NUCLEAR AND HYPERNUCLEAR BINDING ENERGIES

by

Sangarapillai Sampanthar

of

Clare College

Cambridge

A dissertation submitted to the University of Cambridge for the Degree of Doctor of Philosophy

July, 1961.

Preface

This dissertation is in two parts.

Unless otherwise stated in the text Part I of the dissertation is on the original work carried out by me under the direction of Dr. R.J. Eden at the University of Cambridge - during the two Academic years 1957 - 59.

I take this opportunity to express my deep gratitude to Dr. R.J. Eden for his unfailing interest and his constant guidance, and to Dr. V.J. Emery for his patient elucidation of several difficult points. Thanks are also due to Dr.C.F. Clement, Dr. J. Nuttall and Dr. D. Lynden-Bell for several stimulating discussions and for encouragement in moments of doubt.

Unless otherwise stated in the text Part II of the dissertation is on the original work carried out by me under the direction of Dr. A.R. Bodmer at the University of Manchester during the Academic year 1959 - 60.

In this connection I would like to thank Professor B.H. Flowers, F.R.S. for stimulating my interest in hypernuclei and for his interest in my work.

It is with deep gratitude that I acknowledge the long hours spent in conversations with Dr. A.R. Bodmer, out of which emerged the major ideas for my work. To Mrs. A.R.Bodmer

I express my sincere thanks for the patient and understanding manner in which she put up with the restrictions of freedom of movement for herself and her children in her own house, and other inconveniences she suffered whilst this work was taking shape.

Thanks are also due to Mr. D. Wilmore for help in programming for the Mercury Computer and to the members of the Department of Theoretical Physics at the University of Manchester for a very pleasant year I spent in their midst.

Finally I thank Mr. G.E. Kilby for reading through this dissertation for mistakes in English and Miss M. Barlow for typing it.

While the work reported in this dissertation was in progress I was supported by a Government of Ceylon University Scholarship.

எ. சம்பந்தர்.

July, 1961. S. Sampanthar.

Department of Physics,
The University,
Sheffield, 10.

Part I

The Binding Energies of Atomic Nuclei

CONTENTS

CHAPTER IV

Formalism for a Finite Nucleus

CHAPTER V

Further Developments

CHAPTER VI

Self Consistency and the Definition of the t Matrix

CHAPTER VII

Nuclear Forces and the Ground State of ^{16}O .

CHAPTER VIII

Conclusion

Chapter I

1.1 Introduction.

A number of observed properties of an atomic nucleus fit in with the idea of independent particle motion in an average field for the constituent nucleons of the nucleus.[1] On the other hand certain experimental studies, such as those on the photonuclear effect[2] suggest that correlations must exist between pairs of nucleons inside a nucleus. For a long time now, the problem of understanding these and other nuclear properties (e.g. nuclear saturation) has been separated from that of determining nuclear forces.

The accumulation of experimental evidence on the nature of nuclear forces has been rapid during the past few years and several potential models of these forces, which fitted with various degrees of success, not only the low energy scattering and deuteron data but also the high energy data, have been put forward.[3] A feature of many of the more successful models has been the presence of a repulsive hard core interaction term in the inter-nucleon potentials. This is particularly so for the models which fit the high energy data rather well.

Assuming that many body forces within a nucleus are negligible* and that the nuclear forces deduced from two

* This is not an unreasonable assumption when one remembers that nuclear densities are low and nuclear forces short ranged.

body data may be used for inter-nucleon forces within a nucleus one may try to get some understanding of nuclear properties. For this purpose one has to use inter-nucleon forces that also fit the higher energy data, as a simple order of magnitude estimate of the kinetic energies involved when two nucleons in a medium heavy nucleus collide, will show.

The usual forms of perturbation theory cannot be applied for such forces. Recently a method [4) 5) 6)] based on a modification of perturbation theory suitable for these singular potentials has been proposed, and applied with some success to the computation of the properties of nuclear matter [5)]. However at the time the present work was undertaken no applications of the theory to finite nuclei had been made.

Part I of this dissertation will be concerned with the development and application of a practical method for the computation of sizes and binding energies of (real) finite nuclei, from a knowledge of nuclear forces.

In §1.2 we summarise the evidence which compelled the rather complicated potential models of nuclear forces to be put forward. Chapter II introduces the concept of nuclear matter and discusses the inadequacy of the customary perturbation theories to deal with such an infinite medium. In Chapter III we deal with the modifications to perturbation theory that have been put forward by Brueckner and his

collaborators [7], and discuss the application of the method to the determination of the properties of nuclear matter.

Chapter IV is on the formulation of the theory in a way suitable for application to a finite system and follows the work of Goldstone [8] and of Eden [4]. The variational principle postulated by Eden and Emery [9] is also discussed.

In Chapter V certain approximations are introduced. These relate to the choice of the trial single particle potential and the exclusion principle operator, due again to Eden and Emery. Coupled integro differential equations for the perturbed relative wave functions of pairs of nucleons are written down. This is new.

Chapter VI prepares the ground for the application of the method to the O^{16} nucleus. A tensor force approximation [10] is introduced to uncouple the integro differential equations. Self consistency is discussed and an approximate way to deal with it developed. This is an improvement on the customary effective mass approximation.

In Chapter VII the results of the calculations on O^{16} are presented.[11] Six different inter-nucleon potentials were used. Pair correlations are also discussed.

Chapter VIII gives a brief resumé of the method and results. A discussion concludes Part I of the dissertation.

1.2 The Two Nucleon Interaction.

There is really no evidence to show that the two nucleon interaction can in fact be represented by a potential, either static or momentum dependent. But if it were possible a useful Schrödinger equation could be written down. Hence such an approach has often been pursued - and with success.

For low energy phenomena knowledge of certain broad features of the interactions, viz. the scattering length and the effective range, prove sufficient to explain the experimental data. As the energy of the experiments increased more specific forms for the inter-nucleon potential had to be assumed (with more free parameters) to obtain fits also to the newer data. A climax to this approach was reached in 1957 when Gammel and Thaler [12) 13)] on the one hand and Signell and Marshak [14)] on the other, produced essentially similar potentials which besides fitting the usual low energy data also fitted data obtained at energies as high as 310 Mev. At higher energies relativistic effects are important and even the notion of a potential may have to be given up.

Meson theory suggests that the potentials are of Yukawa form.[15)] The large ratio of the binding energy of helium to that of the deuteron suggests that the nuclear forces are strong but short ranged.

Saturation, the property that the nuclear volume is proportional to the number of nucleons in the nucleus, as is the

binding energy, suggests the existence of a hard core repulsion in the potential. So does the sign of the low angular momentum phase shifts.

The anomalous magnetic moment of the deuteron (different from the sum of those of the proton and the neutron) as also its quadrupole moment suggests the existence of a tensor term in the interaction.

The observed low energy n-p scattering cross-sections are large compared to those deduced from deuteron binding energy, suggesting that the interaction is spin dependent.

Symmetry of n-p scattering and angular distribution about 90° shows that the inter-nucleon force is weak in odd parity states (parity dependence).

Studies on mirror nuclei provide strong evidence for charge symmetry of nuclear forces, while the assumption of charge independence is consistent with observed low energy phenomena to a very good approximation.

For the interaction to be physically reasonable the potential must

(i) depend on $\underline{r}$ and $\underline{p}$ (the relative co-ordinate and momentum of the two nucleons) and the spins of the nucleons only. This ensures conservation of total momentum and the separation of the centre of mass motion.

(ii) be invariant under proper Lorentz rotations ensuring the conservation of total angular momentum.

(iii) be symmetric in the co-ordinates of the two particles

and (iv) conserve total electric charge.

Guided by these considerations and with the hopes that the potential depends on $\underline{p}$ at most linearly and that the spin-orbit force is of the type "$\underline{L}.\underline{S}$." (where $\underline{L}$ is the angular momentum and $\underline{S}$ the spin), Gammel and Thaler were able to write down for the inter-nucleon potential an expression

$$V = V_c + V_t S_{12} + V_{ls} \underline{L}\cdot\underline{S} \qquad r > r_c$$

$$= \infty \qquad r < r_c$$

where the V's are spin and parity dependent Yukawa potentials

S_{12} the tensor operator

and r_c the radius of the infinitely repulsive hard core and obtain fits for the data.

In fact several sets of potentials which satisfied the data with various degrees of success were obtained; none satisfying all the data perfectly.

Signell and Marshak's potential is essentially the Gartenhaus' potential [16] to which a spin-orbit term has been added. We do not discuss it here as we do not use it in our calculations. Further discussion of inter-nucleon potentials is postponed to § 7.1.

CHAPTER II

2.1 Nuclear Matter.

The simplest properties of atomic nuclei are their binding energies and their sizes. The former are known accurately from the masses of the nuclei and their constituents. Electron scattering experiments [17] determine reasonably accurately the r.m.s. radius of charge distributions of nuclei. This and the knowledge of the proton charge distribution gives the proton distribution in nuclei. Often the neutron distribution in certain nuclei may be taken to be the same as the proton distribution and one arrives at the r.m.s. radius of the matter distribution in these nuclei.

Any theory which attempts to predict and interpret nuclear properties from a knowledge of nuclear forces must in the first instance predict reasonably well the binding energies and sizes of nuclei. We shall be concerned mainly with these aspects of nuclear structure only.

In a finite nucleus besides the specifically nuclear forces there are the electro-magnetic forces between protons, indeed between all nucleons if we remember that neutrons too have magnetic moments. The forces due to the magnetic moments are small enough to be negligible. The effect of the Coulomb forces is to push the protons apart so that in a large (stable) nucleus there are more neutrons than protons.

CHAPTER II

2.1 Nuclear Matter.

The simplest properties of atomic nuclei are their binding energies and their sizes. The former are known accurately from the masses of the nuclei and their constituents. Electron scattering experiments [17] determine reasonably accurately the r.m.s. radius of charge distributions of nuclei. This and the knowledge of the proton charge distribution gives the proton distribution in nuclei. Often the neutron distribution in certain nuclei may be taken to be the same as the proton distribution and one arrives at the r.m.s. radius of the matter distribution in these nuclei.

Any theory which attempts to predict and interpret nuclear properties from a knowledge of nuclear forces must in the first instance predict reasonably well the binding energies and sizes of nuclei. We shall be concerned mainly with these aspects of nuclear structure only.

In a finite nucleus besides the specifically nuclear forces there are the electro-magnetic forces between protons, indeed between all nucleons if we remember that neutrons too have magnetic moments. The forces due to the magnetic moments are small enough to be negligible. The effect of the Coulomb forces is to push the protons apart so that in a large (stable) nucleus there are more neutrons than protons.

It would be convenient if the smallish Coulomb forces could be taken account of at a later stage and one could consider only the nuclear forces. Then there will be symmetry between the protons and neutrons. Another consequence of the switching off of the Coulomb interactions would be that much larger nuclei than are present in Nature could exist. Near the centre of such nuclei the nucleons will be uniformly distributed unaware of the happenings at the surface, which for all purposes could be taken at infinity. Such a region is often termed nuclear matter and the central regions of large real nuclei are supposed to be good approximations to it.

If we define the surface of real finite nuclei as the region in which the nuclear density is (rapidly) changing, then for a medium sized nucleus (e,g, Al) just about half the constituent nucleons are to be found at the surface. Thus the surface regions of nuclei are expected to be important. Nevertheless it is often useful to consider idealised large nuclei without Coulomb forces and to identify the binding energy per particle of such nuclei with the volume term of the Von Weissäker mass formula [18]. In the remainder of this chapter and in Chapter III we shall be concerned with such an infinite nuclear medium only.

2.2 The Wigner Perturbation Theory.

As we shall see even for non-singular nuclear forces the usual perturbation theories present some difficulties when the

system in infinite.

As outlined in §2.1 our problem is that of N nucleons confined to a volume Ω and interacting by two body forces v_{ij}. When it suits us we shall let $N \to \infty$, $\Omega \to \infty$ such that $\frac{N}{\Omega} = \text{const} = \rho$.

At time $t = -\infty$ the nucleons are supposed to be non-interacting and the system is in the groundstate Φ_0 with energy E_0 of the unperturbed Hamietonian H_0 made up of the kinetic energy operators of the N nucleons

$$H_0 \Phi_0 = E_0 \Phi_0 \qquad (2.1)$$

As time increases the interaction is gradually switched on so as not to cause any transitions to other eigenstates of H_0, until the full interaction is acting at $t = 0$ [19]. The wave function and the energy gradually change to Ψ_0 and $E_0 + \Delta E$ where

$$(H_0 + H')\,\Psi_0 = (E_0 + \Delta E)\,\Psi_0 \qquad (2.2)$$

and $H' = \frac{1}{2}\sum v_{ij}$ is the perturbation. If Ψ_0 is normalised such that

$$\langle \Phi_0 | \Psi_0 \rangle = 1$$

then

$$\Delta E = \langle \Phi_0 | H' | \Psi_0 \rangle \qquad (2.3)$$

and

$$\Psi_0 = \Phi_0 + \frac{Q_0}{E_0 + \Delta E - H_0} H' \Psi_0 \qquad (2.4)$$

where Q_0 is an operator which projects off the wave function Φ_0, are the formal solutions of eqn (2.2).

Now define K such that

$K\phi_o = H'\psi_o$ so that eqns.

(2.3) and (2.4) become

$$\Delta E = \langle \phi_o | K | \phi_o \rangle \tag{2.5}$$

and $$K = H' + H' \frac{Q_o}{E_o + \Delta E - H_o} K \tag{2.6}$$

and we have arrived at the Wigner Perturbation theory.[20]

For ρ sufficiently small (with N and Ω both infinite) it is unlikely that when one nucleon collides with another (which itself occurs only rarely) they will feel the presence of the other nucleons. Hence the energy shift ΔE may be expected to be the sum of the shifts obtained for each pair interaction.

$$\Delta E = \sum_{i<j} \Delta E_{ij}$$

where

$$\Delta E_{ij} = \langle \phi_o(i,j) | K_{ij} | \phi_o(i,j) \rangle$$

in an obvious notation.

The equation for K_{ij}, v_{12}

$$K_{ij} = v_{ij} + v_{ij} \frac{Q_o}{E_o + \Delta E - H_o} K_{ij} \tag{2.7}$$

is very similar to the equation

$$R_{ij} = v_{ij} + v_{ij} \frac{Q_o}{E_o - H_o} R_{ij} \tag{2.7a}$$

of scattering theory and indeed an expression for ΔE_{ij} in terms of the scattering phase shifts,[7] could be written down.

Then why need we go further? It is because in nuclear matter the density though low is not low enough for the above approximation to be useful. The presence of other nucleons

is expected, through the Exclusion principle, to affect the scattering of a pair of nucleons. Either scattering is prevented by the presence of nucleons in neighbouring states, or it takes place with large momentum transfer. This effect is important and has to be taken into account.

First let us meet the other difficulties. Expand ΔE in powers of H' and study the N dependence of the terms

$$\Delta E = \langle H' \rangle + \langle H' \tfrac{1}{a} H' \rangle + \langle H' \tfrac{1}{a} H' \tfrac{1}{a} H' \rangle + \cdots \qquad (2.8)$$

Here $\frac{1}{a}$ denotes $\frac{Q_0}{E_0 + \Delta E - H_0}$ and $\langle \ldots \rangle$, the expectation value in the unperturbed state.

(i) $\langle v_{ij} \rangle$ is of the order $\frac{1}{\Omega}$. Hence $\langle H' \rangle = \Sigma \langle v_{ij} \rangle$ is of order $N^2/\Omega = N\rho$

(ii) Write $\langle H' \tfrac{1}{a} H' \rangle = 2 \sum \langle v_{ij} \tfrac{1}{a} v_{ij} \rangle$

The sum over i, j gives a factor $\propto N^2$

The v's are of order $\frac{1}{\Omega}$

$\frac{1}{a}$ is of order $\frac{1}{N}$

and the intermediate sum contributes a factor $\propto \Omega$.

Hence $\langle H' \tfrac{1}{a} H' \rangle$ is independent of N.

(iii) Similar reasoning shows that the rest of the series too are independent of N.

At first sight it may appear that the first order term is the only one that matters and the other terms could be dropped. This is not so for the series (2.8) does not converge until we go to terms of order N, all terms apart from the first being of the same order.

Another difficulty one encounters is that in a variational calculation for the energy the first order wave function does not improve on the result obtained from the zero order (unperturbed) wave function ϕ_o . Physically one may reason as follows: The ground state represents a state in which no nucleons are excited. The first order wave function $\psi_o^{(1)}$ represents a sum of states in each of which only a pair of particles are excited. However in a large system one would expect a large number of particles to be simultaneously excited. Hence $\psi_o^{(1)}$ cannot be a "physical" wave function any more than ϕ_o .

2.3 The Schrödinger Perturbation Theory.

The Schrödinger [21] series may be obtained by expanding the $\Delta E's$ occurring in the energy denominations of the Wigner series (eqn. (2.8)) to give

$$\Delta E = \sum_n \epsilon_n \qquad (2.9)$$

where

$$\epsilon_1 = \langle H' \rangle$$

$$\epsilon_2 = \langle H' \tfrac{1}{b} H' \rangle$$

$$\epsilon_3 = \langle H' \tfrac{1}{b} H' \tfrac{1}{b} H' \rangle - \epsilon_1 \langle H' \tfrac{1}{b^2} H' \rangle$$

$$= \langle H' \tfrac{1}{b} [H' - \langle H' \rangle] \tfrac{1}{b} H' \rangle$$

etc.

with $\frac{1}{b}$ denoting $\frac{Q_o}{E_o - H_o}$.

The N dependence of the terms can be worked out as before remembering that the energy denominator is now independent of N . We find that ϵ_1 and ϵ_2 are both of order Np. However $\langle v_{ij} \tfrac{1}{b} v_{kl} \tfrac{1}{b} v_{ij} \rangle$ which is a part of ϵ_3 that is not

"linked" (see ref. 24) turns out to be of order $(NP)^2$ and such a dependence cannot have a physical meaning. In fact it turns out that this term is exactly cancelled by a second term (called the renormalisation term) in ϵ_3 . In higher order too one gets "unphysical" terms which are similarly cancelled by other renormalisation terms. One of Goldstone's contributions [8] was to prove this result for every order of perturbation theory [22].

Thus for large N we are left with a series for the energy per particle which is independent of the volume of the system, and which could converge for well behaved weak two body interactions.

However neither the first order wave function nor any higher order wave function gives any improvement to the energy obtained by using the unperturbed wave function in the Ritz variational principle [23]; again for the same reason as mentioned at the end of § 2.2.

It may be asked "How then can the energy series be good in principle"? The answer lies along the following lines: In determining energies we are essentially calculating expectation values of pair interactions. Thus for the calculation of the energy of a pair it is only necessary to know accurately the wave function of the system near that pair - up to the region to which the range of the interaction extends. The perturbed wave function is able to do this.

In the next chapter we go on to discuss the modifications of the theory which not only makes the theory applicable to singular interaction but also one hopes gives a more rapidly convergent energy series.

Chapter III

3.1 Pictorial Representation of the Perturbation Series:

We have already hinted at the method we shall be using in our problem. It consists of solving equations of the type (see § 2.2).

$$K = v + v \frac{Q}{e} K$$

and using the K's to compute interaction energies. The modified Q takes account of the Exclusion Principle and e, the propagation of the nucleons through the nuclear medium.

To see the relation to the perturbation treatment let us begin by representing the terms of the Schrödinger perturbation series by diagrams. We follow Goldstone [8] and regard the unperturbed ground state ϕ_o as the vacuum. Any nucleon in an excited single particle state is considered a particle and represented by a line with an arrow in the 'up' direction. An unoccupied single particle state in ϕ_o is regarded as a hole and represented by a line with an arrow in the 'down' direction. An interaction is represented by a dashed 'horizontal' line.

Without further details which may be found in ref. [8] we assert that the Diagram 1 represents the interaction of nucleons in states m_o and n_o (where these letters specify completely the single particle states that go to make up the Slater determinant wave function ϕ_o). As a consequence

these nucleons are scattered into previously empty states m and n. Diagram 2 gives the representations of the first few terms of the energy series and their order. We have not drawn all diagrams corresponding to ϵ_4, the unlinked diagrams nor the exchange diagrams. We notice that the number of diagrams to each order increases astronomically.

3.2 The Propagator Modification:

Certain types of diagrams could be combined to give (simpler) new kinds of diagram. First notice that the diagrams have no external lines [8], and take the Pauli Principle (with respect to the occupied states of ϕ_o) explicitly into account in the intermediate states. Further when we write down the matrix element corresponding to a given diagram we must introduce a factor $\frac{1}{6}$ for every pair of particle lines in an intermediate state, to represent their "propagation" between interactions.

Consider the diagram 2(d). Since n_o is involved in the intermediate state a summation over it is involved. Also the interaction with the nucleon in state n_o does not alter the intermediate state. Indeed any number of interactions of the type shown in Diagram 3 of particle m do not alter the intermediate state. Hence one may hope to alter the propagator of the diagram 2(b) to enable it to take into account all possible interactions of the type shown in Diagram 4. i.e. all interactions of excited particles with nucleons of the unexcited medium in which only forward

scattering takes place.

Define $$\sum_{n_o \neq \ell_o} v_{m_o n_o} = V_{m_o m_o} \qquad (3.1)$$

and $$\sum_{n_o \neq \ell_o} \langle v_{m_o n_o} \rangle = V^{o}_{m_o m_o} \qquad (3.2)$$

and consider the propagator

$$\frac{1}{b''} = \frac{Q_o}{E_o + V^{e}_{m_o m_o} + V^{o}_{\ell_o \ell_o} - H_o - V_{m_o m_o} - V_{\ell_o \ell_o}} \qquad (3.3)$$

If this propagator is used for Diagram 2 (b) and the corresponding matrix element expanded it can be checked that matrix elements corresponding to Diagram 2 (d) (and a host of other diagrams) as well as the matrix element of Diagram 2 (b) with the old propagator will result. This type of reduction is quite general and henceforth we use $\frac{1}{b''}$ for the propagator.

3.3 Summation of Ladder Diagrams:

Another possible type of reduction is the replacement of the series of diagrams shown in Diagram 5 (a) by Diagram 5 (b). This defines a new interaction $K''_{\ell_o m_o}$ such that

$$\langle K''_{\ell_o m_o} \rangle = \langle v_{\ell_o m_o} \rangle + \langle v_{\ell_o m_o} \frac{1}{b''} v_{\ell_o m_o} \rangle + \cdots$$

i.e. $$K''_{\ell_o m_o} = v_{\ell_o m_o} + v_{\ell_o m_o} \frac{1}{b''} K''_{\ell_o m_o} \qquad (3.4)$$

(The same letters used both for nucleons and single particle states should not cause confusion). Notice the similarity of eqn. (3.4) to the scattering theory equation (eqn. 2.7a). Physically this summation of the interactions amounts to saying that when two nucleons collide they do so several times before separating. Such summation may be necessary for the

strong and short ranged nuclear forces. Also, the knowledge that the scattering from a hard shere potential is well behaved leads us to expect that such summations may also be sufficient to obtain finite answers for hard core potentials.

3.4 The t Matrix:

Thus by altering the propagator and by redefining the interaction operator we have eliminated a large number of diagrams by taking account of their total effect in some of the remaining diagrams (interpreted with the new definitions). We can go further. We can define a new K operator as in § 3.3 to replace all diagrams of the type shown in Diagram 6 (a) by Diagram 6 (b).

A further modification of the propagator is now possible along the lines of § 3.2 to eliminate diagrams of the type shown in Diagram 4 (with "wavy" lines in place of the dashed lines) in favour of Diagram 2 (b) (with "wavy" lines instead of dashed lines).

This process ultimately leads us to the self consistent equations

$$t_{\ell_0 m_0} = v_{\ell_0 m_0} + v_{\ell_0 m_0} \frac{1}{e} t_{\ell_0 m_0} \qquad (3.5)$$

and

$$\frac{1}{e} = \frac{Q_0}{E_0 + \tau^0_{\ell_0 \ell_0} + \tau^0_{m_0 m_0} - H_0 - \tau_{\ell_0 \ell_0} - \tau_{m_0 m_0}} \qquad (3.6)$$

where

$$\tau_{m_o m_o} = \sum_{n_o \neq l_o} t_{m_o n_o} \qquad (3.7)$$

$$\tau^{\circ}_{m_o m_o} = \sum_{n_o \neq l_o} \langle t_{m_o n_o} \rangle \qquad (3.8)$$

and $t_{m_o n_o}$ is termed the t matrix for the nucleons m_o and n_o.

It is in this manner that we take account of the interactions of excited nucleons with the nucleons of the medium. With these definitions Diagram 7 represents the terms (apart from the exchange ones) that are left in the first three orders of the energy series. We remark that there is no second order term left.

Other types of reductions are possible and useful for other many body problems. We refer to ref.[7] for these and for further details.

3.5 Application to Nuclear Matter;

The formalism of the previous section has been used to calculate the energy per particle and saturation density of nuclear matter [5) 25)]. In this section we briefly review the method.

The ground state energy of nuclear matter is taken to be

$$E = \sum_{m_o} \langle m_o | T | m_o \rangle + \frac{1}{2} \sum \langle m_o n_o | t | m_o n_o - n_o m_o \rangle \qquad (3.9)$$

As we remarked there are no second order terms in t and the third order cluster term has been shown to be small [6)].

The equation for the t matrix which determines the interaction of a pair of nucleons in the medium, (taking account of the Exclusion Principle and the binding effects of the nuclear medium) may be written (c.f. eqns. (3.5) and (3.6).

$$\langle m_o n_o | t | m_o n_o \rangle = \langle m_o n_o | v | m_o n_o \rangle + \sum \langle m_o n_o | v | k e \rangle \frac{Q}{e} \langle k e | t | m_o n_o \rangle \qquad (3.10)$$

$Q = o$ if k, e, refer to unexcited states

otherwise $$\frac{Q}{e} = \frac{1}{E_{m_o} + E_{n_o} - E_e - E_k} \qquad (3.11)$$

where

$$E_{m_o} = \langle m_o | T | m_o \rangle + \sum_{n_o} \langle m_o n_o | t | m_o n_o - n_o m_o \rangle \qquad (3.12)$$

with similar definitions for E_{n_o}, E_e, and E_k. The notation is obvious.

E as determined by these equations is dependent on the single particle states $|m_o\rangle$ that go to make up ϕ_o. For nuclear matter spatial homogeneity suggests that the $|m_o\rangle$'s be plane wave states. Usually ϕ_o is taken to represent the fully degenerate Fermi gas state and hence the method can only hope to give the energy of the "normal" ground state of nuclear matter.[26]

Two other corrections to the set of equations written down should be mentioned. Both are concerned with improvements to the single particle (virtual) excitation energies.

The first due to the effects of "off-energy shell" propagation has already been considered approximately in ref.[5]. The results suggest that the resulting correction to t is not probably more than half an Mev per particle.

The more important correction is due to the re-arrangement effects [27] in the single particle energy. Even this alters the binding energy by about 1 Mev per particle only, (though it is of far greater importance to the single particle energies.

Thus it would appear that the method is able to predict the binding energy per particle (and other properties of nuclear matter) to an accuracy of 15 - 20%. Will our method for finite nuclei turn out to be equally good? We shall not answer the question now.

CHAPTER IV

4.1 Goldstone's Energy Series:

For a real finite nucleus certain modifications of the preceding formalism are necessary. This is because we have to give up spatial homogeneity and plane wave states are no longer the unperturbed single particle states. It is therefore useful to incorporate a potential energy term in the unperturbed Hamiltonian, also termed 'model' Hamiltonian. Let

$$H_o = \sum T_i + \sum V_i$$

be the model Hamiltonian for a spherically symmetric closed shell nucleus containing $\frac{A}{2}$ protons and $\frac{A}{2}$ neutrons. T_i and V_i denote the single particle kinetic and potential energy operators for the i^{th} nucleon. For such nuclei Goldstone's theory [8] is readily applicable.[30].

If ϕ_o is the ground state wave function and ϵ_o the corresponding energy

$$H_o \phi_o = \epsilon_o \phi_o$$

and ϕ_o is the Slater determinant wave function formed by the lowest A single particle orbitals $|n^\circ\rangle$ that satisfy

$$(T + V)\,|n^\circ\rangle = E_n^\circ\,|n^\circ\rangle \qquad (4.1)$$

n° denotes all quantum numbers that completely specify the orbital and E_n° is its energy.

Assuming that only two body nuclear interactions v_{ij} exist between the constituent nucleons of the nucleus we may

write the total Hamiltonian as

$$H = \sum T_i + \sum v_{ij} = H_0 + H'$$

where $$H' = \sum v_{ij} - \sum V_i$$

may be regarded as the perturbation (see §2.2).

If now we regard the nuclear ground state as developing out of the model state ϕ_0 as the perturbation is adiabatically switched on Goldstone's result [8] for the ground state energy E is

$$E = \epsilon_0 + \sum_{nL} \langle \phi_0 | H' \{ \frac{1}{\epsilon_0 - H_0} \}^n | \phi_0 \rangle \qquad (4.2)$$

L under the summation sign means that only 'linked' terms are to be included in the sum (§3.1).

We remark that the single particle potential energy V is quite arbitrary at this stage.

4.2 <u>The Hartree Fock (H.F.) Approximation</u>:

As the method we shall be developing resembles the H.F. method it is useful to review the latter first.

Expanding the energy series (eqn. (4.2)) up to the first two orders (this process will have a strict meaning only for well behaved potentials) we obtain

$$E^{(2)} = \epsilon_0 + \frac{1}{2} \sum_{m^0 n^0} \langle m^0 n^0 | v | m^0 n^0 - n^0 m^0 \rangle - \sum_{m^0} \langle m^0 | V | m^0 \rangle$$

$$+ \sum_{m^0 n^0 s} \{ \langle m^0 n^0 - n^0 m^0 | v | m^0 s \rangle - \langle n^0 | V | s \rangle \} \frac{1}{E_n^0 - E_s} \{ \sum_{e^0} \langle e^0 s | v | e^0 n^0 - n^0 e^0 \rangle - \langle s | V | n^0 \rangle \}$$

$$+ \sum_{\substack{m^0 n^0 \\ m' n'}} \langle m^0 n^0 | v | m' n' \rangle \frac{1}{E_{m'}^0 + E_{n'}^0 - E_{m'} - E_{n'}} \langle m' n' | v | m^0 n^0 - n^0 m^0 \rangle \qquad (4.3)$$

$$= \sum_{m^0} \langle m^0 | T | m^0 \rangle + \sum_{m^0 n^0} \langle m^0 n^0 | v | m^0 n^0 - n^0 m^0 \rangle + \Delta E_2^{(1)} + \Delta E_2^{(2)} \qquad (4.4)$$

where $\Delta E_2^{(1)}$ and $\Delta E_2^{(2)}$ are the last two terms of eqn. (4.3). Superscripts are normally used to denote states "within" ϕ_o.

If V is chosen such that

$$\langle r|V|s\rangle = \sum_{\ell^o} \langle \ell^o r|v|\ell^o s - s\ell^o\rangle \qquad (4.5)$$

it appears that the convergence of the energy series is greatly helped. For the second order term $\Delta E_2^{(1)}$ vanishes and what is more higher order terms involving V also vanish.

It is worth remarking that this choice for V also results when the first order terms for the energy are minimised with respect to different trial potentials, giving us the familiar (H.F.) self consistency problem for the determination of $|n^o\rangle$ and V.

4.3 The Brueckner Approximation:

Just as in the case of the infinite medium (§ 3.4) an infinite series of v interactions may be summed to obtain an effective t interaction and the energy series (eqn. (4.2)) expressed in terms of these t interactions (and the V interactions). In operator form we write

$$t = v + v \frac{Q}{\epsilon_o - H_o} t \qquad (4.6)$$

where Q is a projection operator which ensures that the intermediate states satisfy the Pauli Exclusion Principle with respect to the states occupied in ϕ_o. Very little work has so far been done [32] on the difficult questions of the convergence of the original or the new series for the

energy and on the question of the convergence of the series for t [29].

Following Eden [4] the energy series may be written

$$E = \sum_{m^\circ} \langle m^\circ | T | m^\circ \rangle + \sum_{m^\circ n^\circ} \langle m^\circ n^\circ | t | m^\circ n^\circ - n^\circ m^\circ \rangle$$
$$+ \sum_{n^\circ s} \{ \sum_{m^\circ} \langle m^\circ n^\circ - n^\circ m^\circ | t | m^\circ s \rangle - \langle n^\circ | V | s \rangle \} \frac{Q'}{E_{n^\circ} - E_s}$$
$$\times \{ \sum_{e^\circ} \langle e^\circ s | t | e^\circ n^\circ - n^\circ e^\circ \rangle - \langle s | V | n^\circ \rangle \} \qquad (4.7)$$

Once again we could choose V to make the second order terms vanish.

$$\langle s | V | n^\circ \rangle = \sum_{m^\circ} \langle s m^\circ | t | n^\circ m^\circ - m^\circ n^\circ \rangle \qquad (4.8)$$

But, as the definition of t (eqn. 4.6) depends on the excitation of the rest of the nucleons of the nucleus as well as on the two nucleons under consideration, this choice will not automatically make the higher order terms involving V also vanish. We can only hope that the cancellation is nearly complete and it makes the series converge as rapidly as possible.

For considerations of single particle energies and dissociation energies this choice of V is not the best. One should arrange for V to cancel certain higher order terms as well [27]. These are of little interest to the binding energy problem and in any case as we can only hope to approximate to eqn. (4.8) (see §6.4, §6.5) we shall not discuss this point any further.

Hence if we assume that eqn. (4.8) should hold even when

s is a state belonging to ϕ_2 we have the set of eqns. (4.1), (4.6) and (4.8) to solve self consistently to determine the single particle potential and the energy of the system (assuming that a single particle state in the nucleus is not too unstable).

4.4 The Eden Emery Variational Scheme:

At the present stage it may be said that the double self consistency problem formulated above is almost insoluble. Thus Eden and Emery were lead to formulate a variational method [9] in close analogy with the H.F. scheme. Essentially it consists of minimising the first order energy

$$E_1 = \sum_{m^\circ} \langle m^\circ | T | m^\circ \rangle + \Sigma \langle m^\circ n^\circ | t | m^\circ n^\circ - n^\circ m^\circ \rangle \qquad (4.9)$$

as a function of the size of the single particle potential well.

