

# STRATEGY-PROOF STOCHASTIC ASSIGNMENT

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**ABSTRACT.** I study strategy-proof assignment mechanisms where the agents reveal their preference rankings over the available objects. A stochastic mechanism returns lotteries over deterministic assignments, and mechanisms are compared according to first-order stochastic dominance.

I show that non-wasteful strategy-proof mechanisms are not dominated by strategy-proof mechanisms, however nonwastefulness is highly restrictive when the mechanism involves randomization. In fact, the Random Priority mechanism (i.e., the Random Serial Dictatorship), and a recently adopted school choice mechanism, Deferred Acceptance with Random Tie-breaking, are wasteful. I find that both these mechanisms are dominated by strategy-proof mechanisms.

In general, strategy-proof improvement cannot be due to merely reshuffling objects, and therefore must involve assigning more objects.

*Keywords:* random assignment, strategy-proofness, priority based assignment, ordinal efficiency, school choice. (JEL: C78, D61, D63.)

## 1. INTRODUCTION

Consider the problem of assigning a number of heterogenous indivisible objects to individuals, where each individual can receive at most one object. In a wide range of contexts where this problem arises, such as school choice, course assignment at universities, or office allocation, the procedures typically rule out monetary transfers, and rely on agents' preference rankings over the objects. Three properties are crucial for an assignment procedure: efficiency, strategy-proofness, and fairness.

Indivisibilities make it impossible to treat the agents equally when assignments are made in a deterministic way. *Randomization* greatly expands the set of mechanisms and allows *fairness*. For example, the *random priority mechanism* orders the agents randomly, and lets them pick their objects in this order. In the richer

set of stochastic mechanisms, agents' assignments are lotteries over objects. The assignment at the end of the procedure in application will depend on the realization of the lottery, but the agents, when revealing their preferences, are facing lotteries over deterministic outcomes. Therefore, the analysis of the mechanism calls for a stochastic perspective.

Since the agents' preferences are their private information, it is important for the mechanism to provide the agents with the right incentives to reveal their preferences truthfully. Thus efficiency is really a property of the mechanism, rather than of the assignment resulting from the mechanism after the preferences are revealed. In our environment the primitives on preferences are ordinal rankings over the objects, so the appropriate concept for incentive compatibility is *strategy-proofness*, and the natural notion of welfare is *Pareto efficiency*, where lotteries are compared via *first-order stochastic dominance*.

A mechanism *weakly dominates* another if for every preference profile, it assigns to every agent a lottery which she weakly prefers in the first-order stochastic dominance sense. As it is with efficiency, *non-wastefulness* is also extended to this stochastic environment in a natural way: if an agent would rather have more of some object, say  $x$ , instead of another object she has received with positive probability, then it must be that all of  $x$  is already assigned.

I first show that a non-wasteful strategy-proof mechanism cannot be dominated by another strategy-proof mechanism. On the other hand, for randomized mechanisms, this *ordinal*, or *ex ante*, notion of non-wastefulness is a very demanding condition, much stronger than *ex post* non-wastefulness. While every non-wasteful stochastic assignment is a convex combination of non-wasteful deterministic assignments, stochastic mechanisms are often wasteful even when they are randomizations over non-wasteful mechanisms.

My next result, then, sheds light on the nature of strategy-proof improvement: if a strategy-proof mechanism is dominated by another, such Pareto improvement cannot be achieved by merely re-allocating the objects, but must involve assigning *more* objects. Indeed, in two important applications, I show that the *random priority (RP)*, i.e., the *random serial dictatorship* mechanism, and the *randomized deferred acceptance (RDA)* mechanism admit strategy-proof improvement. I give explicit constructions of strategy-proof mechanisms which dominate them.

Bogomolnaia and Moulin (2001) make the critical observation that the RP is dominated, but they also show that there is no strategy-proof and efficient mechanism satisfying equal treatment of equals. Zhou (1990) notes that whether the RP is optimal within the class of symmetric, ex post Pareto optimal, strategy-proof mechanisms remains an open question. My construction shows that it is not.

A question of the same nature recently emerged from another practical market design issue. In a typical school choice program, each school has an exogenous priority ranking over the students. A matching *respects these priorities* if whenever a student prefers a school  $x$  to her own, it must be that only students of equal or higher priority for  $x$  are assigned to school  $x$ . When these priority rankings are strict, the *Deferred Acceptance (DA)* mechanism of Gale and Shapley (1962) returns the *constrained efficient* assignment (i.e., the student-optimal assignment among those assignments which respect the priorities). However, in many applications, large groups of students have equal priority, and therefore the priority rankings have ties. The leading mechanism in this environment *randomly* breaks ties before applying the DA mechanism. Ehlers (2006) observes that some tie-breaking rules even result in constrained inefficiency. Therefore, unlike the DA with strict priorities, the *randomized deferred acceptance (RDA)* mechanism with weak priorities is short of constrained efficiency. While the DA is strategy-proof (Dubins and Freedman, 1981; Roth, 1982), Erdil and Ergin (2008) show that no strategy-proof mechanism is ex post constrained efficient when there are ties in priorities. In contrast with these impossibility results, I explicitly construct a strategy-proof mechanism that dominates the RDA.

In large markets where certain *regularity* conditions hold, Che and Kojima (2010) show that the RP is asymptotically efficient. More generally Liu and Pycia (2011) establish that the RP is the only symmetric, strategy-proof and efficient mechanism. Therefore, the room for strategy-proof improvement over symmetric, strategy-proof, ex post efficient mechanisms vanishes as the market size grows. My results complement these findings for the case of small markets. The constructions of strategy-proof improvement are computationally demanding, and are possibly intractable for large markets. But the ideas are likely to lend themselves to applications when the market size is small, where we establish that such improvement is indeed possible.

One insight that emerges from this paper is that while randomizing over strategy-proof mechanisms preserves strategy-proofness, it does not necessarily preserve the

property of being un-dominated within the strategy-proof class. Starting, instead, with stochastic mechanisms, Pycia and Ünver (2012) investigate whether their various desirable properties survive decomposition as a lottery over deterministic mechanisms each of which also satisfies those properties. They find a sufficient condition and establish positive results on some restricted preference domains.<sup>1</sup>

## 2. STOCHASTIC ASSIGNMENTS

It is common to employ randomization to accommodate various notions of justice, equity, or fairness. Especially when allocating indivisible resources, the discrete nature of the problem implies that mechanisms with deterministic outcomes will treat agents asymmetrically. For example, the serial dictatorship mechanism is highly asymmetric in its treatment of the agents. As Zhou (1990) and Abdulkadiroğlu and Sönmez (1998) highlight, choosing, with equal probability, an exogenous priority order in which the agents get to pick their objects recovers symmetry between the agents. While this *random priority* mechanism satisfies *equal treatment of equals*, is strategy-proof and returns a lottery over efficient assignments, Bogomolnaia and Moulin (2001) show that efficiency, equal treatment of equals and strategy-proofness are incompatible. Therefore, the efficiency loss associated with the random priority mechanism cannot be alleviated in a strategy-proof way. Given the prominence of this mechanism in practice, it is important to know whether it can be improved upon by a strategy-proof mechanism.

Take, for another important application, the school admissions model, where schools typically have different priority rankings with ties. Abdulkadiroğlu et al. (2009), in their study of the New York City school choice system, show that for strict priorities, DA is not dominated by a strategy-proof mechanism. The mechanism used in NYC is of course not the DA with strict priorities, but the following stochastic mechanism: firstly the mechanism, randomly and with equal probability, chooses a linear order over the students. Then all ties in priority rankings are broken according to this order to get strict priorities. And finally the DA algorithm is run with respect to these strict priorities. We know that this mechanism is ex post constrained inefficient, and there is no strategy-proof and ex post student optimal stable mechanism (Erdil and Ergin, 2008). It remains to be answered whether the stochastic mechanism used in NYC is on the *efficient frontier* of strategy-proof mechanisms.

