Effects of Massive Fields on the Early Universe

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Abstract

Cosmology is one of the best tools to understand the physics that governs the universe at high energies. On one hand, inflation is a very robust mechanism to explain the initial conditions of the universe. On the other hand general relativity provides a solid framework for the formation of cosmic structures at cosmological scales. Nevertheless, there are still important issues that remain without a clear answer. For example, inflation still lacks of a concrete microphysical description, and also there is still no satisfactory mechanism to explain the late time acceleration of the universe. This thesis addresses these two topics.

In the first part we discuss the effects of heavy degrees of freedom coupled to inflation. This has been an important topic over the years, because the experimental success might make it possible to detect new degrees of freedom in inflation. In chapter two we discuss the case when non relativistic heavy fields are coupled to the inflaton through a non minimal gravitational coupling. Here we find that, for certain geometries, the heavy field can modify the potential for a few e-folds, either stopping inflation, or setting its initial conditions. In chapter 3 we study the dynamics of fluctuations in holographic inspired models of multi-field inflation. We find that the entropy mass \( \mu \) (the mass of the fluctuation orthogonal to the trajectory of inflation) satisfies an universal upper bound given by \( \mu \leq 3H/2 \). This bound coincides with the requirement of unitarity of conformal operators living on the boundary of the theory.

In the second part of the thesis we study high energy effects on the Cosmic Microwave Background (CMB). In the fourth chapter we study the role of disformal transformation on cosmological backgrounds and its relation to the speed of sound for tensor modes. A speed different from one for tensor modes can arise in several contexts such as Galileons theories, or massive gravity. Nevertheless the speed is very constrained to be one by observations of gravitational wave emission. It has been shown that in inflation a disformal transformation allows the speed for tensor modes, to be set to one without making changes to the curvature power spectrum. We show that on the CMB, after doing the transformation, there is an imprint on the acoustic peaks, and the diffusion damping. This has interesting consequences: for a particular
class of theories the transformation can be used to constrain the parameter space in different regimes. In chapter five we study the impact of gravitons with non-vanishing masses on the polarisation of the CMB. We also focus on putative modifications to the speed of the gravitational waves. We find that a change of the graviton speed shifts the acoustic peaks of the B-mode polarization and then could be easily constrained. In all cases when both massless and massive gravitons are present, we find that the B-mode CMB spectrum is characterised by a low $l$ plateau together with a shifted position for the first few peaks compared to a massless graviton spectrum. This shift depends on the mixing between the gravitons in their coupling to matter and could serve as a hint in favour of the existence of multiple gravitons.
Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specified in the text. However, as it is common in theoretical physics nowadays, most of the material of this thesis is based on results from collaborative projects. In particular, Chapter 2 is based on work done in collaboration with Anne Davis which was presented in [1]. The work of chapter 3 is done in collaboration with Ana Achucarro, Anne Davis, Gonzalo Palma and Fernando Quevedo, and it has not been published yet. Chapter 4 are discussed in [2] in collaboration with Clare Burrage and Anne Davis. Finally, the work of Chapter 5 was presented in [3], in collaboration with Phillipe Brax and Anne Davis. I have been primarily responsible for all the work presented in this thesis.

I hereby declare that my thesis entitled *Effects of Massive Fields on the Early Universe* is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as specified in the text.

Sebastian Cespedes
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Chapter 1

Introduction

One of the scientific landmarks of recent times has been the understanding of the origin of the Universe. It was not long ago that the knowledge about how cosmic structure was formed was, to say the least, precarious. Noteworthy important aspects have come to light both from theory and experiments. To name a few: the existence of dark matter, the formation of nuclear elements, the formation of large and small scale cosmic structures and the existence and understanding of the cosmic microwave background. This has led to a deep understanding of physics of the Universe, which with great precision allows us to keep delving into important questions.

All this progress has been made possible not only by impressive technological advances but also because there is an extensive physical understanding of the dynamics of the universe. Much of what is known in theoretical physics has found in cosmology a firm ground to apply and investigate related ideas. For example, inflation [5–7] the theory that explains why the universe is flat and homogeneous at large scale, has been possible by a combined effort from string theorists, particle physicist and astrophysicists. This kind of symbiosis makes cosmology an interesting, and yet relevant, field of study to hopefully gain an even more profound understanding about the universe and learn more about the theories that govern nature.

Out of the many questions that have been solved there are still many aspects that remain unanswered. First of all there are several proposals for a theory of quantum gravity, none of them still in full form. This has consequences for early cosmology, because it is believed that in its origin the universe was intrinsically quantum. Then, important questions such as the origin of time, have not been solved. Also it implies that there is no concrete model of inflation, so all our knowledge has been based upon making educated guesses about how its microphysics worked. Secondly there is no clear explanation for the late time acceleration of the universe, which has been
now confirmed by several independent observations [8–13]. This problem sometimes called \textit{dark energy}, has been addressed from several angles but none of them makes a sound explanation [14–16]. Lastly, most of the matter of the universe does not seem to interact with photons, hence the name \textit{dark matter}. Even though there is a clear understanding about how dark matter behaves at large scales, there is no knowledge of its particle physics content, and so it has not yet been detected as a particle [17, 18]. This thesis addresses theoretical problems of inflation and some dark energy models and in that sense studies possible proposed solutions to the problems mentioned above.

\textbf{Inflation as a high energy probe}

It is fair to say that inflation allows the possibility of detecting new physics by using cosmological experiments. This is because inflation provides the initial condition for the formation of large scale structures, whose probability distribution at early times can be measured with high precision [4]. Then different mechanisms or particle content of inflation could in principle, lead to different signatures to be measured by studying the structure of the early universe. In particular, the particle responsible for inflation, the \textit{inflaton} seems to be very light compared to the mass scale of inflation $H$. Because the influence of gravity is important during inflation, the normal techniques of particle physics have to be used very carefully. For example, although, from an effective field theory (EFT) perspective, heavier particles would be decoupled from the inflaton, it is known that this is not usually the case [19–25]. Detecting the interaction with other heavier particles will be a clear discovery of new physics. Moreover, there are other interesting effects over the inflaton due to the influence of these heavy particles [26–29]. Then studying deviations on the inflaton trajectory or its speed of sound, to name a few, could be interpreted as the effect of heavier particles [30–32].

Embedding inflation into a concrete model is an important task for the future. The main candidate for quantum gravity, string theory, then represents a relevant framework where to put inflation [33–35]. Nevertheless, there are great difficulties to do so, and there is the possibility that this might not be possible [36–38]. A way to go, is to study particular features inherited by string theory. For example the existence of several scalar fields, or the fact that the geometry of field space might not be flat [39–44]. These features produce distinctive signatures that can be studied in a more model independent way, and then contrasted with experiments. By doing so we might get a guideline on how a hypothetical quantum gravity model of inflation could be.
Over recent years holography has come to light as one of the better ways to understand theories of quantum gravity [45–47]. Initially formulated as a duality between maximally symmetric spacetimes with negative cosmological constant (anti de Sitter) and conformal field theories living on the boundary of those spacetimes, has helped to formalise intuition and gain some understanding about situations where quantum gravity is known to be relevant, e.g., black holes [48–51]. The universe during inflation is well described by a de Sitter geometry, i.e., it had maximally symmetric spacetimes with positive cosmological constant. Moreover a dark energy dominated universe will lead to a de Sitter epoch in the future after all matter and radiation has diluted. Then, it is very natural to ask whether the universe can be understood holographically, or in other words, whether there is a conformal field theory living on some boundary, whose physics can be related to the physics of the inflationary epoch.

This has been a pivotal question over the last twenty years, and although progress has been made, there is still a large debate about the precise details, or even the existence of such dualities [52–57]. Many of the problems arise because de Sitter spacetimes are much harder to understand than the anti de Sitter (AdS) counterparts. For example, the notion of conserved energy is not well defined, because all Killing vectors are spacelike in some regions and timelike in others, then associated conserved charges will be positive or negative depending on their region of spacetime. Moreover the only notion of boundary known is given by the horizon of the de Sitter spacetime. This implies that intuition brought from AdS might lead to wrong physics if not handled carefully [55, 58]. Nevertheless there are still important lessons that have been drawn by doing analogies with the AdS. For example, it is now well understood that the scalar perturbations posses conformal symmetry when their size is compared to the horizon. This implies a strong constraint on their correlation functions, and therefore on what should be expected from observations [59, 60]. More so, the high degree of symmetry, if respected implies constraints on other particles, thus making it possible to narrow down experimental searches to the most well motivated scenarios [20, 61].

**Cosmology as a test for dark energy**

A fruitful way to explain the late time acceleration of the universe has been to modify gravity at large scales. It seems that it is not hard to modify the Einstein equations in such a way that there are self accelerating cosmologies [62, 63]. This idea does not come without detractors: it is not clear why Einstein gravity should be modified in the first place, given that it has not shown any signal of failure, and that there are no motivations from quantum gravity. Nevertheless modifying gravity persists as the best
way to parametrise any possible deviations, and it is fair to say, that Einstein gravity is a non linear theory, whose study has not yet been concluded.

One important modification is to give the graviton a mass [64, 65]. This implies a scalar degree of freedom, thus explaining the late time acceleration. Moreover it implies that the cosmological constant degravitates, \textit{ie}, is not affected by gravity, making the problem potentially more tractable. Achieving a concrete realization of massive gravity has been a huge achievement over the past decade, starting from the first Lorentz violating attempts [61, 66, 64], to the covariants models of [67, 68]. All these models have to satisfy several constraints before being put in a similar standing to general relativity. The presence of an extra scalar field implies that there is a fifth force, which is constrained at a high precision using solar system tests [69]. Massive gravity solves this problem by being screened in the presence of dense sources. This makes the scalar field undetectable on small scales, but still present over large scales, which drives the acceleration of the universe. On the other hand, cosmological tests might prove to be lethal for these theories if primordial gravitational waves are detected. This is because a massive graviton modifies the large angular modes of the polarization power spectrum [70], and if detected could constrain the mass of the graviton to be of order of $10^{-33}$ eV, where it loses all theoretical interest [71].

Another important class of examples were introduced in [72, 73], by drawing analogies with brane models. It was noted that they have to respect an extra shift symmetry that could modify gravity and provide self accelerating cosmological solutions [74–82]. This symmetry is a generalization of the Galilean transformations, so these models are called \textit{Galileons}. The Galilean symmetry allows one to have higher derivative theories which are still second order in the action. The simplicity of this modification provides a motivated and tractable example to study which has been thoroughly generalised and applied to several contexts. It also has allowed us to distinguish the main features that appear when modifying gravity [83, 84]. To mention the most important, the existence of a fifth force and the modifications of the linear equations of motion for the graviton. This one implies, in the case of higher derivative theories, that the graviton has a modified speed of propagations, and so the tighter constraints obtained by the LIGO experiments could be applied [85–88].

**Outline of the thesis**

We start this thesis by outlining the main theoretical framework of cosmology as it is understood today. We first review the Friedmann Robertson Walker (FRW) universe and how it provides a solid background to understand the history of the universe. Then
we introduce inflation as a solution to the horizon and flatness problems that appear when considering a FRW universe. We detail single and multifield inflation, and we consider their perturbations. We finally introduce the physics of the cosmic microwave background (CMB) as a way to detect new physics.

The remainder constitutes the main work of the thesis. The first part deals with massive fields during inflation. In the second chapter we study how non canonical couplings to inflation in the presence of matter, could lead to several modifications for the inflationary potential. This scenario is applied to several examples arising from supergravity theory, where it might be an important effect.

In the third chapter we study holography in the context of inflation. We start by discussing how to translate the dictionary in AdS/CFT to de Sitter and inflation. We focus in particular how to define a well behaved wavefunction of the universe. Then we discuss how to understand different models of inflation in a holographic way, in particular to models motivated by string theory.

In the second part of this thesis, we study how to probe new physics by using the cosmic microwave background. In chapter four we analyse the idea, drawn from inflation, that the graviton speed of sound can be interchanged by a speed of sound for the scalar perturbations. We then focus on realising this idea in Galileon theories. In the fifth chapter we study the modifications of massive gravity on the CMB. There we first study how to understand analytically the modifications of massive gravity on the large scales of the polarisation power spectrum. We apply this to models with more than one graviton, because, they seem to be well behaved in certain cosmological limits. We finally provide conclusions, and a series of appendices with technical details.

1.1 FRW Universe

In this section we will collect the main results of the standard model of cosmology. We start by describing the Friedmann-Robertson-Walker (FRW) metric and how it is useful to model the homogeneous universe. We then introduce inflation as a solution to the causality problems of a matter and radiation dominated universe.

We will pay attention to the inhomogeneities produced early in the Universe, either in inflation, and as at later times, where we we will describe the cosmological microwave background (CMB). We will focus mainly on the physics rather than the explicit calculations, although the main results are all stated.
1.1.1 FRW background

As observations show, our universe is homogeneous and isotropic when looked on scales larger than 100Mpc. This is usually stated as the cosmological principle, and it has been confirmed by several experiments, in particular during the last 20 years. Mathematically this implies that the universe is well described by the Friedmann-Robertson-Walker (FRW) metric,

\[ ds^2 = -dt^2 + a^2(t)d\ell^2, \]  

(1.1)

where \( a(t) \) is called the scale factor, \( t \) runs asymptotically from 0 in the infinite past to \( \infty \) in the infinite future, and \( d\ell^2 \) represents the line element of a 3 dimensionally maximally symmetric space. Depending on the curvature \( k \), this is written as,

\[ d\ell^2 = \frac{dr^2}{1-kr^2} + r^2d\Omega^2, \]  

(1.2)

where \( k = 0 \) represents a flat Euclidean spacetime \( E^3 \), \( k = 1 \) a 3-sphere \( S^3 \) and for \( k = -1 \) a hyperboloid \( H^3 \). Current observations indicate that the universe is nearly flat, so for the rest of the chapter we will assume that \( k = 0 \).

It is useful to define the Hubble parameter \( H \equiv \dot{a}/a \), which also quantifies the growth rate of the universe. By convention \( a(t) \) is set to be 1 at \( t = 0 \). The measured Hubble parameter today is \( H_0 = 70 \pm 0.5 \) Km sec\(^{-1}\) Mpc\(^{-1}\). It is also convenient to define the redshift as \( a^{-1} \equiv 1 + z \), so that local observations are at redshift zero.

Now to determine the dynamics of the universe let us use the symmetries of the FRW metric. Homogeneity requires that all scalar functions depend only on time, and isotropy that any vector component vanishes. Hence, the matter content of the universe can be considered a perfect fluid, whose energy momentum tensor is

\[ T_{\mu\nu} = (\rho + p)U_\mu U_\nu - P g_{\mu\nu}, \]  

(1.3)

where \( \rho \) and \( P \) are the density and pressure of the fluid, and \( U_\mu \) is the four velocity with respect to an observer. The continuity equation is \( \dot{\rho} = -3(\rho + p)H \). For \( (\rho + p) > 0 \), it implies that the density decay was larger at earlier times. The Einstein equations for \footnote{There is a discrepancy about the value of \( H_0 \), whereas for CMB experiments the value is 67.8 \pm 0.9 km/s/Mpc [8], measurements using galactic Cepheids, which are independent of CMB measurements, give 73.52 \pm 1.62 [89].}
1.1 FRW Universe

this system are,

\[ \dot{H} + H^2 = -\frac{4\pi G}{3} 8\rho, \]

\[ H^2 = \frac{8\pi G}{3} \rho. \]  \hspace{1cm} (1.4)

where \( \dot{H} = \frac{dH}{dt} \). This system of equations are usually called Friedmann equations.

To solve the equations we have to assume a particular matter content to be modelled by the perfect fluid. The usual simplification is to consider that all massive particles as dust or matter, hence a pressureless fluid, and that massless particles constitute radiation with an equation of state \( p = \rho/3 \). Also, we can write a cosmological constant \( \Lambda \) as a fluid with equation of state \( p = -\rho \). Then, the solutions for the equations (1.4) are,

\[ a(t) \propto \begin{cases} 
  t^{2/3} & \text{matter domination,} \\
  t^{1/2} & \text{radiation domination,} \\
  e^{H_0 t} & \text{dark energy domination.} 
\end{cases} \]  \hspace{1cm} (1.5)

As it can be seen from the solutions, at earlier times radiation starts dominating the universe, until this cools down enough for matter to take over. It was discovered twenty years ago, by the measurement of the Hubble constant, that the expansion of the universe at redshift 0 does not match matter domination, but it is better explained by considering a dark energy dominated universe. A satisfactory explanation of what constitutes dark energy, is still one of the most important questions of physics.

There have been many attempts to solve this problem, but none of them is still satisfactory. From a phenomenological point of view, an interesting attempt has been to modify gravity at late times. It has been found that a scalar field can provide an accelerating universe. Interestingly, considering a massive graviton also provides a self accelerating solution, besides more interesting dynamics.

In order to quantify distances, we can rewrite the Friedmann equations in terms of the critical densities, \( \Omega_X = \rho_X/\rho_c \), where the critical density, \( \rho_c \), is calculated assuming that \( \Lambda = 0 \) today, so \( \rho_c = \frac{3H_0^2}{8\pi G} \). Then, the Friedmann equation (1.4) become,

\[ H^2 = H_0^2 \left( \Omega_\Lambda + \Omega_M a^{-3} + \Omega_R a^{-4} \right). \]  \hspace{1cm} (1.6)

This equation, can be integrated to obtain the distances and age of the universe. Doing so, we find that the transition to radiation domination occurred at \( z \sim 2200 \) while dark energy starts to dominate around redshift 0. Time is proportional to \( H_0^{-1} \). Integrating
1.6 and assuming that the universe is flat we can calculate the age of the universe to be around 13.6 \textit{Gyr}. Let us analyse the structure of this FRW metric. First, we can write the metric as

\[ ds^2 = a^2(\tau)(-d\tau^2 + dl^2), \]  

(1.7)

where the \textit{conformal time} \( \tau \), is defined by \( \tau = \int_{t_0}^{t} \frac{dt}{a(t)} \) and runs from \(-\infty\) in the past to 0 in the future. The metric clearly implies the existence of an horizon beyond which no possible interaction could be causal. The observable universe can be quantified by the event horizon

\[ \chi(t) \equiv \int_{0}^{t} \frac{dt'}{a(t')}, \]

(1.8)

The symmetries in this case are of the Minkowski space in 4 dimensions \( SO(3,1) \), which corresponds to the 3 killing vectors from isotropy and time translations from homogeneity. For a cosmological constant, we have that \( H \) is constant and the space is maximally symmetric, corresponding to that of a \textit{de Sitter} spacetime. This space has a larger number of symmetries \( SO(4,1) \) and crucially it is conformally invariant. This extra symmetry will play an important role, as it allows us to severely constrain all the observables, and also it guides any possible answer for the early universe.

1.2 Inflation

As we have seen, for a FRW metric there is a horizon that limits information an observer can receive. The lightcone of a particle at a time \( t \) is given by,

\[ \chi(t) \equiv \int_{0}^{t} \frac{dt'}{a(t')}, \]

(1.9)

where we have considered the limit \( t \to 0 \). In terms of the equation of state \( p = \omega \rho \) the distance becomes,

\[ \chi(t) = \frac{2}{1 + 3\omega}(aH)^{-1}. \]

(1.10)

So the distance is proportional to the comoving Hubble horizon \((aH)^{-1}\), which for dust and radiation is finite and grows with time. Then, most of the contribution to the total distance comes from later times. This implies, that for the present size of the horizon, there are regions which were not causally connected when the universe
began. As the universe is very homogeneous at large scales, one should have expected that all the information in the universe was contained in a causal patch at early times. Otherwise no local dynamics would have produced the measured level of homogeneity.

A possible solution to this problem was given by Guth and others in a theory called inflation [5–7]. During inflation, the comoving Hubble horizon \((aH)^{-1}\) shrinks, thus all information contained in the late time observable horizon was in a causal patch at the beginning of inflation. This implies that,

\[
\frac{d}{dt}(aH)^{-1} = -\frac{1}{a} \left( 1 + \frac{\dot{H}}{H^2} \right) < 0. \tag{1.11}
\]

so we can define inflation as the epoch where the parameter \(\epsilon \equiv -\frac{\dot{H}}{H^2} < 1\). Note that this is equivalent to \(\dot{a} > 0\), so during inflation the universe was accelerating. As we mentioned before, a period of accelerated expansion corresponds to a universe filled with a cosmological constant, i.e., a de Sitter universe. There is a crucial difference with pure de Sitter, inflation has to end to enable radiation domination, thus time translations during inflation have to be broken. This means that the Hubble parameter \(H\) has a small time dependence, such that \(0 < \epsilon = -\frac{\dot{H}}{H^2} < 1\).

By exploiting this description, inflation is usually implemented as a scalar field \(\phi\) slowly rolling down a potential \(V(\phi)\). Deviations from de Sitter spacetime are parameterised as the rate of the kinetic energy of the scalar field, \(\dot{\phi}\), with respect to the de Sitter energy scale, \(\epsilon \equiv \dot{\phi}^2/2H^2\), which for exact de Sitter is zero. Inflation will end when its kinetic energy dominates over the quasi constant potential. The equations of motion for the homogeneous scalar field are,

\[
\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \\
3H^2 = \dot{\phi}^2/2 + V(\phi). \tag{1.12}
\]

and also we have that \(\dot{H} = -\frac{1}{2}\dot{\phi}^2\), so, as we mentioned, broken time translation implies that the Hubble parameter has a time dependence, and that the scalar field is a dynamical degree of freedom. Note that both conditions for inflation are,

\[
\epsilon \equiv \frac{\dot{\phi}^2}{2H^2} = -\frac{\dot{H}}{H^2} < 1. \tag{1.13}
\]

Also, the acceleration of the field has to be slow, we then get ,

\[
\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{1.14}
\]
The case when $|\eta| < 1$ and $\epsilon < 1$ is called the slow roll conditions, and are the most commonly used implementation of inflation. Assuming that they are valid we can simplify the equations of motion, and neglect $\ddot{\phi}$. The slow roll condition can be shown to be equivalent to reduce to the following parameters,

$$\epsilon_V \approx \frac{V_{,\phi}}{V}, \quad \eta_V \approx -\frac{V_{,\phi\phi}}{V}. \quad (1.15)$$

being smaller than one. So during slow roll the potential is approximately flat with a negative step. Moreover, slow roll solutions are an attractor so no matter which initial condition, as long as they flow to the attractor solution, the predictions will depend only on the slow roll parameters.

We need inflation to last long enough so that an initial patch can cover all the observable universe. To quantify this is useful to define $N$ as the number of e-folds. Assuming slow roll we get,

$$N = \int_{a_i}^{a_e} d\ln a = \int d\phi \frac{1}{\sqrt{2\epsilon V}}. \quad (1.16)$$

The CMB was formed at a redshift of $z = 1000$ so we need at least that $a_i H_i/a_0 H_0 < 10^{-5}$ which is 60 e-folds of inflation to explain the observable universe. This and the slow roll conditions parameterise most of the models of single field inflation, and are a useful guide to build different models of inflation. Nevertheless there are many other scenarios in which a shrinking Hubble volume is achieved without requiring slow roll conditions, or where inflation is driven by gauge fields. Moreover, a natural extension to slow roll models is to add other scalar fields, and to study their interactions. This regime, called multifield inflation poses several technical difficulties but provides a richer phenomenology. In this introduction we will describe later a particular way of address multifield inflation

### 1.2.1 Inhomogeneities

So far we have used the large scale homogeneity of the universe to describe its properties and time evolution. However, when looked at scales smaller than $10^5$ Mpc inhomogeneities are present in all directions. These then become cosmic structures formed because of gravitational collapse. For the case of inflation, the scalar field when quantised has fluctuations that freeze once they leave the horizon. After inflation stops, these fluctuations re enter the horizon and seed the later formation of structures. Understanding these physical processes has led to a deep understanding of the fundamental physics
underlying the formation of the universe. Moreover, any future experimental probe, will have to come by understanding the small scale dynamics of the early time universe.

To study all these effects we will consider the perturbed Einstein equations. Perturbations to the Friedmann Robertson Walker metric can be decomposed in terms of scalar, vector and tensor components [90, 91].

\[
\text{d}s^2 = a^2(\tau) \left( -(1 + 2\Phi)\text{d}t^2 + 2(B_i + S_i)\text{d}x^i\text{d}\tau + (1 - 2\Psi)\delta_{ij} - 2E_{ij} - F_{(ij)} - h_{ij} \right)\text{d}x^i\text{d}x^j ,
\]

(1.17)

where \( \Phi, \Psi, B \) and \( E \) are scalars, \( S_i \), and \( F_{i} \), vectors and \( h_{ij} \) tensor degrees of freedom. Invariance under diffeomorphisms of the Einstein equations further restrict the above ansatz. So then we have that we can make any two of the scalar functions vanish by means of the gauge transformations. The gauge chosen depends heavily on the type of calculation needed, and although it is possible to write the observables in a gauge independent form, for what remains we will be specifying the gauge used. In particular, Unitary gauge where \( B \) and \( E \) vanish generalises the Newtonian potential. In fact the scalar line element reduces to,

\[
\text{d}s^2 = a^2(\tau) \left( -(1 + 2\Phi)\text{d}t^2 + (1 - 2\Psi)\text{d}x^2 \right).
\]

(1.18)

Furthermore, in the presence of a diagonal stress energy tensor \( \Psi = -\Phi \), and then only one scalar variable is needed.

### 1.2.2 Inflationary perturbations

Now let us consider the case of the perturbations produced during inflation. We can write the action for single field inflation as

\[
S = \frac{1}{2} \int \text{d}^4x \left[ M_{Pl}^2 R - (\partial\phi)^2 - 2V(\phi) \right] ,
\]

(1.19)

where \( R \) is the Ricci scalar and \( (\partial\phi)^2 \equiv \partial_{\mu}\phi\partial^{\mu}\phi \) is the scalar field kinetic energy. Then we can consider perturbations to the scalar field \( \phi(x) \rightarrow \delta\phi(x) + \phi_0(t) \), and to the metric as before. It is very convenient to do an ADM splitting, as it will become clear which parts are the physical degrees of freedom.

\[
\text{d}s^2 = -N\text{d}t^2 + h_{ij}(\text{d}t^2 + N^i\text{d}x^i)(\text{d}x_j + N^j\text{d}t) ,
\]

(1.20)
where \( N_i \) is the lapse and \( N \) is the shift. The action is
\[
S = \frac{1}{2} \int \sqrt{h} \left[ M_{\text{Pl}}^2 N R^{(3)} - 2 N V + N - 1 (E_{ij} E^{ij} - E^2) + N^{-1} \left( \dot{\phi} - N^i \partial_i \phi \right)^2 - N h_{ij} \partial_i \dot{\phi} \partial_j \phi \right].
\] (1.21)

Here the spacetime is decomposed in terms of a 3-submanifold encompassed by constant \( N \), and evolution from there is given by evolving with \( N^i \). Furthermore, by the addition of constraints one is able to isolate the physical degrees of freedom. In unitary gauge the only scalar degree of freedom is the curvature perturbation \( R \). This is written as,
\[
\delta \phi = 0, \quad h_{ij} = a^2 \left[ (1 - 2R) \delta_{ij} + \gamma_{ij} \right], \quad \partial_i \gamma_{ij} = 0, \quad \gamma_{ii} = 0.
\] (1.22)

Then by solving the constraint we find that the lapse and shift are,
\[
N = 1 + \frac{\dot{R}}{H}, \quad N^i = \partial \left( -a^2 \frac{R}{H} + \partial^2 (\epsilon \dot{R}) \right).
\] (1.23)

Replacing back into the action we have,
\[
S = \frac{M_{\text{Pl}}^2}{2} \int d^4 x a^3 \epsilon \left[ \dot{R}^2 - \frac{k^2}{a^2} \dot{R} \right].
\] (1.24)

So note that the action is proportional to \( \epsilon \), as it is expected. In the case of vanishing \( \epsilon \) the spacetime becomes de Sitter and the curvature perturbation \( R \) a pure gauge mode. To specify initial conditions we note that well inside the horizon \( k \ll aH \) we have that the scale of the fluctuations is much smaller than any effect of gravity, so we can consider that well inside the spacetime can be approximated by Minkowski spacetime. This implies that we should match the initial conditions as those for a Minkowski spacetime solution. Hence when we obtain the classical solution \( R_k^{cl}(t) \) from (1.24), we impose that at early times
\[
\lim_{\eta \to \infty} R_k(\eta) = \frac{1}{\sqrt{2k}} e^{-i k \eta}.
\] (1.25)

At large wavelengths \( k \gg aH \) the modes freeze when they leave the horizon. For modes which are closer to the scale of the horizon \( \tau k \sim H \) the symmetries of the de Sitter spacetime play a key role for them. This implies, by dilatation that it cannot be any time dependence on the observables . By employing the special conformal
it has been shown that the curvature modes are exactly constant outside the horizon.

We can quantise these fields to calculate the fluctuations power spectrum [92]. As the action is quadratic we follow the standard procedure for spin 0 actions. The curvature mode can be rewritten in terms of creation and annihilation operators $a^\dagger$, and $a$, as

$$\mathcal{R}_k(t) = \mathcal{R}_k^0(t)a^\dagger_k + \mathcal{R}_k^{2*}(t)a_{-k}. \quad (1.26)$$

where $a$ and $a^\dagger$ satisfy the commutation relations $[a_k, a^\dagger_q] = \delta^3(k + q)$, $[a_k, a_q] = [a^\dagger_k, a^\dagger_q] = 0$. This implies a normalisation condition for $\mathcal{R}_k^0(t)$. A useful quantity is the power spectrum of a field $v_k$ defined as

$$P(v)(k)\delta^3(k - k') \equiv \frac{k^3}{2\pi^2}\langle v_k(\tau)v_k'(\tau) \rangle \quad (1.27)$$

For the curvature mode this is given by

$$P_R(k) = \frac{1}{8\pi^2}\frac{1}{\epsilon}\frac{H^2}{M_{Pl}^2}. \quad (1.28)$$

Let’s make a comment from the solutions. This depends on the slow roll and the Hubble parameter. Therefore it first implies that the power spectrum is nearly scale invariant. We can see calculating

$$\frac{d\ln k^3P_R(k)}{d\ln k} = 1 - 2\epsilon - \eta. \quad (1.29)$$

Therefore for slow roll, indeed, cannot be scale invariant. Also, non Gaussianities are very small so it is very likely that all the important information is only contained in the power spectrum.

Besides the scalar field fluctuations, de Sitter spacetime contains tensor fluctuations which can also been quantised. As they do not depend on the time translation breaking its power spectrum is only proportional to $H$. Hence, a measurement of them will be a direct probe of the energy scale of inflation. This is parameterised in terms of $r \equiv P_R/P_t = 16\epsilon$. As we will see later in this chapter, the primordial gravitational wave seed the polarized CMB. Although it has not been detected yet, current bounds put $r$ to be below $10^{-2}$. By looking at the two parameters, $r$ and $n_s$ is possible to constrain a large class of different inflationary models, as it is seen in Fig1.1.
Fig. 1.1 Planck constraints on inflation, marginalised at 68% and 95% confidence region. Constraints are calculated assume negligible running of the power spectrum. The predictions of a number of models are as a function of the number of e-folds $N_*$ [4].

1.2.3 Non Gaussianity

It can be seen from the power spectrum that inflation produces a Gaussian distribution. This is due to the approximate conformal symmetry respected by the curvature mode at horizon crossing. In fact, the metric $d s^2 = -d t^2 + e^{2Ht}d\vec{x}^2$ is invariant under the isometry given by $t \to t - H^{-1}\log \lambda$ and $\vec{x} \to \lambda \vec{x}$. This symmetry constrains the shape and amplitude of the correlation functions for the curvature mode. For example $\mathcal{R}$ transforms under dilatations in Fourier space as $\mathcal{R}_k \to \lambda^{-3}\mathcal{R}_{k/\lambda}$ so then the power spectrum has to be,

$$\langle \mathcal{R}_k \mathcal{R}_{-k} \rangle = (2\pi)^3 \delta^{(3)}(k + k') \frac{1}{k^3} F(k e^{-Ht}),$$

(1.30)

and then imposing that $\mathcal{R}$ is time independent at horizon crossing implies that $F$ is a constant [59] which is the same result we found in (1.28) but just using symmetry arguments. Higher order correlators are very interesting and usually are called non
1.3 Multifield inflation

Gaussianities. This, as we have seen for the two point function, are very constrained by symmetry. For the case of single field inflation, the amplitude of the three point function is of order $O(\epsilon^2)$ [55]. Moreover, the limit when one of the modes is much larger than the other two, is restricted to be [55, 93],

$$\langle R_k R_{k'} R_q \rangle_{q \to 0} = (2\pi)^3 \delta^{(3)}(k + k' + q)(n_s - 1)P_R(k)P_R(k'),$$  \hspace{1cm} (1.31)

which follows from the Ward identities associated to the symmetries respected by the curvature mode. By the same arguments it is possible to find an infinite set of consistency relations at higher order [59, 60] which will be respected by single field inflation. Moreover, these consistency conditions imply that the larger mode is a gauge mode, so it is not possible to observe [94]. Therefore, detecting any deviation from (1.31), will imply that there other degrees of freedom, or new physics, acting at the horizon.

1.3 Multifield inflation

Single field inflation is a very good approximation to understand how inflation can solve the horizon problems and give rise to a nearly scale invariant power spectrum. Nevertheless it is unsatisfactory in the sense there is no reason why it should be only one scalar field. Moreover, from string theory, all possible candidates contain many scalar fields with different masses. In principle the dynamics of several scalar fields can be very complicated, as in principle there is no symmetry relating the fields. Even so, many effects have been classified, and it is possible to do a systematic study of some of the multifield effects on inflation.

For instance, one of the key aspects of inflation when there are more than one field is the existence of isocurvature modes. The fact that these have not been observed severely constrain the types of models that are available. For the present discussion we will focus on a particular parametrisation of multifield inflation that separates the direction that produces the curvature perturbations from the others that might result in isocurvature modes [95, 96, 27]. The action of $N$ scalar fields $\phi^a$, with $a = 1, \ldots, N$, under a potential $V(\phi^a)$ is given by,

$$S = \frac{1}{2} \int \left[ M_{Pl}^2 R - \gamma_{ab} \partial^\mu \phi^a \partial_\mu \phi^b - 2V(\phi^a) \right]$$  \hspace{1cm} (1.32)
where $\gamma^{ab}$ is the field metric. The equations of motion for the background are,

$$
\frac{D}{dt} \dot{\phi}^a + 3H \dot{\phi}^a + V^a = 0,
$$

$$
3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi).
$$

(1.33)

where $V_a \equiv \partial V/\partial \phi^a$ and we raise indices using the field metric $\gamma_{ij}$. Also we defined the covariant derivative on field space as

$$
\frac{D}{dt} X^a = \dot{X}^a - \Gamma^a_{bc} X^b \dot{\phi}^c,
$$

(1.34)

with $\Gamma^a_{bc}$ is the corresponding Christoffel symbols associated to the field metric $\gamma_{ab}$,

$$
\Gamma^a_{bc} = \frac{1}{2} \gamma^{ad} (\partial_b \gamma_{dc} + \partial_c \gamma_{bd} - \partial_d \gamma_{bc}).
$$

(1.35)

It is very useful to parameterise the action in terms of its trajectory on field space. To do this we define

$$
T^a \equiv \frac{\phi^a}{\phi_0}, \quad N^a \equiv \left( \gamma_{bc} \frac{D T^b}{dt} \frac{D T^c}{dt} \right)^{-1/2} \frac{D T^a}{dt}.
$$

(1.36)

By the use of this parametrisation we can rewrite the equations of motion in terms of its trajectory. For instance, we can make use of the covariant derivative to write $D/dt = \dot{\phi} T^a \nabla_a \equiv \dot{\phi} \nabla_{\phi}$. Using this we can project (1.33) into the tangent trajectory to obtain,

$$
\ddot{\phi} + 3H \dot{\phi} + V_{\phi} = 0,
$$

(1.37)

and also we have

$$
\frac{D T^a}{dt} = -\frac{V_N}{\dot{\phi}} N^a.
$$

(1.38)

The quantity $V_N$ parameterises deviations from the geodesic trajectory, and it can be interpreted as the coupling between other degrees of freedom and the inflaton. For convenience we define

$$
\Omega \equiv -\frac{V_N}{\dot{\phi}}.
$$

(1.39)
The slow roll parameters are,

\[ \epsilon = -\frac{\dot{H}}{H^2} = -\frac{\dot{\phi}^2}{2H^2}, \]

\[ \eta^a = -\frac{1}{H\phi} \frac{D}{dt} \phi^a = -\frac{\ddot{\phi}}{H\phi} T^a + \frac{\Omega}{H} N^a. \]

Where in the last line we have projected \( \eta^a \) in its tangent and normal components. The first gives the usual \( \eta \) parameter while the second is proportional to the coupling between degrees of freedom. This is usually called \( \eta_\perp \). It has been shown that in order to have inflation one requires that the usual slow roll parameters \( \epsilon \) and \( \eta \) to be smaller than 1, where \( \eta_\perp = \Omega/H \) can be larger provided that its variation rate is smaller than the inflation time scale \( 1/H \).

### 1.3.1 Fluctuations

We can use the same parameterisation to calculate the fluctuations of the scalar fields. We then have that the scalar field can be written as

\[ \delta \phi(x)^a = \delta \phi(x) \parallel T^a(t) + \sigma(x) N^a(t). \]

Plugging back into the ADM ansatz we find that the action is

\[ S = \frac{1}{2} \int d^4x a^3 \left[ 2\epsilon \left( \mathcal{R}^2 - \frac{1}{a^2} (\nabla \mathcal{R})^2 + 2\sqrt{2}\Omega \mathcal{R}\sigma \right) + \sigma^2 - \frac{1}{a^2} (\nabla \sigma)^2 - m^2 \sigma^2 \right], \]

where \( m^2 \equiv N^a N^b (V_{ab} - \Gamma^c_{cb} V_c) + \epsilon H^2 R - \Omega^2 \) with \( R \) the Ricci scalar of the fields manifold. Note here that the coupling between the two degrees of freedom \( \Omega \) becomes explicit.

Although, there is no analytical solution for the above action, different limits are well understood. For instance, the case where \( m^2 \sigma \sim H \) and \( \Omega \ll 1 \) is known as quasi single field inflation, where the interactions between the extra degree of freedom and the inflaton can be understood analytically.

Another interesting example is the case where \( m_\sigma \ll H \) and \( \Omega \sim H \). There, the heavy field is stuck but still interacts with the inflaton. Because of the hierarchy, this can be integrated out resulting in an effective action for single field with a modified speed of sound.

One of the important consequences of heavy fields is that they can potentially lead to distinctive shapes of non-Gaussianity. For example in models where the speed of
sound is modified or when the inflaton is coupled to other scalar field with masses $m \sim H$. All these examples lead to modifications to the consitency condition 1.31 (For example see [97, 27, 25].

1.4 Cosmic microwave background

As we have seen, shortly after inflation the Universe was a hot dense plasma. At that stage all particles were tightly coupled together. When the universe expanded and cooled it down, particles started being released from the plasma. This process, assumes also that the expansion of the universe was adiabatic, hence all the interactions occurred slowly compared to cosmic scales.

When the universe started expanding, particles whose interaction rate $\Gamma$ drops below the Hubble rate $H$, are decoupled from the plasma and its abundance freeze out. For standard model particles neutrinos are the first to decouple at around 0.8 MeV. Below 1 eV electrons and protons started to form neutral hydrogen. This process called recombination occurred at a redshift $z = 1100$. Recombination made the density of electrons drop and then Thomson scattering $e + p \rightarrow H + \gamma$ became inefficient. Photons decoupled from the gas and start free streaming. This gas of photons was emitted in all directions forming a black body spectrum that now has a temperature of around $3K$. For the rest of this chapter conformal time will be denoted by $\eta$.

1.4.1 CMB anisotropies

A great achievement of the last decades has been the understanding of the physics of the CMB [98, 99, 99]. We now review the main ideas. As we saw photons became decoupled and started free streaming. These photons were affected by the gravitational infall, and hence have a small anisotropy which is now observable. To calculate the evolution of these photons we need to make use of the Boltzmann equation for the photon distribution $f$,

$$\frac{df}{d\eta} = C[f, f_e], \quad (1.44)$$

where $d/d\eta$ is the derivative along the path, and $C$ is the collision term that depends on the details of the Thompson scattering between photons and electrons. We denote the distribution of electrons by $f_e$. The zeroth order distribution of photons is that of
1.4 Cosmic microwave background

a blackbody of comoving energy $\varepsilon$ and temperature $T(\eta)$,

\[
f(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon}{aT(\eta)}\right) - 1} = \frac{1}{\exp\left(\frac{\varepsilon}{T_0}\right) - 1},
\]

(1.45)

where $T(\eta) = T_0/a$ and $T_0 = 2.7255K$ is the current CMB temperature. Deviations from this distribution are parametrised by the temperature perturbation $\Theta \equiv \delta T/T$ that depends on time, direction $\vec{x}$ and position of the electrons $p$. Plugging this back into the unperturbed distribution (1.45) we get

\[
f(\varepsilon, \eta, \vec{x}, p) = \frac{1}{\exp\left(\frac{\varepsilon}{aT(\eta)(1+\Theta(\eta, \vec{x}, p))}\right) - 1}.
\]

(1.46)

By using this we can obtain the Boltzmann equation for $\Theta$. The collisionless equation is,

\[
-\frac{d \ln \bar{f}(\varepsilon)}{d \ln \varepsilon} \left( \frac{d \Theta}{d \eta} - \frac{d \ln \varepsilon}{d \eta} \right) = 0,
\]

(1.47)

where $\bar{f}$ is the unperturbed distribution function (1.45). In order to interpret this equation we need to calculate the geodesic equation for photons, which in terms of the Newtonian potentials $\Psi$ and $\Phi$, is

\[
\frac{d \ln \varepsilon}{d \eta} = -\frac{d \Psi}{d \eta} + (\dot{\Psi} + \dot{\Phi}).
\]

(1.49)

Thus we obtain that the temperature anisotropy gets modified along the path, by the evolution of the Newtonian potential along the path and by its time evolution. Adding the collision terms introduces a dependence on the electron energy $e$ and speed $v_e$.

The Boltzmann equation is then given by,

\[
\frac{d \Theta}{d \eta} - \frac{d \ln \varepsilon}{d \eta} = -\Gamma \left[ \Theta - \vec{p} \cdot \vec{v}_e - \frac{3}{16\pi} \int d\hat{m} \Theta(\hat{m})(1 + \hat{e} \cdot \hat{m}) \right],
\]

(1.50)

\[\text{Our notation for the derivatives along the path is,}\]

\[
\frac{d}{d \eta} = \frac{\partial}{\partial \eta} + \hat{\vec{p}} \frac{\partial}{\partial \vec{x}} + \mathcal{O}(2),
\]

(1.48)
where $\Gamma = a \bar{n}_e \sigma_T$, with $\bar{n}_e$ is the number of electrons and $\sigma_T$ is the Thomson cross section. The first two terms of the right hand side produce a shift to the total temperature anisotropy, whereas the last terms is angle dependent. Ignoring higher order terms the Boltzmann equation can be written as,

$$\frac{d\Theta}{d\eta} = \frac{d \ln \epsilon}{d\eta} - \Gamma [\Theta - \Theta_0 - \mathbf{p} \cdot \mathbf{v}_e],$$

(1.52)

where we have defined $\Theta_0 \equiv \frac{1}{4\pi} \int d\Omega \Theta(\hat{\mathbf{p}}, \vec{x}, t)$, the temperature monopole. Employing the geodesic equation (1.49) and Fourier transforming, we get,

$$\dot{\Theta} + i k \mu \Theta = \dot{\Phi} - i k \mu \Psi - \Gamma [\Theta - \Theta_0 - i \mu \mathbf{v}_e],$$

(1.53)

where $\mathbf{v}_e = iv_e \hat{k}$ and we have defined $\mu \equiv \hat{k} \cdot \mathbf{p}$. From this equation we can see that the gradient term induces a gravitational redshift on the photons as they travel through the potential well. This is called the Sachs-Wolfe effect. The time dependence of the metric also produces a time dilatation which is called integrated Sachs-Wolfe effect.

To proceed further is useful to decompose the temperature anisotropy in terms of Legendre polynomials as,

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} d\mu \frac{P_l(\mu)}{2} \Theta(\mu).$$

(1.54)

Furthermore it can be noted that by considering the perturbed stress energy tensor for photons $T^\mu_\nu = \int \frac{d^3p}{E(p)} f P^\mu P_\nu$ that

$$\delta_\gamma \equiv 4 \Theta_0, \quad \mathbf{v}_\gamma = -\Theta_1, \quad \sigma_\gamma = \frac{3}{5} \Theta_2.$$

(1.55)

With this in hand we can write the equations for the evolution of the monopole and the dipole,

$$\delta_\gamma + \frac{4}{3} \left( \nabla \cdot \mathbf{v}_\gamma + 3 \dot{\Phi} \right) = 0,$$

(1.56)

$$\dot{\mathbf{v}}_\gamma + \frac{1}{4} \nabla \delta_\gamma + \nabla \Phi = -\Gamma (\mathbf{v}_\gamma - \mathbf{v}_e).$$

(1.57)

---

$\sigma_T$ is the Thomson cross section given by

$$\sigma_T = \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi \epsilon_0 m_e c^2} \right) = 6.665 \times 10^{-29} \text{m}^2$$

(1.51)

with $q_e$ is the electron charge, $m_e$ the mass of the electron, $\epsilon_0$ the permittivity of free space and $c$ the speed of light.
Now these equations need to be supplemented with the evolution of the electrons. This is usually done by considering that the total stress energy tensor is conserved, furthermore that electrons and baryons are tightly coupled together, and can be considered as a single pressureless fluid. The equations for the baryon density $\delta_b$ and speed $v_b$ are,

\begin{align}
\dot{\delta}_b &= -kv_b + 3\Phi, \quad (1.58) \\
\dot{v}_b &= -\mathcal{H}v_b - k\Psi - \frac{\Gamma}{R}(\Theta_1 + v_b), \quad (1.59)
\end{align}

where $R \equiv \frac{\bar{\rho}_b}{\bar{\rho}}$ is the fractional contribution from baryons. It will also be useful to introduce the *optical depth* between times $\tau$ and $\tau_0$ as,

$$
\tau(\eta) \equiv \int_{\eta}^{\eta_0} \Gamma(\eta')d\eta',
$$

which physically is the opacity of the universe at a given time. The *visibility function* is defined as

$$
g(\eta) \equiv -\dot{\tau}(\eta)e^{-\tau(\eta)}
$$

and corresponds to the probability of a photon to last scatter at the time $\eta$. In order to solve this system it is very useful to make use of the fact that before recombination baryons and photons are coupled through Compton scattering and thus $\dot{\tau}$ is very high. This is called *tight coupling limit*, and implies that we can ignore high $l$ terms, $\Theta_l = 0$ for $l \geq 2$. At first order in $\dot{\tau}$ the system of equations reduces to,

$$
\ddot{\Theta}_0 + H \frac{R}{1 + R} \dot{\Theta} + k^2 c_s^2 \Theta_0 = -\ddot{\Phi} - H \frac{R}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi,
$$

where the photon baryon speed of sound is $c_s^2 = \frac{1}{3} \frac{1}{1 + R}$ and the sound horizon is defined as, $r_s(\eta) = \int_{\eta}^{\eta_0} c_s d\eta'$. This is the equation of a damped forced harmonic oscillator. We can distinguish three main regimes depending on the wavelength of the perturbations.