The full scheme may be outlined as follows:

(i) Assume a trial single particle potential V which gives single particle wave functions in terms of which the Exclusion Principle operator Q may be expressed.

(ii) The matrix elements of t between states occupied in ϕ_2 are required. These may be calculated unambiguously in principle and E_1 evaluated.

(iii) The procedure is repeated for other trial potentials The best approximation to the energy is obtained when

E_1 so obtained is a minimum when also eqn. (4.8) which we term the self consistency condition is satisfied as well as possible.

This is the scheme in principle. In practice further approximations are necessary. These are introduced in the next chapter.

CHAPTER V

5.1 The Choice of V:

Because of well known (see below) advantages of Simple Harmonic Oscillator (S.H.O.) wave functions, and because of the successes of the S.H.O. Shell Model we restrict V to be a S.H.O. single particle potential. Later we shall have occasion to comment on the essential limitations of this choice (see § 8.3). The S.H.O. wave functions satisfy

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} K r^2 \right) \langle r | n \rangle = E_n \langle r | n \rangle \qquad (5.1)$$

For convenience of writing we let n stand for (n, ℓ, m) where n is the principal quantum number and ℓ, m, the usual angular momentum quantum numbers.

with $$\alpha^4 = \frac{mK}{4\hbar^2} \quad , \quad \lambda_n = \frac{2E_n}{\hbar \left(\frac{K}{m}\right)^{1/2}}$$

we have the relation between the eigenvalue and the principal quantum number in the form

$$n = \tfrac{1}{2}(\lambda_n - 3) \qquad (5.2)$$

The Schrödinger equation for two nucleons in model (S.H.O.) single particle states $|n_1\rangle\ |n_2\rangle$ is easily written down. Define

$$\underline{R} = \tfrac{1}{2}(\underline{r}_1 + \underline{r}_2) \qquad \underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\underline{Q} = \alpha \underline{R} \qquad \underline{z} = \alpha \underline{r} \qquad (5.3)$$

The relative and centre of mass motions of the two nucleons separate to give, in an obvious notation, two equations similar to eqn. (5.1):

$$(-\nabla_x^2 + x^2)\langle x|n\rangle = \lambda_n \langle x|n\rangle$$

$$\text{and} \quad (-\nabla_Q^2 + 16Q^2)\langle Q|N\rangle = 4\lambda_N \langle Q|N\rangle \tag{5.4}$$

The following relation holds between the single particle and the relative and centre of mass energies:

$$\lambda_N + \lambda_n = \lambda_{n_1} + \lambda_{n_2} \tag{5.5}$$

It is possible to express a product of S.H.O. wave functions of two nucleons as a linear combination of products of (S.H.O.) wave functions for the centre of mass and relative motions of the two particles:

$$|n_1, n_2\rangle = \sum_{nN} |nN\rangle\langle nN|n_1 n_2\rangle \tag{5.6}$$

Details, further properties and a tabulation of some of the coefficients $\langle nN|n_1 n_2\rangle$ may be found in ref.[33].

5.2 Wave Equation for a Pair of Nucleons:

As it is easier to solve for $\langle r_1 r_2|\Omega|n_3 n_4\rangle$, the perturbed wave function for a pair of nucleons. t matrix elements between unperturbed single particle states for pairs of nucleons are most conveniently calculated in terms of a wave matrix for a pair of nucleons defined such that

$$\langle n_1 n_2|t|n_3 n_4\rangle = \iint dr_1\, dr_2\ \langle n_1 n_2|v|r_1 r_2\rangle\langle r_1 r_2|\Omega|n_3 n_4\rangle$$

The integral equation (eqn. 4.6) for t gives the following equation for Ω

$$\langle r_1 r_2|\Omega|n_1^\circ n_2^\circ\rangle = \langle r_1 r_2|n_1^\circ n_2^\circ\rangle + \sum_{n_1' n_2'} \langle r_1 r_2|n_1' n_2'\rangle \frac{Q(n_1')Q(n_2')}{E_1^\circ + E_2^\circ - E_1' - E_2'} \langle n_1' n_2'|v\Omega|n_1^\circ n_2^\circ\rangle \tag{5.7}$$

Here $Q(n_i) = 0$ if n_i denotes a state occupied in the ground state ϕ_0.

Otherwise $Q(n_i') = 1$

In terms of the relative and centre of mass co-ordinates defined in eqn. (5.3) we may write instead of eqn. (5.7)

$$\langle rR|\Omega|n^\circ N^\circ\rangle = \langle rR|n^\circ N^\circ\rangle + \sum_{n_1' n_2'} \langle rR|n_1' n_2'\rangle \frac{Q(n_1')\,Q(n_2')}{E_n^\circ + E_N^\circ - E_1' - E_2'} \langle n_1' n_2'|v\Omega|n^\circ N^\circ\rangle \qquad (5.8)$$

In obtaining this we have used eqns. (5.5) and (5.6).

5.3 Exclusion Principle Approximation:

Further progress could be made with eqn. (5.8) if we could separate the relative and centre of mass motions. To this end we make an approximation to the Exclusion Principle. Though the method is more general we restrict our considerations to the ^{16}O nucleus as it is to this nucleus that the method that is being developed is applied.

Perhaps some comment as to why ^{16}O was chosen is not out of place here. Apart from being a spherically symmetric closed shell nucleus with equal number of protons and neutrons it has the advantage of being neither too large to make the calculations unduly tedious nor too small to make the corrections of order $\frac{1}{A}$ which are inherent in the theory uncomfortably large. It is for this reason too that it was not considered worthwhile to correct for the spurions kinetic energy of the nuclear centre of mass motion due to the wave functions being centred at the origin.

In ^{16}O, only the model levels with $n = 0$ and 1 are occupied.

Hence

$$Q(n_i') = 0 \quad \text{if} \quad n_i' \leq 1$$
$$= 1 \quad \text{if} \quad n_i' > 1 \qquad (5.9)$$

In Fig. 1 the excluded states in the n_1' n_2' plane are marked "x" and the allowed states marked "o". We wish to express $Q(n_1')\,Q(n_2')$ in terms of n' and N'. Since we cannot do so exactly we take a lead from the energy relation (eqn. 5.5) and approximate as follows:

$$Q(n_1')\,Q(n_2') \longrightarrow Q(n_1' + n_2') = Q(n' + N')$$
$$= 0 \quad \text{if} \quad n' + N' \leq 4$$
$$= 1 \quad \text{if} \quad n' + N' > 4 \qquad (5.10)$$

From Fig. 1 we note that eqn. (5.10) excludes correctly all states in which both n_1' and n_2' coincide with occupied levels of ^{16}O. It also excludes correctly the states in which only one particle is excited to the next shell. The second excited shell should have the states

$(n_1', n_2') = (4,0), (3,1), (1,3)$ and $(0,4)$ excluded;

but (2,2) should be in. However our approximation excludes all these states. There is some compensation since our approximation allows all higher states though some of them would be excluded by eqn. (5.9). It has not been feasible to estimate the accuracy of this approximation. However the following result of Eden and Emery[9] may be of interest. They calculated the t matrix elements of pairs of nucleons in the state $n=0$ using three forms of the Exclusion Principle, namely

Hence

$$Q(n_i') = 0 \quad \text{if} \quad n_i' \leq 1$$
$$= 1 \quad \text{if} \quad n_i' > 1 \qquad (5.9)$$

In Fig. 1 the excluded states in the n_1' n_2' plane are marked "x" and the allowed states marked "o". We wish to express $Q(n_1')\,Q(n_2')$ in terms of n' and N'. Since we cannot do so exactly we take a lead from the energy relation (eqn. 5.5) and approximate as follows:

$$Q(n_1')\,Q(n_2') \longrightarrow Q(n_1'+n_2') = Q(n'+N')$$
$$= 0 \quad \text{if} \quad n'+N' \leq 4$$
$$= 1 \quad \text{if} \quad n'+N' > 4 \qquad (5.10)$$

From Fig. 1 we note that eqn. (5.10) excludes correctly all states in which both n_1' and n_2' coincide with occupied levels of ^{16}O. It also excludes correctly the states in which only one particle is excited to the next shell. The second excited shell should have the states

$(n_1', n_2') = (4,0), (3,1), (1,3)$ and $(0,4)$ excluded;

but (2,2) should be in. However our approximation excludes all these states. There is some compensation since our approximation allows all higher states though some of them would be excluded by eqn. (5.9). It has not been feasible to estimate the accuracy of this approximation. However the following result of Eden and Emery[9] may be of interest. They calculated the t matrix elements of pairs of nucleons in the state $n=0$ using three forms of the Exclusion Principle, namely

by excluding all states for which

$$n' + N \quad \leq \quad 2, 3 \text{ and } 4 \text{ respectively}$$

Their result shows that while it is important to exclude correctly the occupied states the number of other states that are excluded is not too important, though there is a weakening of the interaction as more states are excluded.

5.4 Separation of the Centre of Mass Motion:

With the "triangle" form of the Exclusion Principle (eqn. 5.10) we are able to separate out the centre of mass motion in eqn. (5.8). We write it in the form

$$\langle nN|\Omega|n^\circ N^\circ\rangle = \langle nN|n^\circ N^\circ\rangle + \frac{Q(n+N)}{E_n^\circ + E_N^\circ - E_n - E_N} \langle nN|v\Omega|n^\circ N^\circ\rangle \qquad (5.11)$$

Since v depends only on the relative co-ordinates

$$\langle nN|v|n'N'\rangle = \langle N|N'\rangle\langle n|v|n'\rangle \qquad (5.12)$$

and eqn. (5.11) may be written in the form

$$\sum_{n'} \{\langle n|n'\rangle - \frac{Q(n+N)}{E_n^\circ + E_N^\circ - E_n - E_N} \langle n|v|n'\rangle\} \langle n'N|\Omega|n^\circ N^\circ\rangle = \langle nN|n^\circ N^\circ\rangle$$

Using the fact that the r.h.s is diagonal in N and changing the representation we obtain

$$\langle r|\Omega(N^\circ)|n^\circ\rangle = \langle r|n^\circ\rangle + \int dr' \sum_{n'} \langle r|n'\rangle \frac{Q(n'+N^\circ)}{E_n^\circ - E_{n'}} \langle n'|r'\rangle v(r') \langle r'|\Omega(N^\circ)|n^\circ\rangle \qquad (5.13)$$

We emphasise that $v(r')$ may include a tensor and a spin orbit force. Also the dependence of the perturbed relative wave function $\langle r|\Omega(N^\circ)|n^\circ\rangle$, of a pair of nucleons, on their centre of mass motion is only through the occurrence of N°

in the Exclusion Principle operator.

It is convenient to turn the integral equation (5.13) into an integro differential equation by applying the operator $E_n^\circ - \mathcal{H}$ where $\mathcal{H}$ is the model Hamiltonian in relative co-ordinate space (see eqn. (5.4)). We obtain

$$(E_n^\circ - \mathcal{H}) \langle r | \Omega(N^\circ) | n^\circ \rangle = v(r) \langle r | \Omega(N^\circ) | n^\circ \rangle - \int dr' K_{N^\circ}(rr') v(r') \langle r' | \Omega(N^\circ) | n^\circ \rangle \quad (5.14)$$

where

$$K_{N^\circ}(rr') = \sum_{n'+N^\circ \leq 4} \langle r | n' \rangle \langle n' | r' \rangle \quad (5.15)$$

5.5 Evaluation of the Angular Integrations:

We re-write eqn. (5.14) in a more familiar form suppressing all suffixes

$$(E - \mathcal{H}) \psi = v\psi - \int dr' K(rr') v(r') \psi(r') \quad (5.16)$$

If the unperturbed relative wave function for the pair of nucleons under discussion is

$$\langle r | n^\circ \rangle = \varphi = \frac{R_\ell}{r} Y_\ell^m \chi_s^{m_s} \quad ; \quad \frac{R_\ell}{r} = f_\ell$$

where χ is a spin function

Y the usual spherical harmonic

and f a radial wave function

the most general perturbed wave function for a general two body interaction is given by

$$\psi = \sum_{J\ell'} U_{\ell\ell'}^{Js} F_{\ell'}^{Jm_s s} C(J\, m_\ell + m_s,\ \ell\, m_\ell,\ s\, m_s) \quad (5.17)$$

where C is the Clebsch-Gordan coefficient[34], F total angular

momentum eigen functions and U perturbed radial wave function.

The superscripts and subscripts have the usual meaning (see for example the appendix of ref. (3)).

We remark that in general there are 5 terms for Ψ for $S=1$ (triplet states) and that there is only one term for $S=0$ (singlet state).

Substituting (5.17) into eqn. (5.16) and omitting the summation over J (which is a constanant of the motion) we obtain

$$(E-H)\sum_{e'}\frac{1}{r}U^{JS}_{ee'}F^{Jm_Js}_{e'} = V\sum_{e'}\frac{1}{r}U^{JS}_{ee'}F^{Jm_Js}_{e'}$$

$$-\int d\hat{r}'\sum_{LM_L} f_L(r)f_L(r')Y_L^{M_L}(\hat{r})Y_L^{M_L}(\hat{r}')V(r')\sum_{e'}\frac{1}{r'}U^{JS}_{ee'}(r')F^{Jm_Js}_{e'} \qquad (5.18)$$

Introducing the unit operator in spin space, viz

$$\sum_{s'm_s'}\chi_{s'}^{m_s'}(r')\chi_s^{m_s}(r)$$

the integral on the r. h. s. of (5.18) becomes

$$-\int dr'r'^2\sum_{e'LKm_L}\frac{1}{r'}f_L(r')U^{JS}_{ee'}(r')\int d\hat{r}'F_L^{Km_L+m_s's'}(\hat{r}')V(r')F_{e'}^{Jm_Js}(r')$$

$$\times f_L(r)F_L^{Km_L+m_s's'}|C(Km_L+m_s\,Lm_L\,sm_s'|^2$$

$$=\sum_{KLL'm_s'}\int r'^2dr'\frac{1}{r'}f_L(r')U^{JS}_{ee'}(r')v^{JS}_{Le'}(r')f_L(r)F_L^{Km_L+m_s's'}$$

$$C(Jm_J\,Lm_L\,sm_s')C(Km_J\,Lm_L\,sm_s')$$

where

$$v^{JS}_{Le'}(r') = \int d\hat{r}'F_L^{Km_L+m_s's}V(r')F_{e'}^{Jm_Js}$$

and which have been discussed and tabulated for instance by Ashkin and Wu [35].

Thus eqn. (5.18) simplifies to

$$\left(E - H_r - \frac{\ell'(\ell'+1)}{r^2}\right) U^{JS}_{\ell\ell'} = \sum_{\ell''} v^{JS}_{\ell\ell''} U^{JS}_{\ell\ell''}$$

$$- \sum_{\ell''} \int dr' \, G_{\ell''}(r'r) \, v^{JS}_{\ell'\ell''}(r') \, U^{JS}_{\ell\ell''}(r') \qquad (5.19)$$

where $$G_\ell(r'r) = \sum_{n+N^\circ \le 4} r^2 f_{\ell'}(r) f_{\ell'}(r') = \sum R_{\ell'}(r') R_{\ell'}(r)$$

and where H_r denotes the radial part of the relative Hamiltonian.

These are the basic equations of our theory. They are a set of coupled integro differential equations in general. In § 6.2 we shall introduce an approximation to uncouple them.

5.6 Matrix Elements:

It is useful to give the general expression for the t matrix elements in terms of the U's of eqn. (5.19). From the definition of t and ψ ,

$$\langle \varphi | t | \varphi \rangle = \langle \varphi | v | \psi \rangle$$

Hence for the ϕ of the previous section

$$\sum_{m_s} \langle \varphi | t | \varphi \rangle = \sum_{m_s} \langle f_\ell \, Y_\ell^m \, \chi_s^{m_s} | v | \sum_{J\ell'} \frac{1}{r} U^{JS}_{\ell\ell'} \, F^{Jm_J S}_{\ell'} \, C(J m_J \ell m_\ell s m_s)$$

$$= \sum_{J\ell' K m_s} \langle f_\ell \, F_\ell^{K m_J S} \, v \, \frac{1}{r} \, U^{JS}_{\ell\ell'} \, F^{Jm_J S}_{\ell'} \, C(J m_J \ell m_\ell s m_s) \, C(K m_J \ell m_\ell s m_s) \rangle$$

$$= \sum_{J\ell'} \langle r^2 f_\ell \frac{1}{r} U^{JS}_{\ell\ell'} \frac{2J+1}{2\ell+1} \rangle$$

$$= \sum_{J\ell'} \frac{2J+1}{2\ell+1} \int dr \, R_\ell(r) \, v^{JS}_{\ell\ell'}(r) \, U^{JS}_{\ell\ell'}(r) \, .$$

CHAPTER VI

6.1 Potential Energy for ^{16}O:

In our model of the ^{16}O nucleus the lowest s and p states are occupied, there being four nucleons in the s state (neutron and proton with spin up and down) and 4 x 3 = 12 nucleons in the p state (in which the z component of the angular momentum can take the three values 1, 0 and -1). It is convenient to separate the total potential energy (expectation value of t) into three parts

$$P = W(ss) + W(pp) + W(sp) \qquad (6.1a)$$

Here $W(ss)$ is the contribution from pairs of particles both of which are in s states and $W(pp)$ and $W(sp)$ are similarly defined. We saw already (§5.1) that wave functions of pairs of nucleons could be expressed in terms of centre of mass and relative wave functions and the equation for the perturbed relative wave function depends only on the principal quantum number of the centre of mass wave function. (through the number of states excluded) Hence in considering matrix elements we could represent states of pairs of particles by $|\beta, N\rangle$ where β describes the relative wave function (itself a S.H.O. wave function) and N is the principal quantum number of the centre of mass wave function.

Using the transformation coefficients listed in ref. [33] it is straight-forward to obtain

$$W(ss) = 6\langle 1s,0|\ t\ |1s,0\rangle$$

$$W(sp) = 18\langle 1s,1|\ t\ |1s,1\rangle + 30\langle 1p,0|\ t\ |1p,0\rangle$$

$$W(pp) = 18\langle 1s,2|\ t\ |1s,2\rangle + 15\langle 1d,0|\ t\ |1d,0\rangle + 30\langle 1p,1|\ t\ |1p,1\rangle + 3\langle 2s,0|\ t\ |2s,0\rangle \qquad (6.1)$$

The expectation values are all written in the form $\langle\varphi|v|\varphi\rangle$ which $= \langle\varphi|v|\psi\rangle$ and may involve a sum of terms (§5.6).

6.2 Equations for Relative Radial Wave Functions:

We see from eqn. (6.1) that the evaluation of the ground state potential energy of ^{16}O requires the solution of the equations for the perturbed relative wave functions which 'start' from the following unperturbed states.

$|1s,0\rangle$, $|1s,1\rangle$, $|1s,2\rangle$; $|1p,0\rangle$, $|1p,1\rangle$; $|1d,0\rangle$ and $|2s,0\rangle$

Because of the relatively weak dependence of the equations on N (see §5.3) (our calculation also showed this) it was found sufficient to consider the following four states only:

$|1s,2\rangle$ $|2s,0\rangle$ $|1d,0\rangle$ and $|1p,1\rangle$

In principle this involves the solution of 17 integro-differential equations which are easily written down from the general equations already obtained (eqn. 5.19). Some of these are coupled.

We write down explicitly the equations arising out of the $|1s,2\rangle$ state. These are

$$(E_0 - H_r)\, U^{00}_{00} = v^{00}_{00}\, U^{00}_{00} - \int dr'\, G_0(rr')\, v^{00}_{00}(r')\, U^{00}_{00}(r') \qquad (6.2)$$

$$(E_0 - H_r)\, U''_{00} = v''_{00}\, U''_{00} + v''_{02}\, U''_{02} - \int dr'\, G_0(rr') \{ v''_{00}\, U''_{00} + v''_{02}\, U''_{02} \} \qquad (6.3)$$

and $(E_0 - H_r - \frac{6}{r^2}) U''_{02} = v''_{20} U''_{00} + v''_{22} U''_{02} - \int dr' G_2(rr') \{ v''_{20} U''_{00} + v''_{22} U''_{02} \}$ (6.4)

Eqns. (6.3) and (6.4) are coupled.

At this stage we ignore the spin orbit forces and assume that the tensor forces act in s states only. This 'uncouples' all the coupled equations. In ref. [10] the following equations

$$(E_0 - H_r) U''_{00} = v''_{00} U''_{00} - \int dr' G_0(rr') v''_{00} U''_{00} \quad (6.5)$$

and $$(E_0 - H_r - \frac{6}{r^2}) U''_{02} = v''_{20} U''_{00} - \int dr' G_2(rr') v''_{20} U''_{00} \quad (6.6)$$

which are the approximations to eqns. (6.3) and (6.4) were obtained by the use of the generalised perturbation method of Eden and Emery [8]. The present method has the virtue of showing exactly what terms are dropped in arriving at eqns. (6.5) and (6.6) from eqns. (6.3) and (6.4). In principle this allows us to check the tensor approximation by solving eqns. (6.3) and (6.4) exactly, though this has not been achieved.

Thus the number of equations to be solved is reduced to ten. In Table 0 we summarise the details of these equations. Appendix A deals with the method used for the solution of these equations.

6.3 Discussion of the Effective Mass Approximation:

An improvement to the scheme so far developed for the calculation of binding energies of atomic nuclei could be made. In the infinite nuclear case two improvements were

made to the linked cluster perturbation theory. The first dealt with the summation of selected terms, considered for the finite case too. The other was the use of (self consistent) perturbed energies instead of the unperturbed ones for the energy denominators to take account of the excitation of the medium through which selected fermions moved. The self consistency condition (eqn. 4.8) is expected to take account of this effect in the finite case. But, can we do better than merely to hope that in our procedure the self consistency condition is satisfied as well as possible? (See §4.4).

In an infinite medium the level separation between two states of momenta k_o and k may be written

$$\delta E = \frac{\hbar^2}{m}(k_o^2 - k^2) + V(k_o) - V(k)$$

where $V(k)$ is the self consistent single particle potential. For unexcited states $V(k)$ is nearly quadratic and hence

$$\delta E = \frac{\hbar^2}{m^*}(k_o^2 - k^2)$$

where m^* is an effective mass, is a good approximation. However for large k this is not true. In fact $V(k) \to 0$ as $k \to \infty$. Therefore for k_o an occupied state and k an intermediate (excited) state the effective mass approximation cannot be expected to give a reasonable picture.

For a finite nucleus in the S.H.O. independent particle model the level spacing between two states is given by

$$\Delta_{on} = (\lambda_o - \lambda_n)\frac{\hbar}{m}\alpha^2$$

An effective mass approximation would merely increase all

level spacings. In ^{16}O the smallest Δ_{on} we encounter is $4\frac{\hbar}{m}\alpha^2$ while the average is more like $8\frac{\hbar}{m}\alpha^2$. For $\alpha = .4\,f^{-1}$ corresponding to the experimental radius of ^{16}O (see §7.2) the latter energy is about 53 MeV. Further excited states add successive units of 26.5 MeV. In nuclear matter the energy separation between the top and bottom of the occupied levels is about 70 MeV. Hence we see that in ^{16}O, the excitations in intermediate states are similar (as far as the size of the energy denominators are concerned) to those in nuclear matter.

Therefore as for nuclear matter we cannot hope to approximate to the self consistency condition by an effective mass approximation.

6.4 Self Consistency and the State Dependent Potential:

The self consistency condition (eqn. 4.8) implies in particular that

$$\langle n|V|n\rangle = \sum_{m^\circ} \langle nm^\circ|t|nm^\circ - m^\circ n\rangle \qquad (6.7)$$

We now exploit the fact that the restriction to S.H.O. wave functions does not completely determine the unperturbed Hamiltonian. If H_o is a S.H.O. Hamiltonian whose eigenfunctions and eigenvalues are given by $H_o|n\rangle = E_n|n\rangle$ then

$$H = H_o - \sum_{o}^{\infty} \eta_n |n\rangle\langle n|$$

is another Hamiltonian with eigenvalues $E_n - \eta_n$ and the same eigenfunctions $|n\rangle$. The diagonal matrix elements of V become

$$\langle n|V|n\rangle = \tfrac{1}{2}E_n - \eta_n$$

and we can hope to choose the eigenvalues to satisfy eqn.(6.7) self consistently. However there is a snag. With the new Hamiltonian the Exclusion Principle will not become simple for arbitrary η_n since the energy relation, eqn. (5.5) contributed to simplifying the equation for the relative wave function. Hence we redefine the total unperturbed Hamiltonian as the symmetrised sum of pair Hamiltonians, where the Hamiltonian for a pair of particles ij is

$$H_{ij} = H_0(i) + H_0(j) - \sum_{\nu=0}^{\infty} \sum_{m_1+m_2=\nu} \mu_\nu |m_1 m_2\rangle\langle m_1 m_2|$$

The total Hamiltonian will have products of single particle S.H.O. wave functions as eigenfunctions and perturbation theory should go through for this type of Hamiltonian. We assume this.

Now self consistency requires us to choose the μ_ν's so that the calculated and the unperturbed energy levels are the same. We cannot hope to do this for all levels. Hence we put $\mu_\nu = 0$ except for occupied states. This is expected to include the more important effects of self consistency for as we mentioned earlier (§ 6.3) the state dependence of the potential decreases with increasing energy.

The integral equations for the relative wave functions now become

$$\langle r|\Omega(N^0)|n^0\rangle = \langle r|n^0\rangle + \int dr' \sum_{n'} \frac{\langle r|n'\rangle Q(n'+N^0)\langle n'|r'\rangle}{E_n^0 - E_{n'} - \mu_\nu} v(r') \langle r'|\Omega(N^0)|n^0\rangle$$

$$= \langle r|n^0\rangle + \int dr' \sum_{n'} \frac{\langle r|n'\rangle Q(n'+N^0)\langle n'|r'\rangle}{E_n^{00} - E_{n'}} v(r') \langle r'|\Omega(N^0)|n^0\rangle$$

(6.8)

where $\nu = n^\circ + N^\circ$.

Without superscripts and subscripts we write down a typical (uncoupled) integro differential equation for the radial wave function that follow from eqn. (6.8)

$$-u''(x) + x^2 u(x) + \frac{l(l+1)}{x^2} u(x) + v(x) u(x)$$
$$= \lambda_n^{\circ\circ} u(x) + (\lambda_n^{\circ\circ} - \lambda_n^{\circ}) R(x) \int_0^\infty R(x') u(x') dx'$$
$$+ \int_0^\infty dx' G(x,x') v(x') u(x') \qquad (6.9)$$

Here $\lambda_n^{\circ\circ}$ corresponds to $E_n^{\circ\circ}$ (§5.1). We have used the dimensionless variable $x = \alpha r$ and the fact that $u(x)$ is normalised to $R(x)$.

The solution of these equations is again along the lines of Appendix I.

In the actual calculations we choose $\lambda_n^{\circ\circ}$ so that

$$\mu_0 = 2\mu_2 \qquad \text{and} \qquad \mu_1 = 3/2\,\mu_2$$

This restriction amounts to equating the $1s$-$1p$ level separation to the $1d$-$1p$ separation for a single particle Hamiltonian and neglecting the self consistency of the $1d$ level. This reduces the energy level self consistency (as eqn. (6.7) is termed) to a one parameter problem. In practice it is sufficient to vary $\lambda^{\circ\circ}$ for the largest matrix element corresponding to the state $|1s, 2\rangle$.

6.5 Self Consistency and the Variational Procedure:

Energy level self consistency (eqn. 6.7) is not enough.

Eigenvalue self consistency (the more general eqn. (4.8)) is also necessary. This means that the eigenfunctions of the unperturbed Hamiltonian should approximate as well as possible to those of the perturbed Hamiltonian. If t were a fixed two body potential this condition will be the same as the H.F. self consistency condition, obtained by minimising the first order energy (§4.2).

$$E(\alpha) = \sum_{m^\circ} \langle m^\circ | T | m^\circ \rangle + \tfrac{1}{2} \sum \langle m^\circ n^\circ | t | m^\circ n^\circ - n^\circ m^\circ \rangle \qquad (6.10)$$

by varying the wave functions (with t kept fixed). The best value of α if we restrict ourselves to S.H.O. wave functions is given by

$$\frac{\partial E}{\partial \alpha}\Big|_{t \text{ fixed}} = 0 \qquad (6.11)$$

Write $\beta = \sqrt{2}\alpha$. The radial parts of the single particle wave functions that occur in ^{16}O are [33]

$$R_{00} = 2\pi^{-1/4} \beta^{3/2} e^{-\frac{1}{2}\beta^2 r^2}$$

and

$$R_{11} = \pi^{-1/4} \sqrt{3/2}\, \beta^{5/2} r\, e^{-\frac{1}{2}\beta^2 r^2}$$

We check that

$$\frac{\partial R_{00}}{\partial \beta} = \sqrt{3/2}\, R_{20}$$

and

$$\frac{\partial R_{11}}{\partial \beta} = \sqrt{5/2}\, R_{31}$$

Hence we conclude that the off diagonal elements of t give $\frac{\partial E}{\partial \alpha}\Big|_{t \text{ fixed}}$. Indeed eqn. (6.11) gives a new 'self consistency' condition

$$\sum_{\ell^\circ} \{ \langle p | V | \ell^\circ \rangle - \sum_{m^\circ} \langle p\, m^\circ | t | \ell^\circ m^\circ - m^\circ \ell^\circ \rangle \} = 0 \qquad (6.12)$$

where $|p\rangle$ refers to the state obtained from $|e^{\circ}\rangle$ by differentiation as shown above.

The relation (6.12) contains off diagonal elements referring to s state nucleons added to those containing p state nucleons. The s and p state contributions are separately zero for different values of α which bracket the value giving eqn. (6.12).

Can one use different oscillator wells for the s and p state particles? This procedure will give rise to difficulties, associated with non orthogonal wave functions, which can only be overcome by introducing non local single particle potentials. Hence we satisfy ourselves with the condition of eqn. (6.12) which is not quite the condition of eqn. (4.8). Thus we aim for approximate eigenfunction self consistency only.

The error in energy, due to single particle excitations, due to not minimising s and p terms separately can be calculated. Indeed the only non zero second order terms involving V are those of the type encountered in eqn. (6.12), there being no second order terms involving t alone.

This chapter completes the development of the theory. In the next chapter we give the results of applying this theory to the calculation of the binding energy and radius of ^{16}O .

CHAPTER VII

7.1 Selection of Inter Nucleon Potentials:

In § 1.2 we noted that the phenomenological two nucleon interaction contains a large number of parameters chosen to fit the experimental data. These parameters are obviously interdependent and a variation of only one of the parameters to study the effect on a certain property would destory the fit to data and hence will be of little use. Yet there is one parameter - the size of the hard core radius - which is more important than others in nuclear structure problems. Even the existence of a hard core in the potential has been recently questioned.[x]

To study the effects of hard cores of different radii we have therefore used six inter nucleon potentials whose hard core radii range from .2 to .6 f. All the potentials fit the low energy data and some of them also fit the high energy data rather well.

---

[x] See the work of Gammel, Nuttall and Sampanthar on the Effective range expansion parameters both with and without the inclusion of the vacuum polarisation effects. Ref. 3) Sec. 7. Their analysis of data of low energy experiments seem to indicate that the sign of the shape dependent parameter is positive, casting doubt on the hard core hypothesis. However the experimental results are not sufficiently accurate to enable one to draw firm conclusions.

The best Gammel Thaler type potential that fits the high energy data is that of ref.[12] and [13]. However this potential has different hard core radii in different states and strength parameters which vary with the energy. In their study of nuclear matter Brueckner and Gammel used a simpler potential which had the same (.4f) hard core radius in all states and fixed strengths and which gave nearly as good an agreement with data. We use this potential denoted by 4(a) in Tables I and II.

This potential has spin orbit forces which we ignore in our approximation. Hence we also use a similar potential not having a strong spin orbit force from the list of Gammel, Christian, Thaler potentials given in ref.[3] and denote it by 4(b). For core radii .3f and .5f we selected potentials from ref.[36]. These are denoted by (3) and (5). All these potentials have tensor forces.

Odd state potentials were considered only for 4(a). This showed that odd states have relatively little effect on nuclear binding for O^{16}. Further, odd state potentials are not too well known. Taken together we did not think it worthwhile to consider the odd state potentials in the other cases.

We also use two central force potentials thereby avoiding the tensor force approximation. These are due to Ohmura, Morita and Yamada.[37] For a variational calculation of the binding energy of ^{16}O Dabrowski [38] has used these potentials

and in §7.4 we compare the pair correlation function obtained by our method with that used by Dabrowski.

The potentials are listed in Tables I and II. They are all of the form

$$V = \infty \quad , r < r_c$$

$$= -V_0 e^{-\mu r}/\mu r \quad , r > r_c \qquad \text{where}$$

r_c denotes the hard core radius V_0 the strength and μ the inverse range.

7.2 Notation and Main Results:

(a) Notation:

Our units are fermi ($= 10^{-13}$ cm) for length and Mev for energy. In terms of these

$$\frac{\hbar^2}{m} = 41.46 \qquad \text{where}$$

m is mass of nucleon.

α, the inverse radius parameter for harmonic oscillator well (§5.1) is measured in fermi $^{-1}$.

r.m.s. radius of S.H.O. model of ^{16}O = $\frac{1.09}{\alpha}$ fermi.

$E(\alpha)$ = kinetic energy expectation value of t matrix in (model) ground state of ^{16}O

$$= 36 \frac{\hbar^2}{m}\alpha^2 + P(\alpha) \qquad \text{where}$$

$P(\alpha)$ is given by eqn. (6.1a)

$$\delta E = \tfrac{1}{2}\alpha \left[\frac{\partial E(\alpha)}{\partial \alpha}\right]_{t\ fixed} = \delta E_p + \delta E_s$$

where

δE_p = p state contribution to δE (in Mev)

and δE_s = s state contribution to δE (in Mev)

ΔE_1 = second order correction terms arising from single particle excitations from the ground state wave function

$$= 4 \frac{\langle 1s | V - \Sigma \pm | 2s \rangle^2}{E_{1s} - E_{2s}} + 12 \frac{\langle 1p | V - \Sigma \pm | 2p \rangle^2}{E_{1p} - E_{2p}}$$

$$= \frac{(\delta E_s)^2}{6 \frac{\hbar^2}{m} \alpha^2 (\lambda^{00} - 7)} + \frac{(\delta E_p)^2}{30 \frac{\hbar^2}{m} \alpha^2 (\lambda^{10} - 9)}$$

Where λ^{00} and λ^{10} are the s and p state energy parameters shifted to take account of energy level self consistency.

