Aspects of non positive curvature for linear groups with no infinite order unipotents

J. O. Button
Selwyn College
University of Cambridge
Cambridge CB3 9DQ
U.K.
j.o.button@dpmms.cam.ac.uk

2010 *Mathematics Subject Classification:*
Primary 20F67; Secondary 20F65

Keywords: linear, positive characteristic, CAT(0) space

Abstract

We show that a linear group without unipotent elements of infinite order possesses properties akin to those held by groups of non positive curvature. Moreover in positive characteristic any finitely generated linear group acts properly and semisimply on a CAT(0) space. We present applications, including that the mapping class group of a surface having genus at least 3 has no faithful linear representation which is complex unitary or over any field of positive characteristic.

1 Introduction

Knowing that a group G is non positively curved allows us to draw strong conclusions about G, not just about its geometry but also its group theoretic structure. This is explained in the book [6] where two notions of what it means for a group to be non positively curved are examined in detail. First is the class of CAT(0) groups, which are those groups acting geometrically (namely properly and cocompactly by isometries) on some CAT(0) metric space (X, d), whereupon we can conclude:

Theorem 1.1 *([6] Part III Chapter* Γ *Theorem 1.1 Part 1)*
A CAT(0) group Γ *has the following properties:*
(1) Γ *is finitely presented.*
(2) Γ *has only finitely many conjugacy classes of finite subgroups.*
(3) Every solvable subgroup of Γ *is virtually abelian.*
(4) Every abelian subgroup of Γ *is finitely generated.*
(5) If Γ *is torsion-free, then it is the fundamental group of a compact cell complex whose universal cover is contractible.*

Now the focus of this paper is on linear groups G (here meaning that G embeds in the general linear group $GL(d, \mathbb{F})$ for some integer d and some field $\mathbb{F}$ of arbitrary characteristic) which are finitely generated. Such groups also have good properties, for instance they are residually finite and satisfy the Tits alternative. However in general there is no relationship between linearity and non positive curvature. For instance the Burger-Mozes groups in [7] are CAT(0) and even have a geometric action on a 2 dimensional CAT(0) cube complex, but are infinite simple finitely presented groups, so are as far from being linear as can be imagined. As for the other way round, it is not hard to see using well known examples in small dimension that none of the five properties above are true for every finitely generated linear group: taking various subgroups of $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ is enough to show that (1), (2), (5) do not hold and the Heisenberg group of upper unitriangular 3 by 3 matrices over $\mathbb{Z}$ fails (3), whereas wreath products such as $\mathbb{Z} \wr \mathbb{Z}$ in $GL(2, \mathbb{R})$, or in positive characteristic we could take $C_p \wr \mathbb{Z}$ in $GL(2, \mathbb{F}_p(t))$, do not satisfy (4).

Another source of variation is that linear groups have good closure properties, such as being preserved under taking subgroups and commensurability. As for CAT(0) groups, even a finitely presented subgroup of a CAT(0) group need not itself be CAT(0) (see [6] Chapter III Γ Section 5) and it is unknown whether being CAT(0) is preserved under commensurability. However there is a more general notion of non positive curvature: here we will say that a group G is **weak CAT(0)** if G has an isometric action on a complete CAT(0) space X which is proper and semisimple (meaning that for any $g \in G$ the displacement function $x \mapsto d(x, g(x))$ attains its infimum over X). For this notion of non positive curvature (which does indeed hold if we have a CAT(0) group) we obtain:

Theorem 1.2 *([6] Part III Chapter* Γ *Theorem 1.1 Part 2)*
If H is a finitely generated group that acts properly (but not necessarily co-compactly) by semisimple isometries on the CAT(0) space X, then:
(i) Every polycyclic subgroup of H is virtually abelian.
(ii) All finitely generated abelian subgroups of H are undistorted in H.
(iii) H does not contain subgroups of the form $\langle a, t | t^{-1} a^p t = a^q \rangle$ for non zero p, q with $|p| \neq |q|$.
(iv) If $A \cong \mathbb{Z}^n$ is central in H then there exists a subgroup of finite index in H that contains A as a direct factor.

Moreover this class of weak CAT(0) groups is closed under taking subgroups and commensurability. But once again none of the properties (i) to (iv) hold for all finitely generated linear groups: for instance the Heisenberg group above does not satisfy (i), (ii) or (iv) whereas the Baumslag-Solitar group $BS(1, 2)$ fails (ii) and (iii) but embeds in $GL(2, \mathbb{Q})$. However in writing out the obvious matrices that generate these counterexamples, one is struck by how often unipotent matrices (where all eigenvalues equal 1) appear. This might lead us to ask: what if we only consider linear groups with no unipotent elements (other than the identity)? Do such groups share the properties (i) to (iv) of non positive curvature in Theorem 1.2? Indeed are they even weak CAT(0) groups? In this paper we will consider both zero and positive characteristic linear groups. In fact over positive characteristic it is straightforward to see that all unipotent elements have finite order, whereas our counterexamples above used infinite order unipotents. In the next section we prove Theorem 2.3, which states that any finitely generated linear group in positive characteristic is weak CAT(0) and so we conclude by the above theorem that it does possess all of the properties (i) to (iv).