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<sup>1</sup>See also Peters et al. (2013) for related results.

We now turn to a formal treatment of *stochastic assignment mechanisms* in order to answer the above questions. Let  $N$  denote a finite set of agents, and  $X$  a finite set of types of objects. There are  $q_x \geq 1$  copies of type  $x$  for  $x \in X$ . For agent  $i$ , the option of staying unassigned (which can also be interpreted as being assigned one's outside option) is also denoted by  $i$ . Each agent has a **strict preference ranking**  $R_i$  over  $X_i := X \cup \{i\}$ . That is,  $R_i$  is a complete, transitive and antisymmetric relation over  $X_i$ , with  $P_i$  denoting its asymmetric part. An object  $x$  is **acceptable** to  $i$  if  $xP_i i$ . A **preference profile** is a vector  $R = (R_i)_{i \in N}$  of individual preference relations.

A *stochastic assignment*, also called a *probabilistic assignment*, assigns each agent  $i$  a lottery over  $X_i$  such that the total probability share of  $x$  is at most  $q_x$  (where  $q_i = 1$ ). Formally, a **stochastic assignment** is a profile  $\gamma = (\gamma_i)_{i \in N}$  of vectors  $\gamma_i = (\gamma_{ix})_{x \in X_i}$  such that  $\gamma_{ix} \in \mathbb{R}_+$  for all  $i \in N$  and  $x \in X_i$ , and

- (a)  $\sum_{x \in X_i} \gamma_{ix} = 1$  for each agent  $i \in N$ ,
- (b)  $\sum_{i \in N} \gamma_{ix} \leq q_x$  for each object  $x \in X$ .

An assignment  $\gamma$  is called **deterministic** if  $\gamma_{ix} \in \{0, 1\}$  for all  $i$  and  $x$ . For every agent  $i$ , there is at most one  $x$  such that  $\gamma_{ix} = 1$ . For every such  $i$  and  $x$ , we write  $\gamma(i) = x$ . A generalized version of the Birkhoff-von Neumann theorem (see, e.g., Kojima and Manea, 2010) says that any stochastic assignment is a convex combination of deterministic assignments. Hence, every stochastic assignment can be interpreted as a lottery over deterministic assignments.

The total amount of objects allocated by  $\gamma$  will be called the **size of an assignment**  $\gamma$ , and is formally defined as  $|\gamma| = \sum_{x \in X} \sum_{i \in N} \gamma_{ix}$ . Given a preference profile  $R$ , a stochastic assignment  $\gamma$  is called **wasteful** if there exist  $j \in N$  and  $y, z \in X_j$  such that

$$zP_j y, \quad \gamma_{jy} > 0, \quad \sum_{i \in N} \gamma_{iz} < q_z.$$

If  $\gamma$  is not wasteful, it is called **non-wasteful**. Our perspective on mechanisms is that with the assignment procedures involving randomization, the agents are actually facing stochastic mechanisms. Kesten and Ünver (2012) refer to this as the *ex ante* approach. It is common in mechanism design to use the term *ex ante* for evaluating a mechanism before the agents know their preferences (e.g., Holmström and Myerson, 1983). In order to avoid confusion then, throughout the rest of the paper, I will write non-wasteful with the above definition in mind. This stochastic analogue of non-wastefulness is not necessarily the same as the assignment being

a lottery over non-wasteful deterministic assignments. In fact, as we will see in Remark 2, a randomization over non-wasteful deterministic assignments may be wasteful.

Unless otherwise stated, I will restrict my attention to **individually rational** assignments: no agent is assigned a positive amount of an object she likes less than her outside option, that is,  $iP_i x \Rightarrow \gamma_{ix} = 0$ . A **stochastic assignment mechanism** is a function which associates to each preference profile an individually rational stochastic assignment. If its outcome is a deterministic assignment for every preference profile, then the mechanism is said to be **deterministic**.  $f$  satisfies **equal treatment of equals** if  $f_i(R) = f_j(R)$  whenever  $R_i = R_j$ .

Given an agent  $i$ 's ordinal preference relation  $R_i$  over  $X_i$ , one can easily extend it to a relation  $\tilde{R}_i$  over lotteries over  $X_i$  via first-order stochastic dominance (FOSD) so that if  $\omega_i = \sum_{x \in X_i} \omega_{ix} x$  and  $\gamma_i = \sum_{x \in X_i} \gamma_{ix} x$ , then

$$\text{(FOSD)} \quad \gamma_i \tilde{R}_i \omega_i \iff \text{for every } y \in X_i, \sum_{x: x R_i y} \gamma_{ix} \geq \sum_{x: x R_i y} \omega_{ix}.$$

This condition is equivalent to the requirement that  $\text{Eu}_i[\gamma] \geq \text{Eu}_i[\omega]$  for any vNM utility function  $u_i$  representing  $R_i$ .  $\gamma_i$  is said to FOSD  $\omega_i$ , denoted  $\gamma_i \tilde{P}_i \omega_i$ , if in addition to the above, the inequality on the right hand side of (FOSD) is strict for at least one  $y$ .

An assignment  $\gamma$  **weakly dominates** another assignment  $\omega$  if for each agent  $i$ , her assignment under  $\gamma$  weakly first-order stochastically dominates her assignment under  $\omega$ , i.e.,  $\gamma_i \tilde{R}_i \omega_i$ . If we also have  $\gamma_j \tilde{P}_j \omega_j$  for some  $j$ , we say that  $\gamma$  **dominates**  $\omega$ . If an assignment is not dominated by any other assignment, then it is called **efficient**.<sup>2</sup> If  $\mu$  is a convex combination of efficient deterministic assignments, then it is called **ex post efficient**. Every efficient assignment is non-wasteful and ex post efficient, but as Bogomolnaia and Moulin (2001) illustrate, the converse does not hold. A mechanism  $g$  **dominates** another mechanism  $f$ , if for every preference profile, the outcome of the first weakly dominates that of the latter, with strict dominance for at least one preference profile. In other words,  $g_i(R) \tilde{R}_i f_i(R)$  for each  $R$  and for each  $i$ ; and  $g_j(R') \tilde{P}_j f_j(R')$  for some  $j$  and  $R'$ . A mechanism

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<sup>2</sup>As it is with non-wastefulness, we consider *efficiency* as a property of lotteries, because the outcomes of stochastic mechanisms are lotteries, and the realization of these lotteries is not part of the mechanism. Bogomolnaia and Moulin (2001) use the term *ordinally efficient*, whereas Kesten and Ünver (2012) use *ex ante efficient* to emphasize that the assignment is viewed before the lottery has realized.

$f$  is **strategy-proof** if  $f_i(R)\tilde{R}_i f_i(R'_i, R_{-i})$  for every  $R$  and  $R'_i$ . And finally, a mechanism is said to be on the **efficient frontier** of strategy-proof mechanisms if it is strategy-proof and is not dominated by another strategy-proof mechanism.

