figure 6.2.

We shall first work at finite $k$. The coordinate ring is composed of all polynomials in $q,\tilde{q},X,\tilde{X},Y,\tilde{Y}$ that are invariant under the gauge group modulo the $F$-term equations. Before quotienting by the $F$-term relations, we have

$$\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)] = \mathbb{C}[f,\tilde{q}gq | f,g \in \mathbb{C}[X,Y,\tilde{X},\tilde{Y}],\, f \text{ in } \underline{1},\, g \text{ in } \underline{k}\times\overline{\underline{k}} \text{ of } g]\,. \tag{4.83}$$

We define $X_{(0)},\tilde{X}_{(0)},Y_{(0)},\tilde{Y}_{(0)} \equiv 0$. The $F$-term relations are

$$\begin{aligned} X_{(A-1)}\tilde{X}_{(A-1)} + X_{(A)}\tilde{X}_{(A)} &= 0\,, \quad 1 \leqslant A < k-1\,,\\ Y_{(A-1)}\tilde{Y}_{(A-1)} + Y_{(A)}\tilde{Y}_{(A)} &= 0\,, \quad 1 \leqslant A < k-1\,,\\ X_{(k-1)}\tilde{X}_{(k-1)} + Y_{(k-1)}\tilde{Y}_{(k-1)} + q\tilde{q} &= 0\,. \end{aligned} \tag{4.84}$$

We define for $r \in \mathbb{N}_0$ and $i,j = 1,\ldots,N$,

$$\begin{aligned} C_{(r)} &:= \operatorname{tr}\left((\tilde{X}_{(k-1)}X_{(k-1)})^s\right),\\ J_{(r)}{}^{i}{}_{j} &:= \tilde{q}_j(\tilde{X}_{(k-1)}X_{(k-1)})^r q^i\,. \end{aligned} \tag{4.85}$$

We will show that

$$\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)] = \mathbb{C}[C_{(r)},J_{(r)}{}^{i}{}_{j} | i,j = 1,\ldots,N \text{ and } r \in \mathbb{N}_0]\,. \tag{4.86}$$

Note that when $k$ is finite, the ring on the right hand side of equation (4.86) will not be freely generated because of finite rank constraints (this is the classical limit of the truncation in section 6.3.1).

The Hilbert series for finite $k$, as found in chapter 4 equation (7.73), can be written using the Cauchy expansion and the expression for the dot product in equation (1.4) as

$$\mathrm{HS}_k(0,0) = \sum_{\substack{\lambda \\ l(\lambda)\leqslant k}} \frac{1}{b_\lambda(\tau^2)\varphi_{k-l(\lambda)}(\tau^2)} Q'_\lambda(\tau Z;\tau^2) Q'_\lambda(\tau Z^{-1};\tau^2)\,. \tag{4.87}$$

The infinite rank limit[4] of this is given simply by the Cauchy identity as

$$\lim_{k\to\infty} \mathrm{HS}_k(0,0) = \mathrm{PE}\left[\sum_{r=1}^{\infty} \tau^{2r} + \sum_{r=0}^{\infty}\sum_{i,j=1}^{N} \tau^{2+2r} z_i z_j^{-1}\right]\,. \tag{4.88}$$

This then means that in the infinite rank limit, there are no relations in equation (4.86), and it is freely generated as a commutative ring by the elements $J$ and $C$.

We prove equation (4.86) by strong induction on the degree of the monomial. It is clearly true at degree 0. Suppose it is true up to degree $2r-2$ polynomials (all polynomials will be of even degree).

We first show that the degree $2r$ monomial can be made to contain no $Y$ terms. Indeed, if one did then either two things are true, (1) it contains the term $\tilde{Y}_{(k-1)}$ or (2) the polynomials is a trace of a polynomial purely in the $Y,\tilde{Y}$ variables not containing $\tilde{Y}_{(k-1)}$. Assume it is the first case, then our polynomial contains a substring, for some $s = 1,\dots,k-1$, of the form

$$\tilde{Y}_{(k-1)}\tilde{Y}_{(k-2)}\dots\tilde{Y}_{(k-s)}Y_{(k-s)}\,. \tag{4.89}$$

We can use the $F$-term constraints to rewrite this as

$$(-)^r\big(\tilde{X}_{(k-1)}X_{(k-1)} + \sum_{j=1}^{N}\tilde{q}_j q^j\big)\tilde{Y}_{(k-1)}\dots\tilde{Y}_{(k-s+1)}\,. \tag{4.90}$$

This can be clearly continued until there are no more $Y$-terms. In the other case, we have a polynomial of the form

$$\mathrm{tr}\,\tilde{Y}_{(s)}\dots\tilde{Y}_{(s-a)}Y_{(s-a)}\dots Y_{(s-a+b)}\tilde{Y}_{(s-a+b)}\dots Y_{(s)}\,, \tag{4.91}$$

[4] $\mathrm{HS}_{\tilde{k}}(0,0) - \mathrm{HS}_k(0,0) \equiv 0 \pmod{\tau^{2k}}$ for all $\tilde{k} > k$ means that this limit is well defined, though we already showed the limit was well defined in proving theorem 10.

where $s$ is the maximal subscript. Then, using the $F$-term constraints, we commute the first $Y$ term past all the $\tilde{Y}$'s to its left, and then cycle it to the right using the cyclicity of the trace. This gives

$$(-)^{a+1}\operatorname{tr}\tilde{Y}_{(s+1)}\dots\tilde{Y}_{(s-a+1)}Y_{(s-a+1)}\dots Y_{(s-a+b)}\tilde{Y}_{(s-a+b)}\dots Y_{(s)}Y_{(s+1)}\,, \tag{4.92}$$

Thus, we can send $s$ up to $k$ in this way and use our result for case 1.

Our argument for dealing with the $Y$'s can be applied to the $X$'s such that only $X_{(k-1)}$ and $\tilde{X}_{(k-1)}$ appears. The result then follows.

Now we compute the Poisson algebra. We take the holomorphic symplectic form

$$\omega_{\mathbb{C}} = \sum_{A=1}^{k-1}\left(\operatorname{tr} dX_{(A)}\wedge d\tilde{X}_{(A)} + \operatorname{tr} dY_{(A)}\wedge d\tilde{Y}_{(A)}\right) - \sum_{j=1}^{N} d\tilde{q}_j\wedge dq^j\,. \tag{4.93}$$

The Poisson brackets are

$$\begin{aligned}
&\{C_{(r)},C_{(s)}\} = \{C_{(r)},J_{(s)}\} = 0\,,\\
&\{J_{(r)}{}^{i}{}_{j}, J_{(s)}{}^{k}{}_{l}\} = \delta^i_l J_{(r+s)}{}^{l}{}_{j} - \delta^k_j J_{(r+s)}{}^{i}{}_{l}\\
&\qquad + \sum_{m=0}^{r-1}\left(J_{(m+s)}{}^{k}{}_{j} J_{(r-1-m)}{}^{i}{}_{l} - J_{(m)}{}^{k}{}_{j} J_{(s-1+r-m)}{}^{j}{}_{l}\right)\,.
\end{aligned} \tag{4.94}$$

Note that this is, up to quotienting by the ideal generated by the central elements $C_{(r)}$ , the semi-classical limit of the Yangian of $\mathfrak{gl}_N$, for this see [149]. This is exactly what we would expect from section 6.3.1.

## 6.5 Proof of lemma 6

The proof that we give here is a generalisation of the proof of section 7 of [150]. We will follow his argument, with some subtleties arising from the generalisation.

