Profinite properties of RAAGs and special groups

Robert Kropholler and Gareth Wilkes

Abstract

In this paper we prove that RAAGs are distinguished from each other by their pro-
completion for any choice of prime $p$, and that RACGs are distinguished from each other by their pro-$2$
completions. We also give a new proof that hyperbolic virtually special groups are good in the
sense of Serre. Furthermore we give an example of a property of the underlying graph of a RAAG
that translates to a property of the profinite completion.

Right-angled Artin groups (RAAGs) have been the subject of much recent interest, especially
because of their rich subgroup structure; in particular every special group embeds in a RAAG
[7]. Furthermore RAAGs are linear and have excellent residual properties. Here we will
show that RAAGs, and the closely related right-angled Coxeter groups (RACGs), are in fact
completely determined by their finite quotient groups. The proofs will rely principally on the
cohomological rigidity result of Koberda [9].

First let us recall some definitions.

Definition 1. Given a (finite simplicial) graph $\Gamma$, the right-angled Artin group $A(\Gamma)$ is the
group with generating set $V(\Gamma)$ with the relation that vertices $v, w$ commute imposed whenever
$v, w$ span an edge of $\Gamma$. The right-angled Coxeter group $C(\Gamma)$ is the quotient of $A(\Gamma)$ with the
additional constraint that each generator has order 2.

Definition 2. Given a discrete group $G$, the profinite completion of $G$ is the inverse limit
of the system of groups

$$ \hat{G} = \lim_{\leftarrow} G/N $$

where $N$ ranges over the finite index normal subgroups of $G$. This is a compact Hausdorff
topological group. Similarly one may define the pro-$p$ completion $\hat{G}_{(p)}$ as the inverse limit of
all finite quotients of $G$ which are $p$-groups, for $p$ a prime.

The isomorphism type of a right-angled Artin group $A(\Gamma)$ uniquely determines the graph $\Gamma$
up to isomorphism; this fact was first established by Droms [3]. We use a stronger cohomological
criterion proved by Koberda [9], who builds on earlier work of Subalka [16], Droms [4],
Gubeladze [6] and Charney and Davis [2].

Theorem (Koberda [9]). Let $\Gamma, \Gamma'$ be finite graphs. Then $\Gamma \cong \Gamma'$ if and only if there is an
isomorphism of cohomology groups

$$ H^*(A(\Gamma); \mathbb{Q}) \cong H^*(A(\Gamma'); \mathbb{Q}) $$
in dimensions one and two, which respects the cup product.

The proof relied solely on the following fact: each vertex \( v \in \Gamma \) is dual to a cohomology class \( f_v \in H^1(A(\Gamma); \mathbb{Q}) \) for which the map

\[
f_v \sim \bullet : H^1(A(\Gamma); \mathbb{Q}) \to H^2(A(\Gamma); \mathbb{Q})
\]

has rank precisely the degree of the vertex \( v \); moreover a vertex \( w \in \Gamma \) is adjacent to \( v \) precisely if \( f_v \sim f_w \) is non-zero.

Now the class \( f_v \sim f_w \) is dual to an embedded 2-torus in the Salvetti complex of \( A(\Gamma) \), hence gives a primitive element of \( H^2(A(\Gamma); \mathbb{Z}) \). It follows that changing the coefficient field \( \mathbb{Q} \) to a finite field \( \mathbb{Z}/p \) (for \( p \) a prime) changes neither the rank of the above map, nor the adjacency condition following it. Hence Koberda’s cohomological rigidity result also holds with coefficient field \( \mathbb{Z}/p \).

It remains to show that the pro-\( p \) completion of our right-angled Artin group detects the cohomology in dimensions one and two, and the cup product. As we will discuss later, it is frequently the case for groups arising in low-dimensional topology that the cohomology of a group is determined by its profinite completion. We always have substantial control over the cohomology in dimensions one and two. See [18] for definitions and basic properties of profinite cohomology; the definitions largely parallel those for discrete groups. In particular there is a natural notion of cup product and the natural map from cohomology; the definitions largely parallel those for discrete groups. The third point follows from the first two and naturality of the cup product. We give here an explicit proof in dimension two in terms of extensions.