### 3. RANDOM PRIORITY AND RANDOMIZED DEFERRED ACCEPTANCE

A common method to allocate objects is random assignment. For example, when there are  $n$  agents and  $n$  objects, and if each agent is to receive exactly one object, then there are  $n!$  possible deterministic assignments. The *Random Assignment (RA)* mechanism is nothing but a random variable over these assignments, with each assignment having equal probability. As such it satisfies equal treatment of equals. As it disregards preferences, it is strategy-proof, however it also fails to be individually rational for some preference profiles. The *Serial Dictatorship* mechanism fixes an exogenous ranking over the agents, assigns the first agent to her favorite object, assigns the second agent to her favorite from among the remaining objects, and so on. The highly unequal treatment of this mechanism is remedied by the **Random Priority (RP)** mechanism, also called the *Random Serial Dictatorship*, which is defined as a uniform lottery over all possible serial dictatorships.

**Remark 1.** *The RP dominates the RA.* Note that for any agent, and any integer  $k \leq n$ , her total share of her most preferred  $k$  objects under equal distribution is  $k/n$ . Under the RP, with probability  $k/n$ , she receives one of her top  $k$  choices. Hence, the RP weakly dominates the RA. An obvious preference profile at which the dominance is strict is where one agent's top choice is bottom ranked by everyone else.

Note also that the RA is *not* individually rational. Each agent receives some fraction of each object even when she does not find them acceptable. If we modify the RA by allowing free disposal, and thus recover individual rationality, we make the mechanism “more wasteful” than the RP.  $\diamond$

When the demand for some objects exceeds the supply, a common method to decide who receives the objects is to appeal to an exogenous priority ranking over the agents. A **priority structure** is a profile  $\succsim = (\succsim_x)_{x \in X}$  of weak orders (complete and transitive relations) on  $N$ , where for each  $x \in X$ ,  $\succsim_x$  ranks agents with respect to their priority for  $x$ . Let  $\succ_x$  denote the asymmetric part of  $\succsim_x$ . If  $\succsim_x$  is antisymmetric for each  $x \in X$ , then the priority structure  $\succsim$  is called **strict**.

Given priorities  $\succsim$  and preferences  $R$ , let  $\mu$  be a deterministic assignment. We say  $\mu$  **violates the priority of  $i$  for  $x$** , if there is an agent  $j$  such that  $j$  is assigned to  $x$  whereas  $i$  both prefers  $x$  to her assigned object and has strictly higher priority for it, i.e.,  $\mu(j) = x$ ,  $xP_i\mu(i)$ , and  $i \succ_x j$ . If  $\mu$  is non-wasteful and if it does not violate priorities, then it is called **stable**. It is **constrained efficient** if it is stable, and is not Pareto dominated by any other stable assignment. Given  $\succsim$ , a mechanism is called ex post stable if it associates to every preference profile  $R$  a convex combination of assignments which are stable with respect to  $R$  and  $\succsim$ .

The **Serial Dictatorship (SD $^\succsim$ )** mechanism sets the priority rankings over the agents as  $i_1 \succ_x i_2 \succ_x \cdots \succ_x i_n$  for each  $x$ , and at every preference profile  $R$ , it returns the unique stable assignment. It is Pareto efficient and strategy-proof.

For an arbitrary strict priority structure, Gale and Shapley (1962) prove the existence of a stable assignment via the following algorithm: Given strict priorities  $\succ$ , and strict preferences  $R$ , at step 1, every  $i$  applies to her favorite acceptable object. For each  $x$ , the  $q_x$  applicants who have the highest priority for  $x$  are placed on the waiting list of  $x$ , and the others are rejected. At step  $r$ , the applicants who were rejected at step  $r - 1$  apply to their next best acceptable objects. For each  $x$ , the highest priority  $q_x$  agents among the new applicants and those in the waiting list are placed on the new waiting list, and the rest are rejected. The algorithm terminates when every agent is either on a waiting list or has been rejected by every object that is acceptable to her. At the end, objects are assigned to the agents on their waiting lists.

Gale and Shapley (1962) not only establish existence of stable assignments, but also show that the outcome of their algorithm is Pareto superior to any other stable assignment. Fixing strict priorities  $\succ$ , the mechanism which associates to a preference profile the outcome of the above algorithm is called the **Deferred Acceptance** mechanism, and is denoted by  $DA^\succ$ . Moreover, this mechanism is strategy-proof (Dubins and Freedman, 1981; Roth, 1982), but Roth (1982) also shows that the outcome of the DA is not always Pareto efficient. In fact Ergin (2002) shows that the DA is an efficient mechanism only under very restrictive assumptions on the priorities.<sup>3</sup> Since the DA is widely used in real life, it would be important to recover this welfare loss. Kesten (2010) shows that when priorities are strict, and all schools are acceptable to all students, no strategy-proof and efficient

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<sup>3</sup>Erdil and Ehlers (2010) give a general characterization of weak priority structures for which the DA is efficient.



mechanism dominates the DA. Abdulkadiroğlu et al. (2009) show that there is no strategy-proof mechanism which dominates the DA with strict priorities. Kesten and Kurino (2012) show that this impossibility holds also in the more restricted preference domain where all schools are acceptable to all students. In some sense there is a tension between strategy-proofness and efficiency when we take the DA as a benchmark.

In many applications, the priority rankings have ties. Designing a mechanism which respects weak priority rankings involves some form of randomization. For example, in New York City High School Match, large classes of students have equal priorities, and the ties are broken randomly to ensure fairness between such students.

In general, a **tie-breaking rule** is a function  $\tau$  from the set of priority structures to the set of strict priority structures such that denoting  $\succ' = \tau(\succ)$ , we have

$$i \succ'_x j \implies i \succ_x j.$$

According to this definition, a tie-breaking rule is independent of agents' preferences.

Let  $\mathcal{T}$  be a set of tie-breaking rules, and  $\pi$  be a probability distribution over  $\mathcal{T}$ . Given a priority structure  $\succ$ , let us denote by  $\text{DA}^{\tau(\succ)}$  the mechanism which breaks the ties in  $\succ$  according to the tie-breaking rule  $\tau$ , and then runs the DA algorithm. Fixing an exogenously given priority structure  $\succ$ , let us define the **stochastic DA mechanism associated with  $\mathcal{T}$  and  $\pi$**  as

$$\text{RDA}^{\succ, \mathcal{T}, \pi} = \sum_{\tau \in \mathcal{T}} \pi_{\tau} \text{DA}^{\tau(\succ)}.$$

It is strategy-proof, because it is an exogenous randomization over strategy-proof mechanisms.

Consider a simple tie-breaking rule which fixes a linear order on  $N$ , and breaks all the ties according to that order. Denote both this linear order and the associated tie-breaking rule by  $\tau$ . If  $\succ' = \tau(\succ)$ , then we should have for all  $i, j \in N$  and  $x \in X$

$$i \sim_x j \quad \text{and} \quad i \tau j \implies i \succ'_x j.$$

Various school choice mechanisms (such as those in New York City and Boston) adopt the RDA. If  $\succ$  is the prevailing priority structure, the specific mechanism in use is based on  $\text{RDA}^{\succ, \mathcal{L}, \pi}$ , where  $\mathcal{L}$  is the set of linear orders on the set of agents  $N$ , and  $\pi$  is the uniform distribution over  $\mathcal{L}$ . Thus,  $|\mathcal{L}| = n!$ , and  $\pi_{\tau} = 1/n!$  for

every  $\tau \in \mathcal{L}$ . In analogy with the random priority mechanism being denoted by RP, I will use the notation  $\mathbf{RDA}^{\succsim}$  to denote:

$$\mathbf{RDA}^{\succsim} = \frac{1}{n!} \sum_{\tau \in \mathcal{L}} \mathbf{DA}^{\tau(\succsim)}.$$

When all agents are of equal priority for each object, the RDA reduces to the RP. Given the prominence of the RDA (and its special case the RP) in real life applications, it is important to know whether it is on the efficient frontier of strategy-proof mechanisms.