*Proof.* Define $L$ to be the maximal value such that $\nu_L^{(i)} \neq 0$ for some $i \in V$. We introduce fugacities $y_{ai}$ for $i \in V$ , $a = 1,\dots,L$ that count the internal magnetic charges, this is not gauge invariant, and so is not an observable. Define the ring

$$R_L := \mathbb{Q}(\tau)[\![x_i, y_{ai} | i \in V\,,\, a = 1,\dots,L]\!]\,. \tag{5.95}$$

This ring has elements

$$
\begin{aligned}
s_L(\vec{\nu}) &:= \sum_{\vec{\zeta}\in\mathscr{P}^V} x^{|\vec{\zeta}|} \prod_{a=1}^{L} y_a^{\zeta_a-\zeta_{a+1}(1-\delta_{aL})} \prod_{k\geqslant 1}\left(\tau^{2T(\zeta_k)}\left[\sum_{l=1}^{k}(\nu_l-\zeta_l),\zeta_k-\zeta_{k+1}\right]\right), \\
s_L &:= \sum_{\vec{\zeta}\in\mathscr{P}^V} x^{|\vec{\zeta}|} \prod_{a=1}^{L} y_a^{\zeta_a-\zeta_{a+1}(1-\delta_{aL})} \prod_{k\geqslant 1}\left(\tau^{2T(\zeta_k)}\left[\infty,\zeta_k-\zeta_{k+1}\right]\right).
\end{aligned}
\tag{5.96}
$$

Note that

$$
s_L = \lim_{\vec{\nu}\to\infty} s_L(\vec{\nu}), \tag{5.97}
$$

where in this limit we send $\nu_1^{(i)} \to \infty$ for each $i \in V$.

For $\vec{\alpha} \in (\mathscr{P}_L)^V$ , define

$$
\begin{aligned}
\vec{\alpha}' &:= (\alpha_1-\alpha_2,\alpha_2-\alpha_3,\ldots,\alpha_{L-1}-\alpha_L,\alpha_L) \in (\mathbb{N}_0^L)^V, \\
A_a &:= \sum_{l=1}^{a} \alpha_l \in \mathbb{N}_0^V.
\end{aligned}
\tag{5.98}
$$

Fix an $\vec{\alpha} \in (\mathscr{P}_L)^V$ and define[5] $\vec{N},\vec{A} \in \mathscr{P}^V$ for $a \in \mathbb{N}_0$

$$
N_a := \sum_{l=1}^{a} \nu_l \in \mathbb{N}_0^V, \tag{5.99}
$$

For $t \in R_L$ and $\vec{\alpha} \in (\mathscr{P}_L)^V$, define $t_{\vec{\alpha}'} \in \mathbb{Q}(\tau)[\![x_i|i \in V]\!]$ via

$$
t = \sum_{\vec{\alpha}\in(\mathscr{P}_L)^V} t_{\vec{\alpha}'} y^{\vec{\alpha}'}. \tag{5.100}
$$

Define the elements of $\mathbb{Q}(\tau)[\![y_{a\,i}|i \in V]\!]$,

$$
\begin{aligned}
p_a(\nu) &:= \sum_{\alpha\in\mathbb{N}_0^V} [\nu,\alpha] y_a^{\alpha}, \\
p_a &:= \sum_{\alpha\in\mathbb{N}_0^V} [\infty,\alpha] y_a^{\alpha}.
\end{aligned}
\tag{5.101}
$$

[5] The author hopes the use of $N$ here will not be confused with the $N$ used to denote the flavour nodes. Nowhere in this section will we use $N$ to denote the ranks of flavour nodes.

Define

$$
\begin{aligned}
|\vec{\nu}| &:= N_L = \sum_{a=1}^{L} \nu_a \in \mathbb{N}_0^V\,, \\
|\vec{\alpha}| &:= A_L = \sum_{a=1}^{L} \alpha_a \in \mathbb{N}_0^V\,.
\end{aligned}
\tag{5.102}
$$

Define the elements of $\mathbb{Q}(\tau)[\![x_i, y_{Li} | i \in V]\!]$

$$
\begin{aligned}
s_{1,L}(|\vec{\nu}|) &:= \sum_{\vec{\zeta}\in\mathscr{P}^V} x^{|\vec{\zeta}|} y_L^{\zeta_1} \prod_{k\geqslant 1}\left(\tau^{2T(\zeta_k)}[|\vec{\nu}| - \sum_{l=1}^{k}\zeta_l, \zeta_k - \zeta_{k+1}]\right)\,, \\
s_{1,L} &:= \sum_{\vec{\zeta}\in\mathscr{P}^V} x^{|\vec{\zeta}|} y_L^{\zeta_1} \prod_{k\geqslant 1}\left(\tau^{2T(\zeta_k)}[\infty, \zeta_k - \zeta_{k+1}]\right)\,.
\end{aligned}
\tag{5.103}
$$

Note that, just as in equation (5.97), we have that

$$
s_{1,L} = \lim_{|\vec{\nu}|\to\infty} s_{1,L}(|\vec{\nu}|)\,. \tag{5.104}
$$

We define a partial order on $\mathscr{P}^V$ via, for $\vec{\alpha}, \vec{\beta} \in \mathscr{P}^V$,

$$
\vec{\alpha} \leqslant \vec{\beta} \iff \alpha_a^{(i)} - \beta_a^{(i)} \geqslant 0 \quad \forall i \in V \text{ and } a \in \mathbb{N}\,. \tag{5.105}
$$

We extend this to a partial order on $(\mathbb{N}_0^L)^V$ in the obvious fashion.

**Lemma 7.** *We have for all $\vec{\nu} \in \mathscr{P}^V$ and $\vec{\alpha} \in \mathscr{P}_L^V$,*

1. $s_L(\vec{\nu}+\vec{\alpha})_{\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\left(\prod_{a=1}^{L} p_a(N_a)\cdot s_{1,L}(|\vec{\nu}|)\right)_{\vec{\alpha}'}\,.$
2. $s_{L\,\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\left(\prod_{a=1}^{L} p_a s_{1,L}\right)_{\vec{\alpha}'}\,.$

*Proof.* We calculate

$$
\begin{aligned}
s_L(\vec{\nu}+\vec{\alpha})_{\vec{\alpha}'} &= \sum_{\substack{\vec{\zeta}\in\mathscr{P}^V \\ \zeta_a=\alpha_a\,\forall 1\leqslant a\leqslant L}} x^{|\vec{\zeta}|}\prod_{k\geqslant 1}\tau^{2T(\zeta_k)}\left[\sum_{l=1}^{k}(\nu_l+\alpha_l-\zeta_l), \zeta_k-\zeta_{k+1}\right] \\
&= x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\prod_{a=1}^{L-1}[N_a, \alpha_a-\alpha_{a+1}]\sum_{\substack{\vec{\zeta}\in\mathscr{P}^V \\ 0\leqslant\zeta_1\leqslant\alpha_L}} x^{|\vec{\zeta}|}\,[|\vec{\nu}|, \alpha_L-\zeta_1] \\
&\quad \prod_{k\geqslant 1}\left(\tau^{2T(\zeta_k)}[|\vec{\nu}| - \sum_{l=1}^{k}\zeta_l, \zeta_k-\zeta_{k+1}]\right)\,.
\end{aligned}
\tag{5.106}
$$

It then follows from the definitions (5.101) and (5.103) that

$$s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\left(\prod_{a=1}^{L}p_a(N_a)\cdot s_{1,L}(|\vec{v}|)\right)_{\vec{\alpha}'} . \tag{5.107}$$

By taking the limit $\vec{v}\to\infty$ and equation (5.97) we get

$$s_{L\,\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\left(\prod_{a=1}^{L}p_a s_{1,L}\right)_{\vec{\alpha}'} . \tag{5.108}$$

□

Define the ring homomorphism

$$\begin{aligned} S_{\nu,a} &: R_L \to R_L \\ \sum_\beta f_\beta y_a^\beta &\mapsto \sum_\beta \tau^{2(\nu,\beta)} f_\beta y_a^\beta , \end{aligned} \tag{5.109}$$

and then define

$$S_{\vec{\nu}} := S_{\nu_1,1}\circ S_{\nu_2,2}\circ\cdots\circ S_{\nu_L,L} . \tag{5.110}$$

We state some results from [150] (recall the definition of $\overline{*}$ from section 6.1):