In characteristic zero we also consider the class of linear groups with no infinite order unipotent elements, which now is equivalent to saying there are no non trivial unipotents (as in this case all other unipotent matrices have infinite order). This includes any real orthogonal or complex unitary group, as well as much else besides, and we can show similar properties of non positive curvature for these groups too. In particular Section 3 is about splitting theorems for centralisers and we establish property (iv) in

Corollary 3.3 for finitely generated linear groups in characteristic zero with no non trivial unipotents. An immediate application of these results, using the ideas in [5] on centralisers of Dehn twists in mapping class groups, is then given in Corollary 3.6: the mapping class group of an orientable surface of genus at least 3 cannot be linear over any field of positive characteristic, nor can it embed in the complex unitary group of any finite dimension. Indeed any linear representation in any dimension over any field sends each Dehn twist to a matrix that either has finite order or is virtually unipotent.

In Section 4 we examine the difference between a finitely generated group G having all of its infinite cyclic subgroups undistorted in G and the stronger property where this holds for all finitely generated abelian subgroups. The relevance here to linear groups is that it was shown in [16] that if G is a finitely generated linear group with no infinite order unipotents then any infinite cyclic subgroup of G is undistorted. This fact also holds under the same hypotheses for finitely generated abelian subgroups of G, but instead we examine how we might establish this stronger property without using linearity directly. To this end, we adapt some ideas of G. Connor on translation lengths to show by a quick argument in Theorem 4.2 that if a finitely generated group G has "uniformly non distorted" infinite cyclic subgroups then any finitely generated abelian subgroup is also undistorted in G. As an application, consider the group $Out(F_n)$ (which is known not to be a linear group when $n \geq 4$). It was recently shown in [18] that any abelian subgroup of $Out(F_n)$ (which will be finitely generated) is undistorted. This extended the paper [2] which established the same result for infinite cyclic subgroups of $Out(F_n)$. However as this paper actually obtains "uniform non distortion" for infinite cyclic subgroups, we obtain as an immediate corollary of Theorem 4.2 a much shorter proof of the result in [18]. We finish by noting that the property of undistorted abelian subgroups then extends immediately to $Aut(F_n)$ and free by cyclic groups $F_n \rtimes \mathbb{Z}$.

We would like to thank the various referees for their comments on the initial submission of this paper, resulting here in a version which is both stronger and shorter than the original draft.

2 Finitely generated linear groups in positive characteristic

Let k be any field in any characteristic and $d \in \mathbb{N}$ any positive integer. An element g of the general linear group $GL(d,k)$ is said to be **unipotent** if all its eigenvalues (considered over the algebraic closure $\overline{k}$ of k) are equal to 1, or equivalently some positive power of $g - I$ is the zero matrix. It is then an easy exercise (for instance using Jordan decomposition) to show that if k has characteristic $p > 0$ then any unipotent element M in $GL(d,k)$ has finite (indeed p power) order, whereas if k has zero characteristic then the only finite order unipotent element in $GL(d,k)$ is I_d.

Given an arbitrary field k (again in any characteristic for now), a **discrete valuation** on k is a function $\nu : k \to \mathbb{Z} \cup \{\infty\}$ such that
(1) $\nu(x) = \infty$ if and only if $x = 0$
(2) $\nu(xy) = \nu(x) + \nu(y)$
(3) $\nu(x + y) \geq \min(\nu(x), \nu(y))$.
This gives rise to a non archimedean metric on k via setting $d(x,y) = e^{-\nu(x-y)}$. The set of elements $\mathcal{O}_\nu = \{x \in k : \nu(x) \geq 0\}$ forms a subring of k, called the valuation ring, which is a principal ideal domain and an element π with $\nu(\pi) = 1$ is called a uniformiser.

Now suppose that k is a field of characteristic $p > 0$ which is finitely generated over the prime subfield $\mathbb{F}_p$. This means that there are finitely many algebraically independent transcendental elements $t_1, \ldots, t_d$ such that k is a finite extension of the field $\mathbb{F}_p(t_1, \ldots, t_d)$. We now adapt Theorem 1 of [13] to establish a similar result (in fact they worked in arbitrary characteristic whereas here we are only in positive characteristic but we obtain a stronger conclusion for this case).

Proposition 2.1 *If k is any finitely generated field of positive characteristic and R is any finitely generated subring of k then there exist finitely many discrete valuations $\nu_1, \ldots, \nu_b$ on k such that for any $m \in \mathbb{Z}$ the set*

$$\{r \in R : \nu_i(r) \geq m \text{ for all } 1 \leq i \leq b\}$$

is finite.

Proof. First suppose this is true for the field $k = \mathbb{F}_p(t_1, \ldots, t_d)$ and let us add a new algebraically independent transcendental t to obtain the field $k(t)$ which contains the unique factorisation domain $k[t]$. Finite generation of R means that there is a finitely generated subring $S \subseteq k$ and finitely many monic irreducible polynomials $p_1, \ldots, p_u$ (which are prime in $k[t]$) such that R is contained in the finitely generated ring $S[t, 1/p_1, \ldots, 1/p_u]$. We now proceed by induction on the transcendence degree d, so we take the valuations $\nu_1, \ldots, \nu_b$ on k satisfying the required condition for S and we extend each of these to $\nu'_1, \ldots, \nu'_b$ on $k[t]$ and then to the field of fractions $k(t)$ by setting

$$\nu'_i(a_0 + a_1 t + \ldots + a_n t^n) = \min\left(\nu_i(a_0), \nu_i(a_1), \ldots, \nu_i(a_n)\right).$$

We also add valuations $\mu_0, \mu_1 \ldots, \mu_u$ given by $\mu_0(a_0 + a_1 t + \ldots + a_n t^n) = -n$ and for $1 \le i \le u$ we have

$$\mu_i(p_i^n \frac{a}{b}) = n \in \mathbb{Z}$$

for the primes $p_1, \ldots, p_u$ above.