#### 4. RESULTS

My first main result relies on a critical lemma which I state here because of its centrality.

**Reshuffling Lemma.** *Suppose that an assignment  $\gamma$  first-order stochastically dominates another assignment  $\omega$ . If  $\omega$  is non-wasteful, then  $\sum_{x \in X} \gamma_{ix} = \sum_{x \in X} \omega_{ix}$  for every  $i \in N$ .*

Reshuffling Lemma basically says that a first-order stochastic dominance improvement over a non-wasteful assignment is nothing but a reshuffling of the assigned objects (in probability shares) between the agents.

**Proposition 1.** *If a strategy-proof mechanism is non-wasteful, then it is not dominated by any other strategy-proof mechanism.*

While significant applications, such as the RP and the RDA, involve randomizations over non-wasteful deterministic assignments, the above proposition is not necessarily conclusive about such mechanisms, because a randomization over non-wasteful mechanisms can be wasteful.

**Remark 2.** *The RP is wasteful (and, hence, so is the RDA).* In order to see why, suppose there are four agents 1, 2, 3, 4, and three distinct objects  $a, b, c$ . Let  $\succsim$  be a *trivial* priority structure in the sense that  $i \sim_x j$  for all agents  $i, j$ , and objects  $x$ . The preference profile

$R_1$	$R_2$	$R_3$	$R_4$	results in	RP( $R$ )	$a$	$b$	$c$
$a$	$c$	$a$	$c$		1	1/2	3/8	
$b$	$b$				2		3/8	1/2
					3	1/2		
					4			1/2

Not all of  $b$  is assigned, whereas 1 finds  $b$  acceptable, and the sum of her assigned probability shares is less than 1. Hence Proposition 1 is mute about the RDA.  $\diamond$

Bogomolnaia and Moulin (2004), in their study of a two-sided matching problem with dichotomous preferences, find that randomization enables efficiency and strategy-proofness to hold at the same time where deterministic mechanisms fail to do so. In our environment too, in principle, randomization over non-strategy-proof mechanisms might be strategy-proof. For instance, given a priority structure  $\succsim$ , suppose that we have a family  $\mathcal{T}$  of tie-breaking rules, and a family  $\{g^\tau\}_{\tau \in \mathcal{T}}$  of mechanisms such that  $g^\tau$  weakly dominates  $\text{DA}^{\tau(\succsim)}$  for all  $\tau$ , with strict dominance for some  $\tau$ . We know from Abdulkadiroğlu et al. (2009) that  $g^\tau$  cannot be strategy-proof if  $g^\tau$  strictly dominates  $\text{DA}^{\tau(\succsim)}$ . But perhaps an appropriate randomization over these mechanisms is strategy-proof. As Remark 2 illustrates, the size of the assignment  $\text{DA}^{\tau(\succsim)}(R)$  varies depending on the tie-breaking rule  $\tau$ . However for any  $\tau$ , Reshuffling Lemma implies  $|g^\tau(R)| = |\text{DA}^{\tau(\succsim)}(R)|$ . Hence for any probability distribution  $\pi$  on  $\mathcal{T}$ , and preferences  $R$ , we have

$$(1) \quad \left| \sum_{\tau \in \mathcal{T}} \pi_\tau g^\tau(R) \right| = \left| \sum_{\tau \in \mathcal{T}} \pi_\tau \text{DA}^{\tau(\succsim)}(R) \right|.$$

Let us say mechanisms  $f$  and  $g$  are of **equal size** if  $|f(R)| = |g(R)|$  for all  $R$ . If, on the other hand,  $|g(R)| \geq |f(R)|$  for all  $R$  with strict inequality for at least some  $R$ , then we say  $g$  is of **greater size** than  $f$ .

**Proposition 2.** *If a strategy-proof mechanism  $g$  dominates another strategy-proof mechanism  $f$ , then  $g$  is of greater size than  $f$ .*

This result not only applies to several mechanisms studied in the literature and nests various earlier results of a similar spirit, but also provides a strong conclusion regarding mechanisms which rely on improving strategy-proof mechanisms via re-allocating objects or shares of objects. Equation 1 says that the mechanism  $\sum \pi_\tau g^\tau$  is of the same size as the strategy-proof mechanism  $\sum \pi_\tau \text{DA}^{\tau(\succsim)}$ . Hence Proposition 2 applies, and  $\sum \pi_\tau g^\tau$  cannot be strategy-proof if it dominates  $\sum \pi_\tau \text{DA}^{\tau(\succsim)}$ . In particular, when priority rankings have ties, randomization does not recover strategy-proofness whether we use Kesten's (2010) improvement over the DA, or the stable improvement cycles of Erdil and Ergin (2008).

If there is any way to improve upon a strategy-proof mechanism without skewing incentives, Proposition 2 shows us where improvements could come from. The improving mechanism has to be allocating *more* objects at some preference profile.

In fact, we find that wastefulness of the RP (and the RDA) does create an opportunity for strategy-proof improvement. The mechanisms RP [and RDA] are not on the efficient frontier of strategy-proof assignment mechanisms. Moreover, the dominating mechanism is not only strategy-proof, but also *ex post efficient* [*ex post stable*]: the outcome is a lottery over deterministic efficient [stable] assignments.

**Proposition 3.** (i) *The RP is dominated within the class of ex post efficient strategy-proof mechanisms which satisfy equal treatment of equals.*

(ii) *The RDA is dominated within the class of strategy-proof and ex post stable mechanisms.*

*Sketch of proof.* Say there are at least three types objects  $a, b, c$  with a single copy each, and four agents 1, 2, 3, 4. If the preferences of the agents are

$$\begin{array}{c|c|c|c} R_1 & R_2 & R_3 & R_4 \\ \hline a & c & c & c \\ b & a & b & \end{array} ,$$

breaking the ties as  $4 \succ 3 \succ 2 \succ 1$  or  $4 \succ 2 \succ 3 \succ 1$  leads to the assignment  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & a & b & c \end{pmatrix}$ . Tie-breaking according to  $3 \succ 1 \succ 2 \succ 4$  or  $3 \succ 1 \succ 4 \succ 2$  leads to the assignment  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & 2 & c & 4 \end{pmatrix}$ . Therefore with probability at least  $1/12$ , agent 1 is not assigned an object, and  $b$  is not assigned to anyone. By replacing  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & a & b & c \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & 2 & c & 4 \end{pmatrix}$  with  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ b & a & c & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & 2 & b & c \end{pmatrix}$  with probability  $1/12$ , we assign an extra  $1/12$  of  $b$  to agent 1, while holding everything else the same. And crucially if 1's preference ranking were  $bP_1a$ , her assignment would be increased by  $1/12$  of object  $a$  due to the symmetry of  $R_{-1}$ . This will ensure that 1's incentives for truthful reporting are not skewed.

This improvement treats agent 1 differently. To recover equal treatment of equals, we define analogous improvements for other agents when the names of the agents are permuted, and randomize over such mechanisms with equal probability.

Since the RP is a special case of the RDA, our proof also establishes that the RDA is not on the efficient frontier of strategy-proof mechanisms.<sup>45</sup>  $\diamond$

<sup>4</sup>For completeness, we show in the Appendix the existence of non-trivial priority structures  $\succsim$  for which the RDA is dominated by a strategy-proof ex post stable mechanism.