**Lemma 8.**

$$p_a(\nu) = p_a\cdot S_{\nu,a}\overline{p}_a \tag{5.111}$$

*Proof.* Lemma 7.1 of [150]. □

**Lemma 9.**

$$s_{1,L}(\nu) = s_{1,L}\cdot S_{\nu,L}\overline{s}_{1,L} . \tag{5.112}$$

*Proof.* Theorem 7.4 of [150]. □

**Lemma 10.**

$$s_L(\vec{\nu}) = s_L\cdot S_{\vec{N}}\overline{s}_L . \tag{5.113}$$

*Proof.* By lemma 7, we can write

$$s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L}T(\alpha_a)}\left(\prod_{a=1}^{L}p_a(N_a)\cdot s_{1,L}(|\vec{v}|)\right)_{\vec{\alpha}'} . \tag{5.114}$$

Use lemmas 8 and 9 to write

$$s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} = x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L} T(\alpha_a)} \left( \prod_{a=1}^{L} p_a \cdot S_{N_a a}\overline{p}_a \cdot s_{1,L} \cdot S_{|\vec{v}|L}\overline{s}_{1,L} \right)_{\vec{\alpha}'} . \tag{5.115}$$

$S_{v,a}$ is a ring homomorphism and $\overline{s}_{1,L}$ does not contain $y_a$ for $a < L$, so we may write (observing that $\vec{\beta}' \leqslant \vec{\alpha}' \implies \vec{\beta} \leqslant \vec{\alpha}$)

$$\begin{aligned} s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} &= x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L} T(\alpha_a)} \left( \prod_{a=1}^{L} p_a \cdot s_{1,L} S_{\vec{N}} \left( \prod_{a=1}^{L} \overline{p}_a \cdot \overline{s}_{1,L} \right) \right)_{\vec{\alpha}'} \\ &= x^{|\vec{\alpha}|}\tau^{2\sum_{a=1}^{L} T(\alpha_a)} \sum_{0 \leqslant \vec{\beta} \leqslant \vec{\alpha}} \left( \prod_{a=1}^{L} p_a \cdot s_{1,L} \right)_{\vec{\beta}'} \left( S_{\vec{N}} \left( \overline{\prod_{a=1}^{L} p_a \cdot s_{1,L}} \right) \right)_{\vec{\alpha}'-\vec{\beta}'} . \end{aligned} \tag{5.116}$$

Then we use lemma 7 twice to write

$$\begin{aligned} & s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} \\ &= \sum_{0 \leqslant \vec{\beta} \leqslant \vec{\alpha}} x^{|\vec{\alpha}|-|\vec{\beta}|}\tau^{2\sum_{a=1}^{L} \left(T(\alpha_a)-T(\beta_a)\right)} s_{L\vec{\beta}'} \left( S_{\vec{N}} x^{-|\vec{\alpha}|+|\vec{\beta}|} \tau^{-\sum_{a=1}^{L} T(\alpha_a-\beta_a)} \overline{s}_L \right)_{\vec{\alpha}'-\vec{\beta}'} . \end{aligned} \tag{5.117}$$

Rearranging this gives

$$s_L(\vec{v}+\vec{\alpha})_{\vec{\alpha}'} = \sum_{0 \leqslant \vec{\beta}' \leqslant \vec{\alpha}'} \tau^{2\sum_{a=1}^{L}(\alpha_a,\beta_a)} s_{L\vec{\beta}'} \left( S_{\vec{N}} \overline{s}_L \right)_{\vec{\alpha}'-\vec{\beta}'} . \tag{5.118}$$

We will show that $s_L S_{\vec{N}} \overline{s}_L$ equals the right hand side of equation (5.118), and thus they must be equal. Consider

$$\begin{aligned} (s_L \cdot S_{\vec{N}+\vec{A}} \overline{s}_L)_{\vec{\alpha}'} &= \sum_{0 \leqslant \vec{\beta} \leqslant \vec{\alpha}} s_{L\vec{\alpha}'-\vec{\beta}'} \cdot S_{\vec{N}+\vec{A}} \Bigg( \sum_{\vec{\zeta} \in \mathscr{P}^V} x^{|\vec{\zeta}|} \prod_{a=1}^{L} y_a^{\zeta_a - \zeta_{a+1}(1-\delta_{aL})} \\ &\qquad \prod_{k \geqslant 1} \tau^{-2T(\zeta_k)} \overline{[\infty, \zeta_k - \zeta_{k+1}]} \Bigg)_{\vec{\beta}'} \\ &= \sum_{0 \leqslant \vec{\beta} \leqslant \vec{\alpha}} \tau^{2\sum_{a=1}^{L}(A_a, \beta_a - \beta_{a+1}(1-\delta_{aL}))} s_{L\vec{\alpha}'-\vec{\beta}'} \cdot (S_{\vec{N}} \overline{s}_L)_{\vec{\beta}'} \\ &= \sum_{0 \leqslant \vec{\beta} \leqslant \vec{\alpha}} \tau^{2\sum_{a=1}^{L}(\alpha_a,\beta_a)} s_{L\vec{\alpha}'-\vec{\beta}'} \cdot (S_{\vec{N}} \overline{s}_L)_{\vec{\beta}'} . \end{aligned} \tag{5.119}$$

With this we may conclude that

$$s_L(\vec{v}) = s_L \cdot S_{\vec{N}} \overline{s}_L . \tag{5.120}$$

□

Define the ring homomorphisms

$$\Phi_{v,a} : \mathbb{Q}(\tau)[\![x_i, y_{bi} | i \in V\,, b = 1, \dots, a-1]\!][y_{ai} | i \in V] \to \mathbb{Q}(\tau)[\![x_i, y_{bi} | i \in V\,, b = 1, \dots, a-1]\!] \tag{5.121}$$

via

$$\sum_\beta f_\beta y_a^\beta \mapsto \sum_\beta \tau^{-2(v,\beta)} f_\beta\,. \tag{5.122}$$

Define

$$\Phi_{\vec{v}} := \Phi_{v_1,1} \circ \Phi_{v_2,2} \circ \cdots \circ \Phi_{v_L,L}\,. \tag{5.123}$$

With this we are now ready to proof lemma 6. We have that

$$n(\vec{v}) = \Phi_{\vec{N}}(s_L(\vec{v}))\,, \quad r(\vec{v}) = \overline{\Phi_{\vec{N}}(s_L)}\,. \tag{5.124}$$

So using equations (5.113) and (5.124) we get

$$\begin{aligned} n(\vec{v}) &= \Phi_{\vec{N}}(s_L(\vec{v})) \\ &= \Phi_{\vec{N}}(s_L \cdot S_{\vec{N}} \bar{s}_L) \\ &= \Phi_{\vec{N}} s_L \cdot \Phi_{\vec{0}} \bar{s}_L \\ &= \overline{r(\vec{v})} \cdot r(\vec{0})\,. \end{aligned} \tag{5.125}$$

This extends the proof to the case of generic charge. □

## 6.6 Proof of theorem 12

*Proof.* For $\Gamma$ a finite quiver, [117] showed that for any $\alpha \in \mathbb{N}^V$, there exists polynomials $a_\alpha(\Gamma) \in \mathbb{Z}[q]$ and $m_\alpha(\Gamma) \in \mathbb{Z}[q]$ such that for any finite field $\mathbb{F}_q$, $a_\alpha(\Gamma, q)$ (respectively $m_\alpha(\Gamma, q)$) is the number of isomorphism classes of indecomposable representations (respectively all representations) of $\Gamma$ over $\mathbb{F}_q$ of dimensions $\alpha$. Define the generating functions $a(\Gamma, q) = \sum_{\alpha \in \mathbb{N}_0^V} a_\alpha(\Gamma, q) Z^\alpha$ and $m(\Gamma, q) = \sum_{\alpha \in \mathbb{N}_0^V} m_\alpha(\Gamma, q) Z^\alpha$.