On taking these $b+u+1$ valuations on $k(t)$, suppose we are given $m \in \mathbb{Z}$. Any $r \in R$ is of the form

$$r = \frac{a}{p_1^{i_1} \cdots p_u^{i_u}} \qquad \text{for } a \in S[t] \text{ and } i_1, \ldots, i_u \ge 0.$$

If $\nu(r) \ge m$ for every valuation above then we immediately have $i_1, \ldots, i_u \le -m$, giving only finitely many possibilities for the denominator. Thus we can now consider $r = a$ which is a polynomial $s_0 + s_1 t + \ldots + s_n t^n$, but $\mu_0(a) \ge m$ so the degree is also bounded above by $-m$.

Now our inductive hypothesis is that

$$\{s \in S : \nu_i(s) \ge m \text{ for all } 1 \le i \le b\}$$

is finite, and we require polynomials $f(t) = s_0 + s_1 t + \ldots + s_n t^n$ with coefficients in S such that $\nu'_i(f) \ge m$ for each $1 \le i \le b$. This means that for each coefficient s_j, all of $\nu_1(s_j), \ldots, \nu_b(s_j)$ are at least m and consequently by our inductive hypothesis we have only finitely many choices for each of $s_0, \ldots, s_n$. Thus overall we have only finitely many choices for $r \in R$.

This concludes the proof for purely transcendental extensions of $\mathbb{F}_p$. As for finite extensions of these, the argument in [13] Lemma 3 now goes through verbatim.

□

Given any infinite field k with discrete valuation ν, valuation ring $\mathcal{O}_\nu$ and a uniformiser π, we have an action of $SL(n,k)$ on its Bruhat - Tits building, here denoted $\mathcal{B}_\nu$. We will need the following facts (see for instance [1] Section 6.9):
A lattice L in k^n is a free $\mathcal{O}_\nu$ submodule of k^n that contains a basis of k^n and a lattice class $[L]$ is the orbit k^*L under the obvious multiplication action. The Bruhat - Tits building $\mathcal{B}_\nu$ is a contractible $n-1$ dimensional simplicial complex where the j-simplices are subsets of lattice classes

$$\{[L_0],\ldots,[L_j]\} \text{ where } \pi L_j \subset L_0 \subset L_1 \subset \ldots \subset L_j.$$

As $SL(n,k)$ acts on the set of lattices and hence also on the set of lattice classes, it admits a simplicial action on $\mathcal{B}_\nu$ where the stabiliser of a vertex is conjugate (in $GL(n,k)$ but not necessarily in $SL(n,k)$) to $SL(n,\mathcal{O}_\nu)$. Moreover $SL(n,k)$ acts on $\mathcal{B}_\nu$ "without permutations", which is to say that given $g \in SL(n,k)$ and a simplex σ with $g(\sigma)=\sigma$ then g fixes the vertices of σ pointwise.

As $\mathcal{B}_\nu$ has the structure of a Euclidean building, it can be turned into a metric space by putting the correct Euclidean metric on the top dimensional simplices (the chambers), whereupon it becomes a complete CAT(0) space and the simplicial action of $SL(n,k)$ is by isometries. However the action is not proper (in the sense of a discrete group) because the stabiliser of a vertex is isomorphic to the infinite group $SL(n,\mathcal{O}_\nu)$. In order to further examine the stabilisers of vertices, we use the following basic lemma.

Lemma 2.2 *Given any vertex $v \in \mathcal{B}_\nu$, there exists $m \in \mathbb{Z}$ such that for all elements $g \in SL(n,k)$ fixing v we have $\nu(g_{ij}) \geq m$ for every entry g_{ij} of g.*

Proof. By the fact about stabilisers above, there exists $h \in GL(n,k)$ such that $hgh^{-1} \in SL(n,\mathcal{O}_\nu)$. But if we have two matrices $x,y \in GL(n,k)$ and $m,n \in \mathbb{Z}$ such that all entries x_{ij} of x have $\nu(x_{ij}) \geq m$ and similarly $\nu(y_{ij}) \geq n$ then the valuation of any entry of the matrix product xy is at least $m+n$ by axioms (2) and (3). As each entry s_{ij} in an element s of $SL(n,\mathcal{O}_\nu)$ has $\nu(s_{ij}) \geq 0$ by definition and $g = h^{-1}(hgh^{-1})h$, we apply this fact twice.

□

We can now give our main result.