**Proposition 3.** Let \( G \) be a discrete group and \( p \) a prime.

- \( H^1(\hat{\Gamma}(\mathbb{Z})); \mathbb{Z}/p) \to H^1(G; \mathbb{Z}/p) \) is an isomorphism;
- \( H^2(\hat{\Gamma}(\mathbb{Z})); \mathbb{Z}/p) \to H^2(G; \mathbb{Z}/p) \) is injective; and
- if \( H^{1+1} \) denotes that part of second cohomology generated by cup products of elements of \( H^1 \), then \( H^{1+1}(\hat{\Gamma}(\mathbb{Z})); \mathbb{Z}/p) \to H^{1+1}(G; \mathbb{Z}/p) \) is an isomorphism;

where all the maps are the natural ones induced by \( G \to \hat{\Gamma}(\mathbb{Z}) \).

**Proof.** The first point is a trivial consequence of the fact that \( H^1(\mathbb{Z}/p) \) is naturally isomorphic to Hom(\( \mathbb{Z}/p) \) in either the category of discrete groups or the category of pro-\( p \) groups. The third point follows from the first two and naturality of the cup product. The second is a special case of Exercise 2.6.1 of [18]; we give here an explicit proof in dimension two in terms of extensions.

Recall that \( H^2(\mathbb{Z}/p) \) classifies central extensions of \( G \) by \( \mathbb{Z}/p \) both for discrete and pro-\( p \) groups (see Section 6.8 of [15] for the profinite theory). Take a central extension \( H \) of \( \hat{\Gamma}(\mathbb{Z}) \) by \( \mathbb{Z}/p \) representing \( \xi \in H^2(\hat{\Gamma}(\mathbb{Z})); \mathbb{Z}/p) \). Then the pull-back

\[
P = \{(g, h) \in G \times H \text{ such that } \pi(h) = \iota(g)\}
\]

(where \( \pi : H \to \hat{\Gamma}(\mathbb{Z}) \) and \( \iota : G \to \hat{\Gamma}(\mathbb{Z}) \) are the obvious maps) gives a central extension of \( G \) by \( \mathbb{Z}/p \) representing \( \iota^*(\xi) \). If \( \iota^*(\xi) = 0 \) in \( H^2 \) then the extension splits; so there is a group-theoretic section \( s : G \to P \). Now \( s \) induces a map \( \hat{s} : \hat{\Gamma}(\mathbb{Z}) \to \hat{P}(\mathbb{Z}) \). Furthermore \( H \) is a pro-\( p \) group so that the projection \( \text{pr}_2 : P \to H \) induces a map \( \text{pr}_2 : \hat{P}(\mathbb{Z}) \to H \); then \( \text{pr}_2 \hat{s} \) is a section
of $H \to \hat{G}_{(p)}$ and so $\xi$ was a trivial extension also.

\[ \begin{array}{cccccc}
Z/\pi & \to & P & \to & G \\
\downarrow & & \downarrow & & \downarrow \\
Z/\pi & \to & \hat{P}_{(p)} & \to & \hat{G}_{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
Z/\pi & \to & H & \to & \hat{G}_{(p)} \\
\end{array} \]

\[ \begin{array}{cccccc}
p_{\pi_2} & & \hat{\pi}_2 & & \pi \\
& & & \tilde{s} & & \tilde{s} \\
\end{array} \]

Remark. Note that a diagram chase applied to the lower parallelogram in the above diagram shows that in fact $H \cong \hat{P}_{(p)}$. Thus the above analysis also illustrates why the map on $H^2$ may fail to be surjective; for any central extension of $\hat{G}_{(p)}$ yielding a given extension $P$ of $G$ must be $\hat{P}_{(p)}$; however there is no \textit{a priori} reason that the map from $Z/\pi$ to $\hat{P}_{(p)}$ need be injective. See the example at the end of [12] for an example where the map on $H^2$ fails to be surjective.

