Simulating Ising and n-state planar Potts models and external fields with non-equilibrium condensates

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Abstract

Classical spin models with discrete or continuous degrees of freedom arise in many studies of complex physical systems. A wide class of hard real-life optimisation problems can be formulated as a minimisation of a spin Hamiltonian. Here we show how to simulate the discrete Ising and n-state planar Potts models with or without external fields using the physical gain-dissipative platforms with continuous phases such as lasers and various non-equilibrium Bose-Einstein condensates. The underlying operational principle originates from a combination of resonant and non-resonant pumping. Our results lay grounds for the physical simulations of a broad range of Hamiltonians with complex interactions that can vary in time and space and with combined symmetries.

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Understanding of the phase transitions, structures of the ground states, dynamical behaviour of classical spin systems lies at the core of studies of complex physical systems. The interplay of different types of excitations, couplings, continuous and discrete degrees of freedom can lead to complex behaviour that is mostly unaccessible. On the one hand, the majority of such spin problems are computationally impractical for conventional classical computers for sufficiently large system sizes. On the other hand, experimental studies in solid-state systems are challenging for implementing, isolating and controlling such spin model Hamiltonians.

Beyond the investigation of new physical regimes, not realisable in condensed matter systems lies another important area of research where finding the ground state configuration of spin models is associated with classical optimization (NP-hard) problems in vastly different areas such as vehicle routing and scheduling problems, dynamic analysis of neural networks and financial markets, prediction of new chemical materials and machine learning. Recently various analog Hamiltonian optimisers and quantum devices to simulate classical NP-hard problems have been proposed and realised using different types of physical platforms. Regardless of a particular physical system used, the operational principle of these analog simulators is substantially similar: a classical optimisation problem is reformulated as the problem of finding the ground state of a particular spin Hamiltonian with discrete or continuous degrees of freedom that a given simulator can emulate. The discrete NP-hard combinatorial optimization problems such as traveling salesman, graph colouring, partitioning, etc. were explicitly mapped into the Ising Hamiltonian [1] whereas the continuous quadratic optimization problems such as phase retrieval is formulated as finding the global minimum of the XY Hamiltonian [2].

Among the various platforms that aim to simulate the classical spin Hamiltonians a new subclass of simulators has recently emerged – the gain-dissipative simulators – non-equilibrium systems that use a gain process to raise the system above the threshold for a phase transition to a coherent state at the minimum of an associated functional that describes the occupation of this state. Such platforms are mostly optical such as injection-locked laser systems [3], the networks of optical parametric oscillators, [4–7], coupled lasers [8], and non-equilibrium condensates such as polariton condensates [9], and photon condensates [10]. The 'spin' in such systems is represented by the phase θ_i of the complex amplitude Ψ_i that describes the state of the $i-$th laser, oscillator or condensate depending on the nature of the system. We will refer to these states as coherent centres (CCs). The phases of the complex amplitudes take continuous values in $[0, 2\pi)$ and the system may simulate the XY Hamiltonian $H_{XY} = -\sum J_{ij}\cos(\theta_i - \theta_j)$. For a general form of the matrix of couplings J_{ij} finding the ground state of such Hamiltonian is known to be NP-hard [11]; therefore, any NP problem can be mapped into the XY Hamiltonian but with a polynomial overhead on the number of nodes that can be rather significant [12]. Combinatorial optimization problems are usually discrete, so mapping them to a continuous Hamiltonian is not practical. It is important to be able to reduce the overhead by mapping discrete problems into discrete spin Hamiltonians directly.

The nature of the couplings varies between the physical systems, so in this Letter we focus on non-equilibrium condensates and coupled lasers. So far in any realisation of a lattice of condensates and lasers the 'spins' – phases of CCs could take any value between 0 and 2π. There were various attempts to map the continuous phases into Ising discrete spins. For instance, in the injection-locked lasers platform [3] and exciton-polariton systems [13] the phases of the coherent states are projected on the spin up and down polarisation configurations depending on the occupation of left and right circularly polarised states; however, there is no evidence that such projection indeed minimises the Ising Hamiltonian. In this Letter we describe the procedure for implementing the discrete Hamiltonians – the Ising and n-state planar Potts Hamiltonians – in the geometrically coupled non-equilibrium condensates which can be adapted to other gain-dissipative systems. The result is a flexible model system which allows us to find the time and pumping-dependent behaviour and interplay of the discrete and continuous order parameters with different symmetry breaking properties.