- At superhorizon scales $k\eta \gg 1$ the gravitational potential $\ddot{\Phi}$ dominates. This is due to the *integrated Sachs-Wolfe effect*.  

---

4In terms of the total density of baryons $R = 0.6 \left( \frac{\Omega_{b,0} h^2}{0.02} \right) \left( \frac{a}{10^9} \right)$. Note that the parameter is small at early times but grows to be of order one at recombination.
• Near the horizon $k\eta \sim 1$ the gravitational infall $k^2\Psi$ becomes important. This driving force produces an adiabatic growth of the photon-baryon perturbation $\Theta_0$

• Inside the sound horizon $kr_s \gtrsim 1$ the photon pressure produces acoustic oscillations on the fluid.

At smaller scales the approximation breaks down and photons get diffused. This effect can be captured by,

$$\dot{\Theta}_0 + \dot{\Psi} = (\Theta_0 + \Psi)e^{-(k/k_D(\eta))^2},$$

(1.63)

where the right hand side is the $\Theta$ at tight coupling and

$$k_D^{-2} = \frac{1}{6} \int_0^\eta \frac{1}{r} \frac{R^2 + 4(1 + R)/5}{(1 + R)^2}$$,

(1.64)

which can be understood as the distance a photon can random walk by $\eta$. The temperature anisotropy $\Theta$ can be decomposed in spherical harmonics

$$\Theta(n) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(n),$$

(1.65)

where the coefficients $a_{lm}$ respect,

$$C_l = \frac{1}{2l + 1} \sum_m |a_{lm}|^2.$$

(1.66)

Then the $C_l$ represent the two point function for the temperature anisotropy $\langle \Theta_{lm} \Theta_{l'm'}^{*} \rangle = C_l \delta_{ll'} \delta_{mm'}$, and then the angular power spectrum is,

$$C_l = \frac{4\pi}{(2l + 1)} \int d\ln k \Theta_l^2(k) P_R(k).$$

(1.67)

Finally we need to solve for the moments of $\Theta_l$. This is usually understood better by integrating over the line of sight, which is the integral along the photon past light cone. Then $\Theta_l$ becomes [100],

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} d\eta \left[ g(\Theta_0 + \Psi) + (g\nu_e)' + e^{-\tau}(\dot{\Phi} + \dot{\Psi}) \right] (2l + 1) j_l(k(\eta_0 - \eta))$$

(1.68)

This rather simple description fits very well with the observations, as the Fig.1.2 shows. As the Bessel function peak at $k(\eta_0 - \eta) \sim 1$ we can classify different effects by their scale. At large scales the Sachs-Wolfe effect is more relevant. To further simplify we
can neglect the effect of the spherical projection and can consider the projection to be a simpler Fourier transform. We can then simplify 1.68, to get an expression for the anisotropies,

\[
\frac{l(l+1)}{2\pi} C_l \approx P_R(k) \left[ \frac{\Theta(\eta, k) + \psi(\eta, k)}{R_k} \right],
\]

which produces a scale invariant power spectrum as can be seen from Fig. 1.2. This simple dependence is very useful for comparing alternative models of gravity. We are provided with a clean probe of the effects of the gravitational fields at early times. At scales near the sound horizon the gravitational infall competes with the photon pressure resulting in acoustic oscillations whose peaks are at \( k_s = n\pi / r_s(\eta_*) \). At smaller scales photon diffusion is more important and the temperature anisotropy gets exponentially suppressed.

Furthermore, the amplitude of the peaks is proportional to \( \Omega_m \), so the matter density of the universe has been measured by this with great precision. Moreover, we have considered that the initial power spectrum is quasi scale invariant, as from inflation. This fact has allowed us to discard many other modes of structure formation,
such as cosmic defects, and put inflation as the main candidate theory for the origin of the universe.

1.4.2 CMB polarization

Thomson scattering produced a quadrupole mode that polarised the CMB [101, 102]. This can be measured in terms of the Stoke parameters which have the properties of a spin 2 particle. By a linear combination of the two Stokes parameters it is possible to write two spin 0 modes, called \( E \) and \( B \), by their analogy with electromagnetism [103–106]. Crucially the parity odd \( B \)-modes depend only on the primordial tensor power spectrum \( P_h \) and thus are a clear proof of \( r \).

Detecting primordial gravitational waves is one of the more important experimental goals for the next decades [107–110]. Should this happen, it will have important theoretical consequences, some of them are examined through this thesis. Crucially we will have access to the energy scales of inflation.

1.5 Other observables

This is, of course, not the end of the story. There are other very important experimental goals. In particular understanding and measuring the large scale structure, will be of crucial importance to learn about new physics using cosmology. Out of the range of things to be measured, non Gaussianity will be of great importance for inflation. There are now several experiments being released or planned to achieve this goal [111, 109, 112–114], and hopefully we will see exciting results within the next years.
Chapter 2

Non-canonical inflation coupled to matter

In this chapter we discuss the effect on inflationary theories of non-relativistic matter when gravity is non-minimal coupled. This effect arises since matter is coupled via the metric, hence a non minimal coupling to gravity is also a coupling between matter and the inflaton. A large class of theories of inflation are naturally written into non-canonical framework, for example, some supergravity models. Moreover, Jordan frame actions arise in string theory or Kaluza Klein theories due to the existence of a dilaton field [115].

In particular we will investigate whether a version of the symmetron mechanism applies to different models of inflation whose origins is in supergravity [116]. To do so we will work with Jordan frame scalar gravity sectors of supergravity actions. This simplification allows us to study how massive scalar fields affect the inflationary potential because they will remain conformally coupled to the Einstein action once they are integrated out. Scalar fields whose masses are much higher than the scale of inflation are thought to be decoupled, unless they induce a non-trivial geometry in the target space. On the contrary, we find that they may affect the dynamics of inflation by inducing large corrections to the potential. This result is based on the symmetron mechanism as described before. We will begin by considering the Starobinsky model, then move on to \( \alpha \) attractors and string theory motivated models.

This chapter will be structured as follows. In section two we analyse non-canonical models of inflation and introduce some of the main tools needed to proceed. We also give details of the Starobinsky model and possible embeddings in supergravity. We then discuss superconformal \( \alpha \)-attractors, which generalise the Starobinsky model. In section three we analyse how matter is coupled to inflation and its affect on the inflaton
potential via the symmetron mechanism. We explain under which circumstances inflation is spoiled by the presence of non-relativistic matter and we described how matter changes the attractor behaviour of single field inflation. By generalising to the universal attractor models, we described how variations in the coupling may produce different implications. We analyse possible issues that may affect our conclusions. Finally we conclude by giving some possible directions for further work.

2.1 Non-canonical inflation

In this section we first review the standard formalism used when studying non-canonical inflationary theories. We will then focus on explaining in some detail the Starobinsky model of inflation, and give some examples for supergravity theories in which the Starobinsky model can be embedded.

By making some assumptions we will simplify one particular superconformal supergravity model which can give rise to an $SO(1, 1)$ invariant scalar tensor theory. We will show how this action is equivalent to the Starobinsky model. Finally, we extend this framework to include other more general types of models of inflation, including chaotic inflation.

2.1.1 Basic setup

In this chapter we will work with non-canonical models of inflation with action given by,

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \Omega^2(\phi) R - \frac{K(\phi)}{2} \partial_\mu \phi \partial^\mu \phi - V_J(\phi) \right\}, \quad (2.1)$$

where $\Omega(\phi)$ is the non-minimal coupling with the Ricci scalar and $K(\phi)$ is the non-canonical kinetic term. This action is very general, for example, with coupling function of the form $\Omega = 1 + \xi \phi^2$ inflation can be realised using the Higgs boson [117]. Furthermore, requiring the action to be conformally invariant puts constraints on the coupling functions. For Weyl invariance, i.e. scale invariance, one has that $\Omega^2(\phi) = -6K(\phi)$.

The coupling function $\Omega(\phi)$ can be made constant by performing a conformal transformation,

$$g_{\mu\nu} \to \frac{1}{\Omega(\phi)^2} \tilde{g}_{\mu\nu}. \quad (2.2)$$
When $\Omega(\phi) = M^2_{\text{Pl}}$, the action is in the Einstein frame, whereas when $\Omega(\phi)$ is not a constant, it is in Jordan frame. In general, this will introduce new terms in the kinetic energy because of the transformation law for the Ricci scalar\(^1\). The action in Einstein frame is,

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M^2_{\text{Pl}}}{2} R - \frac{1}{2} \left\{ \frac{K(\phi)}{\Omega^2} + 6(\log \Omega(\phi))^2 \right\} \partial_\mu \phi \partial^\mu \phi - V_E(\phi) \right\},
\]

where all the fields are transformed and $V_E = V_J/\Omega^4$. The term $(\log \Omega(\phi))^2$ is shorthand notation for $(\log(\Omega(\phi)))^2 \equiv \frac{1}{\Omega} \frac{\partial \Omega}{\partial \phi}$. Although it is preferable to change to the Einstein frame when working with inflation because observables are simpler to work with, sometimes working in the Jordan frame is useful because it makes the physics clearer. Both frames should make the same predictions because they just correspond to different coordinate systems, a fact that we exploit during the course of this chapter. Finally, note that the kinetic coupling can be made canonical by a field redefinition, provided that $K$ is well defined and non-zero in the domain of interest.

### 2.1.2 Starobinsky model

One of the first models of inflation was based on a modification of the Einstein Hilbert action proposed by Starobinsky [118]. In this framework the new action contains a higher derivative term which is minimally coupled to the metric. Its action is given by

\[
S_S = \frac{M^2_{\text{Pl}}}{2} \int d^4x \sqrt{-g} \left( R + \frac{1}{6M^2_{\text{Pl}}M^2} R^2 \right),
\]

where there is a new coupling constant $M^2$ which may be constrained, for example, using inflation in which case $M^2 \ll M^2_{\text{Pl}}$. To see more clearly how this model can produce an inflationary epoch let us rewrite it as a scalar tensor theory introducing an auxiliary field $\psi$. The action is,

\[
S_S = \int d^4x \sqrt{-g} \left( \frac{M^2_{\text{Pl}}}{2} R + \frac{1}{M} R\psi - 3\psi^2 \right).
\]

Since $\psi$ is just a Lagrange multiplier, the action can be transformed into the original form (2.4) by solving the constraint equation for the field $\psi$.

\(^1\)The Ricci scalar transforms under a conformal transformation $\tilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$ as, $\tilde{R} = \Omega^{-2}(R - 6\partial^\mu \partial_\mu \ln \Omega - 6\partial^\mu \partial_\nu \ln(\partial_\nu \Omega))$. 
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To count the degrees of freedom of the theory let us rewrite the action (2.5) in Einstein frame by means of the transformation,

\[
g_{\mu\nu} \rightarrow e^{-\sqrt{2}/3\phi/M_{Pl}}g_{\mu\nu} = \left(1 + \frac{2\psi}{M_{Pl}}\right)^{-1}g_{\mu\nu}, \tag{2.6}\]

which leads to the more familiar scalar field action,

\[
S = \int d^4x\sqrt{-g^E}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{3}{4}M^2M_{Pl}^4\left(1 - e^{-\sqrt{2}/3\phi/M_{Pl}}\right)^2\right]. \tag{2.7}\]

Therefore, we see that the effect of adding an \( R^2 \) term in (2.4), is to introduce a scalar degree of freedom in addition to the two graviton helicities already present in the Hilbert-Einstein action. Now we have the basic ingredients for single field inflation.

The potential in (2.7) grows exponentially for negative \( \phi \) whereas it has a plateau for positive \( \phi \). Hence it is possible to have inflation if the field starts rolling down from right of the potential. Indeed since inflation will terminate when \( \epsilon = 1 \), we have that there is a critical value \( \phi_c \) for which a field starting to roll down at \( \phi_0 > \phi_c \), can produce enough efolds \( N \) of inflation, as required by CMB observations. This issue will be considered later in this work.

Moreover, one can predict the value for the tilt and the tensor to scalar ratio solely in terms of \( N \), which are,

\[
n_s - 1 \approx -\frac{2}{N}, \quad r \approx \frac{12}{N^2}. \tag{2.8}\]

Thus we have that for \( N \approx 60 \), \( 1 - n_s \approx 0.03 \) and \( r \approx 0.004 \) which are compatible with the Planck data. The recent interest in the Starobinsky model is because it seems to be favoured by recent cosmological experiments. This has motivated further theoretical research. For example, it has been pointed out that Higgs inflation [117] and the Starobinsky model are equivalent for a particular limit of the action (2.3)[119]. Moreover, it was suggested that they form part of a wider family of models, all of which have similar predictions for inflation, called universal attractors [120–122].

2.1.3 Inflation from superconformal gravity

The canonical superconformal supergravity (CSS) approach developed in [122, 123], is particularly useful for model building because it can provide a supergravity context to Jordan frame actions. We will now review part of the framework relevant to our work. Superconformal symmetry has for a long time been used as a method
to build supergravity multiplets. Here the action is invariant under superconformal transformations. These transformations includes, conformal transformations and $U(1)_R$ transformations. Therefore one has to fix the gauge to recover the standard supergravity formulation. Due to a higher degree of symmetry the superconformal action is much simpler than Poincare supergravity and thus more suitable for phenomenological applications. The action composed of $X^I$ bosonic fields, $\Omega^I$ fermionic fields $\lambda$, and $F^I$ auxiliary fields is given by
\[
\mathcal{L} = \mathcal{N}(X, \bar{X})|_D + \mathcal{W}(X)|_F + f_{\alpha\beta} \bar{\lambda}^\alpha L_{L}^\beta|_F.
\] (2.9)

where function $\mathcal{N}(X, \bar{X})$ is related to the Kahler potential and $\mathcal{W}(X)|_F$ to the super potential. The last part is related to the kinetic term for the vector fields and will not be important for us. The subindex F and D denotes the last term of the multiplet. Note that the action is written in direct analogy with the supersymmetric Wess Zumino model. The conformal weights of each function are in order 2, 3, 0. One of the multiplets $X^0, \Omega^0, F^0$, can be viewed as a compensator multiplet, i.e. fields who are fixed to break the conformal symmetry. The scalar fields form a Kahler manifold whose metric is given by
\[
G_{IJ} = \frac{\partial \mathcal{N}(X, \bar{X})}{\partial X^I \partial \bar{X}^J}.
\] (2.10)

We will now describe the the $SU(2, 2|1)$ superconformal model developed in [123, 124].

**Canonical superconformal supergravity**

Usually superconformal symmetry is fixed at very early stages because it is used to formulate Poincare supergravity. However, in this formalism the action has a very simple bosonic sector, it can be written in the Jordan frame; thus it is suitable when studying non-canonical models of inflation. In any case, the action (2.9) is still very complicated and to simplify let us make two assumptions. First, we make a choice of Kahler manifold invariant under $SU(1, N)$,
\[
\mathcal{N}(X, \bar{X}) = -|X^0|^2 + |X^\alpha|^2 \quad \alpha = 1, \ldots, N.
\] (2.11)

This gives a flat Kahler manifold metric, given by
\[
G_{IJ} = \partial_i \partial_j \mathcal{N} = \eta_{IJ}.
\] (2.12)
Also we choose a general cubic superpotential independent of $X^0$,

$$
\mathcal{W}(X) = \frac{1}{3} d_{\alpha\beta\gamma} X^\alpha X^\beta X^\gamma,
$$

$$
\mathcal{W}_0 = 0,
$$

(2.13)

where $d_{\alpha\beta\gamma}$ is a numerical matrix. The scalar gravity part of the action becomes,

$$
\frac{1}{\sqrt{-g}} \mathcal{L}_{SG} = -\frac{1}{6} \mathcal{N}(X, \bar{X}) R - G^{IJ} D_\mu X^I D^\mu \bar{X}^J - G^{IJ} \mathcal{W}_I \mathcal{W}_J, \quad I, \bar{I} = 0, 1, \ldots, (2.14)
$$

We still need to fix the gauge by a change of variables from the basis $\{X^I\}$ to a basis $\{y, z^\alpha\}$, using $X^I = y Z^I(z)$. The dilaton and $U(1)$ symmetry are fixed by choosing,

$$
\mathcal{N}(X, \bar{X}) = -|X^0|^2 + |X^\alpha|^2 = \Phi(z, \bar{z}),
$$

and

$$
X^0 = \tilde{X}^0 = \sqrt{3} M_{Pl}, \quad y = \bar{y} = 1, \quad X^\alpha = z^\alpha.
$$

(2.15)

This choice results in the compensator fields decoupling from the "physical" fields, thus the functions become

$$
\hat{\Phi}(z, \bar{z}) = -3 M_{Pl}^2 + \delta_{\alpha\beta} z^\alpha z^\beta, \quad \mathcal{W}(X) = W(z) = \frac{1}{3} d_{\alpha\beta\gamma} z^\alpha z^\beta z^\gamma,
$$

(2.16)

To recover the usual Poincaré supergravity one has to fix the compensator fields. For example by choosing $X^0 = \tilde{X}^0 = \sqrt{3}$ we get,

$$
\mathcal{N}(X, \bar{X})|_{X^0=\tilde{X}^0=\sqrt{3}} = -3 e^{\frac{1}{2}\kappa(X^I, \bar{X}^\bar{I})},
$$

$$
\mathcal{W}(X^I)|_{X^0=\sqrt{3}} = W(X^I).
$$

(2.17)

(2.18)

This formalism was used to build a supergravity generalisation of Higgs inflation [124], but has been applied to several other models [125–128]. What makes it particularly interesting is the fact that it is simple to set up viable models of inflation in which the fields other the inflaton are rendered stable.

**Starobinsky model**

The Starobinsky model can be embedded in supergravity. As explained before, since the scalar tensor action (2.7) is equivalent to (2.4), it is not necessary to specify the potential, but it is sufficient to find a supermultiplet containing the term $R^2$. This attempt was done by Ceccoti *et al.* during the eighties [129, 130], where it was shown that the supermultiplet could be realised in either old or new supergravity. Whereas the original
models were unstable more accurate versions have became available. Furthermore, one can also try to build an action by specifying the potential. This possibility was developed by Kallosh, Linde, \textit{et al.} using CSS described earlier \cite{131}. In this framework one starts with a superconformal theory which recovers supergravity once the conformal invariance is broken. The theory is composed of three chiral superfields,

\begin{equation}
X^I = (X^0, X^1 = \Phi, X^2 = S),
\end{equation}

where \(X^0\) is a conformon field, \(X^1\) is the inflaton \(\Phi\) and \(X^2 = S\) is a Goldstino superfield. The role of \(S\) is to allow a stable inflationary trajectory. Taking the functions

\begin{equation}
\mathcal{N}(X, \bar{X}) = -|X^0|^2 \exp \left( -\frac{|S|^2}{|X^0|^2} + \frac{1}{2} \left( \frac{\Phi}{X^0} - \frac{\bar{\Phi}}{\bar{X}^0} \right)^2 + \zeta \frac{|S|^4}{|X^0|^4} \right),
\end{equation}

\begin{equation}
\mathcal{W}(X^0, \Phi, S) = \frac{M}{2\sqrt{3}} S(X^0)^2 \left( 1 - e^{-\frac{2\Phi}{\sqrt{3}}} \right)^2.
\end{equation}

The field \(S\) has to be introduced in order to spontaneously break the supersymmetry, which is achieved when the auxiliary field is vanishing, \(F = \mathcal{W}_S = 0\). Following A.19, the corresponding Kahler manifold is given by,

\begin{equation}
\mathcal{K}(\Phi, \bar{\Phi}, S, \bar{S}) = S \bar{S} - \frac{(\Phi - \bar{\Phi})^2}{2} - \zeta (S \bar{S})^2.
\end{equation}

Note that there is a shift symmetry for the field \(\Phi\), and thus large corrections to the mass are forbidden. To get an inflationary action after fixing the conformal symmetry one assume that there is an inflationary trajectory along the space defined by \(S = \text{Im}\Phi = 0\). This direction becomes stable by the term proportional to \(\zeta\). Then one recovers a scalar gravity action for \(\varphi = \text{Re}\Phi\), which is equivalent to the Starobinsky model with potential \(V \sim \left( 1 - e^{-\sqrt{2/3} \varphi/M_{Pl}} \right)^2\). Notice that this embedding is not unique, other possibilities has been explored but using different aproaches to supergravity \cite{132–134}. Nevertheless the one we described has the advantage of producing an action which starts in Jordan frame.

### 2.1.4 Conformal inflation

Using the previous choice of gauge one may just simply assume that the low energy inflationary limit exists and study its phenomenological consequences. In order to do so we fix the field \(S = 0\) but also work with the real part of the fields \(X^0, \Phi\). Following \cite{131} we start by studying the phenomenological Lagrangian given by the action,
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\[ S = \int d^4x \sqrt{-g} \left\{ \left( \frac{\chi^2}{12} - \frac{\phi^2}{12} \right) R + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} \phi^2 (\phi - \chi)^2 \right\}, \quad (2.23) \]

which was shown to be equivalent to the Starobinsky model. This model spontaneously breaks the conformal symmetry and thus there is an extra scalar degree of freedom compared to the conformally invariant De Sitter case,

\[ S = \int d^4x \sqrt{-g} \left\{ \left( \frac{\chi^2}{12} - \frac{\phi^2}{12} \right) R + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - \chi^2)^2 \right\}, \quad (2.24) \]

which is locally conformally invariant under,

\[ \phi \rightarrow e^{\sigma(x)} \phi, \quad \chi \rightarrow e^{\sigma(x)} \chi \quad g_{\mu\nu} \rightarrow e^{-2\sigma(x)} g_{\mu\nu}. \quad (2.25) \]

It is a generalisation of the conformally invariant action,

\[ S = \int d^4x \sqrt{-g} \left\{ -\frac{\phi^2}{12} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right\}, \quad (2.26) \]

which is equivalent to Einstein-Hilbert gravity just by a conformal transformation. In the above action the kinetic term has the wrong sign but the theory is not ill defined because the \( \phi \) field is unphysical. Indeed the above theory just propagates two degrees of freedom corresponding to the two graviton polarizations. This redundancy might be removed by fixing the gauge freedom choosing, \( \phi = \sqrt{6} M_{\text{Pl}}^2 \), which leads to the Einstein-Hilbert action. Moreover, this construction proved to be fundamental when building supergravity actions. As we mentioned before, by requiring invariance under superconformal transformations, it is possible to build a Poincare supergravity upon gauge fixing.

Now coming back to (2.23), this action is well defined for all field space except for the point where \( \phi = \chi \). By doing the transformation \( \phi = r \sinh(\varphi/\sqrt{6}), \quad \phi = r \cosh(\varphi/\sqrt{6}) \) the action becomes the Starobinski model,

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{3}{4} \lambda M_{\text{Pl}}^4 \left( 1 - e^{-\sqrt{2/3} \frac{\varphi}{M_{\text{Pl}}}} \right)^2 \right\}, \quad (2.27) \]

after transforming to Einstein frame. Note that the action does not depend on \( r \), the kinetic terms for \( r \) cancels with the term coming from the transformation of \( R \), whereas the potential does not depend on \( r \). The above condition is implicit in the fact that
going to Einstein frame, \( r \neq 0 \). Also, as the action does not depend on \( r \), there are three degrees of freedom in contrast to (2.26).

Recently, it has been pointed out that this conformal symmetry is a fake symmetry because its current is zero, and thus has no net effect on the physics [135, 136] One then should be careful when considering properties derived from this symmetry.

As we are mainly interested in the gauged case, for us this fact will be unimportant. For our analysis it is more appropriate to proceed by choosing the gauge,

\[
\chi = \sqrt{6} M_{\text{Pl}}^2,
\]

This gauge is related to the rapidity of the fields. One can consider several other gauge choices, for example \( \chi^2 - \phi^2 = 6 \), that lead to equivalent Einstein frame actions for the scalar field. Nevertheless, one has to be very careful when generalising this theory, because some issues will appear later when coupling matter to the action (2.23). The action, in our gauge choice \( \chi = \sqrt{6} M_{\text{Pl}}^2 \), becomes

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M_{\text{Pl}}^2}{2} \left( 1 - \frac{\phi^2}{6 M_{\text{Pl}}^2} \right) R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{3}{2} \lambda M_{\text{Pl}}^2 \phi^2 \left( \frac{\phi}{\sqrt{6} M_{\text{Pl}}} - 1 \right)^2 \right\} (2.29)
\]

which is a scalar tensor theory where the kinetic term has the correct sign. To analyse the inflationary limit we can transform this action into Einstein frame by rescaling the metric, using,

\[
g_{\mu\nu}^J = \frac{g_{\mu\nu}^E}{1 - \frac{\phi^2}{6 M_{\text{Pl}}^2}}. \tag{2.30}
\]

Notice that the transformation is undefined for \( \frac{\phi^2}{6 M_{\text{Pl}}^2} = 1 \), which is equivalent to our earlier observation that the action is defined for all field space except for \( \chi = \phi \). Doing so, we get the following action,

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} \left( \Omega(\phi)^{-1} + \frac{3}{2} \left[ \log \Omega(\phi) \right]^2 \right)^{\prime} \partial_{\mu} \phi \partial^{\mu} \phi - \lambda M_{\text{Pl}}^2 \phi^2 \left( \frac{\phi}{\sqrt{6} M_{\text{Pl}}} - 1 \right)^2 \right], \tag{2.31}
\]

where we defined \( \Omega(\phi) \equiv 1 - \frac{\phi^2}{6 M_{\text{Pl}}^2} \). The logarithms come from the transformation law for the Riemann tensor. Note that the action will have, in general, a non-canonical
kinetic term. Expanding the kinetic term in (2.31),

\[
\left( \Omega(\phi)^{-1} + \frac{3}{2} \log \Omega(\phi)^2 \right) \partial_\mu \phi \partial^\mu \phi = \frac{1}{(1 - \phi^2/6M_{Pl}^2)^2} \partial_\mu \phi \partial^\mu \phi. 
\] (2.32)

The field can be canonically normalised by performing a field redefinition,

\[
\frac{d\phi}{d\varphi} = \frac{1}{1 - \phi^2/6M_{Pl}^2} 
\] (2.33)

This relation can be easily integrated, which leads to

\[
\phi = \sqrt{6}M_{Pl} \tanh(\frac{\varphi}{\sqrt{6}M_{Pl}}), 
\] (2.34)

where the field $\varphi$ goes from minus infinity to infinity. Now, we will have an action for a scalar field which is in the Einstein frame and also has a canonical kinetic term. Replacing the new field into the action, we get

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{M_{Pl}^2}{2} R - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{3}{4} \lambda M_{Pl}^4 \left( 1 - e^{-\sqrt{\frac{2}{3}} \frac{\varphi}{M_{Pl}}} \right)^2 \right\}, 
\] (2.35)

which is the same action for the Starobinsky model described earlier (2.7), with $\lambda = M^2$.

### 2.1.5 $\alpha$ models

The formalism we described before can be used in order to obtain more diverse scenarios which are different from the Starobinsky model. For example, in the $\alpha$ model, proposed by Kallosh and Linde [137], the Kahler potential is a deformation of the simpler flat $SO(1,1)$ model we considered earlier. Its Kahler potential is given by

\[
\mathcal{N}(X_0, X_1, S) = -|X_0|^2 \left[ 1 - \frac{|X_1|^2 + |S|^2}{|X_0|^2} \right]^\alpha, 
\] (2.36)

where we can see $\mathcal{N}(X, \bar{X})$ is no longer flat nor invariant under $SO(1,2)$, with $\alpha$ parametrising deviations from it. Indeed, for $\alpha = 1$ we recover the original theory. Due to the new form of $\mathcal{N}(X_0, X_1, S)$, we also have a non-canonical kinetic term,

\[
G_{IJ} = \partial_I \partial_J \mathcal{N} \equiv \frac{\partial \mathcal{N}(X \bar{X})}{\partial X^I \partial \bar{X}^J}. 
\] (2.37)
The superpotential is now given by
\[
W = S(X^0)^2 f(X^1/X^0) \left[ 1 - \frac{(X^1)^2}{(X^0)^2} \right]^{(3\alpha-1)/2}.
\] (2.38)

where to admit more generality, the potential will depend on the arbitrary scalar function \( f(X^1/X^0) \). For our purpose, it will be enough to assume that the theory has an inflationary limit where the moduli fields are stabilised, therefore the action can be written in a form similar to (2.23). We can use as a guide the same framework we studied at the beginning of this section, where \( \alpha \) will parametrise deviations from the action (2.23). Nevertheless, the action is no longer invariant under \( SO(1,1) \), and we need to set \( X_0 = \sqrt{3}M^2_{\text{Pl}} \) to get inflation. The potential term is then given by
\[
V = G^{SS} \left| \frac{\partial W}{\partial S} \right|^2.
\] (2.39)

To describe inflation, the bosonic action in Jordan frame is,
\[
S(\phi) = \int d^4x \sqrt{-g} \left\{ \frac{M^2_{\text{Pl}}}{2} \left( 1 - \frac{\phi^2}{6M^2_{\text{Pl}}} \right)^\alpha R - \frac{1}{2} \frac{\alpha - \alpha^2\phi^2/6M^2_{\text{Pl}}}{(1 - \phi^2/6M^2_{\text{Pl}})^2 - \alpha} (\partial \phi)^2 - V(\phi) \right\},
\] (2.40)

where the non-canonical kinetic term is due to the shape of the Kahler potential (2.36). In Einstein frame (2.40) reduces to
\[
S = \int d^4x \sqrt{-g} \left\{ M^2_{\text{Pl}} R - \frac{\alpha/2}{(1 - \phi^2/6M^2_{\text{Pl}})^2} (\partial \mu \phi)^2 - f^2(\frac{\phi}{\sqrt{6M^2_{\text{Pl}}}}) \right\},
\] (2.41)

We can canonically normalise the kinetic energy with the transformation
\[
\phi = \tanh\left( \frac{\varphi}{\sqrt{6\alpha M^2_{\text{Pl}}}} \right),
\] (2.42)

to get the action
\[
S = \int d^4x \sqrt{-g} \left\{ M^2_{\text{Pl}} R - \frac{1}{2} (\partial \mu \varphi)^2 - f^2(\tanh(\frac{\varphi}{\sqrt{6\alpha M^2_{\text{Pl}}}})) \right\}.
\] (2.43)

note that the potential depends on the function \( \tanh(\varphi/\sqrt{6\alpha}) \). It has been pointed out in [122]
\[
\Omega = \left( \sinh(\frac{\varphi}{\sqrt{6\alpha M^2_{\text{Pl}}}}) \right)^{2\alpha}.
\] (2.44)
Choosing \( f(\phi/\sqrt{g}) = \frac{\lambda \phi^2}{(\phi/\sqrt{6+1})^2} \), we can recover Starobinsky like potentials of the form,

\[
V(\varphi) \sim \left( 1 - e^{-\sqrt{\frac{\lambda}{3 M_{Pl}^2}} \varphi} \right)^2,
\]

which is the same as that for the Starobinsky model when \( \alpha = 1 \). Nonetheless, note that for larger \( \alpha \) we do not get exponential terms in the potential but something more similar to chaotic inflation. One can calculate the inflationary parameters in terms of \( \alpha \), which turn out to be,

\[
n_s = 1 - \frac{2}{N}, \quad r \approx \frac{12\alpha}{N^2},
\]

We thus see that for large \( \alpha \), \( r \) may be of the same magnitude as that predicted by other models such as chaotic inflation. This particular feature is very important since \( r \), once detected, may be sufficiently large to rule out the Starobinsky model, but not all the \( \alpha \) models. Finally, supersymmetry breaking places further theoretical constraints on \( \alpha \). For example, to avoid tachyonic instabilities that may arise from \( S \) during an inflationary trajectory, \( \alpha > 1/3 \). Furthermore it was found that \( \alpha \) is related to the curvature of the Kahler manifold as,

\[
\mathcal{R}_k = -\frac{2}{\alpha},
\]

Therefore we can identify the different models of inflation as related to a different geometry in field space.

### 2.2 Matter coupled to inflation

We will now examine what happens when matter is coupled to non-canonical inflation. We will also show how the symmetron mechanism can be used to further constrain the possible scenarios for the Starobinsky model. In the low energy approximation several extra fields have been integrated out and no longer participate in the dynamics. Hence, we can write the action as,

\[
S = S_{\text{infl}} + S_{\text{matter}}
\]
where

\[ S_{\text{matter}} = \int d^4x \sqrt{-g^J} \mathcal{L}_M(g_{\mu\nu}^J, \psi^I), \quad (2.49) \]

is the action for the non-relativistic integrated out matter and \( S_{\text{infl}} \) is the action (2.1). Kaluza-Klein towers would contribute to \( S_M \) in a similar way. Now, assuming that a UV complete theory like supergravity gives an inflationary limit in Jordan frame, we can re-express the action in the Einstein frame. To do so, we make a conformal transformation in the non-relativistic matter action as well. It follows that under (2.25), the action transforms as,

\[ \int d^4x \sqrt{-g^J} \mathcal{L}_M(g_{\mu\nu}^J, \psi^I) \to \int d^4x \sqrt{-g^E} \mathcal{L}_M(\Omega^{-2}g_{\mu\nu}^E, \psi^I). \quad (2.50) \]

Varying the action (2.48), with respect to inflaton \( \phi \) we obtain the following equations of motion,

\[ \Box \phi - V_{,\phi} + \Omega(\phi)^{-3} \Omega_{,\phi} T^J = 0, \quad (2.51) \]

where \( T^J = g^J_{\mu\nu} T_{\mu\nu}^J \) is the trace of the energy momentum tensor written in Jordan frame, \( T_{\mu\nu}^J = (-2/\sqrt{-g^J}) \partial \mathcal{L}_m / \partial g_{\mu\nu}^J \). This stress energy tensor is covariantly conserved. Recalling that for non-relativistic matter \( T^J = -\rho^J \), we can rewrite this equation for Einstein frame as \( \rho^E = \Omega^{-3} \rho^J \). This quantity is also conserved in Einstein frame[16]. Hence, the equation (2.51) becomes,

\[ \Box \phi - V_{,\phi} - \frac{\Omega \phi}{\Omega^2} \rho = 0. \quad (2.52) \]

We can think of this equation as a scalar field governed by an effective potential \( V_{\text{eff}}(\phi) = V(\phi) - (\Omega(\phi))^{-1} \rho \), which is dependent on the density of matter. Note that \( \rho \) will decay as \( a^{-3} \) as the universe expands, as it occurs in an inflationary background. Thus it is consistent to assume the universe contained non-relativistic matter before inflation since all such matter would disappear as the universe starts inflating.

However, since there is a coupling dependence the potential could still be modified before the matter was diluted away. This mechanism was first studied by Hinterbichler and Khoury [138] in the context of modified gravity. They found that non-relativistic matter could induce a breaking in the symmetry of the system which is restored once the matter fades away.
This was applied to inflation [116, 139], where it was shown that for a density \( \rho \) larger than the energy density of the inflaton, the system could be governed by a broken symmetry potential. This in turn, could lead to a natural mechanism to set the initial conditions for inflation, because when matter dominates the inflation decays to the bottom of the effective potential but once matter flushes away, the original inflationary potential is restored and the field naturally starts to roll down. Whereas in [139] the conformal coupling was introduce by hand, in [116] the studied model was conformally invariant under \( SO(N) \) from the beginning. This case was inspired by Higgs-Dilaton inflation [140] where it also was applied. In these models the coupling is given by the symmetry and the mechanism works more naturally. We will now apply this mechanism to the Starobinsky model in the context of a conformal field theory.

For our case, let us start from the bosonic sector of the action described in (2.1.3). For simplicity let us assume that there is an inflationary limit which has as an action (2.23). Once we have set the gauge by fixing \( \chi = \sqrt{6}M_{Pl} \) in eq. 2.28 the coupling is \( \Omega^2(\phi) = 1 - \phi^2/6M_{Pl}^2 \). After canonically normalising the fields, the effective potential becomes,

\[
V_{\text{eff}} = \frac{9}{4} \lambda M_{Pl}^4 \left(1 - e^{-\sqrt{2/3} \frac{\phi}{M_{Pl}}}\right)^2 + \rho \sinh^2 \left(\frac{\phi}{\sqrt{6}M_{Pl}}\right).
\]  

(2.53)

We see that the new term behaves as \( \sim \rho e^{\sqrt{2/3} \frac{\phi}{M_{Pl}}} \) for positive \( \phi \), and thus it will overturn the inflationary potential for \( \phi/M_{Pl} > \sqrt{3}/2 \). Whether or not this affects inflation will also be dependent on the initial \( \rho \), but we can in any case assume that \( \rho_0 \sim \lambda M_{Pl}^4 \) and continue our investigation. To analyse the model, we rescale the time by \( t \rightarrow t/\sqrt{\lambda} \). Then (2.53) becomes,

\[
\varphi'' + 3H \varphi' + 3 \sqrt{2} \rho e^{-\sqrt{2/3} \frac{\varphi}{M_{Pl}}} \left(1 - e^{-\sqrt{2/3} \frac{\varphi}{M_{Pl}}}\right) + \frac{\rho}{\sqrt{6}} \sinh \left(\frac{\sqrt{2/3} \varphi}{M_{Pl}}\right) = 0,
\]  

(2.54)

with \( \tilde{\rho} \equiv \frac{\rho}{\sqrt{\lambda}} \). From now on we will call \( \tilde{\rho} \) just \( \rho \). In the Starobinsky model, inflation occurs while the inflaton is rolling down the plateau of the potential until it reaches the point where \( \epsilon \) becomes larger than 1. Once it enters this region the inflaton starts decaying until it reaches the minimum of the potential where reheating starts. We can calculate the amount of e-folds that it would take to cross this threshold by using the slow roll approximation,

\[
N \approx \int_{\varphi_1}^{\varphi_E} \frac{d\varphi}{\sqrt{2\epsilon V}}.
\]  

(2.55)
To simplify this integral we can use the equations of motion to rewrite $\sqrt{2\epsilon V}$ and then expand it in terms of $\rho$. Since we want to study how the matter is affected by the coupling we will assume that $\rho$ is constant. We then get that,

$$
\frac{1}{\sqrt{2\epsilon V}} \approx \frac{1}{\sqrt{2\epsilon V}_{\rho=0}} - \frac{1}{\epsilon V}_{\rho=0} \frac{e^{\sqrt{\frac{7}{6} \pi^2}}} {9\sqrt{6}} \rho, \tag{2.56}
$$

which is valid for small $\rho$ as long as the relation remains positive. Now we can see that for small values of $\rho$ the integrand will decrease when matter is added. This in turn will mean that the number of e-folds for which the inflaton remains in the inflationary sector of the potential severely decreases.

Furthermore, we can integrate eq. (2.55) numerically, as shown in Fig.2.1. Indeed, we see that the field decays in a small number of e-folds and thus inflation is never realised due to the steepness of the potential induced by the coupling term.

We can be more systematic and generalise this result. To do so we note that the system has attractor solutions for fields starting to roll down from the left of the potential due to the Hubble friction, irrespective of the value of $\rho$. 

Fig. 2.1 Number of e-folds before decaying into the attractor solution. Note that the scalar field decays rapidly even if $\rho < 1$. 

![Graph showing the number of e-folds before decaying into the attractor solution](image-url)
In the case of the Starobinsky model, fields falling into the attractor will keep rolling down producing an inflationary expansion until the slow roll condition no longer holds, when inflation terminates. Thus reviewing this attractor behaviour we can see what happens to the system when matter is included. Rewriting the Klein Gordon equation for the inflaton as follows,

\[
\frac{d\dot{\phi}}{d\phi} = -\frac{3H\dot{\phi} + V,\phi}{\phi} \tag{2.57}
\]

\[
= -\frac{3\sqrt{\frac{\phi^2}{3} + V}}{\sqrt{V}} + V,\phi, \tag{2.58}
\]

we can analyse the phase space for the equations of motion. To find the attractor solutions we assume that there is a trajectory for which \( \frac{d\dot{\phi}}{d\phi} \approx 0 \). The solution for this equation is,

\[
\dot{\phi}^2 = V \left( -1 + \sqrt{1 + \frac{(V,\phi)^2}{3V^2}} \right), \tag{2.59}
\]

which gives the behaviour of the kinetic energy in terms of the potential. Firstly consider the case when matter is absent. In order to study inflationary trajectories in the Starobinsky model we assume that the field is far from the origin, thus \( \phi \gg M_{Pl} / \sqrt{3/2} \). Then from (2.58),

\[
\dot{\phi}^2 = \frac{(V,\phi)^2}{3V} = 2e^{-2\sqrt{2/3} \frac{\phi}{M_{Pl}}}, \tag{2.60}
\]

where we see that when \( \phi \) is large the kinetic energy stays very suppressed with respect to the potential, as expected by the slow roll conditions. However, in the case when \( \phi \sim O(1) \) the kinetic energy is the same order as the potential. When this happens the slow roll conditions are violated and inflation has ended. In the case when \( \rho \gg 1 \), we have that \( \frac{(V,\phi)^2}{3V} \approx \frac{3}{5} \coth^2(\phi/\sqrt{6}) \), inserting back into eq. (2.58), we get that,

\[
\dot{\phi}^2 \approx \frac{\sqrt{2}}{6} \rho \sinh(\sqrt{2} \frac{\phi}{3 M_{Pl}}). \tag{2.61}
\]

We now see that, as opposed to the previous case, the kinetic energy increases as we move away from the origin. Thus the slow roll conditions are no longer valid because the kinetic term is not negligible. Indeed, using this relation for \( \dot{\phi}^2 \) we can calculate
the Hubble ratio when $\rho$ is large,

$$H^2 = \frac{1}{3} \rho \sinh \left( \frac{\varphi}{\sqrt{6} M_{\text{Pl}}} \right) \left( \frac{\sqrt{2}}{3} \cosh \left( \frac{\varphi}{\sqrt{6} M_{\text{Pl}}} \right) + \sinh \left( \frac{\varphi}{\sqrt{6} M_{\text{Pl}}} \right) \right), \quad (2.62)$$

which is of the same order as the kinetic energy. This indicates that the effect of the coupling term is to uplift the kinetic energy in such a way that solutions decay to the origin exponentially faster, thus not allowing enough time for inflation to solve the horizon problem. Indeed the Hubble friction will increase exponentially if we move away from the origin, thus the solution decays faster than in the case without matter. We can also calculate the time of decay. When $\rho = 0$,

$$\varphi \propto -\sqrt{\frac{3}{2}} \log \left( \frac{2}{\sqrt{3}} H_0 t \right). \quad (2.63)$$
However, when $\rho \gg 1$,

$$\varphi \propto -2\sqrt{6} \tanh^{-1}(\tan(\frac{5H_0 t}{3\sqrt{3}})).$$

We see that if there is sufficient matter, inflation will not happen because, for any suitable initial condition, the inflaton decays very fast to the bottom of the potential. As we show this result holds even if the matter density $\rho$ is of the same order of the energy density needed for inflation to start. We will now generalise this result to include other types of models.

2.2.1 $\alpha$ model

We start with the $\alpha$ model [137] introduced in section (2.1.5). That action can be written in the Jordan frame as,

$$S(\phi) = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left(1 - \frac{\phi^2}{6M_{Pl}^2}\right)^{\alpha} R - \frac{1}{2} \frac{\alpha - \alpha^2 \phi^2/6M_{Pl}^2}{(1 - \phi^2/6M_{Pl}^2)^{2-\alpha}} (\partial \phi)^2 - V(\phi) \right\},$$

where $\phi$ is the inflaton, we have set the rest of the fields to zero, and fixed the conformal symmetry by setting $X = \bar{X} = \sqrt{6}M_{Pl}$. Also choosing $f(\phi/\sqrt{6}) = \frac{\lambda \phi^2}{(\phi/\sqrt{6}+1)^2}$

the potential in Einstein frame will be given by $V(\varphi) = V_0 \left(1 - e^{-\sqrt{\frac{2}{3\alpha}} \frac{\varphi}{M_{Pl}}} \right)^2$. To study the phenomenology of the $\alpha$ models let us start by noting that for large $\alpha$ and for $\frac{\varphi}{M_{Pl}} \ll \sqrt{\frac{2\alpha}{3}}$, the potential term behaves as,

$$V = \frac{m^2}{2} \phi^2,$$

where $m^2 = \frac{4\lambda_0}{3\alpha}$, whereas for small $\alpha$ it is the usual Starobinsky potential. Therefore we can divide our analysis in terms of the value of $\alpha$. Indeed, for $\alpha < \frac{\varphi^2}{6M_{Pl}^2}$, the inverse of the coupling function is $\Omega^{-1} \sim \sinh(\frac{\varphi}{\sqrt{6}M_{Pl}}) \ll 1$. Therefore for large $\alpha$ we find that non-relativistic matter does not interfere and inflation may take place, provided that the initial conditions are set not far from the origin.