The procedure (§6.4) of minimising $E(\alpha)$ keeping t fixed gives

$$\delta E = 0 \qquad (7.1)$$

α_0 denotes the value of α for which eqn. (7.1) holds

$E_0 = E(\alpha_0)$

α_m denotes the values of α for which $E(\alpha)$ is a minimum and $E_m = E(\alpha_m)$

(b) Experimental Values:

The binding energy of ^{16}O is 127.3 Mev. By fitting electron scattering data with charge distributions obtained from S.H.O. wave functions one obtaines an r.m.s. radius of 2.65 f for ^{16}O. Ravenhall[17] estimates that a more realistic wave function could change this value by about 5%. This is because the electron scattering data is expected to depend more strongly on the nuclear surface region rather than on

average density and the S.H.O. wave functions with their rapid fall off may not represent the surface region accurately enough. Thus we should take the experimental α as lying between .4f and .43f.

(c) Results:

Figures (2) and (3) illustrate the main features of the results for a typical potential (that of code (5)). We observe from Fig. 3 that the s state particles have self consistency at $\delta E_s = 0$ which corresponds to a larger α (smaller r.m.s. radius) than that given by p state particles at $\delta E_p = 0$.

This illustrates the limitations of the S.H.O. potential for the unperturbed system. A more realistic potential will be of finite depth. This will cause the p state to be nearer the continuum than the s state. Consequently the off diagonal terms of the potential will appear to correspond to a wider well for p states than for s states.

Further there will be state dependence not only for the diagonal matrix elements of V but also for its off diagonal elements.

The leading correction term is not large in the region of α between s and p state self consistency, even though the expectation value of the energy varies considerably in this region. All three points of self consistency occur for values of $\alpha > \alpha_m$. These results are typical for all potentials.

Our method chooses α_0 as giving the best value for the model r.m.s. radius. Pair correlations are found not to change the r.m.s. radius of the matter distribution by more than 1%.

From Fig. 2 we observe the tendency for the correction terms to compensate the minimum in $E(\alpha)$. Eden[32] has observed that when $|\Delta E_1|$ is large there will be other large correction terms as well and that neither $E(\alpha)$ nor $E(\alpha) + \Delta E_1$ will have any relation to the actual nuclear energy. We note from the figure that $E(\alpha)$ takes the value E_0 again for a smaller α. However at this point ΔE_1 is large and we conclude that at this point the actual nuclear wave function will have no relation to the model wave function.

Table 3 gives the final results for the different potentials. We note the trend towards a higher density as the hard core radius decreases. The other point of interest is the larger binding energy for potential 4(a) than for 4(b). (The former contains a spin orbit force and the latter does not). However, because of our approximations (§6.2) we cannot immediately draw any conclusions.

Tables 4 to 9 contain more detailed results for the different potentials used.

The Coulomb energy has been calculated in the H.F. approximation. Hence it is the same for all potentials (for each α) as is the kinetic energy. They are therefore listed in Table 4 only. The final column in each table is

for the value of α which gives self consistency.

The potential energy $P(\alpha)$ is also shown in Table 4 only as it has an obvious relation to $E(\alpha)$ and the kinetic energy and Coulomb energies. Fortunately for each potential, $P(\alpha)$ has its point of inflexion in the region where $E(\alpha)$ is a minimum, besides having a fairly constant slope over this range. Hence we were able to interpolate rather accurately when the need arose to obtain $E(\alpha)$ at intermediate points.

As a consequence of the hard core in the potential it is clear that the correlation kinetic energy will dominate over the attractive potential energy at high density. This is reflected in $P(\alpha)$ passing through a minimum at higher values of α than are listed in the Tables and eventually becoming positive and large.

Table 10 lists the separate contributions to the potential energy (see §6.1) for $\alpha = .4f$ (experimental value). The parameter for energy level self consistency was also kept constant at $\lambda^{\circ\circ} = -1$. We note that there is less than 10% variation in the total potential energy for the different potentials used. However when the kinetic and Coulomb energies are added (252 Mev at $\alpha = .4\ f^{-1}$) the resulting binding energy varies from 122 Mev for potential (5) to 160 Mev for potential 4(a).

7.3 Self Consistency and the Matrix Elements:

(a) The Matrix Elements:

The matrix element which dominates the potential energy

is the one due to the $|1s\rangle$ term.

In Tables 11 and 12 we list the contribution of the different matrix elements to the potential energy for the potentials 4(a) and (5) respectively. Their weight factors are also shown. λ^{00} was kept fixed at -1, a value fairly close to the energy level self consistency right across the table.

For potential 4(a) we notice that the potential energy due to the tensor force is somewhat greater than that due to the central force. Further while the tensor contribution increases uniformly with α the central contribution does so more slowly for large α (For larger α it begins to decrease as it must). Hence the minimum of the energy curve is shifted to a higher density.

For potential (5) even though the tensor force is weaker the greater strength of the triplet central force delays the effect of the hard core on the rate of increase of the central potential. In fact both the minimum of $E(\alpha)$ and the point of self consistency occur at a higher density for potential (5) than for potential 4(a).

(b) Energy Level Self Consistency:

In the S.H.O. potential well the energy levels for s and for p states correspond to $\lambda = 3$ and $\lambda = 5$ respectively while the $2s$ level is at $\lambda = 7$. The s-p separation is made up from equal contributions of one unit ($= \frac{\hbar^2}{m}\alpha^2$ Mev) from

the kinetic and potential energies and is one half the s-$2s$ separation.

In our model we required the s and p state nucleons in the unperturbed system to have different well depths and took account of it by allowing a variation in the s and p state energy levels. We kept the $2s$ level fixed at $\lambda = 7$ and shifted the s state energy levels to $\lambda = 1, -1, -3$ or interpolated values. The p level was taken to be halfway between the s and $2s$ levels.

The contribution to the level separation from the kinetic energy is unaltered as we go from the unperturbed to the perturbed system. The contribution from the potential energy to the s-p separation in the unperturbed system changes to

$$\Delta_{sp}(\text{model}) = 3\frac{\hbar^2}{m}\alpha^2, \quad 5\frac{\hbar^2}{m}\alpha^2, \quad \text{or} \quad 7\frac{\hbar^2}{m}\alpha^2$$

as we depress the s state level to $\lambda = 1, -1$, or -3.

The calculated s-p separation in the perturbed system is made up of the unperturbed kinetic energy part and a potential energy part

$$\Delta_{sp}(\text{calculated}) = \frac{1}{6}\{(1\alpha,0) + (2s,0) - \frac{1}{14}(1s,1)\} \tag{7.2}$$

where the terms on the r.h.s. denote the total contribution to $E(\alpha)$ from the corresponding matrix elements including the weighting factors. We remark that the calculated s-$2s$ separation is not quite twice Δ_{sp}(calculated). Hence we must assume that the $2s$ level is also shifted slightly.

In principle to obtain energy level self consistency we choose the unperturbed levels so that

$$\Delta_{sp}(\text{model}) = \Delta_{sp}(\text{calculated}) \qquad (7.3)$$

However in practice since we work with centre of mass and relative levels and not the single particle levels involved in eqn. (7.3) we introduce a further approximation to obtain a simpler procedure.

If the single particle s and p levels are moved respectively to $\lambda = 1, -1, -3$ and $\lambda = 4, 3, 2$ the eigenvalues of the relative state $|1s\,0\rangle$ move to $-1, -5, -9$ and those for the relative state $|1s\,1\rangle$ to $0, -3, -6$ when it occurs in $W(s,p)$ and to $1, -1, -3$ when it occurs in $W(pp)$. In the potential energy term in $E(\alpha)$ the matrix elements of the latter two states are equally weighted and large compared to the other states. Strictly we should calculate Δ_{sp} given in eqn. (7.2) using different values of λ^{00} for the relative s level that are appropriate for the different matrix elements. But we found that the variation in Δ_{sp}(calculated) with λ^{00} is much slower than in Δ_{sp}(model). Hence we calculated all relative s state matrix elements as if they had λ^{00} the same as for single particle states. This procedure is correct for $|1s,1\rangle$ in $W(p,p)$ and a good approximation for $|1s,1\rangle$ in $W(s,p)$. As these matrix elements dominate the variation of Δ_{sp}(calculated) with λ^{00} we did not bother to recalculate $(1\alpha,0)$ and $(2s,0)$ as λ^{00} changed.

In Table 13 for potential (5) we give the variation of

the total potential energy with $\lambda^{\circ\circ}$. The kinetic and Coulomb energies are also listed.

In Table 14 the values of $\lambda^{\circ\circ}$ for which energy level self consistency is obtained are listed for each potential at $\alpha = .5$ which is close to the point for wave function self consistency.

7.4 Wave Functions and Pair Correlations:

Define the pair distribution function for the unperturbed system by

$$g_0(r) = 4\pi \int d\underline{R}\, d\underline{r}_3 \cdots d\underline{r}_6 \, |\Phi_0|^2 \tag{7.4}$$

where $\underline{R} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$ and Φ_0 is the ground state total wave function. Then $r^2 g_0(r)\, dr$ gives the probability that given a particle at the origin another particle is found in the region $(r, r+dr)$.

For the perturbed total wave function the first order pair distribution function may be written in terms of the u's which are solutions of the equations of the type eqn. (5.19). Labelling the wave functions in the same way we labelled the states in §6.1 we obtain for the first order pair distribution function the expression

$$\alpha^2 r^2 g(r) = \frac{7}{40}(u_{1s01}^2 + u_{1s03}^2) + \frac{1}{80}(u_{2s01}^2 + u_{2s03}^2)$$

$$+ \frac{1}{60}(u_{1d01}^2 + u_{1d03}^2) + \frac{1}{20}(9u_{1p13}^2 + u_{1p11}^2)$$

(7.5)

where the fourth suffix (1 or 3) denotes a singlet or triplet state and where we have omitted all 'tensor' wave functions.

Using these expressions we have compared the unperturbed and perturbed pair distribution functions for the potential (4b) at $\alpha = .5\,f^{-1}$ and with $\lambda^{00} = -1$. In Fig. 4 we give a plot of $x^{-2}\,g_0(x)$ and $x^{-2}\,g(x)$ where $x = \alpha r$. In Fig. 5 we give the ratio of the two pair distribution functions.

The perturbed pair correlation function is zero within the hard core region. Outside it increases rapidly to a maximum at about $1\,f$ where it is about 12% greater than the unperturbed function. At greater distances it decreases and beyond about $2.4\,f$ the difference between the perturbed and unperturbed functions is not more than 2%.

It is interesting to compare this result (though such direct comparison is perhaps not quite legitimate) with that of Dabrowski [38] who applied the Jastrow variational method [39] to ^{16}O. Dabrowski restricted his trial functions to have a pair correlation ratio zero inside the hard core region and to rise monotonically to unity outside the region. As Emery has observed [40] such a function gives a smaller value for the binding energy than one which has a maximum (like ours). That is perhaps why Dabrowski obtained r.m.s. radii for ^{16}O using potentials (2) and (6) which are considerably larger than our values for the same potentials. This could explain his largish correction terms too.

As for a binding energy comparison it is perhaps more

appropriate to compare his values with our E_m rather than our E_o . Our E_m's turn out to be larger than the binding energies he obtains.

CHAPTER VIII

In this final chapter of Part I of the dissertation we collect together the main features of our method and our results. A discussion follows:

8.1 The Method:

Our main physical assumptions were:

(i) that there are no significant many body forces between nucleons in the nucleus.

and

(ii) that the forces between two nucleons are not altered by the presence of other nucleons in the nucleus.

This enabled us to use the best available phenomenological two body potentials chosen to fit all relevant two nucleon data.

The main mathematical assumption was that the energy of an atomic nucleus could be determined from a suitably defined t matrix. The convergence problems that arise have already been referred to (§4.3).

We emphasise the following features of the method:

(i) The unperturbed system is taken to be an independent particle model of the nucleus in which the nucleons move in a common S.H.O. potential. This potential is characterised by a parameter α which is inversely proportional to the r.m.s. radius of the model nucleus (§5.1).

(ii) In the definition of the t matrix and hence the wave

matrix the exclusion principle operator is approximated to enable the separation of the relative and centre of mass motions. (§5.3)

(iii) The parameter α is chosen to minimise the first order term in the energy expansion with respect to variations in α in the wave functions with the t operator kept fixed. (§6.5)

(iv) The single particle potential is allowed to be state dependent by varying the low lying energy levels in such a way that the calculated and unperturbed level spacings are equal. (§7.3) This is the analogue of momentum dependence in an infinite system and an improvement on the customary effective mass approximation.

(v) Spin orbit forces are ignored and tensor forces are assumed to act in *s* states only. This enabled us to uncouple the coupled integro differential equations of the theory. (§6.2)

(vi) The Coulomb energy is calculated in the Hartree Fock approximation.

8.2 Results:

The main results may be summarised as follows:-

(i) The r.m.s. radius for self consistency turns out to be 1.9f for potential (4*b*) and 2.0f for potential (5) while the experimental value is 2.65f.

(ii) The theory gives a binding energy of 123 Mev for potential

matrix the exclusion principle operator is approximated to enable the separation of the relative and centre of mass motions. (§5.3)

(iii) The parameter α is chosen to minimise the first order term in the energy expansion with respect to variations in α in the wave functions with the t operator kept fixed. (§6.5)

(iv) The single particle potential is allowed to be state dependent by varying the low lying energy levels in such a way that the calculated and unperturbed level spacings are equal. (§7.3) This is the analogue of momentum dependence in an infinite system and an improvement on the customary effective mass approximation.

(v) Spin orbit forces are ignored and tensor forces are assumed to act in *s* states only. This enabled us to uncouple the coupled integro differential equations of the theory. (§6.2)

(vi) The Coulomb energy is calculated in the Hartree Fock approximation.

8.2 Results:

The main results may be summarised as follows:-

(i) The r.m.s. radius for self consistency turns out to be 1.9f for potential (4b) and 2.0f for potential (5) while the experimental value is 2.65f.

(ii) The theory gives a binding energy of 123 Mev for potential

(4b) and 118 MeV for potential (5). The experimental value is 127.3 MeV.

(iii) The main effect of energy level self consistency is to reduce the potential energy at higher densities and hence to increase the calculated r.m.s. radius and reduce the magnitude of the binding energy. Without self consistency the radii would have been about 10 - 20% smaller and the binding energies more than 20% greater.

(iv) The r.m.s. radius for self consistency is about 10% smaller than the r.m.s. radius for which the energy expectation value is a minimum.

(v) The potentials for self consistency of the s and p state particles are different. The r.m.s. radius (of the whole nucleus) for s state self consistency alone is about 10% smaller than for p state self consistency. Since we could not use different wells we had to take a weighted mean for the r.m.s. radius.

(vi) The importance of the Exclusion Principle lay in that it excluded transitions to other states of equal energy. However it is not too important how many other transitions are disallowed. (§ 5.3)

(vii) The equation for the t matrix was never singular in our case since the energy spectrum was discrete. For the strongest potential (singlet s state) at normal density

($\alpha = \cdot 4 f^{-1}$) and without taking account of energy level self consistency we found that it was necessary to increase the potential threefold before we could obtain a change in sign in the t matrix (indicating a singularity). Had we included energy level self consistency we would have required an even stronger potential.

(viii) The expectation value of the t matrix, the kinetic energy and the energy of the nucleus at experimental density are roughly in the ratio -4: 3: -1: If we remember that the t matrix contains a positive core contribution (correlation kinetic energy) which increases rapidly with density and is of the same order of magnitude to the negative contribution at normal density we are lead to expect an error of about 10% in the binding energy near self consistency for an error of 1% in the potential energy.

8.3 Discussion:

First, the 25% discrepancy between the calculated r.m.s. radius for potentials 4(a), 4(b) and (5) and that deduced from electron scattering data deserve some comment. The reason may lie with the use of S.H.O. wave functions which prevents us from obtaining full wave function self consistency. (§6.3)

The S.H.O. wave functions fall off much too fast unlike

an exponential which is perhaps the form to be expected for a more realistic finite single particle well. For a given r.m.s. radius a S.H.O. wave function will be less peaked near the origin than a wave function for a finite well. Thus the correlation kinetic energy (the core contribution) will be relatively more important for the latter; so also the edge of the wave function. These effects combine to decrease corresponding t matrix elements even allowing for any increased attraction. The attraction will decrease with density while the former effect will increase as the relative size of the core becomes more important. On the whole the t matrix elements may be expected to be reduced at high density and increased at low density. But for a given r.m.s. radius (and a more realistic potential) the kinetic energy may be increased for the unperturbed wave function by as much as 10% or more. For the S.H.O. potential leads to the smallest kinetic energy compatible with a given r.m.s. radius.

Hence we conclude that a more realistic potential would shift the minimum in the energy curve to a smaller value of α. We could expect the self consistent value of α also to shift in the same direction and we do not consider a 20% shift as impossible.

Next we observe that the different inter nucleon potentials gave quite different answers. This we believe is due to the differences in the potentials rather than to our approximations. The possible uncertainties involved in the

use of strong tensor forces as in potential 4(a) to fit the data may have an important effect on the binding energy. For example a 5% change in the tensor potential in potential 4(a) at $\alpha = .4f^{-1}$ leads to a change of about 30 MeV in the binding energy.

The best potentials we used depended rather strongly on the 310 MeV scattering data. Perhaps potentials that fit data at about 150 MeV are the ones most suitable for nuclear structure calculations (§1.2). Indeed if one had such a potential and also knew the uncertainties in the potential parameters arising from the possible errors in the experimental data and the analysis that gave the potential, one could go more confidently to the next steps in refining the formalism for the computation of nuclear binding energies from the knowledge of nuclear forces. Now what are these steps?

Within the present formalism itself two improvements could be suggested. The first is to avoid the s state tensor force approximation, by solving exactly the coupled integro differential equations of the theory. Besides improving the method it will also enable one to calculate other properties of nuclei, such as level spacings and spin orbit splittings, as well. This is comparatively 'simple'.

Much more formidable will be the task of solving the coupled equations for the centre of mass and relative motions that will result if we were to avoid the Exclusion Principle

approximation. In addition if we were to use a finite well it is perhaps a fair comment to make that to achieve progress one may have to await further development in computing techniques. However preliminary calculations could be made with a finite well in relative co-ordinate space to check some of our conjectures. (§ 8.2)

We conclude with a brief discussion of the alternative scheme developed by Brueckner and collaborators [41] for studying the properties of finite nuclei. In their method the binding energy of an atomic nucleus is calculated essentially in the H.F. approximation except that instead of the two nucleon interaction the t matrix appropriate to nuclear matter of the local density is used. This was their basic assumption.

Now this approximation would be expected to be adequate if the correlation range in the t matrix were much smaller than the region in which the density changes appreciably. This is plainly not so. For large nuclei the nuclear density drops from about 90% to 10% of its central value over a region of the order of 2.4f, while the correlation distance is some-what greater than 1f [41].

Hence their approximation cannot be valid particularly for as light a nucleus as ^{16}O. Indeed their calculations also point to this conclusion, in that the binding energies

they obtain are much too low. Their r.m.s. radii are reasonable, though on the low side. Hence their method has to be suitably modified before it can give results in quantitative agreement with experiment.

APPENDIX I

The Solution of the Integro-differential Equations:

We illustrate the method we used for the solution of the different integro-differential equations of the theory by considering the details for the equation

$$-u''(x) + x^2 u(x) + v(x) u(x) - 3 u(x) = \int_0^\infty G_{2s}(x x') v(x') u(x') dx' \qquad A(1)$$

We have omitted the superscripts and the subscripts for convenience of writing. Now assume that $u(x)$ is normalised so that it has unit derivative at the core edge, viz. at $x = a$.

$u(x)$ vanishes on and inside the core region; but $u(x) v(x)$ does not.

Denote it by $g(x)$ so that

$$v(x) u(x) = g(x) + \delta(x - a) \quad \text{for} \quad x \leq a$$

Eqn. A(1) now becomes, for $x < a$

$$g(x) = \int_0^\infty G_{2s}(x x') v(x') u(x') dx'$$

$$= \int_0^a G_{2s}(x x') g(x') dx' + G_{2s}(x, a)$$

$$+ \int_a^\infty G_{2s}(x x') v(x') u(x') dx' \qquad A(2)$$

$$G_{2s}(x x') = x x' \left\{ \frac{4}{\sqrt{\pi}} + \frac{6}{\sqrt{\pi}} (1 - 2/3 x^2)(1 - 2/3 x'^2) \right\} e^{-1/2 (x^2 + x'^2)}$$

$$= x e^{-1/2 x^2} \left\{ \frac{x'}{\sqrt{\pi}} (10 - 4 x'^2) \right\} e^{-1/2 x'^2}$$

$$+ x^3 e^{-1/2 x^2} \left\{ \frac{x'}{\sqrt{\pi}} (-4 + 8/3 x'^2) \right\} e^{-1/2 x'^2}$$

Hence eqn. A(1) may be written

$$-u''(x) + x^2 u(x) + v(x) u(x) - 3 u(x) = B x e^{-1/2 x^2} + C x^3 e^{-1/2 x^2} \qquad A(3)$$

where the numbers B and C are to be determined.

$g(x)$ may also be written

$$g(x) = B x e^{-1/2 x^2} + C x^3 e^{-1/2 x^2}$$

Eqn. A(3) has two linearly independent solutions when $B = C = 0$. Asymptotically they behave like $x e^{-1/2 x^2}$ and $\frac{1}{x^2} e^{1/2 x^2}$. The solution that converges at infinity is the one of interest to us.

This physical solution $u_1(x)$ is found by setting the asymptotic value of the solution and its derivative and integrating inwards using the Runga - Kutta - Gill method. This was done on the Automatic Digital computer, Edsac II at the Cambridge Mathematical Laboratory. It was found sufficient to start at $x = 6 + a$ and use a step length of .01 in the inward integration.

By similar methods two more solutions of the equation A(3) were found, one with $B = 1$, $C = 0$ and the other with $B = 0$, $C = 1$. Call these $u_2(x)$ and $u_3(x)$ respectively.

Then the general (physical) solution of eqn. A(3) may be written

$$u(x) = A u_1(x) + B u_2(x) + C u_3(x)$$

where A is an arbitrary constant.

Equating the coefficients of $x e^{-1/2 x^2}$ and $x^3 e^{-1/2 x^2}$ on either side of the eqn. A(2) we obtain two equations for B and C involving A.

The condition that the wave function must vanish at the core edge gives us a third equation involving A, B, and C. These equations are

$$\sqrt{\pi}\, B = a(10 - 4a^2) e^{-1/2 a^2} + \int_0^a dx'\, x' e^{-1/2 x'^2} (10 - 4x'^2) [B x' e^{-1/2 x'^2} + C x'^3 e^{-1/2 x'^2}]$$

$$+ \int_a^\infty dx'\, x' e^{-1/2 x'^2} (10 - 4x'^2) [A u_1 + B u_2 + C u_3]$$

$$\sqrt{\pi}\, C = a(-4 + 8/3\, a^2) e^{-1/2 a^2} + \int_0^a dx'\, x' e^{-1/2 x'^2} (-4 + \tfrac{8}{3} x'^2) [B x' e^{-1/2 x'^2} + C x'^3 e^{-1/2 x'^2}]$$

$$+ \int_a^\infty dx'\, x' e^{-1/2 x'^2} (-4 + 8/3\, x'^2) [A u_1 + B u_2 + C u_3]$$

and $$A u_1(a) + B u_2(a) + C u_3(a) = 0$$

These determine A, B, C and hence $u(x)$. The integrations that occur were all done by Simpson's rule. An interval of .05 was found to be adequate.

The resulting $u(x)$ is then normalised to the unperturbed wave function by multiplying it by N where

$$\int_0^\infty x\, e^{-\frac{1}{2}x^2}\, u(x)\, dx = \frac{\pi^{1/4}}{2N}$$

The whole process of determining a normalised $u(x)$ took just under two minutes on Edsac II.

APPENDIX II KERNELS FOR THE INTEGRO-DIFFERENTIAL EQUATIONS.

$$G_{2s}(xx') = 4\pi^{-1/2} xx' e^{-1/2(x^2+x'^2)} + 6\pi^{-1/2} xx'(1-2/3x^2)(1-2/3x'^2) e^{-1/2(x^2+x'^2)}$$

$$G_{3s}(xx') = G_{2s}(xx') + \frac{15}{2}\pi^{-1/2} xx'(1-\frac{4}{3}x^2+\frac{4}{15}x^4)(1-\frac{4}{3}x'^2+\frac{4}{15}x'^4) e^{-1/2(x^2+x'^2)}$$

$$G_{2p}(xx') = \frac{8}{3}\pi^{-1/2} x^2x'^2 e^{-1/2(x^2+x'^2)} + \frac{20}{3}\pi^{-1/2} x^2x'^2(1-2/5x^2)(1-2/5x'^2) e^{-1/2(x^2+x'^2)}$$

$$G_{1d}(xx') = \frac{16}{15}\pi^{-1/2} x^3x'^3 e^{-1/2(x^2+x'^2)}$$

$$G_{2d}(xx') = G_{1d}(xx') + \frac{56}{15}\pi^{-1/2} x^3x'^3(1-2/7x^2)(1-2/7x'^2) e^{-1/2(x^2+x'^2)}$$

TABLE 0. KERNELS AND ANGULAR MOMENTA FOR THE INTEGRO-DIFFERENTIAL EQUATIONS.

ℓ is the angular momentum and $G(x, x')$ and $G_2(x, x')$ the kernels for the integro-differential equations (5.19) corresponding to the unperturbed state $|\beta, N\rangle$ whose eigenvalue is E_n. The analytic forms of the kernels G_{3s}, etc., are listed in the appendix II. The use of G_{2s} for state $|1s,0\rangle$ is discussed in §6.2.

state	ℓ	λ_n	$G(x, x')$	$G_2(x, x')$
$\|1s, 0\rangle$	0	3	G_{2s}	G_{2d}
$\|1s, 1\rangle$	0	3	G_{2s}	G_{1d}
$\|1s, 2\rangle$	0	3	G_{2s}	G_{2d}
$\|2s, 0\rangle$	0	7	G_{3s}	G_{2d}
$\|1d, 0\rangle$	2	7	G_{2d}	-
$\|1p, 1\rangle$	1	5	G_{2p}	-
$\|1p, 0\rangle$				

TABLE 1. EVEN-STATE PARAMETERS AND NOTATION FOR POTENTIALS. V IS MEASURED IN MeV AND μ IN INVERSE FERMI.

potential code no.	r_c	$^3V_c^+$	$^3\mu_c^+$	$^1V_c^+$	$^1\mu_c^+$	$^3V_T^+$	$^3\mu_T^+$	$^3V_{LS}^+$	$^3\mu_{LS}^+$
(2)	0.2	181.1	1.085	130.3	1.1	0	-	0	-
(3)	0.3	595.7	1.75	230.0	1.26	26.0	0.6124	0	-
(4a)	0.4	877.4	2.091	434.0	1.45	159.4	1.045	5000	3.7
(4b)	0.4	726.7	1.955	434.0	1.45	121.0	0.978	0	-
(5)	0.5	6395.0	2.936	905.6	1.7	45.0	0.7342	0	-
(6)	0.6	10396.6	2.843	2097.5	2.0	0	-	0	-

TABLE 2. ODD-STATE PARAMETERS FOR POTENTIAL (4a).

potential code no.	r_c	$^3V_c^-$	$^3\mu_c^-$	$^1V_c^-$	$^1\mu_c^-$	$^3V_T^-$	$^3\mu_T^-$	$^3V_{LS}^-$	$^3\mu_{LS}^-$
(4a)	0.4	14.0	1.0	-130.0	1.0	-22.0	0.8	7315	3.7

TABLE 3. THE VALUE α_o FOR SELF-CONSISTENCY, AND THE CORRESPONDING ENERGY E_o IN MeV, AND THE r.m.s. RADIUS R_o.

potential	(2)	(3)	(4a)	(4b)	(5)	(6)
α_o	0.72	0.65	0.61	0.57	0.55	0.54
E_o	256	160	150	123	118	155
R_o	1.5	1.7	1.8	1.9	2.0	2.0

TABLE 4. POTENTIAL (2); NOTATION IS GIVEN IN 7.2(a).

α	0.4	0.5	0.6	0.7	0.8	0.72
$E(\alpha)$	-158.6	-215.2	-255.1	-259.4	-206.6	-256
$\delta E_p = \frac{1}{2}\alpha \frac{\partial E_p}{\partial \alpha}$		-186.0	-69.4	+42.7	+237.5	+70.0
$\delta E_s = \frac{1}{2}\alpha \frac{\partial E_s}{\partial \alpha}$		-157.9	-116.6	-81.3	-6.5	-70.0
$\delta E_p + \delta E_s$		-343.9	-186	-38.0	+231	0
ΔE_1		-71.1	-23.6	-8.3	-12.9	-6.8
T (k.e.)	238.9	373.2	537.4	761.3	854.5	
Coulomb	13.5	16.9	20.2	23.6	27.1	
$P(\alpha)$	-411	-605	-813	-1044	-1288	

TABLE 5. POTENTIAL (3).

α	0.3	0.4	0.5	0.6	0.7	0.65
$E(\alpha)$	-87.4	-134.4	-167.7	-177.0	-121.2	-160
δE_p	-95.2	-112.2	-86.8	-4.8	+167.1	+70
δE_s	-71.7	-98.8	-105.4	-83.1	-13.8	-70
$\delta E_p+\delta E_s$	-166.9	-211.0	-192.0	-87.9	+153.3	0
ΔE_1	-40.4	-41.2	-27.0	-11.3	+8.7	8.5

TABLE 6. POTENTIAL (4a).

α	0.3	0.4	0.5	0.6	0.7	0.61
$E(\alpha)$	-98.3	-159.9	-194.2	-154.5	.	-150
δE_p	-100.8	-116.6	-64.7	+52.0	.	+64
δE_s	-74.8	-101.5	-97.8	-67.6	.	-64
$\delta E_p+\delta E_s$	-175.6	-218.1	-162.5	-15.0	.	0
ΔE_1	-42.6	-40.5	-21.5	-8.1	.	-8.3

TABLE 7. POTENTIAL (4b).

α	0.3	0.4	0.5	0.6	0.57
$E(\alpha)$	-85.9	-137.7	-158.8	-103.8	-123.0
δE_p	-92.1	-106.0	-56.3	+94.9	+32
δE_s	-67.2	-90.6	-81.8	-11.8	-32
$\delta E_s+\delta E_p$	-159.3	-196.6	-138.1	+83.1	0
ΔE_1	-37.8	-35.2	-15.1	-3.5	-2.0

TABLE 8. POTENTIAL (5).

α	0.3	0.4	0.5	0.6	0.55
$E(\alpha)$	-84.2	-121.7	-135.5	-95.1	-118
δE_p	-96.0	-104.0	-43.4	+116.8	+34
δE_s	-68.9	-87.0	-70.2	+2.1	-34
$\delta E_p + \delta E_s$	-174.9	-191.0	-113.6	+118.9	0
ΔE_1	-38.5	-32.8	-11.8	-5.9	-2.8

TABLE 9. POTENTIAL (6).

α	0.3	0.4	0.5	0.6	0.54
$E(\alpha)$	-96.9	-141.4	-166.3	-132.0	-155.0
δE_p	-124.5	-123.0	-49.7	+151.6	+30
δE_s	-75.4	-88.3	-58.9	+45.6	-30
$\delta E_p + \delta E_s$	-199.9	-211.3	-108.6	+197.1	0
ΔE_1	-51.5	-35.6	-8.7	-13.2	-2.2

TABLE 10. COMPARISON OF CONTRIBUTIONS TO THE POTENTIAL ENERGY FROM PARTICLES IN THE s- AND p-SHELLS AT EXPERIMENTAL DENSITY.

potentials	(2)	(3)	(4a)	(4b)	(5)	(6)
α	0.4	0.4	0.4	0.4	0.4	0.4
W (s,s)	-53.4	-50.8	-55.2	-50.4	-49.5	-52.0
W (s,p)	-160.2	-152.4	-162.1	-151.3	-148.5	-156.0
W (p,p)	-197.4	-183.8	-194	-188.3	-176.4	-185.9
Total	-411.0	-387.0	-412.3	-390.0	-374.4	-393.9

TABLE 11. MATRIX ELEMENTS FROM (4a) FOR DIFFERENT VALUES OF α, WITH $\lambda^{oo} = -1$.

matrix element	weight factor	0.3	0.4	0.5	0.6
$^3t_c^+(1s,N)$	x 21	-28.4	-49.1	-62.7	-52.8
$'t_c^+(1s,N)$	x 21	-80.3	-136.7	-191.6	-225.8
$^3t_c^+(2s,0)$	x 1.5	-1.9	-1.8	-0.2	+5.0
$'t_c^+(2s,0)$	x 1.5	-5.6	-7.0	-6.4	-2.5
$^3t_c^+(1d,0)$	x 7.5	-0.3	-1.6	-5.0	-11.9
$'t_c^+(1d,0)$	x 7.5	-1.2	-5.1	-13.7	-28.2
$^3t_c^-(1p,0)$	x 54	-4.1	-8.3	-9.5	-0.1
$'t_c^-(1p,0)$	x 6	+5.5	+14.9	+30.3	+52.0
$^3t_T^+(1s,N)$	x 21	-118.9	-206.5	-302.2	-398.8
$^3t_T^+(2s,0)$	x 1.5	-11.2	-17.5	-23.6	-29.5

TABLE 12. MATRIX ELEMENTS FROM (5) FOR DIFFERENT VALUES OF α , WITH $\lambda = -1$.

matrix element	weight factor	0.3	0.4	0.5	0.6
$^3L_c^+(1s,N)$	x 21	-71.5	-127.3	-187.4	-232.8
$^1L_c^+(1s,N)$	x 21	-78.6	-132.1	-177.9	-187.1
$^3L_c^+(2s,0)$	x 1.5	-6.0	-8.2	-8.5	-4.7
$^1L_c^+(2s,0)$	x 1.5	-5.4	-6.2	-4.2	+3.0
$^3L_c^+(1d,0)$	x 7.5	-0.3	-1.8	-6.4	-16.9
$^1L_c^+(1d,0)$	x 7.5	-1.1	-5.0	-14.4	-31.0
$^3L_T^+(1s,N)$	x 21	-61.8	-87.2	-109.8	-129.3
$^3L_T^+(2s,0)$	x 1.5	-5.9	-6.7	-8.1	-9.1

TABLE 13. VARIATION OF THE POTENTIAL ENERGY FOR POTENTIAL (5) WITH THE ENERGY LEVEL SELF-CONSISTENCY PARAMETER λ^{00} OF THE UNPERTURBED s-LEVEL.