[151] lemma 5 shows that

$$m(\Gamma, q) = \mathrm{PE}[a(\Gamma, q)]\,. \tag{6.126}$$

Results of [112] can be used to show that (shown in theorem 6 of [151])

$$\mathrm{PE}\left[\frac{a(\Gamma,q)}{q-1}\right] = r(\Gamma,q)\,. \tag{6.127}$$

It is known that for $\Gamma$ of $ADE$-type that

$$m(\Gamma,q) = \prod_{\alpha\in Q_+} \frac{1}{1-Z^\alpha}\,. \tag{6.128}$$

From this we see that for $\Gamma$ of $ADE$-type,

$$r(\Gamma,q)r(\Gamma,q^{-1}) = \mathrm{PE}\left[\frac{a(\Gamma,q)}{q-1}\right]\mathrm{PE}\left[\frac{a(\Gamma,q^{-1})}{q^{-1}-1}\right] = \mathrm{PE}[-a] = \frac{1}{m}\,. \tag{6.129}$$

Assume from now that $\Gamma$ is of $ADE$-type. Define the space $V(\vec{\tilde{\lambda}})$ to be the classical limit of the tensor product of $\tilde{\lambda}_a^{(i)}$ lots of the quantum Kirillov-Reshitikhin modules $W_q^{i,a}$ for $i \in V$, $a \in \mathbb{N}$. From the discussion in section 6.3.2 we know that this is equal to the fusion product of the classical Kirillov-Reshitkhin modules[6]

$$V(\vec{\tilde{\lambda}}) = \mathop{\ast}_{\substack{k \\ a_k(1)=c}} \mathrm{KR}_{i,a}^{*\tilde{\lambda}_a^{(i)}}\,. \tag{6.130}$$

This is acted on by $T = \mathbb{C}^\times \times H$, where $H$ is the Cartan subgroup of $G$ the Lie group associated to $\mathfrak{g}$, and the $\mathbb{C}^\times$ action is given by the grading of the fusion product. We know further from section 6.3.2 that

$$\mathrm{ch}_T\left(V\left(\vec{\tilde{\lambda}}\right)\right) = \sum_{\lambda\in P_+} n\left(\vec{\tilde{\lambda}},\lambda,\tau^2\right)\mathrm{ch}_H L_\lambda\,. \tag{6.131}$$

Using equation (3.67), we can write

$$\begin{aligned}
\mathrm{ch}_T V\left(\vec{\tilde{\lambda}}\right) &= \sum_{\lambda\in P} n\left(\vec{\tilde{\lambda}},\lambda,\tau^2\right)\mathrm{ch}_H M_\lambda \\
&= \sum_{\lambda\in P} n\left(\vec{\tilde{\lambda}},\lambda,\tau^2\right) Z^\lambda \prod_{\alpha\in Q_+}\frac{1}{1-Z^\alpha} \\
&= n\left(\vec{\tilde{\lambda}},\tau^2\right)\prod_{\alpha\in Q_+}\frac{1}{1-Z^\alpha}\,.
\end{aligned} \tag{6.132}$$

[6] In that definition there is a dependence on a complex number, but we can ignore this because of the (proven) Feigin-Loktev conjecture.

By lemma 6 and equation (6.129), we see that

$$\mathrm{ch}_T V\left(\vec{\tilde{\lambda}}\right) = \frac{r\left(\vec{\tilde{\lambda}}, \tau^2\right)}{r(0,\tau^2)}\,. \tag{6.133}$$

For $\pi \in \mathscr{P}^V$,

$$\prod_{i\in V}\prod_{a\in\mathbb{N}} \frac{1}{\varphi_{m_a(\pi_i)}(\tau^2)} = \prod_{a\in\mathbb{N}}[\infty, \pi_a^\vee]\,. \tag{6.134}$$

So, equations (2.25), (6.134) and (6.133) tells us that

$$\begin{aligned}
\overline{\mathrm{HS}}[\vec{m}] &= \frac{1}{\varphi_\infty(\tau^2)^{|V|}} Z^{|m|} r(m^{(+)\vee}, \tau^2) r(m^{(-)\vee}, \tau^2)^\dagger \\
&= Z^{|m|}\mathrm{ch}_T V(m^{(+)\vee})\,\mathrm{ch}_T V(m^{(-)\vee})^\dagger r(0,\tau^2) r(0,\tau^2)^\dagger \frac{1}{\varphi_\infty(\tau^2)^{|V|}} \\
&= Z^{|m|}\mathrm{ch}_T V(m^{(+)\vee})\,\mathrm{ch}_T V(m^{(-)\vee})^\dagger \mathrm{HS}(0)\,.
\end{aligned} \tag{6.135}$$

This proves theorem 12. □

# Chapter 7

# Summary and future directions

## 7.1 Summary

In chapter 3, we defined a superconformal quantum mechanical model on a conical hyperKähler manifold that comes from a hyperKähler quotient of affine space, with a homogeneous symplectic form of degree 2. We defined the superconformal index of the theory, and took a regularisation given by the projective symplectic resolution. The index did not depend on the choice of regularisation parameter. We computed the superconformal index in many cases using localisation theorems.

In chapter 5, we computed the superconformal index of the above mentioned quantum mechanical model for the case of its target space being the Nakajima quiver variety associated to $A$- and $\hat{A}$-type quivers, including in the presence of baryonic charge. We use this analysis of the linear quiver to prove theorem 9, which writes the Hilbert series with background charge as a plethystic piece times a symmetric polynomial. We further prove a similar result for the superconformal index.

Chapter 4 contains theorem 7, which is the statement of three dimensional mirror symmetry in the specific case of the ADHM quiver with generic baryonic charge $k$. The proof of this involved the use of the theory of symmetric functions and multi-dimensional contour integrals, one of which we showed in section 4.4.1 is equivalent to a generalised principal specialisation formula. We remarked in section 4.6 that our derivation seems to use the structure of the Cherkis bow of [54, 159]. Finally, we showed that our derivation applies equally well to a certain linear quiver and showed how the gluing procedure of [61] can be applied on the Higgs branch.

In the final chapter 6, we investigate the action of the Poisson algebra of holomorphic functions on a hyperKähler manifold on sections of line bundles on the manifold. We look at the case where the Poisson algebra is the classical limit of the Yangian and find in theorem 12 that the representation theory is given by the fusion product of Kirillov-Reshitkhin modules with the Poisson algebra. We further find in theorem 10 that for a specific choice of line bundle we can produce the generating function for Poincaré polynomials of $ADE$-type quiver varities.

## 7.2 Future directions

In theorem 12 we decomposed the Hilbert series of the Coulomb branch of an infinite rank $ADE$-quiver into its representation theory under the Yangian. An obvious generalisation is how to see the representation theory at finite rank, where instead of the full Yangian one has the truncated (shifted) Yangian (the shift is present when the quiver is not balanced).

It known that the partition function of a three dimensional theory on $S^3$ gives a quantity which has a Coulomb branch Hilbert series and a Higgs branch Hilbert series limit, [173]. It would certainly be of interest to see if one could apply the infinite rank limit to this theory, and see how the quantum group symmetry manifests itself in the full partition function, and then of course how the truncated shifted Yangian manifests at finite rank.

The partition function on $S^2 \times S^1$ is given by a sum over vortices [173] (this is essentially a two dimensional version of the results of [169]), with a contribution for each vortex number given as a $q$-quantised form of the Hilbert series of the moduli space. How does one instead get the $q$-quantised form of the superconformal index? How is the interpretation of the $q$-quantised superconformal index in terms of quantum K-theory? Can one compute the quantum K-theory in the infinite rank limit and get quantum Kirillov-Reshitkhin modules analagous to theorem 12?

It would also be of interest to calculate the superconformal index in the infinite rank limit in a similar fashion to theorem 12. This would be an instructive toy model for the ultimate goal of constructing the infinite rank limit of the ADHM quiver (note that for $A$-type quivers the superconformal index tells us quite a lot about the superconformal index of the ADHM quiver because of equation 3.30 in chapter 5). The reason this is of interest is because the quantum mechanical model with target space instanton moduli space is the discrete light cone quantisation (DLCQ) of the six dimensional (2,0) theory, [3, 4]. The infinite rank limit is the decompactification limit of the null circle, and so one may hope to obtain information about the (2,0) theory via this limit of the quantum mechanics. The presentation of the superconformal index in theorem 9 may be conducive to taking the infinite rank limit.