On the other hand for $\alpha \sim \mathcal{O}(1)$, $\Omega^{-1}$ is large enough to interfere with the potential making it too steep for inflation to take place. Thus we have that as in the previous case, initial matter density can spoil inflation. Notice that this result is related to the curvature of the Kahler manifold, which is parametrized by $\alpha$, then it can be used to constrain possible scenarios.
2.2 Matter coupled to inflation

2.2.2 D-term inflation

D-term inflation [141] is a supersymmetric completion of the hybrid inflation scenario. Whilst the original model suffers problems, the canonical superconformal model can be used to achieve a consistent embedding [126]. For the large field limit, D-term inflation is equivalent to the Starobinsky model [134]. Once the superconformal symmetry has been fixed the Kahler potential is given by,

\[ K(z, \bar{z}) = 3 \ln \Omega(z, \bar{z}), \quad \text{where} \]
\[ \Omega^{-2} = 1 - \frac{1}{3}(|S^2| + |\phi_-|^2 + |\phi_+|^2) - \frac{\xi}{6}(S^2 + \bar{S}^2), \]

with superpotential \( W = \lambda S \phi_+ \phi_- \). The inflaton is \( S \) and the other two fields are the waterfall fields and contribute at the end of inflation. Notice that the term proportional to \( \xi \) breaks superconformal symmetry explicitly. For this model the potential becomes inflationary by radiative corrections. For an inflationary phase one has to consider a trajectory along the direction where just the real part of \( \text{Re}S \equiv \phi \) is non-zero and the rest of the fields are in their minima \(|\phi_-| = |\phi_+| = \text{Im}S = 0\). Then the non-canonical kinetic term in the Einstein frame is given by

\[ K(\phi) = \frac{1}{1 - \frac{1}{6}(1 + \xi)\phi^2} \left( 1 + \frac{(1 + \xi)\phi^2}{6(1 - \frac{1}{6}(1 + \xi)\phi^2)} \right). \]  

(2.69)

The coupling function induces corrections to the potential, because canonically normalising the function is given by,

\[ \Omega^2 \sim 1 - \frac{(1 - \xi)^2}{1 + \xi} \tanh^2 \left( \frac{2\phi}{\sqrt{\frac{6}{1 + \xi}} + \frac{\sqrt{6} \xi}{\sqrt{1 + \xi}}} \right). \]  

(2.70)

Thus we see that this function has the same shape as the one arising in conformal inflation, and therefore one has to expect that inflation cannot take place for a significant initial density of matter.

2.2.3 Universal attractors

The universal attractor model studied in [121, 120] generalises the models we have studied and therefore we can investigate deviations of the conformal case. Here, one has a non minimal coupling to gravity, and a potential given by \( \Omega = 1 + \xi f(\phi), K = 1 \) and a potential \( U \equiv \lambda f^2(\phi)\Omega^2 \), where we used the notation from (2.1). By varying
the value of $\xi$ one can obtain different models of inflation. Thus for certain values the predictions will be equivalent to the ones of the Starobinsky model, whereas for others, to chaotic inflation.

This model can be brought into Einstein frame by a conformal transformation. We will start by considering the case when $\frac{\Omega^2}{\Omega_1} \ll 1$. Then, the kinetic term reduces to,

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{Pl}^2}{2} R - \frac{1}{2\Omega} (\partial\phi)^2 - V(\phi) \right\}. \tag{2.71}$$

with corrections up to order $O(\epsilon^2)$ in slow roll parameters and $V = U/\Omega^2$. Let us specialise to the case $f(\phi) = \phi^2$. This case is equivalent to assuming that $|\xi| \ll 1$. When integrating the kinetic term to obtain the canonically normalised field $\varphi$, we find that depending on the sign of $\xi$,

$$\frac{\varphi}{M_{Pl}} = \frac{1}{\sqrt{\xi}} \sinh(\sqrt{|\xi|} \frac{\phi}{M_{Pl}}), \quad \xi > 0, \tag{2.72}$$

$$\frac{\varphi}{M_{Pl}} = \frac{1}{\sqrt{\xi}} \sin(\sqrt{|\xi|} \frac{\varphi}{M_{Pl}}), \quad \xi < 0. \tag{2.73}$$

which in turn leads to two similar situations. For positive $\xi$, we have a similar case to the one studied in [116] where the coupling function $\Omega$ will induce a change in the shape of the potential in such a way that the initial conditions for inflation are set. Indeed in this case, the potential $V \sim (\sinh(\sqrt{\xi} \varphi))^4$ has a minima at $\varphi = 0$ which is uplifted when there is an initial density of matter by a term $\sim \rho \cosh^{-2}(\sqrt{\xi} \varphi)$. Then, the inflaton will decay to a new minima until matter flushes away and inflation starts.

On the other hand for negative and small $\xi$ the case is analogous to the one studied at the beginning of this section. However it has some variations, here the potential will be $V \sim \sin^4(\sqrt{\xi} \varphi)$, which is similar to a chaotic potential for $|\varphi| < \pi/(2\sqrt{\xi})$. Indeed, the potential has a plateau for small values of $\varphi$ where inflation can occur. In the presence of matter the dominant term will be $\sim \rho \cos^{-2}(\sqrt{\xi} \varphi)$. This term grows faster than the original potential and will uplift the plateau, thus the inflaton will decay quickly to the origin, similar to the case for the conformal coupling $\chi = -1/6$.

For the strong coupling limit $\xi \gg 1$ [142], we have that $\frac{\Omega^2}{\Omega_1} \gg \frac{1}{H^2}$, and then

$$S = \int d^4x \sqrt{-g} \left\{ \frac{M_{Pl}^2}{2} R - \frac{3\Omega^2}{2\Omega_1^2} (\partial\phi)^2 - V(\phi) \right\}. \tag{2.74}$$
Normalising the kinetic term the new variable is \( \varphi = \pm \sqrt{\frac{3}{2}} \log \Omega(\phi) \). Choosing the correct sign to obtain an inflationary potential one finds \( V(\varphi) = \frac{\lambda}{\xi^2} \left( 1 - e^{-\sqrt{2/3} \frac{\varphi}{M_{Pl}}} \right)^2 \), is independent of the choice of \( f(\phi) \). The potential now resembles the Starobinsky model. Note also that this result is independent of the sign of \( \xi \). Moreover, the coupling to gravity is,

\[ \Omega(\varphi) = e^{\sqrt{2/3} \frac{\varphi}{M_{Pl}}} \quad \text{(2.75)} \]

and then, the coupling to matter decays exponentially for large \( \phi \). However, for large non-relativistic matter \( \rho \) the potential will not have a local minima because the function \( \Omega(\phi)^{-1} \) grows for negative values. Hence, the original minimum of the Starobinsky potential is no longer a minimum in this case. We see that this is rather different to the other cases we have considered and seems to be a generic feature for this kind of situation. To further proceed one needs to study the reheating process for this model, and how the change in the minima will affect it.

To examine the intermediate case, we need to use numerical integration. To do so, let us assume again that,

\[ \Omega(\phi) = 1 + \xi \frac{\varphi^2}{M_{Pl}^2} \quad \text{(2.76)} \]

Now, the canonical variable will be given by,

\[ \frac{d\varphi}{d\phi} = \frac{\sqrt{1 + \xi \varphi^2 + 6 \xi^2 \varphi^2}}{1 + \xi \varphi^2} \quad \text{(2.77)} \]

First note that for small \( \xi \) the canonical term is approximately flat and the potential is quartic for both signs. Therefore inflation occurs in the flat section near the origin. However, we see from (2.77) that the result of this integral will change, depending on the sign of \( \xi \) and thus the coupling to matter will also change. This gives the different situations described previously. To see this more clearly, we plot the values of \( \xi \) in Fig. 2.3. We see that when \( \xi < 0 \) the slope changes and the coupling function will be similar to a smoothed \( \cos^{-2} \varphi \), whereas for the other case will be a \( \cosh^{-2} \varphi \). On the contrary, when \( \xi \) is large near the origin the potential will be too steep and inflation occurs in the plateau characteristic, of the Starobinsky model. In the intermediate situation when \( \xi \) is increased the potential near the origin will become steep, but the potential will change to \( V(\varphi) \sim \left( 1 - e^{-\sqrt{2/3} \frac{\varphi}{M_{Pl}}} \right)^2 \), and then inflation will occur in the plateau.
Non-canonical inflation coupled to matter

Fig. 2.3 a) Plot of $\phi$ for various values of $\xi$. Note that there is a change in the curvature depending on the sign of $\xi$. In b) the plot shows the canonically normalise fields for large values of $\xi$ of both signs, and it is noticeable that both are similar in the region where inflation happens.

of this potential. Increasing $\xi$ will diminish the effect of the coupled matter, because the coupling will behave as $e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}}$ and then for $\phi \sim \mathcal{O}(1)$ will be suppressed.

Finally, let us again note, that it is not sufficient to have negative $\xi$, to affect inflation. This is because for very large $\xi$, there is a strong coupling regime in which the overall form of $\Omega$ and therefore the sign of $\xi$ is not important. Indeed, we see from the equation for (2.77) that for large $\phi$ the result is independent of $\xi$, and thus the inverse of the coupling function will behave exactly as we discussed earlier.
2.3 Conclusions

In this chapter we have shown that the presence of non-relativistic matter can have a noticeable affect on non-canonical inflationary theories. This is due to the fact that when coupled matter is taken into consideration it can change the shape of the inflationary potential for enough time to affect its dynamics. We focused on the particular case of conformal models which are equivalent to the Starobinsky model. These models are interesting because the initial action in Jordan frame leads to a unique coupling to matter when transformed to Einstein frame. In this case the resulting potential is too steep for inflation to begin when there is an initial density of matter present. We proved that although an initial density of matter will decay rapidly due to cosmic expansion, coupled matter changed the attractor behaviour of the theory and that generally the inflaton will decay to the origin in a few efolds. This result is consistent with [116, 139], where it was shown that an initial matter density can set the initial conditions before decaying.

We first considered the Starobinsky model. In this case we can study the phenomenology in detail. Here since inflation can only take place in a plateau away of the origin, the field cannot start to produce inflation once matter is coupled because the potential will be too steep. We showed that this situation arises naturally in the superconformal supergravity theories of inflation because there is an extra symmetry required that fixes the coupling to gravity. We showed that in the case of an SO(1,1) symmetry the coupling will be induced by the symmetry. Using this, we generalised to include other models. Thus in this case the coupling is a feature of the symmetry rather than the particular model. We showed that in more general models, such as $\alpha$-models there is a similar behaviour, although this case depends on the geometry of the Kahler manifold.

Moreover, as discussed previously [116], the inclusion of coupled matter can also set the initial conditions for inflation. The difference arises from the geometry of the field space, or in supergravity from the Kahler manifold. We proved that by changing the curvature parameter in the universal attractor one can either spoil inflation or naturally set its initial conditions. It is particularly interesting that the more symmetric cases were the ones with more dramatic consequences. Indeed it is very easy to see that the result we obtain for the conformally invariant model can be generalised for the $SO(1,N)$ case.

Our results hold independently of the frame chosen. For example, if one gauges the theory by choosing $\chi^2 - \phi^2 = \sqrt{6}$ on (2.23), then one is taken directly to the Einstein frame of the theory (2.7) with coupling function $\Omega$ is equal to 1. Because of this, one
may think that there is a possible ambiguity between both frames, but, this is no longer true if the situation is considered more carefully. Taking a step back one can perform a full gauge invariant calculation to show that the equation of motion are,

\[ \Box \phi - V_{\phi} - \Omega_{\phi} \Omega^3 T^J = 0, \quad (2.1) \]

where \( \Omega(\phi) = \Omega(\phi(\phi^2 - \chi^2)) \), ie, in general \( \Omega \) is a function of both fields and it should not be removed by a gauge choice. Then one can see, our results holds in any gauge. Furthermore, the result in \[143\] where a massless curvaton field coupled to inflation was considered, showed physical observables were the same, though there was a different physical interpretation in both frames. Notice also that this result is different to the \( \eta \) problem, where inflation is unviable because the inflaton mass receives large radiative correction. Indeed for the particular supergravity studied a shift symmetry avoids the corrections at all energies and therefore we can isolate this effect from the inclusion of non-relativistic matter. In this sense one can estimate the value of \( \rho \) for which inflation is affected. We showed in the Starobinsky model that for \( \rho \sim \lambda M_{Pl}^4 \) the potential was too steep. Since inflation occurs at energies \( \frac{V}{M_{Pl}^2} \ll \Delta_R \sim 10^9 \), and therefore \( \lambda \sim 10^{-9} \), we have that \( \rho \ll M_{Pl}^9 \), and the theory is stable at this level. We also proved that this effect arises in the more general universal attractor models. Here it is easy to appreciate how the geometry of the target space affects the coupling to matter in such a way that for certain cases inflation cannot take place, while for other cases works as a mechanism to set the initial conditions.
Chapter 3

Holographic inflation

3.1 Introduction

In this chapter we study the dynamics of multifield inflation in a holographic context. In the first part we study the constraints that the Hamilton-Jacobi equations pose to the fields that contribute during inflation. In the second part we study the holographic \( \beta \) function and how one can understand different models of inflation in this language.

Holographic models of inflation are known to have self-interactions constrained by certain requirements on the holographic correspondence connecting the inflationary bulk cosmology and the 3d Euclidean QFT at the boundary [144–146]. More precisely, in this class of theories, the marginally relevant scalar operators deforming the CFT at the boundary are found to be dual to a bulk inflationary potential \( V \) determined by a given superpotential \( W \) as

\[
V = 3W^2 - 2\gamma^{ab}W_aW_b, \quad (3.1)
\]

where \( \gamma^{ab} \) is the inverse of the sigma model metric characterising the geometry of the multi-scalar field target space spanned by \( \phi^a \) (with \( a = 1, \cdots, N \)) and \( W_a = \partial W / \partial \phi^a \).

This structure of the potential admits the following class of background solutions:

\[
\dot{\phi}^a = -2\gamma^{ab}W_b. \quad (3.2)
\]

The trajectory described by this solution is dual to the renormalization group flow of the boundary operators with fixed points representing static de Sitter configurations of the cosmological bulk. In this sense, the entire cosmological history, starting from a static de Sitter universe (inflation), and ending in another static de Sitter universe (our dark
energy dominated universe) may be understood as the consequence renormalization
group flow from the UV-fixed point to the IR-fixed point.

In the first part of this chapter we will study how to calculate correlations for
massive fields in inflation. We will start by studying the analogies between de Sitter
and anti de Sitter spacetimes. When moving to inflation and de Sitter the dynamics
of multi-field fluctuations coming from the constrained structure of the potential of
Eq. (3.1) will be relevant. We find that the mass of scalar fields must respect an upper
bound given by:

\[ m \leq \frac{3}{2} H, \quad (3.3) \]

where \( H \) is the Hubble expansion rate during inflation. We explain this bound in terms
of the local part of the on shell action, defined as the phase of the wavefunction of the
universe [147]. As a consequence, in holographic models of inflation, the isocurvature
modes, if present, are necessarily kinematically active during horizon crossing, and their
interaction with the curvature perturbation could have led to sizeable imprints on the
observable primordial spectra. More importantly, if future observations reveal evidence
of heavy fields with entropy masses larger than \( 3H/2 \), then holographic versions of
inflation would be essentially ruled out.

The fact that inflation can be interpreted holographically as a deformation from a
CFT, automatically induces a flow. This flow defines a \( \beta \) function that is associated
with the roll on field space for the inflaton. In the second part we use this \( \beta \) function to
study the flows of different models of inflation. We find that under general assumptions
single field inflation flows from an IR fixed point. We also show that when embedded
in a more realistic scenarios, this flow might not happen. This might be associated to
the presence of extra degrees of freedom that are still not stabilised at the beginning
of inflation.

This chapter is organised as follows: In Section 3.2 we review the calculation of
the two point function of scalar fields in AdS, dS and inflation. We then show how to
remove the divergences that occur at the boundary of these spacetimes. Afterwards we
explain that in the case of inflation and dS the HJ equations imply a constraint on
the masses of the renormalizable operators. We finally study the consequences of this
finding. In the second part we explain how the \( \beta \) function arises when interpreting
inflation in holography. Later we explain the behaviour of the \( \beta \) function in simple cases
of single field inflation. We also study string theory models of inflation for different
moduli stabilization mechanism. We find that, generically, these models have an IR
fixed point at large \( \phi \), which might be spoiled by the combined action of matter. We
will use the notation \( \langle \rangle \prime = (2\pi)^3 \delta^{(3)}(k - k') \langle \rangle \).
3.2 dS/CFT

In this section we briefly review perturbative holographic calculations. Since the discovery of the AdS/CFT correspondence finding a precise de Sitter counterpart has been a missing corner. Nevertheless, many attempts have give some clues in how to proceed, and there is now a clear idea of the problems and challenges required to solve to find a duality [52–55, 57]. One important step was done by Maldacena [55] where he stated that the correspondence was to be given by

\[ Z[g] = \Psi[g], \]  

where the left hand side is the partition function of a conformal field theory in three dimensions, and the right hand side is the wavefunction of the universe for a three metric \( g \). This statement is valid on a Poincare patch, and thus is relevant for cosmology, but it might not be valid in global coordinates. Here all fields started to evolve from a Bunch-Davies vacuum and are evaluated at horizon crossing. This brings a direct analogy with inflation, and hence it is useful when comparing to observations.

One of the implications of this proposal is that calculating correlation functions in the wavefunction of the universe \( \Psi \) at the limit \( k\tau \to 0 \) is equivalent to calculating correlation functions on a CFT. Then, the dimension of the conformal operators depends on the mass of the scalar fields on (A)dS. Furthermore he mentioned that there is a prescription to go from Euclidean AdS calculations to dS ones by making the analytic continuation,

\[ \tau = iz, \quad R_{dS} = iR_{AdS}, \]  

where \( z \) is the radial coordinate of an AdS space and \( R_{dS} = H^{-1} \) the size of the horizon. In what follows we will examine what these statements mean, first for scalar fields on a background anti de Sitter and de Sitter spacetime, and then for inflation.

3.2.1 From AdS to dS

AdS

We start by reviewing correlation functions for scalar fields on an AdS spacetime. Particularly clear discussions are in [47, 148]. For a Euclidean AdS spacetime we have \( ds^2 = \frac{1}{z^2}(dt^2 + d\vec{x}^2 + dz^2) \), with \( z \) running from 0 to \( \infty \) and the spatial coordinates covering a 3 sphere. We need to impose Dirichlet boundary conditions, and we choose
to pick fields that are independent of $z$ at $z = z_c$ a cutoff, and die off when $z \to \infty$.

Now, the action of a free scalar field on AdS is given by

$$S = \frac{R^2}{2} \int d^3x \int_{z_c}^\infty \frac{dz}{z^2} \left[ (\partial_x \phi)^2 + (\partial \phi)^2 + \frac{m^2}{z^2} \phi^2 \right]. \tag{3.3}$$

The equations of motion are,

$$\left[ z^2 \partial_z (z^{-2} \partial_z) + \partial^2 + \frac{m^2}{z^2} \right] \phi = 0, \tag{3.4}$$

a solution that satisfies the boundary condition we are imposing is given by

$$\phi_k = \varphi_k \left( \frac{z}{z_c} \right)^{3/2} \frac{K_{\nu}(kz)}{K_{\nu}(kz_c)} \quad \text{where} \quad \nu = \sqrt{\frac{9}{4} + m^2 R^2}, \tag{3.5}$$

where $\varphi$ is a field profile independent of $z$ and we have Fourier transformed the solution. This solution contains non analytic parts, which are leading order in $k$ are,

$$K_{\nu}(kz) = \frac{\Gamma(\nu)}{2} \left( \frac{kz}{2} \right)^{-\nu} \left[ 1 + \ldots + \left( \frac{kz}{2} \right)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \right], \tag{3.6}$$

where the dots are higher order in $k$. Now to calculate correlation functions in AdS/CFT we first start by integrating by parts and then using the equations of motion. This leads to the on-shell action in Fourier space

$$S_{\text{on-shell}} = \frac{R}{2} \int \frac{d^3k}{(2\pi)^3} \phi_k \partial_x \phi_{-k} |_{z=z_c}, \tag{3.7}$$

where the term that remains is the boundary that comes from integrating the action by parts. Replacing the solutions we get

$$S_{\text{on-shell}} = \frac{R}{2} \int \frac{d^3k}{(2\pi)^3} \left( \frac{3/2 - \nu}{z_c^3} - \frac{kK_{\nu-1}(kz_c)}{z_c^3 K_{\nu}(kz_c)} \right) \varphi_k \varphi_{-k}, \tag{3.8}$$

so the action contains local and nonlocal terms. Using the approximation (3.6), the on-shell action becomes

$$S_{\text{on-shell}} = \frac{R}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{z_c^3} \left( \frac{3/2 - \nu}{z_c^3} - \frac{2\Gamma(1-\nu)}{\Gamma(\nu)} \left( \frac{kz_c}{2} \right)^{2\nu} \right) \varphi_k \varphi_{-k}, \tag{3.9}$$
where the dots represents higher order in $k$. This action diverges at $\tau_0$ and in order to make it finite one has to first remove the first term $(3/2 - \nu)/z_c^3$ by employing counterterms, and then rescale $\varphi_k \to z_c^{3/2-\nu} \varphi_k$. Correlations functions are calculated by taking derivatives with respect to $\varphi_k$ of the wavefunctional $e^{-S}$. The two point function, for example, is

$$\langle O_k O'_k \rangle = -2\pi^3 \delta^3(k + k') \frac{R^2 \Gamma(1 - \nu)}{2\pi \Gamma(\nu)} \left( \frac{kz_c}{2} \right)^{2\nu} z_c^{-3} + \text{local}, \quad (3.10)$$

with the local term coming from the first term of the action. Since it is local means that their Fourier transform is just a contact term. This is, for example the Fourier transform of a $k^2$, that comes from a $\nabla^2$ operator is a $\nabla^2 \delta(\vec{x} - \vec{y})$. In contrast the Fourier transform of the non analytic part is proportional to

$$\frac{1}{|\vec{x} - \vec{y}|^{3+2\nu}}, \quad (3.11)$$

which corresponds to the two point function for two conformal operators of weight $\Delta = 3/2\nu$, as expected in a CFT.

Now let us to go back to the divergence that arises from $(3/2 - \nu)/z_c^3$. A way to remove this divergence is outlined in [144], where the Hamilton Jacobi formalism is employed. The main idea comes from identifying that only local terms have to be removed. These are generated through the Hamilton Jacobi equations by using the ansatz

$$S_{ct} = \int d^3k \sqrt{h} \left[ 2W(\phi) + M(\phi)k^2 \phi^2 + K(\phi)R + .. \right]. \quad (3.12)$$

As both the AdS action and $S_{ct}$ are solutions for the Hamilton Jacobi constraints one find that,

$$m\phi^2/2 = -3W^2 + 2W_{\phi\phi}^2, \quad (3.13)$$

Solutions to this equation automatically satisfy the requirement of eliminating divergences as $z_c \to 0$. For example for this case we find that by solving (3.13), one finds that $W_{\phi\phi} = \frac{3-2\nu}{2}$. And hence by defining the total action as

$$e^{-S_{ct}} e^{S_{on-shell}}, \quad (3.14)$$

one makes sure that at the boundary there are no divergences.
Some final comments before we move to de Sitter. Our calculation was valid in the case where $\nu$ is not an integer. For integer $\nu$ one should be more careful and consider the Green function first, which is hypergeometric function as is well defined for all $\nu$ [149].

**dS**

Now let’s repeat the above calculation but for a massive scalar field on de Sitter spacetime. We have that the metric on the Poincare patch is given by $ds^2 = \frac{1}{H_0^2}(-d\tau^2 + d\vec{x}^2)$ where the *conformal time* $\tau$ runs from $-\infty$ in the infinite past to 0 in the infinite future. $H_0^{-1}$ is the de Sitter radius. The action for a massive scalar field is

$$S = \frac{1}{2H_0^2} \int d\vec{x} \int_{-\infty}^{\tau_0} d\tau \left( (\partial_\tau \phi)^2 - (\partial \phi)^2 - \frac{m^2}{\tau^2} \phi^2 \right),$$

(3.15)

where we are integrating up to a cutoff $\tau_0 \ll 1$. We will consider scalar fields whose initial state was the Bunch–Davies vacuum. In analogy with the AdS case we will consider the solution at the cut off to be time independent. We then have that

$$\phi_k = \varphi_k \left( \frac{\tau}{\tau_0} \right)^{3/2} \frac{H^{(1)}_\nu(-k\tau)}{H^{(1)}_\nu(-k\tau_0)},$$

where $H^{(1)}_\nu(x) = J_\nu(x) + iY_\nu(x)$ is the Hankel function of the first kind. Note that for dS spacetime there is a regime where the mass makes $\nu$ to be imaginary. This is referred to the fact that massive particles should decay once they leave the horizon. We can expand the Hankel function for small $| -k\tau | \ll 1$ as,

$$H^{(1)}_\nu(-k\tau) \sim \frac{1}{\Gamma(1+\nu)} \left( \frac{-k\tau}{2} \right)^\nu \left( 1 + \mathcal{O}(k^2) \right)$$

(3.17)

$$-i \left( \frac{\cos \pi \nu}{\pi} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left( \frac{-k\tau}{2} \right)^\nu + \frac{\Gamma(\nu)}{\pi} \left( \frac{-k\tau}{2} \right)^{-\nu} \right) \left( 1 + \mathcal{O}(k^2) \right).$$

We can see that for imaginary $\nu$ the field has two oscillatory solutions factorised by a Boltzmann factor $e^{-\pi\mu}$ that appears from the $\Gamma$ functions. We will focus for now on the case $m^2/H^2 < 9/4$ but will return to the other case later. For this range of masses
the on-shell action is,

\[ S_{\text{on-shell}} = \frac{1}{H_0^2} \int \frac{d^3k}{(2\pi)^3} \tau_0^{-2} \phi_k \partial_\tau \phi_k = \frac{1}{H_0^2} \int \frac{d^3k}{(2\pi)^3} \tau_0^{-2} \left( \frac{3 - 2\nu}{2\tau_0} - \frac{k H^{(1)}_{\nu-1}(-k \tau_0)}{H^{(1)}_{\nu}(-k \tau_0)} \right) \phi_k \phi_k. \]  

\[(3.18)\]

At this stage we can now make the claim that both on-shell actions coincide upon the continuation \( R \rightarrow iH^{-1}, \ z \rightarrow i\tau \). Indeed, by using the fact that the modified Bessel function can be written as

\[ K_\nu(kz) = \frac{\pi}{2} i^{\nu+1} H^{(1)}_{\nu}(-k \tau), \]  

\[(3.19)\]

both actions coincide upon an overall phase, \(-S_{\text{AdS}} \rightarrow iS_{\text{dS}}\), which is expected for actions which are Wick rotated. Expanding for \( k \tau \ll 1 \) we have,

\[ i \frac{1}{H_0^2} \int \frac{d^3k}{(2\pi)^3} \tau_0^{-2} \left( \frac{3 - 2\nu}{2\tau_0} - \frac{2}{\tau_0} \frac{\pi e^{-i\nu \tau}}{\pi \Gamma(\nu) \cos(\pi \nu)} \left( \frac{1}{2} \right)^{2\nu} \right) \phi_k \phi_k, \]  

\[(3.20)\]

which has a similar structure to the AdS action, although there is an important difference. Whereas for the former there is a local divergence for the case of de Sitter all local terms appear to be a pure phase. Following [55], we should interpret the current calculation as \( \Psi \sim e^{iS} \) where \( \Psi \) is the wavefunction of the universe, and hence,

\[ \langle \phi^2 \rangle = \int D\phi \phi^2 |\Psi|^2, \]  

\[(3.21)\]

then local terms get cancelled. Nevertheless, a more careful analysis shows that for the range of masses \( m^2 > \frac{9}{4} H^2 \), the action (3.19) becomes,

\[ i \frac{1}{H_0^2} \int \frac{d^3k}{(2\pi)^3} \tau_0^{-2} \left( \frac{3 - 2i\nu}{2\tau_0} - \frac{2i}{\tau_0} \frac{\nu^2 \Gamma(i\nu)}{\pi (1 + \coth(\pi \nu))} \left( \frac{1}{2} \right)^{2\nu} \right) \phi_k \phi_k, \]  

\[(3.22)\]

which implies, that the local divergence at \( \tau_0 \) is not a phase anymore but will contribute to (3.21). To remove it we need to obtain local counterterms, which are generated by the Hamilton Jacobi equations as in the AdS case, but as we will show later this is not possible.

Now restricting the discussion for \( \nu \) real, the 2 point function is, \( \sim \frac{2 \cos \pi \nu}{\tau_0} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left( \frac{-k \tau}{2} \right)^{-2\nu} \) which agrees with the two point function of a conformal operator of weight \( 3/2 + \nu \). There is subtlety with this procedure, as is indicated by [55, 58], because we are losing
the decaying solution from the 2 point correlation function. This might be seen by considering the two point correlation function in de Sitter and then taking the limit \( k\tau \to 0 \)

\[
\langle \phi_k(\tau)\phi_{-k}(\tau') \rangle = \frac{(\tau\tau')^2}{4\pi} \left[ \Gamma(\nu)^2 \left( \frac{k^2\tau\tau'}{4} \right)^{-\nu} + \Gamma(-\nu)^2 \left( \frac{k^2\tau\tau'}{4} \right)^{\nu} \right]. \tag{3.23}
\]

We get a decaying mode which does not appear from the wavefunction perspective. This might be related to the fact that the decaying solution is subleading and will not be observable, but in any case it shows the limitations of our approach. Note that, we have that the two point function can be expressed as a sum of conformal operators \( O \)

\[
\phi(x) \sim \sum_{\Delta} \tau^{\Delta} O_{\Delta}(\vec{x}), \tag{3.24}
\]

with conformal weight \( \Delta = \frac{3}{2} \pm \nu \). The unitarity of the conformal operators depends on their dimension and in this case they are unitary only for masses below \( \frac{3H}{2} \) [150]. It has been argued that there is no need to require unitarity for the CFT part of de Sitter as the existence of non unitary CFT appears in many other applications [53]. We will show that, in inflation, the Hamilton Jacobi equations in fact restrict the conformal operators to be unitary. This is because there is a bound on the mass of the fields coupled to the inflaton and then on the dimension of the conformal operators associated with it.

### 3.2.2 Generalization to inflation

So far our discussion has been based on pure de Sitter, but when considering inflation we need to take into account the fact that the background is evolving, and hence \( H \) has a time dependence. This is parameterised in terms of \( \epsilon = -\frac{\dot{H}}{H^2} \), which has to be smaller than one but nonzero in order to achieve inflation. This time dependence will imply that there is a light curvature mode, that measures the small time differences at the hypersurface of \( k\tau \to 0 \). A powerful approach that makes use of this idea is the effective field theory (EFT) of inflation [151, 152]. Here the diffeomorphisms invariance is recovered by introducing \( t \to t + \pi(x) \) where \( \pi \) is the Goldstone boson of broken time translations. We will first focus on single field inflation but the generalisation to multi field will be direct. The background for the inflationary action

\[
S = -\int d^4x\sqrt{-g} \left[ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) \right], \tag{3.25}
\]
becomes, in unitary gauge, after reintroducing time diffeomorphisms invariance,

\[ S = - \int d^4x a^3(-\dot{H}) \left[ (\partial \pi)^2 - 3\dot{H}\pi^2 \right], \quad (3.26) \]

where we have ignored tadpoles and higher order terms in \( \pi \), and we have used \( V = 3H^2 - \dot{H} \). This action is equivalent to the action for the more commonly used curvature mode \( \zeta \) by the identification \( \zeta = -H\pi \), which comes from expanding the metric. The classical solution for this equation of motion is subject to a Bunch-Davies vacuum at \( \tau \to \infty \). As before we are interested in the on-shell action,

\[ S_{\text{on-shell}} = \int d^3x a^3(-\dot{H})\pi \dot{\pi}, \quad (3.27) \]

which is evaluated at a time \( \tau_0 \). As in the AdS case we will use a profile \( \pi^0_k \) to impose boundary conditions, these are Bunch-Davies vacuum at \( \tau \to -\infty \) and time independence for \( k\tau \to 0 \). The solution is given by,

\[ \pi = \pi^0_k \frac{(1 - i k \tau)}{(1 - i k \tau_0)} e^{i k \tau} \quad (3.28) \]

Let’s Fourier transform this and also write it in terms of the more familiar curvature perturbation \( \zeta = -H\pi \).

\[ S_{\text{on-shell}} = \int \frac{d^3k}{(2\pi)^3} a^2 \epsilon \zeta^0_k \partial^\tau \zeta^0_{-k} \mid_{\tau_0} \sim \int \frac{d^3k}{(2\pi)^3} \epsilon \frac{k^2}{H^2} \left[ \frac{k^2}{\tau_0} + i k^3 + O(k^4) \right] \zeta^0_k \zeta^0_{-k}, \quad (3.29) \]

which is the same as that for a massless scalar field on de Sitter (3.19), which is expected due to the symmetries of inflation at short wavelengths. Also note that the local term is a pure phase of the wavenfunctional \( e^{iS} \), hence does not contribute to the total probability. Now let us study multifield inflation. We will consider cases where the field metric is flat, but the generalisation is straightforward and won’t change our results. The action is

\[ S = - \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a + V(\phi^a) \right]. \quad (3.30) \]

To obtain the perturbations we follow [152], where perturbations are separated into tangent and normal directions to the trajectory on field space. The idea is that the curvature perturbation defines a geodesic trajectory in which inflation happens, whereas isocurvature directions are perpendicular to it. In order to do so we decompose field
space in terms of the basis $\gamma_{ij} = T_a T_b + N_a N_b + \delta_{ij} e^a e^b$, where $T^a$ and $N^a$ are defined as,

$$T^a = \frac{\dot{\phi}^a}{\dot{\phi}_0}, \quad N_a = -\sqrt{\gamma\epsilon} e_a T^b. \quad (3.31)$$

We have that in the case of two fields, we can write,

$$\frac{D}{dt} T^a = -\Omega N_a, \quad (3.32)$$

$$\frac{D}{dt} N^a = \Omega T^a, \quad (3.33)$$

where $\Omega \equiv \dot{T}^a N_a$ is the coupling between curvature and isocurvature modes. We reintroduce time diffeomorphisms as in single field inflation, through the reparametrization, $t \rightarrow t - \pi(x)$. Then, the scalar field transforms as,

$$\phi(t)^a \rightarrow \phi^a(t + \pi(x)) + N^a(t + \pi(x)) \sigma, \quad (3.34)$$

where $\sigma$ is a massive scalar field representing the tangent perturbations. At second order in perturbations the action (3.30), becomes,

$$S = \int d^4 x a^3 \left[ \epsilon_\sigma^2 - \frac{\epsilon}{a^2} (\nabla \zeta)^2 + 2\sqrt{2\epsilon} \Omega \dot{\zeta} \sigma + \frac{1}{2} \left( \dot{\sigma}^2 - \frac{1}{a^2} (\nabla \sigma)^2 \right) - \frac{1}{2} m^2 \sigma \right]. \quad (3.35)$$

where $m^2 = V_{NN} - \epsilon H \mathcal{R}$, with $\mathcal{R}$ the curvature of the field manifold. We have changed the Goldston boson $\pi$ for the curvature mode $\zeta = -H^{-1} \pi$. At linear order the equations of motions are,

$$\frac{d}{dt} \left( \dot{\zeta} + \frac{2\Omega}{\sqrt{2\epsilon}} \sigma \right) + (3 + \eta) \left( \dot{\zeta} + \frac{2\Omega}{\sqrt{2\epsilon}} \sigma \right) + \frac{k^2}{a^2} \dot{\zeta} = 0, \quad (3.36)$$

$$\ddot{\sigma} + 3H \dot{\sigma} + \frac{k^2}{a^2} \sigma + \mu^2 \sigma + \sqrt{2\epsilon} \lambda H \left( \frac{2\Omega}{\sqrt{2\epsilon}} \sigma \right) = 0. \quad (3.37)$$

These equations do not have an analytical solution but we can follow the approach of [20, 153], and study the regime where $\Omega < H$. Introducing the interaction between the fields $2\sqrt{2\epsilon} \Omega \dot{\zeta} \sigma$ as a perturbation. We first have that at zeroth order in $\Omega/H$,

$$\ddot{\sigma} + 3H \dot{\sigma} = 0, \quad (3.38)$$
which after imposing Bunch-Davies vacuum has the solution

\[ \sigma_k = e^{-i\pi \nu / 2} \sqrt{\pi / 2} H(-\tau)^{3/2} H^{(1)}_\nu (-k\tau) \]

(3.39)

Now integrating by parts and ignoring the interactions between the fields, we have that (3.35) can be rewritten as

\[ S = -\frac{1}{2} \int d^3x a^3 (\dot{\phi}^2 \pi^2 + \sigma \dot{\sigma}) + \frac{1}{2} \int d^4x a^3 \pi (\text{eom}) \pi + .. \]

(3.40)

where the last part of the action is proportional to the equations of motion and thus vanishes. In order to make sense of the on-shell action we are required to describe \( \sigma \) as

\[ \sigma_k = \sigma_k^0 \left( \frac{\tau}{\tau_0} \right)^{3/2} \frac{H^{(1)}_\nu (-k\tau)}{H^{(1)}_\nu (-k\tau_0)}, \]

(3.41)

where \( \sigma_k^0 \) is time independent. Replacing back we get in the long wavelength limit, that the on-shell action is,

\[ iS_{\text{on-shell}} \sim \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\epsilon}{H^2} \left( \frac{k^2}{\tau_0} + i k^3 \right) \sigma_k^0 \sigma_{-k}^0 \right. \\
+ \left. \left( \frac{3/2 - \nu}{H \tau_0^3} - \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{3/2 + \nu}{H \tau_0^3} \frac{e^{i\pi \nu}}{2} \right) \right\} \sigma_k^0 \sigma_{-k}^0 \],

(3.42)

so our result is a straightforward generalisation of the results for de Sitter spacetime (3.19). Interaction between two fields can be introduced perturbatively and we will do so later in this chapter. As expected there are local divergences that in the case of \( m \leq \frac{3}{2} H \) are pure imaginary and hence are cancelled on the calculations. For the case \( m > \frac{3}{2} H \) they could in principle be removed by a suitable choice of counterterms but we will show this is not possible.

To find a way to interprete the divergences, let us remember that our interpretation of the wavefunction of the universe \( \Psi \sim e^{iS} \) implies that \( S \) is a solution of the Hamilton Jacobi equations [154, 155]. For the case of inflation these are defined on a particular hypersurface at fixed time, whose action is given by \( e^{iU} \)

\[ U = U_0 + \int d^3x \sqrt{h} \left( 2W(\phi) + \Phi(\phi) R + M_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b \right). \]

(3.43)
where by construction all the functions inside the integral are local, and higher order
will be at higher order in $k$. All non local information is encoded in $U_0$. The local
part of the action can be related to the background equations of motion through the
Hamilton Jacobi constraints. This leads to the following equations,

\begin{align}
V &= 3W^2 - W^a W_a, \quad W_a = -\dot{\phi}^a/2, \quad (3.44) \\
-2 &= W\Phi^a - 2W_a\Phi^a, \quad (3.45) \\
0 &= \frac{1}{2} - WM - 4U_a\Phi^a + \frac{1}{4}(W_aM^a + 2M\nabla^2 W). \quad (3.46)
\end{align}

which implies that $W = H$. By expanding (3.43) in terms of the $\phi(t)^a \to \phi^a(t + \pi(x)) +
N^a(t + \pi(x))\sigma$ we can relate the functions (3.43) to the local part of (3.42). We have
that the expansion at second order in perturbations and lowest order in (3.43) becomes,

\[ S_{ct} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} a^3 \left( \epsilon W_{TT}\zeta_k^2 + 2W_{NN}\sigma_k^2 + \sqrt{\epsilon} W_{NT}\zeta_k\sigma_{-k} \right). \quad (3.47) \]

where we have expanded $W$ in terms of the orthonormal basis (3.31) so for instances,
$W_{TT} = T^a T^b \nabla_a \nabla_b W$. Equating both (3.43) and (3.42) we can get then expressions for
the second derivatives of $W$. This implies for example that,

\[ 2a^3 W_{NN} = \frac{1}{H^2 a^3} \left( \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \right). \quad (3.48) \]

or that $m^2 = 6HW_{NN} - 4W_{NN}^2$. In the case of complex $\nu$ we have that $m^2 =
4W_{NN}^2 - 6HW_{NN} + 9/4$.

Now let us follow another approach by making use of the equations (3.44). We will
project the scalar potentials $V$ and $W$ on field space by using the basis (3.31) to find
relations between them and thus expressions for the functions $W_{NN}$ and $W_{TT}$. The
gradients of $V$ in terms of (3.44) are,

\begin{align}
\nabla_a V &= 6W_a W_b - 4W_b \nabla_a W^b, \\
\nabla_a \nabla_b V &= 6W_a W_b + 6W \nabla_b W_a - 4\nabla_b W_c \nabla_a W^c - 4W^c \nabla_a \nabla_b W_c. \quad (3.49)
\end{align}

\footnote{See Appendix B.1 for details on the Hamilton Jacobi formulation}
Projecting the second derivative of the potential into the basis (3.31) we get the following identities,

\begin{align}
T^a T^b \nabla_a V_b &= 3 H^2 (2 \epsilon - \eta/2) + \Omega^2, \quad (3.50) \\
N^a T^b \nabla_a V_b &= \Omega H \left( 3 + \frac{\dot{\Omega}}{H \Omega} + \eta - 2 \epsilon \right), \quad (3.51) \\
N^a N^b \nabla_a V_b &= M^2 + \Omega^2 - \epsilon H^2 \mathcal{R}. \quad (3.52)
\end{align}

where \( \epsilon \) and \( \eta \) are the usual slow roll parameters but we haven’t assumed them to be small. Now we also have that projecting \( \nabla_a \nabla_b W_c \) into the normal and tangent directions we get,

\begin{align}
N^a T^b \nabla_a W_b &= \frac{1}{2} \Omega, \quad (3.53) \\
T^a T^b \nabla_a W_b &= \frac{1}{4} H (2 \epsilon - \eta), \quad (3.54) \\
N^a N^b \nabla_a W_b &= W_{NN}. \quad (3.55)
\end{align}

In order to relate the last parameter \( W_{NN} \) to other known parameters we can calculate

\begin{align}
N^a N^b \nabla_a \nabla_b V &= 6 W W_{NN} - 4 (W_{NN})^2 - \epsilon H^2 \mathcal{R} + 2 \dot{W}_{NN} - 3 \Omega^2, \quad (3.56)
\end{align}

Now using the entropy mass \( \mu^2 = M^2 + 4 \Omega^2 \) and comparing with (3.52) we get that,

\begin{align}
\mu^2 &= 6 W W_{NN} - 4 (W_{NN})^2 + 2 \dot{W}_{NN}. \quad (3.57)
\end{align}

Interestingly this expression does not depend on \( \Omega \) or any slow roll parameter so it will hold in general for any solution of the Hamilton Jacobi eqs. (3.44). Neglecting the last term we find that \( \mu \) has a maximum at

\begin{align}
\mu_{\text{max}}^2 &= \frac{9}{4} H^2, \quad (3.58)
\end{align}

Surprisingly we have obtained a generalisation of (3.44) and we have shown that the spectrum of particles coupled to the inflaton has to respect the bound (3.58). This result also suggests that particles with masses above the threshold \( m^2 \leq 9/4 \ H^2 \) cannot make sense from a wavefunction of the universe point of view, because of the local divergences that will appear. Hence our results show that these cannot be removed, as they are not solutions of the Hamilton Jacobi equations.
Our result seems to suggest that particles above the bound are not possible to express as the local part of the wavefunction $e^{iS}$. Although this result seems to indicate that they are forbidden, it does not imply that they might not be observed. This will indeed, mean that our approach is limited and cannot describe the whole spectrum of particles coupled to inflation. All our calculations rely in the validity of the Hamilton Jacobi formalism. It has been shown that there are cases in holographic renormalisation when this is valid [156, 58], but those are cases well beyond the semiclassical limit and is beyond our interest.

We will now generalise the bound to $N$ fields. Let us start by considering the simplest case: a straight trajectory in a model with $N$ fields with canonical kinetic terms $\gamma_{ab} = \delta_{ab}$. In this case the dynamical “entropy” mass coincides with the naive mass (we will later discuss the most general situation). Without loss of generality we can take the inflationary trajectory to be along the $\phi^1 \equiv \phi$ direction, with all other fields stabilized: $\dot{\phi}^i \equiv \dot{\sigma}^i = \sigma^i_0$ for $i = 2, \cdots, N$. Note that the equations (3.2) imply $W_{\sigma^i} = 0$ on the inflationary trajectory, and we can expand the superpotential as

$$W(\phi, \sigma^i) = w(\phi) + \frac{1}{2} \sum_{i=2}^{N} h_i(\phi)(\sigma^i - \sigma^i_0)^2 + \cdots,$$

(3.59)

where $w(\phi)$ and $h_i(\phi)$ are given functions of $\phi$. Inserting this expression into (3.1) gives

$$V = 3w^2 - 2(w')^2 + \frac{1}{2} \sum_i m^2_i(\phi)(\sigma^i - \sigma^i_0)^2,$$

(3.60)

where the masses $m^i(\phi)$ of the fields $\sigma^i$ at each point on the trajectory are found to be given by

$$m^2_i(\phi) = 6wh_i - 4h_i^2 - 4w'h_i'. $$

(3.61)

We can rewrite this expression in a more useful way by noticing from Eq. (3.2) that $w = H$ and $w' = -\dot{\phi}/2$. Then, we obtain $m^2_i = 6Hh_i(1 + \delta_i/3) - 4h_i^2$, where we have defined $\delta_i = \dot{h}_i/Hh_i$. From the previous expression, it follows that $m^2_i$ has a maximum value given by

$$m_{\text{max}} = \frac{3H}{2} (1 + \delta/3).$$

(3.62)

Notice that $\delta_i$ measures the running of $h_i$, which near the maximum satisfies $h_i \sim H$. If background quantities evolve slowly, then we expect $\delta \sim \mathcal{O}(\epsilon)$, implying that masses stay almost constant during slow roll. On the other hand, unless $h_i \ll H$, a large value of $\delta_i$ could make the field $\sigma^i$ tachyonic.
The bound of Eq. (3.58) coincides with the analytic continuation of the Breitenlohner-Freedman bound encountered in scalar field theories in AdS spacetimes [157]. This is in fact not a coincidence: well behaved holographic constructions in dS/CFT are usually the result of the analytic continuation of AdS/CFT. In the particular case of holographic inflation, because the inflationary trajectory is dual to the renormalization group flow in the CFT side of the duality, the potential driving inflation must always admit monotonic solutions of the form (3.2), regardless of the initial conditions. This is satisfied for flows that are solutions of the Hamilton–Jacobi equations. This restricts the value of the masses of the fields, simply because a field with mass larger than $3H/2$ has non-monotonic trajectories. To appreciate this, let us disregard the motion of the inflaton $\phi$ and focus on the background evolution of one of the massive fields $\sigma$ with a mass $m$. Its background equation of motion is given by

$$\ddot{\sigma} + 3H\dot{\sigma} + m^2(\sigma - \sigma_0) = 0. \quad (3.63)$$

The general solution is of the form $\sigma(t) = \sigma_0 + A_+ e^{\omega_+ t} + A_- e^{\omega_- t}$ with:

$$\omega_{\pm} = -\frac{3}{2}H \pm \frac{3}{2}H\sqrt{1 - \frac{4m^2}{9H^2}}. \quad (3.64)$$

If $m < m_{\text{max}}$, the solutions are overdamped, and the field $\sigma$ reaches $\sigma = \sigma_0$ monotonically at a time $t \gg H^{-1}$. On the other hand, if $m > m_{\text{max}}$ the underdamped solutions are oscillatory, and not of the desired form $\dot{\sigma} = f(\sigma)$.