Recall that for a RAAG, the dimension two cohomology is in fact generated by cup products; thus in dimensions one and two, the algebra $H^*(A(\Gamma); Z/p)$ is determined by the pro-$p$ completion $\hat{A}(\Gamma)_{(p)}$; hence we have proved

\textbf{Theorem 4.} Let $\Gamma, \Gamma'$ be finite graphs and $p$ a prime. Then $\hat{A}(\Gamma)_{(p)} \cong \hat{A}(\Gamma')_{(p)}$ if and only if $\Gamma \cong \Gamma'$.

Note that \textit{a fortiori} an isomorphism of profinite completions also forces the graphs to be isomorphic. In fact much more is true about the cohomology of the pro-$p$ completion of $A(\Gamma)$; in particular:

\textbf{Theorem 5 (Lorensen [11], [12]).} The map from a right-angled Artin group to its pro-p completion (or profinite completion) induces an isomorphism of mod-p cohomology for any prime $p$.

We can extend Theorem 4 to right-angled Coxeter groups by noting that there are natural isomorphisms

$$H^1(C(\Gamma); Z/2) \cong H^1(A(\Gamma); Z/2)$$

and

$$H^2(C(\Gamma); Z/2) \cong H^2(A(\Gamma); Z/2) \oplus (Z/2)^{|V(\Gamma)|}$$

where the second summand above derives from the relations $v^2 = 1$. The quotient map on $H^2$ which restricts to the isomorphism of the first summand with $H^2(A(\Gamma); Z/2)$ is induced by the natural map $A(\Gamma) \to C(\Gamma)$. This quotient map is unique (i.e. does not depend on the presentation of $C(\Gamma)$ as a particular right-angled Coxeter group) in the following sense. Modulo 2, we have the relations $(a + b)^2 = a^2 + b^2$ so that the image of the squaring map $a \to a \cdot a$ is a subgroup $\Sigma$ of $H^2(C(\Gamma))$, the image of the diagonal subgroup of $(H^1(C(\Gamma)))^2$. The second summand $(Z/2)^{|V(\Gamma)|}$ is precisely this subgroup $\Sigma$. 
Thus the structure of the algebra $H^*(A(\Gamma))$ in dimensions one and two is determined by the behaviour of $H^*(C(\Gamma);\mathbb{Z}/2)$ in those dimensions, with the cup product map being given by the canonical map

$$(H^1(A(\Gamma)))^2 \xrightarrow{\cup} (H^1(C(\Gamma)))^2 \xrightarrow{\cup} H^2(C(\Gamma)) \rightarrow H^2(C(\Gamma))/\Sigma \cong H^2(A(\Gamma))$$

described above. Proposition 3 shows this algebra to be an invariant of the pro-2 completion, so that we have:

**Theorem 6.** Let $\Gamma, \Gamma'$ be finite graphs. Then $\hat{C}(\Gamma)(2) \cong \hat{C}(\Gamma')(2)$ if and only if $\Gamma \cong \Gamma'$.

Proposition 3 was sufficient to prove the Theorem; in fact right-angled Coxeter groups are 2-good, so that we have an isomorphism on cohomology in all dimensions. This follows from extension properties of 2-goodness applied to Proposition 9 of [10] or from the work on grpah products in [17].

We made heavy use of the cohomology of the profinite completions of RAAGs and RACGs, so let us digress and study the following property. A group $G$ is **good** if the natural map on cohomology induced by $G \rightarrow \hat{G}$ is an isomorphism

$$H^n(\hat{G}; M) \cong H^n(G; M)$$

for every finite $G$-module $M$ and every $n \geq 1$.