<sup>5</sup>If the priority structure is trivial in the sense that every agent has the same priority for every object, and if there is a single copy of every object, the RDA boils down to the RP. Abdulkadiroğlu and Sönmez (1998) show that the RP is equivalent to the core of the market where the agents are initially allocated the objects with a uniform randomization. Pathak and Sethuraman (2011) and Carroll (2013) generalize this equivalence to the case of multiple copies

The fact that the RP is not on the efficient frontier of strategy-proof mechanisms relies on the environments in which objects are scarce and the number of agents exceeds three. Obviously, if there are sufficiently many copies of each object, the RP is efficient. Moreover for  $n \leq 3$ , Bogomolnaia and Moulin (2001) show that the RP is efficient.

By Proposition 2, any strategy-proof mechanism that dominates the RP is necessarily *less wasteful* than the RP, i.e., for some realization of preferences, such a mechanism assigns more objects than the RP. However we find that not all of the waste can be recovered in a strategy-proof way:

**Proposition 4.** *If a strategy-proof mechanism  $g$  dominates the RP mechanism, then  $g$  is wasteful.*

Thus, if we take the RP as a starting point on which to improve, it is impossible to avoid waste within the class of strategy-proof mechanisms. Among the strategy-proof mechanisms which dominate the RP, there must be one which is un-dominated, and hence is on the efficient frontier of strategy-proof mechanisms. Since this mechanism is necessarily wasteful, I have also established that the converse of Proposition 1 does not hold: being un-dominated in the strategy-proof class does not imply being non-wasteful.

## 5. CONCLUDING REMARKS

I have shown that randomizations over un-dominated strategy-proof mechanisms admit strategy-proof improvements. Specifically, both the Random Priority and the Randomized Deferred Acceptance are dominated within the strategy-proof class. While I explicitly demonstrate strategy-proof improvement over the aforementioned mechanisms, the full extent of such improvement is yet to be explored. One key insight that emerges from my analysis is that for a strategy-proof mechanism to improve on another, it must allocate more objects at some preference profile. A general and computationally efficient method to find strategy-proof improvements over the aforementioned popular mechanisms would readily find its use in real-life market design applications. Designing fair mechanisms on the efficient frontier of strategy-proof mechanisms remains an important challenge. Taking the widely popular Random Priority mechanism as a benchmark to improve upon, any

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and non-trivial priorities, where the core is replaced by an appropriate formulation of the top trading cycles mechanism.

strategy-proof mechanism that dominates it must be wasteful. In connection with this finding, an open question is whether there exists a non-wasteful strategy-proof mechanism which satisfies equal treatment of equals.

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## APPENDIX A. PROOFS

**Lemma 1.** *Given  $R_i$  over  $X \cup \{i\}$ , say there are exactly  $m$  objects in  $X$  acceptable to agent  $i$ , and enumerate them as  $x^1, x^2, \dots, x^m \in X$  such that  $x^1 P_i x^2 P_i \dots P_i x^m$ . If the assignment  $\gamma_i = \sum_{t=1}^m \gamma_{it} x^t$  strictly first-order stochastically dominates another assignment  $\omega_i = \sum_{t=1}^m \omega_{it} x^t$ , then there exists  $s$  such that  $\gamma_{is} > \omega_{is}$ , and  $\gamma_{it} = \omega_{it}$  for all  $t < s$ .*

**Proof of Lemma 1.**  $\gamma_i$  strictly FOSD  $\omega_i$ , i.e.,  $\gamma_i \tilde{P}_i \omega_i$ , therefore

$$\begin{aligned} \gamma_{i1} &\geq \omega_{i1} \\ \gamma_{i1} + \gamma_{i2} &\geq \omega_{i1} + \omega_{i2} \\ &\vdots \\ \gamma_{i1} + \gamma_{i2} + \dots + \gamma_{im} &\geq \omega_{i1} + \omega_{i2} + \dots + \omega_{im} \end{aligned}$$

with at least one of the inequalities strict.

If  $\gamma_{it} \leq \omega_{it}$  for all  $t = 1, \dots, m$ , then the above inequalities all have to be equalities, and therefore we must have  $\gamma_i = \omega_i$ , contradicting the assumption that  $\gamma_i$  FOSD  $\omega_i$ . Then, let  $s$  be the smallest positive integer for which  $\gamma_{is} > \omega_{is}$ . If  $s > 1$ , then  $\gamma_{i1} = \omega_{i1}$ , since we already know that  $\gamma_{i1} \geq \omega_{i1}$ . But then, we have  $\gamma_{i2} \geq \omega_{i2}$  from the second inequality above, and if  $s > 2$ , then  $\gamma_{i2} = \omega_{i2}$ . Arguing inductively in the same fashion, we conclude that  $\gamma_{it} = \omega_{it}$  for  $t < s$ .  $\square$

**Proof of Reshuffling Lemma.** Let  $N'$  be the set of agents whose assignments improved from  $\omega$  to  $\gamma$ , and let  $X'$  be the set of objects such that there is some agent in  $N'$  whose share of such an object increased from  $\omega$  to  $\gamma$ .

First, let us show that no object is assigned less at  $\omega$  than at  $\gamma$ . In order to see this, assume the contrary. Then there is an object  $x$  assigned less than  $q_x$  at  $\omega$ , and an agent  $i$  who gets more of  $x$  at  $\gamma$  than  $\omega$ . That is,

$$\sum_{j \in N} \omega_{jx} < q_x \quad \text{and} \quad \gamma_{ix} > \omega_{ix}.$$

$\gamma$  dominates  $\omega$ , therefore  $i$  prefers her assignment  $\gamma_i$  to  $\omega_i$ . Since  $\gamma_i$  FOSD  $\omega_i$ , we have

$$\sum_{y P_i x} \gamma_{iy} \geq \sum_{y P_i x} \omega_{iy}.$$

This, in turn, implies

$$\gamma_{ix} + \sum_{x P_i y} \gamma_{iy} \leq \omega_{ix} + \sum_{x P_i y} \omega_{iy},$$

and after re-arranging, and using the fact that  $\gamma_{ix} - \omega_{ix} > 0$ , we get

$$0 < \gamma_{ix} - \omega_{ix} + \sum_{x P_i y} \gamma_{iy} \leq \sum_{x P_i y} \omega_{iy}.$$

Thus,  $i$  receives a positive amount of objects she likes less than  $x$ , whereas not all of  $x$  is assigned under  $\omega$ , a contradiction with  $\omega$  being non-wasteful. Hence for each  $x \in X$

$$(2) \quad \sum_{i \in N} \omega_{ix} \geq \sum_{i \in N} \gamma_{ix}.$$

Adding up across  $x$ , we get

$$(3) \quad \sum_{x \in X} \sum_{i \in N} \omega_{ix} \geq \sum_{x \in X} \sum_{i \in N} \gamma_{ix}.$$

Secondly, since each agent is weakly better off at  $\gamma$ , and  $\omega$  is individually rational, each agent receives at least as much share of acceptable objects at  $\gamma$  as she did at  $\omega$ , i.e., for each  $i \in N$ :

$$(4) \quad \sum_{x \in X} \omega_{ix} = \sum_{x: x P_i i} \omega_{ix} \leq \sum_{x: x P_i i} \gamma_{ix} = \sum_{x \in X} \gamma_{ix},$$

which, adding up across  $i$ , lead to

$$(5) \quad \sum_{i \in N} \sum_{x \in X} \omega_{ix} \leq \sum_{i \in N} \sum_{x \in X} \gamma_{ix}.$$

Hence the inequalities (3) and (5) both must be equality. This, in turn, implies the weak inequalities (2) and (4) are indeed equality for each  $x$  and  $i$ , respectively.