λ^{00} \ α	0.3	0.4	0.5	0.6
3	-263.6	-419.2	-587.4	-737.2
+1	-244.3	-394.0	(-548)	(-666)
-1	-230.6	-374.5	-516.7	-607.9
-3	-220.4	-358.6	-487.8	-551.0
k.e.	134.4	238.9	373.2	537.4
Coulomb	10.1	13.5	16.9	20.2

TABLE 14. THE VALUE OF THE s-STATE ENERGY PARAMETER λ^{00} FOR ENERGY LEVEL SELF-CONSISTENCY AT THE VALUE $\alpha = 0.5$.

potential	(2)	(3)	(4a)	(4b)	(5)	(6)
λ^{00}	-0.7	-0.8	-1	-1	-0.4	-0.6

References

1) See for example M.G. Mayer and J.H.D. Jenson: Elementary Theory of Nuclear Shell Structure. New York (1955).

2) J.S. Levinger, Nuc. Phy. 8 (1958) 428 where references to earlier work may also be found.

3) The review article of J.L. Gammel and R.M. Thaler in Elementary Particle and Cosmic Ray Physics (1959), Amsterdam, gives exhaustive references.

4) For a summary see R.J. Eden, Nuclear Reactions Vol. I, (1958) Amsterdam.

5) K.A. Brueckner and J.L. Gammel, Phy. Rev. 109 (1958), 1023, where references to the earlier papers of Brueckner and his collaborators may be found.

6) H.A. Bethe, Phy. Rev. 103 (1956) 1353.

7) A connected account may be found in -
K.A. Brueckner, The Many Body Problem. (1958).

8) J. Goldstone, Proc. Roy. Soc., A.239 (1957) 267.
R.J. Eden, Phy. Rev. 99 (1955) 1418.

9) R.J. Eden and V.J. Emery, Proc. Roy. Soc. A.248 (1958) 266.

10) R.J. Eden, V.J. Emery and S. Sampanthar, Proc. Roy. Soc. A.253 (1959) 177.

11) R.J. Eden, V.J. Emery and S. Sampanthar, Proc. Roy. Soc. A. 253 (1959) 186.

12) J.L. Gammel and R.M. Thaler, Phy. Rev. 107 (1957) 291.

13) J.L. Gammel and R.M. Thaler, Phy. Rev. 107 (1957) 1337.

14) P. Signell and R. Marshak, Phy. Rev. 109 (1958) 1229.

15) Besides the review article of 3) that of R.J.N. Phillips in Reports on Progress in Physics, Vol. XXII (1959) 562, may be consulted for references to the literature on the two nucleon interaction.

16) S. Gartenhaus, Phy. Rev. 100 (1955) 900.

17) See for example D.G. Ravenhall, Rev. Mod. Phy. 30 (1958) 430.

18) J.L. Blatt and V. Weisskopt, Theoretical Nuclear Physics (1952) New York.

19) This assumption is termed the Adiabatic Hypothesis in the literature: See for example Schweber, Bethe and Hoffman, Mesons and Fields, Vol I (1954) Row Peterson.

20) E. Wigner, Math u naturw. Anz Ungur. Akad Wiss. 53 475.

21) E. Schrödinger, Ann. Phy. Lpz. 80 (1926) 437.

22) See also N.M. Hugenholtz, Physica 23 (1957) 481.

23) See for example L. Schiff, Quantum Mechanics. McGraw - Hill (1955).

24) K.A. Brueckner, Phy. Rev. 100 (1953) 36.

25) In unpublished work C.F. Clement, J. Nuttall and S. Sampanthar confirm the main conclusions of Ref.[5]

26) That the presence of attractive forces near the Fermi surface could change the level structure has been shown for instance by A Bohr, B.R. Mottelson and D. Pines, Phy. Rev. 110 (1958) 936.

27) K.A. Brueckner, Phy. Rev. 110 (1958) 597.
N.M. Hugenholtz and L. Van Hove, 24 (1958) 363.

28) K.A. Brueckner, J.L. Gammel and J.T. Kubis, Phy. Rev. 118, (1960) 1438.

29) V.J. Emery has shown that the partial summation which defines the t matrix could be singular.

30) For a critique of the Goldstone method see W. Kohn and J.M. Luttinger, Phy. Rev. 118 (1960) 41.

31) V. Fock, Z. Phys. 61 (1930) 126.
D.R. Hartree. The calculation of Atomic Structures (1957) Chapman and Hall.

32) R.J. Eden, Nuc. Phy. 9 (1958) 65.

33) I. Talmi, Helv. Phy. Acta. 25 (1952) 185.

34) These are defined for instance in A.R. Edmond's Angular Momentum in Quantum Mechanics, Princeton (1957).

35) J. Ashkin and T.Y. Wu, Phy. Rev. 73 (1948) 973.

36) J.L. Gammel, R.S. Christian and R.M. Thaler, Phy. Rev. 105, (1957) 311.

37) T. Ohmura, M. Morita and M. Yamada, Prog. Theor. Phys. 17 (1957) 326.

38) J. Dabrowski, Proc. Phy. Soc. 71 (1958) 658.

39) R. Jastrow, Phy. Rev. 98 (1955) 1479.

40) V.J. Emery, Nuc. Phy. 6 (1958) 585.

41) K.A. Brueckner, J.L. Gammel and H. Weitzner, Phy. Rev. 110 (1958) 431.

K.A. Brueckner and D.T. Goldman, Phy. Rev. 117 (1960) 207.

K.A. Brueckner, A.M. Lockett and M. Rotenberg, Phy. Rev. 121 (1961) 255.

FIGURE CAPTIONS

Figure 1. n_1' and n_2' are the total quantum numbers for the intermediate states of particles 1 and 2. Points marked with a cross indicate sets of excluded states and points marked with a circle indicate sets of allowed states in the usual form of the exclusion principle. The modified form of exclusion principle excludes only those states below and on the line $n_1' + n_2' = 4$ which is indicated

Figure 2. $E(\alpha)$ denotes the total expectation value of the energy; ΔE_1 denotes the second order correction term to the energy; α_0 denotes the value of α for self-consistency.

Figure 3. δE_s denotes the derivative energy for the s-state particles. δE_p denotes the derivative energy for the p-state particles. $\delta E_p + \delta E_s = 0$ determines the best self-consistency, at α_0.

Figure 4. $\varkappa^{-2}$ times pair correlations for (1) perturbed and (2) unperturbed systems; $\varkappa = \alpha r$, $\alpha = 0.5\ f^{-1}$. Solutions are for (4b).

Figure 5. The ratio perturbed to unperturbed pair correlation function for potential (4b) ($\varkappa = \alpha r$, $\alpha = 0.5\ f^{-1}$).

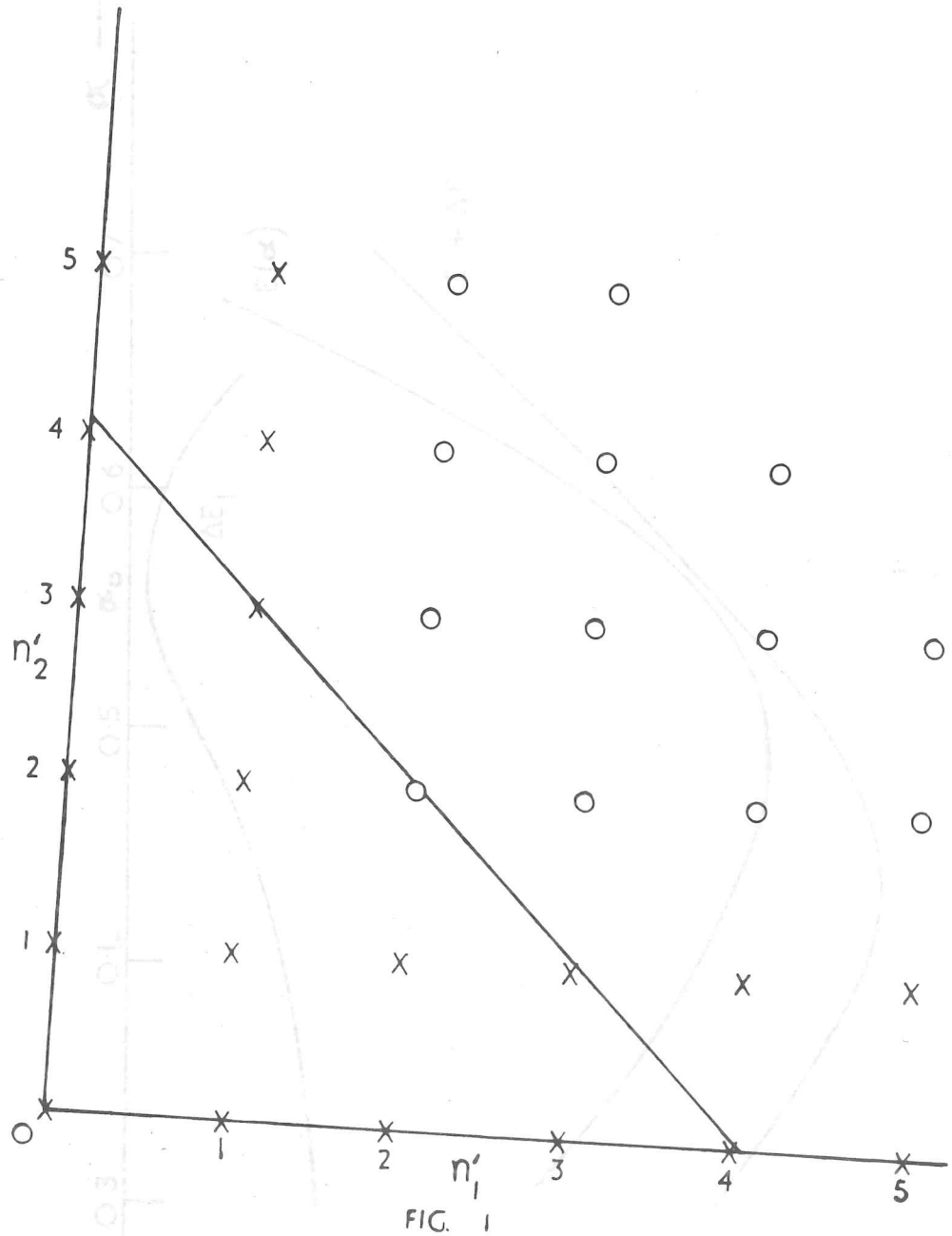


FIG. 1

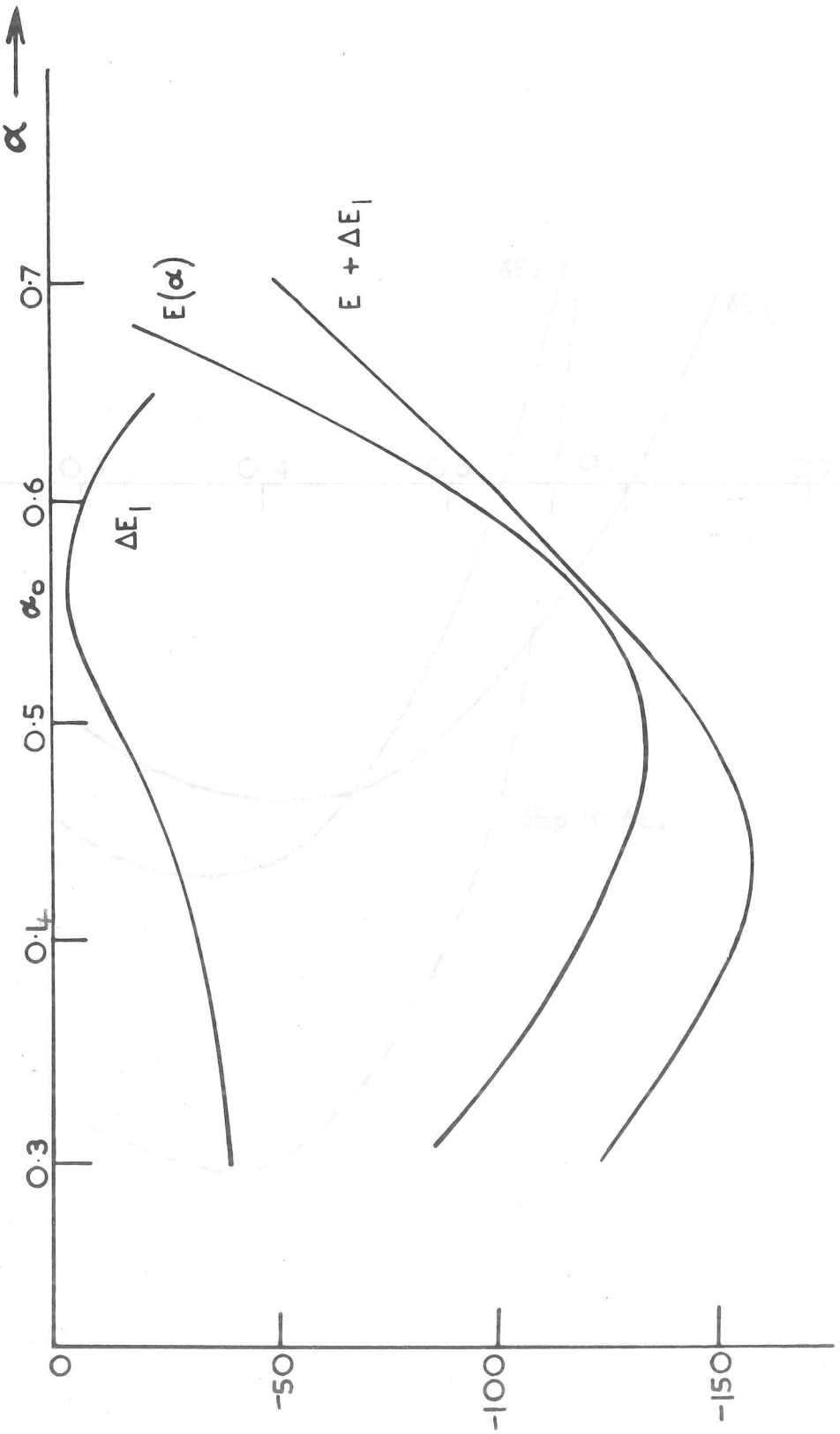


FIG 2

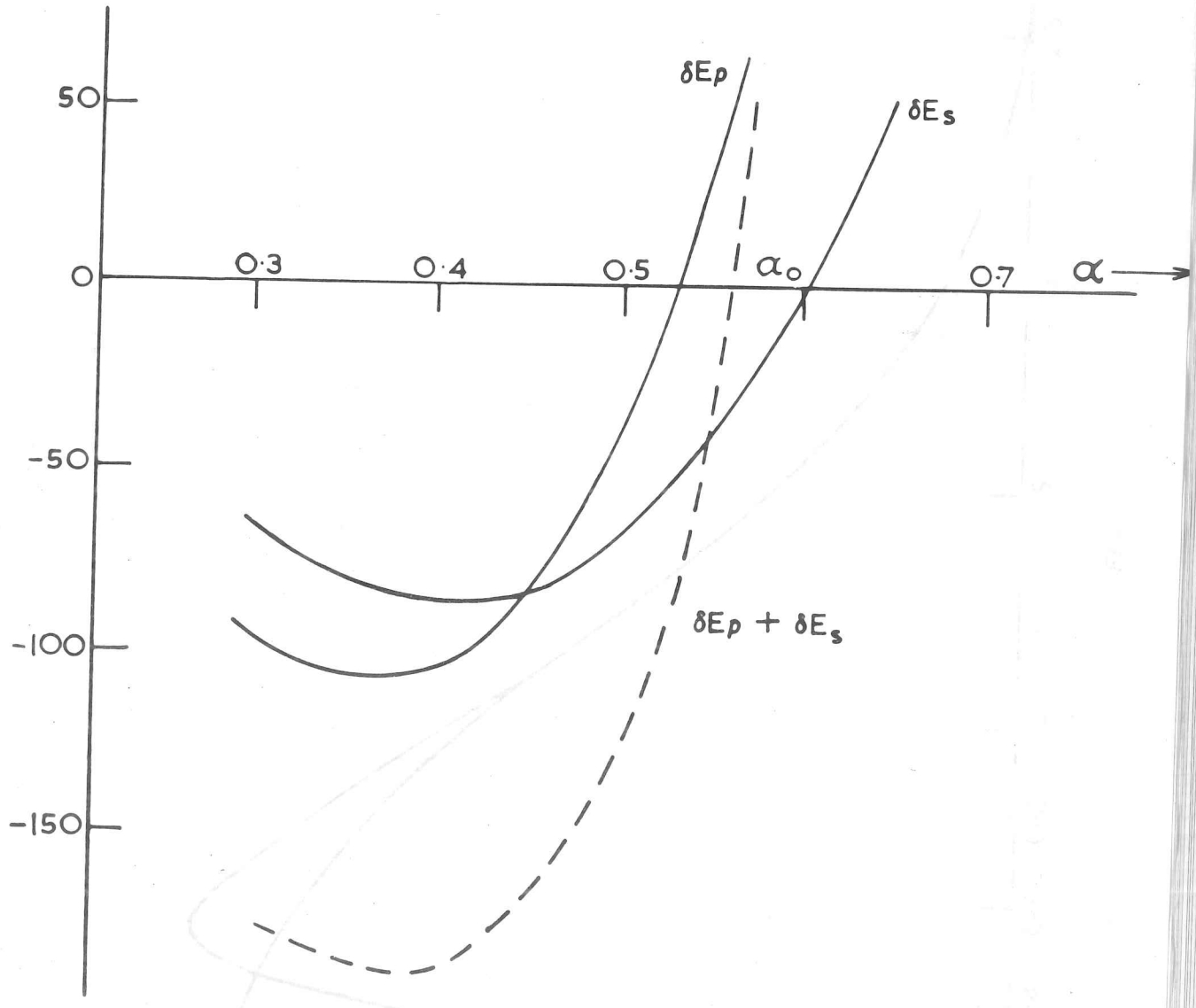


FIG. 3

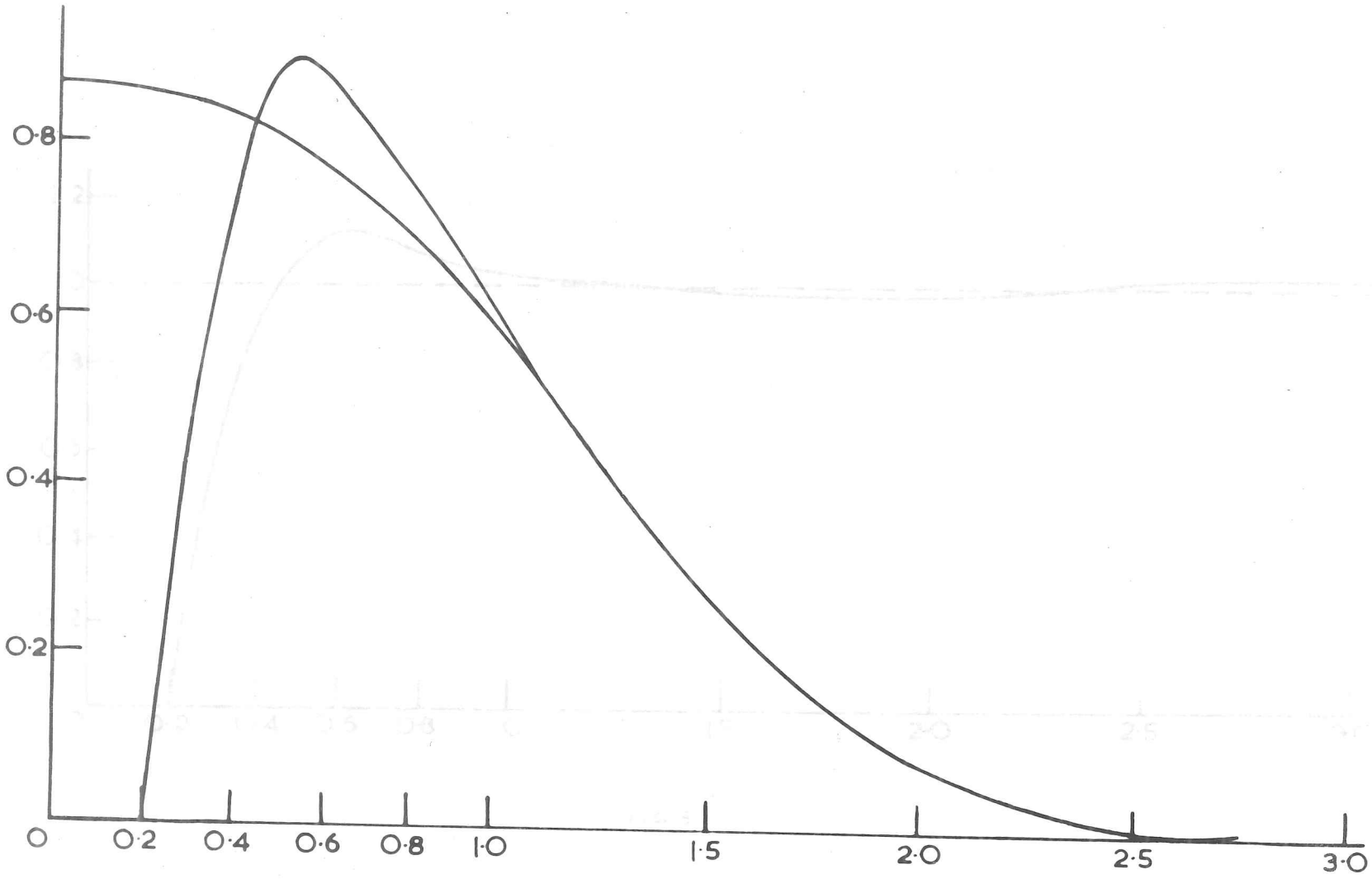


FIG. 4

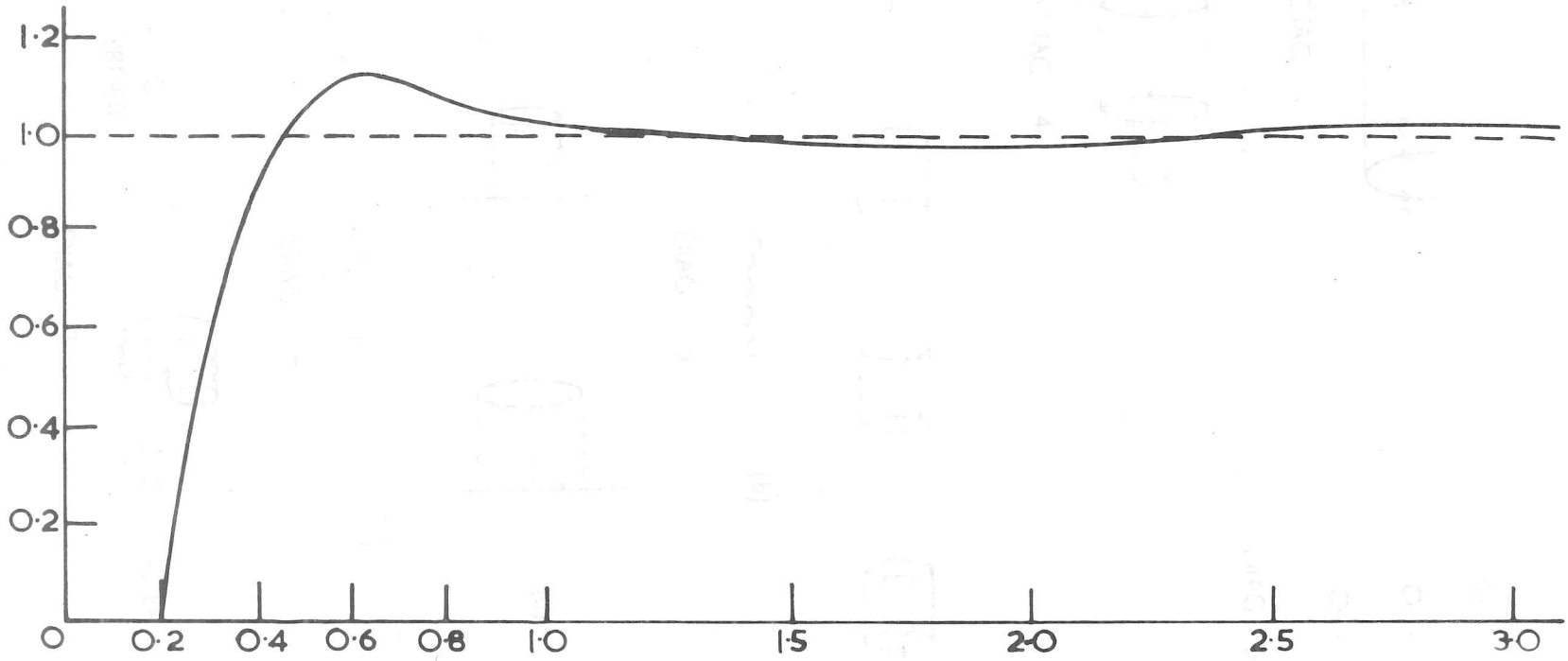
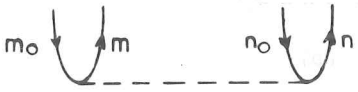
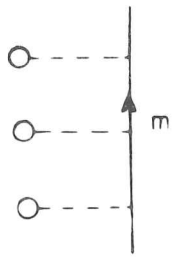


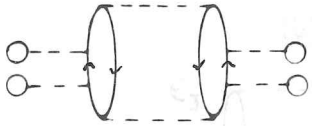
FIG. 5



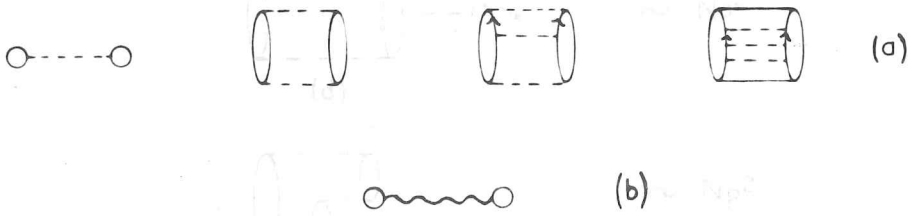
DIAG. 1.



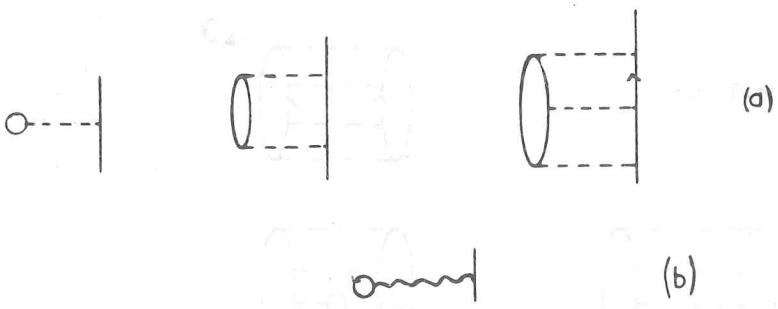
DIAG. 3.



DIAG. 4.

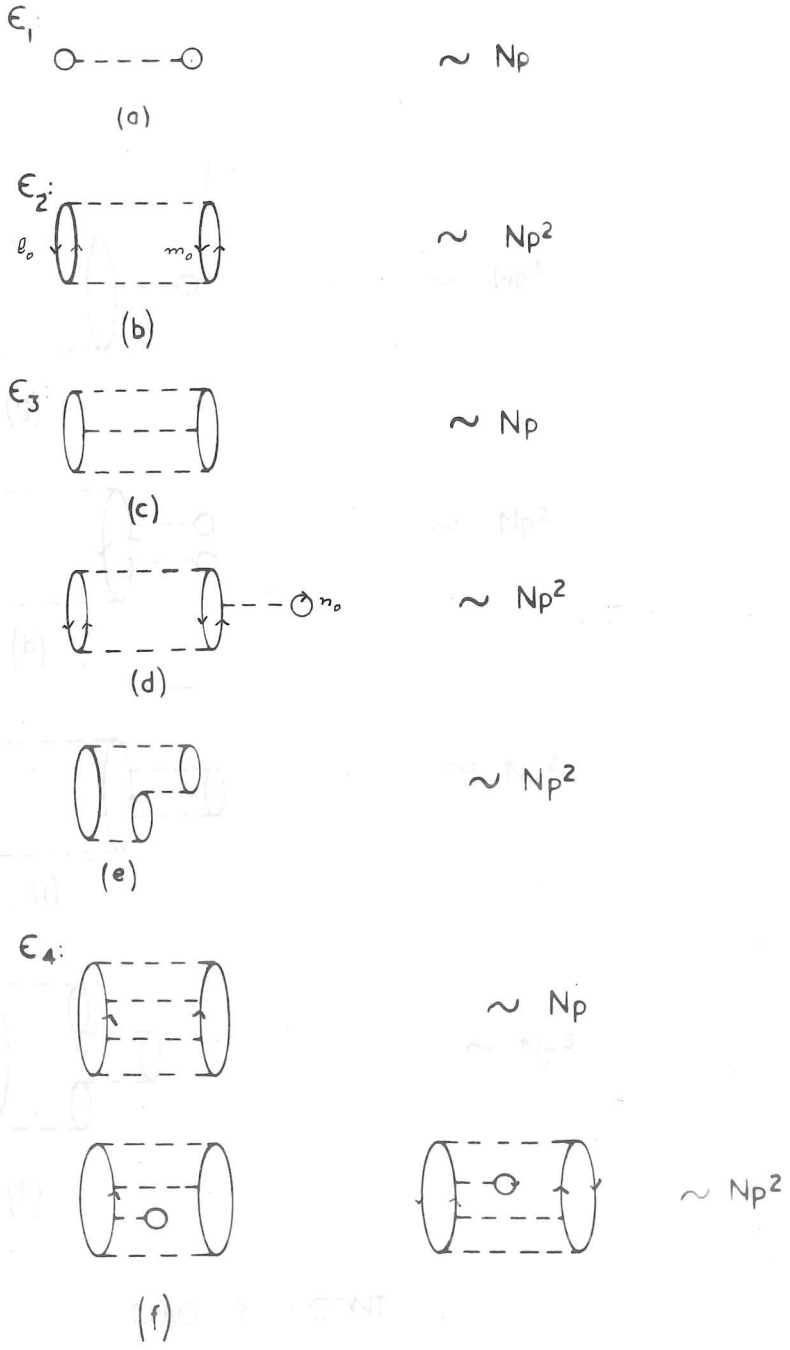


DIAG. 5.



DIAG. 6.

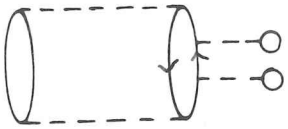
DIAG. 7.


ϵ_1
$\sim Np$
(a)
ϵ_2:
ℓ_o
m_o
$\sim Np^2$
(b)
ϵ_3
$\sim Np$
(c)
n_o
$\sim Np^2$
(d)
$\sim Np^2$
(e)
ϵ_4:
$\sim Np$
$\sim Np^2$
(f)

DIAG. 2

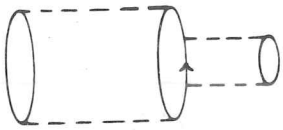
$\sim Np^3$

(g)



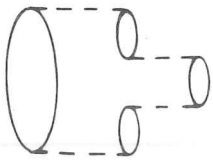
$\sim Np^2$

(h)



$\sim Np^2$

(i)



$\sim Np^3$

(f)

DIAG. 2. CONT.

Part II

Binding Energies of Λ Particles and the Λ-Nucleon Interaction

Contents

CHAPTER V

Λ POTENTIAL IN NUCLEAR MATTER DUE TO TWO-BODY FORCES

CHAPTER VI

Λ POTENTIAL IN NUCLEAR MATTER DUE TO THREE-BODY FORCES

CHAPTER VII

THE Λ-N INTERACTION AND THE Λ POTENTIAL IN NUCLEAR MATTER

CHAPTER I

1 Introduction:

Several authors [1 - 6], and in particular Dalitz and Downs [1 - 3], have analysed the binding energies of Λ particles in the light hypernuclei in order to determine the Λ- N interaction. This interaction has mostly been assumed to be charge symmetric and two-bodied. In these hypernuclei with $A \leq 5$, where A denotes the total mass number inclusive of the Λ, the Λ- N interactions occur essentially only in relative s-states. Hence only central forces need be considered and there is no way of distinguishing between ordinary and exchange forces.

In accordance with the accepted conservation laws of strong interactions, and meson theory,[7] the direct Λ- N interaction is expected to have a maximum Yukawa range $\mu_{2\pi}^{-1} = \frac{\hbar}{2 m_\pi c} = .7$ f corresponding to an exchange of two pions while the exchange interaction has a maximum range $\mu_K^{-1} = \frac{\hbar}{m_K c} = .397$ f corresponding to the exchange of a K meson. The volume integrals of the Λ - N interaction obtained from the analysis of the hypernuclei with $A \leq 5$ depend significantly on the range assumed for the interaction, i.e. whether $\mu_{2\pi}^{-1}$ or μ_K^{-1}.

This range dependence, together with the contributions from higher relative angular momentum states for a Λ bound in a heavy nucleus, makes us believe that a knowledge of Λ

binding energies in heavier hypernuclei may enable us to distinguish between ordinary and exchange forces. It appears that experimental information on such binding energies may emerge from the use of Uranium loaded emulsions.[8] Less direct information has already been obtained both by Filimonov[9] and by Walecka[10] who have extrapolated to large A the presently known Λ binding energies for the hypernuclei with $A \leq 12$[11]. There is a trend towards a limiting value V_o, expected for large A because of nuclear saturation and because the Λ-particle being distinct from the nucleons can always be in the lowest possible (1 s) state. For large A the kinetic energy of the Λ will be approximately proportional to $A^{-2/3}$ and hence V_o will correspond to the potential depth felt by a Λ of zero momentum in nuclear matter. Walecka obtained $V_o \approx 23$ MeV and has discussed the implications of this value for two body Λ - N interactions with a hard core.