Note that this map $G \rightarrow \hat{G}$ for any group $G$ respects the cup product; for the cup product is defined for cohomology of profinite groups by precisely the same formulae as for abstract groups. Thus for a good group $G \rightarrow \hat{G}$ will not only induce an isomorphism of groups, but an isomorphism of graded algebras $H^*(\hat{G}; M) \cong H^*(G; M)$ under the cup product.

Goodness is preserved under taking extensions by good groups (under some mild conditions; see [18]), and passing to finite index subgroups or overgroups. Surface groups are good, as follows immemdiately from the exercises in [18]. Hence all virtually-fibred 3-manifold groups are good; due to work of Agol [1] and Wise [19] this includes all finite volume hyperbolic 3-manifolds.

Goodness is also preserved under taking amalgamated free products and HNN extensions, given suitable conditions. Recall that the full profinite topology on a group $H$ is the topology induced by the map $H \rightarrow \hat{H}$. An amalgamated free product $G = A *_{C} B$ is said to be **efficient** if $G$ is residually finite, if $A, B, C$ are closed in the profinite topology on $G$, and if $G$ induces the full profinite topology on $A, B$ and $C$. In particular if $G$ is LERF (and $A, B, C$ are finitely generated) this condition will be satisfied. Similarly one may define efficient HNN extensions. Then:

**Theorem 7** (Proposition 3.6 of [5]). An efficient amalgamated free product or HNN extension of good groups is good.

We may now prove that all hyperbolic virtually special groups are good. This was proved by Schreve [17] and later by Minasyan and Zalesskii [13], both using virtual retraction properties; we give another proof using cube complex hierarchies. Hierarchies have also been used to prove goodness in other contexts, in particular in [5].

**Definition 8.** Let $QVH$ be the smallest class of hyperbolic groups closed under the following operations.

(i) The trivial group is in $QVH$. 
(ii)
(ii) If $A, B, C \in \mathcal{QVH}$ and $C$ is quasiconvex in $G = A \ast_C B$, then $G \in \mathcal{QVH}$.

(iii) If $A, B \in \mathcal{QVH}$ and $B$ is quasiconvex in $G = A \ast B$, then $G \in \mathcal{QVH}$.

(iv) If $H$ is commensurable to $G$ and $G \in \mathcal{QVH}$, then $H \in \mathcal{QVH}$.

We have the following useful characterisation of the groups in $\mathcal{QVH}$.

**Theorem (Wise [19]).** A hyperbolic group is in $\mathcal{QVH}$ if and only if it is virtually special.

Furthermore recall

**Theorem (Haglund-Wise [7]).** Quasi-convex subgroups of hyperbolic virtually special groups are separable.

With this in mind we can now prove.

**Theorem 9.** Hyperbolic virtually special groups are good.

**Proof.** As noted above, we are free to pass to an arbitrary finite index subgroup of $G$ and prove goodness there. We define a measure of complexity for a special group $H$. Set $n(H)$ to be the minimal dimension of a CAT(0) cube complex $X$ on which $H$ acts with special quotient. After subdividing, a hyperplane in this complex is an embedded 2-sided cubical subcomplex, and $H$ splits as an HNN extension or amalgamated free product over the stabiliser of this hyperplane. Iterating this process yields a rooted tree of groups in which each vertex has either two or three descendants (depending on whether the vertex splits as an HNN extension or amalgamated free product). Because $H$ is in $\mathcal{QVH}$, this tree is finite and each branch terminates in the trivial group. Set $m(H)$ to be the length of a longest branch over all such trees with minimal diameter; that is, the length of a shortest hierarchy for $H$. Now define the quasiconvex hierarchy complexity $\mu(G)$ of a special group $G$ to be the pair of integers $(n(G), m(G))$; order the pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ lexicographically.