Theorem 10, relating the Hilbert series of the infinite rank Coulomb branch of a balanced $ADE$-quiver to the generating function of the Poincaré polynomial of the finite rank Higgs branch of the same $ADE$-quiver remains mysterious. It is certainly not directly due to three dimensional mirror symmetry, because the graphs of the quivers are the same, which is not true for the mirror duals. It is possibly a consequence of symplectic duality, [36, 35], a mathematical duality between the Higgs branch and the Coulomb branch of the same theory, but this requires further investigation.

An obvious generalisation of the work in chapter 3 is to four supercharge quiver varieties. For example the work of [154] already provides an answer for the superconformal index of the handsaw quiver (defined in that paper). However the exact geometrical interpretation of this quantities remains to be found. It is known that the handsaw quiver variety is the moduli space of vortices in a $T_\sigma(SU(N))$ theory, [44]. In terms of quantum group symmetry, one finds a similar story to the linear quiver, but a step behind. By this I mean that the linear quiver gives a spin chain symmetry in the infinite rank limit, and then for all equal charge would give a finite dimensional affine algebra character in the infinite charge length limit, corresponding to a thermodynamic limit of the spin chain, see theorem 5.4 [101] . While for the handsaw quiver one sees the spin chain character at finite rank and gets the affine Lie algebra character in the infinite rank limit [76, 65]. Of course this is just one of many four supercharge quivers that one could investigate.

The superconformal index of chapter 3 is one side of the not-well-understood $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. The superconformal index should be comparable to the black hole entropy of a black hole in $\mathrm{AdS}_2 \times K$, for $K$ some eight dimensional manifold. See [181] and the references therein.

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# Appendix A

# The Jeffrey-Kirwan pole procedure

In this chapter we give a brief description of the Jeffrey-Kirwan localisation and residue procedure.

The Jeffrey-Kirwan pole procedure is the method by which we may evaluate the Molien-like integrals of, for example, sections 2.2.3 and 5.3.4, in order to compute the Hilbert series/superconformal index we are interested in.

In this appendix we shall briefly why the Molien integral is relevant for our calculation, and how to take the Jeffrey-Kirwan residue of it.

Our emphasis on this appendix is on calculation as opposed to theory, and in this light we work through two examples. One being the derivation of the Molien integral for the non-equivariant Euler character of line bundles on $\mathbb{CP}^1$ and the other being the Jeffrey-Kirwan pole procedure applied to line bundles on a simple linear quiver. Furthermore, we refer to the literature on equivariant cohomology and characteristic classes for the definitions of these objects, see for example the excellent lecture notes [139].

The Jeffrey-Kirwan procedure was first developed by Witten, [203], Jeffrey and Kirwan, [116], and Guillemin and Kalkman, [95], using rather complicated analytic formulae. The subsequent work by Brion and Vergne, [39], further elucidated by Szenes and Vergne, [189], provided a simpler algebraic presentation and coined the term “Jeffrey-Kirwan residue".

The procedure was first used in localisation computations in the physics literature in [25, 26] to compute the partition function of the Ramond-Ramond sector of two dimensional gauged linear $\sigma$-models on $T^2$ with $\mathcal{N} = (2,0)$ supersymmetry (this quantity is known as the elliptic genus), and then in further works such as [58] and [113] to compute quantum mechanical partition functions.

One of the crucial ingredients for the Jeffrey-Kirwan residue is the Jeffrey-Kirwan parameter, $\eta \in \mathfrak{t}^*$ (living in the dual of the Cartan subalgebra of the gauge group's Lie algebra). In general, the residue is not independent of this parameter. Despite living in a different space to the Fayet-Ilioupoulos parameter, $\zeta_{\mathbb{R}} \in \pi_1(G)^\vee$, they are often identified (here we set $\zeta_{\mathbb{C}} = 0$, this identification is, in fact, generally done for symplectic quotients). There are justifications in the literature for this identification from path integral derivations of the residue formula, see for example [58, 113]. Assuming this identification, we are able to compute wall crossing phenomena by varying the Jeffrey-Kirwan parameter pass a wall.

One can understand, from a path integral derivation, the Molien integral over the gauge fugacities $W$ as an integral over the values of the vector multiplet scalars. The poles then correspond to the fixed points of the Higgs branch[1].

In section A.1, we give a brief description of how one would expect the Molien integral to emerge. We do not work equivariantly with respect to isometries, but the results can all be lifted. In section A.1.1, we give the relevant calculation for the simple case of non-equivariant line bundles on $\mathbb{CP}^1$. In section A.2, we explicitly describe how to take the Jeffrey-Kirwan residue of a meromorphic form. In section A.2.1, we work through the Jeffrey-Kirwan pole procedure for line bundles on a particular quiver.

## A.1 Motivation and statement

In chapter 3, we found ourselves interested in the equivariant Euler character of equivariant coherent sheaves on a smooth hyperKähler quotient $\mathfrak{M}_{\vec{\zeta}}$ (a quasiprojective variety), with $\vec{\zeta} = (\zeta_{\mathbb{R}}, 0)$ a regular value. Let $\mathscr{V}$ be an equivariant coherent sheaf on $\mathfrak{M}_{\vec{\zeta}}$. Its Euler character can be thought of as its image under the K-theoretic push forward (also known as transfer homomorphism) of the map $\Pi : \mathfrak{M}_{\vec{\zeta}} \to \{\mathrm{pt}\}$, i.e.

$$\chi(\mathscr{V}) = \Pi_!(\mathscr{V}) . \tag{1.1}$$

There is a map from equivariant $K$-theory to equivariant cohomology called the Chern character, denoted Ch. This character commutes with pullbacks. The Grothendieck-Riemann-Roch theorem tells us to what extent it commutes with the pushforward. It says that

$$\mathrm{Ch}\, \Pi_!(\mathscr{V}) = \Pi_* \left( \mathrm{Ch}(\mathscr{V}) \mathrm{td}(\mathfrak{M}_{\vec{\zeta}}) \right) , \tag{1.2}$$

where $\mathrm{td}(\mathfrak{M}_{\vec{\zeta}})$ is the Todd genus for the tangent bundle of $\mathfrak{M}_{\vec{\zeta}}$. We have that

$$\mathrm{Ch}\, \Pi_!(\mathscr{V}) = \Pi_!(\mathscr{V}) = \chi(\mathscr{V}) , \tag{1.3}$$

[1] Recall from chapter 2 that when a generic complex hypermultiplet mass parameter and a generic real Fayet-Ilioupoulos parameter are turned on, the vacua lift to a finite number of isolated fixed points, which have both non-zero hypermultiplet and vector multiplet scalar vacuum expectation values.

as $\Pi$ maps to a point. We further have that (by definition)

$$\Pi_*\left(\mathrm{Ch}(\mathscr{V})\mathrm{td}(\mathfrak{M}_{\vec{\zeta}})\right) = \int_{\mathfrak{M}_{\vec{\zeta}}} \mathrm{Ch}(\mathscr{V})\mathrm{td}(\mathfrak{M}_{\vec{\zeta}}). \tag{1.4}$$

So, we see that the evaluation of the Euler class of $\mathscr{V}$ is reduced to the problem of integrating a certain cohomology class over the hyperKähler quotient manifold $\mathfrak{M}_{\vec{\zeta}}$. Specifically,

$$\chi(\mathscr{V}) = \int_{\mathfrak{M}_{\vec{\zeta}}} \mathrm{Ch}(\mathscr{V})\mathrm{td}(\mathfrak{M}_{\vec{\zeta}}). \tag{1.5}$$

In the work [116], Jeffrey and Kirwan showed us exactly how to evaluate such integrals[2]. We lift the integrand to an equivariant cohomology class of the big space, then take a certain residue of this class called the Jeffrey-Kirwan residue.