### 3.2.3 Multifield example

Based on the past example we can generalise to the case of a curved trajectory. Let’s first consider the following two field model,

$$W(\phi, \sigma) = a(\phi) + b(\phi)(\sigma - \sigma_0)^2, \quad (3.65)$$

where we have that $\gamma_{11} = f(\sigma)$, $\gamma_{22} = 1$, $\gamma_{21} = \gamma_{12} = 0$, so $R = f''/2f - f''/f$. This model describes a trajectory with $\sigma = \sigma_0$, so $W_\phi = a'$ and $W_\sigma = 0$. This model is a solution by construction of the HJ with $\sigma = \sigma_0$. The potential in this trajectory is

$$V = 3a^2 - \frac{2a'^2}{f}. \quad (3.66)$$
Then, the orthogonal direction has a non vanishing mass. To see this first let us calculate the second derivative of the potential with respect to $\sigma$

$$V_{\sigma\sigma} = 12ab - 8\frac{a'b'}{f} - 16b^2 - 2(a')^2 \frac{d}{d\sigma} \left( \frac{f'}{f^2} \right). \tag{3.67}$$

We have that $W_{NN}^2 = N^a N^b W_{ab} = N^\sigma N^\sigma W_{\sigma\sigma} = 2b$ so then the entropy mass is

$$\mu^2 = 6WW_{NN} - 4W_{NN}^2 + 2\dot{W}_{NN} = 12ab - 16b^2 - 8a'b'/f. \tag{3.68}$$

On the other hand we have that the turning rate is $\Omega = \frac{f'a'}{f^3}$, so the entropy mass is also given by,

$$\mu^2 = V_{NN} + \epsilon H^2 \mathcal{R} + 3\Omega^2,$$

$$= 12ab - 8a'b'/f - 16b^2 + 2(a')^2 \frac{d}{d\sigma} \left( \frac{f'}{f^2} \right) + 3\frac{f'^2}{f^3} a^2 + \epsilon H^2 \mathcal{R}. \tag{3.69}$$

We have that $\epsilon = \frac{2a'^2}{f'^2}$, so $\epsilon H^2 \mathcal{R} = a^2 \left( \frac{f'^2}{f^3} - 2\frac{f''}{f^2} \right)$, so finally we get,

$$\mu^2 = 12ab - 16b^2 - 8a'b'/f, \tag{3.70}$$

which coincides with the expression derived using the formula (3.57). Note that the last term can be written as $4\sqrt{2} \epsilon a b^2$, so it’s expected to be a correction to first two terms.

### 3.2.4 Observational consequences

To study the observational consequences of the bound (3.58) we will study how a massive field modifies the two and three point functions of the curvature mode $\zeta$. To do so, we will introduce the coupling $\Omega < H$ to calculate the corrections to the curvature power spectrum. Apart from considering $\sigma_k$ we will also use $u_k = a\sqrt{2}\epsilon k \zeta_k$, whose initial conditions are given by the Bunch-Davies vacuum. Its classical solutions is given by,

$$u_k(\tau) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) e^{-ik\tau}. \tag{3.71}$$

$$\tag{3.72} \label{eq:uk_classical}$$
We quantise the fields by promoting them to quantum operators as,

\[
\hat{\sigma}_k(\tau) = \sigma_k b_k + \sigma^*_k \tilde{b}_k, \\
\hat{u}_k(\tau) = u_k a_k + u^*_k \tilde{a}_k,
\]

(3.73)

where \(a \) and \(b \) obey the usual commutation relations \([a_k, a^\dagger_{k'}] = [b_k, \tilde{b}^\dagger_{k'}] = (2\pi^3)\delta^3(k-k')\).

To calculate the effect of the massive field \(\sigma_k\) over the power spectrum we introduce the interaction Hamiltonian,

\[
H_I = -\int \frac{d^3k}{(2\pi^3)} \frac{\Omega}{H^2} \sigma_k \left( \frac{du_k}{d\tau} + \frac{u_k}{\tau} \right).
\]

(3.74)

Where \(H\) is constant as we have used that \(a(\tau) = -1/H\tau\). Correlation functions are calculated using the \(\text{in-in}\) formalism \cite{[158]}

\[
\langle O \rangle = \left\langle \left[ \bar{T} \exp \left( i \int_{-\infty}^{\tau} d\tau' H_I(\tau') \right) \right] O_0 \left[ \bar{T} \exp \left( i \int_{-\infty}^{\tau} d\tau' H_I(\tau') \right) \right] \right\rangle,
\]

(3.75)

where the operator \(O_0\) is in the interaction picture and \(\bar{T}, T\) are anti time ordered and time ordered operators. The two point correlation function is given by

\[
\langle u_k u^*_k \rangle = \langle u^0_k u^0_{-k} \rangle + \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau} d\tau'' \langle H_I(\tau') u^0_k u^0_{-k} H_I(\tau'') \rangle
- 2\text{Re} \left[ \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau} d\tau'' \langle u^0_k u^0_{-k} H_I(\tau') H_I(\tau'') \rangle \right].
\]

(3.76)

We write this as

\[
\langle u_k u^*_k \rangle = P_u(k) \left[ 1 + \frac{\Omega^2}{H^2} (I_1 + I_2) \right],
\]

(3.77)

where \(P_u(k) = 1/2k^3\tau^2\) is the power spectrum for \(u^0_k\). The first integral gives

\[
\frac{\pi}{8} \left| \int_0^{\infty} \frac{dx}{\sqrt{x}} H^{(1)}_\nu(x) e^{ix} \right|^2.
\]

(3.78)

The second integral is

\[
-\frac{\pi}{4} \text{Re} \left[ \int_0^{\infty} \frac{dx}{\sqrt{x}} H^{(1)}_\nu(x) e^{-ix} \int_x^{\infty} \frac{dx'}{\sqrt{x'}} H^{(2)}_\nu(x') e^{-ix'} \right].
\]

(3.79)

The first integral diverges at 0 but the result of the sum is finite. It can be done
Holographic inflation

Fig. 3.1 Integral evaluated for $\nu = \sqrt{\frac{9}{4} - \frac{\mu^2}{H^2}}$.

numerically, as has been shown by [153], the result being in Fig3.1. We note that the correction is order 1 and grows exponentially as the mass of the field goes to zero. This case has to be taken with care as the solutions for $\sigma_k$ are different. See [159], where this was done. The result is

$$\langle u_k u^*_k \rangle = P_u(k) \left( 1 + \frac{\Omega^2}{H^2} [A_1 - A_2 (\ln(-k\tau)) + \ln^2(-k\tau)] \right). \quad (3.80)$$

where are numerical coefficients, $A_1 \simeq -2.11$, $A_2 \simeq 1.46$

Now for cases outside the mass bound the results are interesting to compare. This is achieved by writing $\nu = i\mu$ where $\mu$ is real. We have that the phase of the solution for $\omega_k$ is now a damping factor as expected for massive fields in de Sitter. The power spectrum is now

$$\langle u_k u^*_k \rangle = P_u(k)[1 + \frac{\Omega^2}{H^2} (I_A + I_B)], \quad (3.81)$$

where

$$I_A = \frac{\pi}{4} e^{-\pi \mu} \left| \int_0^\infty \frac{dx}{\sqrt{x}} H^{(1)}_{i\mu}(x)e^{ix}, \right|^2 \quad (3.82)$$

$$I_B = -2e^{-\pi \mu} \text{Re} \left[ \int_0^\infty \frac{dx}{\sqrt{x}} H^{(1)}_{i\mu}(x)e^{-ix} \int_x^\infty \frac{dx'}{\sqrt{x'}} H^{(2)}_{i\mu}(x')e^{-ix'} \right]. \quad (3.83)$$

And, as expected the correction starts to decay for $-k\eta \sim \mu$

Three point function

We first start by considering the calculation of quasi single field inflation for $m^2/H^2 \leq 9/4$. We have, following, [25, 160] that the main contribution will come from the
interaction Hamiltonian,

\[ H_I = H_1 + H_2 = \dot{\zeta}^2 \sigma + \dot{\zeta} \sigma. \] (3.84)

We will be interested in the effect of \( \sigma \) over the squeezed limit of the three point function. Thus we need to calculate

\[
\langle \zeta_q \zeta_{k_1} \zeta_{k_2} \rangle_{\sigma} = -2\text{Re} \left\{ \zeta_q(\tau_0) \zeta_{k_1}(\tau_0) \zeta_{k_2}(\tau_0) \int_{-\infty}^{\tau_0} \frac{d\eta_1}{\tau_1} \int_{-\infty}^{\tau_1} \frac{d\tau_2}{\tau_2} H_1(\tau_1) H_2(\tau_2) \right\} + \left\langle \int_{-\infty}^{\tau_0} \frac{d\eta_1}{\tau_1} H_1(\tau_1) \zeta_q(\tau_0) \zeta_{k_1}(\tau_0) \zeta_{k_2}(\tau_0) \int_{-\infty}^{\tau_1} \frac{d\tau_2}{\tau_2} H_2(\tau_2) \right\} + H_1(\tau_1) \leftrightarrow H_2(\tau_2),
\] (3.85)

where we have introduced a regulator \( \tau_0 \) to account for any time dependence that can appear in \( \sigma \). We will consider the limit where \( q \ll k_1, k_2 \). Hence the main contribution comes from,

\[
\zeta_q(\tau_0) \zeta_{k_1}(\tau_0) \zeta_{k_2}(\tau_0) \int_{-\infty}^{\tau_0} \frac{d\eta_1}{\tau_1} \int_{-\infty}^{\tau_1} \frac{d\tau_2}{\tau_2} \zeta_{k_1}^*(\tau_1) \zeta_{k_2}^*(\tau_1) \sigma_q(\eta_1) \sigma_{q_1}^*(\eta_2) \zeta_q^*(\tau_2) + \text{cc.}
\]

\[
+ \zeta_q(\tau_0) \zeta_{k_1}^*(\tau_0) \zeta_{k_2}^*(\tau_0) \int_{-\infty}^{\tau_0} \frac{d\eta_1}{\tau_1} \zeta_{k_1}(\tau_1) \zeta_{k_2}(\tau_1) \sigma_q(\eta_1) \int_{-\infty}^{\tau_1} \frac{d\tau_2}{\tau_2} \sigma_{q_1}^*(\eta_2) \zeta_q^*(\tau_2) + \text{cc.} \quad (3.86)
\]

Then as we want to have that \( 1/k \ll 1/q \) we can write the integral as

\[
\langle \zeta_q \zeta_{k_1} \zeta_{k_2} \rangle_{\sigma} \sim \frac{\langle \zeta_q \sigma-q(\eta_0) \rangle \langle \zeta_{k_1} \zeta_{k_2} \sigma_q(\eta_0) \rangle}{P_\sigma(q, \eta_0)} \propto P_q P_k \left( \frac{q}{k} \right)^{\frac{3}{2} - \nu},
\] (3.87)

where there is a no analytic dependence on the mass of the field. This distinctive signature could be observed by looking at the dark matter distribution. Check [161] for observational prospects. For the case of massive fields \( m^2/H^2 > 9/4 \) there is an interference effect and the squeezed limit,

\[
\langle \zeta_q \zeta_{k_1} \zeta_{k_2} \rangle_{\sigma} \propto P_\zeta(q) P_\zeta(k) \left( \frac{q}{k} \right)^{3/2} \cos(\mu \log \frac{q}{k} - \phi_0)
\] (3.88)

where as before, \( \nu = i\mu \). This result enters into what is also known as cosmological collider. These signatures could be observed with future surveys by looking, for example, at the dark matter distribution or the 21 cm line [162, 109, 161].
3.3 Inflation from a conformal field theory

The main difference between inflation and pure de Sitter space time is that inflation requires time translations to be broken. This implies an evolving background that differs from the fixed radius of de Sitter. Because the curvature mode from inflation is frozen after it leaves the horizon, all its physical effects will be proportional to a massless de Sitter field. To be more precise, correlation functions after leaving the horizon are still dictated from a CFT and thus inflation behaves as if it were dual to a 3 dimensional conformal field theory at the horizon. Now as inflation implies a broken de Sitter spacetime it is also possible to break the symmetries of a CFT and study the flow of those operators to describe inflation. This is done by deforming the CFT by a nearly marginal operator \[163\],

\[
S = S_{\text{CFT}} + \phi \int d^3x O(x), \tag{3.89}
\]

where \(\phi\) is a scalar coupling and the operator has dimension \(\Delta \equiv 3 - \lambda\). Because we know the dimension \(\Delta\) of \(O\), its 2 point function is completely determined up to a constant \(c\),

\[
\langle O_k O'_k \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') c k^{3-2\lambda}. \tag{3.90}
\]

The correlation functions for inflation are contained in the correlation function of the stress energy tensor of the full theory \(T_{ij}\), with the trace \(T\) corresponding to the scalar fields, and the trace free part \(T^s\) to the graviton. For the conformally invariant case \(\langle TT \rangle = 0\) and \(\langle T^s T^s \rangle = ck^3\) which is expected as there is no curvature mode, so the degrees of freedom come only from the graviton. Diffeomorphisms invariance implies that \(k^i \langle T_{ij} T_{kj} \rangle = 0\) so we have that,

\[
\langle T_{ij}(\vec{k}) T_{kj}(-\vec{k}) \rangle' = \frac{1}{4} \left[ \delta^i_k \delta^j_l \langle T_k T_{-k} \rangle' + \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k - \delta^i_l \delta^j_k \right) \langle T^s_k T^s_{-k} \rangle \right], \tag{3.91}
\]

where \(\delta^i_j \equiv \delta_{ij} - k_i k_j / k^2\). The two point function can be calculated as,

\[
\langle T_{ij} T_{kj} \rangle = \langle T_{ij} T_{kj} e^{-\phi \int d^3x O} \rangle = \langle T_{ij} T_{kj} \rangle_0 - \phi \int d^3x \langle T_{ij} T_{kj} O(\vec{x}) \rangle_0
\]

\[
+ \frac{\phi^2}{2} \int d^3 \vec{x} d^3 \vec{y} \langle T_{ij} T_{kj} O(\vec{x}) O(\vec{y}) \rangle_0, \tag{3.92}
\]
where the subindex 0 implies that the correlator is evaluated in the CFT. The correlation functions are constrained by the Ward identities and then we have that

\[
\langle T^i_i(x) T^j_j(y) O(z) \rangle_0 = \lambda (\delta(x - z) \langle T^k_l(y) O(z) \rangle_0) + \chi \leftrightarrow \tilde{w},
\]

(3.93)

\[
\langle T^i_i(x) T^j_k(y) O(z) O(w) \rangle_0 = \lambda (\delta(z - w) \langle O(z) O(w) \rangle_0) + \chi \leftrightarrow \tilde{w},
\]

(3.94)

\[
\langle T^i_i(x) T^j_k(y) O(z) O(w) \rangle_0 = \lambda (\delta(x - z) \delta(y - w) \langle O(z) O(w) \rangle_0) + \chi \leftrightarrow \tilde{w},
\]

(3.95)

Using this we have that the scalar part receives the first correction at order \( \phi^2 \) and thus

\[
\langle T^i_k T^j_k \rangle' = \phi^2 \lambda^2 \langle O_k O'_k \rangle' = c \phi^2 \lambda^2 k^{3-2\lambda}.
\]

(3.96)

The scalar to tensor ratio is then \( r = \phi^2 \lambda^2 = 16 \epsilon \) which implies that we have \( \phi \lambda = \pm 4 \sqrt{\epsilon} \).

### 3.3.1 \( \beta \) functions

Conformal field theories are fixed points in RG flows. Then, any deformation to the CFT will generate a flow that can be characterised by a \( \beta \) function, for example in (3.89), for the coupling \( \phi \). Interpreting inflation holographically implies that the RG flow is equivalent to time translations in the bulk. This follows from the following, let us recall that a de Sitter background \( ds^2 = -dt^2 + e^{Ht} d\vec{x}^2 \) is invariant under the transformations,

\[
t \rightarrow t - H^{-1} \log \lambda, \quad \vec{x} \rightarrow \lambda x.
\]

(3.97)

Hence, time translations are associated with scale transformations on the boundary theory. Furthermore then we have that \( t \rightarrow \infty \) \((\tau \rightarrow 0)\) is associated with the UV \((\text{or } \vec{x} \rightarrow 0)\), whereas \( t \rightarrow -\infty \) implies flowing to the IR. The running of the coupling can be written as a \( \beta \) function equation by identifying the scale of the coupling \( \mu \) with the scale factor \( a \). Then,

\[
\beta = \frac{d\phi}{d \log \mu} = \frac{\dot{\phi}}{H},
\]

(3.98)

where in the last line we have replaced \( \mu \propto a \). Zeros of this \( \beta \) function are interpreted as de Sitter fixed point, then we have that at \( t \rightarrow -\infty \) inflation starts flowing from a fixed IR point towards a UV fixed point at \( t \rightarrow \infty \), so time evolution generates an inverse RG flow. As the flow starts at the UV fixed point at \( \tau \rightarrow 0 \), degrees of freedom
are integrated in, as the theory flows towards the IR. This is consistent with inflation, where we are only allowed to observe correlation functions after they have crossed the horizon at a conformal time $|\tau_e| \ll 1$. At that time, the spacetime is very close to the boundary and all the physics is very constrained by the conformal symmetry. Then, all the physics happening at earlier times, for example the initial conditions, have been washed out due to the inflationary expansion.

Then, the statement from (3.89) implies that a nearly marginal deformation will start a flow towards an IR point, with inflation happening within that flows. When the theory is deformed by relevant operators, the flows does not go to a fixed point on the IR, although it does not necessarily mean that inflation is not achieved. We will study several examples where the flows is modified due to action of matter and the geometry of the target space, in order to understand the implications of different RG flows for inflation.

In order to do so we will be studying different models of inflation by their $\beta$ functions. Progress was made in [164] where the running of the inflaton at large values $\phi \gg 1$ was used to categorise different models of inflation into universality classes. We will focus on understanding the main features of the $\beta$ functions in order to apply this to $\alpha$ attractors [122, 120] and then to study different models of string inflation.

To make the RG flow quantitatively precise it is important that the evolution on the bulk matches the flow of the $\beta$ function. This is achieved through the Hamilton Jacobi equations, in fact the above equation can be written as

$$\beta_\phi = \frac{2W_\phi}{W} \propto \sqrt{t},$$

and henceforth all inflationary trajectories can be understood in terms of their holographic RG flows, provided that they are solutions to the Hamilton Jacobi eqns. (3.44). We will first start by examining the equations (3.44) in the case of single field inflation. Assuming that the potential has a minima implies that,

$$V' = 0 = 9W' \left(\frac{3}{2}W - W''\right).$$

Hence either $W' = 0$ or $W = A \cos(\sqrt{\frac{3}{2}}\phi + B)$. The latter case appears as we allow the field to flow past the minima, as we will see later. For the case where $W' = 0$, we can write the second derivative of the potential as

$$\frac{V''}{V} = 3W'' \left(\frac{3}{2}W - W''\right).$$
This result implies a similar bound for the mass of a scalar field, although we have imposed an extra condition. It is related to the existence of supersymmetric solutions in (A)dS slices of domain wall [165, 166]. Moreover in this case the Hamilton Jacobi equations solutions are Killing spinors defined by the potential so solutions with $W' = 0$ are stable. As this implies a regime where $\beta = 0$, this can be interpreted as the existence of a supersymmetric fixed point.

**Constant potential**

We will continue by studying the case of a scalar field under a constant potential $V = V_0$. This example can actually achieve inflation and is called *ultra slowroll* inflation[167, 168]. For us, in particular, it will be interesting because the Hamilton Jacobi equations can be exactly solved. We have, indeed, that the equations of motions are

\[
\ddot{\phi} + 3H\dot{\phi} = 0, \\
H^2 = V_0 + \frac{1}{2}\dot{\phi}^2.
\]  

(3.102)

for $V_0 > 0$. The first equation implies that there is a conserved quantity $d(a\dot{\phi}^3)/dt = 0$, and the second can be integrated if we identify that $W = H$ and $2W_{,\phi} = \dot{\phi}$. This gives,

\[
W = W_0 \cosh \left( \sqrt{\frac{3}{2}}(\phi - \phi_0) \right) + \sqrt{\frac{2}{3}}(W_0)_{,\phi} \sinh \left( \sqrt{\frac{3}{2}}(\phi - \phi_0) \right),
\]  

(3.103)

where the solution depends on the initial conditions, $W_0$ and $(W_0)_{,\phi}$. We can relate them through $(W_0)^2 = \frac{3}{2}W_0^2 - 2V$, and use the conserved quantity to fix $W_0$. Now let us calculate the beta function, $\beta = -W_{,\phi}/W$

\[
\beta = \sqrt{3} \tanh \left( \sqrt{\frac{3}{2}}(\phi - \phi_0) \right) \left( \sqrt{\frac{2}{3}}(W_0)_{,\phi} \coth \left( \sqrt{\frac{3}{2}}(\phi - \phi_0) \right) + 1 \right) \left( \sqrt{\frac{3}{2}}(W_0)_{,\phi} \tanh \left( \sqrt{\frac{3}{2}}(\phi - \phi_0) \right) + 1 \right).
\]  

(3.104)

So the zeros of this function depend on the initial conditions. To make some progress let us examine this function, we have that $V > 0$ then $\frac{2}{3} \frac{(W_0)^2}{W_0^2} = 1 - \frac{4V_0}{3W_0^2} < 1$ as long as we admit $W$ to be real. Then the solution moves between $\pm \frac{3}{2}$ at large $\pm \phi$ going through zero at $\phi_0 + \frac{1}{3} \log \left( \sqrt{\frac{2}{3}}\frac{W_0^2 - 2(W_0)^2}{2(W_0)^2 + 2W_0} \right)$. We can expand on the parameter $\frac{2}{3} \frac{(W_0)^2}{W_0^2}$

\[
\phi_0 + \frac{1}{3} \log \left( \left( \frac{\sqrt{2}}{\sqrt{3}} \frac{W_0^2 - 2(W_0)^2}{2(W_0)^2 + 2W_0} \right) \right).
\]
to get the point where \( \beta \) is zero,
\[
\phi = \phi_0 - 2\sqrt{\frac{2}{3}} (W_0)_{,\phi} + \mathcal{O}\left(\sqrt{\frac{2}{3}} \frac{(W_0)_{,\phi}}{W_0}\right). \tag{3.105}
\]

So the shift of the fixed point is proportion to powers of \( \sqrt{\frac{2}{3}} \frac{(W_0)_{,\phi}}{W_0} \). Now for large \( |\phi| \) we assume that there is no initial kinetic energy, or where the constant of motion has been set to 0, we have that
\[
\beta = \sqrt{\frac{3}{2}} \tanh \left( \sqrt{\frac{3}{2}} (\phi - \phi_0) \right). \tag{3.106}
\]

Then we see that the \( \beta \) function grows to a constant at large \( \phi \), with \( \beta \) being positive or negative depending on the sign of \( (\phi - \phi_0) \). If we consider small deviations from the initial \( \phi_0 \) we get,
\[
\beta = \frac{(W_0)_{,\phi}}{W_0} + \left( \frac{3}{2} - \left( \frac{(W_0)_{,\phi}}{W_0} \right)^2 \right) (\phi - \phi_0) \]
\[
- \frac{(W_0)_{,\phi}}{W_0} \left( \frac{3}{2} - \left( \frac{(W_0)_{,\phi}}{W_0} \right)^2 \right) (\phi - \phi_0)^2 + \mathcal{O}(\phi^3). \tag{3.107}
\]

Because we have that \( V > 0 \) then we have that \( \frac{3}{2} - \left( \frac{(W_0)_{,\phi}}{W_0} \right)^2 = \frac{2V}{W_0^2} > 0 \). Then notice that for a given initial condition the system will evolve to the point where \( \beta = 0 \) which has to be interpreted as rolling towards an IR fixed point, as happens when there is no kinetic energy. One can think of this setup as a potential barrier where a particle will lose its initial kinetic energy and then decay to an equilibrium point.

Importantly, we have confirmed that the theory doesn’t flow from an IR fixed point which makes it sensitive to details about the initial conditions. One can imagine embedding ultraslow roll inflation on an effective field theory, and obtaining similar conclusions. Hence rewriting this example in terms of \( \beta \) function formalises the differences between slow roll and ultra slow roll models. With this in mind we can also calculate the central charge defined by Strominger [52] as
\[
c = \frac{1}{\kappa H^2} = \frac{2}{\kappa W^2} = \frac{2}{(W_0 \cosh \left( \sqrt{\frac{2}{3}} (\phi - \phi_0) \right) + \sqrt{\frac{2}{3}} (W_0)_{,\phi} \sinh \left( \sqrt{\frac{2}{3}} (\phi - \phi_0) \right))^2}, \tag{3.108}
\]
which peaks at $\phi = \phi_0 - 2\sqrt{\frac{7}{3} \frac{W_0}{W}}$ and decays at large $\pm \phi$, so its biggest value is given by the zero of the $\beta$ function. According to the proposed duality the $c$ function should increase as we go along the time direction because it is an inverse RG flow, this means that one is integrating in degrees of freedom as we move towards the UV fixed point. The section of the $\beta$ function for $\phi < \phi_0$ has to be interpreted as it does not add any new information, because there is no physical mechanism to remove the field from its unstable minimal. Any post inflationary evolution will depend on the same details as the inflationary evolution. One can imagine a situation where after inflation new operators kick in modifying the flow, leading to a new flow towards dark energy.

To finish this section let us calculate the perturbations from the constant potential. We first expand the action in terms of a perturbation $f$.

$$\phi(\tau, x) = \phi(\tau) + \frac{f(\tau, \vec{x})}{a(\tau)}$$

(3.109)

where we have switched to conformal time $\tau$. The inflationary action is

$$S = \frac{1}{2} \int d\eta d^3x \left[ (f')^2 - (\nabla f)^2 + \frac{a''}{a} f^2 \right],$$

(3.110)

which is the action of a massless scalar field on a de Sitter background. This is expected from the last calculation where assuming a constant potential implies a massless mode on a de Sitter background. The predictions for this model are similar to inflation but with $n_s = 1$, which is almost ruled out by Planck [4]. A way to circumvent this still using a constant potentials is in the constant roll models [169, 170].

### 3.3.2 $\alpha$-attractors

We will now move to study slow roll inflation starting from $\alpha$ attractors. These, discussed in the last chapter, are simple generalisations of the Starobinsky model [118]. They have the property that by tuning a parameter $\alpha$ its geometry can change, ranging from chaotic inflation to the flatter potential of the Starobinsky model. This is reflected in the predictions, which in terms of $\alpha$, are

$$r = 12\frac{\alpha}{N^2}, \quad n_s - 1 = \frac{1}{N}.$$

(3.111)
Then, there is a map between large $\alpha$ and short field inflation and small $\alpha$ and large field inflation. The origin from this can be seen from the model given by the action,

$$S = \int d^4x \sqrt{-g} \left[ R - \frac{\alpha/2}{(1 - \phi^2/6)^2} (\partial \phi)^2 - \frac{V(\phi)}{\Omega(\phi)} \right].$$

(3.112)

The kinetic energy can be canonically normalised by the change of coordinates, $\phi = \tanh \left( \frac{\sqrt{2}}{\sqrt{3} \alpha} \right)$, which for example for the potential $V = m^2 \phi^2$, leads to the potential,

$$V = V_0 (1 - e^{-\sqrt{2/3} \phi})^2.$$  

(3.113)

For $\alpha \sim \mathcal{O}(1)$ the potential has a plateau for $\phi \sim \sqrt{2/3}$ which corresponds to the Starobinsky model. For large $\alpha$ the first terms of the exponential dominates and we recover chaotic inflation. The $\beta$ function for this model is,

$$\beta = \frac{2 \sqrt{\frac{2}{3\alpha}}}{-1 + e^{-\sqrt{2/3} \phi}} \rightarrow_{\phi \gg 1} \frac{1}{\phi} - \sqrt{\frac{2}{3\alpha}}.$$  

(3.114)

where the last limit is only valid when $\alpha \gg 1$. We see that the $\beta$ functions goes to 0 at large field values. This can be interpreted as the field starting inflation from an UV fixed point, and rolling down the potential breaking the conformal invariance.

We can also calculate the $\beta$ function by solving the equations of motion, which is plotted in Fig. 3.2. Here we have set the initial conditions as $\phi(0) = \phi_0$ and $\phi'(0) = 0$

![Fig. 3.2 $\beta$ function for an RG flow decaying to the origin. Note that the amplitude does not change, but the range of $\phi$ becomes smaller until it reaches zero](image)

which is a fixed point of the $\beta$ function. Then, the inflaton will start rolling down the potential reaching the attractor solutions. There are two main features to notice: first, after starting its evolution from a fixed point there is a flow towards the attractor solution which depends on the geometry of the potential. Secondly, after reaching the
minima at $V' = 0$ the field keeps flowing until it will eventually reach $V = 0$. There it oscillates with the same amplitude but for shorter times for each oscillation due to the friction terms. Eventually it loses all its kinetic energy and freezes at the minima. This comes from the second solution we found in (3.100) that contained an oscillatory piece, and hence it might be interpreted as a fixed point that is reached asymptotically in the future.

To understand the slow roll regime in terms of the $\beta$ function we can solve the equations (3.44) by assuming that there are attractor trajectories. This is achieved when,

$$0 \approx \frac{d\dot{\phi}}{d\phi} = 0,$$

(3.115)

which corresponds to the slow roll conditions, then we can write the $\beta$ function as,

$$\beta = -\frac{1}{\kappa} \frac{1 - (V, \phi^2)}{1 + \sqrt{1 + \frac{(V, \phi^2)}{3V^2}}},$$

$$\approx \frac{1}{\kappa} \frac{(V, \phi^2)}{12V^2} + \frac{1}{8\kappa} \left( \frac{(V, \phi^2)}{3V^2} \right)^2 + \mathcal{O}(\epsilon^3).$$

(3.116)

Hence during the slow roll we have that $\beta \approx \sqrt{\epsilon}$. This approximation breaks down at the end of inflation when the $\beta$ function grows, reaching a maxima at $V = 0$. As this is an attractor solution any dependence on the initial conditions is washed out, hence the flow from the IR fixed point is not imprinted on the observables. On the contrary the ultra slow roll case, the flows depends on its initial conditions.

We can make our model a bit more complicated by coupling to matter as we did in the last chapter. To do so we rewrite the action (3.112) in the Jordan frame and we add a tower of non relativistic heavy fields. In contrast to the past chapter the masses of the tower of states have to be bounded by $\frac{9}{4}H^2$ to be a solution of the HJ equations. We will impose that all the fields saturate the bound. Furthermore we will assume that the massive fields remain decoupled through the inflationary trajectory. Then, we have,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( 1 - \frac{\phi^2}{6} \right)^{2\alpha} R - \frac{1}{2} \left( \frac{\alpha - \alpha^2 \phi^2/6}{(1 - \phi^2/6)^{2-\alpha}} \right) \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right]$$

$$+ \int d^4x \sqrt{-g} \mathcal{L}_m,$$

(3.117)
which after a conformal transformation and canonically normalise the kinetic term the action in Einstein frame becomes,

\[ S = \int d^4x \sqrt{-g} \left[ R - (\partial \varphi)^2 - V(\varphi)/\Omega(\varphi) \right] + \int d^4x \sqrt{-g} \Omega(\varphi) L_m. \]  

(3.118)

Now when varying the action with respect to the inflaton \( \varphi \) we also need to take into account the matter fields which are coupled through the Einstein frame metric. The inflationary potential can be rewritten as,

\[ V_{\text{eff}} = V_0 (1 - e^{-\sqrt{2/3} \varphi})^2 + \rho \sinh^2(\varphi/\sqrt{6\alpha}), \]  

(3.119)

where \( \rho \) is the matter density \( -\rho = T = g_{\mu \nu} T^{\mu \nu} \) obtained by varying the matter action in the Jordan frame. As was explained in the last chapter the influence of the heavy fields is larger at the beginning of inflation. We can thus examine changes on the IR fixed point by studying the \( \beta \) functions. Assuming that the field is away from the origin \( \phi > 3/2 \) we have,

\[ \dot{\phi}^2 = \frac{\sqrt{2}}{6\alpha} \rho \sinh \left( \sqrt{\frac{2}{3\alpha}} \phi \right). \]  

(3.120)

Then replacing this expression into (3.44), we get \( W = -\sqrt{3} e^{-2\sqrt{2/3} \varphi} \). Hence we have for the case of \( \alpha = 1 \) that the \( \beta \) function is,

\[ \beta = \sqrt{\frac{2}{3}} \left( -\frac{4}{\rho e^{\sqrt{2/3} \phi} + 4} + \frac{2}{e^{\sqrt{2/3} \phi} - 1} + 1 \right), \]  

(3.121)

which tends to a constant different from zero when \( \rho \neq 0 \), in fact for \( \rho \gg 1 \), \( \beta \approx \sqrt{2/3} \). The \( \beta \) function is plotted in fig 3.3. This result suggests that for \( \alpha \lesssim 1 \) there it might not be an IR fixed point when including matter. Another way of seeing this result is by calculating the \( \beta \) function in the slow roll approximation. This gives,

\[ \beta \approx \sqrt{\frac{2}{3\alpha}} \coth \left( \frac{\phi}{\sqrt{6\alpha}} \right) + \mathcal{O}(1/\rho), \]  

(3.122)

which tends to \( \sqrt{2/3} \) for large \( \phi \). So for large density the result does not depend on the amount of matter. We see that the solution at the beginning of inflation are modified. In this sense, the non existence of an IR fixed point, might mean that there is no de Sitter solution previous to inflation.
In the last chapter we saw that for the case when $\alpha \gg 1$ we have that all corrections become subdominant, and any effect of adding matter is negligible. As expected this is translated to the $\beta$ function. In fact for $\alpha \gg 1$, the flow from the IR fixed point is not modified in the presence of matter.

### 3.3.3 Multifield $\alpha$ attractors

Most of the simplicity of $\alpha$ attractors models stem from the fact that they inherit symmetries from supergravity. For example the complex part of the Kahler manifold is set to 0 by hand, and thus the models only have one scalar field. By relaxing this assumption, there is an interesting regime that has drawn recent attention [171, 172].

Its action is given by

$$g^{-1}L = \frac{1}{2}((\partial \phi)^2 + \frac{3\alpha}{4} \sinh^2(\sqrt{\frac{2}{3\alpha}} \phi)(\partial \theta)^2 - V(\phi, \theta),$$

whose equations of motion are,

$$\ddot{\phi} + 3H \dot{\phi} + V_{\phi} - \frac{1}{2} \sqrt{\frac{3\alpha}{2}} \sinh \left( \frac{3\alpha}{2} \phi \right) \dot{\theta}^2 = 0,$$

$$\ddot{\theta} + 3H \dot{\theta} + \frac{V_{\theta}}{\frac{3\alpha}{2} \sinh^2(\sqrt{\frac{3\alpha}{2}} \phi)} + \frac{2\dot{\theta}^2}{\frac{3\alpha}{2} \tanh(\sqrt{\frac{3\alpha}{2}} \phi)} = 0,$$

and the Friedmann equation

$$3H^2 = \frac{1}{2} \left( \dot{\phi}^2 + \frac{3\alpha}{2} \sinh^2(\sqrt{\frac{3\alpha}{2}} \phi) \dot{\theta}^2 \right) + V(\phi, \theta),$$
The dynamics of these kind of models depends on the geometry mostly because the

angular field gets a mass that depends on the \( \alpha \). For the simpler class of models near the region where \( \theta = 0 \) the mass is given by 
\[
m_\theta^2 = 2V(1 - 1/3\alpha)
\]
For large \( \alpha \) the mass is larger than \( m^2 \geq H^2 \) and then the angular field gets stuck and it does not participate in the dynamics. For \( \alpha \sim \mathcal{O}(1/3) \) the system behaves as multifield but interestingly the predictions does not depend on the second field, but remain unchanged from \( \alpha \) attractors.

We can understand this from the point of view of the \( \beta \) functions, as is shown from Fig.3.4, where we solve the equations of motion of the system. There, we see that during inflation, the \( \beta \) functions for both fields depends heavily on the choice of \( \alpha \). Interestingly for the case when \( \alpha = 1/3 \) the flows coincide from the end of inflation towards the IR. For the other cases we see that the \( \beta \) function of the fields

Fig. 3.4 \( \beta \) function for multified \( \alpha \) attractors for different values of \( \alpha \). The blue line is the radial direction whereas the orange line is the angular one.
that do not participate on inflation goes to zero. This is expected as they are not
dynamic at horizon exit. The fact that the flows coincide makes the fields effectively
indistinguishable and is another way of understanding the results of [171]. This result
then suggest that fields with similar flows are indistinguishable, and hence one can
think of a situation where several fields produce inflation with the same flow. So,
irrespective the microscopic details of the different fields they will produce the same
predictions.

As we saw in the first part of this chapter Hamilton Jacobi equations are only valid
for scalar fields with masses $m \leq 3H/2$. This implies that if we want to interpret
multifield $\alpha$ attractors holographically then the entropy mass cannot be arbitrary. This
is in fact the case, as it is [171],

$$\mu^2 \sim e^{-\sqrt{\frac{3\pi}{\alpha}}} H^2.$$  \hspace{1cm} (3.127)

Hence as expected for small values of $\alpha$ this is supressed enough to allow an RG flow
between the two fixed pointes whereas for large $\alpha$, whereas for large $\alpha$ might not be
towards a fixed point. This can be understood as the dimension of the operators
perturbing the CFT will be as,

$$\Delta = \frac{3}{2} \pm \sqrt{\frac{9}{4} + e^{-\sqrt{\frac{3\pi}{\alpha}}}}.$$  \hspace{1cm} (3.128)

Then, when the exponential does not decay fast enough relevant operators will move
the flow away from a fixed point, whereas in the case when the entropy mass is
negligible, the flow will be towards an IR fixed point.

### 3.4 String theory examples

We will now review several examples of of inflation in string theory where we apply the
$\beta$ function formalism. Achieving inflation in string theory is a still pending issue, and
there many difficulties faced when building (meta) stable de Sitter vacuum solutions
(see [34] for a review of the topic).

One important aspect of inflation in string theory is the presence of many moduli
fields, which arise generically when compactifying the 10 dimensional manifold to a
4 dimensional one. Furthermore besides the classical superpotential there are large
corrections that might spoil the solutions, giving a large mass to the inflaton and
destabilising the moduli. Achieving control over these effects is a key part of the
stabilisation mechanisms to obtain de Sitter solutions. One important step was done in [173] where the de Sitter vacuum is achieved by uplifting a known stable AdS solution, which is present with D-branes. In these models the inflaton is given by the separation between two brane or between a brane and an anti brane. Another alternative model is the large volume scenario (LVS) [43, 174], where all the corrections are negligible due to a large volume moduli. Here the inflaton is the modulus field that is displaced from the stable solution.

In what follows we will explain different examples of inflation in string theory in both scenarios described above. We will also consider the coupling of matter, because as we have seen can modify the $\beta$ functions in a drastic way. As we discussed previously, the Hamilton Jacobi impose a bound over the masses of all the scalar fields. Masses with values $\sim H$ can still contribute to inflation and modify its dynamics. In what follow we will take ignore those effects unless mentioned. This is justified in the case of fields saturating the bound, and then we will assume that moduli are stabilised at that value.

### 3.4.1 Kahler Moduli Inflation

We first start with the Kahler moduli inflation model [175]. These are based on the LVS, and as such we first start by reviewing that procedure for moduli stabilisation. The main ingredients of these model are a IIB compactified Calabi Yau whose superpotential is given by

$$ W = W_0 + \sum A_i e^{-\alpha T_i}, $$

where $W_0$ is the integrated out complex structure moduli and $T_i, i = 1, .., N$ are Kahler moduli fields. We have for each $T_i = \tau_i + i c_i$ where $\tau_i$ is a four-cycle volume and $c_i$ is the associated axion. The classical Kahler potential depends only on the volume as

$$ K = -2 \log(V), $$

where $V$, the volume is given by the Kahler moduli. For Kahler moduli inflation these are written as

$$ V = \tau_b^{3/2} - \sum \tau_i^{3/2}. $$

where $\tau_b \gg \tau_i$ controls the size of the volume while the other moduli fields parameterise the holes of the compactification. There are other ways of writing the volume and
we will get back to those for other examples, but meanwhile let us stick to this
configuration. In any case, for any of these models the main quantum corrections to
the Kahler potential are proportional to the Euler number. These can be written as,

$$
\mathcal{K} = -2 \log(\mathcal{V} + \frac{\chi}{2g_s^{3/2}}).
$$

The scalar potential for supergravity is given by

$$
V = e^{\mathcal{K}} \left[ G^{ij} \partial_i W \partial_j \bar{W} + G^{ij} \left( (\partial_i \mathcal{K}) W \partial_j \bar{W} + \text{c.c.} \right) \right].
$$

(3.3)

Now in the case where $\mathcal{V} \gg 1$, up to leading order in the volume, the scalar potential
is,

$$
V = \sum_i \frac{8(a_i A_i)^2}{3\mathcal{V} \lambda_i} \sqrt{\tau_i} e^{-2a_i \tau_i} - \sum_i 4 \frac{a_i A_i}{\mathcal{V}^2} W_0 \tau_i e^{-a_i \tau_i} + \frac{3\xi W_0^2}{4\mathcal{V}^3}.
$$

(3.4)

The potential can be minimised with respect to the 4 cycles $\tau_i$, giving the relation
$$
a_i \tau_i e^{-a_i \tau_i} = \frac{3\lambda_i}{2\mathcal{V}} (\frac{1-a_i \tau_i}{\frac{1}{2} - 2a_i \tau_i}) \sqrt{\tau_i},
$$
so then one finds that in the large $\mathcal{V}$ limit $a_i \tau_i \approx \ln \mathcal{V}$. Replacing into the potential one finds that

$$
V = -\frac{3W_0^2}{2\mathcal{V}^3} \left( \sum_{i=1}^n \left( \frac{\lambda_i}{a_i^{3/2}} \right) (\ln \mathcal{V})^{3/2} - \frac{\chi}{2} \right),
$$

(3.5)

which clearly is an AdS minima. In order to brings this solution into a Minkowski vacuum one is required to add other elements into play [174]. This effect can be captured by the uplift potential $V = \frac{D}{\mathcal{V}^2}$, with $1 \leq \gamma \leq 3$ depending on the mechanism employed. The masses of the moduli are

$$
m_{\tau_i}^2 = \frac{W_0^2}{\mathcal{V}^2} \ln \mathcal{V}^2, \quad m_{\mathcal{V}}^2 = \frac{W_0^2}{\mathcal{V}^3 \ln \mathcal{V}},
$$

(3.6)

so we have that the volume mode is the lightest particle. Now we are in position to
discuss Kahler moduli inflation. This is achieved by displacing one of the small moduli
away from its minima, and leaving the rest to stay stabilised. Being the inflaton the
modulus $\tau_n$ with $a_n \tau_n \gg \mathcal{V}^2$ the potential can be written as,

$$
V = -\frac{3W_0^2}{2\mathcal{V}^3} \left( \sum_{i=1}^n \left( \frac{\lambda_i}{a_i^{3/2}} \right) (\ln \mathcal{V})^{3/2} - \frac{\chi}{2} \right) + \frac{\gamma W_0^2}{\mathcal{V}^2} - \frac{4\tau_n A_n e^{-a_n \tau_n}}{\mathcal{V}^2}.
$$

(3.7)
Now in order to achieve enough inflation the volume modulus has to be heavy enough to not destabilise during inflation. This is achieved for \( D = \gamma W_0^2 / V^2 \) and for values of the volume of order \( V \sim 10^5 - 10^6 \) [175]. After minimising with respect to the volume modulus the resulting potential is the following,

\[
V_{\text{infl}} = V_0 - \frac{4 \tau_n W_0 a_n A_n e^{-a_n \tau_n}}{V^2}, \tag{3.8}
\]

where, \( V_0 = \frac{\beta W_0^2}{V^2} \). In order to get the inflationary trajectory we have to canonically normalise this moduli field. The kinetic term coupling is given by \( K_{\bar{n}n} = \frac{3 \lambda}{8 \sqrt{\tau_n v}} \) then the canonically normalise field is,

\[
\tau_n^c = \sqrt{\frac{4 \lambda}{3 V \tau_n}}, \tag{3.9}
\]

and the potential becomes,

\[
V = V_0 - \frac{4 W_0 a_n A_n}{V^2} \left( \frac{3 V}{4 \lambda} \right)^{2/3} \left( \tau_n^c \right)^{4/3} e^{-a_n \left( \frac{3 V}{4 \lambda} \right)^{2/3} (\tau_n^c)^{4/3}}. \tag{3.10}
\]

This potential gives slow roll inflation, provides that the volume modulus remain stable. In terms of the number of efolds \( N_e \) the slow roll parameters are given by,

\[
\epsilon \simeq \left( \frac{3 \lambda_n}{8 a_n^{3/2} V} \right) \frac{1}{N_e^2 \sqrt{a_n \tau_n}}, \quad \eta \simeq -\frac{1}{N_e}, \tag{3.11}
\]

so \( \epsilon \ll \eta \) for \( \eta \ll 1 \) and large \( V \ll 1 \). This in turn implies, under general assumptions, that \( r \ll 5 \times 10^{-10} \) which is undetectable.