Now assume that all hyperbolic special groups $H$ with $\mu(H) < \mu(G)$ are good. We have a splitting of $G$ either as $A \ast_C B$ or $A \ast C$. Note that $A, B, C$ are hyperbolic. Now $C$ is the stabiliser of a hyperplane; this is a CAT(0) cube complex whose quotient by $C$ is special and for which the intersections with other hyperplanes of $X$ give a quasiconvex hierarchy; hence $n(C) < n(G)$, so $\mu(C) < \mu(G)$ and so $C$ is good. Furthermore $A$ and $B$ have shorter hierarchies than $G$, so whether or not $n(A) = n(G)$, the complexities $\mu(A) < \mu(G)$ and $\mu(B) < \mu(G)$ do strictly decrease. Thus $A, B, C$ are good. Furthermore quasiconvex subgroups of $G$ are separable. All finite index subgroups of $A, B, C$ are quasiconvex so the splitting is efficient and we may apply Theorem 7 to conclude that $G$ is good. Note that the base case for the induction is simply the trivial group.

Recalling that Haglund and Wise [8] proved that all Coxeter groups are virtually special, we have:

**Corollary 10.** Hyperbolic Coxeter groups are good.

For right-angled Artin groups, Theorem 5 guaranteed that in fact the mod-p cohomology is determined by the pro-p completion. This property, which is sometimes called $p$-goodness, is rather rarer than straightforward goodness; in particular proofs will often require strong
separability constraints in which only $p$-group quotients are available. These constraints are difficult to obtain in general.

We move now to a result of a rather different flavour. Often, properties of the underlying graph of a right-angled Artin or Coxeter group are expressible as group theoretic properties. As an example of such a property carrying over to the profinite world, we prove the following Theorem.

**Theorem 11.** Let $\Gamma$ be a graph. Then $\hat{A}(\Gamma)$, respectively $\hat{C}(\Gamma)$, splits as a non-trivial profinite free product $H_1 \amalg H_2$ if and only if $\Gamma$ is disconnected.

The proof will call upon the theory of actions on profinite trees developed by, among others, Ribes and Zalesskii. The theory is contained in the unpublished book [14]; the closely related pro-$p$ version may be found in published form in [15].

**Proof of Theorem 11.** If $\Gamma$ is disconnected the result follows directly from the abstract case. So suppose that $\Gamma$ is connected and that $G = \hat{A}(\Gamma)$ splits as a profinite free product $H_1 \amalg H_2$. The case when $\Gamma$ is a point is easy, so suppose that $\Gamma$ is not a point.

The splitting of $G$ as a free profinite product induces an action of $G$ on a profinite tree $T$, where vertex stabilisers are precisely the conjugates of the $H_i$ (see Lemma 5.3.1 of [14]). All edge stabilisers are trivial, so that no element of $G$ can fix more than one point of $T$. By Proposition 3.2.3 of [14], any abelian group acting on $T$ either fixes a point or is a subgroup of $\mathbb{Z}$. Each of the standard generators of $\hat{A}(\Gamma)$ is contained in a copy of $\mathbb{Z}^2$ as there is some edge adjacent to the corresponding vertex; and these copies of $\mathbb{Z}^2$ are retracts of the whole RAAG, hence give an inclusion of $\mathbb{Z}^2$ in the profinite completion. Hence every generator of $G$ fixes some (unique) vertex of $T$, and so is contained in a (unique) conjugate of $H_1$ or $H_2$.

Note that for every edge $e = [v, w]$ of $\Gamma$, the subgroup $<v, w>$ is a rank 2 free abelian group so that $v, w$ fix the same vertex of $T$; by connectedness of $\Gamma$, it follows that all of $\hat{A}(\Gamma)$ fixes a vertex of $T$ and so $H_1 = 1$ or $H_2 = 1$.

The case of a right-angled Coxeter group is similar; indeed it is easier, as all the generators have finite order, and therefore fix a vertex of $T$ by Theorem 3.1.7 of [14].

**Acknowledgements.** The authors would like to thank Marc Lackenby for helpful discussions regarding the paper. We also thank the referee for making useful comments and suggestions.

**References**