Theorem 2.3 *If G is any finitely generated linear group over a field of positive characteristic then G acts properly and semisimply by isometries on a complete CAT(0) space.*

Proof. We are given $G \leq GL(n.k)$ where k is an arbitrary infinite field of positive characteristic but we can increase n by 1 so that we can assume G is actually a subgroup of $SL(n,k)$. Now finite generation of G means that the ring R generated by the entries of G is also finitely generated. We then replace k by the field of fractions of R, henceforth also called k, thus now k is finitely generated as a field but is still infinite and of positive characteristic. This means that $G \leq SL(n,R) \leq SL(n,k)$ and the hypothesis of Proposition 2.1 is satisfied. Consequently we obtain discrete valuations $\nu_1,\ldots,\nu_b$ on k with the aforementioned property.

We now show that $SL(n,R)$ acts properly and semisimply by isometries on an appropriate complete CAT(0) space X, thus G does too as this property passes to arbitrary subgroups. The space X will be the product of the Bruhat - Tits buildings $\mathcal{B}_{\nu_1} \times \ldots \times \mathcal{B}_{\nu_b}$ for each valuation on k that we took. As each of these factors is complete and CAT(0), our space X will be too on being given the Euclidean product metric. Moreover $SL(n,k)$ acts on each factor by isometries so it also does so on X via the diagonal action.

We now need to establish the two properties of this isometric action. Starting with properness, first consider the stabiliser $T_{\mathbf{v}}$ not in $SL(n,k)$ but rather in $SL(n,R)$ of a product of vertices

$$\mathbf{v} = (v_1,\ldots,v_b) \in \mathcal{B}_{\nu_1} \times \ldots \times \mathcal{B}_{\nu_b}.$$

Thus if g is in $T_{\mathbf{v}}$ so that $g(v_1) = v_1,\ldots,g(v_b) = v_b$ then by Lemma 2.2 we have integers $m_1,\ldots,m_b$ with every entry g_{ij} of g having $\nu_1(g_{ij}) \geq m_1,\ldots,\nu_b(g_{ij}) \geq m_b$. Hence for $m = \min(m_1,\ldots,m_b)$ we see that every entry g_{ij} satisfies $\nu_1(g_{ij}) \geq m,\ldots,\nu_b(g_{ij}) \geq m$. This means that by Proposition 2.1 there are only finitely many possibilities for the entries of g and hence the stabiliser $T_{\mathbf{v}}$ is a finite subgroup.

Now consider the stabiliser H in $SL(n,R)$ of a general point $(x_1,\ldots,x_b)$. Take any $h \in H$ with $h(x_1) = x_1,\ldots,h(x_b) = x_b$ and first consider the action of $SL(n,R)$ on the Bruhat - Tits building $\mathcal{B}_{\nu_1}$. Let d be the dimension such that x_1 lies in the d-skeleton of $\mathcal{B}_{\nu_1}$ but not the $(d-1)$-skeleton. Then the d-simplex σ_d in which x_1 lies must be sent to itself by h and so, as $SL(n,R)$ acts without permutations, we have that h fixes the vertices of σ_d pointwise.

Applying this also to $x_2, \ldots, x_b$, we see that h also fixes a vertex in every building $\mathcal{B}_{\nu_i}$ thus $h \in T_{\mathbf{v}}$ for some $\mathbf{v}$.

In order to obtain properness of the action, we take any point $(x_1, \ldots, x_b) \in X$ and consider the simplex σ_d for x_1 as above. Then there exists $\epsilon_1 > 0$ such that the ball $B(x_1, \epsilon_1)$ in $\mathcal{B}_{\nu_1}$ intersects the d-skeleton of $\mathcal{B}_{\nu_1}$ only in the interior of the simplex σ_d. Thus as any element $g \in SL(n, R)$ sends σ_d to some d-simplex, if g moves x_1 by less than ϵ_1 then we have that $g(\sigma_d) = \sigma_d$ and so g fixes a vertex as above. In particular we obtain $\epsilon_1, \ldots, \epsilon_b$ and $\epsilon = \min(\epsilon_1, \ldots, \epsilon_b)$ such that if g moves the point $(x_1, \ldots, x_b)$ by less than ϵ in the product metric on X then g moves each x_i by less than ϵ_i. Consequently g fixes a vertex in every building and so lies in the finite subgroup $T_{\mathbf{v}}$ for some $\mathbf{v}$.

Finally to show the action is semisimple, we first quote [6] Part II 6.6 (2) which states that an action by simplicial isometries of a group on a complete non positively curved simplicial complex having a finite set of shapes (which here holds, as all d dimensional simplices are isometric) is a semisimple action. Next we use Proposition 6.9 in the same part of the same volume, where it is shown that an isometric action on a product of CAT(0) spaces is semisimple if the action on each component is too.

□

Note that in general this CAT(0) space is not a proper metric space because the Bruhat - Tits building associated to the valuation ν of the field k is only locally finite if the residue field of ν is finite. This is the case for (finite extensions of) $k = \mathbb{F}_p(t)$ but fails as soon as we have more than one independent transcendental.

We finish this section with a few words on related results and on trying to extend this to the characteristic zero case. Of course there are fields of characteristic zero possessing discrete valuations, such as the p-adic valuations on $\mathbb{Q}$. In [3] a similar result to our Proposition 2.1 in characteristic zero is obtained (though the resulting sets consist of algebraic integers and need not be finite) and then the diagonal action on the resulting product of Bruhat - Tits buildings is utilised to establish exactly which finitely generated linear groups in characteristic zero have finite virtual cohomological dimension. Equivalent cohomological results for linear groups in positive characteristic occur in [10] which proceeds along the lines of Theorem 2.3, though the stabiliser of a product of vertices under this action is shown to be locally finite (which is not strong enough here as finitely generated linear groups in positive characteristic need not be virtually torsion free).