□

**Proof of Proposition 1.** Let  $f$  be non-wasteful and strategy-proof, and suppose, for a contradiction, that a strategy-proof mechanism  $g$  dominates  $f$ . Then there must be an agent  $i$  and a preference profile  $R$  such that  $g_i(R)\tilde{P}_i f_i(R)$ .

Enumerate elements of  $X$  so that  $x^1 P_i x^2 P_i \cdots P_i x^m$ , where  $x^1, \dots, x^m$  are the only objects acceptable to  $i$ . If the stochastic assignments above are written explicitly as

$$g_i(R) = \sum_{t=1}^m \gamma_{it} x^t \quad \text{and} \quad f_i(R) = \sum_{t=1}^m \omega_{it} x^t,$$

then we know from Lemma 1 that there exists  $s$  with  $\gamma_{is} > \omega_{is}$  such that  $\gamma_{it} = \omega_{it}$  for  $t < s$ .

Let the preference relation  $R'_i$  be the linear order derived from  $R_i$  by declaring everything less preferable than  $x^s$  according to  $R_i$  unacceptable in  $R'_i$  and keeping the order the same for everything else<sup>6</sup>, i.e.,  $x^1 P'_i x^2 P'_i \cdots P'_i x^s$ , and  $i P'_i x$  if  $x \neq x^t$  for some  $t = 1, \dots, s$ .

Strategy-proofness of  $f$  implies that  $f_i(R)\tilde{R}_i f_i(R'_i, R_{-i})$ . Since  $f$  is individually rational,  $f_i(R'_i, R_{-i})$  does not involve any  $x^t$  for  $t > s$ , and hence can be written explicitly as  $\sum_{t=1}^s \omega'_{it} x^t$ , to get

$$\sum_{t=1}^m \omega_{it} x^t \quad \tilde{R}_i \quad \sum_{t=1}^s \omega'_{it} x^t,$$

which implies

$$\begin{aligned} \omega_{i1} &\geq \omega'_{i1} \\ \omega_{i1} + \omega_{i2} &\geq \omega'_{i1} + \omega'_{i2} \\ &\vdots \\ \omega_{i1} + \omega_{i2} + \cdots + \omega_{is} &\geq \omega'_{i1} + \omega'_{i2} + \cdots + \omega'_{is} \end{aligned}$$

On the other hand, strategy-proofness of  $g$  implies that  $g_i(R'_i, R_{-i})\tilde{R}'_i g_i(R)$ . Writing  $g_i(R'_i, R_{-i})$  explicitly as  $\sum_{t=1}^m \gamma'_{it} x^t$ , we get

$$\sum_{t=1}^m \gamma'_{it} x^t \quad \tilde{R}'_i \quad \sum_{t=1}^m \gamma_{it} x^t,$$

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<sup>6</sup>If  $R'_i = R_i$ , then  $s = m$ , and  $\sum_{t=1}^m \gamma_{it} > \sum_{t=1}^m \omega_{it}$ , contradicting Reshuffling Lemma.



which implies

$$\begin{aligned}\gamma'_{i1} &\geq \gamma_{i1} \\ \gamma'_{i1} + \gamma'_{i2} &\geq \gamma_{i1} + \gamma_{i2} \\ &\vdots \\ \gamma'_{i1} + \gamma'_{i2} + \cdots + \gamma'_{is} &\geq \gamma_{i1} + \gamma_{i2} + \cdots + \gamma_{is}\end{aligned}$$

Combining the sets of inequalities above, we reach the following:

$$\begin{aligned}\gamma'_{i1} &\geq \gamma_{i1} &\geq \omega_{i1} &\geq \omega'_{i1} \\ \gamma'_{i1} + \gamma'_{i2} &\geq \gamma_{i1} + \gamma_{i2} &\geq \omega_{i1} + \omega_{i2} &\geq \omega'_{i1} + \omega'_{i2} \\ &\vdots &&\vdots \\ \gamma'_{i1} + \cdots + \gamma'_{is} &\geq \gamma_{i1} + \cdots + \gamma_{is} &\geq \omega_{i1} + \cdots + \omega_{is} &\geq \omega'_{i1} + \cdots + \omega'_{is}\end{aligned}$$

$g$  dominates  $f$ , so it must be that  $g(R'_i, R_{-i})$  weakly dominates  $f(R'_i, R_{-i})$ . Since  $f$  is non-wasteful, Reshuffling Lemma implies

$$(6) \quad \omega'_{i1} + \omega'_{i2} + \cdots + \omega'_{is} = \gamma'_{i1} + \gamma'_{i2} + \cdots + \gamma'_{is}.$$

This equation combined with the last inequality above imply that

$$\gamma_{i1} + \gamma_{i2} + \cdots + \gamma_{is} = \omega_{i1} + \omega_{i2} + \cdots + \omega_{is}.$$

Since  $\gamma_{it} = \omega_{it}$  for  $t = 1, \dots, s-1$ , the above equation implies that we also have  $\gamma_{is} = \omega_{is}$ , which contradicts with the earlier choice of  $s$  such that  $\gamma_{is} > \omega_{is}$ . Hence  $g$  cannot be strategy-proof.  $\square$

**Proof of Proposition 2.** We will first prove the following variation of Reshuffling Lemma for assignments of equal size. Suppose that an assignment  $\gamma$  FOSD an individually rational assignment  $\omega$ . If  $|\omega| = |\gamma|$ , then  $\sum_{x \in X} \gamma_{ix} = \sum_{x \in X} \omega_{ix}$  for every  $i \in N$ .

Since  $\gamma$  dominates  $\omega$ , and the latter is individually rational,  $\gamma$  is also individually rational. Moreover each agent's total assignment of objects in  $X$  under  $\gamma$  is at least as much as it is under  $\omega$ :

$$(7) \quad \sum_{x \in X} \gamma_{ix} = \sum_{x: xP_i i} \gamma_{ix} \geq \sum_{x: xP_i i} \omega_{ix} = \sum_{x \in X} \omega_{ix} \quad \text{for each } i.$$

Adding up across  $i$ :

$$|\gamma| = \sum_{i \in N} \sum_{x: xP_i i} \gamma_{ix} \geq \sum_{i \in N} \sum_{x: xP_i i} \omega_{ix} = |\omega|.$$

Since we assumed that  $|\gamma| = |\omega|$ , each weak inequality (7) must indeed be equality.

Now, the proof of Proposition 1 above can be reproduced to conclude.  $\square$

**Proof of Proposition 3.** (i) Given  $|N| \geq 4$  and  $|X| \geq 3$ , let  $1, 2, 3, 4 \in N$  stand for four distinct agents, whereas  $a, b, c \in X$  denote three distinct objects. Consider the preference profile  $R$ , where every agent  $i \in N \setminus \{1, 2, 3, 4\}$  finds every object  $x \in X$  unacceptable. Suppose also that  $R_2, R_3$  and  $R_4$  are as follows

$R_2$	$R_3$	$R_4$	.
$c$	$c$	$c$	
$a$	$b$		

We can express the preference domain for agent 1 as the disjoint union of the following three sets:

$$\begin{aligned} \mathcal{R}_1^a &= \{R_1 \mid aP_i bP_i d \text{ for all } d \in X_i \setminus \{a, b, c\}\} \\ \mathcal{R}_1^b &= \{R_1 \mid bP_i aP_i d \text{ for all } d \in X_i \setminus \{a, b, c\}\} \\ \hat{\mathcal{R}}_1 &= \{R_1 \mid \text{There exists } d \in X_i \setminus \{a, b, c\} \text{ such that } dP_i a \text{ or } dP_i b\}. \end{aligned}$$

Note that when preferences are  $R$  with  $R_1 \in \hat{\mathcal{R}}_1$ , we will treat  $d$  as agent 1's outside option since there is no one else who finds  $d$  acceptable.