**Including matter**

We want to study the effect of chiral matter on the action. To do so we need to modify the Kahler metric and the superpotential. We also need to take care of the fact that the Kahler potential is of no-scale type, and that all the terms in the superpotential are non perturbative. It follows that the minimal coupling of a matter field \( U \) is given by,

\[
\mathcal{K} = -2 \log(V + \frac{\chi}{2 \delta x}) + f(\tau_s, \bar{\tau}_s) U \bar{U},
\]

\[
W = W_0 + A_s e^{-a_s T_s} (1 + U^2), \tag{3.12}
\]
where \( f(\tau_s, \bar{\tau}_s) \sim \frac{\tau_s^{1/3}}{\sqrt{V_{2/3}}} \) as indicated in [176]. To calculate the contribution to the scalar potential we assume that the minima of the chiral matter field does not destabilise the de Sitter minima. We get that, at leading order in the volume, the contribution to the potential is,

\[
V_{\text{infl}} = \left( V_0 - \frac{4 \tau_n W_0 a_n A_n e^{-a_n \tau_n}}{\lambda^2} \right) (1 + U^2) - \frac{4 V_0 W_0 a_n A_n e^{-a_n \tau_n}}{\lambda^2} U^2 + \frac{f_s(\tau_s)}{V_{2/3}^2} W_0^2 U^2,
\]

(3.13)

where the first term comes from \( G^{\tau_s \bar{\tau}_s} \partial_{\tau_s} W \partial_{\bar{\tau}_s} \bar{W} \), and the second term from

\[
K^{U \bar{U}} \partial_U K W \partial_{\bar{U}} \bar{W} = \frac{2}{f_s(\tau_s)} f_s(\tau) U W \partial_{\bar{U}} \bar{W},
\]

(3.14)

The last term appears from

\[
K^{U \bar{U}} \partial_U K W \partial_{\bar{U}} K \bar{W} = f_s(\tau_s) W_0^2 U^2.
\]

(3.15)

Fig. 3.5 \( \beta \) function for Kahler moduli inflation with and without matter.

The rest of the contributions are subdominant in volume or are suppressed by \( \exp(-a_n \tau_s) \). Assuming that \( f_s(\tau_s) = \frac{\tau_s^{1/2}}{V_{2/3}} \), the potential in terms of the canonically
normalised field $\tau_n^c$ is,

$$
V = V_0 - \frac{4W_0 a_n A_n}{\sqrt[2/3]{4\lambda}} \left( \frac{3V}{4\lambda} \right)^{2/3} (\tau_n^c)^{4/3} e^{-a_n \left( \frac{3V}{4\lambda} \right)^{2/3} (\tau_n^c)^{4/3} (1 + U^2)} - \frac{4V_0 W_0 a_n A_n}{\sqrt[2/3]{4\lambda}} e^{-a_n \left( \frac{3V}{4\lambda} \right)^{2/3} (\tau_n^c)^{4/3} U^2} + \frac{W_0^2}{\sqrt[8/3]{4\lambda}} \left( \frac{3V}{4\lambda} \right)^{1/3} (\tau_n^c)^{2/3} U^2. \quad (3.16)
$$

Where we include the last term as it is of order of $V_0$. Let us note that the action has a minima for $U = 0$ as expected. Now to analyse the effect of adding matter, first observe that the first term is just a rescaling of the potential, so it does not affect inflation, whereas the last term is new and can change the slope of the potential. The second term is suppressed by the volume when we change to canonically normalised variables.

We are in position to study the $\beta$ function to see whether this potential; has an holographic UV fixed point. As we the inflationary trajectories are well approximated by the slow roll regime we have that

$$
\beta = -\frac{d \log V}{d \phi_c} \propto \sqrt{\epsilon}, \quad (3.17)
$$

where $\phi_c$ is a canonically normalised field. We see from (3.11) that $\beta \propto \phi_c^{-1/2}$. Given that the field range is quite small in this model, it is hard to assume that all trajectories will flow from a fixed point. Indeed, it has been mentioned that large values of $\phi$ could destabilise the other moduli. We have included matter in order to study possible modifications before massive fields relax. We have that including matter the fixed points moves to a larger $\phi$, as it is plotted in fig 3.5. Although this is not a problem for the effective descriptions we are studying, other possible effects could kick in [177] at those field displacements making the flow to an IR fixed point very unlikely.

### 3.4.2 Fibre Inflation

In order to continue investigating the LVS let us study another class of models. Fibre inflation is another example using the LVS to stabilise the moduli, with the important difference that the geometry of the volume is given by a fibration of a submanifold. The simplest realisation of fibre inflation includes three Kahler moduli, where the volume is given by,

$$
V = \alpha \left( \sqrt{\tau_1 \tau_2} - \gamma \tau_3^{3/2} \right), \quad (3.18)
$$
3.4 String theory examples

The Kahler potential and the superpotential, similarly for all the LVS models, are,

\[ K = -2 \ln (\mathcal{V} + \frac{\zeta}{g^2}), \]
\[ W = W_0 + Ae^{-aT_3}. \] (3.19)

Note that only \( T_3 \) appears on the superpotential, whereas \( T_2 \) and \( T_3 \) will appear only through the volume on the scalar potential. This will mean that these directions will be flat during inflation. The scalar potential is generated by Kaluza Klein modes and one and two loop corrections. For the canonically normalised case this looks like,

\[ V = \frac{V_0}{\sqrt{10/3}} \left( 3 - 4e^{-k\phi c/2} \right). \] (3.20)

Which is similar to the Starobinsky potential, although it’s a displaced version because the terms inside the parenthesis do not match. To include matter we will first assume that the action can be written in the Jordan frame. Although first let us note that this case is less straightforward than Kahler moduli inflation because here the corrections become important and there is not a clear way of adding chiral matter to the superpotential. We could focus on the classical action, and by using reverse engineering determine the value of the non-canonical coupling to gravity. Because we have that the Kahler metric is usually,

\[ K_{tt} = \frac{1}{t^2} dt d\bar{t}. \] (3.21)

We could simply assume that the Jordan frame action is of the form \( t^2 R \). This is justified because the moduli fields have to be of \( \mathcal{O}(1) \), so higher order terms are negligible. Then we couple matter the usual way, but after transforming to Einstein frame again we have that,

\[ V = V_{\text{inf}} + e^{-2\phi} \rho. \] (3.22)

With this we can appreciate the differences between the two realisation of the LVS we have described. For this case we recover an intermediate situation between the Starobinsky inflation, in the sense that there is a flow to an UV fixed point that occurs at a short range away from the inflationary trajectory, furthermore we have shown that coupling matter does not affect this flow, as it is only a rescaling of the potential. This stems from the fact that the different configuration for the volume, allows inflation on a shorter range \( \Delta \phi \) while not being affected by the coupling. For this case a flow to an
IR fixed point seems more likely than for the first example. This is mainly because small field inflation is more robust under corrections, so when considering the effects of the heavy fields this are decoupled.

### 3.4.3 Examples in KKLT

We have seen that within the LVS several effects come into play when considering a description in terms of $\beta$ functions. To continue our study let us now consider a different framework, developed in [173], generally called KKLT. To do so we will base our investigation on the discussion presented at [178], where the idea was to study the effect of integrating out heavy moduli fields in string inspired models of inflation. These corrections depend on the geometry of the potential so might change the predictions of inflation or render inflation unsuitable. Such an effect might be considered as a concrete realisation of the effect of adding matter we studied for $\alpha$ attractors, and as we will see, their effects are indeed very similar.

Now let us start by studying how to build de Sitter solutions using KKLT. The simplest setup contains a no scale Kahler potential,

$$K = -3 \ln(T + \bar{T}),$$  \hspace{1cm} (3.23)

where the moduli $T$ and $\bar{T}$ control the volume of the compactification. The superpotential contains integrated out fluxes, and non perturbative corrections. This is written as,

$$W = W_0 + A e^{-aT},$$  \hspace{1cm} (3.24)

The scalar potential has two extrema at $D_T W = 0$, one at infinity which is when the volume decompactifies. At the other minima $T_0$ the superpotential becomes,

$$W_0 = -A e^{-aT_0} \left(1 + \frac{2}{3}aT_0\right),$$  \hspace{1cm} (3.25)

which leads to an AdS potential $V = -\frac{a^2 A^2 e^{-2aT_0}}{6T_0}$. To uplift it to have a Minkowski minima, one alternative is to use the $F$ term of a Polony field $X$ with,

$$K_{\text{up}} = k(|X|^2), \hspace{1cm} W_{\text{up}} = fX,$$  \hspace{1cm} (3.26)
It is assumed that the field $X$ stabilise at a high scale and is completely decoupled from the dynamic of inflation and moduli stabilisation. Its only contribution is then

$$V_{up} = e^K f^2,$$  \hfill (3.27)

which can be fine tuned to cancel the negative contribution given by (3.25). The potential has now two minima. The derivative $\partial_T V = 0$ gives,

$$D_T W = -\frac{3W}{4T} \left( 1 \pm \sqrt{1 - \frac{2f^2}{(aT + 2)W^2}} \right),$$  \hfill (3.28)

which for the AdS minima $T_0$, implies

$$D_T W = -\frac{3f^2}{4aT_0^2 W|_{T_0}} + O(T_0^{-3}),$$  \hfill (3.29)

where the combination $aT_0$ has to be large for consistency with the single instanton approximation. We can calculate the effect of small deviations $\delta T = \bar{T}_0 - T_0$ about the stabilised moduli because of the new minima. It follows that

$$D_T W|_{T_0} = D_T W_0|_{T_0} + \delta T \partial_T D_T W|_{T_0},$$

$$= D_T W|_{T_0} - \delta T a_k T W|_{T_0} + O(T_0^{-2}),$$  \hfill (3.30)

which implies that,

$$\frac{\delta T}{T_0} \approx \frac{f^2}{2a^2 T_0 W_0^2} + O(T_0^{-2}).$$  \hfill (3.31)

This effect will be crucial when considering specific models of inflation as we will now see. In fact, integrating out heavy moduli seems analogous to consider matter coupled through a non canonical term, as we saw for $\alpha$ attractors.

### 3.4.4 Chaotic inflation

Now let us study different examples using this framework. First, we will examine the case of chaotic inflation. Here we need to add to (3.23) and (3.24)

$$W = -\frac{1}{4} m \phi^2, \quad K = \frac{1}{2} (\phi + \bar{\phi})^2,$$  \hfill (3.32)
The addition of these terms displace again the minima by

\[ D_T W = -\frac{3W}{4T} \left( 1 \pm \sqrt{1 - \frac{2f^2 + m^2 \phi^2 / 2}{(aT + 2)W^2}} \right), \tag{3.33} \]

which implies that

\[ \frac{\delta T}{T_0} \approx \frac{1}{2aT_0W_0^2} + \mathcal{O}(T_0^{-2}) \approx \frac{\tilde{m}\phi^2}{4aT_0 m_{3/2}}, \tag{3.34} \]

Now it is useful to parameterise all these deviations in terms of the gravitino mass. This is given by,

\[ m_{3/2} = e^{K/2}W_0 = \frac{W_0}{(2T_0)^{3/2}} \left( 1 - \frac{3}{2aT_0} + \mathcal{O}(aT_0)^{-2} \right) \approx \frac{W_0}{(2T_0)^{3/2}}, \tag{3.35} \]

and \( \tilde{m} = m/(2T_0)^{3/2} \). Then inserting back the corrections we get the following potential.

\[ V(\phi) = \frac{1}{2} \tilde{m}^2 \phi^2 + \frac{3}{2} \tilde{m}m_{3/2} \phi^2 - \frac{3}{16} \tilde{m}^2 \phi^4 - \frac{3}{4aT_0} \left( 3\tilde{m}m_{3/2} \phi^2 + \frac{3}{4} \tilde{m}^2 \phi^4 \right). \tag{3.36} \]

When neglecting higher order terms in \((aT_0)^{-1}\), we obtain the following scalar potential,

\[ V \approx \frac{3}{2} \tilde{m}m_{3/2} \phi^2 \left( 1 - \frac{1}{8} \frac{\tilde{m}}{m_{3/2}} \phi^2 \right), \tag{3.37} \]

whose inflationary predictions are,

\[ n_s = 0.996 \tag{3.38} \]
\[ r = 0.106 \tag{3.39} \]

which are slightly similar to chaotic inflation, with the difference due to the effect of the backreaction. Interestingly, these effects are not crucial, and inflation will survive for enough efolds. Adding matter to this model will have the same effect that we described for \( \alpha \) attractors, namely, the matter will only push up the amplitude of the potential. These two effects are because the flat geometry of the target space that makes them negligible.
3.4 String theory examples

3.4.5 Starobinsky model

Now let us study a realisation of the Starobinsky model [179]. We can see the same KKLT moduli stabilisation mechanism to produce a Starobinsky like potential by,

\[
K = -3 \log \left( T + \bar{T} - \frac{1}{3} |\Phi|^2 \right),
\]

\[
W = M \left( \frac{1}{2} \Phi^2 - \frac{b}{3\sqrt{3}} \Phi^3 \right) + W_0 + Ae^{-aT}.
\] (3.40)

The main difference here is with chaotic inflation, is that the scalar field has a non-canonical kinetic term. This will change the trajectory on field space. It is given by,

\[
K_{\Phi \bar{\Phi}} = -\frac{3}{(2T_0 - \phi^2/3)^2},
\] (3.41)

which implies that the normalised field is \( \varphi = \sqrt{6T_0} \tanh \frac{\varphi}{\sqrt{6}} \). The potential including, the backreaction effect for this model is

\[
V = \frac{3M^2}{8T_0} \left( 1 - e^{-\sqrt{2/3} \varphi} \right)^2 + \frac{3Mm_3/2}{\sqrt{8T_0}} \sinh^2 \sqrt{\frac{2}{3}} \varphi - \frac{3M^2}{8T_0} \sinh^4 \varphi \sqrt{6} + \mathcal{O} \left( \frac{M}{m_T} \right).
\] (3.42)

So we see that the first terms is a Starobinsky like potential but there are other terms that might threaten inflation. On the contrary to the case of chaotic inflation the backreaction terms are important, and they might not allow inflation for a large range in parameter space. This effect is similar to what we describe for \( \alpha \) attractors with small \( \alpha \). The hyperbolic geometry of the target space makes integrated out fields important, because they modify the inflationary potential and hence its dynamics. Now let’s turn to study the effect of matter and the \( \beta \) function for these two examples.

\( \beta \) functions

Before calculating the \( \beta \) functions for both models we will also add matter to these models. This could be done by simply adding the Kahler and superpotential,

\[
W = \frac{1}{4} M^2 \psi^2, \quad K = -(\psi + \bar{\psi})^2.
\] (3.43)
to the KKLT potentials we are considering. For the chaotic inflation example the potential get’s modified as,

$$V \approx \frac{3}{2} \tilde{m} m_{3/2} \phi^2 \left( 1 - \frac{1}{8} \frac{\tilde{m}}{m_{3/2}} \phi^2 \right) + \tilde{m} \tilde{M} \psi^2 \phi^2. \quad (3.44)$$

The $\beta$ function in this case is,

$$\beta = \frac{2}{\phi} - \frac{2m_{3/2}}{\tilde{m} \phi} \frac{1}{\rho}. \quad (3.45)$$

This implies that it goes to zero for large $\phi$ and large $\rho$, which is the same behaviour we have seen for large field models. As we mentioned before, the flat geometry of target space allows to control the corrections, making this model robust.

For the Starobinsky model, we add the same terms to the Lagrangian, as we have that the new potential is

$$V = \frac{3M^2}{8T_0} \left( 1 - e^{-\sqrt{2/3} \phi} \right)^2 + \frac{3M^2}{T_0} \cosh^4(\varphi/\sqrt{6}) \sinh^2(\varphi/\sqrt{6}), \quad (3.46)$$

Which is a very similar result to the potential (3.42) we calculated before. Now the $\beta$ function for the Starobinsky model is,

$$\beta = \left( \frac{2}{\sqrt{3}} \frac{\phi}{\sqrt{6}} \right) \left( \coth^2 \left( \frac{\phi}{\sqrt{6}} \right) + 2 \right) + O(\rho^{-1}), \quad (3.47)$$

which tends to $3\sqrt{2/3}$ for large $\phi$. Hence we have the same situation that for (3.122), where the matter will not allow the flow to the IR fixed point. This effect is interesting as we see the difficulties that a non trivial field metric poses over the inflationary potential.

### 3.4.6 Higgs-otic inflation

There is an interesting class of string theory inspired models of inflation under the name of Higgs-otic inflation [180, 181]. The idea is to achieve Higgs Inflation, or to use a Higgs like field, whose origin is from string theory, to achieve inflation. We will revise some aspects and will make some comments about a $\beta$ function implementation.
Using the same KKLT moduli stabilisation procedure, we have that,

\[ K = -\log \left[ (S + \bar{S})(U + \bar{U}) - \frac{1}{2}(\Phi + \bar{\Phi})^2 \right] - 3\log[T + \bar{T}], \quad (3.48) \]

\[ W = \mu\Phi^2 + W_0 + Ae^{-\alpha T}, \quad (3.49) \]

where \( S \) and \( U \) are the axio-dilation and complex structure modulus respectively, which we are going to assume are stabilised. It will convenient to simplify the Kahler potential to,

\[ K = -\log \left[ s - \frac{1}{2}(\Phi + \bar{\Phi})^2 \right] - 3\log[T + \bar{T}]. \quad (3.50) \]

The potential is given by,

\[ V(t, \phi) = \frac{1}{8s t^3} \left[ \Delta^2 + \frac{4}{3} \alpha t Ae^{-2\alpha t}(3A + \alpha t A + 3W_0 e^{\alpha t}) + 2 (\alpha t Ae^{-\alpha t} + s\mu) s\mu \phi^2 \right] \quad (3.51) \]

This potential has an AdS minima for the moduli fields that can be uplifted to Minkowski by fine tuning \( \Delta \) and \( A \). This implies that \( \alpha t_0 \gg 1 \). The values for free parameters are,

\[ A = -\frac{3W_0(\alpha t_0 - 1)e^{\alpha t_0}}{2\alpha t_0(\alpha t_0 + 2) - 3}, \quad (3.52) \]

\[ \Delta^2 = \frac{12\alpha^2 t_0^2 W_0^2(\alpha t_0 - 1)(\alpha t_0 + 2)}{3 - 3\alpha t_0(\alpha t_0 + 2)^2}, \quad (3.53) \]

Now we assume that the moduli reach its minima adiabatically and that it is heavy enough that it can be integrated out. This is done by minimising the potential up to second order in \( \delta t = t - t_0 \). Therefore we require that \( \partial_t V|_{t_0 + \delta t} = V|_{t_0} + \delta t = 0 \), which leads to

\[ \frac{\delta t(\phi)}{t_o} = -\frac{s\mu \phi^2}{2\alpha t_0 W_0} + \mathcal{O}(H^2/m^2). \quad (3.54) \]

Replacing back into the potential to leading order in \( H^2/m^2 \) and \( 1/\alpha t_0 \),

\[ V_{\text{eff}} = \frac{1}{4t_0^3} \left( s\mu \phi^2 + \frac{3}{2} \mu W_0 \phi^2 - \frac{3}{8} s\mu^2 \phi^4 \right) + \ldots \quad (3.55) \]
Adding matter

We will modify the superpotential to add an scalar field coupled to the Higgs, as,

\[ W = \mu \Phi^2 (1 + \lambda \Psi^2) + W_0 + Ae^{-\alpha T}, \]  

(3.56)

whereas the Kahler manifold is

\[ K = -\frac{\langle \Psi \bar{\Psi} \rangle}{(s - \frac{1}{2}(\Phi + \bar{\Phi})^2)}. \]  

(3.57)

Assuming that \( \Psi = i\psi \) we have that the potential becomes

\[ V(t, \phi) = \frac{1}{8s^2 t^3} \left[ \Delta^2 + \frac{4}{3} \alpha t A e^{-2\alpha t} (3A + \alpha t A + 3W_0 e^{\alpha t}) \right] \]

\[ + 2 \left( \alpha t A e^{-\alpha t} (1 + \psi^2) + s\mu \right) s\mu \phi^2 + \mu \phi^2 \psi^2 \].

(3.58)

Minimizing with respect to value of the moduli field \( t \) after inflation we get that at leading order in \( 1/(\alpha t_0) \)

\[ \frac{\delta l(t)}{t_0} = \frac{s\mu \phi^2}{2\alpha t_0 W_0} (1 + \psi^2) + O(H^2/m^2). \]  

(3.59)

Replacing in the potential we have that, and neglecting terms that are subleading in \( W_0 \gg \mu \), we get

\[ V_{\text{eff}} = \frac{1}{4t_0^3} \left( + \frac{3}{2} \mu W_0 \phi^2 (1 + \psi^2) - \frac{3}{8} s\mu^2 \phi^4 (1 + \psi^2)^2 + \mu \phi^2 \psi^2 + \lambda W_0^2 \psi^2 \right). \]

(3.60)

So we have a very similar situation for chaotic inflation in (3.44). One could expect that the effect of matter does not modify the potential. Nevertheless it is still important to consider the kinetic term. This is given by

\[ \mathcal{L}_{\text{kin}} = -\left( \frac{1}{2} + 3a \frac{\mu W_0 \phi^2}{16t_0^3} \right) (\partial \phi)^2. \]  

(3.61)

We cannot invert this function analytically but we can solve this numerically, where it can be shown that it flattens the potential. We can now calculate the \( \beta \) function.
3.5 Conclusions

Notice that at large densities the $\beta$ function

$$\beta(\phi) = -\frac{d \ln V(\phi)}{d\phi}. \quad (3.62)$$

decays as $1/\phi$ which seems to not change much when including matter. Interestingly this decay is slow and might imply that there is not enough available range on field space to flow towards a fixed point.

We have studied three different models of inflation using KKLT to stabilise the moduli. Here we see that when looked from a holographic perspective the main effect is the curvature on field space. This implies that flowing towards an IR fixed point is more realistic in large field scenarios such as chaotic inflation. This is contrary to the expectations from effective field theories [182], where for large field models quantum corrections could spoil the potential. The main reason we have found there is not a flow towards a fixed point is when the effect of the geometry is more important.

These results could be seen also in the light of the swampland criteria [183, 38, 184–186], where it was conjectured that there are no dS vacua if $|\nabla V| \geq c \cdot V$, which written in terms of the $\beta$ function is $\beta \geq c$. Interpreted in those terms, theories without IR fixed points satisfy the swampland criteria, meaning that they are not theories with de Sitter vacua. This will imply a way out, where as we can see restricting the theory and the matter content of the theory could allow for a fixed point in the IR that will violate the swampland criteria.

This might have important consequences, as requiring a holographic RG flows implies that the masses of the fields are bounded by $\frac{9}{4} H^2$. Then for example for KKLT, an analysis of integrated out fields might have to be considered. In this case the stability of the solution might not be guaranteed.

3.5 Conclusions

In this chapter we have analysed some relations between holography and inflation. In the first part we started by discussing the dS/CFT conjecture as stated in [55]. One of the main calculational tools is that late time correlators in dS are related by an analytic continuation in AdS. We analysed these procedures very carefully and we showed that in order to make consistent predictions one has to restrict the masses of the scalar fields to be $m^2 < \frac{9}{4} H^2$. This implies for inflation that the particles interacting with the inflaton, at the time of horizon crossing have to respect that bound. We also showed that this implies that the associated conformal operators are unitary. We also
studied the observational consequences of the bound. We showed that the squeezed limit is sensitive to the mass of the coupled massive fields, but for fields whose masses are \( m^2 \geq 9/4H^2 \), the bispectrum becomes scale dependent with a frequency given by the mass. Hence by detecting these kind of features one could rule out any kind of construction based on the Hamilton Jacobi equations.

We also studied the holographic \( \beta \) function. This arises because the broken time symmetries of inflation can be interpreted as a broken conformal field theory. This implies a running whose scale is identified with the scale factor \( a \). The \( \beta \) function associated with this flow is then used to classify different models of inflation. The common lore is that there is a reverse flow between two fixed points, with inflation starting at an IR fixed point while observables are measured at the end of inflation on the UV fixed point. This has to be interpreted as that slow roll inflation is an attractor solution, so initial conditions are washed out progressively as time happens, analogous to a flow from the UV will imply that degrees of freedom are integrated in as the theory evolves towards the IR.

We investigated the existence of IR fixed points. These are initial states of inflation where the initial conditions are not important. We showed that there are two main elements that come into play: the geometry of the target space, and the matter content of the theory. When combined they can modify the existence of IR fixed points. In particular this is the case for hyperbolic geometries, such as \( \alpha \) attractors, or supergravity models arising from no-scale superpotentials. Here, matter which is coupled through the metric, has a drastic effect modifying the potential at the beginning of inflation. This might be interpreted as marginal operators becoming relevant, due to the combined effect of matter and non trivial target spaces, driving the flow out of a fixed point. Interestingly, this might be related to the existence of de Sitter vacua on string theory. Hence, by restricting the geometry and the matter content it might be possible to look for de Sitter vacua in a systematic way.
Chapter 4

Disformal Transformations on the CMB

In chapter two and three we have studied different effects that arise in inflation when considering massive degrees of freedom. We will now slightly change the focus and for the remaining part of this thesis we will study how to observe different effects of massive fields on the cosmic microwave background (CMB). In particular in this chapter we study the possibility of modifying the speed of light of tensor modes during inflation and at later times. Despite the tight constraints from the observation of gravitational waves [85], it is still valid to ask whether at later times we have a situation analogous to inflation where the speed is removed by a special coordinate transformation, called disformal transformation [187]. Now the situation during epochs where matter and radiation are dominating can be quite different to inflation. This is because during inflation all matter is washed out by de Sitter expansion and therefore it is only necessary to consider the evolution of the inflaton. On the other hand during the matter and radiation domination epoch the coupled matter introduces characteristic time scales which may break time invariance. More specifically we investigate the invariance of the Einstein equations and the Boltzmann equations under a disformal transformation. It turns out that the change on the slope of the light cone changes the speed of propagation with respect to the untransformed frame. Thus by a choice of parameters any speed appearing can be removed. Nevertheless, by assuming that the speed in either the tensor or scalar sector differs from one then, after the transformation there will be an effective speed different from one for the other sector, since the transformation affects the time and scale factor. We focus on the case where the graviton has an original speed of sound \( c_T \neq 1 \). Removing it by deforming the light cone changes the speed of the scalar degree of freedom.
In order to investigate whether the CMB gets modified, we examine the evolution of photons coupled to gravity. Since the Boltzmann equations are invariant, the characteristic timescales of scattering processes that might occur do not change. There is a characteristic time scale, the Silk damping scale, characterizing the interactions of matter and photons which is invariant under disformal transformation. Therefore the comparison between the rates of these interactions and the graviton speed will not be invariant under a disformal transformation. By solving the equations of motion the different speed of propagation shifts the peaks for the CMB anisotropy power spectrum, moreover because the Silk damping scale does not change compared to the untransformed case the damping tail is different in comparison with Planck results. Hence, since any such modification leads to traceable effects, that modification cannot be applied to remove the speed of sound for the tensor modes.

As an example we study Galileon theories because they capture an important aspect and are simple enough to be solved. In particular, an approximate Galileon symmetry can enhance curvature operator which allow the speed of gravitons to vary. We first show how curvature operators are enhanced during inflation leading to a modified speed for the gravitational waves. By using a disformal transformation this speed is removed from the tensor action, but it appears as a new parameter on the scalar action, we showed that this parameter dependence is removed from the power spectrum. By means of this symmetry the Galileon theory is further constrained. We also focus on Galileons during radiation domination, we showed here that the speed for the gravitons cannot be removed because the transformation significantly changes the anisotropy power spectrum.

The outline of this chapter is as follows. We first explain what disformal transformations are and their application to inflation. To do so we detail the EFT of inflation where the transformation can be better understood. We then calculate the transformation rules for the Einstein and the Boltzmann equations. By analysing how matter couples to it we then show that the Silk damping scale is not modified, and thus when solving the coupled system its power spectrum will be damped at a different scale. Furthermore the new sound horizon changes the peak structure and then small changes can be easily detected. We illustrate this by solving the temperature anisotropy in the two fluid approximation [188]. While this is not very useful to get strong constraints it shows clear examples of how easy it is to modify the power spectrum.

Then, we show how a non-unitary speed arises in Galileon theories on a cosmological background. We first detail Galileon inflation and calculate the speed for the tensor modes. By applying the disformal transformation we show explicitly that this can
be removed. We perform a similar analysis for Galileon theories during radiation and matter domination. In this case the speed of the tensor modes is modified but observations of the CMB mean that it is strongly constrained to be close to one. Finally we discuss our results and analyse its implications.

4.1 EFT of inflation

4.1.1 Scalar and tensor fluctuations

To study modifications to the speed of light it is convenient to define an EFT because then it may be possible to organise the contributions to the two point function in terms of slow roll parameters. This can be done by using the framework first introduced in [151] where the realisation of inflation is parametrised as a broken quasi de Sitter symmetry and the inflaton is the resulting pseudo Goldstone boson. Formally the action for perturbations is given by,

\[
\Delta S = \int d^4x \sqrt{-g} \left[ \frac{M_2^4(t)}{2} (\delta g^{00})^2 + \frac{M_3^4(t)}{3!} (\delta g^{00})^3 + \frac{M_4^4(t)}{4!} (\delta g^{00})^4 + ... \right.
\]

\[
- \frac{\dot{M}_1^2(t)}{2} \delta g^{00} \delta K - \frac{\dot{M}_2^3(t)}{2} (\delta K)^2 - \frac{\dot{M}_3^3(t)}{2} (\delta K^\mu)^2 + ... \]

\[
- \frac{\dot{M}_2^3(t)}{2} \delta g^{00} R + ... \right],
\]

where \( \delta g \) is the metric variation and \( \delta K \) is the 3-curvature variation. We can see that contributions to the quadratic part of the gravitational section will be contained in the second order curvature operator. Although one should expect them to be a perturbative series and then small compared to the first order operators, it can be the case that some higher order curvature operators are enhanced by a symmetry such as in DBI or Galileon theories.

The action for the perturbations will be given by first identifying the Goldstone boson \( \pi \) with the broken time invariance. Then this Goldstone boson will be the inflaton perturbation on a de Sitter background. In the limit where the mixing of \( \pi \) with gravity is effectively decoupled the action is given by,

\[
S = \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R - M_{Pl}^2 \dot{H} \left( \dot{\pi}^2 - \frac{(\partial \pi)^2}{a^2} \right) + 2M_2^4 \left( \dot{\pi}^2 + \dot{\pi}^3 - \frac{\partial \pi}{a^2} \right) - \frac{4}{3} 2M_3^4 \pi^3 ... \right].
\]
Note that that higher order operators modify the dispersion relation for the Goldstone boson. Indeed, the speed of sound can be written as

\[ c_s^{-2} = 1 - \frac{2M_2^4}{M_{\text{Pl}}^2 H}. \]  

(4.3)

Moreover, \( c_s \) parameterises the amount of equilateral non-Gaussianity, \( f_{\text{NL}}^{\text{equil}} \propto c_s^{-2} \), as it relates how large \( M_3^4 \) is with respect to the first terms, hence it can be quantified how much the theory departs from a harmonic oscillator. In cases with more symmetry \( c_s \) can be used to further constrain the theory. For example in DBI inflation [31, 30], the next order parameter is found to be \( M_3^4 \sim M_{\text{Pl}}^2 |\dot{H}| c_s^4 \). Other interesting examples include effects coming from integrating out heavy degrees of freedom where it is possible that the inflation interchanges energy with other fields, then reducing its speed of propagation [27].

The action for the gravitons is obtained by replacing the ansatz with a traceless and divergenceless graviton \( ds^2 = -dt^2 + a^2 h_{ij} dx^i dx^j \) in (4.1). We can decompose the Ricci scalar as, \( R = (3) R + K_{ij} K^{ij} - K^2 + ... \) The first part yields the second order action, 

\[ S = \frac{M_{\text{Pl}}^2}{8} \int d^4 x a^3 \left[ \dot{\gamma}_{ij}^2 - (\partial \gamma_{kj})^2 \right]. \]  

(4.4)

where higher order operators have the following contribution

\[(\delta K^\mu_\nu)^2 - (\delta K)^2 = \frac{1}{4} (\dot{\gamma}_{ij})^2.\]  

(4.5)

And thus only the combination \( \tilde{M}_3^2(t) - \tilde{M}_2^2(t) \), from (4.2), can contribute to modifications of the speed of sound up to second order. The modifications for the speed of sound at first order on the slow roll parameters may arise from considering higher order curvature terms, such as the Weyl tensor squared \( W^2 \) or the parity violating Weyl tensor \( \tilde{W}^2 \) [189].

### 4.1.2 Disformal transformation

As we have seen modifications to the two point function for tensor and scalar modes can written in terms of an effective speed of sound for each sector. Since this is the speed at which these degrees of freedom propagate through the lightcone, a redefinition of the null vectors could set the speed to one. This, also called a disformal
transformation [190, 187], is given by,
\[ g_{\mu\nu} \rightarrow \frac{1}{c} \left( g_{\mu\nu} + (1 - c^2(t))n_\mu n_\nu \right). \] (4.6)

where \( n_\mu \) is the unit vector perpendicular to surfaces of constant time. Let us consider how the action for the EFT of inflation transforms under (4.6). We first calculate how it affects the background components,
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R - \frac{M_{\text{Pl}}^2}{c} \left( 3H^2 + \dot{H} \right) + M_{\text{Pl}}^2 \dot{H} g^{00} \right], \] (4.7)

where the only change is a re-scaling of Newton’s constant, this is because the background is time dependent and thus it will be subject to a conformal rescaling which is equivalent to changing \( M_{\text{Pl}} \) in the Einstein equations. More interesting are the perturbations where gradients get a different rescaling. The second order action for the Goldstone boson \( \pi \) becomes,
\[ S = \int d^4 x \frac{a^3}{c^2} M_{\text{Pl}}^2 \dot{H} \left\{ \dot{\pi}^2 - \frac{c^2}{a^2} (\nabla \pi)^2 \right\}. \] (4.8)

Then it acquires a speed of propagation just due to the change of coordinates. We could have allowed a time dependent speed of sound, this would have included second order corrections but we are mostly interested in first order in slow roll parameter terms.

**Tensor modes**

We now consider the effect of the transformation (4.6) on tensor perturbations. By means of this transformation the tensor modes changes as \( h_{ij} \rightarrow c^{-1} h_{ij} \). Therefore the second order action becomes,
\[ S = \frac{M_{\text{Pl}}^2}{8} \int d^4 x a^3 \left[ \frac{1}{c_F^2} \dot{\gamma}_{ij}^2 - \frac{\left( \partial_k \gamma_{ij} \right)^2}{a^2} \right]. \] (4.9)

where again we see that gravitons acquires a speed of sound from the rescaling of the parameters, similar to the action for the Goldstone boson \( \pi \). We can also consider the corrections that give rise to a non-unity speed of sound. These rescale as
\[ (\delta K^\mu_\nu)^2 - (\delta K)^2 = \frac{1}{4c} (\dot{\gamma}_{ij})^2, \] (4.10)
Then choosing that $\bar{M}^3_3(t) - \bar{M}^3_2(t) = (1 - c^{-2})$ the previous action becomes,

$$S = \frac{M_{Pl}^2}{8} \int d^4xa^3 \left[ \frac{1}{c^4} \bar{\gamma}^2_{ij} - \frac{(\partial_k \bar{\gamma}_{ij})^2}{a^2} \right].$$  \hspace{1cm} (4.11)

**Removing the speed of sound**

We have seen that the transformation (4.6) sets a different speed of propagation for each mode. Therefore by choosing $c$ equal to $c_s$ or $c_T$ we can remove either of these speeds and move it to an effective speed for the other sector.

$$ds^2 = -d\bar{t}^2 + \bar{a}(\bar{t})d\bar{x}^2,$$ \hspace{1cm} (4.12)

where $d\bar{t} = c_T^{1/2}dt$ and $\bar{a}(\bar{t}) = c_T^{-1/2}a(t)$ For the transformed values the action becomes

$$S = \int d^4\bar{x}\bar{a}^3 M_{Pl}^2 \hat{H} \left\{ c_T^2 \bar{\pi}^2 - \frac{1}{a^2} (\nabla \bar{\pi})^2 \right\}.$$ \hspace{1cm} (4.13)

Now rewriting (4.11) in terms of the rescaled variables we get that,

$$S_{\gamma\gamma} = \frac{M_{Pl}^2}{8} \int d\bar{t}d^3\bar{x}\bar{a}^3 \left[ \frac{\dot{\gamma}^2_{ij}}{\bar{a}^2} - \frac{(\partial_k \bar{\gamma}_{ij})^2}{a^2} \right].$$ \hspace{1cm} (4.14)

Thus in the rescaled frame the graviton has speed of sound equal to one and the Goldstone mode propagates with a speed $c_s = 1/c_T$. It remains to be seen however if these modifications appear in observables such as the power spectrum. This can be calculated by rescaling momenta by $k/c_s$, thus,

$$\Delta_R \equiv \frac{k^3}{2\pi} P_R(k) = \frac{c_T H^4}{4M_{Pl} H}.$$ \hspace{1cm} (4.15)

In the transformed frame where $\hat{H} = c_T H$ the power spectrum is equivalent to the case when $c_s = 1$. Note that allowing a $c_s \neq 1$ does not change the results as the next term in the EFT expansion gets the same rescaling than the tree level terms and then the second order action (4.8) just get a physical speed $c_s$, which cannot be removed.

**4.2 Einstein equations**

We now want to see whether applying a disformal transformation can be used to remove any graviton speed different from one throughout cosmological evolution. One could
rescale the time and the Hubble parameters from the Einstein equations but we prefer
to keep track of all the speed coefficients appearing to see when these can be safely
removed and when they could not.

Hence we first apply the disformal transformation to the perturbed Einstein equa-
tions in Newtonian gauge for both scalar and tensor sectors. It would be interesting to
consider vector perturbations but since they decay fast during matter and radiation
eras we will neglect them. We start by considering the following transformed metric,

\[ ds^2 = \frac{a^2(\tau)}{c_T}\left(-c_T^2(1 + 2\psi)d\tau^2 + (1 - 2\phi)\delta_{ij}dx^i dx^j\right), \]  

(4.16)

which is the perturbed FRW metric in Einstein frame under a disformal followed by a
conformal transformation (4.6). Assuming that the stress energy tensor is a perturbed
perfect fluid, defined as,

\[ T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \]  

(4.17)

where \( u_\mu = \delta_{0\mu}/\sqrt{-g^{00}} \). Since the perturbed stress energy tensor will have a spatial
velocity we assume that this is of the form \( \delta u^i = \frac{v_i}{c_T^{1/2}} \), thus \( \delta u^i \delta u_i = 1 \). Now the
perturbed 4–velocity is,

\[ u_\mu = \frac{a}{c_T^{1/2}}(c_T(1 + \psi), v_i). \]  

(4.18)

Then we calculate the Einstein equations of motion. Its zero component are the
Friedmann equations,

\[ 3\dot{H}^2 = 8\pi G a^2 c_T \bar{\rho}, \]

\[ -\dot{H}^2 - 2\dot{H}' = 8\pi G a^2 c_T \bar{P}. \]  

(4.19)

which we see are dependent on the conformal factor as doing a conformal transformation
is equivalent to reescaling Newton’s constant. The Euler equations for the perfect fluid
are,

\[ \bar{\rho}' = -3\dot{H}(\bar{P} + \bar{\rho}) \]  

(4.20)

The perturbed equations, are given by
\( c_T^2 \nabla^2 \phi - 3H(\mathcal{H}\psi + \phi') = 4\pi G a^2 c_T \delta \rho, \)
\( \mathcal{H}\psi + \phi' = -4\pi G a^2 (\bar{P} + \bar{\rho}) v, \)
\( \phi'' + \mathcal{H}(\psi' + 2\phi') + 2\mathcal{H}'\psi + \mathcal{H}^2 \psi - \frac{1}{3} c_T^2 \nabla^2 (\phi - \psi) = 4\pi G a^2 c_T \delta P. \)  

(4.21)

where \( v^i = \partial^i v. \)

The perturbed Euler equation and continuity equations are,

\[
\begin{align*}
\mathbf{v}' + 3\mathcal{H}(\frac{1}{3} - \frac{\bar{P}'}{\bar{\rho}})\mathbf{v} &= -\frac{c_T \nabla \delta P}{\bar{\rho} + \bar{P}} - c_T \nabla \psi, \\
\delta' + 3\mathcal{H} \left( \frac{\delta P}{\delta \rho} - \frac{P}{\rho} \right) \delta &= - \left( 1 + \frac{\bar{P}}{\bar{\rho}} \right) (c_T \partial_i v^i - 3\phi').
\end{align*}
\]

(4.22)

Hence the disformal transformation also introduces a \textit{speed of sound} which can be different from one for the perturbations in addition rescaling the Newton constant. It can also be noticed that the second order in perturbations equations obtained by combining (4.21) and (4.22) will be wave equations with a speed of propagation \( c_T. \) It can also be remarked that we could have obtained these results by doing a rescaling of time and the scale factor, but it is preferable to keep track of where the \( c_T \) factor appears.

### 4.2.1 Gravitational waves

To calculate the power spectrum for gravitational waves we first apply the disformal transformation to the line element,

\[
ds^2 = -a^2 d\tau^2 + a^2 (\delta_{ij} + 2E_{ij}) dx^i dx^j, \]

(4.23)

where \( E_{ij} \) parameterises the two tensor degrees of freedom, thus \( E_{i}^{i} = E_{j}^{j} = 0, \) which becomes

\[
ds^2 = -c_T a^2 d\tau^2 + \frac{a^2}{c_T} (\delta_{ij} + 2E_{ij}) dx^i dx^j. \]

(4.24)

Where similarly to the case for the scalar perturbations (4.16) the transformation factorised from the spatial and the temporal part of the line element. Now we can
calculate the perturbed Einstein tensor which is,
\[ \delta G_{ij} = \frac{E''_{ij}}{c_T^2} - \nabla^2 E_{ij} + \frac{2\mathcal{H} E'_{ij}}{c_T^2} - \frac{2(2\mathcal{H}' + \mathcal{H}^2) E_{ij}}{c_T^2}. \] (4.25)

To calculate the first order perturbation to the stress energy tensor we notice that for the spatial part \( u_i \) is first order in perturbation, therefore it will only contribute at second order in perturbations in the expansion of the stress energy tensor. Thus the only first order term is be given by,
\[ \delta T_{ij} = \frac{2\mathcal{P} a^2 E_{ij}}{c_T^2}. \] (4.26)

Summing both RHS of (4.25) and (4.26) and using the background Einstein equations we get that
\[ E''_{ij} - \frac{c_T^2}{c_T^2} \nabla^2 E_{ij} + 2\mathcal{H} E'_{ij} = 0, \] (4.27)

which, as in inflation depends only on the Hubble parameter because we have not considered anisotropic matter. In order to do so we would have to modify (4.26) to include such contribution. This would have led to a source term for the above equation. Furthermore without matter the (4.27) corresponds to a rescaling of time and the scale factor, as for the scalar.

### 4.2.2 Modified speed of tensor modes

Now that we know how the Einstein equations transform under a disformal transformation, the next question would be to apply this framework to the case when there is a modified \( c_T \). We can imagine that by higher derivative corrections the gravitational wave equations get modified in such a way that there is an effective speed of sound,
\[ E''_{ij} - \frac{c_T^2}{c_T^2} \nabla^2 E_{ij} + 2\mathcal{H} E'_{ij} = 0. \] (4.28)

Which is the same equation as (4.27), but in this case we have not applied any transformation to the metric. By applying an inverse disformal transformation we can cancel the \( c_T \) out of the equations. Then equation (4.28) can be written as,
\[ \tilde{E}''_{ij} - \nabla^2 \tilde{E}_{ij} + 2\tilde{\mathcal{H}} \tilde{E}'_{ij} = 0. \] (4.29)
Where we have expressed the functions in the new frame as tilded. This transformation will have an effect on the scalar modes, by inducing a tensor speed $c_S = c_T^{-1}$. For example the Einstein equations, (4.21) become,

$$c_T^{-2} \nabla^2 \ddot{\phi} - 3 \ddot{\mathcal{H}} (\dot{\mathcal{H}} \psi + \ddot{\phi}) = 4 \pi G \ddot{a}^2 c_T^{-1} \ddot{\rho},$$

$$\ddot{\mathcal{H}} \psi + \ddot{\phi} = -4 \pi G \ddot{a}^2 (\ddot{\bar{P}} + \ddot{\bar{\rho}}) \bar{v},$$

$$\ddot{\phi}'' + \ddot{\mathcal{H}} (\psi' + 2 \dddot{\phi}) + 2 \dddot{\mathcal{H}} \psi + \dddot{\mathcal{H}}^2 \psi - \frac{1}{3} c_T^{-2} \nabla^2 (\ddot{\phi} - \ddot{\psi}) = 4 \pi G a^2 c_T^{-1} \delta \bar{P}. \quad (4.30)$$

To see whether there is an observable effect we have to first solve these equations on a suitable background. We will show that, unlike inflation observables will depend on $c_T$ because we need to consider how matter couples to the Einstein potential.

### 4.3 CMB

To get an insight of the effect of $c_T$ we will solve Einstein equations to calculate the anisotropy power spectrum. There it will be a $c_T$ factor in the equations, because we made a disformal transformation that rescaled the time and the scale factor, this will change the particle horizon $\int dt/\dot{a}$ by a factor of $c_T$.

Before recombination electron and baryons are tightly coupled and they can be considered as a single fluid. This fluid will interact with photons through Thompson scattering, which is very efficient at large scales so it will be useful to consider a system of two fluids, photons and baryons, tightly coupled, hence moving at the same speed. Since the Thomson cross section is finite there will be scales affected by it. To take account of this we also need to transform the Boltzmann equation under a disformal transformation.

We will solve the system numerically by using the two fluid approximation [188] which solves the equations until the scales on which Thomson scattering is very efficient and then interpolates to further epochs. This approach will be useful to get an insight on how important are the modifications introduced by a disformal transformation.

### 4.3.1 Boltzmann equation

To keep track of any modification to the propagation of photons at early times it will be convenient to calculate whether the Boltzman equation changes under a disformal transformation. This also includes a contribution coming from the null geodesics which we need to take into account. Considering a perturbed perfect fluid with an energy
perturbation given by \( E = \varepsilon / a \), its geodesic equation transforms under (4.6) as\(^1\),
\[
\frac{d \log \varepsilon}{d \eta} = - \frac{d \psi}{d \eta} + \psi' + \phi',
\]
(4.31)
where conformal time is now denoted by \( \eta \). Note that any effect of the rescaling will be on the total derivative \( d/d\eta = \partial_\eta + c_T e^i \partial_i \). Hence, there is no modification in the null geodesics which is in agreement to what one can expect as there it has not been a modification to the Einstein equations but a change of coordinates.