However three-body Λ - N interactions (arising in lowest order through the exchange of one pion with each of a pair of nucleons) are expected to be of comparable importance to the two-body direct interactions. To see what extra information knowledge of the Λ binding energies in heavier hypernuclei will give it is therefore necessary to consider also three-body interactions and to extract (from the light hypernuclei) as much information as possible about these and the two-body interactions.

Part II of the dissertation will therefore be concerned with the implications of Λ binding energies for the Λ-nucleon interaction when three-body forces are also included.

In § 2 we discuss the nature of our assumed $\Lambda - N$ interaction.

Chapters II and III are devoted to the two hypernuclei ${}_{\Lambda}H^{3}$ and ${}_{\Lambda}He^{5}$ which give the most reliable information. For ${}_{\Lambda}H^{3}$, the effect of three-body forces is small and is obtained by a perturbation procedure. The Λ binding is almost entirely due to two-body forces. For ${}_{\Lambda}He^{5}$ the core is assumed to be undistorted by the Λ.

In Chapter IV other light hypernuclei are considered, showing that ${}_{\Lambda}H^{4}$ and ${}_{\Lambda}He^{4}$ do not seem capable of yielding any significant information while the two-body system ${}_{\Lambda}H^{2}$ is not bound unless the three-body forces are very strongly attractive. This chapter completes the work on light hypernuclei.

Chapters V and VI are devoted to the calculation in perturbation theory of the potential felt by a Λ in nuclear matter, this being considered mainly as a Fermi gas. Both direct and exchange two-body forces are considered as the second order contribution, the effect of nuclear pair correlations and the velocity dependence are all small.

The nucleon exchange factor in the three-body forces strongly suppresses their effect in nuclear matter.

Chapter VII concludes Part II of the dissertation. Here the implications of the empirical values of the Λ potential depth are considered. In particular if the pion-baryon couplings predominate then strongly attractive three-body forces of not too large a range are favoured. This corresponds to a larger singlet than triplet two-body volume integral. ✱

✱ It is hoped that most of the material of Part II of the dissertation will appear in Nuclear Physics in a joint article with Dr. A.R. Bodmer.

---

2 The Λ- N Interaction:

We assume conventional two-body interactions like those of ref. [1 - 3] and three-body interactions having shape and exchange character suggested by meson theory [12]. Even though such Λ- N interactions are considerably simpler than the actual interactions they may be expected to give a fairly reasonable representation of the latter when Λ binding energies are considered. These simplified interactions may be investigated in a fairly systematic way, enabling us to obtain some insight into those aspects of the actual Λ- N interactions represented by the simplified forces used. Perhaps the most essential simplification is the neglect of the Σ channel: $\Lambda \rightleftarrows \Sigma + \pi$

The conventional forces assumed enable us to use straightforward perturbation theory and by going to the second order we are able to investigate the usefulness of the first order or independent particle approximation, in the calculation of the potential felt by a Λ in nuclear matter. Furthermore there is little uncertainty with soft forces in using the interaction strengths obtained from the light hypernuclei for predicting the potential depths in nuclear matter. With hard core forces the difficulties will be much greater. This is mainly because the relative Λ-Nucleon wave function in the deep attractive region outside the hard core must be accurately known if reliable

results are to be obtained for the binding energy, which is essentially a small difference between the large correlation kinetic energy due to the hard core and the large potential energy due to the attractive tail. This balance may well be different in light and in heavy hypernuclei leading to additional uncertainties when the results for these are considered in relation to each other. Finally, short of much more Λ-nucleon scattering data, the existence of a hard core in the $\Lambda - N$ interaction and even less the value of its radius can hardly be considered as established. For these reasons we have not considered hard core $\Lambda - N$ interactions in our work.

We shall almost exclusively consider charge symmetric spin dependent two-body Yukawa interactions

$$V(r) = -V \frac{e^{-\mu r}}{\mu r} = -U v(r) \qquad (1.1)$$

where U, the volume integral is given by

$$U = -\int_{\text{all space}} dr\, V(r) = \frac{4\pi}{\mu^3} V$$

and

$$v(r) = \frac{\mu^3}{4\pi} \frac{e^{-\mu r}}{\mu r} \qquad (1.2)$$

is the normalised interaction of unit strength.

For the short ranges ($\mu_{2\pi}^{-1}$ and μ_{κ}^{-1}) considered interactions of different shape but having the same intrinsic range were found by Dalitz and Downs to have approximately the same volume integrals (to within about 10% [4]). The

shape used for the two-body interactions is thus relatively unimportant.

For three-body interactions we use the following form consistent with the meson theoretically expected central interactions for large separations [12]

$$W(1,2,\lambda) = w(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)\,W(r_1, r_2, r_\lambda) \qquad (1.3)$$

where $1, 2, \lambda$ denote the co-ordinates of the nucleons and of the Λ and w measures the strength of the interaction. Meson theoretical considerations [12] prefer the exponential for the shape factor W with range $\nu^{-1} = \nu_\pi^{-1} = \frac{\hbar}{m_\pi c} = \frac{1}{.72}\,f$ corresponding to an exchange of one pion with each of two nucleons. Higher order processes, appropriate to an exchange of more mesons will correspond to a shorter range and may be expected to have a smaller effect. Nevertheless because of these terms the precise form and range used for W cannot be too significant. Therefore for the shape factor we consider both an exponential and a Yukawa form:

$$= W_E(r_{1\lambda}, r_{2\lambda}) = e^{-\nu(r_{1\lambda}+r_{2\lambda})} \qquad (1.4a)$$

$$W(r_1 r_2 r_\lambda) = W_Y(r_{1\lambda}, r_{2\lambda}) = \frac{e^{-\nu(r_{1\lambda}+r_{2\lambda})}}{\nu^2(r_{1\lambda}+r_{2\lambda})} \qquad (1.4b)$$

where $r_{i\lambda} = |r_i - r_\lambda|$, $i = 1, 2$, and also consider three values of ν for each shape, viz.

$$\nu = .5,\ .72 \text{ and } 1f$$

For a given ν the exponential form has a relatively longer

shape used for the two-body interactions is thus relatively unimportant.

For three-body interactions we use the following form consistent with the meson theoretically expected central interactions for large separations [12]

$$W(1,2,\lambda) = w(\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) W(r_1, r_2, r_\lambda) \qquad (1.3)$$

where $1, 2, \lambda$ denote the co-ordinates of the nucleons and of the Λ and w measures the strength of the interaction. Meson theoretical considerations [12] prefer the exponential for the shape factor W with range $\nu^{-1} = \nu_\pi^{-1} = \frac{\hbar}{m_\pi c} = \frac{1}{.72}\,f$ corresponding to an exchange of one pion with each of two nucleons. Higher order processes, appropriate to an exchange of more mesons will correspond to a shorter range and may be expected to have a smaller effect. Nevertheless because of these terms the precise form and range used for W cannot be too significant. Therefore for the shape factor we consider both an exponential and a Yukawa form:

$$= W_E(r_{1\lambda}, r_{2\lambda}) = e^{-\nu(r_{1\lambda} + r_{2\lambda})} \qquad (1.4a)$$

$$W(r_1 r_2 r_\lambda) = W_Y(r_{1\lambda}, r_{2\lambda}) = \frac{e^{-\nu(r_{1\lambda} + r_{2\lambda})}}{\nu^2(r_{1\lambda} + r_{2\lambda})} \qquad (1.4b)$$

where $r_{i\lambda} = |r_i - r_\lambda|$, $i = 1, 2$, and also consider three values of ν for each shape, viz.

$$\nu = .5,\ .72 \text{ and } 1f$$

For a given ν the exponential form has a relatively longer

range than the Yukawa form. Since 'physically' the maximum range corresponds to an exponential with $\nu = .72\ f^{-1}$ the results for W_E with $\nu < \nu_\pi$ cannot be regarded as physically meaningful.

For even relative angular momentum states of the two nucleons one has $(\sigma_1 \cdot \sigma_2)(\tau_1 \cdot \tau_2) = -3$. Instead of w it is therefore convenient to use the effective s state strength $w_S = -3w$, since for $A \leq 5$ the three-body interactions occur predominantly with pairs of nucleons in relative s-states. It is also useful to define

$$U_3 = \begin{matrix} U_{3E} = \left(\frac{8\pi}{\nu^3}\right)^2 w_S \\ U_{3Y} = \left(\frac{4\pi}{\nu^3}\right)^2 w_S \end{matrix} \qquad (1.5)$$

These three-body volume integrals are the analogues of the two-body volume integrals in the sense that for very short three-body ranges the corresponding potential energy will depend on U_3 and not on ν.

CHAPTER II

1 Volume Integrals for the Light Hypernuclei:

Dalitz and Downs [1-3] have carried out an extensive analysis of the light hypernuclei particularly of ${}_{\Lambda}H^{3}$ and ${}_{\Lambda}He^{5}$. From the empirical Λ binding energies $B_{\Lambda}^{(A-1)}$, assuming two-body $\Lambda - N$ interactions of given Yukawa range they have determined the volume integrals $U_2^{(A-1)}$ of the effective interactions in these hypernuclei. The superscript $(A-1)$ refers to the hypernucleus ${}_{\Lambda}Z^{A}$ of mass number A. With spin 1/2 for Λ and assuming that the interactions are effective only in relative s-states, the ground state volume integrals $U_2^{(A-1)}$ for $A \leq 5$ are given in terms of the singlet and triplet $\Lambda - N$ volume integrals U_s and U_t, by the following expressions:

Case ___:	$U_s > U_t$	J
$U_2^{(2)}$	$= \frac{3}{2} U_s + \frac{1}{2} U_t$	$\frac{1}{2}$
$U_2^{(3)}$	$= \frac{3U_s}{2} + \frac{3}{2} U_t$	0
$U_2^{(4)}$	$= U_s + 3 U_t$	$\frac{1}{2}$
Case ___:	$U_t > U_s$	J
$U_2^{(2)}$	$= 2U_t$	$\frac{3}{2}$
$U_2^{(3)}$	$= \frac{1}{2} U_s + \frac{5}{2} U_t$	1
$U_2^{(4)}$	$= U_s + 3 U_t$	$\frac{1}{2}$

(2.1)

J is the appropriate ground state spin for the hypernucleus.

It seems most probable that $U_s > U_t$, this being consistant with zero spin for $_{\Lambda}He^4$ and $_{\Lambda}H^4$ and a pseudo-scalar K meson. Though we shall emphasise this case, the results for $U_t > U_s$ will also be discussed.

Dalitz and Downs [1-2] have been able to treat the three-body system $_{\Lambda}H^3$ by a variational method using a sufficiently flexible trial wave function. For $_{\Lambda}He^5$ because of the stability of the core nucleus He^4 the Λ may be considered to a good approximation as moving in the potential well produced by the nucleons of the undistorted core nucleus. The density distribution of this is reasonably well determined from electron scattering data.

Since the density distributions of the core nuclei He^3 and H^3 of $_{\Lambda}He^4$ and $_{\Lambda}H^4$ are not known with adequate accuracy[3] and because the distortion of the core nuclei by the Λ for these hypernuclei are expected to be important the results from these hypernuclei will be uncertain. From eqn. (2.1) we deduce

$$U_2^{(3)} = \tfrac{3}{4} U_2^{(2)} + \tfrac{3}{8} U_2^{(4)} \quad \text{for } U_s > U_t$$

$$\text{and } U_2^{(3)} = \tfrac{1}{2} U_2^{(2)} + \tfrac{1}{2} U_2^{(4)} \quad \text{for } U_t > U_s \qquad (2.2)$$

Thus for only two-body forces the values of $B_{\Lambda}^{(3)}$ provide only a check which is approximately satisfied [3]. Although the inclusion of three-body forces could in principle give new information, because of the uncertainties mentioned, we do not consider these hypernuclei in any detail. The rest of

this chapter is devoted to the hypertritron and in Chapter III we consider $_{\Lambda}He^{5}$.

2 Dalitz and Downs' Analysis for $_{\Lambda}H^{3}$:

The trial function used in ref.[2] for studying the loosely bound three-body system $_{\Lambda}H^{3}$ was

$$\psi = \prod_{i=1}^{3} \sum_{\alpha=1}^{2} \lambda_{i\alpha} e^{-a_{i\alpha} r_i} \qquad (2.3)$$

where

$$\lambda_{i1} = 1 \quad \text{all } i \; ; \quad \lambda_{12} = \lambda_{22} = x \; ; \quad \lambda_{32} = y \; ;$$

$$a_{11} = a_{21} = a \; ; \qquad a_{31} = a_3$$

$$a_{12} = a_{22} = b \; ; \qquad a_{32} = b_3$$

and where a, b, a_3, b_3, x and y are six parameters. The r_i are interparticle distances, r_3 being the internucleon separation. The r_i are triangular co-ordinates and hence satisfy the usual triangular inequalities: $r_1 + r_2 \geq r_3$, etc.

The requirement that the minimum energy be equal to the empirical binding energy then determines the volume integral $U^{(2)}$ and the value of the six parameters which specify the 'best' wave function. We use $U^{(2)}$ in contrast to $U_2{}^{(2)}$ to denote the volume integral obtained in the absence of three-body interactions.

The wave function (2.3) is sufficiently flexible to allow for (a) the strong correlations between the Λ and the nucleons necessary because of the short range of the Λ - N interactions, (b) the long tail of the wave function for separation of the Λ

from the nucleons and (c) the lack of symmetry between the Λ and the nucleons, the separation energies of the latter being considerably larger than that of the Λ. The wave function (2.3) probably gives a very good description of ${}_{\Lambda}H^3$ since the much less flexible two parameter (α, β) wave function ¥

$$\psi = e^{-\alpha(r_1+r_2)-\beta r_3} \qquad (2.4)$$

of ref.[1] already gives results for $U^{(2)}$ which are only about 15% larger than those from the wave function (2.3).

In ref.[2] the N-N interaction was chosen to be of Yukawa form and to have the necessary strength and range to reproduce the s-wave triplet data. Dalitz and Downs [2] give results for $U^{(2)}$ and for the six parameters specifying the wave function (2.3) for $B_{\Lambda}^{(2)} = 0$, .25 and 1 MeV and for the two ranges $\mu_{2\pi}^{-1}$ and μ_{K}^{-1}.

The best value of $B_{\Lambda}^{(2)}$ given in ref.[11] is .12 $\pm$ .34 MeV. The results obtained for $U^{(2)}$ with the wave function (2.4) were found to be insensitive to the shape of the Λ - N interaction if this has a given intrinsic range.

3 Our Perturbation Procedure:

${}_{\Lambda}H^3$, because of its small binding energy is a rather open

---

¥ This is in fact a special case of the wave function (2.3) obtained from it when $x = y = 0$, $a = \alpha$ and $a_3 = \beta$.

system. Hence the probability of finding all three particles sufficiently close together for the three-body forces to be effective is small. Hence for comparable strengths of the two and three-body Λ – N interactions the potential energy due to the latter will be small compared with that due to the former.

We may therefore obtain to a good approximation the relation between $U_2^{(2)}$ and U_3 consistent with a given $B_\Lambda^{(2)}$ by the following perturbation procedure. We assume, that for a given $B_\Lambda^{(2)}$ the values of the parameters of the wave function (2.3) which minimise the energy when only two-body forces are used remain unchanged to a good approximation when also three-body forces are included. Then the total potential energy due to the Λ – N interactions, when three-body forces are also included, must to a good approximation be the same as the potential energy obtained by Dalitz and Downs with only two-body forces.

The potential energy due to the two-body Λ – N interactions is for the wave function (2.3),

$$N^{-1}\left(\frac{\mu^3}{4\pi}\right) U_2^{(2)} \times$$

where

$$\times = \int \psi^+ \left(\frac{e^{-\mu r_1}}{\mu r_1} + \frac{e^{-\mu r_2}}{\mu r_2} \right) \psi \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \qquad (2.5)$$

Here $N = \int |\psi|^2 \, r_1 \, r_2 \, r_3 \, dr_1 \, dr_2 \, dr_3$ is the normalisation integral and $U_2^{(2)}$ is given by eqn. (2.1).

Similarly the three-body potential energy is $N^{-1} \omega_3 \gamma$

where $$Y = \int \psi^{+} W(r_1 r_2) \psi \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3 \qquad (2.6)$$

The details of the evaluation of these integrals are given in the Appendix .

Our perturbation procedure then gives

$$\frac{\mu^3}{4\pi} U^{(2)} X = \frac{\mu^3}{4\pi} U_2^{(2)} X + \omega_s Y$$

Here the l.h.s. is proportional to the total potential energy when no three-body forces are present. In terms of U_3 this relation may be written

$$U_3 = Z^{(2)} \left(U^{(2)} - U_2^{(2)} \right) \qquad (2.7)$$

with

$$Z^{(2)} = \begin{cases} Z_E^{(2)} = \frac{X}{Y_E} \left(\frac{16\pi\mu^3}{\nu 6} \right) & \text{for } W_E \\ Z_Y^{(2)} = \frac{X}{Y_Y} \left(\frac{4\pi\mu^3}{\nu 6} \right) & \text{for } W_Y \end{cases}$$

In Table I are displayed the values of $Z^{(2)}$ and $U^{(2)}$ for the various values of $B_\wedge^{(2)}$ and the ranges considered. The results for $B_\wedge^{(2)} = .12$ MeV have been obtained by suitable interpolation. The values of $U^{(2)}$ are those of ref.[2]. ✱

✱ The values of $Z^{(2)}$ and $U^{(2)}$ for the simpler wave function (2.4) and for W_Y with ν_π are:

For $B_\wedge^{(2)} = 0$ MeV: $Z^{(2)}(\mu_{2\pi}) = 257\ f^3$, $Z^{(2)}(\mu_K) = 309.8\ f^3$

$U^{(2)}(\mu_{2\pi}) = 706\ \text{MeV}\, f^3$, $U^{(2)}(\mu_K) = 489\ \text{MeV}\, f^3$

For $B_\wedge^{(2)} = .4$ MeV: $Z^{(2)}(\mu_{2\pi}) = 246.8\ f^3$, $Z^{(2)}(\mu_K) = 302\ f^3$

$U^{(2)}(\mu_{2\pi}) = 786\ \text{MeV}\, f^3$, $U^{(2)}(\mu_K) = 494\ \text{MeV}\, f^3$

Consistent with the above procedure the ratio of the three-body potential energy to the total potential energy due to the Λ – N interactions is

$$\xi^{(2)} = \frac{U^{(2)} - U_2^{(2)}}{U^{(2)}} \qquad (2.8)$$

Even for $\xi^{(2)}$ not particularly small the above perturbation procedure may still be expected to be fairly accurate. Thus for a given $B_\Lambda^{(2)}$ the wave function for large separations of the Λ from the nucleons will be determined by $B_\Lambda^{(2)}$ and hence because of the diffuseness of ${}_\Lambda H^3$ the wave function as a whole will not be expected to depend too critically on $\xi^{(2)}$ so long as this is not too large. Since from the variational principle it follows that the first order changes in the wave function affect our perturbation procedure only to second order, it follows that this procedure is expected to be quite accurate especially since $\xi^{(2)}$ will in fact turn out to be quite small. ✱

✱ Rather direct supporting evidence for the accuracy of our perturbation procedure comes from using this to obtain for just two-body forces the value of $U^{(2)}$ for, say, the range μ_K^{-1} using the wave function appropriate to the range $\mu_{2\pi}^{-1}$ (or vice versa). Thus with the simpler wave function, (2.4), and with the appropriate parameters for $\mu_{2\pi}^{-1}$ given in ref.[1]), the value for μ_K^{-1} so found (for $B_\Lambda^{(2)} = 0$ MeV) is 524 MeV f^3 compared with 489 MeV f^3 obtained in ref.[1]) by the variational method. The corresponding results for $\mu_{2\pi}^{-1}$ are 822 MeV f^3, using the perturbation procedure based on the wave function for μ_K^{-1}, as against 766 MeV f^3 obtained by the variational method. Thus even though in this case there is quite a large difference between the potential energies and the wave function for the two ranges, the perturbation procedure nevertheless gives results accurate to better than 10%.

However for large $\xi^{(2)}$ of the order of unity the perturbation procedure will be of limited accuracy. Thus if eqn. (2.7) were exact, the values of $Z^{(2)}\ U^{(2)}$, i.e. the value of U_3 in the absence of two-body interaction - should be independent of the two-body interaction (i.e. of μ), and should depend only on the shape and range of the three-body interaction. From Table I we see that this is not so. The ratio of $Z^{(2)}U^{(2)}$ for $\mu_{2\pi}$ to that for μ_K varies from approximately .75 for the three-body interaction of the longest range (i.e. W_E with ν = .5 f^{-1}) to approximately 1.28 for that of the shortest range (i.e. W_Y with $\nu = 1\ f^{-1}$). Hence we conclude that the exact relation corresponding to eqn. (2.7) is not in general linear. This point is discussed further in §3.3.

CHAPTER III

1 Potential for Λ in He^4 core:

For $_{\Lambda}He^5$, as in ref.[3] we consider the Λ as moving in the potential well produced by the nucleons of the undistorted core nucleus, He^4. The effect due to radial compressions of the core nucleus is small[3]. ¥

¥ Thus with a reasonable value of the stiffness coefficient $K^{(4)}$ for He^4 the effect of a radial compression was found by Dalitz and Downs[3], with only two-body forces, to decrease $U^{(4)}$ by $\pm$ 1.5% for $\mu_{2\pi}$ and $\pm$ 3% for μ_K. The coefficient $K^{(4)}$ was defined by $E_{He}4(\alpha) = E_{He}4(1) \frac{1}{2}K^{(4)}(1-\alpha)^2$ where $\alpha = R/Ro$ and Ro is the equilibrium r.m.s. radius. A recent calculation by Mang and Wild [13] of the binding energy of He^4, using hard core forces (with $r_c = 0.4$ f), gives reasonable agreement both for the binding energy and the r.m.s. radius. Their calculation gives $K^{(4)} = 180$ MeV (see ref.[14]), which is roughly consistent with the values considered by Dalitz and Downs ($\pm$ 280 MeV). It should be noted that the compressibility coefficient K as usually defined is related to $K^{(4)}$ by $K^{(4)} = AK = 4K$. Thus Mang and Wild's value corresponds to K = 45 MeV which is considerably smaller than the value of approximately 170 MeV obtained by Brueckner and Gammel [15] for nuclear matter using hard core forces. This large difference, although in the expected direction, may indicate that Mang and Wild's value is somewhat on the small side. This foot-note is due to Dr. A.R. Bodmer.

A Gaussian charge distribution of r.m.s. radius 1.61 $\pm$ .05 f [16] fits rather well the electron scattering data for He^4. Allowing for the proton charge distribution (Gaussian of r.m.s. radius .72 $\pm$ .05 [17]) we obtain for the proton distribution in He^4, a Gaussian of r.m.s. radius R = 1.44 $\pm$ .07 f. Assuming the same neutron distribution, the nucleon distribution, normalised to unity becomes

$$\rho\left(\frac{r}{a}\right) = \frac{1}{\pi^{3/2} a^3} e^{-r^2/a^2} \qquad (3.1)$$

where $a^2 = \frac{2}{3} R^2$. This Gaussian distribution is expected to be fairly realistic since the nucleons in He^4 are tightly bound.

The potential, $\mathcal{V}(r)$, felt by the Λ will be written as the sum of the potentials due to the two and three-body Λ - N forces:

$$\mathcal{V}(r) = \mathcal{V}_2(r) + \mathcal{V}_3(r) \qquad (3.2)$$

where r is the distance of the Λ from the centre of mass of the He^4 core.

For $\mathcal{V}_2(r)$ we obtain [3]

$$\mathcal{V}_2(r) = U_2^{(4)} \int v(r_{1\Lambda})\, \rho\left(\frac{r_1}{a}\right) dr_1 \qquad (3.3)$$

where $r_{1\Lambda} = |r_1 - r|$, $U_2^{(4)}$ is given by eqn. (2.1), v(r) by eqn. (1.2) and ρ by eqn. (3.1).

Eqn. (3.3) simplifies to

$$\mathcal{V}_2(r) = U_2^{(4)} \left(\frac{\mu^3}{4\pi}\right) I_y(r; \mu a) \qquad (3.4)$$

where

$$I_Y(r;\mu,a) = \int \frac{e^{-\mu r_{1\lambda}}}{\mu r_{1\lambda}} \rho\left(\frac{r_1}{a}\right) d\underline{r}_1$$

$$= \frac{e^{a^2\mu^2/4}}{2\mu r}\left\{ e^{-\mu r}\left[1 - \mathrm{Erf}\left(\frac{\mu a}{2} - \frac{r}{a}\right)\right] - e^{\mu r}\left[1 - \mathrm{Erf}\left(\frac{\mu a}{2} + \frac{r}{a}\right)\right]\right\} \qquad (3.5)$$

For $V_3(r)$, we obtain

$$V_3(r) = w_3 \sum_{i<j=1}^{4} \int F(r_i, r_j)\, W(r_i, r_j, r)\, d\underline{r}_i\, d\underline{r}_j$$

$$= 6\, w_3 \int F(r_1, r_2)\, W(r_1, r_2, r)\, d\underline{r}_1\, d\underline{r}_2$$

where $6F(r_1\ r_2)$ is the pair distribution function and where it has been assumed that the $\wedge$ interacts predominantly with pairs of nucleons in relative s-states consistent with the He^4 wave function being predominantly a $^{11}S_0$ state. Further if short range nucleon pair correlations may be neglected. ¥

$$F(r_1, r_2) = \rho(r_1)\,\rho(r_2)$$

and
$$V_3(r) = 6\, w_3 \int W(r_{1\lambda}, r_{2\lambda})\, \rho\left(\frac{r_1}{a}\right) \rho\left(\frac{r_2}{a}\right) d\underline{r}_1\, d\underline{r}_2 \qquad (3.6)$$

For the Yukawa shape W_Y, we obtain

$$V_{3Y}(r) = U_{3Y} \frac{3\nu^6}{8\pi^2} \left[\int \frac{e^{-\nu r_{1\lambda}}}{\nu r_{1\lambda}} \rho\left(\frac{r_1}{a}\right) d\underline{r}_1 \right]^2$$

$$= U_{3Y}\left(\frac{3\nu^6}{8\pi^2}\right) \left[I_Y(r;\nu,a) \right]^2 \qquad (3.7)$$

¥ Short range nucleon pair correlations will be shown to have negligible effect on the three-body potential in nuclear matter at normal density (see § 6.2). Since the average density of He^4 is even less our neglect of nucleon pair correlations is perhaps justified.

For the exponential shape W_E we obtain

$$V_{3E}(r) = U_{3E}\left(\frac{3\nu^6}{32\pi^2}\right)\left[I_E(r;\nu,a)\right]^2 \tag{3.8}$$

where

$$I_E(r;\nu a) = \frac{a}{2r}\, e^{\nu^2a^2/4}\left\{e^{\nu r}\left(\frac{\nu a}{2}+\frac{r}{a}\right)\left[1-\mathrm{Erf}\left(\frac{\nu a}{2}+\frac{r}{a}\right)\right] - e^{\nu r}\left(\frac{\nu a}{2}-\frac{r}{a}\right)\left[1-\mathrm{Erf}\left(\frac{\nu a}{2}-\frac{r}{a}\right)\right]\right. \tag{3.9}$$

2 The Eigenvalue Equation:

The Schrödinger equation for the Λ in ${}_{\Lambda}He^5$ is

$$\left[-\frac{\nabla^2}{2M_{\Lambda r}} + V_2(r) + V_3(r)\right]\psi_\Lambda(r) = B_\Lambda^{(4)}\psi_\Lambda(r) \tag{3.10}$$

where in an obvious notation

$$M_{\Lambda r} = \frac{M_\Lambda M_{He^4}}{M_\Lambda + M_{He^4}} \qquad \text{and} \qquad B_\Lambda^{(4)} = 3.08 \pm .22 \text{ MeV}\ ^{11)}.$$

The numerical solution for the ground state (1s) wave function then gives a relation between $U_2^{(4)}$ and U_3, consistent with $B_\Lambda^{(4)}$ for each set of ranges and shapes of W considered. For all cases this relation was found to be linear to a very good approximation. Corresponding to eqn. (2.7) for ${}_{\Lambda}H^3$ we may write the relation as

$$U_3 = Z^{(4)}\left(U^{(4)} - U_2^{(4)}\right) \tag{3.11}$$

where $U^{(4)}$ is the value of $U_2^{(4)}$ in the absence of three-body interactions. Thus $U^{(4)}$ depends only on μ. Similarly $Z^{(4)}\ U^{(4)}$ - i.e. U_3 in the absence of two-body interactions

depends only on ν and the shape W. This is in contrast to the analogous quantity $Z^{(2)}\ U^{(2)}$ for $_{\Lambda}H^{3}$.

The results for $U^{(4)}$ and $Z^{(4)}$ are given in Table II, the former agreeing essentially with the results of Dalitz and Downs [3] who used Gaussian two-body interactions ¤ (see §3.3).

The linearity found between U_3 and $U_2^{(4)}$ for a given $B_{\Lambda}^{(4)}$ will follow from the variational principle if the wave function does not depend too much on the relative amounts of the two and three-body interactions. The numerical results obtained confirm this. Thus the maximum variation of $\psi_{\Lambda}(r)$ between different cases is about 10% and occurs for $r \doteq 1$ f. The differences rapidly become very small for larger distances. The following reasons may be ascribed to the small dependence of $\psi_{\Lambda}(r)$ on the details of $v(r)$.

Firstly the shapes and extensions of $v_2(r)$ and $v_3(r)$ are not too different for the various shapes and ranges of the

---

¤ Preliminary results for $U^{(4)}$ and $Z^{(4)}$ were obtained using a variational method, a Gaussian trial wave function and Gaussian two and three-body interactions. Thus with only two-body Gaussian interactions (eqn. 3.13) we got $U^{(4)}(\mu_{2\pi}) = 1004$ MeV f^3 and $U^{(4)}(\mu_K) = 792$ MeV f^3 as compared with 932 and 725 MeV f^3 obtained by solving the eigenvalue problem exactly.

interactions used, being to a large extent determined by $\rho(r)$. ¥

Secondly because $B_{\Lambda}^{(4)}$ is small $\psi_{\Lambda}(r)$ has a long tail in the region where $v(r)$ has effectively become zero and $\psi_{\Lambda}(r)$ is thus largely determined for all r by the value of $B_{\Lambda}^{(4)}$. ⊙

In view of these considerations, we may expect the ratio of the three-body to the total potential energy in $_{\Lambda}He^{5}$ to be

---

¥ The variations involved are indicated by the following values obtained for the central depths, remembering that these multiplied by the squares of the corresponding extensions are expected to be roughly constant. Thus with only two-body interactions: $v_2(0) = 46.5$ and 55.3 MeV for $\mu_{2\pi}$ and μ_K respectively. With only three-body Yukawa interactions: $v_3(0) = 44.6$, 66.6 and 74 MeV for $\nu = .5$, .72 and 1f respectively, while the corresponding values for the exponential shape are $v_3(0) = 28.6$, 37.4 and 47.3 MeV. The appropriate potential wells may be obtained from these values and the relevant expressions for $v_2(r)$ and $v_3(r)$ given in the text. However it may be noted that with only two-body interactions and for $\mu_{2\pi}$: $v_2(r = 1.45\ f) = \frac{1}{2}\, v_2(0)$.

⊙ The wave functions attain their maximum value for $r \simeq 1.5 - 1.7$ f and fall to half this value for $r \simeq 4 - 4.4$ f.

given to a good approximation by

$$\xi^{(4)} = \frac{U^{(4)} - U_2^{(4)}}{U^{(4)}} \qquad (3.12)$$

The reason why the exact relation for $_{\Lambda}H^3$ corresponding to the approximate relation (2.7) will not in general be linear, in contrast to eqn. (3.11) for $_{\Lambda}He^5$ is because $_{\Lambda}H^3$ is treated as a three-body problem while $_{\Lambda}He^5$ has been treated essentially as a two-body problem. Thus for $_{\Lambda}H^3$ the important short range correlations between the nucleons and the Λ will depend quite strongly on the details of the Λ - N interaction and are not largely determined by the tail of the wave function for large separations between the Λ and the nucleons.

3 Comparison with Results for a Gaussian Interaction:

It may be of some interest to consider, for $_{\Lambda}He^5$, interactions of different shapes but having the same intrinsic range. Thus for an intrinsic range b the Gaussian interaction normalised to unity is

$$v_2(r) = \left(\frac{2.06}{\pi b^2}\right)^{3/2} \exp\left(-2.06\, r^2/b^2\right) \qquad (3.13)$$

Hence the potential $v_2(r)$ becomes

$$v_2(r) = U_2^{(4)}\, \rho(r/a') \qquad (3.14)$$

where

$$a'^2 = a^2 + b^2/2.06$$

For a Yukawa range $\mu_{2\pi}^{-1}$ the use of eqn. (3.14) together with the relation $b = 2.12\, \mu^{-1}$ gives $U^{(4)} = 932$ MeV f^3.

given to a good approximation by

$$\xi^{(4)} = \frac{U^{(4)} - U_2^{(4)}}{U^{(4)}} \qquad (3.12)$$

The reason why the exact relation for ${}_\Lambda H^3$ corresponding to the approximate relation (2.7) will not in general be linear, in contrast to eqn. (3.11) for ${}_\Lambda He^5$ is because ${}_\Lambda H^3$ is treated as a three-body problem while ${}_\Lambda He^5$ has been treated essentially as a two-body problem. Thus for ${}_\Lambda H^3$ the important short range correlations between the nucleons and the Λ will depend quite strongly on the details of the Λ - N interaction and are not largely determined by the tail of the wave function for large separations between the Λ and the nucleons.

3 Comparison with Results for a Gaussian Interaction:

It may be of some interest to consider, for ${}_\Lambda He^5$, interactions of different shapes but having the same intrinsic range. Thus for an intrinsic range b the Gaussian interaction normalised to unity is

$$v_2(r) = \left(\frac{2.06}{\pi b^2}\right)^{3/2} \exp\left(-2.06\, r^2/b^2\right) \qquad (3.13)$$

Hence the potential $v_2^e(r)$ becomes

$$v_2^e(r) = U_2^{(4)}\, \rho(r/a') \qquad (3.14)$$

where

$$a'^2 = a^2 + b^2/2.06$$

For a Yukawa range $\mu_{2\pi}^{-1}$ the use of eqn. (3.14) together with the relation $b = 2.12\, \mu^{-1}$ gives $U^{(4)} = 932$ MeV f^3.