The operation of the gain-dissipative simulators follows from the complex Ginzburg-Landau equation (cGLE) [14–16] that describes the time evolution of the system of N spatially separated non-equilibrium condensates. As we have recently shown [17] the spatial degree of freedom can be integrated out so that such evolution can be described by the coupled rate equations on the time evolution of the complex amplitudes $\Psi_i(t)$ of these condensates (CCs). For such system to reach the global minimum of the XY Hamiltonian one needs to establish a feedback connection between the gain, coupling strengths and densities of the condensates [17]. The resulting system is

$$\frac{d\Psi_i}{dt} = \Psi_i(\gamma_i - \sigma|\Psi_i|^2) - \mathrm{i}U|\Psi_i|^2\Psi_i + \sum_{j,j\neq i}^{N} \Upsilon_{ij}\Psi_j + \xi_i(t), \tag{1}$$

$$\frac{d\gamma_i^{\mathrm{inj}}}{dt} = \epsilon(\rho_{\mathrm{th}} - \rho_i), \quad \frac{d\Upsilon_{ij}}{dt} = \hat{\epsilon}(J_{ij} - \Upsilon_{ij}), \tag{2}$$

where $\Psi_i(t) = \sqrt{\rho_i(t)}\exp[\mathrm{i}\theta_i(t)]$, ρ_i is the number density and θ_i is the phase of the i-th CC, $\gamma_i = \gamma_i^{\mathrm{inj}} - \gamma_c$ is the effective gain where γ_i^{inj} is the pumping rate, γ_c and σ are rates of linear and nonlinear dissipation respectively, U is the strength of the in-site particle interactions, ρ_{th} is the specified threshold number density, Υ_{ij} is the coupling strength between $i-$th and $j-$th CCs. In the derivation of Eqs. (1-2), we assumed that the total wavefunction $\psi(\mathbf{r}, t)$ of the system of N condensates can be approximated by the sum of the wavefunctions of the individual condensates. The integrals of the overlap between the spatially separated condensates were neglected in favour of the integrals involving the reservoirs of hot excitons that populate the condensates at the pumping sites. The constants ϵ and $\hat{\epsilon}$ characterise the rates of the density and coupling adjustments respectively: if the number density (the coupling strength) of the $i-$th CC is below [above] ρ_{th} (J_{ij}) it has to be increased [decreased]. Such a feedback can be achieved by measuring photoluminescence of the signal and holographically reconfiguring the injection via the spatial light modulator or mirror light masks (e.g. by DLP high-speed spatial light modulators). Finally, $\xi_i(t)$ represents the Langevin noise that allows the system to span various phase configurations when approaching the threshold. If the pumping rates are driven sufficiently slow from below the threshold, the fixed points of Eqs. (1,2) coincide with the global minimum of the XY Hamiltonian H_{XY} as we have previously suggested [17].

The system Eqs. (1-2) has stable fixed points $\rho_i = \rho_{\mathrm{th}}$ at which the phases acquire the global frequency (and therefore the global coherence across all N condensates) $\omega_0 = U\rho_{\mathrm{th}}$ and minimise the XY Hamiltonian. To implement the external fields and the discrete versions of the XY model such as the Ising and n-state planar Potts models we need to break the symmetry of Eq. (1) to phase rotations which can be done by forcing the system parametrically at a frequency ω_c resonant with ω_0. In polariton condensates such resonant forcing has recently been used in combination with a non-resonant pumping [18]. To make the sample accessible to resonant excitation without a backscatter from the non-resonant pumping the GaAs substrate was chemically etched to form a thin membrane across the sample that allowed resonant excitation from the back side of the cavity. A narrow linewidth CW laser was used to excite some of the condensates selectively resonantly. Both resonant and nonresonant excitation lasers were synchronized and their pumping intensities were varied on demand. For an integer ratio $n = \omega_c/\omega_0$ the Eq. (1) becomes in analogy with [19, 20]