There is no change in the phase space due to the disformal transformations because we have not changed the variables of the problem. Hence, the Boltzmann equation for the photon distribution \( f \) is,
\[
\frac{df}{d \eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial \eta} + \frac{\partial f}{\partial \log \varepsilon} \frac{\partial \log \varepsilon}{\partial \eta} = \frac{df_{\text{sct}}}{d \eta} \equiv C[f, f_i],
\]
(4.32)
where the RHS represents the scattering of photons with other species \( f_i \), which we will detail later. Allowing only first order terms on \( \epsilon \) we can reduce the Boltzmann equation to,
\[
\frac{df}{d \eta} = \frac{\partial f}{\partial \eta} + c_T e^i \partial_i f + \frac{\partial f}{\partial \log \varepsilon} \frac{\partial \log \varepsilon}{\partial \eta}.
\]
(4.33)
Assuming that the background distribution is a black body, we have that at first order in the temperature perturbation \( \Theta(\eta, \vec{x}) \),
\[
f = \bar{f}(\varepsilon) \left[ 1 - \Theta(\eta, \vec{x}) \frac{\partial \bar{f}}{\partial \log \varepsilon} \right].
\]
(4.34)
Then up to first order the Boltzmann equation becomes,
\[
\frac{d \bar{f}}{d \log \varepsilon} \left( \frac{\partial \Theta}{\partial \eta} + c_T e^i \partial_i \Theta - \frac{\partial \log \varepsilon}{\partial \eta} \right) = c_T C[f, f_i].
\]
(4.35)
Where the only modification arise in the spatial derivative for the temperature perturbations \( \Theta(x) \). The dominant contribution to the scattering term will be the Thomson scattering, which is the diffusion of photons from the electron plasma. It is characterised by the Thomson cross section \( \sigma_T = \frac{8\pi}{3} \left( \frac{m_e c^2}{4\pi\epsilon_0 m_e c} \right)^2 \). The scattering term is not modified, because it relies on the Lorentz invariance of the distribution function, but the total contribution to the Boltzmann equation will be rescaled as \( df_{\text{sct}}/dc_T \eta \). Hence the

\(^1\)See Appendix for details on the geodesic equations.
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The comoving mean free path is defined by \( \Gamma = a \bar{n}_e \sigma_T \), where \( \sigma_T \) is the Thomson cross section, is invariant under the transformation. The scattering rate in the expanding background can be written as,

\[
C[f(\epsilon, e)] = \frac{df}{d \ln \epsilon} \frac{\Gamma}{c_T} \left[ \Theta - e \cdot v_e + 3 \frac{1}{16\pi} \int d\epsilon_{\text{in}} f(\epsilon, \epsilon_{\text{in}}) [1 + (\epsilon_{\text{in}} \cdot e)^2] \right].
\]

This last equation completes (4.35) and the \( c_T \) factor is absorbed in the space derivative. Hence the Boltzmann equation remains invariant under the disformal transformation. This will affect the propagation of the perturbation when their speed is other than one as Thomson scattering will introduce a different diffusion scale than in the untransformed case.

Now it is useful to decompose the temperature fluctuation in terms of Legendre Polynomials \( \Theta(\eta, k, e) = \sum_l (-i)^l \Theta_l(\eta, k) P_l(k \cdot e) \). In which case the Boltzmann equation reduces to,

\[
\frac{d\Theta}{d\eta} - \frac{d \ln \epsilon}{d\eta} = -\Gamma \left[ \Theta - \Theta_0 - i\mu v_e + \frac{1}{10} \Theta_2 P_2(\mu) \right].
\]

With \( \mu \) the angle formed between the observation line and \( v_e \). It can be shown that the first Legendre multipoles are related to the photon fluid variables as \( \Theta_0 = \frac{1}{4} \delta_\gamma \), \( \Theta_1 = -v_\gamma \) and \( \Theta_2 = -\frac{5}{3} \sigma_\gamma \), thus we can relate the last equation to the variables used in (4.21) and (4.22).

As we want to evaluate the perturbations at the surface of last scattering it is useful to define the optical depth along the light of sight between \( \eta \) and \( \eta_0 \),

\[
\tau(\eta) \equiv \int^\eta_{\eta_0} \Gamma(\eta') d\eta'.
\]

Which is invariant under disformal transformations because any factor coming from a rescaled time will be canceled with a corrected version of \( \Gamma \). Using the line of sight parameterisation and replacing the geodesic equation for the photon energy, we then have that (4.35) is rewritten as,

\[
\frac{d}{d\eta} \left( e^{-\tau}(\Theta + \psi) \right) = S_{\text{scal}},
\]

where the scalar source term is

\[
S_{\text{scal}} = e^{-\tau}(\phi' + \psi') + g\psi + \frac{3}{16\pi} g \int d\hat{m} \Theta(\hat{m}) [1 + (\hat{m} \cdot \hat{m})] + g e \cdot v_b,
\]

(4.40)
where \( g \equiv -\dot{\tau} e^{-\tau} \) and \( \hat{m} \) is a unit vector. Integrating along the line of sight and approximating last scattering as sharp, then the last equation reduces to,

\[
\Theta(\eta_0, \vec{x}_0, e) + \psi(\eta_0, \vec{x}_0) = \Theta_0 + \psi + e \cdot \vec{v}_b + \int_{\eta_*}^{\eta_0} \mathrm{d}\eta'(\psi' + \phi').
\] (4.41)

Where all terms are evaluated at the surface of last scattering \( \eta_* \). Crucially, this last equation is invariant under disformal transformations.

### 4.3.2 Photon fluid

We will consider a fluid composed of photons and baryons where electrons are tightly coupled to photons, hence that two fluids will move at the same speed. As baryons have no pressure we can define the two-fluid momentum as,

\[
q = (\bar{\rho}_\gamma + \bar{P}_\gamma) v_\gamma + \bar{\rho}_b v_b \approx \frac{4}{3} (1 + R) \bar{\rho}_\gamma v_\gamma,
\] (4.42)

where \( R \equiv \bar{\rho}_b / (\bar{\rho}_\gamma + \bar{P}_\gamma) \) (using that \( P/\rho = 1/3 \) for radiation). Considering that there are no anisotropic contributions to the stress energy tensors, the continuity equation (4.22) for baryons can be written as,

\[
v_\gamma' + \frac{\mathcal{H} R}{1 + R} v_\gamma + \frac{c_T}{4(1 + R)} \nabla \delta_\gamma + c_T \nabla \psi = 0,
\] (4.43)

\[
\delta_\gamma' + \frac{4}{3} c_T \nabla \cdot v - 4 \phi' = 0.
\] (4.44)

Combining both equations we get

\[
\delta_\gamma'' + \frac{\mathcal{H} R}{1 + R} \delta_\gamma' - \frac{c_T^2}{3(1 + R)} \nabla^2 \delta_\gamma = 4 \phi'' + \frac{\mathcal{H} R}{1 + R} \phi' + \frac{4}{3} c_T^2 \nabla^2 \psi,
\] (4.45)

which relates photon pressure to gravity effects. Note that there is a speed of sound \( c_T \) induced on both photons and gravitational potential, which indicates that all scalar quantity are moving on a different lightcone. To solve this equation we need to solve first for the potential during radiation dominated epoch. Assuming a photon fluid with \( a \propto \eta \), then the equation of motion is

\[
\phi'' + \frac{5}{\eta} \phi' - \frac{c_T^2}{3} \nabla^2 \phi = 0,
\] (4.46)
which has as solutions

$$\phi(\eta, k) = A(k) \frac{j_1(c_T k \eta/\sqrt{3})}{c_T k \eta/\sqrt{3}} + B(k) \frac{n_1(c_T k \eta/\sqrt{3})}{c_T k \eta/\sqrt{3}}$$

(4.47)

Where $A(k)$ and $B(k)$ are dependent on the initial conditions, which we take to be adiabatic initial conditions given by inflation.

Curvature perturbation

In order to fix the initial conditions it is better to express the Newtonian potential $\phi$ in terms of the curvature perturbation $R$, which is constant out of the horizon. For Newtonian gauge the relation is given by,

$$R = -\phi - \frac{H \psi}{c_T}.$$  

(4.48)

Where the $c_T$ factors comes because the disformal transformation (4.6) changes the spatial part of the gauge transformation. Using the Einstein equations and the equation of state this can be expressed as

$$R = -\phi - \frac{2}{3(1 + \omega)} \left( \psi + \frac{\phi'}{H} \right).$$

(4.49)

Note that the dependence on $c_T$ has gone, as we are now considering a particular combination that should be invariant under coordinate transformations. Nevertheless, there is still an implicit dependence on $c_T$, which comes from the Newtonian potential.

To fix the initial conditions we will make some assumptions: First that $R(k)$ is adiabatic and scale invariant, also we will not consider any contribution to the anisotropic stress energy tensor, hence $\phi = \psi$. Furthermore for modes outside of the horizon the last term is negligible, then we have,

$$R = -\frac{5 + 3\omega}{3 + 3\omega} \phi.$$  

(4.50)

Radiation domination

For a photon fluid during radiation domination the above equation reduces to $R = -\left(\frac{3}{2}\right) \phi$. Using this we can now fix the initial conditions. For $k \eta \ll 1$, since $R_k$ is
constant outside the horizon, the solution for $\phi(k, \eta)$ is,

$$
\phi(\eta, k) = -2R_k \frac{j_1(k\eta c_T/\sqrt{3})}{k\eta c_T/\sqrt{3}}, \tag{4.51}
$$

which in the small scale limit ($k\eta \gg 1$) reduces to

$$
\phi(\eta, k) = -6R_k \frac{\cos(k\eta c_T/\sqrt{3})}{(k\eta c_T)^2}. \tag{4.52}
$$

Note that there will be a $c_T$ dependence on the evolution for the fields. To obtain the time evolution for the density we can use (4.21), which for radiation domination and using the background equations of motion becomes,

$$
\delta_\gamma = -\frac{2}{3} \eta^2 c_T k^2 \phi - 2\eta \dot{\phi} - 2\phi. \tag{4.53}
$$

Hence we have that for large scales $c_T k\eta \ll 1$, $\eta \dot{\phi} \ll \phi$, the last equation simplifies to,

$$
\delta_\gamma \approx -2\phi = \frac{1}{3} R_k. \tag{4.54}
$$

On large scales $c_T k\eta \gg 1$ only the first term dominates, thus,

$$
\delta_\gamma(\eta, k) \approx -\frac{2}{3}(k c_T \eta)^2 \phi(\eta, k) = -2R_k \cos(k\eta c_T/\sqrt{3}). \tag{4.55}
$$

### 4.3.3 Acoustic oscillations during matter domination

Now we move to solve the Einstein equations for radiation during matter domination. First, during this epoch the solution of equation (4.45) is

$$
\delta_\gamma(\eta, k) = C(k) \cos(c_T k r_s) + D(k) \sin(c_T k r_s) - 4(1 + R)\psi, \tag{4.56}
$$

where $r_s(\eta) = c_T \int_0^\eta c_s(\eta') d\eta'$ is the sound horizon, $c_s = \frac{1}{\sqrt{3(1+R)}}$, and $C(k)$, $D(k)$ are functions to be fixed by the initial conditions. Note that the sound horizon is factorised by $c_T$ as now the horizon for a FRW is modified by the transformation. We could have also modified the speed of sound to include $c_T$, but for clarity we prefer to factor it out. From the definition of the curvature perturbation (4.49) during matter domination we have that,

$$
R_k = \frac{5}{3} \phi. \tag{4.57}
$$
The Einstein equations imply that $\delta \gamma - 4\phi = \text{const.}$ for all epochs. Using this we can rewrite this as, $\delta \gamma - 4\phi = 4R_k$ for modes out of the horizon. Therefore to obtain $\delta \gamma$ when $kr_s = 0$, we need to replace the value of $\phi$ during matter domination. Doing so results in

$$
\delta \gamma = \frac{6}{5} R_k, \\
\dot{\delta} \gamma = 0,
$$

which in turn implies that $D(k) = 0$ and, $C_k = -\frac{4(1 + 3R)}{5} R_k$. Finally the density perturbation is,

$$
\delta \gamma(\eta, k) = -\frac{4}{5} (1 + 3R) R_k \cos(c_T kr_s) + \frac{12(1 + R)}{5} R_k. \quad (4.59)
$$

Using this we can calculate the Sachs Wolfe contribution to the anisotropy power spectrum. Using that, $\Theta_0 = \delta \gamma/4$, we obtain,

$$
\Theta_0(\eta, k) + \psi(\eta, k) = -\frac{1}{5} R_k [(1 + 3R) \cos(c_T kr_s) - 3R]. \quad (4.60)
$$

From the above equation we have that $\Theta_0(\eta, k) + \psi(\eta, k)$ oscillates between $-\frac{R_k}{5}$ and $(1 + 6R) \frac{R_k}{5}$. So the amplitude of the peaks will not be changed due to the effect of $c_T$.

Now we can calculate the statistical anisotropies which are given by,

$$
l(l + 1)C_l \approx P_R(k) \left[ \frac{\Theta_0(\eta, k) + \psi(\eta, k)}{R_k} \right]^2, \quad (4.61)
$$

with $\Theta_0(\eta, k) + \psi(\eta, k)$ given by (4.60). This means that the acoustic peaks will be given by $c_T kr_s = n\pi$. Therefore we should see a shift of them in the power spectrum due to the reescaling.

**Diffusion damping**

At small scales Thomson scattering is not so efficient and the photons diffuse off the plasma. This can be calculated by relaxing the tight coupling assumption in (4.45), in which case $R$ is a now a function of time. Considering that $R$ varies on cosmological time, thus $R' = H R$ leads to a damped oscillator whose solution can be written as, [191],

$$
(\Theta + \psi) = (\Theta + \psi)e^{(k/k_D)^2}, \quad (4.62)
$$
where

$$k_D^{-2} = \frac{1}{6} \int_0^\eta \frac{1}{F} \frac{R^2 + 4(1 + R)/5}{(1 + R)^2}. \quad (4.63)$$

To calculate how (4.3.3) is modified under a disformal transformation we can look at how (4.45) transformed, in which case (4.3.3) is

$$\left( \Theta + \psi \right) = \left( \Theta + \psi \right) e^{(c_T k / k_D \eta)^2}, \quad (4.64)$$

and (4.63) stays invariant. This is because Thomson scattering relies on the same physics as in the untransformed case and all the changes due to the new speed are contained in the propagation of the perturbations. This different scaling will affect the damping of the anisotropy power spectrum as scales will be damped at $k/c_T$ with respect to the untransformed case.

**Two fluid approximation**

To get a better insight we will solve numerically the set of equations. We will use the two fluid approximation outlined in [188], where the equations for the tight coupling approximation are solved until the a cut off scale in which the approximation is still valid. Then this solution is extrapolated to smaller scales. Finally the damping factor is included which leads to a rough approximation from the correct power spectrum. This approximation is not useful to estimate parameters but it can give us an idea of the effects of the disformal transformation on the power spectrum. In Figure 4.1 we have plotted some cases where the speed varies around ten percent from one. Note how the position of peaks gets shifted as the dispersion relation is modified. Furthermore, we see that the damping tail is modified. As we mentioned before, this can be understood because all the modes get a rescaling with the speed $c_T k$. Therefore as modes reach the horizon at a different time they will be affected by the damping scale differently.

**Modified tensor speed**

For comparison we analyse the case when the modes propagate with a speed different from the speed of light. This scenario could arise in a variety of modified gravity theories and only makes sense when the modification occurs in the early Universe as a very tight bound on the deviation from the speed of light has been set by LIGO/VIRGO in
Fig. 4.1 *TT* anisotropy power spectrum where the metric is under a disformal transformation (4.6).

our local environment at small redshift [85]. In this case the equation is simply,

\[ h'' + 2aHh' + (c_T^2 k^2)h = 0. \]  \hspace{1cm} (4.65)

The solution is the same as in the massless case with a rescaled wavenumber. Hence the gravitational wave will reach the horizon at a shifted time. For example during matter domination we have that

\[ h'' + \frac{4}{\tau} h' + (c_T^2 k^2)h = 0, \]  \hspace{1cm} (4.66)

whose solution is given by the spherical Bessel function

\[ h = 3 \frac{j_1(c_T k \tau)}{c_T k \tau}. \]  \hspace{1cm} (4.67)

It is constant until \( c_T k \tau = 1 \) after which it decays to zero. As the field enters the horizon at a different time, it leaves a modified signature on the CMB. All the acoustic peaks are shifted as the source function, see below, is rescaled by the same factor,
which is shown in figure 4.2. Notice that the location of the peaks are proportional

to $l/c_T$, so by constraining their positions one could infer $c_T$. This is supposed to be at the sub-percent level for CMB stage 4 experiments [109], which is several orders of magnitude worse than the LIGO/VIRGO bounds. One could try to analyse the modification of the tensor speed by performing a disformal transformation, which changes the slope of the gravitational light-cone. When doing this the speed of the scalar part of the perturbations is modified [2]. This would imply that the acoustic peaks of the temperature power spectrum would be shifted.

To summarise this section, we have shown that coupling matter to the Einstein equations does not allow us to remove the speed of sound for the tensor modes by means of a disformal transformation. This is because different scales are invariant under a disformal transformation, but the observables are not invariant as they can trace information contained in the transformation.
4.4 Galileons

So far we have been studying model-independent signatures that might arise when performing a disformal transformation, but it will be interesting to study what we can learn from applying this framework to theories where a $c_T \neq 1$ arise. It is very hard to classify the large amount of theories that modify gravity and thus allow for a speed of tensor modes different from one, furthermore some of these theories might have some other features that we have ignored, such as a non vanishing anisotropic stress energy tensor [69]. Nevertheless by requiring any theory to be able to give account of inflation and any later cosmological evolution highly restricts the class of available models.

We will now study gravitational waves on Galileon theories, which can be embedded in inflation and also allows cosmological solutions [192, 74]. These are defined as the field equations invariant under the Galileon transformation $\phi \rightarrow \phi + bx$ where $b$ is a constant. They have a number of interesting properties but we will focus on the fact that the covariantised Galileon theory allows large curvature operators to modify the speed of sound of the gravitational waves. Since in this theory the speed of the tensor modes is not necessarily one, it will demonstrate that the disformal transformation is a helpful way to constrain and better understand theories of gravity.

Furthermore Galileon theories are related to the weakly coupled regime of massive gravity, and can be generalised to Horndeski theories. This allow us to understand analytically the behaviour of higher order curvature corrections in a large class of well motivated scenarios.

Galileon Action

A curved background generally brakes the Galileon symmetry producing higher order equation of motions. Since we will focus on cosmological backgrounds, it is necessary to consider an alternative to the original Galileon theories. Insisting on second order equations of motion, and accepting a breaking of the shift symmetry proportional to the background curvature yields the covariant formulation, whose action is [193],

$$ S \supset \int d^4x \sqrt{-g} \left[ R + c_2 \mathcal{L}_2 + c_3 \mathcal{L}_3 + c_4 \mathcal{L}_4 + c_5 \mathcal{L}_5 \right], $$

(4.68)
where

\[
\mathcal{L}_2 = \frac{1}{2} (\nabla \phi)^2, \\
\mathcal{L}_3 = \frac{1}{\Lambda^2} \Box \phi (\nabla \phi)^2, \\
\mathcal{L}_4 = \frac{(\nabla \phi)^2}{\Lambda^6} \left[ (\Box \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi - \frac{R}{4} (\nabla \phi)^2 \right], \\
\mathcal{L}_5 = \frac{(\nabla \phi)^2}{\Lambda^9} \left[ (\Box \phi)^3 - 3 \Box \phi \nabla_\mu \phi \nabla^\mu \phi + 2 \nabla_\mu \nabla_\nu \phi \nabla^\rho \phi \nabla^\mu \phi - G_{\mu \nu} \nabla_\rho \nabla^\nu \nabla_\rho \phi \right].
\] 

This action allows a modified speed of sound for the tensor modes because there are second order corrections to the Ricci scalar in the quartic and quintic Galileon.

We will study the propagations of gravitational waves in two cases. First Galileon Inflation where the background has a broken de Sitter symmetry which can be related to an inflationary epoch, and then we will study the Galileon on a FRW background during matter and radiation eras.

### 4.4.1 Galileon inflation

To allow inflation we consider a quasi de Sitter background representing an inflationary phase [192]. Expanding in terms of the metric \(ds^2 = -dt^2 + e^{Ht}dx^2\) we have that,

\[
S_0 = \int d^4 x a^3 \left[ \frac{c_2}{2} \dot{\phi}^2 + \frac{2c_3 H^2}{\Lambda^6} \dot{\phi}^3 + \frac{9c_4 H^2}{2 \Lambda^6} \dot{\phi}^4 + \frac{6c_5 H^3}{\Lambda^9} \dot{\phi}^5 \right].
\] 

To study how non-linearities affect the dynamics it is useful to define the parameter \(Z \equiv \frac{H \dot{\phi}}{\Lambda^3}\). Hence for \(Z \ll 1\) there is a weakly coupled regime where the upshot is very similar to slow roll inflation. For \(Z \gg 1\) non-linearities are present and the inflationary dynamics is modified, thus a more careful analysis has to be done. This goes beyond the reach of this work.

To see how gravitons propagate we will focus on the quartic Galileon. Expanding in terms of the metric, \(ds^2 = -dt^2 + a(t)^2 (\delta_{ij} + \gamma_{ij}) dx^i dx^j\) we find that

\[
(\Box \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \subset -\dot{\phi}^2 \left( \hat{\gamma}_{ij} \right)^2.
\]
where the terms not included are proportional to the background equations of motion. Then, substituting back into the action (4.73) we find

\[
S_g = \frac{1}{4} \int d^4x a^3(t) \sigma(t) \left[ (\dot{\gamma}_{ij})^2 - \frac{c_T^2}{a^2} (\nabla \gamma_{ij})^2 \right],
\]

(4.75)

where,

\[
2\sigma(t) = M_{Pl}^2 - 3c_4 \dot{\phi}^4 / \Lambda^2, \\
\sqrt{c_T} = 1 + \frac{4c_4 \dot{\phi}^2 / \Lambda^2}{M_{Pl}^2 - 3c_4 \dot{\phi}^2 / \Lambda^2}.
\]

(4.76)

In the decoupling limit, where gravitational degrees of freedom can be ignored, we have that the action for small scalar fluctuations can be obtained from (4.73) by using the Stuckelberg trick (Check Appendix C),

\[
S \supset \int d^4xa^3 \left[ \alpha \left( \frac{\dot{\pi}^2 - c_s^2}{a^2} (\partial \pi)^2 \right) + g_1 \dot{\pi}^3 + \frac{g_3}{a^2} \dot{\pi} (\partial \pi)^2 + \frac{g_4}{a^4} (\partial \pi)^2 \partial^2 \pi \right],
\]

(4.77)

where \( \alpha, c_s, g_1, g_2, g_3 \) are functions of the background. As we are going to be interested in effects where the nonlinearities are small we can expand the above functions in terms of \( Z \). We will restrict our attention to \( \alpha \) and \( c_s \) which are

\[
\alpha = \frac{\dot{\phi}^2}{2} \left( c_2 + 12c_5 Z + 54c_4 Z^2 + 120c_5 Z^3 \right), \\
c_s^2 = 1 - \frac{4c_3}{c_2} Z + \left( \frac{48c_3^2}{c_2^2} - \frac{28c_4}{c_2} \right) Z^2.
\]

(4.78)

The wavefunctions can be solved in the case for small \( Z \). Neglecting slow roll corrections, the power spectrum for superhorizon scales is given by,

\[
\Delta_R = \frac{1}{4\pi^3} \frac{H^4}{4\alpha e^3}.
\]

(4.79)

To summarise during Galileon inflation there are terms which modify the dispersion relation in both the scalar and tensor sectors through its nonlinearities. As in slow roll inflation the graviton action is much simpler than its scalar counterpart. Although only \( c_4 \) appears explicitly on the action, all the Galileon dynamics will be present on the background functions.
Conformal and disformal transformation

We will now study whether it is possible to remove this tensor speed from the action through a disformal transformation,

$$g_{\mu\nu} \rightarrow \frac{1}{c_T} \left( g_{\mu\nu} + (1 - c_T^2(t))n_\mu n_\nu \right). \quad (4.80)$$

It is simpler if we first focus on the terms $(\nabla \varphi)^2$ and $\Box \varphi$ to then derive how the Galileons actions transform. Under equation (4.80) these become

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = - \frac{\dot{\varphi}^2}{c_T} + c_T a^{-2} (\partial_i \varphi)^2; \quad (4.81)$$

$$\Box \varphi = g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi = - \frac{\ddot{\varphi}}{c_T} - \frac{3H}{c_T} \dot{\varphi} + a^{-2} c_T \partial^2 \varphi. \quad (4.82)$$

Specialising to the case of a homogeneous and isotropic background the Galileon action will then be rescaled as $\sqrt{-g} \mathcal{L}_n \rightarrow \sqrt{-g} \mathcal{L}_n/c_T^2$. The Galileon action for a de Sitter background then is written as,

$$S_0 = \int d^4 x a^3 \left[ \frac{c_2}{2c_T} \dot{\varphi}^2 + \frac{2c_3 H}{\Lambda^3 c_T^2} \dot{\varphi}^3 + \frac{9c_4 H^2}{2\Lambda^6 c_T^4} \dot{\varphi}^4 + \frac{6c_5 H^3}{\Lambda^9 c_T^6} \dot{\varphi}^5 \right]. \quad (4.83)$$

To keep track of how the perturbations transform it is useful to redefine $Z \equiv \frac{H \dot{\phi}}{\Lambda^3}$, as $\tilde{Z} = Z/c_T$. Now for the fluctuations the time derivatives add a $c_T^{-1/2}$ whereas spatial derivatives add $c_T^{1/2}$ to the corresponding terms in the Lagrangian. Since we defined the perturbations by the map, $t \mapsto t + \pi(x)$, we have that, $\phi \mapsto \phi + \dot{\phi} \tau$, hence the combination $\dot{\phi}\xi$ does not come with any factor of $c_T$. The coefficients for the Lagrangian (4.77) depends on the background as a function of the parameter $\tilde{Z} = Z/c_T$. Then the second order action for the perturbation is,

$$S \int d^4 x a^3 \alpha \left[ \frac{1}{c_T^2} \dot{\pi}^2 - \frac{c^2_s}{a^2} (\partial \pi)^2 \right], \quad (4.84)$$

where $\alpha$ and $c_s^2$ are now given by,

$$\alpha = \frac{\dot{\phi}^2}{2} (c_2 + 12c_3 \tilde{Z} + 54c_4 \tilde{Z}^2 + 120c_5 \tilde{Z}^3), \quad (4.85)$$

$$c_s^2 = 1 - \frac{4c_3}{c_2} \tilde{Z} + \left( \frac{48c_3^2}{c_2^2} - \frac{28c_4}{c_2} \right) \tilde{Z}^2 + ... \quad (4.86)$$
Following the same arguments the action for the gravitational waves is then given by,
\[
S_g = \frac{1}{4} \int d^4 x a^3(t) \tilde{\sigma}(t) \left[ \frac{(\dot{\gamma}_{ij})^2}{c_T^2} - \frac{c_T^2 (\nabla \gamma_{ij})^2}{a^2} \right], \tag{4.87}
\]
where,
\[
2\tilde{\sigma}(t) = M_{Pl}^2 - \frac{3c_4 \dot{\phi}^4}{c_T^2 \Lambda^2},
\]
\[
c_T^2 = 1 + \frac{4c_4 \dot{\phi}^2 / \Lambda^2}{c_T^4 M_{Pl}^2 - 3c_4 \dot{\phi}^2 / \Lambda^2}. \tag{4.88}
\]

**Removing \( c_T \)**

Now, as in Sec. 4.1 we want to remove the physical \( c_T \) by means of a conformal transformation. To do so we tune the parameter of (4.80) to be the inverse of the tensor speed. Doing so, the tensor action becomes,
\[
S_g = \frac{1}{4} \int d^4 x a^3(t) \tilde{\sigma}(t) c_T^2 \left[ (\dot{\gamma}_{ij})^2 - \frac{(\nabla \gamma_{ij})^2}{a^2} \right], \tag{4.89}
\]
whereas the scalar action (4.77) turns into,
\[
S = \int d^4 x a^3 \alpha c_T \left[ \dot{\pi}^2 - \frac{c_s^2}{c_T^2 a^2} (\partial \pi)^2 \right]. \tag{4.90}
\]

Now let us analyse this action. On the scalar part the background dependent functions gets corrected as as \( \alpha c_T \) and \( c_T^2 / c_s^2 \), hence the power spectrum (4.79) is now,
\[
\Delta_R = \frac{1}{4\pi^3} \frac{\tilde{H}^4}{4 \alpha c_s^2 c_T^2} = \Delta_R, \tag{4.91}
\]
because \( \tilde{H} = c_s^{-1/2} H \). Therefore, as we indicated before, inflationary observables are invariant under this transformation. Also this transformation imposes a constraint to the tensor speed as 4.88 becomes,
\[
c_T^2 = 1 + \frac{4c_4 \dot{\phi}^2 / \Lambda^2}{c_T^4 M_{Pl}^2 - 3c_4 \dot{\phi}^2 / \Lambda^2}. \tag{4.92}
\]
If we request \( c_T \) to be real, then \( c_4 \dot{\phi}^2 / \Lambda^2 > -10^{-2} M_{Pl}^2 \). Since the cut off scale is \( \Lambda \sim \phi \), the former bound can be rewritten as \( c_4 H^2 > -0.01 M_{Pl}^2 \). This strongly constraints
negative values for $c_4$. This is interesting as having a positive $c_4$ leads to superluminal propagation.

### 4.4.2 Galileons on a cosmological background

Now we would like to investigate how Galileon symmetry modifies the speed of tensor modes during the matter and radiation epochs. We will consider the quartic Galileon as we are interested in how the Galileon symmetry can enhance operators which modify the dispersion relation for the tensor modes.

During matter and radiation era the Galileon component will modify the background through the Einstein equation $3\dot{H}^2 = 8\pi G a^2 \sum_i \dot{\rho}_i$, where $i$ counts all the matter species which in this case includes the Galileon density as well as matter and radiation. Hence we need to consider a suitable evolution for the Galileon during this epoch. In the case of a quartic Galileon this is given by imposing that the Galileon density dominates at late times.

#### Tensor modes

For a FRW background $ds^2 = -dt^2 + a^2 dx^2$ the equations for tensor fluctuations will be very similar to those for inflation because any matter component will be coupled only to scalars. Therefore we have that for metric perturbations given by the line element $ds^2 = -a^2 d\tau^2 + 2a^2 E_{ij} dx^i dx^j$, the Einstein equations simplifies to,

$$E''_{ij} - c_T^2 \nabla^2 E_{ij} + 2\dot{H}E'_{ij}, \quad (4.93)$$

where

$$c_T^2 = 1 + \frac{4c_4 \dot{\phi}^2 / \Lambda^2}{M_{Pl}^2 - 3c_4 \dot{\phi}^2 / \Lambda^2}. \quad (4.94)$$

Hence, as a result of the curvature operators allowed in the Galileon theory we have a modified speed of sound. When confronted with experiments we could expect to get strong constraints for $c_4$. We can instead, try to keep $c_T$ large by removing it from this action by a disformal transformation.

#### Scalar modes

We need to obtain the perturbation equations during matter and radiation epochs, assuming that Galileons are only coupled to matter. Moreover we will be interested
in the case of a dominating quartic Galileon $c_4$, hence there it will not be anisotropic stress energy tensor and $\phi = \psi$. We can then modify what we did in Section 3 for scalar perturbations because the Galileons will act as a source term to (4.22), while their equation are given by varying (4.73). At late time this yields a self accelerating solution, but we are more interested in the case before decoupling. Then we need to see whether there is a modification to the perturbation growth equation at early times. There are many ways of parameterising deviations from GR. Let us follow the approach of [194], where the Galileon contributions are parameterised as fifth-forces. It follows that the continuity equations (4.22) become,

$$\nabla \cdot v' + 3H \nabla \cdot v + \nabla \psi = F_\phi,$$  \hspace{1cm} (4.95)

$$\delta' + (\nabla \cdot v - 3\phi') = j_\phi,$$  \hspace{1cm} (4.96)

where $F_\phi$ and $j_\phi$ are functions of the Galileon background only. In the case where $F_\phi$ is proportional to $\nabla \cdot v$, as it is for the quartic Galileon we can redefine the Hubble parameter. Hence the equation becomes,

$$\nabla \cdot v' + 3\tilde{H} \nabla \cdot v + \nabla \psi = 0.$$  \hspace{1cm} (4.97)

With this in mind we can now build the second order equation. Since there are effects on the background which are similar for both scalar and tensor modes it is useful to write the equations in such a way that we can identify what corresponds to the disformal transformation. We wrote a two fluid equation (4.45), which demonstrated that the speed for the photon density was slowed down because of baryons. The equation now becomes

$$\delta''_{\gamma} + \tilde{H}R \delta'_{\gamma} - \frac{1}{3(1 + R)} \nabla^2 \delta_{\gamma} = 4\phi'' + \frac{\tilde{H}R}{1 + R} \phi' + \frac{4}{3} \nabla^2 \psi.$$  \hspace{1cm} (4.98)

It follows from this, that the main contribution from a Galileon background will be a different Hubble parameter. Whereas this effect will certainly affect the solution of equation (4.98), it does not change the speed of the scalar and density perturbations $\phi$ and $\delta_{\gamma}$. Moreover a change on the Hubble parameter can be removed by redefining Newton’s constant through Equation (4.19). This will set a new scale for the whole set of Einstein equations but the scalar perturbation will still have speed one on the new frame.

Therefore we can apply the conformal transformation and rely on the results we have obtained for inflation for the speed of the tensor modes. Since the transformation
4.5 Conclusions

will rescale a subdominant contribution to the Hubble scale we can neglect this effect. The transformed equation (4.98) is,

$$\delta'' + \frac{\dot{H}R}{1+R}\delta' - \frac{c_T^2}{3(1+R)}\nabla^2\delta = 4\phi'' + \frac{\dot{H}R}{1+R}\psi' + \frac{4}{3}c_T^2\nabla^2\psi. \tag{4.99}$$

With this we can use what we detailed in Section 4. Then, the disformal transformation will change the temperature power spectrum in a significant way and the physical speed of sound cannot be removed from the tensor modes. Although this result assumes that the Galileon evolution does not affect significantly the Einstein equations this is a valid assumption to make, as the matter and radiation considered is well within the Vainshtein radius, and thus significant effects are screened. On the other hand we have allowed $c_4$ to be large compared to the other terms $c_3$ and $c_2$. Although this does not hold in general, it allows us to study the effect of the disformal transformation more transparently as we do not have to take into account the complicated equation that these terms induce.

In any case it will be interesting to study numerically how these terms are constrained due to requirement of $c_T^2 \approx 1$. For example (4.94) reduces to $|c_4\dot{\phi}^2/\Lambda^2| \lesssim 10^{-7}M_{Pl}^2$.

4.5 Conclusions

In this chapter we have studied how gravitational waves can vary their speed during different cosmological epochs. This is an interesting question as modifications to the speed of sound for tensor modes can arise in a large class of models. As there are strong constraints on the speed of the tensor modes we have focused on the possibility to set it to 1 by means of a disformal transformation. This has been shown to be possible in inflation but we have shown that at later times this is more subtle.

The disformal invariance is dependent on the spatial invariance of inflation, it can then be argued that in scenarios where these symmetries are broken, e.g. solid inflation, there will be an imprint on the equations of motion because the vevs will not transform covariantly under a change of coordinates. Also it has been shown that when there is a feature in the potential, applying a disformal transformation does not remove the speed of the tensor modes but exchanges it with another effect [195].

In the case of multifield inflation it could be the case that isocurvature modes are coupled to the inflationary direction modifying the observables. In principle, these directions might not respect any symmetry and thus a time scale could be introduced.
Therefore, a disformal transformation will leave an imprint on the observables and a speed of sound for the tensor modes can not be removed. It will be interesting to study which symmetry breaking patterns can produce physical gravitational waves.

By studying the equations that give rise to the CMB we have shown that the anisotropy power spectrum cannot remove a $c_T \neq 1$ with a disformal transformation, more precisely the scalar potential acts as if it were propagating at a speed other than one. This effect is present in the baryoacoustics oscillations and also in the diffusion damping. On the first it shifts the acoustic peaks as the modes propagates at a different speed. The relevant scales for the damping are invariant under a transformation, hence the scales are affected differently as modes became smaller or larger compared to the horizon size.

This result is interesting and it can be used to constrain the role of higher curvature operators. We have shown that they can arise in Galileon theories because there are higher order terms, containing corrections to the Ricci scalar, which can be very large. By using this we have found bounds on the parameter space which can shed some light on the realisation of a large class of theories.
Chapter 5

Signatures of graviton masses on the CMB

5.1 Introduction

In this chapter we will focus on the effects of a modified graviton on the B-modes from the cosmic microwave background (CMB). In order to analyse the effects of a modified gravity sector we will assume that the initial conditions are adiabatic and that there is a detectable tensor to scalar ratio $r$. Thus all the important effects will be produced by a modification of gravity during matter and radiation domination. The presence of either a mass or a different speed from the speed of light for gravitons, and the changes in the dispersion relation lead to interesting effects on the low $l$ B-modes. For masses above $10^2 H_0$ this yields an observable plateau [70, 196, 3] in the CMB spectrums. Since this effect cannot be produced by any other known effect, one may hope to constrain gravity very precisely in this manner.

We have found an analytical solution showing that the source function of the B-modes has a plateau until recombination, instead of being zero and then peaking at recombination as happens in the massless case. This plateau is then projected onto the CMB power spectrum producing a plateau for the small $l$ modes. Moreover the amplitude of the plateau oscillates with the mass of the graviton, so this effect could be used to constrain the mass of the graviton.

We also focus on the existence of multiple gravitons which could couple to matter with different strengths. This leads to a richer phenomenology but it also requires a more careful treatment. We have been able to extend our analysis to models with more than one graviton and will assume that there is no hierarchy between the masses or the couplings of the gravitons to matter. We can diagonalise the coupled system.
Signatures of graviton masses on the CMB of equations during matter and radiation domination and thus show how the signal behaves in different configurations. We use this to show that even in the cases with more than one graviton the signal is still similar to that with one graviton, i.e. qualitatively different from the one in massless gravity. The effect of the mixing between a massive and a massless graviton would also be characterised by a shift of the first few peaks of the B-mode CMB spectrum.

We also study the effect of the instability problem of doubly coupled bigravity on the B-modes [197, 198]. As a result the effects on the B-modes of the CMB spectrum that we present when an instability occurs in the radiation era only apply to a toy model where the description with two coupled gravitons subject to the tachyonic instability is extrapolated all the way to the end of inflation. In this regime, strictly speaking, one may expect that the UV completion of doubly coupled bigravity would give differing results which are beyond this chapter.

We first derive analytical solutions for massive gravity during matter and radiation domination in section 5.2 and study the effects of a change of the speed of gravitons. We then calculate in section 5.3 how the B-mode power spectrum behaves for massive gravity, where we get an analytical solution at low \( l \). We then examine the effects of adding another graviton coupled to matter in section 5.4 and we include the case of a pressure instability in radiation domination [197–199]. Finally we conclude.

## 5.2 General results

### 5.2.1 Massive graviton

We want to investigate the propagation of a massive graviton, with mass \( m \), when the background cosmology is described by a FRW (Friedmann-Robertson-Walker) Universe. We will focus on the gravitational waves during matter and radiation domination, as these are the relevant ones for the CMB. The graviton equation is

\[
E'' + \left( k^2 + m^2 a^2 - \frac{a''}{a} \right) E = 0, \tag{5.1}
\]

where we have suppressed the indices and \( E_{ij} = ah_{ij} \) where \( h_{ij} \) represents the transverse and traceless part of the tensor perturbation. Note that, \( a''/a = (aH)^2 + (aH)' \), where \( aH \) is the size of the horizon. Then for \( k^2 + m^2 a^2 \ll \frac{a''}{a} \) modes are out of the horizon and evolve with constant amplitude \( h_{ij} \). The re-entry of the modes inside the horizon

\[\text{For a more thorough discussion see [200].}\]
depends on the mass of the graviton now, contrary to the case of massless gravitons. In the following we will consider that the mass of the graviton is of order $H_0$ or larger. When $k^2 + m^2a^2 \gg \frac{a''}{a}$ the modes start oscillating with a frequency given by $\omega^2 \propto k^2 + m^2a^2$ in the WKB approximation, which leads to imprints on the B-mode spectrum. In order to investigate the precise nature of these oscillations we will solve (5.1) during matter and radiation domination.

**Matter domination**

During matter domination we have that $a = \tilde{H}_0^2 \tau^2 \propto H_0^2 \tau^2$, where $\tilde{H}_0 = O(H_0)$ and the graviton equation becomes

$$E'' + \left( k^2 + m^2 \tilde{H}_0^4 \tau^4 - \frac{2}{\tau^2} \right) E = 0. \quad (5.2)$$

Let us rewrite the equation in terms $h_{ij}$ which reads

$$h'' + 4 \tau h' + (k^2 + m^2 \tilde{H}_0^4 \tau^4)h = 0. \quad (5.3)$$

Using the variable $x = \frac{m\tilde{H}_0^2 \tau^3}{3}$, the wave equation becomes

$$\ddot{h} + \frac{2}{x} \dot{h} + (1 + \frac{k^2}{m^2 \tilde{H}_0^2 \tau^2})h = 0, \quad (5.4)$$

where the dot means $d/dx$.

Notice that in the case where $m^2a^2 \gg k^2$ the solution is a spherical Bessel function of order 0 which is constant at early times until it enters the horizon and then decays. On the other hand when $m^2a^2 \ll k^2$, the field behaves as a massless spin-2 field. An approximate solution which is valid for a wide range of times is given by

$$h = 3 \frac{j_1(k\tau)}{k\tau} \times j_0\left(\frac{1}{3}m\tilde{H}_0^2 \tau^3\right). \quad (5.5)$$

as can be seen in fig.5.1 where the approximation is very accurate up to horizon re-entry. On the other hand the amplitude of the oscillations within the horizon is suppressed due to the combined oscillatory behaviour of the two spherical Bessel functions at small scales. Here the approximate solution differs from the numerical one, but we still capture the most important features for our analysis.
Radiation domination

To find the solutions of the wave equation during radiation domination we follow the same procedure as for the matter era. During this epoch $a = \hat{H}_0 \tau$ where $\hat{H}_0 = \hat{H}_0^2 \tau_{eq}$ where $\tau_{eq}$ is the conformal time at matter-radiation equality, the wave equation is now

$$h'' + \frac{2}{\tau} h' + \left( k^2 + m^2 \hat{H}_0^2 \tau^2 \right) h = 0. \quad (5.6)$$

Defining $x = m \hat{H}_0 \tau^2 / 2$ leads to

$$\dot{h} + \frac{3}{2x} \ddot{h} + \left( \frac{k^2}{2m \hat{H}_0 x} + 1 \right) h = 0. \quad (5.7)$$

When $m^2 a^2 \gg k^2$, the solution is $\propto (m \hat{H}_0 \tau^2 / 2)^{1/4} j_{-1/4}(m \hat{H}_0 \tau^2 / 2)$ and when $m^2 a^2 \ll k^2$ it behaves like $j_0(k \tau)$. A good solution that can interpolate between both regimes is given by

$$h \propto (m \hat{H}_0 \tau^2 / 2)^{1/4} j_{-1/4}(m \hat{H}_0 \tau^2 / 2) j_0(k \tau). \quad (5.8)$$
where again we have that the approximation is very accurate outside the horizon but it oscillates too fast after re-entry. The matching between the solutions in the radiation and matter eras is presented in the Appendix.

**Superhorizon $k\tau \ll 1$ modes**

At wavelengths much larger than the horizon the solution during matter domination is more relevant, as the super horizon modes enter later and make the most important contribution to the B-mode power spectrum. We can expand the matter era solution as

$$h \sim \frac{3 \sin \left( \frac{mH_0^2 \tau^3}{3} \right)}{mH_0^2 \tau^3} \left( 1 - \frac{(k\tau)^2}{10} \right) + O \left( k^4 \right).$$

(5.9)

Note that the first term goes to one when the mass is zero, and we recover the massless solution which is constant out of the horizon. As $\tau$ grows, we can have two possibilities depending on whether $\frac{m\tau^3}{3} > k\tau$ or not. A given mode enters the horizon earlier when the mass of the graviton is big enough to satisfy $\frac{m\tau^3}{3} > k\tau$. In the contrary case the massive part will not lead to a significant modification of the gravitational waves.

These new oscillation due to the mass introduce an imprint on the B-modes which will differ from the one coming from the massless modes. As the contribution of the gravitational waves to the B-modes is dominated by the modes evaluated at $\tau = \tau_{rec}$, for all wavenumbers such that $\frac{m\tau_{rec}^3}{3} > k\tau_{rec}$ the power spectrum will be dominated by the massive part of $h$.

### 5.3 Polarisation and massive gravitons

It is instructive to rewrite the polarisation equations, and how they are modified in the presence of a massive graviton and a change in the speed of propagation of tensor modes. We will focus only on B modes for now as they give primordial information, although the results for E modes are similar.

Thomson scattering in the early universe generates a linear polarisation that can be best described by a $2 \times 2$ traceless tensor involving the $Q$ and $U$ Stokes parameters. It is convenient to pick up a particular combination of these two parameters which only depends on the tensor modes. The $B$ modes have the parity of a magnetic field. The $E$ modes depend on the scalar modes and will not be useful for our analysis.
The polarisation tensor state $\Psi$ can be expressed in term of temperature and polarisation multipoles as [104],

$$
\Psi \equiv \left[ \frac{1}{10} \Delta^{(T)}_{T0} + \frac{1}{35} \Delta^{(T)}_{T2} + \frac{1}{210} \Delta^{(T)}_{T4} - \frac{3}{5} \Delta^{(T)}_{P0} + \frac{6}{35} \Delta^{(T)}_{P2} - \frac{1}{210} \Delta^{(T)}_{P4} \right].
$$

(5.10)

The B modes power spectrum is given by

$$
C_{B\ell} = (4\pi)^2 \int k^2 dk P_h(k) \left| \int_0^{\tau_0} d\tau g(\tau) \Psi(k, \tau) \left( 2 j'_l(k\tau) + \frac{4j_l}{k\tau} \right)^2 \right|,
$$

(5.11)

where $g(\tau)$ is the visibility function which is defined in terms of the optical depth for Thomson scattering $\kappa$ as

$$
g(\tau) = \dot{\kappa}e^{-\kappa}.
$$

(5.12)

$P_h(k) \propto k^{n_T-3}$, with $n_T \sim 0$ to leading order, is the primordial tensor power spectrum and the integral is between the initial time and now at $\tau_0$. We denote by

$$
\Delta_{B\ell} = \int_0^{\tau_0} d\tau g(\tau) \Psi(k, \tau) \left( 2 j'_l(k\tau) + \frac{4j_l}{k\tau} \right)^2,
$$

(5.13)

the source function.