Since the action of $G$ on $\vec{\mu}^{-1}(\vec{\zeta})$ is free (we choose a regular value of $\vec{\zeta}$), there is an isomorphism

$$\varpi : H^*_G(\vec{\mu}^{-1}(\vec{\zeta})) \xrightarrow{\cong} H^*(\mathfrak{M}_{\vec{\zeta}}). \tag{1.6}$$

This map, known as the Borel-Cartan isomorphism, along with the restriction map induced from the inclusion $i : \mu_{\mathbb{C}}^{-1}(0) \to \mu^{-1}(\vec{\zeta})$, give the map

$$\kappa := \varpi \circ i^* : H^*_G(\mu_{\mathbb{C}}^{-1}(0)) \to H^*(\mathfrak{M}_{\vec{\zeta}}). \tag{1.7}$$

It was proved that for compact symplectic quotients that $\kappa$ is surjective in [132], and it was proved that the map is surjective for Nakajima quiver varieties in [146].

Define the forms $\beta_0(\mathscr{V}) := \mathrm{Ch}(\mathscr{V})\mathrm{td}(\mathfrak{M}_{\vec{\zeta}}) \in H^*(\mathfrak{M}_{\vec{\zeta}})$ and $\beta(\mathscr{V}) \in H^*_G(\mu_{\mathbb{C}}^{-1}(0))$, such that $\kappa(\beta(\mathscr{V})) := \beta_0(\mathscr{V})$. We know such a form exists by the surjectivity of $\kappa$.

If we let $\mathscr{F}$ be the set of fixed points of $\mu_{\mathbb{C}}^{-1}(0)$ under $G$, and assume that the fixed points are isolated, then the localisation theorem of Jeffrey and Kirwan tells us that

$$\begin{aligned}\chi(\mathscr{V}) &= \int_{\mathfrak{M}_{\vec{\zeta}}} \beta_0(\mathscr{V}) \\ &= \frac{1}{(2\pi)^{\dim G}|W_G|} \sum_{F\in\mathscr{F}} \text{JK-Res}_\eta \left( \prod_{\alpha\in R_G} \alpha(W) e^{i\mu_{\mathbb{R}}(F)(W)} \frac{i_F^*(\beta(\mathscr{V}))}{e_F(W)} dW \right),\end{aligned} \tag{1.8}$$

[2] They in fact only showed this for compact manifolds, but works such as [145, 190] tell us how to lift to the non-compact setting in, at least some of, the examples we care about.

where $e_F(W)$ is the $G$-equivariant Euler class of the normal bundle to $F$ in $\mu_{\mathbb{C}}^{-1}(0)$, $R_G$ is the set of roots of $G$, $i_F : \{F\} \hookrightarrow \mu^{-1}(0)$ is the inclusion map, $W_G$ is the Weyl group of $G$, and JK-Res$_\eta$ is a residue procedure that we shall describe in section A.2.

This residue is exactly of the integrand of the Molien integral.

### A.1.1 A simple example

To give a sense of the derivation above, we work through one of the simplest non-trivial examples: line bundles on $\mathbb{CP}^1$. One can understand this manifold as the symplectic resolution of the Kähler quotient of $\mathbb{C}^2$ by the Hamiltonian action of $U(1)$, with moment map $\mu = |z_1|^2 + |z_2|^2$, where $(z_1, z_2)$ are the standard coordinates on $\mathbb{C}^2$. One can understand this in four supercharge quiver language as being the Higgs branch of the quiver in figure A.1.

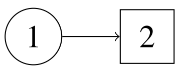


Fig. A.1 The quiver corresponding to $\mathbb{CP}^1$. Note the arrow is directed.

$\mathbb{CP}^1$ has open cover $\mathscr{U} = \{U_0, U_1\}$, where $U_0 = \mathbb{CP}^1 \backslash \{N\}$ and $U_1 = \mathbb{CP}^1 \backslash \{S\}$ ($N$ and $S$ are the North and South poles of $\mathbb{CP}^1$). $U_0 \cong \mathbb{C}$, with coordinate $z$. On $U_0 \cap U_1$, the transition function is $z \mapsto 1/z$.

Kähler metric and form on $\mathbb{CP}^1$ (using coordinates on $U_0$) are respectively given by

$$h = \frac{dz \otimes d\bar{z}}{(1+z\bar{z})^2}\,, \qquad \omega = i\frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}\,. \tag{1.9}$$

The non-trivial vector bundles on $\mathbb{CP}^1$ are the line bundles $\mathscr{O}(B)$, for $B \in \mathbb{Z}$, such that $\mathscr{O}(B)|_{U_i} \cong U_i \times \mathbb{C}$, with gluing along $U_0 \cap U_1$ via the isomorphism $(s, v) \mapsto (s^{-1}, s^{-B}v)$. It is standard to show that all vector bundles decompose into a direct sum of such line bundles.

We want to compute the Euler character of $\mathscr{O}(B)$, using the map $\Pi : \mathbb{CP}^1 \to \{\mathrm{pt}\}$. By equation (1.5), we have that

$$\chi(\mathscr{O}(B)) = \int_{\mathbb{CP}^1} c(\mathscr{O}(B)) \wedge \mathrm{td}(\mathbb{CP}^1)\,. \tag{1.10}$$

Using Chern-Weil theory, one can compute that

$$c(\mathscr{O}(B)) = 1 + \frac{Bi}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\,. \tag{1.11}$$

$T\mathbb{CP}^1 \cong \mathscr{O}(2)$, as can be seen by a direct calculation of the transformation of $\frac{\partial}{\partial z}$ under the map $z \mapsto 1/z$. It is a standard result that

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2 + c_2) + \dots . \tag{1.12}$$

So, over $\mathbb{CP}^1$, we have $\mathrm{td}(E) = 1 + \frac{1}{2}c_1(E)$, and thus

$$\mathrm{td}(\mathbb{CP}^1) = 1 + \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} . \tag{1.13}$$

So we have that

$$\begin{aligned} \chi(\mathscr{O}(B)) &= \int_{\mathbb{CP}^1} \left(1 + \frac{Bi}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\right) \wedge \left(1 + \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\right) \\ &= (B+1)\int_{\mathbb{CP}^1} \frac{i}{2\pi}\mathrm{tr}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \\ &= (B+1) . \end{aligned} \tag{1.14}$$

Note that for $B$ positive, $H^1(\mathbb{CP}^2; \mathscr{O}(B)) = 0$, and so we have reproduced the well-known fact that there are $B+1$ linearly independent global sections of $\mathscr{O}(B)$.

Now we shall evaluate our integral over $\mathbb{CP}^1$ using the Jeffrey-Kirwan localisation procedure. To do this, we must lift our cohomology class to $H^*_{U(1)}(\mathbb{C}^2)$. We use the Borel-Cartan isomorphism, see for example [96]. This maps $H^*_{U(1)}(S^3) \mapsto H^*(S^2)$ via

$$\alpha \otimes w^n \mapsto \alpha_{\mathrm{hor}} \otimes \Omega^n , \tag{1.15}$$

where hor is projecting onto the horizontal component induced from the $U(1)$ principal bundle $S^3 \to S^2 \cong \mathbb{CP}^1$ (note this is exactly the Hopf fibration), $\Omega$ is the curvature of this principal bundle, and $w$ is the fugacity corresponding to the equivariant grading. The $U(1)$-principal bundle has curvature[3] $\frac{i}{4\pi}\omega$. Using the metric $g = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$, we have normal vector to $S^3$, n, and vertical vector in the bundle $v$:

$$\begin{aligned} n &= z_1\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial \bar{z}_1} + z_2\frac{\partial}{\partial z_2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2} , \\ v &= -iz_1\frac{\partial}{\partial z_1} + i\bar{z}_1\frac{\partial}{\partial \bar{z}_1} - iz_2\frac{\partial}{\partial z_2} + i\bar{z}_2\frac{\partial}{\partial \bar{z}_2} . \end{aligned} \tag{1.16}$$

[3] The coefficient is needed to ensure that this is a generator for $\mathbb{Z}$-valued cohomology.