Below we will construct a mechanism  $g$  which improves over the RP for agent 1 when others' preferences realize as  $R_{-1}$  specified above.

First of all, if  $R_1 \in \hat{\mathcal{R}}_1$ , then  $\text{RP}(R)$  is non-wasteful. For such preferences of agent 1,  $g$  is identical to the RP. Hence we look at the cases when  $R_1 \in \mathcal{R}_1^a \cup \mathcal{R}_1^b$ .

Secondly, for all agents  $i \neq 1$ , the assignments are kept identical to what they are under the RP:

$$g_i(Q) = \text{RP}_i(Q) \quad \text{for all } i \neq 1 \quad \text{and} \quad \text{for all } Q.$$

Moreover,

$$\text{if } Q_{-1} \neq R_{-1} \text{ or } Q_1 \in \hat{\mathcal{R}}, \text{ then } g(Q) = \text{RP}(Q).$$

Note that agent 1's total assignment at  $(R_1, R_{-1})$  is  $11/12$  for all  $R_1 \in \mathcal{R}_1^a \cup \mathcal{R}_1^b$ . For such preferences we would like to improve  $i$ 's assignment as follows:

$$\begin{aligned} g_1(R_1, R_{-1}) &= \text{RP}(R) + \epsilon b \quad \text{if } R_1 \in \mathcal{R}_1^a, \\ g_1(R_1, R_{-1}) &= \text{RP}(R) + \epsilon a \quad \text{if } R_1 \in \mathcal{R}_1^b. \end{aligned}$$

We first need to show that the RP indeed wastes  $b$  at  $(R_1, R_{-1})$  for all  $R_1 \in \mathcal{R}_1^a$ , and wastes  $a$  at  $(R_1, R_{-1})$  for all  $R_1 \in \mathcal{R}_1^b$ .

If the serial dictatorship realizes as  $3 \succ 1 \succ 2 \succ 4$  or as  $3 \succ 1 \succ 4 \succ 2$ , and the preferences are  $(R_1, R_{-1})$  with  $R_1 \in \mathcal{R}_1^a$ , then object  $b$  is not assigned. This is because, agent 3 gets  $c$ , then agent 1 gets her favorite object from  $\{a, b\}$ , which is not  $b$ , as  $aP_1b$ . And  $b$  is not acceptable to anyone else. Thus  $b$  is wasted with probability at least  $1/12$ .

Likewise,  $a$  is not assigned when preferences are  $(R_1, R_{-1})$  with  $R_1 \in \mathcal{R}_1^b$ , and the serial dictatorship realizes as  $2 \succ 1 \succ 3 \succ 4$  or  $2 \succ 1 \succ 4 \succ 3$ , because 2 would get  $c$ , 1 would get  $b$ , and  $a$  is not acceptable to agents 3 or 4. Therefore  $a$  is wasted with probability at least  $1/12$ .

Thus choosing  $\epsilon = 1/12$  above is feasible, and agent 1 is strictly better off under the mechanism  $g$ .

What remains to be shown is strategy-proofness of  $g$ . Note that if agent 1 prefers both  $a$  and  $b$  to  $d$ , the mechanism  $g$  improves upon the RP by assigning her more of her less desired object from  $\{a, b\}$  when possible. If she prefers  $d$  over either of them, she will get as much of  $d$  as she likes since she is the only agent interested in  $d$ , and she has no reason to manipulate her preferences to move  $d$  below  $a$  and  $b$ . Therefore, if  $R_1 \in \hat{\mathcal{R}}_1$ , she has no incentive to report a preference order in  $\mathcal{R}_1^a \cup \mathcal{R}_1^b$ . For those preferences in  $\hat{\mathcal{R}}_1$ , she is facing the RP, which is strategy-proof, so it is a weakly dominant strategy for her to be truthful.

Now we need to verify that if  $R_1 \in \mathcal{R}_1^a \cup \mathcal{R}_1^b$ , she has no incentive to misreport. By being truthful, she is doing at least as well as she would be under the RP. If she were to report a ranking in  $\hat{\mathcal{R}}_1$ , the mechanism is assigning just like the RP, which is strategy-proof. Therefore, she cannot do better by reporting a ranking in  $\hat{\mathcal{R}}_1$ , when her actual preference is in  $\mathcal{R}_1^a \cup \mathcal{R}_1^b$ . Finally we need to show that she has no incentive to report some other preference order  $R'_1 \in \mathcal{R}_1^a \cup \mathcal{R}_1^b$ . Let us look at agent 1's assignments at such announcements, when others' preferences are  $R_{-1}$  as specified above. Recall that we can treat  $d$  as her outside option since she is the only agent who finds  $d$  acceptable.

If  $aP_1bP_1c$ , or  $aP_1bP_1cP_11$ , or  $aP_1cP_1bP_11$ , then her assignment is  $(18a + 6b)/24$ .

If  $bP_1aP_1c$ , or  $bP_1aP_1cP_11$ , or  $bP_1cP_1aP_11$ , then her assignment is  $(18b + 6a)/24$ .

If  $cP_1aP_1bP_11$ , then her assignment is  $(6c + 12a + 6b)/24$ .

If  $cP_1bP_1aP_11$ , then her assignment is  $(6c + 12b + 6a)/24$ .

All these announcements lead to the same total amount of objects assigned to agent 1. A straightforward pairwise comparison of these four outcomes verifies that at each preference ranking  $R_1$  in  $\mathcal{R}_1^a \cup \mathcal{R}_1^b$ , truthful revelation (weakly or strongly) first-order stochastically dominates untruthful revelation.

Finally we verify ex post efficiency of  $g$ . Note that the RP is ex post efficient to start with. Improving 1's assignment when  $R_1 \in \mathcal{R}_1^a$  involves increasing her share of  $b$  by  $1/12$ , while holding everyone else's assignment the same. This can be done by replacing the assignments on the left below with the assignments on the right with probability share  $1/12$  each. Each assignment on the left is realized with probability  $1/12$  or more, therefore this improvement is indeed feasible. And the assignments on the right are efficient, therefore ex post efficiency is preserved.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & a & b & c \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & 2 & c & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & a & c & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & 2 & b & c \end{pmatrix}$$

The argument for the case when  $R_1 \in \mathcal{R}_1^b$  is analogous.

This mechanism  $g$  treats agent 1 differently. In order to recover equal treatment of equals, we will appeal to randomization again. Let  $\pi$  be a permutation of the agents, i.e., let  $\pi : N \rightarrow N$  be a bijection. Then  $g^\pi$  is defined via changing the roles of the agents in mechanism  $g$  according to the permutation  $\pi$ . Denoting with  $\Pi$  the set of all permutations of  $N$ , we define  $h$  as

$$h = \frac{1}{n!} \sum_{\pi \in \Pi} g^\pi.$$

Due to symmetry,  $h$  satisfies equal treatment of equals. By construction, for each agent, there exists a preference profile at which the outcome of  $g$  is strictly better than the outcome of the RP. Moreover, the outcome is always a convex combination of deterministic efficient assignments. Hence  $g$  is a strategy-proof, ex post efficient mechanism which satisfies equal treatment of equals and dominates the RP.