In order to obtain proper actions of finitely generated linear groups in characteristic zero, one could also throw in some (non discrete) archimedean absolute values (namely embeddings in $\mathbb{R}$ and $\mathbb{C}$) along with the action on the appropriate symmetric space which will take the place of the Bruhat - Tits building, though some care is needed with the archimedean embeddings if the resulting finitely generated field contains transcendental elements. However the analogous result in characteristic zero would require that the action is semisimple whenever our finitely generated linear group has no non trivial unipotent elements. But by [6] Section II Proposition 10.61 the elements of $SL(n,\mathbb{R})$ that act semisimply on its (real) symmetric space are exactly those matrices which are diagonalisable over $\mathbb{C}$, which is certainly a more stringent restriction even for subgroups of $SL(n,\mathbb{Z})$.

3 Abelianisation of centralisers in linear groups with no infinite order unipotents

For groups H which possess some form of non positive curvature, there are various splitting results in the literature for centralisers, which generally take the form that if an infinite order element h (or even a finitely generated free abelian group) is central in H which is finitely generated then, up to replacing H with a finite index subgroup also containing h, we have that H splits as a direct product $S \times \langle h \rangle$. This is achieved by considering the translation part of an element commuting with h when restricted to an invariant axis of h. For our class of finitely generated linear groups with no infinite order unipotents, we have already shown in Theorem 2.3 that this property holds in positive characteristic and now we do the same in characteristic zero. We first reduce this to:

Lemma 3.1 *Suppose that H is a finitely generated group and $A \cong \mathbb{Z}^n$ is central in H. If we have a homomorphism θ from H to some abelian group C which is injective on A then there exists a subgroup of finite index in H that contains A as a direct factor.*

Proof. By dropping to the image $\theta(H)$, we can assume that θ is onto and so without loss of generality C is also finitely generated. By the classification of finitely generated abelian groups, we have that $C = \mathbb{Z}^m \oplus$Torsion for $m \geq n$

and we can compose θ with a homomorphism ϕ from C onto $\mathbb{Z}^n$ in which A still injects, so the image $B = \phi\theta(A)$ will have finite index.

Thus if we set $K = \text{Ker}(\phi\theta)$ then the pullback $(\phi\theta)^{-1}(B) = KA$ has finite index in H. Also K and A are normal subgroups of H with $K \cap A = \{e\}$, giving $KA \cong K \times A$.

□

We can now use the determinant as our homomorphism with abelian image, enabling us to work in complete generality.

Theorem 3.2 *Suppose that G is a linear group over any field $\mathbb{F}$ of any characteristic and A is an abelian subgroup which is central in G. (Here neither G nor A is assumed to be finitely generated.) Let π be the homomorphism from G to its abelianisation G/G' and $\pi|_A$ the restriction of π to A. If G, or even A, contains no infinite order unipotent element then ker$(\pi|_A)$ is a torsion group.*

Proof. We first replace our field by its algebraic closure, which we will also call $\mathbb{F}$. Then it is true that any abelian subgroup of $GL(d,\mathbb{F})$ is conjugate in $GL(d,\mathbb{F})$ to an upper triangular subgroup of $GL(d,\mathbb{F})$, for instance by induction on the dimension and Schur's Lemma.

For any $g \in G$ and $a \in A$ we have $ga = ag$. This means that g must map not just each eigenspace of a to itself, but each generalised eigenspace

$$E_\lambda(a) = \{v \in \mathbb{F}^d : (a - \lambda I)^n v = 0 \text{ for some } n \in \mathbb{N}\} \text{ where } \lambda \in \mathbb{F}$$

and together these span, so that if a has distinct eigenvalues $\lambda_1^{(a)}, \ldots, \lambda_{d_a}^{(a)}$ then $\bigoplus_{i=1}^{d_a} E_{\lambda_i^{(a)}}(a)$ is a G-invariant direct sum of $\mathbb{F}^d$.

We now take a particular (but arbitrary) non identity element a of A and restrict G to the first of these generalised eigenspaces $E_{\lambda_1^{(a)}}(a)$, so that here a only has the one eigenvalue $\lambda_1^{(a)}$. If this property also holds on $E_{\lambda_1^{(a)}}(a)$ for every other $a' \in A$ then we proceed to $E_{\lambda_2^{(a)}}(a)$, $E_{\lambda_3^{(a)}}(a)$ and so on. Otherwise there is another $a' \in A$ such that we can split $E_{\lambda_1^{(a)}}(a)$ further into pieces where a' has only one eigenvalue on each piece. Moreover this decomposition is also G-invariant because it can be thought of as the direct sum of the generalised eigenspaces of a' when G is restricted to $E_{\lambda_1^{(a)}}(a)$.

We then continue this process on all of the pieces and over all elements of A until it terminates (essentially we can view it as building a rooted tree where every vertex has valency at most d and of finite diameter). We will now find that we have split $\mathbb{F}^d$ into a G-invariant sum $V_1 \oplus \ldots \oplus V_k$ of k blocks, where any element of A has a single eigenvalue when restricted to any one of these blocks.