### 5.3.1 Boltzmann equations for tensor perturbations

The temperature and polarisation multipoles result from the Boltzmann equations associated with the Thomson scattering [104],

$$
\dot{\Delta}^{(T)}_T + i k \mu \Delta^{(T)}_T = -\dot{h} - \dot{\kappa}(\Delta^{(T)}_T - \Psi),
$$

(5.14)

$$
\dot{\Delta}^{(T)}_P + i k \mu \Delta^{(T)}_P = -\dot{\kappa}(\Delta^{(T)}_P + \Psi).
$$

(5.15)

where metric perturbations evolve according to

$$
\ddot{h} + 2\dot{H}\dot{h} + (k^2 + m^2a^2)h = 0.
$$

(5.17)
In the tight coupling approximation, they read

\[
\dot{\Delta}_{T0} = -\dot{h} - \kappa [\Delta_{T0} - \Psi],
\]
\[
\dot{\Delta}_{P0} = -\dot{\kappa} [\Delta_{P0} + \Psi],
\]
\[
\dot{\Delta}_{Tl} = 0, \quad l \geq 1,
\]
\[
\dot{\Delta}_{Pl} = 0, \quad l \geq 1.
\]

(5.18)

Using \(\Psi = \frac{1}{10} \Delta_{T0}^T\) this leads to

\[
\Psi(\tau) = -\frac{1}{10} \int_0^\tau d\tau' \dot{h}(\tau') e^{-\frac{2}{10}(\kappa(\tau') - \kappa(\tau))}.
\]

(5.19)

Assuming that the visibility function is a peaked Gaussian distribution function during recombination, i.e. \(g(\tau) \equiv \kappa e^{-\kappa} = g(\tau_{rec}) e^{-\frac{(\tau - \tau_{rec})^2}{\Delta_{rec}^2}}\), implies that during recombination \(\dot{\kappa} \approx \frac{\kappa}{\Delta_{rec}}\). Also assuming that \(h\) varies slowly during recombination, we have then the approximation

\[
\Psi(\tau) \sim -\frac{\dot{h}(\tau_{rec})}{10} e^{\frac{1}{10} \kappa(\tau)} \Delta_{rec} \int_1^\infty dx \frac{dx}{x} e^{-\frac{2}{10} \kappa x},
\]

(5.20)

where in the last integral we have introduced the variable \(x = \kappa(\tau')/\kappa(\tau)\). This changes the integration limits to 1 and \(e^{\tau/\Delta_{rec}}\) which can be approximated to be infinity as long \(\Delta_{rec}\) is very small.

In the above calculation we have assumed that \(h(\tau)\) peaks around the value \(\tau_{rec}\) as the visibility function is sharply peaked. A better approximation can be obtained by averaging \(h(\tau)\) through recombination. By assuming that the visibility function is a Gaussian as before we have

\[
\dot{h}(\tau) \rightarrow \int_0^\tau d\tau' g(\tau') \dot{h}(\tau') \sim \dot{h}(\tau_{rec}) e^{-\frac{(k\Delta_{rec})^2}{2}},
\]

(5.21)

for modes inside the horizon.

### 5.3.2 Large angular scales

We have seen that there are important changes to the wave equation when the graviton is outside the horizon. As this effect is predominant at large angular scales we will proceed to study this sector in more detail. We first start with the massless case. Here, the solution of the graviton during matter domination is given by

\[
h(\tau) = 3 \frac{j_1(k\tau)}{k\tau},
\]

(5.22)
where the factor of 3 appears to normalise the wave function to 1 at $k\tau = 0$. This solution is constant initially and then decays in an oscillating fashion after $k\tau = 1$. We are interested in modes that enter the horizon around the time of recombination. These modes correspond to scales which are so large that when they enter the horizon the universe is in matter domination. Now on such large scales the effects of recombination are not relevant so we can approximate $g(\tau) = \delta(\tau - \tau_{\text{rec}})$. The spectrum (5.11) of B-modes becomes

$$C_{\text{BB},l}^T \propto \int k^2 dk P_h(k) \left| h(k\tau_{\text{rec}}) \left[ 2j'_l(k(\tau_0 - \tau_{\text{rec}})) + 4\frac{j_l(k(\tau_0 - \tau_{\text{rec}}))}{k(\tau_0 - \tau_{\text{rec}})} \right] \right|^2. \quad (5.23)$$

In the massless case the graviton wave function is constant until it enters the horizon. As the behaviour of the integral is dominated by $k \approx l/(\tau_0 - \tau_{\text{rec}})$, we have the approximation

$$C_{\text{BB},l}^d \approx (k^5 P_h(k, \tau_{\text{rec}}))|_{k\approx l/(\tau_0 - \tau_{\text{rec}})} \int d\ln x \left| 2j'_l(x) + 4\frac{j_l(x)}{x} \right|^2, \quad (5.24)$$

where $h(k\tau_{\text{rec}}) \propto k\tau_{\text{rec}}$ on large scales. The integral of the spherical Bessel function scales as $\propto l^{-2}$ for small $l$, then we have that for large scales,

$$l(l+1)C_{\text{BB},l}^d \sim l(l+1), \quad (5.25)$$

which grows linearly with $l$ for small $l$. In the massive case, the large scale behaviour is very different, see figure 5.2 for instance.

**Massive graviton case**

In the case of a massive graviton during matter domination the solution is approximately given by

$$h = 3\frac{j_1(k\tau)}{k\tau} \times j_0\left(\frac{1}{3} m H_0^2 \tau^3\right). \quad (5.26)$$

We will focus on the case where $ma \gg k \approx l/(\tau_0 - \tau_{\text{rec}})$ as the B-mode spectrum will be stronger at small $l$, and we comment on the other cases. In this regime the graviton has a constant amplitude until it enters the horizon. After entering the horizon the graviton will behave as if massless. On large angular scales the main difference with the massless case springs from the source function. Assuming a Gaussian visibility function, and that projection factors involving spherical Bessel functions vary slowly
over the last scattering surface, using (5.20) and (5.21), the source function becomes

\[
\Delta_{Bl}(k) = P_{Bl}[k(\tau_0 - \tau_{rec})]\frac{\dot{h}(\tau_{rec})}{10} \Delta_{\tau_{rec}} e^{-(k\Delta_{\tau_{rec}})^2/2} \int_0^\infty d\kappa e^{-\kappa} \int_1^\infty dx \frac{\dot{\kappa}}{x} e^{-\frac{2}{3} \pi \kappa x},
\]  

(5.27)

where

\[
P_{Bl}[k(\tau_0 - \tau_{rec})] = |2j'_l(k(\tau_0 - \tau_{rec})) + 4\frac{j_l(k(\tau_0 - \tau_{rec}))}{k(\tau_0 - \tau_{rec})}|^2,
\]  

(5.28)

which implies that the source function can be written as,

\[
\Delta_{Bl}(k) = P_{Bl}[k(\tau_0 - \tau_{rec})] \dot{h}(\tau_{rec}) \Delta_{\tau_{rec}} e^{-(k\Delta_{\tau_{rec}})^2/2} \left( \frac{1}{10} \log \frac{19}{9} \right).
\]  

(5.29)

As we are considering the regime where \(m^2\tilde{H}_0^4 \tau_{rec}^4 \gg k^2\) we can use the approximate solution (5.5) to calculate \(\dot{h}(\tau_{rec})\),

\[
\dot{h}(\tau_{rec}) \approx \cos\left(\frac{m\tau_{rec}^3}{3}\right) (1 - \frac{(k\tau_{rec})^2}{10}).
\]  

(5.30)

Now we can calculate the power spectrum of B-modes at large scales. As the source function falls off inside the horizon, the integral over \(k\) is dominated by large scales corresponding to the low \(l\) modes. In this case the spherical Bessel functions are constant at low \(l\). Therefore the B-mode spectrum on large scales is essentially sensitive to (5.30) yielding

\[
C_l^B \propto \cos^2\left(\frac{m\tau_{rec}^3}{3}\right) \tau_{rec}^2 \propto \frac{1}{2\tau_{rec}^2} \left( 1 + \cos\left(\frac{2m\tilde{H}_0^4 \tau_{rec}^4}{3}\right) \right).
\]  

(5.31)

which is a constant amplitude corresponding to a low \(l\) plateau whose amplitude oscillates with the mass \(m\).

We can evaluate the maximal angular multipole \(l\) where effects of a massive graviton can be observed. The dispersion relation of a massive graviton changes from the massless case when \(k \approx m \times a\) then at recombination this corresponds to

\[
l \approx \frac{m}{\tilde{H}_0} (1 + z_r)^{-1} \int_0^{l(1+z_r)^{-1}} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_m x + \Omega_r}} \approx 3.3(1 + z_r^{-1}) \frac{m}{\tilde{H}_0},
\]  

(5.32)

where for \(z_R \approx 1080\) we have that the B-modes are not modified for masses below 300\(\tilde{H}_0\). The bounds obtained from the LIGO/VIRGO detection are \(m_g < 10^{-22}eV\) [201, 202],
Fig. 5.2 The spectrum of B modes for different values of the graviton mass.

hence a detection of B-modes could test masses 8 orders of magnitude below the current experimental bounds.

5.3.3 Flat sky limit

In order to calculate the $l$ dependence of the B-modes angular power spectrum we use the flat sky approximation, which implies that we Fourier expand instead of projecting using spherical harmonics. In the flat sky limit the amplitude of the scalar and polarisation modes can be written as,

$$a_T^c = \int_0^{\tau_0} d\tau \frac{d^3k}{(2\pi)^3} \zeta(k) e^{-ik\cdot D} S_T^c(k, \tau)(2\pi)^2 \delta^{(2)}(k^\parallel D - 1), \quad (5.33)$$

$$a_B^h = \int_0^{\tau_0} d\tau \frac{d^3k}{(2\pi)^3} \sum \pm h^\pm e^{-ikz D} (2\pi)^2 \delta^{(2)}(k^\parallel D - 1), \quad (5.34)$$

where $S^X$ are the source functions which in the case of B modes is $-g^2 \Psi(k, \tau)$ and in the case of scalar modes is given by the sum of the two scalar gravitational potentials.
Also $D = \tau - \tau_0$. The $\langle BB \rangle$ correlation is given by

$$\langle a_B^h(\lambda) a_B^h(\lambda') \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} 2 \langle h h \rangle \Delta_B^h(k_1^z, l_1) \Delta_B^h(k_2^z, l_2)(-4) \frac{k_1^z k_2^z}{k_1 k_2} (2\pi)^4 \delta^{(3)}(k_1 + k_2),$$

(5.35)

where we have defined the transfer function as,

$$\Delta_B^h(k^z, l) = \int_0^{\tau_0} d\tau S_B^h(k)e^{-ik^z D} \frac{1}{(k^2 + l^2/D^2)^{1/2}} \delta^{(2)}(k \| D).$$

(5.36)

The integral is dominated by $D \sim D'$ and can be rewritten as

$$\langle a_B^h(\lambda) a_B^h(\lambda') \rangle = \int \frac{dk^z}{(2\pi)^2} \int_0^{\tau_0} d\tau \frac{S_B^h(k)}{\sqrt{(k^2 + l^2/D^2)^{1/2}}} \frac{1}{(k^2 + l^2/D^2)^{1/2}} \Delta_B^h(\tau_{\text{rec}}) \frac{1}{10} e^{-\sqrt{(k^2 + l^2/D^2)^2} \Delta_{\tau_{\text{rec}}}^2/2} \times g(\tau_{\text{rec}}) e^{-\tau_{\text{rec}}/\Delta_{\tau_{\text{rec}}}} e^{-\kappa \int_1^\infty \frac{dx}{x} e^{-\alpha x} e^{-ik^z D}},$$

(5.37)

where we have replaced $S_B^h$ by its value from (5.20). The integral gets most of its contribution around $\tau_{\text{rec}}$ because of the exponential function of the opacity which
vanishes before recombination. Using $D \sim \Delta \tau_{\text{rec}}$ we get
\[
\tilde{\Delta}(k, l) \sim \frac{\sqrt{8 \langle hh \rangle (k^z)}}{\sqrt{(k^z)^2 + l^2 / D_r^2}} \frac{g(\tau_{\text{rec}}) \dot{h}(\tau_{\text{rec}})}{10} e^{-(\sqrt{(k^z)^2 + l^2 / D_r^2})^2 \Delta \tau_{\text{rec}}^2 / 2} e^{-ik_z D_r}.
\]
and finally
\[
\langle a_B^h(1) a_B^h(1') \rangle = \int \frac{dk^z}{(2\pi)^2} \frac{8 \langle hh \rangle (k^z)^2}{(k^z)^2 + l^2 / D_r^2} \frac{g(\tau_{\text{rec}}) \dot{h}^2(\tau_{\text{rec}})}{100} e^{-(\sqrt{(k^z)^2 + l^2 / D_r^2})^2 \Delta \tau_{\text{rec}}^2} e^{-2ik_z D_r} \delta^{(2)}(1 + 1').
\]
Using the stationary phase method with a saddle point at $k^z \approx -i D_r \Delta \tau_{\text{rec}}$, the power spectrum becomes
\[
\langle a_B^h(1) a_B^h(1') \rangle = \frac{A_s r^4}{4\pi^{3/2}} \frac{g(\tau_{\text{rec}})^2 \cos^2 \left( \frac{m H_0^2 \tau_{\text{rec}}^3}{3} \right)}{100 \tau_{\text{rec}}^2} e^{-\frac{p_r^2}{\Delta \tau_{\text{rec}}^2} - \frac{l^2 \Delta \tau_{\text{rec}}^2}{D_r^2}} \delta^{(2)}(1 + 1').
\]
where we have replaced $\langle hh \rangle = A r k^{-3}$ with $A$ the amplitude of the primordial scalar perturbations and $r$ the tensor to scalar ratio.

This result implies that the modes will stay constant until $l \approx \frac{D_r}{\Delta \tau_{\text{rec}}} \approx 100$. Although this result is very simplified it shows that the oscillations due to the graviton mass modifies strongly the power spectrum. Note that the damping effect is independent of the mass of the graviton, see figure 5.2 for instance.

### 5.4 Bigravity

#### 5.4.1 Propagating modes

In this section we consider the case of bigravity [198, 197, 199] where one graviton is massless and the other is massive in a Minkowski background. This is inspired by the bigravity case of massive gravity although we only use the gravitational sector of this model and do not deal with problems such as the range of validity of the model related to the existence of strong coupling issues and vectorial instabilities in the radiation era. We only consider masses of cosmological relevance, and we only use this model as an illustration for the type of physics induced by the presence of two gravitons. The

\footnote{We have $e^{-\kappa} \int_1^\infty \frac{dx}{x} e^{-\frac{9\kappa}{10} x} = e^{-\kappa} \Gamma(0, \frac{9\kappa}{10})$, where $\Gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt$ is the incomplete Gamma function. The integral decays for large values of $\kappa$ and is order one for $\kappa \sim \kappa_r$ so we can neglect it in the final expression.}
CMB signal in this case involves the two gravitons implying that

$$\Psi \approx \left[ \beta_1 \times \dot{h}_1 + \beta_2 \times \dot{h}_2 \right]^2,$$

(5.43)

where \( \beta_{1,2} \) are coupling constants. In the generic case the gravitational waves propagate in different FRW metrics characterised two scale factors \( a_{1,2} \) and the ratio between the two lapse functions \( b \) leading to the two wave equations in vacuum

$$E''_1 + k^2 E_1 + (M^2_{11} a_1^2 - \frac{a''_1}{a_1})E_1 + M_{12} a_1 a_2 E_2 = 0,$$

$$E''_2 + b^2 k^2 E_2 + (M^2_{22} a_2^2 - \frac{a''_2}{a_2})E_2 + M_{21} a_1 a_2 E_1 = 0,$$

(5.44)

where we have dropped the tensorial indices so \( E_1 \) should be understood as \( E^i_{1j} \), and \( M_{12} = M_{21} \). Notice that in the following the mass matrix \( M^2_{ij} \) is not restricted to a particular form. In the following we will concentrate on such toy models inspired by doubly coupled bigravity. Typically we will focus on models with the same background cosmology but, contrary to doubly coupled bigravity where the mass matrix depends on the bigravity potential term, here we will take it as a phenomenological input. At the background and as in the case of doubly coupled bigravity we have

$$b = \frac{a_1 H_1}{a_2 H_2},$$

(5.45)

for one of the two branches of cosmological backgrounds. Moreover \( b \equiv 1 \) in both matter and radiation eras which we will use when studying the gravitational waves in such epochs. The conformal time is such that the Hubble rates are defined by

$$H_{1,2} = \frac{a a_{1,2}}{a^2_{1,2} d \eta},$$

(5.46)

implying that when \( b = 1 \) the two scale factors are proportional with \( \beta_1 a_2 = \beta_2 a_1 \) [197]. Then during matter domination since \( a_i = \frac{\beta_i}{\beta_1 + \beta_2} H^2_0 \tau^2 \) such that \( a = \beta_1 a_1 + \beta_2 a_2 = H^2_0 \tau^2 \), we have that the above equations become

$$E''_1 + \left( k^2 + M^2_{11} H^4_0 \tau^4 - \frac{2}{\tau^2} \right) E_1 + M_{12} H^4_0 \tau^4 E_2 = 0,$$

$$E''_2 + \left( b^2 k^2 + M^2_{22} H^4_0 \tau^4 - \frac{2}{\tau^2} \right) E_2 + M_{21} H^4_0 \tau^4 E_1 = 0,$$

(5.47)
where we have redefined $M_{ij}^2 \to \frac{\beta_i \beta_j}{(\beta_i^2 + \beta_j^2)} \tilde{M}_{ij}^2$. We want to find a particular combination of the two gravitons which satisfies an equation for a massive graviton

$$f'' + \left( k^2 + \tilde{M}_{ij}^2 \hat{H}_0^2 \tau^4 - \frac{2}{\tau^2} \right) f = 0, \quad (5.48)$$

where $f = \lambda_1 E_1 + \lambda_2 E_2$ and the coefficients $\lambda_1$ and $\lambda_2$ are constant. This implies that

$$\lambda_1 (M_f^2 - M_{11}^2) - \lambda_2 M_{12}^2 = 0,$$

$$\lambda_2 \left( \frac{k^2}{\tau^4} (1 - b^2) + M_f^2 - M_{22}^2 \right) - \lambda_1 M_{21}^2 = 0, \quad (5.49)$$

which admits non-trivial solutions for $M_f$ given by

$$2M_f^2 = (b^2 - 1) \frac{k^2}{\tau^4} + M_{11}^2 + M_{22}^2 \pm \sqrt{\Delta + (M_{11}^2 + M_{22}^2)^2}, \quad (5.50)$$

In the following we assume that $b = 1$ corresponding to a single speed for the two gravitons as happens in doubly coupled bigravity in matter and radiation domination. We can rewrite the expression for $M_f$ as,

$$2M_f^2 = M_{11}^2 + M_{22}^2 \pm \sqrt{\Delta + (M_{11}^2 + M_{22}^2)^2}, \quad (5.51)$$

where $\Delta = 4M_{12}^2 M_{21}^2 - 4M_{11}^2 M_{22}^2$. There are no tachyonic instabilities when $\Delta$ takes values between 0 and $-(M_{11}^2 + M_{22}^2)^2$. In the latter one of the modes is massless and we have that, $M_{11}^2 M_{22}^2 = M_{12}^2 M_{21}^2$, which implies that the other mode has a mass

$$M_f^2 = M_{11}^2 + M_{22}^2. \quad (5.52)$$

In the generic case, there is always a light and a more massive mode.

### 5.4.2 Coupling to matter

The impact of two propagating gravitons on the CMB B-mode spectrum depends on how they source the polarisation terms. The coupling to matter is of the form

$$\delta S_{\text{matter}} = \frac{1}{2m_{\text{Pl}}} \int d^4x a_{ij} \left\{ \beta_1 E_i^{(1)} + \beta_2 E_j^{(2)} \right\} T^{ij}, \quad (5.53)$$
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Fig. 5.3 Gravitational coupling $\Psi$ (5.20), for the massless case and a combination of a massive and a massless graviton with $\kappa = 0.1$ and the mass of the graviton is $1000H_0$. Notice that $\Psi$ does not vanish on large scales in the massive case.

where $a_J = \beta_1a_1 + \beta_2a_2 \equiv a$ is the Jordan frame scale factor, i.e. the scale factor of the FRW metric which couples to matter. We have also introduced the Planck scale $m_{Pl}$ for the normalised gravitons $E_{1,2}$. In the following we focus on the case with one massless graviton after diagonalisation corresponding to the constraint $\Delta = 0$. When diagonalising the two graviton equations, the eigenmodes are

$$
\begin{align*}
  f_0 &= \alpha_1 E_1 + \alpha_2 E_2, \\
  f_m &= \alpha_3 E_1 + \alpha_4 E_2,
\end{align*}
$$

(5.54)

where $f_0$ is the massless mode in the matter era, and $f_m$ the massive one. Imposing that the gravitons remain normalised for the eigenmodes of the mass matrix implies that the change of basis is a two dimensional rotation as $(\partial f_0)^2 + (\partial f_m)^2 = (\partial E_1)^2 + (\partial E_2)^2$ and therefore $\alpha_1 = \alpha_4 = \cos \theta$ and $\alpha_3 = -\alpha_2 = \sin \theta$. As $\Delta = 0$ there is a massless graviton and a massive one of squared mass $M_{11}^2 + M_{22}^2$ with $\alpha_1 = -\frac{M_{22}^2}{M_{11}^2}\alpha_2$, $\alpha_3 = \frac{M_{22}^2}{M_{11}^2}\alpha_4$. As a result we have that

$$
\begin{align*}
  E_1 &= \frac{1}{\sqrt{1 + \tan^2 \theta}}(f_0 + \tan \theta f_m), \\
  E_2 &= \frac{1}{\sqrt{1 + \tan^2 \theta}}(f_m - \tan \theta f_0),
\end{align*}
$$

(5.55) (5.56)
where \( \tan \theta = \frac{M_{11}^2}{M_{12}^2} = \frac{M_{12}^2}{M_{22}^2} \). Then in the Jordan frame the gravitons are coupled to matter as

\[
\delta S_{\text{matter}} = \frac{1}{2 m_{\text{Pl}} \sqrt{1 + \frac{M_{12}^2}{M_{22}^2}}} \int d^4 x a \lambda \left\{ \beta_1 \left( f_{ij}^0 + \frac{M_{11}^2}{M_{12}^2} f_{ij}^m \right) + \beta_2 \left( f_{ij}^m - \frac{M_{12}^2}{M_{22}^2} f_{ij}^0 \right) \right\} T^{ij},
\]

which implies that,

\[
\delta S_{\text{matter}} = \frac{1}{2 m_{\text{Pl}} \sqrt{1 + \frac{M_{12}^2}{M_{22}^2}}} \int d^4 x a^2 \beta_2 \left\{ \left( \beta_1 - M_{12}^2 \frac{1}{M_{22}^2} \right) \frac{f_{ij}^0}{a} + \left( 1 + \frac{M_{12}^2}{M_{22}^2} \right) \frac{f_{ij}^m}{a} \right\} T^{ij},
\]

where we have reintroduced the tensorial indices. Now in bigravity the physical Planck scale is given \( M_{\text{Pl}} = \frac{m_{\text{Pl}}}{\sqrt{\beta_1 + \beta_2}} \) [197] so that the coupling to matter becomes

\[
\delta S_{\text{matter}} = \frac{1}{2 M_{\text{Pl}}} \int d^4 x \frac{1}{\sqrt{1 + \frac{M_{12}^2}{M_{22}^2} \beta_1 + \beta_2^2}} a \beta_2 \left\{ \left( \beta_1 - M_{12}^2 \frac{1}{M_{22}^2} \right) \frac{f_{ij}^0}{a} + \left( 1 + \frac{M_{12}^2}{M_{22}^2} \right) \frac{f_{ij}^m}{a} \right\} T^{ij}.
\]

It is easy to see that the power spectrum of the graviton coupled to matter goes to the one of a single graviton coupled to matter at the end of inflation. Indeed taking both \( f_0/a \) and \( f_m/a \) go to one at the end of inflation, and upon using the statistical independent of \( f_0 \) and \( f_m \), the power spectrum of the coupled graviton \( \mathcal{P}_h \) is

\[
\mathcal{P}_h = \frac{\beta_2^2}{(1 + \frac{M_{12}^2}{M_{22}^2})(\beta_1^2 + \beta_2^2)} \left( \frac{\beta_1}{\beta_2} - \frac{M_{12}^2}{M_{22}^2} \right)^2 \mathcal{P}_0 + \left( 1 + \frac{\beta_1 M_{12}^2}{\beta_2 M_{22}^2} \right)^2 \mathcal{P}_m.
\]

where \( \mathcal{P}_0,m \) are the power spectra of the massless and massive gravitons. Outside the horizon we normalise the massless and massive spectra similarly implying that

\[
\mathcal{P}_h = \mathcal{P}_0 = \mathcal{P}_m,
\]

and therefore the spectrum of the coupled graviton obtained from (5.59) is automatically normalised in the same fashion as in General Relativity. To analyse the effect of the
5.4 Bigravity

We introduce the parameter $\kappa$ as

$$\kappa = \frac{1 + \beta_1 \frac{M_2^2}{M_1^2}}{\beta_2 - \frac{M_1^2}{M_2^2}}.$$  \hfill (5.62)

The gravitational source becomes of the form $\Psi \propto f_0' + \kappa f_m'$. In figure (5.4) we have plotted the power spectrum produced by (5.59) for different absolute values of the coupling $\kappa$. Notice that the effects of the massive graviton cannot be removed unless $\beta_2/\beta_1 < 0$, which would lead to instabilities, or if there is no coupling $M_{12} = 0$. In the generic case, the characteristic plateau of massive graviton at low values of $l$ is always present. Moreover the position of the first peaks is shifted when the coupling $\kappa$ varies.

Fig. 5.4 The power spectrum $C_{ll}^B$ for different values of $\kappa$ in the case of two gravitons, one being massive whilst the other one is massless. The mass is taken to be $1000H_0$. Notice that the plateau at low $l$ is always present and that the first few peaks shifts with the value of $\kappa$. 
5.4.3 Instabilities

In this section we pursue the study of a toy model with two gravitons which reproduces another feature of doubly coupled bigravity at low energy. In this simplified setting and mimicking what happens in doubly coupled bigravity models below the strong coupling scale\(^3\), there is an instability which appears in the radiation era coming from pressure terms such that \[197\]

\[
\delta S_p = \frac{1}{8} \int d^4x \sqrt{-g} \delta T_{ij} \delta g^{ij}.
\] (5.63)

It turns out that this yields a pressure-dependent mass matrix of the form,

\[
\begin{pmatrix}
M_{12} & M_{11} \\
M_{22} & M_{22}
\end{pmatrix} = \begin{pmatrix}
1/2 & M_{12} \\
M_{22} & 1/10
\end{pmatrix}.
\]

Fig. 5.5 The gravitational coupling \(\Psi\) for a massive graviton together with an unstable massless one. We have taken the mass to be 1000 \(H_0\) and the ratio \(M_{12}/M_{22}\) to be 1/10 and 1/2 respectively. The amplitude for the unstable mode is multiplied by \(10^{-11}\), i.e. this corresponds to the very low initial conditions of the unstable mode which preserves the perturbativity of the model and gives a non-negligible effect on the B-mode spectrum.

\[
\Delta M_p^2 = \frac{3\omega a_j^2 H_j^2}{\beta_1^2 + \beta_2^2} \begin{pmatrix}
-\beta_2 & \beta_1 \\
\beta_2 & -\beta_1^2
\end{pmatrix},
\] (5.64)

\(^3\) When the Hubble rate becomes larger than the typical strong coupling scale \(\Lambda_3 = (m^2 M_{Pl})^{1/3}\), our results cannot be trusted anymore as a viable approximation to doubly coupled bigravity. Indeed, the analysis would have to carried out in the UV completed theory. As this is beyond the present treatment in this thesis, we will only consider the following as a toy model inspired by doubly coupled bigravity and extend its description all the way up to the end of inflation. It would be only in the late radiation era for reasonable masses \(m \sim 10^3 H_0\) that the equations of this toy model would coincide with the one of doubly coupled bigravity.
which has a zero mass eigenstate and an eigenmode of negative mass squared, i.e. a tachyon,

$$m_G^2 = -3\omega_j^2 a_j^2 H_j^2 < 0. \quad (5.65)$$

This is only present during radiation domination where it produces a mild instability. To analyse its effect we can solve the equation for a massive graviton during radiation domination

$$h'' + 2aH h' + (k^2 + m^2 a^2 - a^2 H^2) h = 0. \quad (5.66)$$

where we have that $a \propto \tau$ and the equation reduces to

$$h'' + \frac{2}{\tau} h' + (k^2 + m^2 \hat{H}_0^2 \tau^2 - \frac{1}{\tau^2}) h = 0, \quad (5.67)$$

whose solution can be approximated by,

$$h \propto (m \hat{H}_0 \tau^2 / 2)^{1/4} j_{-1/4}(m \hat{H}_0 \tau^2 / 2) \left( j_0(k\tau) + j_{1/2(-1+\sqrt{5})}(k\tau) \right). \quad (5.68)$$

The new spherical Bessel mode function $j_{1/2(-1+\sqrt{5})}(k\tau)$ arises because of the instability in the radiation era. This can be matched to the matter era using

$$h = \begin{cases} 
(m \hat{H}_0 \tau^2 / 2)^{1/4} j_{-1/4}(m \hat{H}_0 \tau^2 / 2) \left( j_0(k\tau) + j_{1/2(-1+\sqrt{5})}(k\tau) \right) & \tau < \tau_{eq}, \\
3 \frac{1}{k^2} \times j_0(\frac{1}{3} m \hat{H}_0^2 \tau^3) \left( A j_1(k\tau) + B y_1(k\tau) \right) & \tau > \tau_{eq}, 
\end{cases} \quad (5.69)$$

where $A$ and $B$ could be found by matching the solution and its derivative at $h(\tau_{eq})$. For other more realistic approaches see [203], where the WKB approximation is used. This new solution, i.e. its effect on the gravitational coupling $\Psi$, is plotted in fig.5.5. We see that the amplitude of the modes is very large for small $k$ and that this contribution is of similar shape as for the stable mode, albeit with a much higher amplitude. As a result, the power spectrum for these modes has an amplitude of several orders of magnitude higher than for the stable case, which has to be compensated by the choice of very low initial amplitudes for the unstable mode [198, 204].

Now to analyse the effect of the instability in the case of two gravitons we need to include (5.64) in (5.44) and then diagonalise the new set of equations. We will do this in radiation domination, and then we will match to matter domination. The equations
are,
\begin{align*}
E_1'' + \left( k^2 + M_{11}^2 \dot{H}_0^2 \tau^2 - \frac{\beta_2^2}{\beta_1^2 + \beta_2^2} \frac{1}{\tau^2} \right) E_1 + \left( M_{12}^2 \dot{H}_0^2 \tau^2 + \frac{\beta_1 \beta_2}{\beta_1^2 + \beta_2^2} \frac{1}{\tau^2} \right) E_2 &= 0, \\
E_2'' + \left( k^2 + M_{22}^2 \dot{H}_0^2 \tau^2 - \frac{\beta_1^2}{\beta_1^2 + \beta_2^2} \frac{1}{\tau^2} \right) E_2 + \left( M_{21}^2 \dot{H}_0^2 \tau^2 + \frac{\beta_1 \beta_2}{\beta_1^2 + \beta_2^2} \frac{1}{\tau^2} \right) E_1 &= 0.
\end{align*}

Notice again that for now on the matrix $M_{ij}^2$ is an arbitrary positive definite symmetric matrix. As before we try to find solutions of the form $f = \lambda_1 E_1 + \lambda_2 E_2$ where the equation for $f$ satisfies

\begin{equation}
E'' + \left( k^2 + M_f^2 \dot{H}_0^2 \tau^2 - \frac{1}{\tau^2} \right) f = 0. \tag{5.71}
\end{equation}

Let us define $\tilde{M}_{ij}^2 = M_{ij}^2 + \frac{\beta_i \beta_j}{\beta_1^2 + \beta_2^2} \frac{1}{H_0^2 \tau^4}$ in which case we have that

\begin{align*}
\lambda_1 (M_f^2 - \tilde{M}_{11}^2) - \lambda_2 \tilde{M}_{12}^2 &= 0, \\
\lambda_2 (M_f^2 - \tilde{M}_{22}^2) - \lambda_1 \tilde{M}_{12}^2 &= 0,
\end{align*}

which is similar to the case treated previously without an instability. The expression for $M_f$ reduces to,

\begin{equation}
2M_f^2 = \tilde{M}_{11}^2 + \tilde{M}_{22}^2 \pm \sqrt{(\tilde{M}_{11}^2 + \tilde{M}_{22}^2)^2 + 4(M_{12}^2 - \tilde{M}_{12}^2 \tilde{M}_{22}^2)}. \tag{5.73}
\end{equation}

In the following we focus on the case where $M_{12}^2 = M_{11}^2 M_{22}^2$. Notice that deep in the radiation era, as was already the case in the matter era, we have two solutions corresponding to a massless graviton $M_f = 0$ which becomes unstable in the radiation era and a massive graviton of mass $M_f^2 = M_{11}^2 + M_{22}^2$ in both the radiation and matter eras.

Similarly we can use the results (5.56) to diagonalise

\begin{align*}
E_1 &= \frac{1}{\sqrt{1 + \frac{M_{11}^2}{M_{12}^2}}}(f_0 + \frac{\tilde{M}_{11}^2}{M_{12}^2} f_m), \tag{5.74} \\
E_2 &= \frac{1}{\sqrt{1 + \frac{M_{11}^2}{M_{22}^2}}}(f_m - \frac{\tilde{M}_{12}^2}{M_{22}^2} f_0). \tag{5.75}
\end{align*}

where we have denoted by $f_0$ the mode with $M_f = 0$ and $f_m$ the massive one. Notice that the diagonalisation is only valid when the rotation matrix is time independent,
5.4 Bigravity

i.e. at all times apart from the transitory regime where both $M_{ij}^2$ and $\frac{\beta_i \beta_j}{\beta_1^2 + \beta_2^2} \frac{1}{H_0^2}$ are of the same order. In the following, we will neglect this intermediate regime as we are either interested in the early times where the instability is present or later when it has disappeared.

We can again use the results from the previous sections and then analogously to (5.59) we have that the coupling to matter reads

$$\delta S_{\text{matter}} = \frac{1}{2 M_{Pl}} \int d^4x \frac{1}{\sqrt{1 + \frac{M_{12}^2}{M_{22}^2} \beta_1^2 + \beta_2^2}} a \beta_2 \left\{ \left( \frac{\beta_1}{\beta_2} - \frac{\tilde{M}_{12}^2}{M_{22}^2} \right) f_{ij}^0 + \left( 1 + \frac{\beta_1 \tilde{M}_{12}^2}{\beta_2 M_{22}^2} \right) f_{ij}^m \right\} T^{ij}.$$ (5.76)

which is also naturally normalised.

Fig. 5.6 The B-mode spectrum $C_B^{l0}$ for a massive stable graviton and a linear combination of a massive stable graviton and a massless unstable graviton, i.e. a massless graviton with a tachyonic instability in the radiation era. The mass is taken to be $m = 1000 H_0$ and on the right $M_{12}^2/M_{22}^2 = 1/10$ whilst on the left $M_{12}^2/M_{22}^2 = 1/2$. The tensor to scalar ratio is set to $10^{-2}$. Notice that the plateau at low $l$ is hardly modified by the unstable mode whilst the first few peaks are shifted compared to a purely massive graviton.

This result is a generalisation of (5.59) in radiation domination. It can be extrapolated to the transition region by redefining

$$\tilde{M}_{ij}^2 = M_{ij}^2 + \frac{\beta_i \beta_j}{\beta_1^2 + \beta_2^2} 3 \omega H^2,$$ (5.77)
where $\omega = 0$ in the matter era and $\omega = 1/3$ in the radiation era.

During radiation domination the instability dominates and we can approximate

$$\tilde{M}_{22}^2 \approx \frac{\beta_2^2}{\tau^2}, \quad \tilde{M}_{12}^2 \approx \frac{\beta_1 \beta_2}{\tau^2},$$

so we have that $\tilde{M}_{12}^2/\tilde{M}_{22}^2 \approx \beta_1/\beta_2$. Then (5.76) reduces to

$$\delta S_{\text{matter}} = \frac{1}{2M_{\text{Pl}}} \int \frac{d^4 x a^2 f_m}{a} T^{ij},$$

which implies that during radiation domination there is no coupling of the unstable mode to matter. To analyse how this changes the power spectrum of B-modes we set $\beta_2 = \beta_1 = 1$ and we choose a large mass for the graviton of 1000 $H_0$, i.e. leading to a plateau at low $l$. We also vary the ratio $M_{12}^2/M_{22}^2$. We find that the effect of the instability is very mild only affecting the first few peaks of the B-mode spectrum, see figure 5.6. We have imposed that the initial conditions for the unstable modes are such that at recombination its magnitude does not exceed the one of the massive graviton. In principle, the effects of the unstable mode can be reduced by choosing even lower initial values. This choice guarantees that the mode does not go non-linear before recombination. As a result, the power spectrum even in the presence of a tachyonic instability, here tamed by the initial conditions, is characterised by the typical plateau of massive gravitons at low $l$ and a shifted structure of peaks compared to a purely massive graviton case.

### 5.5 Conclusions

In this chapter we have studied the effect that a modification of gravity has on the B-mode power spectrum. Our results suggest that if $r$ becomes observable, the constraints on modified gravity theories will improve greatly. In particular we have studied the effect that massive gravity has on the B mode power spectrum. We have found analytical expressions for the massive tensor modes valid during matter and radiation domination. With this result we have found that the most important effect a massive graviton has on the the CMB is the presence of a plateau at small $l$, as the source function for gravitational waves is constant outside the horizon.

We have also studied multiple gravitons. Using our analytical results we have shown how the effects of massive gravity arise in the presence of a combined coupling to gravity. In general the massive graviton is always coupled to matter and its effect
cannot be removed by tuning the mass parameters of the models as long as one massless graviton is present. Moreover we have also included the effects of a tachyonic instability similar to the one of doubly coupled bigravity, which arises in the radiation era. As the unstable mode does not couple to matter during radiation domination, its effect is very mild and does not alter the existence of a plateau at low $l$, a feature of massive gravitons. This result is valid as long as the initial solutions of the unstable mode are reduced at the end of inflation [198, 204].
Chapter 6

Discussion

Throughout this thesis we have addressed several topics of theoretical cosmology. Our approach has been to study different problems, where aspects of theoretical physics are relevant. The classical example is the application of quantum field theory techniques to inflation, which has led, between other things, to understand the role of symmetries on constraining the theory. This kind of approach has proved to be very powerful as it has made possible to make model independent predictions, which, for example, have been used by Planck to put constraint on a large class of models. This is particularly important for theories such as inflation, where there is no knowledge of the precise microphysics. Also, for theories at later times, such a set of ideas and techniques, has been important on recent years. The use of effective field theories to parameterise dark energy models resulted in a particular set of predictions that could be contrasted with data coming from gravitational wave emission.

This endeavour will continue, as the role of theory will be crucial for the next generation of cosmological experiments. This is because, in order to be able to discover or rule out new physics, it is key to understand what is to be measured. The precision obtained will have to be paired with the technical knowledge about the physics originated in such situation. Nowadays, after the final release of Planck it is the task of theorists to make sharp predictions and understand new theories that can motivate new searches on the data that will be collected. In this thesis we have addressed some ideas, in this direction, that hopefully will contribute to this knowledge.

In this section we summarise the main findings of this thesis, and we give an outlook of future research.
6.1 Summary of the main results

This thesis addressed two main topics: In the first part we discussed the effect of matter coupled to inflation, and in the second part we discuss effects of new physics on the cosmic microwave background.

In the second chapter we discussed the case when non relativistic matter is non canonical coupled through gravity. This situation arises in modified gravity, but also is recurrent in string theory or supergravity. In this chapter we showed, that assuming that there is an initial density of matter, implies that the potential at the start of inflation might be drastically modified. The new potential depends on the geometry of the field space. This is illustrated in several examples. For instance, we considered the case of superconformal realisations of the Starobinsky model [137, 131]. In particular the coupling to gravity can be written as $SO(1,N)$ invariant scalar coupling. This hyperbolic geometry is then translated to the particular potential of the Starobinsky model, but also induces large corrections to the potential through matter which is also coupled to gravity. We showed that for large enough densities this effect could stop inflation within a few e-folds. We generalised this effect to other models. The $SO(1,N)$ geometry can be originated in superconformal supergravity, so the same effect is achieved in supergravity models whose Kahler potential have a similar geometry. We studied examples in supergravity where, generically there are many heavy scalar fields that appear after compactifying the theory to four dimensions, and thus provide a good setting to study this kind of mechanism.

We also studies generalisations of the Starobinsky model where a parameter, $\alpha$, controls the geometry of the target space. For small $\alpha$ the potentials have plateaus, similar to the Starobinsky model potential, whereas for large $\alpha$ the potential is steeper with inflation occurring near the minima, similar to chaotic inflation. There we see that the effects of the geometry happening at small $\alpha$ disappears when $\alpha$ is increased. This might have interesting phenomenological consequences as models where $\alpha$ is large are almost ruled out by observations, so it is important to study in detail models of small field inflation.

In the third chapter we discussed holography within the context of inflation. It is an important goal to try to understand what are the implications and limits when applying holography to cosmology, in particular when thinking about inflation in a holographic way. We discussed two main ideas, first we examine the proposal of [55], where the duality was formulated as an equivalence between a late time Hartle-Hawking wavefunctional and the partition function of a three dimensional Euclidean conformal field theory. This idea is known to have some limitations, and we showed that it is only
well defined for scalar fields whose masses are no bigger than $3/2H$. This is because the wavefunctional is defined as the on-shell action of a weakly coupled gravitational theory, such as inflation. This could be separated into local and nonlocal components, with the nonlocal components giving rise to the required correlators. The local parts are in general divergent but, because they are a pure phase, do not contribute to any correlation function. Moreover the Hartle-Hawking functional obeys the Wheeler de Witt equation, and thus, the local part can be rewritten as the solution of the Hamilton Jacobi equations for certain scalar potentials. Then the local divergent part is equivalent to the new scalar potential, this potential can be used as a counterterm and so the local divergence can be removed. We showed that this is not possible for scalar fields with masses above $3H/2$, because the potential terms have restricted solutions that do not allow masses beyond this bound. This bound also corresponds to requiring unitarity on the euclidean CFT.

This is strong evidence on the limits of the prescription of [55], because, a priori, there is no reason to forbid massive fields in inflation. We state the observational consequences of this finding. Non Gaussianity is affected by fields of masses $\sim H$ interacting with the inflaton at horizon crossing. This is manifested as an oscillatory part for the squeezed limit of the three point function of the scalar field, that depends on the mass of the extra field. This effect is found for fields whose masses are above $3H/2$, and disappears for fields with masses below the bound. Moreover, this effect exists for all higher order correlation functions. Therefore by finding such signal holographic models of inflation will be very unlikely to be realised in nature.

Inflation can also be understood as a RG flow starting from an UV de Sitter fixed point. In the second part of chapter three we explore the $\beta$ functions associated with this RG flowing for different models. The existence of an IR fixed point is linked to marginal deformations of the UV fixed point at horizon crossing. We then showed that this situation can arise in several situations in string theory models of inflation. Also by using ideas drawn from chapter two we study the effect of matter. We found that the effect of matter modifying the potential can be interpreted as relevant operators acting over the CFT. This translates to a RG flow that gets to a point where $\beta \neq 0$. We examined the examples of $\alpha$ attractors of chapter two in the light of holographic inflation. There we showed that for large $\alpha$ there is a suppression of the mass of the heavy fields which translates into marginal operators acting over the UV fixed point producing a RG flow towards an IR fixed point. In the case of small $\alpha$ the suppression is not efficient and the marginal operators become relevant thus the RG flow sweeps
away the fixed point. We exemplified this behaviour on different models of string
theory with similar conclusions.

The release of Planck results has brought in the possibility of testing interesting
theoretical models with great precision. We address this topic in the second part of the
thesis. In particular we focus on the effects of modified theories of gravity. Generally
they are built to solve problems of late time cosmology, such as the current acceleration
of the universe. Nevertheless, some of their features can be tested by analysing its
physics on the early Universe.

In chapter four we address modifications to the speed of tensor modes. This appears
when the action contains higher derivative operators coupled to gravity, for example
in Galileon theories. A speed different from one modifies the peak structure of the
polarisation power spectrum, because the time that the perturbations are within the
horizon is modified. Different tensor speeds also arise in inflation when including higher
curvature theories. Whilst in inflation it is possible to remove any speed for tensor
modes by a coordinate transformation, we showed that for late time cosmology this is
not possible. We showed that a disformal transformation, which changes the lightcone,
indeed exchanges the speed for the tensor modes for a speed in the scalar action. The
effect of the transformations during recombination happens to be very drastic, where
not only the peaks are shifted but also the damping scale is modified. This result is
interesting because, due to the recent observation of gravitational waves, theories that
predict modified tensor speed are very constrained. Then, discarding the possibility
that a disformal transformation could make the theory valid is an important step.

There are also other interesting modifications to gravity that might have an effect
on the cosmic microwave background. In the fifth chapter we addressed the effect of
massive gravity on the polarisation power spectrum. We showed, analytically how the
wave equation of massive gravity modifies the large scale modes of the polarisation
CMB. This is because the transfer function is suppressed until the modes reenter the
horizon. This might be because there are interactions between the massive graviton
and the photons that are now suppressed. We also focus on realistic models of massive
gravity where there are two gravitons coupled to matter. These models are known to
have an instability at radiation domination. We showed that, in general, there are a
massless and a massive mode, with the instability coupling to the massless mode. This
implies that the effect of the instability is not observable but the final power spectrum
acts as a well behaved massive gravity power spectrum. As massive gravity is still a
viable model, constraining the theory is an important step for future observables. We
have helped in this task by analysing carefully how to solve the equations of motion
of the massive graviton for different regimes, and also by understanding how theories with multiple gravitons produce B-modes power spectrum.