$$\frac{d\Psi_i}{dt} = \Psi_i(\gamma_i - \sigma|\Psi_i|^2) - \mathrm{i}U|\Psi_i|^2\Psi_i + \sum_{j,j\neq i}^{N} \Upsilon_{ij}\Psi_j + h_{ni}\Psi_i^{*(n-1)} + \xi_i(t). \tag{3}$$

where h_{ni} is the pumping strength of $i-$th CC at the resonant frequency n. We substitute $\Psi_i(t) = \sqrt{\rho_i(t)}\exp[\mathrm{i}\theta_i(t)]$ in Eq. (3), separate real and imaginary parts, and drop the noise term to get

$$\frac{1}{2}\dot{\rho}_i(t) = (\gamma_i - \sigma\rho_i)\rho_i + \sum_{j;j\neq i} \Upsilon_{ij}\sqrt{\rho_i\rho_j}\cos\theta_{ij} + h_{ni}\rho_i^{\frac{n}{2}}\cos(n\theta_i), \tag{4}$$

$$\dot{\theta}_i(t) = -U\rho_i - \sum_{j;j\neq i} \Upsilon_{ij}\frac{\sqrt{\rho_j}}{\sqrt{\rho_i}}\sin\theta_{ij} - h_{ni}\rho_i^{\frac{n}{2}-1}\sin(n\theta_i), \tag{5}$$

where $\theta_{ij} = \theta_i - \theta_j$. The fixed point of the dynamical system given by the Eqs. (4,5,2) satisfies

$$\rho_i = \rho_{\rm th} = [\gamma_i + \sum_{j;j\neq i} J_{ij}\cos\theta_{ij} + h_{ni}\rho_{\rm th}^{\frac{n}{2}-1}\cos(n\theta_i)]/\sigma. \tag{6}$$

Since for each CC we choose the smallest γ_i by raising it slowly from below then at the threshold the global minimum of

$$H = -\sum_{i=1}^{N}\sum_{j=1;j\neq i}^{N} J_{ij}\cos\theta_{ij} - \rho_{\rm th}^{\frac{n}{2}-1}\sum_{i=1}^{N} h_{ni}\cos(n\theta_i) \tag{7}$$

is achieved while Eq. (5) describes the gradient decent to that minimum. By taking the resonance $n = 1$ we introduce the effective external "magnetic" field $\mathbf{F} = \{h_{1i}/\sqrt{\rho_{\rm th}}\}$ into the model. For $n > 1$ the forcing term in Eq. (3) reduces the invariance to a global phase shift to a discrete symmetry $\theta_i = 2\pi i/n$ and for a sufficiently large $h_{ni}\rho_{\rm th}^{\frac{n}{2}-1} > \sum_j |J_{ij}|$ introduces the penalty term in the Hamiltonian H for the deviation of phases from the discrete values $2\pi i/n$. For $n = 2$ and a uniform strength of the resonant pumping $h_2 = h_{2i}$, Eqs. (3,2) realise