Next we conjugate within each of these blocks so that the restriction of A to this block is upper triangular, using the comment at the start of this proof. Under this basis so obtained for $\mathbb{F}^d$, we have that any $a \in A$ will now be of the form

$$a = \begin{pmatrix} \boxed{T_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{T_k} \end{pmatrix}$$

where each block T_i is an upper triangular matrix with all diagonal entries equal (as these are the eigenvalues of a within this block). More generally any $g \in G$ will be of the form

$$\begin{pmatrix} \boxed{M_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{M_k} \end{pmatrix}$$

for various matrices $M_1, \ldots, M_k$ which are the same size as the respective matrices $T_1, \ldots, T_k$ because we know g preserves this decomposition.

Consequently we have available as homomorphisms from G to the multiplicative abelian group $(\mathbb{F}^*, \times)$ not just the determinant itself but also the "subdeterminant" functions $\det_1, \ldots, \det_k$, where for $g \in G$ the function $\det_j(g)$ is defined as the determinant of the jth block of g when expressed with respect to our basis above, and these are indeed homomorphisms as is

$$\theta : G \to (\mathbb{F}^*)^k \text{ given by } \theta(g) = (\det_1(g), \ldots, \det_k(g)).$$

As θ is a homomorphism from G to an abelian group, it factors through the homomorphism π from G to its abelianisation because this is the universal abelian quotient of G. This means that $\ker(\pi)$ is contained in $\ker(\theta)$ and so we can replace π with θ for the rest of the proof.

Thus suppose that there is some $a \in A$ which is in the kernel of θ. We know that

$$a = \begin{pmatrix} \boxed{T_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{T_k} \end{pmatrix}$$

for upper triangular matrices T_i and as all diagonal entries within each T_i are equal, say μ_i for $\mu_1, \ldots, \mu_k \in \mathbb{F}^*$, we conclude that $\mu_i^{d_i} = 1$ where $d_i = \dim(V_i)$. In other words a is virtually unipotent (namely some positive power of a is unipotent), implying under our hypotheses that a has finite order.

□

We now immediately obtain the same conclusion of Theorem 1.2 Part (iv) for finitely generated linear groups in characteristic zero, provided only that our abelian subgroup is unipotent free.

Corollary 3.3 *If H is any finitely generated linear group in characteristic zero and $A \cong \mathbb{Z}^n$ is central in H and does not contain a non identity unipotent element then there exists a subgroup of finite index in H that contains A as a direct factor.*

Proof. Theorem 3.2 gives us a homomorphism θ from H to some abelian group C whose restriction to A has kernel consisting only of torsion elements, thus θ is injective on A and Lemma 3.1 applies.

□

3.1 Applications to the mapping class groups

Of course any finitely generated group G which fails to satisfy any of the four conditions in Theorem 1.2 cannot act properly and semisimply by isometries on a CAT(0) space. An important example of this is the mapping class group $Mod(\Sigma_g)$ where here Σ_g will be an orientable surface of finite topological type having genus g at least 3 (which might be closed or might have any number of punctures or boundary components). In [5] Bridson shows that for all the surfaces Σ_g mentioned above, the mapping class group $Mod(\Sigma_g)$ is not a weak CAT(0) group, a result first credited to [14]. This is done using the following obstruction which is similar to Theorem 1.2 Part (iv).

Proposition 3.4 *([5] Proposition 4.2)*
If Σ is an orientable surface of finite type having genus at least 3 (with any number of boundary components and punctures) and if T is the Dehn twist about any simple closed curve in Σ then the abelianisation of the centraliser in $Mod(\Sigma)$ of T is finite.

As this is covered by Theorem 3.2, we have

Corollary 3.5 *Suppose that Σ is an orientable surface of finite type having genus at least 3 (with any number of boundary components and punctures) and $\rho : Mod(\Sigma) \to GL(d,k)$ is any linear representation in any dimension over any field. Then for every Dehn twist $T \in Mod(\Sigma)$, the matrix $\rho(T)$ either has finite order or is virtually unipotent.*

Proof. Set A and G to be the images under ρ of $\langle T\rangle$ and the centraliser in $Mod(\Sigma)$ of T respectively. Thus A will be abelian and central in G and we know by Proposition 3.4 that the abelianisation of the centraliser in $Mod(\Sigma)$ of T is finite, so the abelianisation of G is also finite. Thus on applying Theorem 3.2 to G and A, either $A = \langle\rho(T)\rangle$ contains some infinite order unipotent element or $\ker(\pi|_A)$ is a torsion group. As π maps to the finite group G/G', $\rho(T)$ has finite order in the second case.

□

This immediately gives us:

Corollary 3.6 *If Σ is an orientable surface of finite type having genus at least 3 (with any number of boundary components and punctures) and $\rho : Mod(\Sigma) \to GL(d,k)$ is any linear representation of the mapping class group of Σ in any dimension d where the field k either has positive characteristic or $k = \mathbb{C}$ and the image of ρ lies in the unitary group $U(d)$ then $\rho(T)$ has finite order for T any Dehn twist and thus ρ cannot be faithful.*

We note that there are "quantum" linear representations with infinite image but where every Dehn twist has finite order. However linearity of the mapping class group in genus $g \geq 3$ is a longstanding open question, although in genus 2 linearity over $\mathbb{C}$ was established in [4] and in [15] by applying results on braid groups.