6.2 Outlook

There are still several open problems in cosmology. First and foremost, it will be nice to have an explicit realisation of inflation in a quantum gravity theory. String theory remains the best candidate but there are many difficulties to be solved. One approach is to understand what are the features that could make a healthy theory of inflation valid. For example one could impose that the putative quantum theory of gravity has a semiclassical limit. This will imply that Hartle-Hawking wavefunction is a good approximation, but, as we have seen, this puts several constraints over the spectrum of masses. Were this not to be the case then, examining different models could bring some light over this idea. Recently many approaches to solve the Hartle-Hawking wavefunction have been put into light. It will be interesting to see whether it is possible to obtain useful predictions from them.

Over recent years there has been an intense discussion about de Sitter solutions in string theory. The idea is that the landscape of stable solutions are small islands within a vast swampland. Finding correct criteria for when a theory is in the swampland has become an important goal. Up to now there are two main proposals. First that there has to be a direction with \( \nabla V/V = c \) with \( c \) of \( \mathcal{O}(1) \) and that the field excursions are of order one, \( \Delta \phi \sim \mathcal{O}(1) \). Were this criteria true it will certainly have important consequences for inflation. For example it will restrict all large field inflation models, but also it might imply that there are not metastable de Sitter vacua in string theory. An interesting possibility, might be to use the \( \beta \) function formalism discussed in chapter three. The first criteria can be reformulated as \( \beta = c \) which implies that there are relevant operators deforming the inflationary fixed point. Investigating this idea in more detail could possibly give valuable insight on the problem of the swampland.

Holography in de Sitter has the problem that there are not many explicit models where one can apply general knowledge. The exception being higher spin gravity. This theory admits a positive cosmological constant and is dual to \( Sp(N) \) \[57, 205–208\]. Particularly interesting is that it only contains a conformally coupled scalar of mass \( m^2 H^2 = 2 \), hence is stable. A possible idea might be to investigate the deformation of this theory by single or double trace operators. Double trace operators are ubiquitous in AdS but have an interpretation has not been found for de Sitter. Interestingly it has been found recently that double trace deformations of a particular model of AdS/CFT,
ABJM [209], can lead to an RG flow towards a non supersymmetric AdS vacuum. If something similar happens for de Sitter, which is similar to the RG flows discussed on chapter 3, could signal that the swampland criteria is not good enough to constrain the space of solutions. Moreover higher spin perturbations have specific signatures over large scales structures, so it might even be possible to constrain this set of ideas in the next decades.

The detection of gravitational waves from binary systems allows a large class of theories of modified gravity to be ruled out. This was an important step, because the space of possible solutions to the cosmological constant problem is much more constrained. On the way a new class of interesting quantum field theories was thoroughly studied. It will be important to continue on this direction, and an important goal will be to build a stable massive gravity theory that can achieve cosmological evolution without instabilities. Also it is interesting to study massive gravity black hole solutions and its consequences for gravitational waves.

If primordial tensor modes are detected it will open a huge window to learn more about physics. In particular massive gravity will be ruled out. In any case it is interesting to keep investigating solutions of massive gravity to other scenarios. To do so it will be important to develop more technical advances to test gravity using large scale structure. For example it might be a good idea to apply our solutions to the work done by [210–212]. A possible approach might be to use the effective field theory for large scale structure techniques, where different physical effects are encoded in operators, that are potentially measurable. This is still in development, but it has proven, for simple scenarios to be the most promising technique.

Hopefully, we will learn more about the universe within the next few decades. The impressive experimental observations, need to be paired with a deeper understanding of the physics at early and late times. By a combined effort, the intricacies of the universe might keep being unraveled.
Appendix A

Supergravity in inflation

Since inflation started to appear as a viable theory it has been very appealing to embed it in either string theory or supergravity. This has technical difficulties due to the form of the potential,

\[ V = e^K \left( R^{IJ} \partial_I W \partial_J \bar{W} - 3|W|^2 \right) \]  \hspace{1cm} (A.1)

which is too steep due to the exponential of the Kahler potential. This has been called the \( \eta \) problem because \( \eta \equiv M_{\text{Pl}}^2 \frac{V''}{V} \simeq \frac{m_\phi^2}{H^2} \) which is naturally larger than one. In this note we will focus mainly on the Starobinsky model which circumvents the problem by defining the supermultiplet through superconformal symmetry. This allow us to build the supermultiplet as if it were rigid supersymmetry, where this problem does not exist. Thus by requiring the extra symmetry unwanted features of supergravity are removed. Still this has limitations, and it has been used merely to build small field theories of inflation.

A.1 Conformal supergravity

In this section we summarise the main results of conformal supergravity focusing on their applications to cosmology. First let us start by examining the theory of a conformally invariant field coupled to gravity,

\[ S = \frac{1}{2} \int d^4x \left\{ \frac{\phi^2}{6} R - \left( \partial_{\mu} \phi \right)^2 \right\}. \]  \hspace{1cm} (A.2)
Which is invariant under dilatations, because we are including just the scalar field. This theory is equivalent to the action

\[
S = -\frac{1}{2} \int d^D x e \phi \Box^C \phi, \tag{A.3}
\]

where \( \Box^C \) stands for the conformally invariant D’Alambertian. This is a rather baroque way to build conformally invariant actions but it will prove to be very useful when applied to supergravity theories. The procedure will be to first gauge all the parameters of the conformal group and then impose constraints and fix the gauge to get rid of the unwanted degrees of freedom. This is necessary because the original conformally invariant theory has more functions or degrees of freedom many of which are not physical. For example the gauge fields associated to special conformal transformations. Therefore, if we want to obtain the scalar field theory from above we should gauge fix this field. It will also be constraints on the equations of motion, for example we will assume torsion less. First let us remember that the conformal group is the group of transformations that preserve the metric up to a rescaling factor, \( g_{\mu \nu} \rightarrow \Omega(x)^2 g_{\mu \nu} \). The group consists of Lorentz transformations, rotations, dilations, and special conformal transformations. Their gauged transformations are given by,

\[
\begin{align*}
\delta e^{a}_\mu &= -\lambda^{ab} e^{b}_\mu - \lambda^{a}_D e^{a}_\mu, \\
\delta \omega^{ab}_\mu &= \partial_\mu \lambda^{ab} + 2 \omega^{[a}_{\mu} \lambda^{b]c} - 4 \lambda^{[a}_K \lambda^{b]}_e \mu, \\
\delta b^{a}_\mu &= \partial_\mu \lambda^{a}_D + 2 \lambda^{a}_K e^{a}_\mu, \\
\delta f^{a}_\mu &= \partial_\mu \lambda^{a}_K - b^{a}_\mu \lambda^{a}_K + \omega^{a}_{\mu} \lambda^{a}_K b^{a}_\mu - \lambda^{ab} f^{a}_\mu + \lambda^{ab}_D f^{a}_\mu. \tag{A.4}
\end{align*}
\]

The Weyl weight of the vielbein is \(-1\), the field \( f^{a}_\mu \) has Weyl weight 1, and the rest of the fields have Weyl weight 0. Not all these fields are physical but they are composite fields. This is similar to the case of the spin connection \( \omega^{ab}_\mu \) in Hilbert Einstein gravity, where it can be written in terms of the vielbein by imposing a torsionless condition. Here we have to impose other constraints. For the case of special conformal transformations we have that,

\[
\epsilon^{a}_\mu R^{a}_{\mu \nu}(M^{ab}) = 0, \quad R_{\mu \nu}(M^{ab}) = R^{ab}_{\mu \nu} + 8 f^{[a}_{\mu} \epsilon^{b]}_\nu \tag{A.5}
\]

And then we are able to obtain \( f^{a}_\mu \) in terms of the Ricci curvature,

\[
2(D - 2) f^{a}_\mu = -R^{a}_\mu + \frac{1}{2(D - 1)} \epsilon^{a}_\mu R. \tag{A.6}
\]
Furthermore the special conformal transformations can be gauged by fixing $b_\mu = 0$, which leads to the vielbein as the only independent field. Moreover the previous gauge condition leads to

$$\lambda_{K\mu} = -\frac{1}{2} \partial_\mu \lambda_D$$  \hspace{1cm} (A.7)

We now proceed to build the action by coupling the gauge fields to a scalar field. To do so we need to determiate the Weyl weight of the scaling transformation, $\delta \phi = \omega \lambda_D \phi$. We have that the D’Alambertian is given by,

$$\Box^C \phi \equiv \eta^{ab} D_a D_b \phi,$$  \hspace{1cm} (A.8)

where $D$ is the covariant derivative associated to the field $\phi$. We now need to determine the Weyl weight $\omega$ for the scalar field to obtain to make the D’Alambertian invariant under conformal transformations. Expanding we get that,

$$D_a \phi = e_\mu^a (\partial_\mu - \omega b_\mu) \phi$$  \hspace{1cm} (A.9)

Since this is a covariant derivative its transformation only involve terms depending on the vielbein in (A.4). Therefore,

$$\delta D_a \phi = (\omega + 1) \lambda_D D_a \phi - \lambda_a^b D_b \phi - 2 \omega \lambda_K \phi.$$  \hspace{1cm} (A.10)

This in turns implies that the D’Alambertian is,

$$\Box^C \phi \equiv \eta^{ab} D_a D_b \phi = e^{a\mu} \left( \partial_\mu D_a \phi - (\omega + 1) b_\mu D_a \phi + \omega_\mu b D^b \phi + 2 \omega_f b a \phi \right).$$  \hspace{1cm} (A.11)

The transformation of this quantity is,

$$\delta \Box^C \phi = (\omega + 2) \lambda_D \Box^C \phi + (2D - 4\omega - 4) \lambda_K^a D_a \phi.$$  \hspace{1cm} (A.12)

The last term drops out when,

$$\omega = \frac{1}{2} D - 1.$$  \hspace{1cm} (A.13)

Now notices that the Weyl weight of the determinant of the vielbein $e$ is $-D$, and then the action ,

$$S = -\frac{1}{2} \int d^D x e \phi \Box^C \phi,$$  \hspace{1cm} (A.14)
has Weyl weight 0, and therefore is conformally invariant. Inserting back the conditions we found previously we have that,

\[ S = -\frac{1}{2} \int d^{D}xe \left( (\partial_{\mu}\phi)^{2} + \frac{D-2}{4(D-1)}R\phi^{2} \right) \]  

(A.15)

To recover Einstein gravity we fix the compensator field,

\[ \phi = \sqrt{\frac{4(D-1)}{(D-2)\kappa}} \]  

(A.16)

Two things to notice, first by changing to Einstein frame the field disappears and then one recovers the Einstein Hilbert action. This is because the only degrees of freedom (d.o.f) are the two graviton polarizations. Secondly, the ill defined kinetic terms is non-physical because it does not propagate any d.o.f and therefore the action is well defined.

A.2 Superconformal supergravity

The construction of a supergravity multiplet works in analogy with the discussion we had before. In this case we require the action to be invariant under a superconformal group. The algebra of the superconformal group includes the conformal group, two supersymmetries and an \( R \)-symmetry. As in the former case by fixing gauge symmetries imposing constraints the Poincaré action is recovered. The matter coupled superconformal action can be written in analogy to the rigid supersymmetry case,

\[ \mathcal{L} = [\mathcal{N}(X, \bar{X})]_{D} + [\mathcal{W}(X)]_{F} + \left[ f_{AB}(X)\bar{\lambda}^{A}P_{L}\lambda^{B} \right]_{F}. \]  

(A.17)

To recover the usual Poincare supergravity one has to fix the compensator fields. For example by choosing \( X^{0} = \bar{X}^{0} = \sqrt{3} \) we get,

\[ \mathcal{N}(X, \bar{X})|_{X^{0}=\bar{X}^{0}=\sqrt{3}} = -3e^{\frac{1}{2}(X^{I},\bar{X}^{I})}, \]  

(A.18)

\[ \mathcal{W}(X^{I})|_{X^{0}=\sqrt{3}} = W(X^{I}). \]  

(A.19)

which will lead to the supergravity multiplet.
Example: Ceccoti model

We will now use the above framework to build one of the first supergravity realisations of the Starobinsky done by Ceccoti [129]. First let us recall that the Starobinsky model Lagrangian,

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R + \alpha R^2 \right] \]  

(A.20)
can be written using the Lagrange multiplier \( \phi \) as,

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R + 2\alpha \phi R - \alpha \phi^2 \right] . \]  

(A.21)

By performing a conformal transformation we can rewrite the former as,

\[ S = \int d^4x \sqrt{-g} E \left\{ \frac{M_{\text{Pl}}^2}{2} R - 1 \right\}, \]  

(A.22)

Its supersymmetric generalisation is in analogy to the former description,

\[ \mathcal{L}_C = -\frac{3}{2} [S_0 \bar{S}_0]_D + \frac{3}{2} \lambda [R \bar{R}]_D \]  

(A.23)

where \( S_0 \) is a chiral multiplet with Weyl weight 1, and \( R \) is the chiral curvature multiplet given by,

\[ R = S_0^{-1} \nabla^2 \bar{S}_0 . \]  

(A.24)

Doing so the action for the Starobinsky model is contained in the bosonic sector of (A.23). The trick we mentioned before can be used here by rewriting the action as,

\[ \mathcal{L}_C = -\frac{3}{2} [S_0 \bar{S}_0]_D + \frac{3}{2} \lambda [A \bar{A}]_D + \left( \frac{3}{2} [\Lambda (A - R)]_F + \text{h.c} \right) \]  

(A.25)

By using superconformal calculus the last action can be written as,

\[ \mathcal{L}_C = -\frac{3}{2} [S_0 \bar{S}_0]_D + \frac{3}{2} \lambda [A \bar{A}]_D - \frac{3}{2} [\Lambda S_0 \bar{S}_0^{-1}]_D - \frac{3}{2} [\bar{\Lambda} S_0 + \bar{S}_0^{-1}]_D + \left( \frac{3}{2} [\Lambda A]_F + \text{h.c} \right) . \]  

(A.26)

This action can be simplified by defining the chiral multiplets with zero weights,

\[ C = \sqrt{\lambda} S_0^{-1} A, \]  

\[ T = S_0^{-2} \Lambda . \]  

(A.27)
The action for the Cecotti model becomes,

$$\mathcal{L}_C = -3[\Phi(T, C)S_0\bar{S}_0]_D + (\{g(T, C)S_0^3\}_F + \text{h.c.})$$

(A.28)

where,

$$2\Phi = 1 + T + \bar{T} - C\bar{C},$$

$$g = \frac{3}{2}\sqrt{\lambda}CT$$

(A.29)

Upon fixing the compensator field $S_0$, the corresponding Kahler potential is given by,

$$K = -3\ln(T + \bar{T} - \bar{C}C) + \ln\left(\frac{g}{2\lambda}|C|^2|T - \frac{1}{2}|^2\right)$$

(A.30)
Appendix B

Hamilton Jacobi solutions

B.1 Origin of HJ equations

We will start by obtaining the Hamilton Jacobi equations we used from the action. We start with the Lagrangian density,

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - \gamma_{ab} \partial_\mu \phi^a \partial^\mu \phi^b - 2V(\phi^a) \right]. \]  

(B.1)

In order to write the Hamiltonian one starts by writing the action in terms of ADM variables. We will first consider only the background, so the metric will be \( \text{ds}^2 = -dt^2 + h_{ij}dx^idx^j \), with \( h_{ij} = a^2 \delta_{ij} \). Then, the action reads

\[ S = \frac{1}{2} \int dt d^3x \sqrt{h} \left[ K_{ij} K^{ij} - K^2 + (\gamma_{ab} \partial_\mu \phi^a \partial^\mu \phi^b - 2V(\phi^a)) \right], \]  

(B.2)

where for our metric, \( K_{ij} \equiv \frac{1}{2} \dot{h}_{ij} = H h_{ij} \) and the 3 curvature vanishes. The conjugate momenta are,

\[ \pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = \frac{\sqrt{h}}{2} (K^{ij} - h^{ij} K), \]  

(B.3)

\[ \pi^\phi \equiv \frac{\delta L}{\delta \dot{\phi}^I} = \sqrt{h} \dot{\phi}^I. \]  

(B.4)

Then, the Hamiltonian density is defined as,

\[ \mathcal{H} = \pi^{ij} \dot{h}_{ij} + \pi^I \dot{\phi}^I - L, \]

\[ = -\hbar^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - \frac{\hbar^{-1/2}}{2} \gamma^{ij} \pi^\phi \pi^\phi - \sqrt{\hbar} V(\phi). \]  

(B.5)
The Hamilton Jacobi is obtained by varying the on shell action with respect the dynamical fields and imposing an ansatz for its shape. Schematically,

$$S_{HJ}[\phi, g_{\mu\nu}] = S_{\text{on shell}}[\phi, g_{\mu\nu}]. \quad (B.6)$$

So, assuming that the value of the fields on the endpoints is not fixed leads to

$$\frac{\delta S_{\text{on shell}}}{\delta \phi^I} = \sqrt{h}\pi^I, \quad (B.7)$$

$$\frac{\delta S_{\text{on shell}}}{\delta h_{ij}} = \sqrt{h}\pi_{ij}, \quad (B.8)$$

We have used the equations of motion to eliminate the rest of the variation, \( \int \sqrt{-h}\phi(\Box\phi - V,\phi)\delta\phi \). Then, due to diff invariance \( \mathcal{H} = 0 \) we have that replacing \( (B.8) \) into the Hamiltonian we get the Hamilton Jacobi equation,

$$\frac{1}{\sqrt{h}} \left( \left( \frac{\delta S}{\delta h_{ij}} \right)^2 - \frac{1}{2} \left( h_{ij} \frac{\delta S}{\delta h_{ij}} \right)^2 + \frac{1}{2} \left( \frac{\delta S}{\delta \phi^I} \right)^2 \right) + \sqrt{h}V(\phi) = 0, \quad (B.9)$$

where we have dropped the on shell from \( S \). We now need to impose an ansatz for the on shell action. Usually this is assumed to be a expansion in gradients and the inverse metric. A common choice is,

$$S_{\text{on shell}} = \int d^3x \sqrt{h} \left( 2W(\phi) + \Phi(\phi)R + M_{ij}(\phi)\partial_I\phi^I\partial_J\phi^J \right). \quad (B.10)$$

We have then that at first order in gradients only the first term remain. Replacing into the conjugate momentum we find,

$$\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \phi^I} = \partial^I W(\phi) = 2W^I, \quad (B.11)$$

$$\frac{1}{\sqrt{h}} \frac{\delta S}{\delta h_{ij}} = \delta^{ij}W. \quad (B.12)$$

Replacing we get the following relations

$$3W^2 - 2W_\alpha W^\alpha = V,$$

$$W\Phi - 2W_\alpha \Phi^\alpha = -2,$$

$$\frac{1}{2} - WM - 4U_\alpha \Phi^\alpha + \frac{1}{4}(W_\alpha M^\alpha + 2M\nabla^2W) = 0. \quad (B.13)$$
Also we have that $\pi_{ij} = -H\delta_{ij} = W\delta_{ij}$ and $\pi_\phi = 2W$ which is what we had at the beginning of this appendix. Some comments, for the background equations only considering the $W$ respects homogeneity and isotropy, further corrections will be equivalent to consider perturbations, and so, as long as we consider the background it should be enough with $W$. For the case of perturbations $W$ will dominate as long as there are no large gradient terms present, as in slow roll inflation where fluctuations freeze at late times. If this is not be this the case, then we do need to consider the other terms. This implies that considering $W$ as the sole function is more valid for modes out of the horizon, hence maybe the constraints we are finding.

**B.2 Orthogonal directions**

We want to prove that if we write

$$\dot{\phi}^a = \alpha W^a + \beta N^a, \quad (B.14)$$

where $H = W$ and $W_aN^a =$, then $\beta = 0$. By assuming the above decomposition it follows that

$$\frac{\dot{\phi}^2}{2} = -\dot{H} = \alpha^2 W^aW_a + \beta^2 N^aN_a. \quad (B.15)$$

We also have that,

$$\frac{D\dot{\phi}^a}{dt} + 3H\dot{\phi}^a + V^a = 0, \quad (B.16)$$

and,

$$\frac{D\dot{\phi}^a}{dt} = \alpha\frac{DW^a}{dt} + \beta\frac{DN^a}{dt} = \alpha\nabla_bW^a\dot{\phi}^b + \beta\nabla_bN^a\dot{\phi}^b$$

$$= -3H\dot{\phi}^a - V^a. \quad (B.17)$$

Also using the constraint equation

$$2V = 6H^2 - \dot{\phi}^2 = 6H^2 - a^2 W_a W^a - \beta^2 N^aN_a$$

$$2V^a = 12HH^a - 2\alpha^2 W^b\nabla^a W_b - 2\beta^2 N^b\nabla^a N_b, \quad (B.18)$$
so mixing the two equations we get,

\[ \alpha \nabla_b W^a \dot{\phi}^b + \beta \nabla_b N^a \dot{\phi}^b = -3H \dot{\phi}^a - 6HH^a + \alpha^2 W^b \nabla^a W_b + \beta^2 N^b \nabla^a N_b \]

\[ = \alpha \nabla_b W^a (\alpha W^b + \beta N^b) + \beta \nabla_b N^a (\alpha W^b + \beta N^b) \] (B.19)

contracting with \( N_a \), this becomes,

\[ 3 \beta N^a N_a = 0, \] (B.20)

where we have used that \( T^a (\nabla_a T^b - \nabla^b T_a) = 0 \). Contracting with \( W^a \) we find,

\[ \alpha = -2. \] (B.21)

All of this proves that \( \dot{\phi}^a = -2W^a \).

### B.3 Calculation of \( W_{NN} \)

We want to calculate \( N^a T^b \nabla_a \nabla_b V \). Defining \( W_{NN} = N^a N^b \nabla_a \nabla_b W \), and using that \( \nabla_a \nabla_b \nabla_c W = \nabla_a \nabla_c \nabla_b W = [\nabla_a, \nabla_c] \nabla_b W + \nabla_c \nabla_a \nabla_b W \), we have that

\[ N^a N^b \nabla_a \nabla_b V = 6W_{NN} - 4N^a \nabla_a W c \gamma^{cd} N^b \nabla_b W_d - 4N^a N^b W^c [\nabla_a, \nabla_c] \nabla_b W - 4N^a N^b W^c \nabla_c \nabla_a \nabla_b W \]

where we used that \( \gamma^{cd} = T^c T^d + N^c N^d \), and \([\nabla_a, \nabla_c] \nabla_v W = -R_{bac}^d \nabla_d W \), then,

\[ V_{NN} = 6W_{NN} - 4(N^a T^c \nabla_a \nabla_c W)^2 - 4(N^a N^c \nabla_a \nabla_c W)^2 + 4N^a N^b W^c W^d R_{dbac} + 4 \sqrt{W^c W^d N^b T^c \nabla_c (\nabla_a \nabla_b W)} \]

\[ = 6W_{NN} - \Omega^2 - 4(W_{NN})^2 + 4W^c W^d T^d N^a T^b N^c R_{dbac} + 2N^a N^b \frac{D}{dt} (\nabla_a \nabla_b W) \]

\[ = 6W_{NN} - \Omega^2 - 4(W_{NN})^2 - \frac{\dot{\phi}^b \beta}{H^2} R_{TNTN} H^2 + 2 \left[ \frac{D}{dt} (N^a N^b \nabla_a \nabla_b W) - \nabla_a \nabla_b W \frac{D}{dt} (N^a N^b) \right]. \] (B.22)

We have used that \( R_{TNTN} = T^a N^b T^c N^d R_{abcd} \), we also have that,

\[ \mathcal{R} = \gamma^{ac} \gamma^{bd} R_{abcd} = T^a T^b N^c N^d R_{abcd} + N^a N^c T^b T^d R_{abcd} = 2R_{TNTN}. \] (B.23)
so then
\[
N^a N^b \nabla_a \nabla_b V = 6 W W_{NN} - \Omega^2 - 4(W_{NN})^2 - \epsilon H^2 \mathcal{R} + 2 \dot{W}_{NN} - 4 \nabla_a \nabla_b W [N^a \frac{DN^b}{dt}] \\
= 6 W W_{NN} - 4(W_{NN})^2 - \epsilon H^2 \mathcal{R} + 2 \dot{W}_{NN} - 3 \Omega^2. \tag{B.24}
\]

But we also have that
\[
V_{NN} = M^2 + \Omega^2 - \epsilon H^2 \mathcal{R}, \quad M^2 = \mu^2 - 4 \Omega^2. \tag{B.25}
\]

So finally
\[
\mu^2 = 6 W W_{NN} - 4(W_{NN})^2 + 2 \dot{W}_{NN}. \tag{B.26}
\]

We can neglect the last term, which then implisest that there is an upper bound for \(\mu\)
\[
\mu_{\text{max}}^2 = \frac{9}{4} H^2. \tag{B.27}
\]

We want to compute this in the cases of several fields, we have that,
\[
\frac{DT^a}{dt} = -\Omega N^a, \tag{B.28}
\]
\[
\frac{DN^a}{dt} = \Omega T^a + \beta^i e^a_i, \tag{B.29}
\]

and \(\gamma_{ij} = T_a T_b + N_a N_b + \delta_{ij} e^i_a e^j_b\).

### B.4 Removal of divergences.

To start let’s consider the action of a massive scalar field on a De Sitter spacetime,
\[
S = \frac{1}{2} \int d\tau d^3 x a^4 \left[ \frac{\phi'^2}{a^2} + m^2 \phi^2 \right]. \tag{B.30}
\]

Integrating by parts we have that,
\[
S = \frac{1}{2} \int d^3 x a^2 \phi \phi' \bigg|_{t_i}^{t_f} - \frac{1}{2} \int d\tau d^3 x \phi \left[ \partial_\tau (a^2 \partial_\tau \phi) + a^4 m^2 \phi \right] \\
= \frac{1}{2} \int d^3 x a^2 \phi \phi' \bigg|_{t_i}^{t_f}, \tag{B.31}
\]
where the second term vanishes as it’s proportional to the eqs. of motion. Now to obtain the on-shell functional we need to replace into the solution of the eqs of motion. We have that the solution at small wavelength and constant $H = -\frac{1}{\alpha^2}$ is

$$\phi(\tau) = c\tau^\lambda,$$  \hspace{1cm} (B.32)

where $\lambda = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$, and we have killed the other solution because it diverges at $\tau \to -\infty$. Replacing into (B.31) we have

$$S(\phi, \tau) = \frac{1}{2H^2} \int d^3x \tau^3 c^2 \tau \lambda \tau^{2\lambda} = -\frac{1}{2} \int d^3x a^3 \phi^2,$$  \hspace{1cm} (B.33)

Which is HJ functional. Note that the integrand scales as,

$$a^3 \phi^2 \sim \tau^{-2\nu},$$  \hspace{1cm} (B.34)

where

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}.$$  \hspace{1cm} (B.35)

So the HJ function diverges as the modes exit the horizon. Now let’s analyse the role of the ansatz, that in this case reduces to

$$S = \int d^3x \sqrt{h} [W(\phi) + M(\phi) \partial_a \phi \partial^a \phi].$$  \hspace{1cm} (B.36)

The equations are,

$$V = 3HW - 2W^2, \hspace{1cm} -\frac{1}{2} = HM + 4MW_{,\phi\phi} + M_{,\phi}\phi_{,\phi},$$  \hspace{1cm} (B.37)

where $V = m^2 \phi^2$. It’s solutions is,

$$W = -\frac{1}{2} H \lambda \phi^2,$$  \hspace{1cm} (B.38)

$$M = -\frac{1}{2H(1 - 2\lambda)} + ..$$  \hspace{1cm} (B.39)
So replacing back into the ansatz we find that it has the same shape that the HJ functional. This means that a renormalized action would be

$$S_{\text{tot}} = S + S_{\text{ct}},$$

with

$$S_{\text{ct}} = - \int d^3x \sqrt{h} [W(\phi) + M(\phi) \partial_a \phi \partial^a \phi].$$

Indeed when computing the fluctuations around the homogeneous solutions $\phi(x) = \phi_0(t) + \delta \phi(x)$, we find that

$$S^{(2)} = \int d^4x \sqrt{-g} \delta \phi (\nabla^2 - \partial^2 V) \delta \phi + \int_{\partial M} d^3x \sqrt{h} \frac{1}{a} \delta \phi \delta \phi',$$

where the first term is proportional to the eqs of motion for the fluctuations and hence vanishes and the second term contains a derivative evaluated at the cut off, which when Fourier transform becomes

$$S^{(2)} = \int d^3k d^3p \delta ^{(3)}(k + p) \delta \phi_p (\tau_0) \delta \phi_k (\tau_0) F_k (\tau_0),$$

and $F_k (\tau_0) = a(\tau_0)^2 \lim_{\tau \to \tau_0} \frac{\partial}{\partial \tau} \left( \frac{\delta \phi_k (\tau)}{\delta \phi_k (\tau_0)} \right)$. We haven’t written the first order fluctuations because, as expected contains only tadpoles. We need the solution for the fluctuation which is,

$$\delta_k (\tau) = |\tau|^{3/2} H \sqrt{\frac{\pi}{4}} \left( J_\nu(|k\tau|) - e^{\pi i \nu} J_\nu(|k\tau|) \right),$$

which implies that

$$F_k (\tau_0) = \lambda \frac{k^2}{H^2 \tau_0^3} + \frac{k^2}{H^2 (1 - 2\lambda) \tau_0} + \frac{i k^3}{H^2} (k\tau_0)^{(-2\lambda)} + O((k\tau_0)^{2-2\lambda}).$$

Now, varying the action for the counterterms we get

$$S^{(2)}_{\text{ct}} = - \int_{\partial M_0} d^3x \sqrt{h} \left( \delta \phi^2 \partial^2 W + 2M \partial_i \delta \phi \partial^i \delta \phi \right)$$

Inserting back the expressions for $W$ and $M$ we found we get that,

$$S^{(2)}_{\text{ct}} = - \int d^3k d^3p \delta ^{(3)}(k + p)a(\tau_0)^3 \left( H \lambda - \frac{k \cdot p}{H(1 - 2\lambda) a(\tau_0)^2} \right) \delta \phi_p \delta \phi_k.$$
Hence the final action is
\[ S_{\text{tot}} = \int d^3p d^3k \delta^{(3)}(k + p) \frac{i k^3}{2 H^2} (k\tau_0)^{-2\lambda} \delta \phi_k \delta \phi_p \] (B.48)
with \( \lambda = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \). We can extract the power spectrum from this action by the relation
\[ \Psi[\delta \phi] \sim \exp(i S_{\text{tot}}[\delta \phi]) \] (B.49)
so, the two point correlation function is
\[ \langle \delta \phi_k \delta \phi_p \rangle = \int D\delta \phi \delta \phi_k \delta \phi_p |\Psi[\delta \phi]|^2 \] (B.50)
so the power spectrum \( P_{\delta \phi}(k) = \frac{k^3}{2\pi^2} \langle \delta \phi_k \delta \phi_{-k} \rangle \), is,
\[ P_{\delta \phi}(k) = \left( \frac{H}{2\pi} \right)^2 (k\tau_0)^\nu \] (B.51)
By replacing \( \tau_0 \sim 1/aH \) which is the longest scale appearing we have
\[ P_{\delta \phi}(k) = \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^\nu \] (B.52)
which corresponds to the power spectrum of a massive field on De Sitter.

**B.4.1 Renormalization**

Let’s implement the holographic renormalization procedure from [156, 213]. The idea is to divide the action in two parts up to a scale \( \Lambda^{-1} = l < 0 \)
\[ \psi = \int \mathcal{D}\phi_{z<l} \mathcal{D}\phi_{l} \mathcal{D}\phi_{z>l} e^{-S} \] (B.53)
where \( \phi_l = \phi(l) \) is the field evaluated at the scale \( \Lambda^{-1} \), which is not required to be a solution of the equations of motion for \( S \). Then one defines,
\[ \Psi_{UV}(\phi_k) = \int \mathcal{D}\phi_{z\leq l} e^{-S} \] (B.54)
\[ \Psi_{IR}(\phi_k) = \int \mathcal{D}\phi_{z\geq l} e^{-S} \] (B.55)
so that the full path integral is written as,
\[
Z = \int D\phi_l \Psi_{UV} \Psi_{IR}
\] (B.56)

The idea is that \( \phi_l \) minimises the saddle point of the wavefunctions. We will now check this procedure for the scalar field on Ads we studied in 3.2.1. In contrast to what we did before we will have to split the solution in three parts, and so imposing that for both sides \( \phi_{UV,IR}(z = l, k) = \phi_l(k) \). For \( \Psi_{UV} \) we use the same Dirichlet boundary condition as before meaning that the functions is defined up to \( z_c \) that we then remove by a renormalization procedure. A solution satisfying both boundary conditions is
\[
\phi_k(z) = \varphi_k \frac{z^{3/2}}{z_c^{3/2}} \frac{I_\nu(kz)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz)}{I_\nu(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz_c)} + \phi_l \frac{z^{3/2}}{z_c^{3/2}} \frac{I_\nu(kz_c)I_{-\nu}(kz) - I_{\nu}(kz)I_{-\nu}(kz_c)}{I_\nu(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz_c)}
\] (B.57)
\[
\phi_k = \varphi_k \left( \frac{z}{z_c} \right)^{3/2} \frac{K_\nu(kz)}{K_\nu(kz_c)}
\] (B.58)

Plugging back into the action 3.3 we get
\[
S_{UV} = \int \frac{d^3k}{(2\pi)^3} \left[ \left( \frac{3 - 2\nu}{2z_c} + k \frac{I_{\nu-1}(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{1-\nu}(kz_c)}{I_\nu(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz_c)} \right) \varphi_k \varphi_{-k} \right.
\]
\[
+ \left( \frac{3 + 2\nu}{2l} + k \frac{I_{\nu-1}(kl)I_{-\nu}(kz_c) - I_{-\nu-1}(kl)I_{\nu}(kz_c)}{I_\nu(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz_c)} \right) \phi_l \phi^*_l \right]
\]
\[
+ \frac{2}{\Gamma(\nu)\Gamma(1-\nu)} \left( \frac{\sqrt{\nu}}{z_c^{3/2}} - \frac{\sqrt{\nu}}{l^{3/2}} \right) \frac{1}{I_\nu(kz_c)I_{-\nu}(kl) - I_{\nu}(kl)I_{-\nu}(kz_c)} \phi_l \varphi_k \right]
\] (B.59)

Then, we need to remove the divergencies at \( z_c \), which is done by first reescaling \( z_c \rightarrow z_c^{3-\nu} \) and applying the Hamilton Jacobi formalism to remove the contact term. Then \( \Psi_{UV} \) reduces to
\[
S_{UV} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \left( \frac{2^{1-2\nu}\Gamma(1-\nu)k^{2\nu}I_{-\nu}(kl)}{\Gamma(\nu)I_\nu(kl)} \right) \varphi_k \varphi_{-k} \right.
\]
\[
- \left( \frac{3 + 2\nu}{2l} + k \frac{I_{\nu-1}(kl)}{I_\nu(kl)} \right) \phi_l^2 - \frac{2^{1-\nu}k^{\nu}}{\Gamma(\nu)l^{3/2}I_\nu(kl)} \left( \varphi_k \phi_l^* + \phi_l \varphi_k \right) \right]
\] (B.61)
for $\Psi_{IR}$ we impose that the soliton is finite at $z = \infty$ so we have that

$$\phi(z, k) = \left(\frac{z}{l}\right)^{3/2} \frac{K_\nu(kz)}{K_\nu(kl)} \phi_l$$ \hspace{1cm} (B.62)

then

$$S_{IR} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{3/2 - \nu}{l^3} - \frac{k K_{\nu-1}(kl)}{l^3 K_\nu(kl)} \right] \phi_l^2$$ \hspace{1cm} (B.63)
Appendix C

Effective field theory of inflation

C.1 Effective field theory for the scalar modes

A good choice to study perturbations on a cosmological background is by foliating the spacetime into hypersurfaces of constant time. This naturally introduces the ADM decomposition where we have that,

$$\text{d}s^2 = -N^2 \text{d}t^2 + h_{ij}(N^i \text{d}t + \text{d}x^i)(N^j \text{d}t + \text{d}x^j),$$  \hspace{1cm} (C.1)

with functions $N$ and $N^i$ called lapse and shift, parameterising time evolution and its projection on the hypersurface. They are constraint which value is fixed by the Einstein equations, when imposing a metric, eg. a FRW universe. For this case, furthermore, at zero order they may only be time dependent because they are functions of an isotropic and homogeneous solution. Since we will perturb the metric slightly, we can build order parameters, based on background quantities, which allow to have control on how accurate is our description.

We can think of inflation as a quasi de Sitter spacetime which has broken time diffeomorphism and because it ends. Then, an effective field theory (EFT) for deviations from this broken de Sitter spacetime can be written in term of the remaining symmetries of this spacetime. In the foliation that we described, functions of the induced metric $h_{\mu\nu}$ on the hypersurface $\Sigma_t$ will be allowed, such as the the extrinsic curvature tensor $K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu$, and the Riemann curvature $\hat{R}_{\alpha\beta\gamma\delta}$ on the induced metric $\Sigma_t$. Also products of four dimensional covariant tensors with free upper 0 indices, but with all spatial indices contracted are allowed operators ($g^{00}$, $R^{00}$)
The most general action constructed with these elements is

$$S = \int d^4x \sqrt{-\tilde g} \mathcal{L}\left[R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \tilde R_{\mu\nu}, \nabla_\mu; t\right],$$

(C.2)

### C.1.1 Scalar action

Since we are describing perturbations around a given background, the time dependent part of this action will be a tadpole term equivalent to the Friedmann equations.

$$S = \int d^4x \sqrt{-\tilde g} \left[\frac{M_{\text{Pl}}^2}{2} R + M_{\text{Pl}}^2 (3H^2 + \dot{H}) + M_{\text{Pl}}^2 \dot{H} g^{00}\right].$$

(C.3)

The rest can be described in terms of the fluctuations. After fixing the gauge invariance, the full action will be described in terms of two independent functions. In the Newtonian gauge, where the hypersurface of constant time $\Sigma_T$ are orthogonal to the worldlines of observers at rest, the perturbations are described as functions of $\delta g^{00} = g^{00} + 1$ and $\delta K_{\mu\nu} = K_{\mu\nu} - Hh_{\mu\nu}$. The action is formally

$$\Delta S = \int d^4x \sqrt{-\tilde g} \left[\frac{M^2(t)}{2} (\delta g^{00})^2 + \frac{M^4(t)}{3!} (\delta g^{00})^3 + \frac{M^6(t)}{4!} (\delta g^{00})^4 + \ldots \right.$$

$$- \frac{\dot{M}_1^2(t)}{2} \delta g^{00} \delta K^2 - \frac{\dot{M}_3^2(t)}{2} (\delta K^\nu)^2 + \ldots \right],$$

(C.4)

where the time dependent functions are functions of the background, and vary slowly on time. Furthermore for slow roll inflation, they will be functions of the slow roll parameters. By reintroducing time invariance by means of the transformation

$$t \mapsto t' = t + \pi(t, x^i),$$

(C.5)

$$x^i \mapsto x'^i = x_i,$$
where $\pi(t, x^i)$ is the Goldstone boson, the action for the scalar part can be extracted from the EFT by identifying $\pi$, with the inflaton. Then, the action (C.4), becomes,

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R + M_{\text{Pl}}^2 (3H^2(t + \pi) + \dot{H}(t + \pi)) + M_{\text{Pl}}^2 \dot{H}(t + \pi)(-1 - \frac{\pi}{a^2}) \right. \left. + \sum_n \frac{M_n^4(t + \pi)^n}{n!} \left(1 + \frac{\pi}{a^2} + 2\partial_\mu \pi \frac{\partial_{\nu} g^{\mu \nu}}{a^2} \right)^n \right],$$

where as we have anticipated the first function depends on the slow roll parameter $\epsilon = -\frac{\dot{H}}{H^2}$ and thus vary slowly on time. To further simplify it is useful to take the limit where Goldstone modes decouple from gravity. This happens where,

$$M_{\text{Pl}} \rightarrow \infty, \quad \dot{H} \rightarrow 0 \quad M_{\text{Pl}}^2 \dot{H} = \text{const},$$

Then we have to evaluate if mixed terms are small compared to the kinetic energy $M_{\text{Pl}}^2 \dot{H}(\partial_\mu \pi)^2$. For example at background level the leading mixing with gravity comes from the term $M_{\text{Pl}}^2 \dot{H} \delta g^{00}$. Then the action becomes

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R + M_{\text{Pl}}^2 (3H^2(t + \pi) + \dot{H}(t + \pi)) + M_{\text{Pl}}^2 \dot{H}(t + \pi) \left(-1 - 2\pi + (\partial_\mu \pi)^2\right) \right. \left. + \sum_n \frac{M_n^4(t + \pi)^n}{n!} \left(-2\pi^2 + (\partial_\mu \pi)^2\right)^n \right].$$

Up to third order we have that,

$$S = \int d^4x \sqrt{-g} \left[ M_{\text{Pl}}^2 R - M_{\text{Pl}}^2 \dot{H} \left(\pi^2 - \frac{(\partial_\mu \pi)^2}{a^2}\right) + 2M_2^4 \left(\pi^2 + \pi^3 - \frac{\pi}{a^2} (\partial_\mu \pi)^2\right) - \frac{4}{3} 2M_3^4 \pi^3 \ldots \right].$$

Note that that higher order operators modify the dispersion relation for the Goldstone boson. Indeed, the speed of sound can be written as

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\text{Pl}}^2 \dot{H}}.$$

Where this can vary from one by a different set of effects as we described in section 2.
C.1.2 Tensor fluctuations

We define the scalar perturbations as $\zeta$ and tensor perturbations as $\gamma_{ij}$ as

$$h_{ij} = a^2 e^{2\zeta} (e^\gamma)_{ij} \quad \text{with} \quad \gamma_{ii} = \partial_i \gamma_{ij} = 0.$$ (C.14)

We will be interested in corrections to the two point function. Then the relevant contribution to the second order equation of motion for the tensor modes comes from $M_2^4(t)$, $\tilde{M}_1^3(t)$, $\tilde{M}_2^3(t)$ and $\tilde{M}_3^3(t)$. Expanding in the usual ADM decomposition,

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j),$$ (C.15)

where the extrinsic curvature is given by $K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)$. Now expanding we will have that when looking to tensor perturbations at first order in curvature perturbations $N_i = 0$ and $N = 1$. In ADM the Ricci scalar is decomposed as,

$$R = (3)R + K_{ij}K^{ij} - K^2 + ...$$ (C.16)

where the dots corresponds to boundary terms. We then have that up to second order in $h_{ij}$

$$(3)R = -\frac{1}{4a^2}(\partial_j \gamma_{kj})^2,$$ (C.17)

$$K_{ij}K^{ij} - K^2 = -6H^2 + \frac{1}{4}(\dot{\gamma}_{ij})^2.$$ (C.18)

And then the background action for the gravitons is

$$S = \frac{M_{pl}^2}{8} \int d^4x a^3 \left[ \dot{\zeta}_{ij}^2 - (\partial_j \gamma_{kj})^2 \right].$$ (C.19)

Furthermore we have that

$$(\delta K_{\mu}^\nu)^2 - (\delta K)^2 = \frac{1}{4}(\dot{\gamma}_{ij})^2.$$ (C.20)

Hence the combination of terms given by $\tilde{M}_3^3(t) - \tilde{M}_2^3(t)$, can modify the speed of sound up to second order in perturbations.
Appendix D

Photons geodesics

We have to find the geodesic equations for photons. First we parametrise the photons momentum as,

\[ p^\mu = E(E_0^\mu) + p^i(E_i^\mu), \tag{D.1} \]

We also define

\[ E = \frac{\epsilon}{a}, \quad p^i = \frac{\epsilon}{a} e^i \quad \text{with} \quad \delta_{ij} e^i e^j = 1. \tag{D.2} \]

given that

\[ g_{\mu\nu} E_0^\mu E_0^\nu = -1 \quad \text{and} \quad g_{\mu\nu} E_i^\mu E_j^\nu = \delta_{ij}, \tag{D.3} \]

we have thus,

\[ E_0^0 = \frac{1}{a \sqrt{c_T}} (1 - \psi), \tag{D.4} \]
\[ E_i^i = \frac{\sqrt{c_T}}{a} (1 + \phi). \tag{D.5} \]

then,

\[ p^\mu = \frac{\epsilon}{a^2} \left[ \frac{1 - \psi}{\sqrt{c_T}}, \frac{\sqrt{c_T} (1 + \phi)}{a} e^i \right], \tag{D.6} \]

which satisfies the constraint \( p^2 = 0 \). The geodesic equations are,

\[ \frac{dx^\mu}{d\lambda} = p^\mu, \quad \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0 \tag{D.7} \]
Replacing using (D.6) it follows,

\[ \frac{dx^0}{d\lambda} = \frac{\epsilon}{a^2} \frac{1 - \psi}{\sqrt{cT}}, \quad (D.8) \]

\[ \frac{dx^i}{d\eta} = \frac{p_i}{p^0} = c_T(1 + \phi + \psi)e^i. \quad (D.9) \]

The geodesic equations for the 0 component is,

\[ \frac{d \log \epsilon}{d\eta} = \frac{d\psi}{d\eta} - \psi' + \phi' - 2c_T e^i \partial_i \psi, \quad (D.10) \]

Noticing that de derivative of \( \psi \) is along the photon trajectory and thus it is written as,

\[ \frac{d\psi}{d\eta} = \frac{\partial \psi}{\partial \eta} + \frac{\partial \psi}{\partial x^i} \frac{\partial x^i}{\partial \lambda} \frac{\partial \lambda}{\partial \eta} = \psi' + c_T e^i \partial_i \psi. \quad (D.11) \]

The last equation becomes,

\[ \frac{d \log \epsilon}{d\eta} = -\frac{d\psi}{d\eta} + \psi' + \phi', \quad (D.12) \]

where let us notice that the derivatives of the temperature perturbation are along the photon trajectory and thus the last equation can be written as,

\[ \frac{\partial \bar{f}}{\partial \log \epsilon} \frac{d}{d\eta} (\Theta - \log \epsilon) = \frac{df}{d\eta_{\text{set}}}. \quad (D.13) \]
References


References