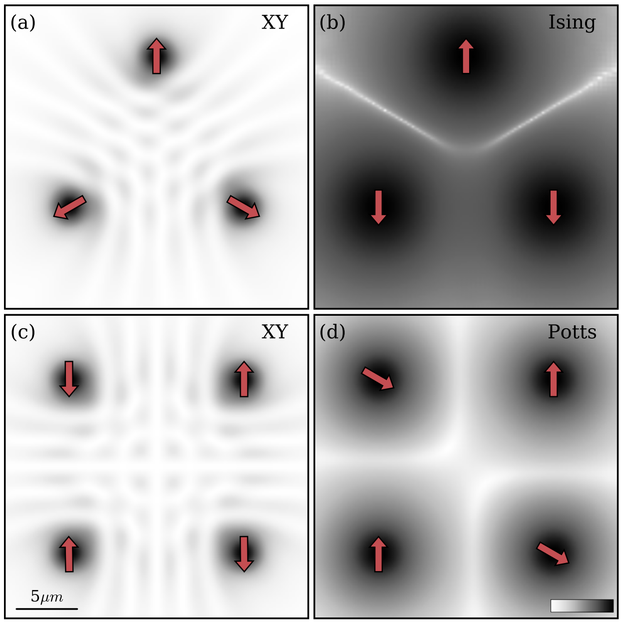

FIG. 1: Contour plots of the density $|\psi(\mathbf{r})|^2$ of the polariton condensates with the nearest neighbours anti-ferromagnetically coupled realising the ground state of (a,c) the XY (b) the Ising and (d) the 3-state planar Potts Hamiltonians at the condensation threshold. The densities for the Ising and 3-state planar Potts Hamiltonians are shown in log scale to emphasise the standing matter waves between the condensates, and, therefore, the phase differences. The densities are obtained by numerical integration of Eqs. (9) with the parameters given in [26].

the minimum of Eq. (7) with θ_i restricted to 0 or π, and therefore, the minimum of the Ising Hamiltonian, whereas for $n > 2$ and $h_n = h_{ni}\rho_{\rm th}^{\frac{n}{2}-1}$ Eqs. (3,2) realise the n-state planar Potts Hamiltonian with $\theta_i = 2\pi i/n$ [21].

The Ising Hamiltonian has been previously realised in optical networks, most notably in the degenerate optical parametric oscillators (DOPOs) [22], where the phase projection to 0 and π is due to the second order nonlinear crystal placed in an optical cavity. However, only resonant forcing is present there, which limits the range of couplings. Our proposal is the first scheme to realise the Ising Hamiltonian using lattices of non-equilibrium Bose-Einstein condensates such as polariton or photon condensates and the first proposal to realise the n-state planar Potts model as well as the combination of continuous and discrete spins in any optical system. In other systems, for instance in ultracold optical lattices [23] the combination of XY and Ising models has been realised using tunable artificial gauge fields, but such systems have no means to address the ground state of the spin Hamiltonian.

To simulate the behaviour of the gain-dissipative system one can use Eqs. (3,2); however, since simplified assumptions were made to arrive at Eq. (3) it is important to verify that a real system shows the transition from achieving the minimum of the continuous (XY) model to the minimum of the discrete model when the resonant forcing is introduced. Therefore, we consider, as an illustrative example, the exciton-polariton system modelled by the cGLE coupled to the rate equation describing the hot exciton reservoir [14, 15]. In such a system polaritons are excited non-resonantly with a single-mode continuous wave laser using a spatial light modulator to generate patterns of laser spots on the sample surface as in our experimental work [9] and with the same parameters. As we established in [9], such a model reliably represents the behaviour of polariton lattices. The resonant drive is achieved by another single-mode continuous wave laser tuned to the multiple of the frequency of the system at the Hopf biburcation [19, 24]. The resulting equations read

$$
\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\left(1-i\eta_d\mathcal{R}\right)\nabla^2\psi + U_0|\psi|^2\psi + \hbar g_R\mathcal{R}\psi \\
&+ i\frac{\hbar}{2}\bigg(R_R\mathcal{R}-\gamma_C\bigg)\psi + P_n(\mathbf{r},t)\psi^{*(n-1)}, \qquad (8)\\
\frac{\partial\mathcal{R}}{\partial t} &= -\left(\gamma_R + R_R|\psi|^2\right)\mathcal{R} + P_0(\mathbf{r}), \qquad (9)
\end{aligned}
$$