The comparable value for the Yukawa shape is 1028 MeV f^3. The corresponding values for $\mu_{1\kappa}$ are 725 and 780 MeV f^3 respectively, the percentage difference being somewhat less than for $\mu_{2\pi}$. These results agree with the conclusions of ref.[3] that interactions of the same intrinsic range are approximately equivalent for $_{\Lambda}He^5$. We emphasise that this equivalence holds only if the range is reasonably small compared with the extension of the nucleon distribution. An examination of the expression for $v_2(r)$ shows that only for short ranges is the range of $v_2(r)$ reasonably close to that of $\rho(r)$ and hence not too dependent on the shape of the interaction. The above results for $\mu_{2\pi}$ and $\mu_{1\kappa}$ tend to confirm this conclusion. Further, the Gaussian interaction which reproduces essentially the same results for $v_2(r)$ as a Yukawa interaction of range $\mu_\pi = 1.4$ f (note that R = 1.44 f) is found to have an intrinsic range b = 2.3 f ($a' = 4.0$ f^2) which is considerably different from the intrinsic range of the Yukawa interaction, this being b = 2.97 f ($a' = 5.65$ f^2). Now since $v_3(r)$ is essentially proportional to the square of $v_2(r)$ for the range ν_π^{-1}, which is comparable to R, there is no approximate equivalence in the sense of an intrinsic range between different shapes of the three-body interactions for the ranges of interest.

Finally as a measure of the dependence of our results on the values of $B_{\Lambda}^{(4)}$ and R we give the following results for small variations about the values $B_{\Lambda}^{(4)} = 3.08$ MeV and R = 1.44f:

$$\left(\frac{\Delta U^{(4)}}{\Delta B_{\Lambda}^{(4)}}\right)_{R\ const} = 78\ \text{MeV f}^3$$

$$\left(\frac{\Delta U^{(4)}}{\Delta R}\right)_{B_{\Lambda}^{(4)}\ const} = 51\ \text{MeV f}^3$$

CHAPTER IV

1 The Volume Integrals of the Λ - N Interaction:

In the last two chapters we have specified the strength of the Λ - N interaction by the three volume integrals U_s, U_t and U_3. Since only $B_\Lambda^{(3)}$ and $B_\Lambda^{(5)}$ were made use of there will be one undetermined volume integral. This we take to be $\Delta = U_s - U_t$ for $U_s > U_t$ and $\delta = U_t - U_s$ for $U_t > U_s$.

Table III is obtained from eqns. (2.1), (2.7) and (3.11), and Tables I and II. Though only the results for $U_s > U_t$ are explicitly given the results for $U_t > U_s$ which can easily be obtained will also be discussed. The dependence of the tabulated quantities on Δ is linear. For the three-body Yukawa shape W_Y only the results for $\nu = .5\ f^{-1}$ and $1\ f^{-1}$ are given, those for $\nu = .72\ f^{-1}$ being very nearly the mean of these. The dependence on ν is not very appreciable for the range of values considered. For W_E, only the results for $\nu = .72\ f^{-1}$ are given. The results for $\nu = 1\ f^{-1}$ are fairly close to those for W_Y with $\nu = .5\ f^{-1}$, while the range $\nu = .5\ f^{-1}$ seems unphysically long to be of interest.

We get $U_3 = 0$ when $\Delta = U^{(2)} - \frac{1}{2} U^{(4)}$ (for $U_s > U_t$)
and when $\delta = 2U^{(2)} - U^{(4)}$ (for $U_t > U_s$)

Thus $U_3 = 0$ when $\Delta = 140$ and 26 MeV f^3 for $\mu_{2\pi}$ and μ_K (if $U_s > U_t$) and when $\delta = 280$ and 52 MeV f^3, (if $U_t > U_s$). For larger Δ, or δ, U_3 becomes positive (corresponding to

an attractive three-body interaction). .[*]

We also note that for $U_s > U_t$,

$U_t < 0$ for $\Delta > U_s$ ($\Delta = U_s = 380$ and 240 MeV f^3 for $\mu_{2\pi}$ and μ_K respectively), while

$U_2^{(4)} < 0$ for $\Delta > \frac{4}{3} U_s$ ($\Delta = \frac{4}{3} U_s = 510$ and 320 MeV f^3 for $\mu_{2\pi}$ and μ_K respectively).

The ratio $\xi^{(2)}$ is seen to be quite small for all cases considered, confirming that three-body interactions are not very effective for ${}_\Lambda H^3$ (and thus justifying our perturbation procedure). $\xi^{(4)}$ however can become quite large. This increase of $\xi^{(A-1)}$ with A can be largely attributed to the dependence of the three-body potential energy for $A < 5$, on the number of nucleon pairs $(A-1)(A-2)/2$.

The smallness of $\xi^{(2)}$ implies that $U_2^{(2)}$ is not very different from $U^{(2)}$ and hence not too dependent on Δ - again demonstrated by the results of Table III. Thus for a given Δ and μ the volume integral U_s (and therefore also U_t) is to a first approximation determined by just $B_\Lambda^{(2)}$. For $U_t > U_s$ it can be seen that to the same approximation $B_\Lambda^{(2)}$ determines just U_t ($= U_2^{(2)}/2$) independently of δ. The volume integral $U_2^{(4)}$ and therefore the two-body potential

---

[*] This is the case for the theoretical calculation of ref.[12] according to which the values of Δ or δ, for $U_3 = 0$ may be considered as lower limits.

energy in $_{\Lambda}He^5$ is also determined to a first approximation. Therefore $\xi^{(4)}$, and thus the three-body potential energy in $_{\Lambda}He^5$, is then determined to a similar approximation by $B_{\Lambda}^{(4)}$ independently of the nature of the three-body interactions. This situation is reflected by the slight dependence of the tabulated results for $\xi^{(4)}$ and $U_2^{(4)}$ on the three-body interactions. For a given shape and range of these interactions, i.e. for a given W the corresponding value of U_3 is then fixed by $\xi^{(4)}$. Thus for a given Δ and μ the tabulated values of U_3 are those which give approximately the same three-body potential energy for $_{\Lambda}He^5$. Such essentially equivalent values of U_3 are seen to depend quite strongly on the shape and range of W. This is because, similarly as for $U^{(A-1)}$, the range of W is not small enough for the three-body interactions to be regarded as effectively of zero range. For such very short ranges $v_3(r)$ becomes (see eqn.(3.6)),

$$v_3(r) = \frac{(A-1)(A-2)}{2} U_3 \rho^2(r)$$

and U_3 would be determined independently of the range and shape of W.

In Table III is also given values of

$$\chi = \frac{U^{(4)} - U_2^{(4)}}{U^{(2)} - U_2^{(2)}} = \frac{\xi^{(4)} U^{(4)}}{\xi^{(2)} U^{(2)}} \qquad (4.1)$$

which is a measure of the ratio of the three-body potential energy in $_{\Lambda}He^5$ to that in $_{\Lambda}H^3$. For given two-body interactions (i.e. given μ and approximately given Δ) χ depends

strongly on W (increasing with the range of W), nearly all this variation coming from $\xi^{(2)}$. Thus three-body interactions which give the same three-body potential energy in ${}_{\Lambda}He^{5}$ may give very different three-body potential energies in ${}_{\Lambda}H^{3}$. This may be ascribed to the importance of the ratio of the range to the nuclear extension (as reflected in the strong dependence of U_3 on this ratio) coupled with the greater diffuseness of ${}_{\Lambda}H^{3}$ as compared with ${}_{\Lambda}He^{5}$; hence also the increase of χ with the range ν^{-1}.

Denoting the results for the case $U_t > U_s$ and $U_s > U_t$ by the suffixes t and s respectively it can be seen that

$$\frac{\xi_t^{(4)}(\delta)}{\xi_s^{(4)}(\Delta)} = \frac{\xi_t^{(2)}(\delta)}{\xi_s^{(2)}(\Delta)} = \frac{2U^{(2)} - U^{(4)} - \delta}{2U^{(2)} - U^{(4)} - \Delta} \qquad (4.2)$$

Thus in particular $\chi_t = \chi_s$. The values of $\xi_t^{(4)}$ together with the corresponding value of U_3 are then obtained from $\xi_s^{(4)}$, when we remember that U_3 is determined by only $\xi^{(4)}$. The other quantities for the case $U_t > U_s$ are then determined from eqns. (2.7), (3.12) and $U_t = U_2^{(2)} / 2$.

An important difference compared to the case $U_s > U_t$ is that both $\xi_t^{(2)}$ and $\xi_t^{(4)}$ are more restricted in magnitude than $\xi_s^{(2)}$ or $\xi_s^{(4)}$. Hence the three-body forces are less important for the case $U_t > U_s$ than for $U_s > U_t$. Thus for $\delta = \Delta = 400$ MeV f^3 we get $\xi_t^{(4)} / \xi_s^{(4)} = .23$ and .465 for $\mu_{2\pi}$ and μ_K respectively. The small values of $\xi_t^{(2)}$ and

$\xi_t^{(4)}$ are essentially due to the fact that $U_2^{(4)}$ is quite strongly restricted for $U_t > U_s$. We have

$$\frac{3}{2} U_2^{(2)} \leq U_2^{(4)} \leq 2U_2^{(2)}$$

the limits corresponding to $\delta = \infty$ and to $\delta = 0$ respectively; also $U_2^{(2)} \simeq U^{(2)}$ for small $\xi_t^{(2)}$.

2 Comparison with Dalitz's Method:

The effect of three-body interactions may be included quite generally by making the following replacement in eqn.(2.1)

$$U_2^{(A-1)} \rightarrow U^{(A-1)} = U_2^{(A-1)} + \frac{(A-1)(A-2)}{2} \bar{W}^{(A-1)}$$

with

$$\bar{W}^{(A-1)} = \xi^{(A-1)} U^{(A-1)} \Big/ \frac{(A-1)(A-2)}{2} \qquad (4.3)$$

The volume integral $\bar{W}^{(A-1)}$ is thus proportional to the three-body potential energy per nucleon pair and involves an average of the three-body interactions over the nucleon pair distribution function and the position of Λ. Dalitz [4] then assumes that $\bar{W}^{(A-1)}$ has the same value $\bar{W}$ for all $A \leq 5$. For the case $U_s > U_t$, the volume integral U_t is then replaced by $U_t + 2\bar{W}$ and it is this combination together with U_s which is then uniquely determined. From eqns. (4.1) and (4.2) we obtain $\varkappa = 6\bar{W}^{(4)} / \bar{W}^{(2)} = 6$ (= the number of nucleon pairs in $_{\Lambda}He^5$) if $\bar{W}^{(A-1)} = \bar{W}$. Thus the deviation of $\varkappa$ from the value six measures the lack of proportionality of the three-body potential energy on the number of nucleon pairs. We have

already seen that such deviations arise from both the finite range of W and the differing structures of ${}_{\Lambda}H^3$ and ${}_{\Lambda}He^5$.

The unique value of U_s obtained from $U^{(2)}$ and $U^{(4)}$ for $x = 6$ is just the value obtained neglecting three-body interactions altogether (i.e. for $U_3 = 0$) and is

U_s = 362 and 215 MeV f^3 for $\mu_{2\pi}$ and μ_K respectively. However it is worth remarking that even if $x \neq 6$ U_s is still only rather weakly dependent on Δ as is seen from Table III. This is yet another consequence of the smallness of $\xi^{(2)}$. Thus for $\xi^{(2)} = 0$ we get $U_s = \frac{1}{2} U^{(2)} + \frac{1}{4} \Delta$ for $U_s > U_t$. This gives for $\mu_{2\pi}$: U_s = 320, 377 and 427 MeV f^3 for Δ = 0, 200, and 400 MeV f^3 respectively, while the corresponding values for μ_K are U_s = 208, 258 and 308 MeV f^3. Thus U_s is not very dependent on Δ even when three-body interactions are neglected for ${}_{\Lambda}H^3$. The values of U_s in Table III correspond to a situation intermediate between $\xi^{(2)} = 0$ (i.e. $x = \infty$ assuming $\xi^{(4)} \neq 0$) and $x = 6$. We therefore conclude that for $U_s > U_t$ and for a given μ, the value of U_s is already approximately determined by only $B_{\Lambda}^{(2)}$, being largely independent of other hypernuclear data. For $U_t > U_s$ we have $U_2^{(2)} = 2U_t$ and hence for $\xi^{(2)} = 0$ just $B_{\Lambda}^{(2)}$ by itself already determines U_t uniquely. This value of U_t is 327 MeV f^3 for $\mu_{2\pi}$ and 208 MeV f^3 for μ_K. Because of the smallness of $\xi^{(2)}$ these values are only slightly changed when three-body interactions are included for ${}_{\Lambda}H^3$.

3 The Two-Body Λ- N System: ¥

For a bound Λ- N system to exist, s, the well depth parameter, defined as the ratio of the potential strength to the strength necessary to give zero binding [18], must be greater than unity. In our units

$$s = 1.25 \times 10^{-1} \; U^{(1)}$$

where $U^{(1)} = U_s$ for $U_s > U_t$ $(J = 0)$

$= U_t$ for $U_t > U_s$ $(J = 1)$

For $U_t > U_s$ and $\xi^{(2)} = 0$ we obtain $s = .58$ and $.65$ for $\mu_{2\pi}$ and μ_K respectively. Thus in view of the previous discussion no bound state is possible for $U_t > U_s$ even when three-body forces are included.

For $U_s > U_t$ the conditions most favourable for a bound state are $\mu = \mu_K$, $\xi^{(2)} = 0$ and Δ as large as possible. With $\Delta = 4000$ MeV f^3 we obtain $s = .96$ (the corresponding value for $\mu_{2\pi}$ is .76). Since, as is discussed in § 4.5, $\Delta = 300$ MeV f^3 is probably an upper limit for μ_K and since U_s is overestimated by putting $\xi^{(2)} = 0$, it seems very unlikely that there should be a bound state for $U_s > U_t$. The existence of such a bound state would imply a repulsive triplet interaction ($U_t < 0$) such that $U_2^{(4)} < 0$ and hence very strongly

---

¥ This section is due to Dr. A.R. Bodmer.

attractive three-body forces for which $\xi^{(4)} > 1$. Our arguments against the existence of a bound Λ - N system are essentially a refinement of those of Dalitz and Downs Iwao [6] using a shell-model approach and effective two-body interactions has also predicted that no bound Λ - N system should exist. However Iwao's approach neglects three-body interactions and also assumes that the structure of all the hypernuclei with $A \leq 5$ is essentially the same.

4 The Hypernuclei $_{\Lambda}H^4$ and $_{\Lambda}He^4$:

These hypernuclei could in principle give us additional information about the Λ- N interaction. However the radii of the core nuclei H^3 and He^3 must be accurately known and their distortions by the Λ carefully taken into account if any significant new information is to be obtained. Furthermore because of the dependence of $\varkappa$ on W and the nuclear extension, such information (e.g. the value of Δ) will be quite sensitive to W, i.e. to the shape and range of the three-body interaction. To see this let us put $\overline{W}^{(A-1)} = \alpha^{(A-1)}\overline{W}$ in eqn. (4.3) and without loss of generality also put $\alpha^{(4)} = 1$. Then $\xi^{(4)} = 6\,\overline{W}/U^{(4)}$ and eqn. (2.1) may be solved for U_s, U_t and $\overline{W}$ in terms of the $U^{(A-1)}$ and $\alpha^{(A-1)}$. For $U_s > U_t$ we obtain

$$\overline{W} = (4U^{(3)} - 3\,U^{(2)} - \tfrac{3}{2}\,U^{(4)}) / D$$

where

$$D = 3\alpha^{(2)} - 12\,\alpha^{(3)} \quad 9\,\alpha^{(4)}$$

(Note that for $\bar{W}^{(A-1)} = \bar{W}$ we have $\alpha^{(A-1)} = 1$ and thus $D = 0$).

Similarly for $U_t > U_s$ we obtain

$$\bar{W} = (U^{(4)} + U^{(2)} - 2U^{(3)}) / D$$

where

$$D = 6\alpha^{(3)} - \alpha^{(2)} - 6\alpha^{(4)}$$

($D = -1$ for $\alpha^{(A-1)} = 1$).

The condition $D\bar{W} = 0$ gives eqn. (2.2) for $U^{(A-1)}$. Because these relations are approximately satisfied already, the product $D\bar{W}$ will depend quite critically on $U^{(3)}$. Thus for $\mu_{2\pi}$ Dalitz and Downs[3] obtained $820 \lesssim U^{(3)} \lesssim 965$ MeV f^3, the spread coming from uncertainties in the radii and compressibilities of the core nuclei. Corresponding to this spread we obtain for $U_s > U_t$:

$$-214 \lesssim D\bar{W} \lesssim 365 \text{ MeV f}^3$$

(or $-1.25 \lesssim D\xi^{(4)} \lesssim 2$, putting $\alpha^{(4)} = 1$)

For $U_t > U_s$: $-40 \lesssim D\bar{W} \lesssim 250$ MeV f^3

(or $-.25 \lesssim D\xi^{(4)} \lesssim 1.45$).

Thus even if D, i.e. the $\alpha^{(A-1)}$, were well known $U^{(3)}$ would still have to be known to great accuracy in order to give significant information about the three-body interaction and hence about Δ. In fact D depends quite sensitively on the $\alpha^{(A-1)}$ (i.e. on the relative amounts of the three-body potential energy in the different hypernuclei) which, as is evident from the results for χ, are expected to depend fairly strongly on W.

(Note that for $\bar{W}^{(A-1)} = \bar{W}$ we have $\alpha^{(A-1)} = 1$ and thus $D = 0$).

Similarly for $U_t > U_s$ we obtain

$$\bar{W} = (U^{(4)} + U^{(2)} - 2U^{(3)}) / D$$

where

$$D = 6\alpha^{(3)} - \alpha^{(2)} - 6\alpha^{(4)}$$

($D = -1$ for $\alpha^{(A-1)} = 1$).

The condition $D\bar{W} = 0$ gives eqn. (2.2) for $U^{(A-1)}$. Because these relations are approximately satisfied already, the product $D\bar{W}$ will depend quite critically on $U^{(3)}$. Thus for $\mu_{2\pi}$ Dalitz and Downs[3] obtained $820 \lesssim U^{(3)} \lesssim 965$ MeV f^3, the spread coming from uncertainties in the radii and compressibilities of the core nuclei. Corresponding to this spread we obtain for $U_s > U_t$:

$$-214 \lesssim D\bar{W} \lesssim 365 \text{ MeV } f^3$$

(or $-1.25 \lesssim D\xi^{(4)} \lesssim 2$, putting $\alpha^{(4)} = 1$)

For $U_t > U_s$: $\quad -40 \lesssim D\bar{W} \lesssim 250$ MeV f^3

(or $-.25 \lesssim D\xi^{(4)} \lesssim 1.45$).

Thus even if D, i.e. the $\alpha^{(A-1)}$, were well known $U^{(3)}$ would still have to be known to great accuracy in order to give significant information about the three-body interaction and hence about Δ. In fact D depends quite sensitively on the $\alpha^{(A-1)}$ (i.e. on the relative amounts of the three-body potential energy in the different hypernuclei) which, as is evident from the results for χ, are expected to depend fairly strongly on W.

Thus it will be difficult to get significant information about the volume integrals of the Λ - N interaction from the Λ binding energies of ${}_{\Lambda}H^4$ and ${}_{\Lambda}He^4$.

Dalitz and Downs [3] found that for $U_s > U_t$ the first excited state of ${}_{\Lambda}He^4$ or ${}_{\Lambda}H^4$, which has $J = 1$, may just be bound, when there are no three-body forces. With attractive three-body forces Δ is larger than for $U_3 = 0$ and the binding energy of the excited state becomes less. For $U_t > U_s$ the first excited state has $J = 0$ and was found not to be bound in the absence of three-body forces [3]. With attractive three-body forces δ will be larger than for $U_3 = 0$ and the excited state will be even less bound. Thus the existence of a bound excited state of ${}_{\Lambda}He^4$ and/or ${}_{\Lambda}H^4$ would indicate $U_s > U_t$ and either very weak attractive or else repulsive three-body interactions.

5 Hypernuclei with $5 < A \leq 12$:

Can these hypernuclei give us additional information on the Λ - N interaction? Ref. [11] gives a comprehensive tabulation of these hypernuclei, some of which have already been considered in ref. [3]. ${}_{\Lambda}Be^9$ is considered also in ref. [19]. Iwao [6] assuming effective two-body interactions has considered these hypernuclei on the basis of the j-j coupling shell-model [20] to predict some properties like γ transition probabilities. However no significant new information about the Λ - N interaction has resulted from all this work. Dalitz [4]

has pointed out that the difference in the volume integrals for $_{\Lambda}He^{7}$ and $_{\Lambda}Li^{7}$ will be very nearly Δ if one assumes L - S coupling and that the core nuclei He^{6} and Li^{6} are both undistorted by the Λ and have the same nucleon density distribution. Then with $B_{\Lambda}(_{\Lambda}Li^{7}) = 5.5 \pm .3$ MeV and $B_{\Lambda}(_{\Lambda}He^{7}) = 3.0 \pm .7$ MeV[11] and with the r.m.s. radius for the nucleon density distribution, as determined by the electron scattering data for Li^{6} [21] and for the proton, given by $\langle r^2 \rangle^{\frac{1}{2}} = 2.6$ f we obtain $\Delta = 370$ and 310 MeV f^3 for $\mu_{2\pi}$ and μ_{K} respectively. These values of Δ imply that U_t and U_2[4] are quite small in magnitude ($U_t < 0$ for μ_K), corresponding to large values of ξ[4] and hence U_3. However even an increase of $\langle r^2 \rangle^{\frac{1}{2}}_{He6}$ by only .4 f relative to $\langle r^2 \rangle^{\frac{1}{2}}_{Li6}$ gives $\Delta \approx 0$ for $\mu_{2\pi}$. Since H_e^{6} is less stable than Li^{6} we could expect that the difference in the effective core radii is in this direction and the above values of Δ are therefore most probably upper limits. Hence only a careful analysis of this pair of nuclei can give us Δ with any degree of accuracy. Other hypernuclei with $A > 5$ may also give additional information about the Λ - N interaction if three-body forces are also included. Thus the ratio of the three to the two-body potential energy in these nuclei may be sufficiently different from the corresponding ratio in $_{\Lambda}He^{5}$ to provide some information about U_3 and hence about Δ.

We shall not pursue these matters here any further but shall go on to consider the binding of a Λ particle in nuclear matter in the next chapters.

CHAPTER V

1 Binding Energy of a Λ Particle in a Heavy Nucleus:

The calculation of the binding energy of a Λ particle in a heavy nucleus may to a good approximation be reduced to the calculation of the potential depth V_o felt by a Λ of zero momentum in nuclear matter having a density equal to the central density of heavy nuclei, since V_o is just the limiting value of the Λ binding energy for large A. The 'empirical' value V_o^1 of V_o given in refs. [9] [10] was obtained by extrapolating from the known Λ binding energies for $A \leq 12$, assuming a square well potential of constant depth V_o and radius $r_o A^{1/3}$. This procedure gives $V_o^1 = 23$ MeV for $r_o = 1.2$ f and $V_o^1 = 26$ MeV for $r_o = 1.0$ f [10]. However these values cannot be considered very reliable since for the hypernuclei with $A \leq 12$ details of structure will be important and conditions generally will be rather different than for heavier hypernuclei.

In the effective mass approximation the real part of the potential felt by a Λ of momentum $\underline{q}$ [*] in nuclear matter may be written as

$$V(q) = -V_o + \frac{1}{2}\left(\frac{1}{M^*_\Lambda} - \frac{1}{M_\Lambda}\right) q^2 \qquad (5.1)$$

---

[*] For $\underline{q} \neq 0$ there will also be an imaginary part to the potential. This, though of interest to the problem of the scattering of Λ particles in nuclear matter, is of no interest to the binding energy problem.

where M_Λ^* is the effective mass.

We treat the nuclear matter mostly as a Fermi gas of independent nucleons containing equal numbers of protons and neutrons. For a density ρ the Fermi momentum is then given by

$$k_F = \left(\frac{3\pi^2}{2}\right)^{1/3} \rho^{1/3} = \left(\frac{9\pi}{8}\right)^{1/3} \frac{1}{r_0}$$

where r_0 is the usual radius constant defined by

$$\rho = \frac{3}{4\pi r_0^3}$$

We evaluate V(q) using Perturbation Theory. The contributions due to two-body forces both direct (associated with range $\mu_{2\pi}^{-1}$) and exchange (associated with range μ_K^{-1}) are considered in this chapter, while the contribution due to three-body forces is considered in the next chapter.

2 Perturbation Calculation of the Λ Potential in Nuclear Matter:

We take for the unperturbed hypernuclear wave function the following product form, normalised in a volume Ω:

$$\Psi_{qA} = \psi_q(\lambda)\ \Phi_A(1\ldots A) \qquad (5.2)$$

where λ denotes the co-ordinates of the Λ particle and 1.....A, those of the nucleons.

$\psi_q(\lambda)$ corresponds to a plane wave state of momentum $\underline{q}$:

$$\psi_q(\lambda) = \frac{1}{\Omega^{1/2}} e^{i\underline{q}\cdot\underline{r}_\lambda} \times \text{spin function} \qquad (5.3)$$

ϕ_A is a determinantal wave function corresponding to A/2 neutrons and A/2 protons:

$$\phi_A = \frac{1}{(A!)^{1/2}} \text{ Det } [\varphi_n(i)] \qquad (5.4)$$

where

$$\varphi_n(i) = \frac{1}{\Omega^{1/2}} e^{i\underline{n}\cdot\underline{r}_i} \times \text{spin, i-spin function} \qquad (5.4a)$$

is a nucleon plane wave state of momentum $\underline{n}$.

The nuclear ground state is obtained when all states up to the Fermi level are filled. The spin and isobaric spin summations occuring in our calculations are quite trivial and for convenience of writing will often not be shown explicitly.

Only the $\Lambda - N$ interactions are treated as the perturbation. This means that the determinantal wave functions and the associated energies are considered to be reasonable approximations to the exact nuclear wave functions and energies of the full nuclear Hamiltonian including the nucleon-nucleon interactions. Thus the best energies to be associated with the ϕ_A are those corresponding to a self consistent single particle Hamiltonian. In the effective mass approximation these energies will be given by specifying the effective nucleon mass M_N^*. Thus no account is taken of effects due to possible correlations in the nuclear wave functions. To include such effects in a perturbation calculation based on the wave function (5.4) we would have to include the N-N interactions also in the perturbation. Nucleon correlation effects

on the Λ potential would then be given by those terms in the corresponding perturbation expansion which depend on both the Λ – N and N–N interactions.

3 The Λ Potential due to Direct Two-Body Interactions:

For the Λ potential due to two-body Yukawa interactions the relevant matrix elements may all be obtained from the following matrix element of the normalised interaction v(r) of eqn. (2.2)

$$\langle \underline{q}' \underline{n}' | v | \underline{q}\, \underline{n} \rangle = \int \psi^*_{\underline{q}'}(\underline{r}_1)\, \varphi^*_{\underline{n}'}(\underline{r}_2)\, v(\underline{r}_1 - \underline{r}_2)\, \psi_{\underline{q}}(\underline{r}_1)\, \varphi_{\underline{n}}(\underline{r}_2)\, d\underline{r}_1\, d\underline{r}_2$$

$$= \frac{\mu^2}{\Omega} \frac{\delta(\underline{q} + \underline{n} - \underline{q}' - \underline{n}')}{\mu^2 + |\underline{q} - \underline{q}'|^2} \qquad (5.5)$$

For an exchange interaction $v P_x$, where P_x is the space exchange operator for a Λ and a nucleon, the corresponding matrix element is $\langle \underline{n}'\underline{q}' | v | \underline{q}\, \underline{n} \rangle$.

We consider the following static direct Λ – N interaction corresponding to the singlet and triplet s – state volume integrals U_s and U_t:

$$v_d = -(U_s P_s + U_t P_t)\, v(r) \qquad (5.6)$$

where P_s and P_t are the projection operators on the singlet and triplet spin states of the Λ – N system. The corresponding potential depth $V_d(q)$ is then to first order

$$V_d^{(1)} = -\left(\Psi_{qA} \sum_i^A v_d(r_{i\Lambda})\, \Psi_{qA}\right)$$

$$= \frac{1}{4}(U_s + 3U_t)\rho = (U_s + 3U_t)\frac{k_F^3}{6\pi^2} \qquad (5.7)$$

This result is quite independent of the correlation structure of the nuclear wave function and thus does not depend on the use of a determinantal wave function. The second order contribution to $V_d(q)$ is

$$V_d^{(2)}(q) = \sum_{n<k_F} \sum_{n'>k_F} \frac{|\langle qn|v_d|q'n'\rangle|^2}{E_{qn} - E_{q'n'}} \qquad (5.8)$$

where

$$E_{qn} = \frac{q^2}{2M_\Lambda} + \frac{n^2}{2M_N^*}$$

The sum over n is over all occupied nucleon states while that over n^1 is restricted to nucleon states outside the Fermi sea because of the Exclusion principle. There is no corresponding restriction on the intermediate state sum for q^1.

Using eqn. (5.5) we obtain

$$V_d^{(2)}(q) = \frac{M_{\Lambda N}^* \mu^4}{64\pi^6}(U_s^2 + 3U_t^2)\int_{n<k_F} d\underline{n} \int_{|n-s|>k_F} d\underline{s} \frac{1}{(\mu^2+s^2)^2\, \underline{s}\cdot(\underline{s}+\underline{q}-\underline{n})} \qquad (5.9)$$

with

$$M_{\Lambda N}^* = \frac{M_\Lambda M_N^*}{M_\Lambda + M_N^*} \qquad (5.10)$$

and where $\underline{s} = \underline{q}^1 - \underline{q} = \underline{n} - \underline{n}^1$ is the momentum transfer to

an intermediate state. If the Λ were to be replaced by a nucleon there would be the additional restriction $|\underline{q} + \underline{s}| > k_F$.

It is possible to carry out the angular integrations and that over n by using methods similar to those of Euler [22]. The resulting one dimensional integral over $|s|$ is very lengthy and complicated. We shall not give the general expression here.

For $q = 0$ we obtain

$$V_d^{(2)}(q=0) = \frac{M_{\Lambda N}^{*}\,\mu^4}{8\pi^4}\,(U_s^2 + 3U_t^2)\; I^{(2)}\left(\frac{k_F}{\mu}\right) \tag{5.11}$$

where, with $t = s/k_F$

$$I^{(2)}\left(\frac{k_F}{\mu}\right) = \int_0^2 \frac{t\,dt}{\left(t^2 + \left(\frac{\mu}{k_F}\right)^2\right)^2}\left[\tfrac{1}{2}(1-t^2)\log(1+t) + t^2\log t/2 + \tfrac{1}{4}t^2 + \tfrac{1}{2}t\right]$$

$$+ \int_2^\infty \frac{t\,dt}{\left(t^2 + \left(\frac{\mu}{k_F}\right)^2\right)^2}\left[\tfrac{1}{2}(1-t^2)\log\left|\frac{t+1}{t-1}\right| + t\right] \tag{5.12}$$

is a dimensionless integral. Fig. I gives a plot of $I^{(2)}$. For the range of values of k_F/μ considered $V_d^{(2)}(q=0)$ is approximately proportional to $\rho^{1/3}$. Hence $V_d^{(2)}(q=0)/V_d^{(1)}$ is approximately proportional to $\rho^{-2/3}$. Thus at high densities the second order correction is relatively less important; corresponding to the increasing effectiveness of the Exclusion Principle in reducing the second order relative to the first order contribution.

An effective mass $M_N^{*} < M_N$ reduces $V_d^{(2)}(q)$. In eqns.

(5.9) and (5.11) the values of M_N^x occurring through eqn. (5.10) is to be interpreted as an appropriate average for the occupied and the relevant excited states. For a range μ^{-1} the latter states will have typical excitations corresponding to momenta of magnitude μ above k_F. Thus for $\mu_{2\pi} = 1.4\ f^{-1}$ and $1.3 \leq k_F \leq 1.5\ f^{-1}$ ($1.2 \geq r_o \geq 1.0$ f), the appropriate average of M_N^x is expected to be not very different from the effective mass at the Fermi surface which may be taken as $M_N^x \doteq .7\ M_N$ [15].

We have already stated that there is no effect due to nucleon correlations on the expression (5.7) for $V_d^{(1)}$, even when the nuclear interactions $\sum_{i<j} v_N(ij)$ are included in the perturbation. This is so because these interactions have to excite at least two nucleons (because of momentum conservation) which can then be de-excited only by the Λ - N interaction acting at least twice, once for each nucleon or the other way about. Thus the lowest order terms which contain both v_d and v_N are of order $v_d^2 v_N$ and will contribute to $V_d^{(2)}$. Iterating v_N in these terms corresponds to the use of Brueckner's t-matrix (Part I, Chapter III) and arises from nuclear pair correlations in the ground and excited nuclear states. These contributions to $V_d^{(2)}$ may be expected to be of the same order - about 10% of $V_d^{(2)}$ - obtained neglecting nucleon correlations - as the contributions due to pair correlations to the first order three-body Λ potential (see § 6.2). Since $V_d^{(2)}/V_d^{(1)}$ is already small, the neglect of the pair correlations in the

evaluation of $V_d^{(2)}$ seems justified. Higher order terms involving v_N (or t), corresponding to three or more nucleon clusterings, will be even smaller.