4 Undistorted cyclic and abelian subgroups

Having shown that the non curvature properties in Theorem 1.2 are satisfied by any finitely generated linear group in positive characteristic, we have also seen that proposition (iv) holds for finitely generated linear groups in characteristic zero if they contain no non trivial unipotent matrices. We might also wonder about the other three properties for this class of groups; indeed they all hold too (see [8]). In this section we examine property (ii), which is that all finitely generated abelian subgroups are undistorted. Our proof of this in [8] involves taking a finite number of different absolute values on the field, in a similar fashion to Proposition 2.1, and then using the operator norm of the matrix elements with respect to each absolute value in order to show that there are no finitely generated abelian subgroups which are distorted. In the special case of infinite cyclic subgroups, this fact was already proved in [16] Proposition 2.4 where an argument also using the operator norm was provided and moreover this is a short proof because only one absolute value is required there. Now if it were true for a finitely generated group G that having all infinite cyclic subgroups undistorted implies all finitely generated abelian subgroups are undistorted then that short argument could also be used to establish property (ii) in Theorem 1.2. However this does not hold in general, for instance some groups of the form $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$ provide counterexamples. Consequently in this section we examine what further conditions can be placed on a finitely generated group in order to ensure that all of its finitely generated abelian subgroups are undistorted. This is achieved by examining the work of G. Connor (in [9] and other papers cited there) on translation length.

We proceed as follows: if G is any finitely generated group then put the word length l_S on G (with respect to some finite generating set S). Next let τ be the associated translation length function from G to $[0, \infty)$, that is $\tau(g) = \lim_{n\to\infty} l_S(g^n)/n$. Now having $\tau(g) > 0$ for all infinite order elements g is equivalent to saying that every cyclic subgroup of G is undistorted. This suggests the following definition:

Definition 4.1 *We say a finitely generated group G has* **uniformly undistorted cyclic subgroups** *if there exists $c > 0$ with $\tau(g) \geq c$ for all infinite order $g \in G$.*

Note that, just as for the property of having undistorted cyclic subgroups, having uniformly undistorted cyclic subgroups is invariant under change of generating set S (though the constant c varies), because both l_S and τ will be replaced by Lipschitz equivalent functions.

Our result is as follows.

Theorem 4.2 *Let G be a finitely generated group with uniformly undistorted cyclic subgroups. Then any finitely generated abelian subgroup A of G is undistorted in G.*

Proof. Any finitely generated abelian subgroup will have a finite index subgroup A which is isomorphic to some free abelian group $\mathbb{Z}^m$ and it is enough to show that A is undistorted in G. First pick out any free abelian basis $a_1, \ldots, a_m$ for A. Now by an $\mathbb{R}$-norm $||\cdot||$ on $\mathbb{R}^m$ we mean the standard definition from normed vector spaces, that is it satisfies the triangle inequality with $||v||$ being zero if and only if v is the zero vector, and also $||\lambda v|| = |\lambda|\cdot||v||$ for $|\cdot|$ the usual modulus on $\mathbb{R}$. We will also define a $\mathbb{Z}$-norm on $\mathbb{Z}^m$ to be a function $f : \mathbb{Z}^m \to [0, \infty)$ having the same properties, except the last becomes $f(na) = |n| \cdot ||a||$ for all $n \in \mathbb{Z}$ and $a \in \mathbb{Z}^m$. We also have $\mathbb{R}$- and $\mathbb{Z}$-seminorms where we remove the $||\cdot|| = 0$ implies $\cdot = 0$ condition.

On considering word length l_S on G with respect to some finite generating set S and the associated translation length τ, we have $0 \leq \tau(g) \leq l_S(g)$ for any $g \in G$ by repeated use of the triangle inequality and also $\tau(g^n) = |n|\tau(g)$. We further have $\tau(gh) \leq \tau(g) + \tau(h)$ for commuting elements g, h (but not in general). Thus on restricting τ to the abelian subgroup A we see that τ is a $\mathbb{Z}$-seminorm on A, hence also a $\mathbb{Z}$-norm as $\tau(a) > 0$ for all $a \in A \setminus \{id\}$. Indeed τ is actually a discrete $\mathbb{Z}$-norm, given that we have $c > 0$ with $\tau(a) \geq c$ for all $a \in A \setminus \{id\}$.

On regarding $A \cong \mathbb{Z}^m$ as embedded in $\mathbb{R}^m$ via the integer lattice points, we can extend τ to $\mathbb{Q}^m$ by dividing through and to $\mathbb{R}^m$ by taking limits, so that τ is also an $\mathbb{R}$-seminorm on $\mathbb{R}^m$. However it is not too hard to see that τ is in fact a genuine $\mathbb{R}$-norm, as explained carefully in [17], and we denote this by $||\cdot||_\tau$.

Now take $a = a_1^{n_1} \ldots a_m^{n_m} \in A$ which is also the lattice point $(n_1, \ldots, n_m) \in \mathbb{R}^m$. We have the ℓ_1 norm $||\cdot||_1$ on $\mathbb{R}^m$ but all norms on $\mathbb{R}^m$ are equivalent, so there is $k > 0$ such that

$$l_S(a) \geq \tau(a) = ||(n_1, \ldots, n_m)||_\tau \geq k||(n_1, \ldots, n_m)||_1 = k(|n_1| + \ldots + |n_m|)$$

so A is undistorted in G.