where $\psi(\mathbf{r},t)$ is the condensate wavefunction, $\mathcal{R}(\mathbf{r},t)$ is the density profile of the hot exciton reservoir, m is the polariton effective mass, U_0 and g_R are the strengths of the effective polariton-polariton interaction and the blue-shift due to interactions with non-condensed particles, respectively, R_R is the rate at which the exciton reservoir feeds the condensate, γ_C is the decay rates of condensed polaritons, γ_R is the rate of the redistribution of hot excitons in the reservoir, η_d is the rate of the energy relaxation, and $P_0(\mathbf{r},t)$ is the rate of non-resonant pumping into the exciton reservoir [26]. The last term on the right-hand side of Eq. (8) is the resonant forcing with resonance n. The polariton lattice of N condensates at the positions $\mathbf{r}=\mathbf{r}_i$ is formed by taking the non-resonant pumping profile as $P_0(\mathbf{r},t)=\sum_{i=1}^N f_i(t)p(|\mathbf{r}-\mathbf{r}_i|)$, where $p(r)=\exp(-\alpha r^2)$, α characterizes the inverse width of the incoherent pumping profile and f_i describes the strength of the pumping centred at the position $\mathbf{r}=\mathbf{r}_i$. The resonant pumping profile $P_n(\mathbf{r},t)=\sum_{i=1}^N \tilde{h}_{ni}p(|\mathbf{r}-\mathbf{r}_i|), n>0$ follows the lattice spatial profile but with different pumping intensities $\tilde{h}_{ni}\ll f_i$. As we have discussed above and shown in detail in [17] the spatial degrees of freedom of Eqs. (8, 9) without the resonant terms can be integrated out to yield the rate equations on the complex amplitudes of the CCs leading to Eqs. (1). Similarly, Eqs. (8, 9) with the resonant forcing yield Eqs. (3) with $h_{ni} = \tilde{h}_{ni} Re[\int p(|\mathbf{r}-\mathbf{r}_i|)\phi^{*n}(|\mathbf{r}-\mathbf{r}_i|)\,d\mathbf{r}]/\int|\phi(|\mathbf{r}-\mathbf{r}_i|)|^2\,d\mathbf{r}$, where $\phi(r)$ is the wavefunction of a single condensate pumped with $p(r)$.

To illustrate how the polariton lattice minimizes the discrete Ising or n-state planar Potts models with or without external fields we study the behaviour of the unit polariton lattice cells when they are subjected to the effect of the resonant forcing. In what follows we explore the behaviour of the system without density and coupling adjustments described by Eqs. (2) to see how the introduction of the resonant forcing changes the couplings. The effect of such adjustments is elucidated elsewhere [25]. First, we consider a simple lattice of three condensates arranged at the corners of the equilateral triangle [27–29] coupled antiferromagnetically with $J = J_{ij} < 0$. Without the resonant forcing, $P_n = 0, n > 0$, the phases arrange themselves with $2\pi/3$ phase differences to minimize $H_{XY} = -J(\cos\theta_{12} + \cos\theta_{23} + \cos\theta_{31})$ as Fig. 1a illustrates. This agrees with the experimental findings [29]. When the resonant forcing is introduced with $n = 2$ the system described by Eqs. (8, 9) finds the global minimum of the Ising Hamiltonian $H_I = -\tilde{J}(s_1s_2 + s_2s_3 + s_1s_3)$, $s_i = \cos\theta_i = \pm 1$, for $\tilde{J} < 0, \tilde{J} \neq J$. The system is frustrated: spins $0, \pi, \pi$ or $0, 0, \pi$ are realised as Fig. 1b depicts.

To illustrate the transition from solving the XY model to the n-state planar Potts model we consider four condensates arranged at the corners of a square with antiferromagnetic coupling between the nearest neighbours and ferromagnetic coupling along the diagonal. Figure 1c shows the solution of Eqs. (8, 9) without the resonant terms ($P_n = 0, n > 0$). Four condensates realise the global minimum of the XY model with $0, \pi, 0, \pi$ phase differences as been also observed in experiments [9, 28]. The same configuration would result from the Ising model, but the 3-state planar Potts model with θ_i restricted to $0, 2\pi/3$ and $4\pi/3$ is minimized by $0, z, 0, z, 0$ configurations where $z = 2\pi/3$ or $z = 4\pi/3$. This is what we observe implementing $n = 3$ resonant forcing in Eqs. (8, 9) as Fig. 1d illustrates.

Finally, we combine two resonant forcing terms: the resonance $n = 1$ and either resonance $n = 2$ or $n = 3$ in Eq. (8) to combine the effect of an external "magnetic" field in the Ising or 3-state planar Potts models respectively. We take $\tilde{h}_{11} = \tilde{h}_{12} > \max|J_{ij}|$ and $\tilde{h}_{13} = \tilde{h}_{14} = 0$. Such external field penalises the objection function if the phases of the bottom two condensates on Fig. 2 (b-c) are not zeros and leads to the phase configurations as shown in Fig. 2 (d-e).