4 The Λ Potential due to Exchange Two-Body Interactions:

The static exchange interaction corresponding to U_s and U_t is

$$v_e = v_d P_x \tag{5.13}$$

where v_d is given by eqn. (5.6). The first order potential depth is then

$$V_e^{(1)} = -\sum_{n<k_F} \langle nq | v_d | qn \rangle$$

which simplifies to

$$V_e^{(1)}(q) = (U_s + 3U_t)\frac{\mu^3 k_F}{16\pi^2}\left[4 - 4\frac{\mu}{k_F}\tan^{-1}\frac{2\mu k_F}{\mu^2+q^2-k_F^2} + \frac{k_F^2-q^2}{qk_F} + \frac{\mu^2}{qk_F}\log\frac{\mu^2+(q+k_F)^2}{\mu^2+(q-k_F)^2}\right] \tag{5.14}$$

Expanding in powers of q^2

$$V_e^{(1)}(q) = V_e^{(1)}(q=0) - V_{e2}^{(1)} q^2 + \cdots \tag{5.15}$$

where

$$V_e^{(1)}(q=0) = (U_s + 3U_t)\frac{\mu^3}{2\pi^2}\left(\frac{k_F}{\mu} - \tan^{-1}\frac{k_F}{\mu}\right) \tag{5.16}$$

and

$$V_{e2}^{(1)} = (U_s + 3U_t)\frac{1}{6\pi^2}\frac{\mu^2 k_F^2}{(\mu^2+k_F^2)^2} \tag{5.17}$$

The corresponding effective mass M_Λ^* is given by

$$\frac{M_\Lambda}{M_\Lambda^*} = 1 + 2M_\Lambda V_{e2}^{(1)} \tag{5.18}$$

For short ranges or low densities, i.e. $k_F/\mu \ll 1$, $V_e^{(1)}(q) \rightarrow V_d^{(1)}$ as expected. In powers of k_F/μ

$$V_e^{(1)}(q=0) = (U_s + 3U_t)\frac{k_F^3}{6\pi^2}\left[1 - \frac{3}{5}\left(\frac{k_F}{\mu}\right)^2 + \frac{3}{7}\left(\frac{k_F}{\mu}\right)^4 - \ldots\right] \qquad (5.19)$$

while $V_{e2}^{(1)} = 0$ for $k_F/\mu = 0$.

For $\mu_K = 2.5\ f^{-1}$, and $k_F = 1.5\ f^{-1}$ $k_F/\mu_K = .6$ and the first three terms of eqn.(5.19) give a good approximation to $V_e^{(1)}(q = 0)$.

In contrast to the expression for $V_d^{(1)}$ that for $V_e^{(1)}$ depends on the assumption of a determinental wave function for the nuclear ground state. If nuclear interactions are also included in the perturbation we will obtain terms due to nuclear correlations which are of the first order in v_e and thus contribute to $V_e^{(1)}$. A term of the lowest order ($v_e \times v_N^2$) in v_N is represented by Fig. II. As before we could use a "t" matrix instead of the v_N^1 if it is a singular potential. Since v_N (or t) must occur at least twice we may expect the correlation contributions to $V_e^{(1)}$ to be small. ✱

✱ It is interesting to observe that the first order potential due to the exchange interaction $v(r_{1\lambda})P_x$ may be quite generally written for plane wave states $\psi_q(\lambda)$ as

$$\Omega^{-1}\int e^{i q\, r_{1\lambda}}\, v(r_{1\lambda})\, \gamma(1,\lambda)\, dr_1\, dr_\lambda$$

where $\gamma(1,2)$ is the first order nuclear density matrix. For nuclear matter this expression becomes

$$\int e^{i\underline{q}\cdot\underline{r}}\, v(r)\, \gamma(r)\, d\underline{r}$$

where $\underline{r} = \underline{r}_1 - \underline{r}_\lambda$, since $\gamma(1,2)$ can depend only on $|r_1 - r_\lambda|$. For a Fermi gas of nucleons

$$\gamma(r) = \frac{2k_F^3}{\pi^2}\,\frac{j_1(k_F r)}{k_F r}$$

and we again get the expressions already obtained.

In the same manner as for $V_d^{(2)}$ we get for the second order contribution to $V_e(q)$

$$V_e^{(2)}(q) = \frac{M_{NN}^6 \mu^4}{64\pi^6}(U_s^2 + 3U_t^2)\int_{n<k_F}\int_{|n-s|>k_F} d\underline{n}\, d\underline{s}\, \frac{1}{(\quad)(\mu^2+(\underline{s}+\underline{q}-\underline{n})^2)^2\, \underline{s}\cdot(\underline{s}+\underline{q}-\underline{n})}$$

(5.20)

The difference between this and eqn. (5.9) is in the presence in the denominator of the factor $(\mu^2 + (\underline{s} + \underline{q} - \underline{n})^2)^2$ instead of $(\mu^2 + \underline{s}^2)^2$, which prevents its reduction to a one dimensional integral.

However for the same value of μ, the integral in eqn.(5.20) will be less than that in eqn. (5.9). Thus for $q = 0$ the only case we consider $\mu^2 + (\underline{s} - \underline{n})^2$ must always be greater than $\mu^2 + k_F^2$ whereas $\mu^2 + s^2$ need only be greater than μ^2. Indeed in the long range limit (i.e $\mu \to 0$) the direct term diverges whereas the exchange term remains finite. For the N-N interactions $\mu \doteq \mu_\pi = .72\ f^{-1}$ which is considerably smaller than k_F ($\doteq 1.5\ f^{-1}$) and one is closer to the long range limit. Corresponding to this is the small ratio of the second order exchange to the second order direct contribution obtained by Euler [22] for the total energy of nuclear matter. ✠

✠ For the nuclear case there is a further restriction on the intermediate states which, however, does not affect our general argument.

In the short range limit (i.e. $\mu \gg k_F$) the exchange term will approach the direct term (if the potential parameters are the same) since the leading term μ^4 in the denominators is the same for both. For the $\Lambda - N$ exchange force we have $\mu_\kappa = 2.5\ f^{-1}$ and as this is considerably larger than k_F we are closer to the short range limit. For the second order exchange interactions we have thus simply replaced eqn. (5.20) by eqn. (5.9) but with the parameters appropriate for the exchange interaction. This gives us an upper limit for $V_e^{(2)}$ $(q = 0)$ which we hope is not too different from the exact result.

CHAPTER VI

1 The Λ Potential due to Three-Body Interactions:

The Λ potential depth in nuclear matter due to the three-body interaction $\omega(1, 2, \lambda)$ is to first order

$$V_3^{(1)} = -\left(\Psi_{qA} \sum_{i<j} \omega(i,j,\lambda)\, \Psi_{qA}\right) \tag{6.1}$$

With Ψ_{qA} given by eqns. (5.2) and (5.4) we obtain

$$V_3^{(1)} = -\int \psi_q^*(\lambda) \left\{ \tfrac{1}{2} \sum_{i,j=1}^{A} \varphi_i^*(1)\, \varphi_j^*(2)\, \omega(1,2,\lambda) \times \left[\varphi_i(1)\,\varphi_j(2) - \varphi_i(2)\,\varphi_j(1)\right] \right\} \psi_q(\lambda)\, d\underline{r}_1\, d\underline{r}_2\, d\underline{r}_\lambda \tag{6.2}$$

For three-body interactions of the type we consider (see §1.2) $V_3^{(1)}$ is independent of q. For the nuclear ground state we have

$$\sum_{i,j=1}^{A} \varphi_i^*(1)\, \varphi_j^*(2)\, (\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2) \left[\varphi_i(1)\,\varphi_j(2) - \varphi_i(2)\,\varphi_j(1)\right] = -\tfrac{9}{4}\rho^2 D^2(k_F r) \tag{6.3}$$

where

$$D(x) = \frac{3j_1(x)}{x} \tag{6.4}$$

and j_1 denotes the first order spherical Bessel function. Hence eqn. (6.2) simplifies to

$$V_3^{(1)} = \frac{9}{8} w \frac{\rho^2}{\Omega} \int D^2(k_F r_{12})\, W(r_{1\lambda}, r_{2\lambda})\, d\underline{r}_1\, d\underline{r}_2\, d\underline{r}_\lambda = \frac{3}{8} w_s \rho^2 \int D^2(k_F r_{12})\, W(r_{1\lambda}\, r_{2\lambda})\, d\underline{r}_{1\lambda}\, d\underline{r}_{2\lambda} \tag{6.5}$$

where W is the shape function and $w_s = 3w$ is the s-state strength.

2 Effect of Nucleon Pair Correlations:

Short range nucleon pair correlations due to a hard core in the N-N interaction may be expected to be of some importance for $V_3^{(1)}$ since $\omega(1,2,\lambda)$ involves a pair of nucleons. To consider the effect of pair correlations we use a Jastrow wave function [23] for the nuclear ground state:

$$\phi_A(1\ldots A) \prod_{i<j} g(r_{ij}) \qquad (6.6)$$

where g(r) is a correlation function and ϕ_A is the ground state determinantal wave function. In the lowest order of the cluster expansion (i.e. neglecting terms involving products of two or more of the g) the only effect due to g(r) is to replace $D(k_F r_{12})$ in eqn. (6.3) by

$$D_c(r_{12}) = D_F(k_F r_{12})\ g(r_{12}) \qquad (6.7)$$

Thus $V_3^{(1)}$ is now given by

$$V_{3c}^{(1)} = \frac{3}{8} w_s \rho^2 \int D_c^2(r_{12})\, W(r_{1\lambda} r_{2\lambda})\, d\underline{r}_{1\lambda}\, d\underline{r}_{2\lambda} \qquad (6.8)$$

In contrast to the binding energy problem in nuclear matter [24] the precise form of g(r) is not important in our case. This is because the Λ is considered as interacting with a given nuclear wave function and g(r) is thus considered as a given function and is not, as for the nuclear matter calculation,

determined by some variational condition.

The integrals (6.5) and (6.8) can be evaluated by transforming to the triangular co-ordinates $x = r_{12}$, $y = r_{2\lambda}$, $z = r_{1\lambda}$ We get

$$\int F(r_{12})\, W(r_{1\lambda}, r_{2\lambda})\, d\underline{r}_{1\lambda}\, d\underline{r}_{2\lambda} = 8\pi^2 \int_0^\infty dx\, x\, F(x) \int_0^\infty dy\, y \int_{|x-y|}^{|x+y|} dz\, z\, W(y,z) \tag{6.9}$$

This can be reduced to a one dimensional integral for an exponential or a Yukawa shape.

For the latter in terms of k_F and U_{3Y},

$$V_{3Y}^{(1)} = U_{3Y} \frac{\nu^3 k_F^3}{12\pi^4} I_{3Y} \tag{6.10}$$

where

$$I_{3Y} = \int_0^\infty dx\, x^\nu D_c^\nu(x)\, e^{-\alpha x} \quad \text{with} \quad \alpha = \nu/k_F \tag{6.11}$$

With a correlation function $g(\beta(r - r_c))$ where r_c is the N-N hard core radius and β^{-1} a measure of the range of the correlations, we obtain for $D_c(x)$ in eqn. (6.11)

$$D_c(x) = D(x)\, g^2(\gamma(x - x_c)) \tag{6.12}$$

with $\gamma = \beta/k_F$ and $x_c = k_F r_c$

The results for the exponential shape are

$$V_{3E}^{(1)} = U_{3E} \frac{\nu^3 k_F^3}{96\pi^4} I_{3E} \tag{6.13}$$

where

$$I_{3E} = \int_0^\infty dx\ x^2\ D_c^2(x)\ e^{-\alpha x}\left(1+\alpha x+\frac{\alpha^2 x^2}{3}\right) \qquad (6.14)$$

In the absence of pair correlations, i.e. for $g = 1$, the dimensionless integrals I_{3Y} and I_{3E} are functions only of $\alpha = \nu/k_F$. These are shown in Fig. 3. The low density or short range limit corresponds to $\alpha \gg 1$ and the following series expansions then result

$$V_{3Y}^{(1)} = U_3\ \frac{3}{8}\ \rho^2\left\{1 - \frac{12}{5}\left(\frac{k_F}{\nu}\right)^2 + \frac{216}{35}\left(\frac{k_F}{\nu}\right)^4 \cdots\right\}$$

$$V_{3E}^{(1)} = U_3\ \frac{3}{8}\ \rho^2\left\{1 - \frac{24}{5}\left(\frac{k_F}{\nu}\right)^2 + \frac{216}{35}\left(\frac{k_F}{\nu}\right)^4 \cdots\right\} \qquad (6.15)$$

The first three terms are adequate for $\alpha \gtrsim 3$. The leading term has the dependence on U_3 and ρ^2 corresponding to δ function forces. For the ranges and densities of interest $\alpha \doteq .5$ and thus rather unfortunately we are quite far from the short range limit.

Also shown in Fig. 3 are I_{3Yc} and I_{3Ec} for $\nu_\pi = .7\ f^{-1}$ and for the correlation function

$$\begin{aligned} g(r) &= 0 \qquad r < r_c \\ &= 1 - e^{-\beta(r - r_c)} \end{aligned} \qquad (6.16)$$

with $\beta = .5\ f^{-1}$ and $r_c = .4\ f$.

These values are reasonably representative for nuclear matter and considering I_3 as a function of α correspond to

$\gamma = 7.1\,\alpha$ and $X_c = .28/\alpha$ respectively. The corresponding values of I_3 for two values of k_F are given in Table IV together with the relative reduction $(V_3^{(1)} - V_{3c}^{(1)})\,/V_3^{(1)}$ due to correlations. The effect of these is quite small, leading to a reduction of $V_3^{(1)}$ by approximately 10 - 15%. This smallness, incidentally, also justifies the use of the lowest order in the cluster expansion. As expected, the reduction becomes large with large ρ (i.e. as the range decreases if we remember that for a given ν the range of W_E is effectively larger than that of W_Y).

3 Comparison with 'Ordinary' Three-Body Interaction:

To consider the effect of the spin isobaric spin factor $(\sigma_1 \cdot \sigma_2)\,(\tau_1 \cdot \tau_2)$ in eqn. (1.3) we consider an 'ordinary' three-body interaction $w_{ord}(1,2,\lambda)$ given by eqn. (1.3) without both spin factors. The appropriate sum corresponding to eqn. (6.3) is now just the pair correlation functions $\rho^2(1 - \frac{1}{4} D^2(k_F r_{12}))$.

For both W_E and W_Y we obtain for $g = 1$

$$V_{3\,ord}^{(1)} = U_3 \frac{\rho^2}{2} - \frac{1}{3} V_3^{(1)} \qquad (6.17)$$

where $V_3^{(1)}$ is given by eqns. (6.10) and (6.13) and where for U_3 the strength W_s is now replaced by w_{ord} in eqn. (1.5). For $\alpha \gg 1$ we get

$$V_{3\,ord}^{(1)} = \left(\frac{3\rho^2}{8}\right) \cdot U_3$$

Thus as expected $V_3^{(1)}_{ord}$ and $V_3^{(1)}$ become the same in the short range limit. However $V_3^{(1)}_{ord}$ rapidly becomes larger than $V_3^{(1)}$ for smaller, more realistic values of α. For light hypernuclei however all the results in terms of U_3 remain unchanged, since the interactions are assumed to take place only in relative s-states for the nucleons.

Second order contributions to V_3 can arise from either the three-body interaction acting twice as in Fig. 4 or with the three-body interaction acting in conjunction with the two-body Λ - N interaction. ~~The latter contributions are, however, only of order 1/A relative to the former and may therefore be neglected.~~ The ratio of the second order contributions corresponding to Fig. 4 to $V_3^{(1)}$ is expected to be smaller than $V_d^{(2)} / V_d^{(1)}$ since in the intermediate states two nucleons must be excited above the Fermi level, except for contributions of order 1/A which arise when $n^1 = n$ or $m^1 = m$. In view of the smallness of $V_d^{(2)} / V_d^{(1)}$ we may expect the second order contributions to be unimportant.

4 Self-Consistency Effects: ¥

For a Λ interacting with nuclear matter the Hartree-Fock self-consistency problem does not arise. However for a finite hypernucleus when both the nuclear and the Λ - N interactions

---

¥ This section is due to Dr A.R. Bodmer.

are included the wave functions must in principle be obtained by a self-consistent procedure. Due to the presence of the Λ, the requirement of self-consistency will lead to a change of the nuclear wave function, i.e. to a distortion of the core nucleus - with a consequent effect on the Λ binding energy. Since there are A nucleons and one Λ such effects may be expected to be proportional to $\frac{1}{A}$ and thus be small.

Assuming a spherically symmetric self-consistent potential an estimate of the effects of self-consistency may be obtained by considering the A nucleons as a sphere of radius R, of nuclear matter which can be compressed (or dilated) by the Λ. This is similar to the assumption of Dalitz and Downs [3] in their calculation of the distortion of the He^4 core in $_{\Lambda}He^5$. The total energy for a density $\rho = A/\frac{4}{3}\pi R^3$ is

$$E(\rho) = E_A(\rho) - V_o(\rho) \qquad (6.18)$$

where E_A is the nuclear energy and $V_o(\rho)$ is the Λ binding energy, i.e. the potential depth for nuclear matter of density ρ.

For ρ close to ρ_o, the equilibrium density for nuclear matter we have to a good approximation

$$E_A(\rho) = E_A(\rho_o) + \frac{AK}{18}\frac{(\rho-\rho_o)^2}{\rho_o^2} \qquad (6.19)$$

$$V_o(\rho) = V_o(\rho_o) + \eta\frac{(\rho-\rho_o)^2}{\rho_o} \qquad (6.20)$$

where K is the usual compressibility coefficient and $\eta = \left[\frac{d V_o(\rho)}{d\rho}\right]_{\rho=\rho_o}$

Substituting eqns. (6.20) and (6.19) in eqn. (6.18) and minimising this with respect to ρ gives

$$\frac{\rho_\Lambda - \rho_0}{\rho_0} = \frac{9\eta}{AK} \quad \text{or} \quad \frac{\Delta R_\Lambda}{R_0} = \frac{R_\Lambda - R_0}{R_0} = \frac{3\eta}{AK} \tag{6.21}$$

$$\frac{\Delta B_\Lambda}{V_0} = \frac{E(\rho_\Lambda) - E(\rho_0)}{V_0(\rho_0)} = \frac{-9}{2A}\frac{\eta^2}{KV_0} \tag{6.22}$$

where the subscript $_\Lambda$ denotes the new equilibrium values when the Λ is present. The Λ binding energy is therefore

$$B_\Lambda = E(\rho_\Lambda) - E_\Lambda(\rho_0) = V_0(\rho_0) + \Delta B_\Lambda$$

and ΔB_Λ is thus the contribution to B_Λ due to the effect of compressibility. As expected $\Delta B_\Lambda/V_0$ and $\Delta R_\Lambda/R_0$ are proportional to $1/A$.

An estimate of η may be obtained from our calculation of V_0. Alternatively η may be estimated very simply by assuming that eqn. (6.20) is approximately valid for all $\rho \leq \rho_0$ ¥ and using the fact that $V_0(\rho = 0) = 0$. This gives $\eta \simeq V_0$ and thus $\Delta B_\Lambda/V_0 \approx -9V_0/2AK$ and

$$\Delta R_\Lambda/R_0 \approx 3V_0/AK$$

With $V_0 \approx 30$ MeV corresponding roughly to the empirical

---

¥ Eqn. (6.20) would hold exactly if $V_0 = V_d^{(1)}$ and to a fair approximation if $V_0 = V_e^{(1)}$.

Substituting eqns. (6.20) and (6.19) in eqn. (6.18) and minimising this with respect to ρ gives

$$\frac{\rho_\Lambda - \rho_0}{\rho_0} = \frac{9\eta}{AK} \quad \text{or} \quad \frac{\Delta R_\Lambda}{R_0} = \frac{R_\Lambda - R_0}{R_0} = \frac{3\eta}{AK} \tag{6.21}$$

$$\frac{\Delta B_\Lambda}{V_0} = \frac{E(\rho_\Lambda) - E(\rho_0)}{V_0(\rho_0)} = \frac{-9}{2A} \frac{\eta^2}{K V_0} \tag{6.22}$$

where the subscript $_\Lambda$ denotes the new equilibrium values when the Λ is present. The Λ binding energy is therefore

$$B_\Lambda = E(\rho_\Lambda) - E_\Lambda(\rho_0) = V_0(\rho_0) + \Delta B_\Lambda$$

and ΔB_Λ is thus the contribution to B_Λ due to the effect of compressibility. As expected $\Delta B_\Lambda / V_0$ and $\Delta R_\Lambda / R_0$ are proportional to 1/A.

An estimate of η may be obtained from our calculation of V_0. Alternatively η may be estimated very simply by assuming that eqn. (6.20) is approximately valid for all $\rho \leq \rho_0$ [X] and using the fact that $V_0(\rho_0 = 0) = 0$. This gives $\eta \simeq V_0$ and thus $\Delta B_\Lambda / V_0 \approx -9V_0/2AK$ and

$$\Delta R_\Lambda / R_0 \approx 3V_0 / AK$$

With $V_0 \approx 30$ MeV corresponding roughly to the empirical

---

[X] Eqn. (6.20) would hold exactly if $V_0 = V_d^{(1)}$ and to a fair approximation if $V_0 = V_e^{(1)}$.

value obtained in ref.[9) 10)] and K = 180 MeV as obtained in ref.[15)] we get

$$\Delta B_\Lambda / V_0 \approx -3/4\ A$$

and $$\Delta R_\Lambda / R_0 \approx 1/2\ A$$

Thus ΔB_Λ and ΔR_Λ are quite small even for fairly small A.

Other types of self-consistency effects may arise from a coupling of the Λ with the nuclear surface and will be associated with deformations of this. All such effects on the binding energy are expected to be quite small.

CHAPTER VII

1 Results for the Λ Potential:

For the numerical values of the Λ potential depths in nuclear matter we consider either wholly direct or wholly exchange two-body interactions. In Table V the results for $V_d^{(1)}$, $V_e^{(1)}$ (for $q = 0$) and $V_3^{(1)}$ for ($g = 1$) and for the corresponding first order total depth $V_o^{(1)} = V_{d \text{ or } e}^{(1)} + V_3^{(1)}$ are given, for $U_s > U_t$ and for $k_F = 1.3\ f^{-1}$ and $1.5\ f^{-1}$. These results as a (linear) function of Δ are obtained from the relevant expressions in Chapters V and VI, together with the results from Table III.

The values of $V_d^{(1)}$, $V_e^{(1)}$ are almost independent of the shape and range of the three-body interaction, i.e. of W. This is not surprising since $V_d^{(1)}$ and $V_e^{(1)}$ depend on the same combination $U_2^{(4)} = U_s + 3U_t$, which occurs for ${}_{\Lambda}He_e^5$ and we have already seen in § 4.1 that $U_2^{(4)}$ is largely independent of W. An important closely associated feature follows if it is remembered that for a given $U_2^{(4)}$ and μ the three-body potential energy in ${}_{\Lambda}He^5$ and hence also U_3 for a given W is completely determined by $B_{\Lambda}^{(4)}$. Thus for a given W and depending on whether the two-body interactions are direct or exchange, not only $V_d^{(1)}$ or $V_e^{(1)}$ but also the value of $V_3^{(1)}$ are completely determined as a function of $U_2^{(4)}$ by $B_{\Lambda}^{(4)}$. Looked at in this way the role of $B_{\Lambda}^{(2)}$ is merely to deter-

mine the dependence of $U_2^{(4)}$ on Δ (on δ if $U_t > U_s$). For a given Δ and depending on whether the two-body interactions are direct or exchange the values of $V_3^{(1)}$ in Table V thus correspond to approximately equivalent three-body interactions in the sense of giving almost the same potential energy in ${}_{\Lambda}He^5$. ✱

Table VI gives the values of the effective mass M_{Λ}^{*} and the second order contributions $V_d^{(2)}$ $(q = 0)$ and $V_e^{(2)}$ $(q = 0)$ as a function of Δ for W_E with range ν_π^{-1}.

The effective mass, obtained using eqns. (5.17) and (5.18) is relevant only for exchange forces. The difference $M_{\Lambda} - M_{\Lambda}^{*}$ is small mainly because of the short range μ_K^{-1} which implies that only a small proportion of the interactions occur in relative angular momentum states with $\ell \gg 1$. The use of an effective mass is in any case only of significance for fairly light hypernuclei since only for these will the kinetic energy make an appreciable contribution to the Λ binding energy.

The values for $V_e^{(2)}$ $(q = 0)$ obtained using eqns. (5.11) and (5.12) are to be regarded only as upper limits. It is

---

✱ The value of $V_3^{(1)}$ corresponding to exactly equivalent three-body interactions may be obtained by adjusting the values in Table V in the proportions required to give the same values of $\xi^{(4)}$ in Table III.

seem that the ratios $V_d^{(2)}$ $(q = 0)$ / $V_d^{(1)}$ and $V_e^{(2)}$ $(q = 0)$/ $V_e^{(1)}$ are thus probably of comparable magnitude and are quite small (about .05 to .1) except when $V_d^{(1)}$ (or $V_e^{(1)}$) are very small in which case the three-body interactions and hence $V_3^{(1)}$ will be large. Since, as discussed in §6.3, the ratio $V_3^{(2)}/ V_3^{(1)}$ is expected to be even smaller than $V_d^{(2)}/ V_d^{(1)}$, for comparable magnitudes of $V_3^{(1)}$ and $V_d^{(1)}$, the first order perturbation theory is expected to be a rather good approximation for a Λ in nuclear matter. This also provides justification for the use of a single particle Λ wave function in the case of a finite hypernucleus. The ratio $V_d^{(2)}/ V_d^{(1)}$ for a nucleon is probably somewhat larger (about .2) than for a Λ if soft static N-N forces reproducing the low energy data are used (see, for example, ref. [25]) where further references are also given. Thus the smaller effective Λ- N interaction probably more than compensates the lesser effectiveness of the Exclusion Principle in reducing the second order contribution for a Λ as compared for a nucleon. However the ratio $V_e^{(2)}/ V_e^{(1)}$ is most probably much smaller (about .01) for a nucleon than for a Λ because of the longer range of the N-N interaction.

If we calculate $V_3^{(1)}$ from the values of U_3 given in Table III then the effect of pair correlations will turn out to be less than that corresponding to the values of $(V_3^{(1)} - V_{3c}^{(1)}) / V_3^{(1)}$ given in Table IV. This is because the values of U_3 in Table III which are determined essentially by

$_{\Lambda}He^{5}$ were obtained neglecting pair correlations in He^{4}. The effect of these is expected to be comparable to that for nuclear matter, though slightly less because of the smaller average density of He^{4}. Since the reduction of $V_3^{(1)}$ by pair correlations is already small even without allowing for this compensation the values of $V_3^{(1)}$ for $g = 1$ given in Table V may be considered as reasonably accurate predictions from the values of U_3 given in Table III.

In view of the above discussions we conclude that the first order results of Table V give reasonably accurate predictions (to within about 10%) from the volume integrals of Table III.

2 Discussion of the Results for $V_d^{(1)}$ and $V_e^{(1)}$:

An essential difference between the effect of a given Λ - N interaction in $_{\Lambda}He^{5}$ and in nuclear matter is due to the finite range of the interaction. Thus $V_d^{(1)}$ depends on the interaction only through $U_2^{(4)}$ while the Λ potential for $_{\Lambda}He^{5}$ (eqn. 3.4) also depends on μ as is reflected in the different values of $U^{(4)}$ obtained for $\mu_{2\pi}$ and μ_{K}.

In fact most of the difference between $V_d^{(1)}$ and $V_e^{(1)}$ is due to this difference in $U^{(4)}$. Thus the contribution to $V_e^{(1)}$ of the second, range dependent, term in the expansion (5.19) is quite small, while that of the higher terms are almost negligible. This corresponds to a predominance of

interactions in relative s-states. Table VII gives the proportions of interaction energies in relative s and p-states to the total interaction energy. Even for the longer range $\mu_{2\pi}^{-1}$ the interactions are seen to occur predominantly in s-states. The main difference between direct and exchange forces is thus due to the difference in the s-state interactions which arise from the range effect in ${}_{\Lambda}He^{5}$, the specific effect of the exchange character of the interactions being relatively unimportant for a Λ in nuclear matter.

Also shown in Table VII are the potential depths in the absence of three-body forces (i.e. for $U_3 = 0$) together with the corresponding depths when only interactions in s-states are included. What are the implications of these results, if the somewhat doubtful empirical value V_o^1 of $V_o^{10)}$ quoted in § 5.1 are accepted? These values together with those of Table VII seem to suggest for $U_3 = 0$, a predominantly exchange interaction. Even allowing for a considerable error in the empirical values V_o^1 it is difficult to reconcile these with a predominantly direct Λ - N interaction even if this were to be strongly weakened in p-states due to a strong velocity dependence. ✱ On the other hand it is not easy to reconcile predominantly exchange interactions with the known strong pion

---

✱ A very strong velocity dependence is however hardly likely since in lowest order the direct interaction is due to the exchange of two relatively light pions.

hyperon couplings.

Our conclusions about the predominance of exchange interactions agree with those of Walecka [10] who with the hard core (direct) forces of ref.[26] obtains $V_s^{(1)} \doteq 23$ MeV (for $k_F = 1.5\ f^{-1}$) but a considerably larger contribution than ours from p-states. In fact a hard core is expected to enhance the importance of states with $\ell \geq 1$, and hence the difference between ordinary and exchange forces, since the repulsive contributions due to the hard core will be relatively less effective for states with $\ell \geq 1$ than for s-states because of the centrifugal barrier. However, since Walecka did not consider the distortion of the relative Λ- nucleon wave function by the deep attractive tail outside the hard core it is difficult to judge the reliability of his value of $V_s^{(1)}$. An increase of the attractive contribution due to the tail by about 20% would bring Walecka's value into agreement with ours. We consider an increase of such an order of magnitude quite possible in view of the approximately 10% second order contributions we have found for soft forces and because of the large depth of the attractive tail for hard core forces.

3 Discussion of First Order Potential Depths, including Three-Body Interactions:

The potential depth $V_3^{(1)}$ depends not only on U_3 but also quite strongly on the range of W, since we are quite far from

the short range limit ($\alpha \gg 1$). This is in contrast to the analogous situation for $V_d^{(1)}$ and $V_e^{(1)}$; the former is independent of μ while the latter depends only weakly on μ for a given $U_2^{(4)}$. The range dependence of $V_3^{(1)}$ is to some extent compensated by that of U_3 (Table III) which arises from the range dependence in ${}_\Lambda He^5$. From Table V we note that for a given Δ and μ (i.e. for three-body forces which are almost equivalent for ${}_\Lambda He^5$) the magnitude of $V_3^{(1)}$ increases with the range of W, the variation of $V_3^{(1)}$ between $\Delta = 0$ and 400 MeV f^3 becoming larger as this range increases. The dependence of $V_3^{(1)}$ on the range of W is rather less than the difference between $V_d^{(1)}$ and $V_e^{(1)}$.

The variations with Δ of $V_3^{(1)}$ and $V_d^{(1)}$ or $V_e^{(1)}$ tend to compensate each other. However for the larger values of Δ there is now very little difference, for given three-body interactions, between the total depth $V_o^{(1)}$ for direct and exchange forces. Thus with Δ = 400 MeV f^3 for $\mu_{2\pi}$ and Δ = 300 MeV f^3 for μ_K (as obtained from the rough analysis of ${}_\Lambda Li^7$ and ${}_\Lambda He^7$ in §4.5) it is no longer possible to distinguish between direct and exchange forces from a knowledge of V_o. However this could now give information about the range of the three-body interactions. It should be noted that since both the direct two-body interactions as well as the three-body interactions arise from the pion-baryon couplings, a large value of Δ, because it implies a strong

three-body interaction, is expected to be consistent only with a predominantly direct two-body interaction. For small Δ, $V_3^{(1)}$ is fairly small and the situation is similar to that already discussed for $U_3 = 0$. If $U_t > U_s$ the three-body interactions are also relatively small and this case is therefore also similar to that for $U_3 = 0$, since $V_d^{(1)}$ or $V_e^{(1)}$ are determined by $U_2^{(4)}$ (and thus by U^4) for $U_3 = 0$) independently of whether $U_s \gtrless U_t$. Thus both for small Δ and for $U_t > U_s$ it would be possible to distinguish between direct and exchange forces from a knowledge of V_o.

It is seen then that when three-body forces are included the empirical value V_o^1 of V_o is no longer inconsistent with predominantly direct two-body forces so long as Δ is sufficiently large. The empirical value V_o^1 then favours the smaller values of the three-body range without discriminating between direct and exchange forces. However in view of the large three-body interactions for large Δ this case is expected to be more consistent with predominantly direct two-body forces. If in fact this is the case, as is likely in view of the strength of the pion-baryon couplings, then V_o^1 is only consistent with the smaller values of ν^{-1} and with large Δ ($\simeq$ 400 MeV f^3). This may be taken as evidence for $U_s > U_t$ and therefore for zero spin for $_\Lambda He^4$ and thus a pseudoscalar K meson.

For small Δ as well as for $U_t > U_s$, V_o^1 is consistent

only with predominantly exchange interactions. For the predominantly strong K - Λ couplings, which are implied by mainly exchange interactions, the associated weak direct interactions are expected to be linked in turn with rather weak three-body interactions. This is so for small Δ and for $U_t > U_s$ and these two cases are therefore internally consistent if predominantly strong K - Λ couplings are accepted as a possibility.

With ordinary three-body interactions the potential depth $V_{3\ ord}^{(1)}$ given by eqn.(6.17) is much larger in magnitude than $V_3^{(1)}$. Thus for W_Y with ν_π one has for $\Delta = 0$ and for direct two-body forces ($\mu_{2\pi}$)

$$U_3 \times \rho^2/2 = -110 \text{ and } -47 \text{ MeV}$$

for $k_F = 1.5$ and 1.3 f respectively the corresponding value for exchange forces (μ_K) being - 29 and - 12 MeV while for $\Delta =$ 400 MeV f^3 the values are 223 and 94 MeV for $\mu_{2\pi}$ and 432 and 182 MeV for μ_K. For longer three-body ranges the corresponding values are even larger in magnitude. Thus the first term in eqn. (6.17) completely determines the second term and unless U_3 is quite small an 'ordinary' three-body interaction is excluded by V_o^1. The magnitude of $V_{3\ ord}^{(1)}$ is so much larger than that of $V_3^{(1)}$ for the same U_3 (i.e. for the same three-body potential energy in the light hypernuclei with $A \lesssim 5$) because of the opposite sign of ($\sigma_1 . \sigma_2$) x ($\tau_1 . \tau_2$) for even and odd states. The latter make a substantial contribution for nuclear matter because of the relatively long ranges which

correspond to the small values of α which are of interest. Only for short ranges with $\alpha \gg 1$, would $V_3^{(1)}$ and $V_{3\,ord}^{(1)}$ be approximately the same.

4 A Better Method for the Λ Binding Energy in a Heavy Nucleus:

The Λ binding energy for a heavy but finite nucleus may be obtained more accurately by using the calculated results for the potential depths as a function of ρ to obtain the Λ potential $V_\Lambda(r)$ as a function of r from the empirical nuclear density distribution $\rho(r)$. The binding energy is then obtained by solving the 1s state eigenvalue problem for the potential $V_\Lambda(r)$, ¥ this procedure corresponding to the extreme statistical approximation. Because of the short range of the Λ– N interactions this approximation may be expected to be quite good, since its validity depends essentially on the range of the Λ - N interactions being small compared to the nuclear surface thickness.