□

4.1 $Out(F_n)$ and related results

We finish with an application of the above theorem to the outer automorphism group $Out(F_n)$ of the rank n free group, as well as to some related groups. Now, at least for $n \geq 4$, $Out(F_n)$ is not linear over any field by

[11] nor is it a weak CAT(0) group by [12]. Thus we cannot apply earlier results directly to $Out(F_n)$ in order to establish that every finitely generated abelian subgroup of $Out(F_n)$ is undistorted, which was recently proven in [18]. But previously the paper [2], which ostensibly had shown that all cyclic subgroups of $Out(F_n)$ are undistorted, actually gave us more:

Theorem 4.3 *([2] Theorem 1.1)*
Every infinite order element g of $Out(F_n)$ has positive translation length $\tau(g)$. Furthermore there exists a constant $c_n > 0$ such that $\tau(g) \geq c_n$ for all $g \in Out(F_n)$ (using the generating set consisting of permutations, inversions and Nielsen twists), so that $Out(F_n)$ has uniformly undistorted cyclic subgroups.

Thus combining that (short) paper with Theorem 4.2, we immediately obtain a quick proof of [18]:

Corollary 4.4 *Every finitely generated abelian subgroup of $Out(F_n)$ is undistorted.*

We end by pointing out that $Out(F_n)$ having undistorted abelian subgroups can be used to establish some further consequences.

Corollary 4.5 *If A is any abelian subgroup of the automorphism group $Aut(F_n)$, or of any free by cyclic group $G = F_n \rtimes_\alpha \mathbb{Z}$ for $\alpha \in Aut(F_n)$ then A is finitely generated and undistorted in $Aut(F_n)$ or in G.*

Proof. If S, H, G are all finitely generated groups with $S \leq H \leq G$ and S is undistorted in G then S is undistorted in H (else extend the generating set of H to one of G, whereupon the distortion persists). The converse also holds if H has finite index in G. Now an observation dating back to Magnus is that $Aut(F_n)$ embeds in $Out(F_{n+1})$ by considering automorphisms of F_{n+1} which fix the last element of the basis.

Moreover for the free by cyclic group $G = F_n \rtimes_\alpha \mathbb{Z}$, we have that if α has infinite order in $Out(F_n)$ then G embeds in $Aut(F_n)$. This can be seen on taking the copy F_n of inner automorphisms in $Aut(F_n)$ and then G is isomorphic to $\langle \alpha, F_n \rangle \leq Aut(F_n)$. If however α has finite order then G contains the finite index subgroup $H = F_n \times \mathbb{Z}$ and this certainly has undistorted finitely generated abelian subgroups.

□

References

[1] P. Abramenko and K. S. Brown, Buildings, GTM 248, Springer, 2008.

[2] E. Alibegovic, *Translation Lengths in* $Out(F_n)$, Geom. Dedicata **92** (2002) 87–93.

[3] R. C. Alperin and P. B. Shalen, *Linear groups of finite cohomological dimension*, Invent. Math. **66** (1982) 89–98.

[4] S. J. Bigelow and R. D. Budney, *The mapping class group of a genus two surface is linear*, Algebr. Geom. Topol. **1** (2001) 699–708.

[5] M. R. Bridson, *Semisimple actions of mapping class groups on CAT(0) spaces*, Geometry of Riemann surfaces, 1–14, London Math. Soc. Lecture Note Ser., 368, Cambridge Univ. Press, Cambridge, 2010.

[6] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer, 1999.

[7] M. Burger and S. Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Ser. I Math. **324** (1997) 747–752.

[8] J. O. Button, *Properties of linear groups with restricted unipotent elements*, `http://arxiv.org/1703.05553`

[9] G. R. Conner, *Discreteness properties of translation numbers in solvable groups*, J. Group Theory **3** (2000) 77–94.

[10] J. Cornick and P. H. Kropholler, *Homological finiteness conditions for modules over group algebras*, J. London Math. Soc. **58** (1998) 49–62.

[11] E. Formanek and C. Procesi, *The automorphism group of a free group is not linear*, J. Algebra **149** (1992) 494–499.

[12] S. M. Gersten, *The automorphism group of a free group is not a CAT(0) group*, Proc. Amer. Math. Soc. **121** (1994), 999–1002.

[13] E. Guentner, N. Higson and S. Weinberger, *The Novikov conjecture for linear groups*, Publ. Math. Inst. Hautes tudes Sci. **101** (2005), 243–268.

[14] M. Kapovich and B. Leeb, *Actions of discrete groups on nonpositively curved spaces*, Math. Ann. **306** (1996) 341–352.

[15] M. Korkmaz, *On the linearity of certain mapping class groups*, Turkish J. Math. **24** (2000) 367–371.

[16] A. Lubotzky, S. Mozes and M. S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, Inst. Hautes tudes Sci. Publ. Math. No. 91 (2000) 5–53.

[17] J. Steprans, *A characterization of free abelian groups*, Proc. Amer. Math. Soc. **93** (1985) 347–349.

[18] D. Wigglesworth, *Distortion for abelian subgroups of* $Out(F_n)$, Internat. J. Algebra Comput. **28** (2018) 573–603.