In conclusion, we formulated an implementation of the discrete spin Hamiltonians such as

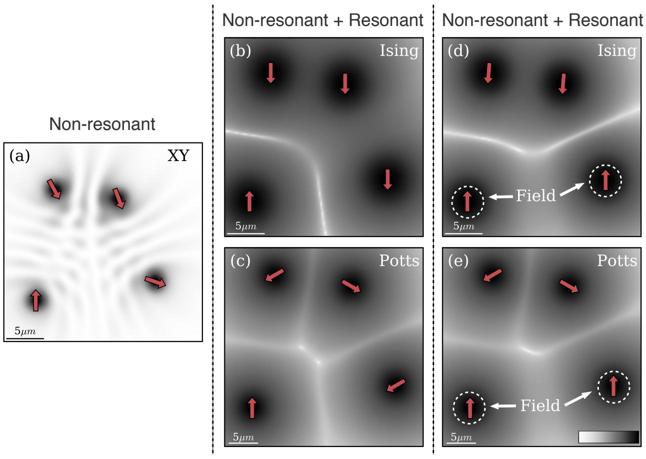

FIG. 2: Contour plots of the density $|\psi(\mathbf{r})|^2$ of the polariton condensates in the corners of quadraterial realising the ground state of (a) the XY (b) the Ising and (c) the 3-state planar Potts Hamiltonians without the external fields; (d) the Ising and (e) the 3-state planar Potts Hamiltonians with the external fields forcing the bottom condensates to acquire phase θ_i. The densities for the Ising and 3-state planar Potts Hamiltonians are shown in log scale to emphasise the standing matter waves between the condensates, and, therefore, the phase differences. The densities are obtained by numerical integration of Eqs. (9) with the parameters given in [26].

the Ising and n-state planar Potts models with or without the external fields in a lattice of non-equilibrium Bose-Einstein condensates. We show that if a resonant pumping is employed together with a non-resonant excitation then depending on the nature of the resonance the phases of the condensates in a lattice take discrete values and at the threshold implement the minimum of the corresponding discrete spin Hamiltonian. This approach opens rather exciting possibilities for simulating complex physical systems, solving combinatorial optimisation problems and developing new computational algorithms. Such lattices can be used to study the dynamical phase transitions and novel regimes of spin systems and the proposed approach offers a unique platform with wealth of controllable and tuneable parameters. The strengths of the resonant and non-resonant pumping can be varied in both time and space, between lattice sites, and even the co-existing lattices with different 'magnetic' properties can be easily simulated. The use of the spatial light modulator to create a lattice makes the system to be easily scalable [9] with the number of condensates/spins being limited by the sample size. Taking the sample area to be of the order of 1cm^2, the width of the condensate as 1μm, and the characteristic distance between two condensates as $3-5\mu$m, brings the number of CCs to $10^6 - 10^7$ with the potential to simulate the system on a millisecond time scale [25]. Our results allow the investigation of new physical regimes, not before realisable in condensed matter systems.

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[26] The numerical evolution of Eqs. (9) were performed using the 4th-order Runge-Kutta integration in time and 4th order finite difference scheme in space starting with many random complex noise configurations. The parameters were the same as in our previous work [9]. In addition, the following parameters were used to simulate the resonant pumping: $P_2(\mathbf{r},t) = 0.5(\tanh(6t/t_{max}-3)+1)\sum_{i=1}^N p(|\mathbf{r}-\mathbf{r_i}|)$, $P_3(\mathbf{r},t) = 0.25(\tanh(6t/t_{max}-3)+1)\sum_{i=1}^N p(|\mathbf{r}-\mathbf{r_i}|)$, and to simulate the field $P_1(\mathbf{r},t) = 0.25(\tanh(6t/t_{max}-3)+1)(p(|\mathbf{r}-\mathbf{r_1}|)+p(|\mathbf{r}-\mathbf{r_2}|))$, where $t_{max}\approx 100$ is the time when a steady state is achieved.

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