Assuming $V(\rho) \propto \rho$ and using a Fermi distribution for $\rho(r)$ gives

$$V_\Lambda(r) = -V_o \Big/ \left(1 \quad \exp \frac{r-c}{a}\right)$$

where c is the half density radius, a determines the surface

¥ A further refinement for somewhat lighter nuclei would be to use an effective mass M_Λ^* instead of M_Λ.

thickness and V_o is very nearly the potential depth for nuclear matter having the central density $\rho(0)$. With the electron scattering values for c and a [16] we obtain for the binding energy $B_\Lambda^{(A)}$:

$$B_\Lambda^{(100)} = 43.2,\ 33.6 \text{ and } 24 \text{ MeV}$$

for V_o = 50, 40 and 30 MeV respectively,

and $B_\Lambda^{(200)}$ = 35.5, 26 MeV,

for V_o = 40 and 30 MeV respectively.

The difference $V_o - B^{(A)}$ is due to the finite nuclear size. For predominantly two-body forces $V(\rho) \propto \rho$ is a good approximation since this proportionality holds exactly for direct forces and very nearly so for exchange forces. For three-body forces $V_3^{(1)}(r)$ may be obtained from our results for $V_3^{(1)}(\rho)$ and will fall off more rapidly than $\rho(r)$ since for small ρ $V_3^{(1)} \propto \rho^2$.

5 Conclusion:

The neglect of two-body Λ - N tensor forces seems justified for nuclear matter since in first order these will average out to zero and the leading, second order, contributions will be quite small in view of the small second order contributions found for central forces and of the predominance of s-state interactions for the short ranges involved. Four-body forces involving an interaction of the Λ with three nucleons will have an asymptotic dependence of the type

$$\exp - \tau (r_{1\lambda} + r_{2\lambda} + r_{3\lambda}) \quad \text{with} \quad \tau = \mu_\pi$$

arising in lowest order from an exchange of one pion with each of three nucleons. For these interactions to be effective the probability of finding three nucleons within a volume corresponding to a radius $\tau^{-1}/2 = \mu_{2\pi}^{-1}$ must be appreciable. But we know that three-body clusterings corresponding to a radius μ_{π}^{-1} is small [27] for nuclear matter. Thus four- (and also more) body forces are expected to be unimportant even if of comparable strength to three-body forces.

A neutron excess will have no effect for charge symmetric two-body Λ - N interactions if these are direct and only a small effect if these are exchange, because of the predominance of interactions in s-states. For three-body interactions the effect of a neutron excess is also expected to be small (of relative order of magnitude (N - Z)/A) due to the presence of the spin part of the operator $(\underline{\sigma}_1 \cdot \underline{\sigma}_2)(\underline{\tau}_1 \cdot \underline{\tau}_2)$. It should be noted that the empirical value V_o' [9) 10)] refers essentially to nuclear matter with N = Z since it is based on an extrapolation from light hypernuclei for which the neutron excess is either zero or else small and of either sign.

We conclude by pointing out the desirability of obtaining direct knowledge about the Λ binding energies in heavier hypernuclei. Such knowledge would be useful in elucidating the Λ - N interaction. However to make full use of it the spin dependence of the Λ - N interaction would have to be obtained from the lighter hypernuclei. Conditions in the hyper-

nuclei with A > 5 may already show some of the features characteristic for a Λ in nuclear matter. Thus if the three-body interactions are large it may be difficult to decide between direct and exchange two-body forces from the Λ binding energies for these hypernuclei, while if the three-body interactions are small this may be possible.

APPENDIX

Potential Energy Integrals for $_{\Lambda}H^3$

The evaluation of the two-body potential energy integral X in eqn. (2.5) may be found in ref.[2] but we give the results here for the sake of completeness. The evaluation of the three-body integrals Y in eqn. (2.6) is made in an analogous manner.

We define the following integrals as in ref.[2]

$$I_{\ell m n}(x, y, z) = (-1)^{\ell+m+n} \left(\frac{\partial}{\partial x}\right)^{\ell} \left(\frac{\partial}{\partial y}\right)^{m} \left(\frac{\partial}{\partial z}\right)^{n} I_{000}(x, y, z)$$

where

$$I_{000}(x, y, z) = \int e^{-(x r_1 + y r_2 + z r_3)} dr_1 dr_2 dr_3 = \frac{2}{(x+y)(y+z)(z+x)} \qquad (A.1)$$

All the integrations must be done with r_1, r_2, r_3 satisfying the usual triangular inequalities. Then with ψ given by eqn. (2.3)

$$X = \int \psi^{+} \left\{ \frac{e^{-\mu r_1}}{\mu r_1} + \frac{e^{-\mu r_2}}{\mu r_2} \right\} \psi \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= \sum_{\substack{\alpha\beta\gamma \\ \alpha'\beta'\gamma'}} \lambda_{1\alpha} \lambda_{2\beta} \lambda_{3\gamma} \lambda_{1\alpha'} \lambda_{2\beta'} \lambda_{3\gamma'} \, u_{\alpha\beta\gamma, \alpha'\beta'\gamma'}$$

with

$$u_{\alpha\beta\gamma, \alpha'\beta'\gamma'} = \int e^{-(a_{1\alpha} r_1 + a_{2\beta} r_2 + a_{3\gamma} r_3)} \left(\frac{e^{-\mu r_1}}{\mu r_1} + \frac{e^{-\mu r_2}}{\mu r_2} \right) e^{-(a_{1\alpha'} r_1 + a_{2\beta'} r_2 + a_{3\gamma'} r_3)} r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= \frac{1}{\mu} \left[I_{011}(A+\mu, B, C) + I_{101}(A, B+\mu, C) \right]$$

where

$$A = a_{1\alpha} + a_{1\alpha'} \quad , \quad B = a_{2\beta} + a_{2\beta'} \quad , \quad C = a_{3\gamma} + a_{3\gamma'}$$

and where the sum is over all the 64 possible combinations of α , β , γ with γ' , α', β' .

For a three-body Yukawa interaction, i.e. for W_Y given by eqn. (1.4b), one has for the three-body integral Y of eqn. (2.6)

$$Y_Y = \int \psi^+ \frac{e^{-\nu r_1}}{\nu r_1} \cdot \frac{e^{-\nu r_2}}{\nu r_2} \psi \, r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= \sum_{\substack{\alpha\beta\gamma \\ \alpha'\beta'\gamma'}} \lambda_{1\alpha} \lambda_{2\beta} \lambda_{3\gamma} \lambda_{1\alpha'} \lambda_{2\beta'} \lambda_{3\gamma'} \chi_{\gamma\alpha\beta\gamma\alpha'\beta'\gamma'}$$

with

$$\chi_{\gamma\alpha\beta\gamma,\alpha'\beta'\gamma'} = \int e^{-(a_{1\alpha} r_1 + a_{2\beta} r_2 + a_{3\gamma} r_3} \left(\frac{e^{-\nu r_1}}{\nu r_1} \cdot \frac{e^{-\nu r_2}}{\nu r_2} \right) e^{-(a_{1\alpha'} r_1 + a_{2\beta'} r_2 + a_{3\gamma'} r_3)} r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= \frac{1}{\nu^2} I_{001}(A+\nu, B+\nu, C)$$

For a three-body exponential interaction, i.e. for W_E given by eqn. (1.4a)

$$Y_E = \sum_{\substack{\alpha\beta\gamma \\ \alpha'\beta'\gamma'}} \lambda_{1\alpha} \lambda_{2\beta} \lambda_{3\gamma} \lambda_{1\alpha'} \lambda_{2\beta'} \lambda_{3\gamma'} \chi_{E\,\gamma\beta\gamma,\alpha'\beta'\gamma'}$$

with

$$\chi_{\alpha\beta\gamma,\alpha'\beta'\gamma'} = \int e^{-(a_{1\alpha}r_1 + a_{2\beta}r_2 + a_{3\gamma}r_3)} e^{-\nu r_1} e^{-\nu r_2} e^{-(a_{1\alpha'}r_1 + a_{2\beta'}r_2 + a_{3\gamma'}r_3)} r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= I_{111}(A+\nu, B+\nu, C)$$

The I_{lmn} relevant for the above expressions are easily evaluated and are

$$I_{111}(x,y,z) = \frac{8\left[x(x+y)(x+z) + y(y+z)(y+x) + z(z+x)(z+y) + 2(x+y)(y+z)(z+x)\right]}{(x+y)^3(y+z)^3(z+x)^3}$$

$$I_{001}(x,y,z) = \frac{2(x+y+2z)}{(x+y)(y+z)^2(z+x)^2}$$

$$I_{011}(x,y,z) = I_{110}(z,y,x)$$

$$I_{101}(x,y,z) = I_{110}(x,z,y)$$

$$I_{110}(x,y,z) = \frac{4\left[(x+y)(x+y+z) + (x+z)(y+z)\right]}{(x+y)^3(y+z)^2(z+x)^2}$$

Using these expressions the numerical evaluation of X and Y is straightforward.

TABLE I

Volume Integrals for $_{\Lambda}H^{3}$

		$\mu_{2\pi} = 1.43\ f^{-1}$			$\mu_{K} = 2.52\ f^{-1}$		
$B_{\Lambda}^{(2)}$ (MeV)	ν (f^{-1})	$U^{(2)}$ (MeV f^3)	$Z_E^{(2)}$ (f^3)	$Z_Y^{(2)}$ (f^3)	$U^{(2)}$ (MeV f^3)	$Z_E^{(2)}$ (f^3)	$Z_Y^{(2)}$ (f^3)
0	0.5	615	9558	812	405	20082	1128
	0.72	615	2328	291	405	4449	373
	1	615	726	125	405	1248	148
0.25	0.5	667	10698	758	420	22452	1091
	0.72	667	2454	261	420	4761	353
	1	667	726	109	420	1290	137
1	0.5	738	11854	692	444	25602	1037
	0.72	738	2555	233	444	5136	327
	1	738	716	94	444	1332	124
0.12	0.5	654	10200	776	416	21500	1106
	0.72	654	2420	270	416	4600	362
	1	654	730	116	416	1275	142

TABLE I

Volume Integrals for $_{\Lambda}H^{3}$

		$\mu_{2\pi} = 1.43\ f^{-1}$			$\mu_{K} = 2.52\ f^{-1}$		
$B_{\Lambda}^{(2)}$ (MeV)	ν (f^{-1})	$U^{(2)}$ (MeV f^3)	$Z_E^{(2)}$ (f^3)	$Z_Y^{(2)}$ (f^3)	$U^{(2)}$ (MeV f^3)	$Z_E^{(2)}$ (f^3)	$Z_Y^{(2)}$ (f^3)
0	0.5	615	9558	812	405	20082	1128
	0.72	615	2328	291	405	4449	373
	1	615	726	125	405	1248	148
0.25	0.5	667	10698	758	420	22452	1091
	0.72	667	2454	261	420	4761	353
	1	667	726	109	420	1290	137
1	0.5	738	11854	692	444	25602	1037
	0.72	738	2555	233	444	5136	327
	1	738	716	94	444	1332	124
0.12	0.5	654	10200	776	416	21500	1106
	0.72	654	2420	270	416	4600	362
	1	654	730	116	416	1275	142

TABLE II

Volume Integrals for $_{\Lambda}He^{5}$ for

$B_{\Lambda}^{(4)} = 3.08$ MeV

	$\mu_{2\pi} = 1.43\ f^{-1}$			$\mu_{K} = 2.5\ 2f^{-1}$		
ν (f^{-1})	$U^{(4)}$ (MeV f^3)	$Z_E^{(4)}$ (f^3)	$Z_Y^{(4)}$ (f^3)	$U^{(4)}$ (MeV f^3)	$Z_E^{(4)}$ (f^3)	$Z_Y^{(4)}$ (f^3)
0.5	1028	656.7	70.2	780	865.5	92.5
0.72	1028	160.6	27.0	780	211.7	35.6
1	1028	52.1	12.8	780	68.7	16.9

TABLE III

Volume Integrals of the - N Interaction

Three-Body Interaction	Δ (MeV f^3)	$\mu_{2\pi} = 1.43$ f^{-1}, $U^{(2)} = 654$ MeV f^3, $U^{(4)} = 1028$ MeV f^3						
		$U_2^{(2)}$ (MeV f^3)	$U_2^{(4)}$ (MeV f^3)	U_s (MeV f^3)	U_3 (MeV f^6)	$\xi^{(2)}$	$\xi^{(4)}$	$\varkappa$
W_Y with $\nu =$ 1.0 f^{-1}	0	694	1388	347	-4615	-0.061	-0.35	9
	200	637	874	368	1978	0.026	0.15	9
	400	580	360	390	8571	0.113	0.65	9
W_Y with $\nu =$ 0.5 f^{-1}	0	685	1370	342	-24.000	-0.047	-0.33	11.1
	200	641	881	370	10286	0.020	0.143	11.1
	400	597	393	398	44573	0.087	0.618	11.1
W_E with $\nu =$ 0.72 f^{-1}	0	675	1351	338	-51858	-0.033	-0.314	14.9
	200	645	890	372	22225	0.014	0.135	14.9
	400	614	428	407	96308	0.061	0.583	14.9

TABLE III

Volume Integrals of the Λ- N Interaction

Three-Body Interaction	Δ (MeV f^3)	μ_K = 2.52 f^{-1}, $U^{(2)}$ = 416 MeV f^3, $U^{(4)}$ = 780 MeV f^3						
		$U_2^{(2)}$ (MeV f^3)	$U_2^{(4)}$ (MeV f^3)	U_s (MeV f^3)	U_3 (MeV f^6)	$\xi^{(2)}$	$\xi^{(4)}$	χ
W_Y with	0	424	848	212	-1155	-0.020	-0.088	8.2
ν =	200	362	323	231	7728	0.113	0.586	8.2
1.0 f^{-1}	400	299	-202	250	16612	0.286	1.259	8.2
W_Y with	0	421	842	211	-5778	-0.013	-0.080	11.5
ν =	200	381	361	241	38671	0.087	0.536	11.5
0.5 f^{-1}	400	341	-118	270	83120	0.187	1.152	11.5
W_E with	0	419	837	209	-12124	-0.006	-0.073	22.8
ν =	200	398	397	249	81137	0.040	0.491	22.8
0.72 f^{-1}	400	378	-44	289	174397	0.087	1.056	22.8

TABLE IV

Effect of short range pair correlations on $V_3^{(1)}$ for

$$\nu = 0.7\ f^{-1}$$

k_F , $\alpha = \nu_\pi / k_F$		Yukawa shape W_Y			Exponential shape W_E		
		I_{3Y}	I_{3Yc}	$\frac{V_3^{(1)} - V_{3c}^{(1)}}{V_3^{(1)}}$	I_{3E}	I_{3Ec}	$\frac{V_3^{(1)} - V_{3c}^{(1)}}{V_3^{(1)}}$
$1.3\ f^{-1}$	0.554	1.22	1.06	0.13	2.8	3.06	0.08
$1.5\ f^{-1}$	0.48	1.46	1.24	0.15	3.0	3.32	0.09

TABLE V

First Order Potential Depths (in MeV) for a Λ in Nuclear Matter

Three-body Inter-actions	Δ (MeV f^3)	Direct two-body interactions ($\mu_{2\pi} = 1.43\,f^{-1}$) $k_F = 1.5\,f^{-1}$ $v_d^{(1)}$	$v_3^{(1)}$	$v_o^{(1)}$	$k_F = 1.3\,f^{-1}$ $v_d^{(1)}$	$v_3^{(1)}$	$v_o^{(1)}$	Exchange two-body interactions ($\mu_K = 2.52\,f^{-1}$) $k_f = 1.5\,f^{-1}$ $v_e^{(1)}$	$v_3^{(1)}$	$v_o^{(1)}$	$k_f = 1.3\,f^{-1}$ $v_e^{(1)}$	$v_3^{(1)}$	$v_o^{(1)}$
W_Y	0	79.1	-13.8	65.3	51.5	-7.5	44	40.1	-3.5	36.6	27.2	-1.9	25.3
$\nu =$	200	49.8	5.9	55.7	32.4	-3.2	35.6	15.3	23.1	38.4	10.4	12.6	23.0
$1.0\,f^{-1}$	400	20.5	25.6	46.1	13.4	14.0	27.4	-9.5	49.1	39.6	-6.5	27.1	20.6
W_Y	0	78.1	-17.0	61.1	50.8	-9.9	40.9	39.8	-4.1	35.7	27.0	-2.4	24.6
$\nu =$	200	50.2	7.3	57.5	32.7	4.3	37.0	17.1	27.3	44.4	11.6	16.0	27.6
$0.5\,f^{-1}$	400	22.4	31.5	53.9	14.6	18.5	33.1	-5.6	58.7	53.1	-3.8	34.4	30.6
W_E	0	77	-23.2	53.8	50.1	-14.2	35.9	39.6	-5.4	34.2	26.9	-3.3	23.6
$\nu =$	200	50.7	10	60.7	33	6.1	39.1	18.8	36.3	55.1	12.7	22.3	35.0
$0.72\,f^{-1}$	400	24.4	43.1	67.5	15.9	26.1	4.2	-21	78.1	76	-1.4	48.6	47.2

TABLE VI

Second Order Contributions to the Potential Depths and the Effective Mass for a Λ in Nuclear Matter

k_F (f^{-1})	Δ (MeV f^3)	Direct two-body interactions ($\mu_{2\pi} = 1.43\ f^{-1}$)				Exchange two-body interactions ($\mu_K = 2.52\ f^{-1}$)				$\frac{M_\Lambda^*}{M_\Lambda}$
		$V_d^{(2)}(q=0)$ in MeV		$V_d^{(2)}(q=0)/V_d^{(1)}$		$V_e^{(2)}(q=0)$ in MeV		$V_e^{(2)}(q=0)/V_e^{(1)}(q=0)$		
		$M_N^x = M_N$	$M_N^x = 0.7\,M_N$	$M_N^x = M_N$	$M_N^x = 0.7 M_N$	$M_N^x = M_N$	$M_N^x = 0.7 M_N$	$M_N^x = M_N$	$M_N^x = 0.7\,M_N$	
1.5	0	7.5	5.4	0.097	0.07	4.7	3.3	0.118	0.083	0.81
	200	3.5	2.4	0.069	0.047	1.8	1.3	0.095	0.069	0.90
	400	2.1	1.5	0.086	0.061	3.2	2.2	-1.52	-1.0	1.02
1.3	0	5.5	3.9	0.11	0.078	3.5	2.5	0.13	0.093	0.85
	200	2.5	1.8	0.076	0.055	1.4	1.0	0.11	0.079	0.93
	400	1.5	1.0	0.094	0.063	2.4	1.7	-1.72	-1.21	1.01

TABLE VII

Interaction Energies in Different Relative Angular Momentum States

	$k_F(f^{-1})$	Total: s-state: p-state	Potential depths (MeV) for $U_3 = 0$ $V_d(1)$	$V_s(1)$
$\mu_{2\pi}$	1.3	1 : 0.86 : 0.14	38	32.7
	1.5	1 : 0.78 : 0.18	59	46.1
			$V_e(1)$	
μ_K	1.3	1 : 1.190 : 0.065	25	26.5
	1.5	1 : 1.190 : 0.08	37.5	40.5

REFERENCES

1) R.H. Dalitz and B.W. Downs, Phys. Rev. 110 (1958) 985.

2) R.H. Dalitz and B.W. Downs, Phys. Rev. 114 (1959) 593.

3) R.H. Dalitz and B.W. Downs, Phys. Rev. 111 (1958) 967.

4) R.H. Dalitz, Ninth Annual International Conference on High Energy Physics, Kiev. 1959 (unpublished).

5) D.B. Lichtenberg, Nuovo. Cim. 8 (1959) 463.

6) S. Iwao and E.C.G. Sudarshan, Phys. Rev. Letters 4 (1960) 140.
S. Iwao, Nuovo. Cim. 17 (1960) 491.

7) G. Wentzel, Phys. Rev. 101, 835 (1956).
D.B. Lichtenberg and M. Ross, Phys. Rev. 103 (1956) 1131 and 107 (1957) 1714.
N. Dallaporta and F. Ferrari, Nuovo. Cim. 5 (1957) 111.
F. Ferrari and L. Fonda, Nuovo. Cim. 9 (1958) 842.

8) R. Levi Setti and W.E. Slater, Nuovo. Cim. 14 (1959) 895.

9) V.A. Filimonov, JETP, 36 (1959) 1113 (English Translation).

10) J.D. Walecka, Nuovo. Cim. 16 (1960) 342.

11) R.G. Ammar, R. Levi Setti, W.E. Slater, S. Limentani, P.E. Schlein and P.H. Steinberg, Nuovo. Cim. 15 (1960) 181.

12) R. Spitzer, Phys. Rev. 110 (1958) 1190.
H. Weitzner, Phys. Rev. 110 (1958) 593.

13) H.J. Mang and W. Wild, Z.f. Phys. 154 (1959) 182.

14) L.D. Pearlstein, Y.C. Tang and K. Wildermuth, Nuclear Physics 18 (1960) 23.

15) K.A. Brueckner and J.L. Gammel, Phys. Rev.109 (1958) 1023.

16) R. Hofstadter, Revs. Mod. Phys. 28 (1956) 214.

17) R. Hofstadter, Ann. Rev. Nucl. Sci. 7 (1957).

18) J.D. Jackson and J. M. Blatt, Revs. Mod. Phys. 22 (1950) 77.
J.M. Blatt and V.F. Weisskopf, Theoretical Nuclear Physics (John Wiley, New York, 1952).

19) L.M. Brown and M. Peshkin, Phys. Rev. 107 (1957) 272.
M. Wilhelmson and P. Zielinski, Nuclear Physics 6 (1958) 219.
K.S. Suh, Phys. Rev. 111 (1958) 941.

20) S. Iwao, Preprint , University of Rochester, (1960).

21) U. Meyer-Berkhout, K.W. Ford and A.E.S. Green, Annals of Phys. 8 (1959) 119.

22) H. Euler, Z.f. Physik 105 (1937) 553.

23) R. Jastrow, Phys. Rev. 98 (1955) 1479.

24) V.J. Emery, Nuclear Physics, 6 (1958) 581.

25) J. Levinger, Nuclear Physics, 19 (1960) 370.

26) J. Kovacs and D.B. Lichtenberg, Nuovo. Cim. 13 (1959) 37.

27) S.D. Drell and K. Huang, Phys . Rev. 91 (1953) 1527.
A.De Shalit and V.F. Weisskopf, Annals of Phys. 5 (1958) 282.

FIGURE CAPTIONS

Figure 1. The dimensionless integral $I^{(2)}(\mu/k_F)$, given by eqn. (5.12) and occurring for the second order potential depth $V_d^{(2)}$ ($q = 0$) in eqn. (5.11) is shown as a function of μ/k_F and also as a function of k_F for $\mu_{2\pi} = 1.43\ f^{-1}$.

Figure 2. The diagram represents a contribution to $V_e^{(1)}$ of the second order in the nuclear interaction v_N but of the first order in the $\Lambda - N$ exchange interaction.

Figure 3. The dimensionless integrals I_3, occurring for the first order three-body potential depth $V_3^{(1)}$, are shown as a function of $\alpha = \nu/k_F$. Without nuclear pair correlations, i.e. for $g = 1$, the appropriate integrals I_{3E} and I_{3Y} are functions only of α. The curves for I_{3Ec} and I_{3Yc} are for $\nu_\pi = 0.7 f^{-1}$ and for the correlation function of eqn. (6.16) with $\beta = 0.5\ f^{-1}$ and $r_c = 0.4$ f.

Figure 4. The diagram represents a second order contribution due to the $\Lambda - N$ three-body interaction.

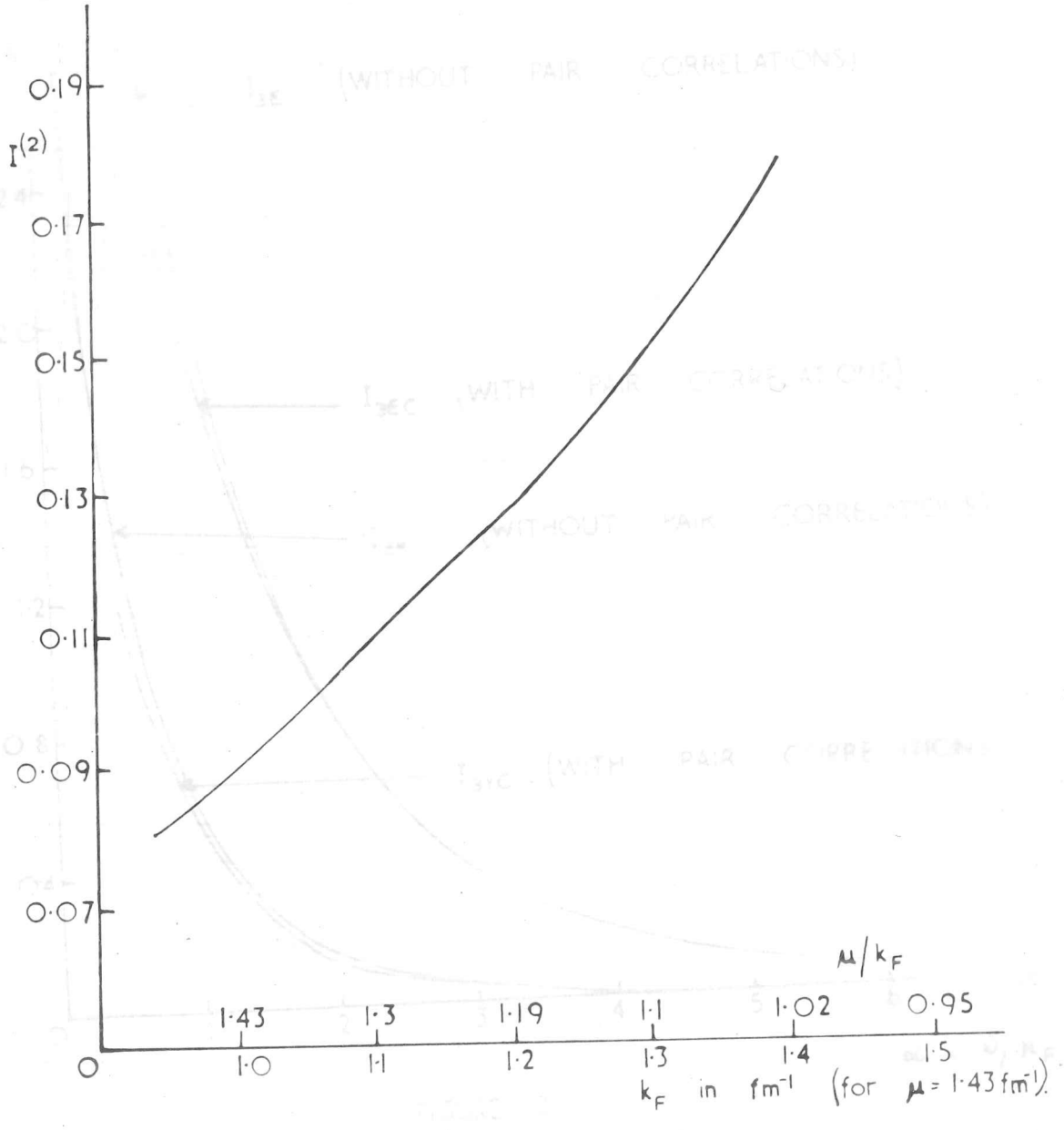

0·19
I(2)
0·17
0·15
0·13
0·11
0·09
0·07
0
μ/kF
1·43
1·3
1·19
1·1
1·02
0·95
1·0
1·1
1·2
1·3
1·4
1·5
kF in fm-1 (for μ = 1·43 fm-1).

FIGURE 1.

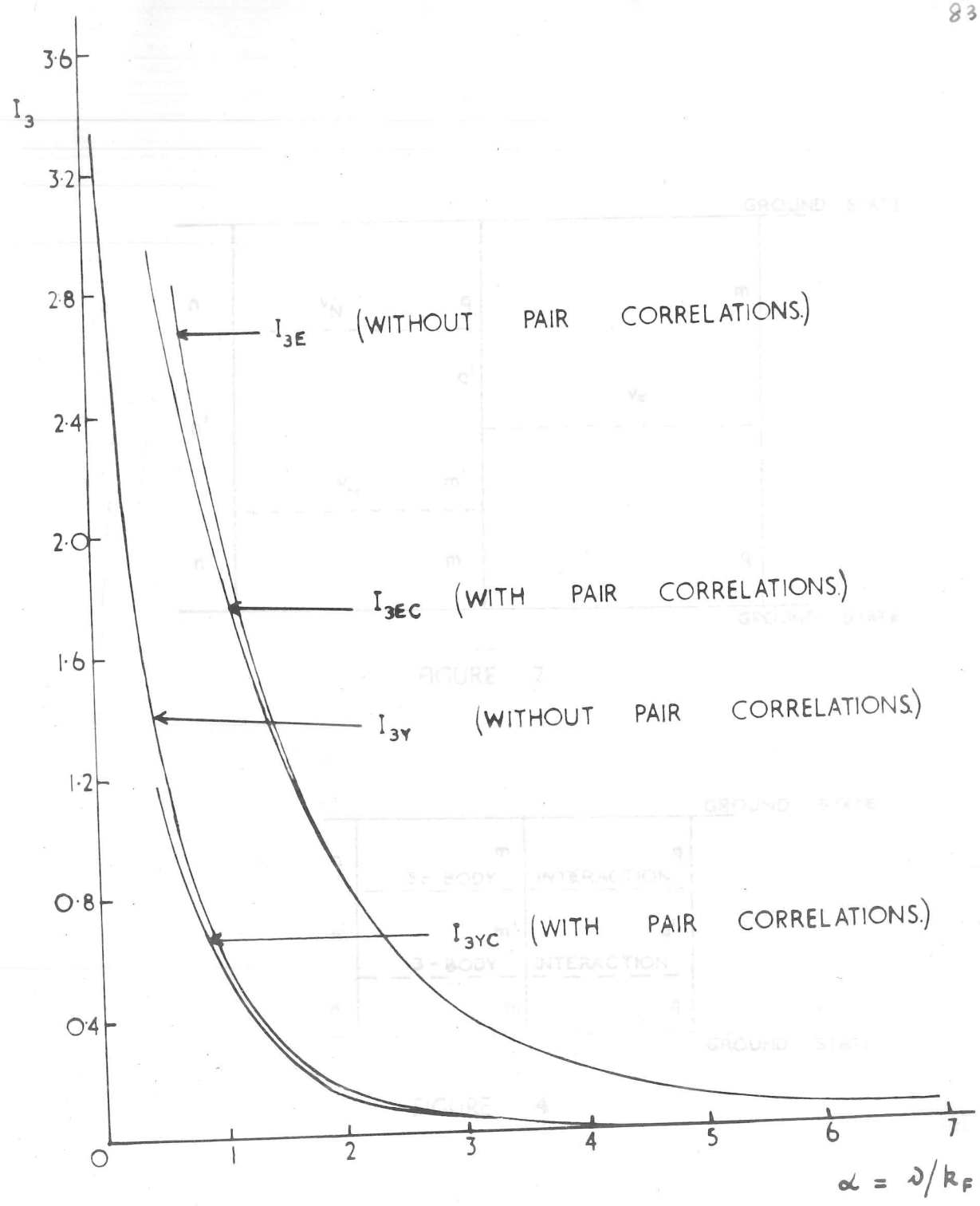

3·6
I_3
3·2
2·8
2·4
2·0
1·6
1·2
0·8
0·4
0
1
2
3
4
5
6
7
I_{3E} (WITHOUT PAIR CORRELATIONS.)
I_{3EC} (WITH PAIR CORRELATIONS.)
I_{3Y} (WITHOUT PAIR CORRELATIONS.)
I_{3YC} (WITH PAIR CORRELATIONS.)
$\alpha = \nu/k_F$

FIGURE 3.

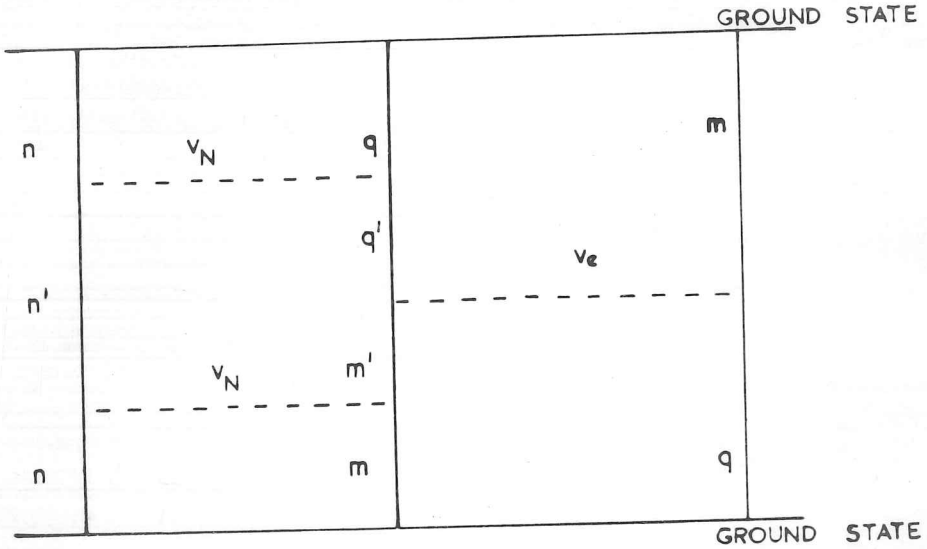


FIGURE 2.

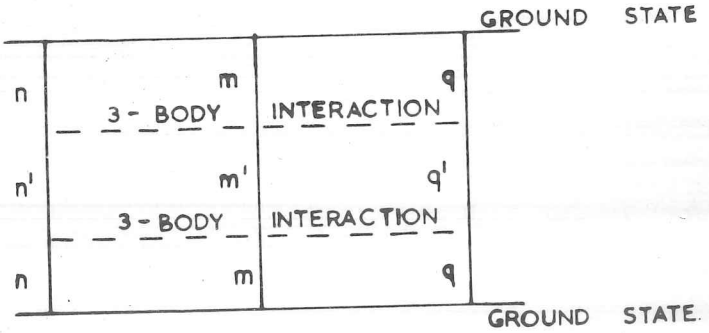


FIGURE 4.