So, the vertical bundle has fibres

$$V_{(z_1,z_2)}(S^3) = \mathbb{R}v. \tag{1.17}$$

The horizontal bundle has fibres

$$H_{(z_1,z_2)}(S^3) = \mathbb{R}(\bar{z}_2\partial_{z_1} - \bar{z}_1\partial_{z_2} + z_2\partial_{\bar{z}_1} - z_1\partial_{\bar{z}_2}) \oplus i\mathbb{R}(\bar{z}_2\partial_{z_1} - \bar{z}_1\partial_{z_2} z_2\partial_{\bar{z}_1} - z_1\partial_{\bar{z}_2}). \tag{1.18}$$

We see that under (1.15), the appropriate preimage of $\frac{i(B+1)}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$ is

$$\alpha = (B+1)w. \tag{1.19}$$

This lives in $H_{U(1)}(S^3)$, but since the map from $H_{U(1)}(\mathbb{C}^2)$ to this is just restriction, we trivially extend it to a cohomology class on the big space.

To finally apply the formula (1.8), we need to sum over the $U(1)$ fixed points of $\mathbb{C}^2$. There is only one, and it is $(0,0) \in \mathbb{C}^2$. Given that $e_{U(1)}(T_{(0,0)}\mathbb{C}^2) = w^2$, we have

$$\chi(\mathscr{O}(B)) = \mathrm{Res}_{w=0}\frac{(B+1)w}{w^2}dw = B+1\,. \tag{1.20}$$

## A.2 How to take Jeffrey-Kirwan residue

Here we review the iterated residue procedure of [189]. We defer to that paper for proofs (it is proven in that paper that this method is equivalent to other definitions of the Jeffrey-Kirwan residue). We shall then do the procedure for a simple linear quiver. We do it for all non-singular choices of the Jeffrey-Kirwan parameter, and observe wall-crossing for appropriate choices of baryonic charge.

The gauge fugacities $W$, live in the real $r$-dimensional vector space, $\mathfrak{a}$. Our integral is over a meromorphic $(r,0)$-form, given by a rational function, $\phi$, times $d\mu := dw_1 \wedge \ldots dw_r$. We shall think of $d\mu$ as having positive orientation. In our rational function, we must pick the poles, corresponding to hyperplanes in $\mathfrak{a}$, that we take the residue of (this corresponds to the choice of linear denominators). We pick these to be all the possible poles in the Hilbert series, ignoring all poles in the superconformal index that have a factor of $y$ in them[4]. We define the set $\mathfrak{A}$ in terms of this choice of poles as follows: any denominator of the form $\frac{1}{1-\tau^{\#}Z^{\#}\prod_i w_i^{n_i}}$, where we arrange the rational function such that the power of $\tau$ is positive, corresponds to the element $\sum_i n_i Q_i$, where $Q_1,\ldots,Q_r$ is an ordered basis for $\mathfrak{a}$. The size of $\mathfrak{A}$

[4]The geometric interpretation for this is clear, since we think of the fixed points as living on the base-manifold, and thus should ignore all poles containing powers of $y$ in the coordinates.

is equal to the number of poles that we have decided to consider for our residue, we shall define this number to be $n$ in this section.

For a choice of $\mathfrak{A}$, define $\mathrm{BInd}(\mathfrak{A})$, the set of *basis index sets*, to be the set of index subsets $\sigma \subseteq \{1,\ldots,n\}$ for which $\{\alpha_i | i \in \sigma\}$ is a basis of $\mathfrak{a}^*$. Use the notation

$$\gamma^\sigma = (\gamma_1^\sigma, \ldots, \gamma_r^\sigma) \tag{2.21}$$

for the basis associated to $\sigma \in \mathrm{BInd}(\mathfrak{A})$. We assume an ordering of the basis elements has been fixed.

Denote by $\mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$ the union of the boundaries of the simplicial cones $\mathrm{Cone}(\gamma^\sigma)$, for $\sigma \in \mathrm{BInd}(\mathfrak{A})$. Elements of $\mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$ are called *singular*, while the other points of $\mathfrak{a}$ are called *regular*. A connected component of $\mathrm{Cone}(\mathfrak{A}) \backslash \mathrm{Cone}_{\mathrm{sing}}(\mathfrak{A})$ is called a *chamber*. For a chamber $\mathfrak{c}$, we define $\mathrm{BInd}(\mathfrak{A}, \mathfrak{c})$ to be the set of $\sigma \in \mathrm{BInd}(\mathfrak{A})$ for which $\mathrm{Cone}(\gamma^\sigma) \supset \mathfrak{c}$. We choose such a chamber $\mathfrak{c}$, this will turn out to be equivalent to the choice of Jeffrey-Kirwan parameter.

There are a finite number of points in $\mathfrak{a}$ where $\geqslant r$ of the hyperplanes defined by $\mathfrak{A}$ intersect. Call these points *full poles*. We label them $p_1, \ldots, p_f$. If $f = 0$, then the integral will be zero. We define $\mathfrak{A}_{p_i}$ to be the set of elements of $\mathfrak{A}$ whose corresponding hyperplanes pass through $p_i$.

For $i = 1, \ldots, f$, let $\mathscr{F}\mathscr{L}(\mathfrak{A}_{p_i})$ be the finite set of flags

$$F = [F_0 = \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r = \mathfrak{a}], \quad \dim F_j = j, \tag{2.22}$$

such that $\mathfrak{A}_{p_i}$ contains a basis of $F_j$ for each $j = 1\ldots, r$. For each $F \in \mathscr{F}\mathscr{L}(\mathfrak{A}_{p_i})$, we choose an ordered basis $\gamma^F = (\gamma_1^F, \ldots, \gamma_r^F)$ of $\mathfrak{a}$ with the following properties:

1. $\gamma_j^F \in \mathrm{span}_{\mathbb{Q}} \mathfrak{A}_{p_i}$ for $j = 1, \ldots, r$;
2. $\{\gamma_m^F\}_{m=1}^j$ is a basis of $F_j$ for $j = 1, \ldots, r$;
3. the basis $\gamma^F$ is positively oriented;
4. $d\gamma_1^F \wedge \cdots \wedge d\gamma_r^F = d\mu$.

To each flag $F \in \mathscr{F}\mathscr{L}(\mathfrak{A}_{p_i})$, one can associate a linear functional $\mathrm{Res}_F$ on the rational function. This is called an *iterated residue*. Consider the elements of $\gamma^F$ as defining coordinates on $\mathfrak{a}$, and write $w_j = \gamma_j^F(w)$, for each $w \in \mathfrak{a}_{\mathbb{C}}$. Then any rational function on $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ can be written as a rational function, $\phi^F$, of these coordinates. $\phi(u) = \phi^F(u_1, \ldots, u_r)$. We write the pole $p_i$ in these coordinates as $p_i = (p_{i,1}, \ldots, p_{i,r})$.

We define the iterated residue associated to the flag $F \in \mathscr{FL}(\mathfrak{A}_{p_i})$ as the functional $\mathrm{Res}_F$ given by the formula

$$\mathrm{Res}_F \phi d\mu = \mathrm{Res}_{u_r=p_{i,r}} du_r \mathrm{Res}_{u_{r-1}=p_{i,r-1}} du_{r-1} \dots \mathrm{Res}_{u_1=p_{i,1}} du_1 \phi^F(u_1,\dots,u_r)\,. \tag{2.23}$$

Pick an element $\eta \in \mathfrak{c}$, this is the *Jeffrey-Kirwan parameter*. Denote by $\Sigma\mathfrak{A}$ the set of elements of $\mathfrak{a}^*$ obtained by partial sums of elements of $\mathfrak{A}$. $\eta$ is *sum-regular*, if it does not belong to any hyperplane generated by elements of $\Sigma\mathfrak{A}$.

For $j = 1,\dots,r$ and $F \in \mathscr{FL}(\mathfrak{A}_{p_i})$, introduce the vectors

$$\kappa_j^F := \sum_{\substack{\ell=1 \\ \alpha_\ell \in F_j \cap \mathfrak{A}_{p_i}}}^{n} \alpha_\ell\,. \tag{2.24}$$

$\kappa_j^F \in \Sigma\mathfrak{A}$, and $\kappa_r^F =: \kappa = \sum_{i=1}^n \alpha_i$ is independent of $F$.