(ii) The RP is a special case of the RDA where there is a single copy of each object, and all agents have equal priority for every object. Therefore any non-wasteful deterministic assignment is stable. Hence the above proof also shows that the RDA admits strategy-proof improvement for some priority structure. For completeness we note a “non-trivial” priority structure  $\succsim$  for which the RDA is

dominated by a strategy-proof mechanism:

$\succsim_a$	$\succsim_b$	$\succsim_c$	$\succsim_{\text{others}}$	$R_1$	$R_2$	$R_3$	$R_4$	$R'_1$
1 – 4	1 – 4	2 – 4	others	$a$	$c$	$c$	$c$	$b$
others	others	1	1 – 4	$b$	$a$	$b$		$a$
		others						

As in the above construction, agent 1's assignment can be improved by adding  $1/12$  of  $b$  at  $(R_1, R_{-1})$ , and by adding  $1/12$  of  $a$  at  $(R'_1, R_{-1})$  without skewing incentives.  $\square$

**Proof of Proposition 4.** Let  $f$  stand for the RP. For any realization of the tie-breaking lottery, we implement a serial dictatorship, so if an object  $x$  is top-ranked by someone at  $R$ , then it is assigned to someone. Hence  $x$  cannot be wasted at  $f(R)$ , i.e.,  $\sum_i f_{ix}(R) = 1$ . Let the preference profile  $R$  be such that the agents other than 1, 2, 3, 4 consider all objects unacceptable. Suppose also that  $R_1, R_2, R_3, R_4, R'_1$ , and  $\hat{R}_2$  are

$R_1$	$R_2$	$R_3$	$R_4$	$R'_1$	$\hat{R}_2$
$a$	$c$	$a$	$c$	$b$	$b$
$b$	$b$			$a$	$c$

$f$  is wasteful at  $R$  since  $f(R)$  assigns the objects as

$$\begin{aligned} 1 &: \frac{1}{2}a + \frac{5}{12}b \\ 2 &: \frac{1}{2}c + \frac{5}{12}b \\ 3 &: \frac{1}{2}a \\ 4 &: \frac{1}{2}c \end{aligned}$$

and it wastes  $b$ .

On the other hand,  $f$  is not wasteful at  $R' = (R'_1, R_{-1})$  nor at  $\hat{R} = (\hat{R}_2, R_{-2})$ , because at both profiles, each acceptable object is top-ranked by someone. If  $g$  dominates  $f$ , then  $g(R')$  and  $g(\hat{R})$  weakly dominate  $f(R')$  and  $f(\hat{R})$ , respectively. Then, by Reshuffling Lemma, we have

$$|g_i(R')| = |f_i(R')| \quad \text{and} \quad |g_i(\hat{R})| = |f_i(\hat{R})|.$$

$R'_1$  and  $\hat{R}_2$  are derived from  $R_1$  and  $R_2$ , respectively, by permuting acceptable objects, so strategy-proofness of  $f$  implies

$$|f_1(R)| = |f_1(R')| \quad \text{and} \quad |f_2(R)| = |f_2(\hat{R})|,$$

whereas strategy-proofness of  $g$  implies

$$|g_1(R)| = |g_1(R')| \quad \text{and} \quad |g_2(R)| = |g_2(\hat{R})|.$$

Combining the three sets of equalities above, we get

$$|f_1(R)| = |g_1(R)| \quad \text{and} \quad |f_2(R)| = |g_2(R)|.$$

Since  $g$  dominates  $f$ , we know that  $g(R)$  weakly dominates  $f(R)$ , and in particular,  $|g_i(R)| \geq |f_i(R)|$  for each  $i$ . All agents get 1/2 of their top choices at  $f(R)$ , and these objects, namely  $a$  and  $c$ , are completely assigned. Therefore  $g(R)$  has to assign the same fractions of these top choices as did  $f(R)$ :

$$f_{ix}(R) = g_{ix}(R) \quad \text{for all } i = 1, 2, 3, 4 \quad \text{and} \quad x = a, c.$$

$b$  is not acceptable to agents 3 and 4, implying that  $g_3(R) = f_3(R)$  and  $g_4(R) = f_4(R)$ . On the other hand  $|f_1(R)| = |g_1(R)|$  and  $|f_2(R)| = |g_2(R)|$ , and hence agents 1 and 2 are assigned the same amount of  $b$  under  $g(R)$  as they were under  $f(R)$ . Therefore  $g(R) = f(R)$ , and thus  $g$  is wasteful at  $R$ .  $\square$

## REFERENCES

- [1] A. Abdulkadiroğlu, P. Pathak, and A. Roth, Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match, *Amer. Econ. Rev.* 99 (2009) 1954–1978.
- [2] A. Abdulkadiroğlu, T. Sönmez, Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems, *Econometrica* 66 (3) (1998) 689–701.
- [3] A. Bogomolnaia, H. Moulin, A New Solution to the Random Assignment Problem, *J. Econ. Theory* 100 (2001) 295–328.
- [4] A. Bogomolnaia, H. Moulin, Random Matching Under Dichotomous Preferences, *Econometrica* 72 (1) (2004) 257–279.
- [5] G. Carroll, A General Equivalence Theorem for Allocation of Indivisible Objects, Mimeo, Stanford University, 2013.
- [6] Y-K Che, F. Kojima, Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms, *Econometrica* 78(5) (2010) 1625–72.
- [7] L. Dubins, D. Freedman, Machiavelli and the Gale-Shapley algorithm, *Amer. Math. Monthly* 88 (1981) 485–494.
- [8] L. Ehlers, Respecting Priorities when Assigning Students to Schools, Mimeo, University of Montreal, 2006.
- [9] L. Ehlers, A. Erdil, Efficient Assignment Respecting Priorities, *J. Econ. Theory* 145 (2010) 1269–1282.
- [10] A. Erdil, H. Ergin, What’s the matter with tie-breaking? Improving efficiency in school choice, *Amer. Econ. Rev.* 98 (3) (2008), 669–689.

- [11] H. Ergin, Efficient resource allocation on the basis of priorities, *Econometrica* 70 (2002) 2489–2497.
- [12] D. Gale, L. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* 69 (1962) 9–15.
- [13] B. Holmström, R. Myerson, Efficient and Durable Decision Rules with Incomplete Information, *Econometrica* 51(6) (1983) 1799–1819.
- [14] O. Kesten, School Choice with Consent, *Quart. J. Econ.* 125 (2010) 1297–1348.
- [15] O. Kesten, M. Kurino, On the (Im)possibility of Improving upon the Student-proposing Deferred Acceptance Mechanism, WZB Discussion Paper, 2012.
- [16] O. Kesten, U. Ünver, A Theory of School-Choice Lotteries, Mimeo, Carnegie Mellon University and Boston College, 2012.
- [17] F. Kojima, M. Manea, Incentives in Probabilistic Serial Mechanism, *J. Econ. Theory* 145 (2010) 106–123.
- [18] Q. Liu, M. Pycia, Ordinal Efficiency, Fairness, and Incentives in Large Markets, Mimeo, Columbia University and UCLA, 2012.
- [19] P. Pathak, J. Sethuraman, Lotteries in student assignment: An equivalence result, *Theoretical Economics* 6 (2011) 1–17.
- [20] H. Peters, S. Roy, A. Sen, T. Storcken, Probabilistic Strategy-Proof Rules over Single-Peaked Domains, Mimeo, Maastricht and Indian Statistical Institute (New Delhi), 2013.
- [21] M. Pycia, U. Ünver, Decomposing Random Mechanisms, Mimeo, UCLA and Boston College, 2012.
- [22] A. Roth, The economics of matching: Stability and incentives, *Math. Oper. Res.* 7 (1982) 617–628.
- [23] L. Zhou, On a Conjecture by Gale about One-Sided Matching Problems, *J. Econ. Theory* 52 (1990) 125–135.