A flag is *proper* if the elements $\kappa_j^F$, for $j = 1,\dots,r$, are linearly independent.

For each $F \in \mathscr{FL}(\mathfrak{A}_{p_i})$, define $\nu(F) \in \{0,\pm 1\}$ as follows:

- $\nu(F) = 0$ if $F$ is not a proper flag;
- if $F$ is a proper flag, then $\nu(F)$ is equal to 1 or $-1$, depending on whether the ordered basis $(\kappa_1^F,\dots,\kappa_r^F)$ of $\mathfrak{a}^*$ is positively or negatively oriented respectively.

For a proper flag $F \in \mathscr{FL}(\mathfrak{A}_{p_i})$, introduce the closed simplicial cone $\mathfrak{s}^+(F,\mathfrak{A}_{p_i})$ as

$$\mathfrak{s}^+(F,\mathfrak{A}_{p_i}) = \sum_{j=1}^r \mathbb{R}_{\geqslant 0} \kappa_j^F\,. \tag{2.25}$$

For $\eta \in \mathfrak{a}^*$, denote by $\mathscr{FL}^+(\mathfrak{A},\eta)$ the set of flags, $F$, such that $\eta \in \mathfrak{s}^+(F,\mathfrak{A})$.

If $\eta$ is sum-regular, then every flag $F \in \mathscr{FL}^+(\mathfrak{A}_{p_i},\eta)$ is proper, and thus, for such $F$ we have $\nu(F) = \pm 1$.

We define the Jeffrey-Kirwan residue as:

$$\text{JK-Res}_{\mathfrak{c}}(\phi) := \sum_{i=1}^{f} \sum_{F \in \mathscr{FL}^+(\mathfrak{A}_{p_i},\eta)} \nu(F) \mathrm{Res}_F \phi\,. \tag{2.26}$$

One can show that this is independent of the choice of $\eta$, up to the choice of chamber $\mathfrak{c}$ that it belongs to, hence the notation $\text{JK-Res}_{\mathfrak{c}}$.

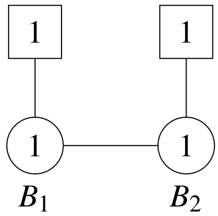


Fig. A.2 The simple length two quiver, with baryonic charge $B_1$ and $B_2$.

### A.2.1 An example

In this subsection, we work through the Jeffrey-Kirwan procedure for a quiver with background charge. The quiver we work with is pictured in figure A.2. Due to the tediousness of the procedure, we only show the details for two poles, before summarising all the results. These calculations can then be applied for $\phi$ the integrand of either the Hilbert series or the superconformal index.

We take $\mathfrak{a} = \mathbb{R}Q_1 \oplus \mathbb{R}Q_2$,

$$d\mu = dQ_1 \wedge dQ_2 \,, \tag{2.27}$$

and

$$\eta = a_1 Q_1 + a_2 Q_2 \,, \tag{2.28}$$

for $a_1, a_2 \in \mathbb{R}$.

There are twelve poles whose coordinates are neither zero or infinity. They are

$$\begin{aligned} &(\tau/z_1, \tau/z_2), \quad (\tau/z_1, 1/\tau z_2), \quad (1/\tau z_1, \tau/z_2), \quad (1/\tau z_1, 1/\tau z_2), \\ &(\tau/z_1, \tau^2/z_1), \quad (\tau/z_1, 1/z_1), \quad (1/\tau z_1, 1/z_1), \quad (1/\tau z_1, 1/\tau^2 z_1), \\ &(\tau^2/z_2, \tau/z_2), \quad (1/z_2, \tau/z_2), \quad (1/z_2, 1/\tau z_2), \quad (1/\tau^2 z_2, 1/\tau z_2) \,. \end{aligned} \tag{2.29}$$

We do the analysis for a single one of these poles explicitly here.
$p := (1/\tau z_1, 1/\tau z_2)$:

$$\mathfrak{A}_p = \{-Q_1, -Q_2\} \,. \tag{2.30}$$

Defines flags

$$\mathscr{F}\mathscr{L}(\mathfrak{A}_p) = \{[0 \subset \langle Q_1 \rangle \subset \mathbb{R}^2], [0 \subset \langle Q_2 \rangle \subset \mathbb{R}^2]\} \,. \tag{2.31}$$

We choose bases

$$\gamma^{-Q_1} = (Q_1, Q_2), \quad \gamma^{-Q_2} = (-Q_2, Q_1) \,. \tag{2.32}$$

Flags have vectors

$$\kappa_1^{-Q_1} = -Q_1, \quad \kappa_1^{-Q_2} = -Q_2, \quad \kappa = -Q_1 - Q_2 . \tag{2.33}$$

The appropriate signs for the flags are

$$\nu(F^{-Q_1}) = +1, \quad \nu(F^{-Q_2}) \ = -1. \tag{2.34}$$

Residue at $p$ is non-zero iff $a_1, a_2 < 0$ and $a_1 \neq a_2$. If this is the case, then the contribution of this pole is

$$\mathrm{Res}_{w_2=1/\tau z_2} dw_2 \mathrm{Res}_{w_1=1/\tau z_1} dw_1\, \phi \tag{2.35}$$

**Summary**[5]

- $a_1, a_2 > 0$ poles are $(\tau/z_1, \tau/z_2)$ $(\tau/z_1, \tau^2/z_1)$ $(\tau^2/z_2, \tau/z_2)$;
- $a_1, a_2 < 0$ poles are $(1/\tau z_1, 1/\tau z_2)$ $(1/\tau z_1, 1/\tau^2 z_1)$ $(1/\tau^2 z_2, 1/\tau z_2)$;
- $a_1 > 0, a_2 < 0$ pole $-(\tau/z_1, 1/\tau z_2)$;
- $a_1 < 0, a_2 > 0$ pole $-(1/\tau z_1, \tau/z_2)$;
- $a_1 > 0, a_2 < 0, a_1 > -a_2$ pole $(\tau^2/z_2, \tau/z_2)$;
- $a_1 > 0, a_2 < 0, -a_2 > a_1$ pole $(1/\tau z_1, 1/\tau^2 z_1)$;
- $a_1 < 0, a_2 > 0, -a_1 > a_2$ pole $(1/\tau^2 z_2, 1/\tau z_2)$;
- $a_1 < 0, a_2 > 0, a_2 > -a_1$ pole $(\tau/z_1, \tau^2/z_1)$;
- $a_1 < 0, a_2 > 0$ $-a_1 > a_2$ $(-a_1 \neq 2a_2)$ pole $-(1/\tau z_1, 1/z_1)$;
- $a_1 > 0, a_2 < 0$ $a_1 > -a_2$ $(a_1 \neq -2a_2)$ pole $-(\tau/z_1, 1/z_1)$;
- $a_1 > 0, a_2 < 0$ $-a_2 > a_1$ $(-a_2 \neq 2a_1)$ pole $-(1/z_2, 1/\tau z_2)$; and
- $a_1 < 0, a_2 > 0$ $a_2 > -a_1$ $(a_2 \neq -2a_1)$ pole $-(1/z_2, \tau/z_2)$.

Note that in each chamber one has contributions from exactly three of the twelve contributing poles, each clearly manifesting the expected coloured Young tableaux structure from chapter 5.

From calculations on Mathematica, one can see that for any choice of $\eta$ such that $a_1 \cdot a_2 \cdot (a_1 + a_2) \cdot (a_1 + 2a_2) \cdot (2a_1 + a_2) \neq 0$ (i.e. $\eta$ is $\Sigma\mathfrak{A}$-sum regular), the Hilbert series and the superconformal index without baryonic charge are independent of $\eta$.

T We now consider the effect of turning on $B_1$ and $B_2$. Through Mathematica, one sees that the value of the Hilbert series and the superconformal index has a wall crossing along

[5] A minus sign in front of the pole's position means that the iterated residue is to be subtracted.

the line $a_1 = -a_2$ for $|B_1 + B_2| > 1$. The change in answer for either the Hilbert series or the superconformal index from wall crossing is a finite polynomial that depends on the charge.