A criterion for residual p-finiteness of arbitrary graphs of finite p-groups

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Abstract

We establish conditions under which the fundamental group of a graph of finite p-groups is necessarily residually p-finite. The technique of proof is independent of previously established results of this type, and the result is also valid for infinite graphs of groups.

Introduction

Residual properties of graphs of groups have long been a subject of study, and have for instance been particularly important in relation to 3-manifold groups [3, 8, 1, 7]. Any study of such properties almost inevitably involves a reduction to the study of graphs of finite groups. The fundamental group of any finite graph of finite groups is well-known to be residually finite [5, Proposition II.2.6.11]. However the situation for properties of *residual p-finiteness* is rather more subtle. Throughout the paper, let p be a prime.

Definition 1. A group G is *residually p-finite* if for any $g \in G \smallsetminus 1$ there exists a homomorphism from G to a finite p-group whose kernel does not contain g; or equivalently, if there exists a normal subgroup of G with index a power of p which does not contain g.

It is emphatically *not* the case that the fundamental group of any finite graph of finite p-groups is residually p-finite, and one must impose a condition on the graph of groups for this to be the case.

Higman [4] studied this problem in the case of amalgamated free products, and proved that the existence of chief series for the p-groups satisfying a certain compatibility condition (see condition I of Theorem 4) is necessary and sufficient for an amalgamated free product of p-groups to be residually p-finite. A similar criterion for HNN extensions was proved by Chatzidakis [2].

In both of these papers the strategy of the proof is to use wreath products in the following manner. For $G = A_1 *_B A_2$ or $G = A_1 *_B$, iterated wreath products are used to construct an explicit finite p-group P and a map $G \to P$ whose restriction to the A_i is an injection. This map then has free kernel which is a normal subgroup of index a power of p. Since free groups are residually

p-finite, this implies (see Lemma 2) that the original fundamental group of the graph of groups is also residually p.

Since finite graphs of groups can be constructed step-by-step as iterated amalgamated free products and HNN extensions, these two papers together could be applied to prove that a condition on chief series implies that a finite graph of p-groups is residually p-finite. However such arguments cannot access infinite graphs of groups.

The purpose of this note is to give a new proof of a chief series condition for residual p-finiteness. There are no wreath products, and we will analyse all graphs of groups directly rather than building them up from one-edge graphs of groups (i.e. amalgams and HNN extensions). In particular, our proof is also valid for infinite graphs of groups. The criterion studied is given in Theorem 4.

The scheme of the proof is to use the language of Bass–Serre theory to give a reformulation of our criterion in terms of an action on the tree dual to the graph of groups. In this formulation we can pass to an index p normal subgroup given by a graph of groups which is 'simpler' in a certain sense. This process concludes with a free group, which is well-known to be residually p-finite, thus proving the theorem.

Graphs of groups and trees

For this paper we will use the notions of graphs and graphs of groups as given by Serre [5, Section I.5.3]. We recall these notions and set up notation as follows. A graph $X = VX \sqcup EX$ consists of a set VX of vertices and a set EX of edges. Each edge y has an opposite edge $\bar{y}$, and has endpoints $o(y)$ and $t(y)$ with $o(\bar{y}) = t(y)$ and $\bar{\bar{y}} = y$.

A graph of groups $(X, G_\bullet)$ consists of the following data:

- a connected graph X;
- a group G_x for each $x \in VX \cup EX$, with $G_y = G_{\bar{y}}$ for $y \in EX$; and
- monomorphisms $f_y \colon G_y \hookrightarrow G_{t(y)}$ for all $y \in EX$.

We fix a maximal subtree T of X, which exists by Zorn's lemma if X is infinite. Choose also an orientation E^+X of X—that is, a subset $E^+X \subseteq EX$ such that for all $y \in EX$ exactly one of y and $\bar{y}$ is in E^+X. Define a function $\epsilon \colon EX \to \{0, 1\}$ by

$$\epsilon(y) = \begin{cases} 0 & \text{if } y \in E^+X \\ 1 & \text{if } y \notin E^+X \end{cases}$$

The fundamental group $G = \pi_1(X, G_\bullet)$ is then defined to be the group obtained from the free product of the G_x (for $x \in VX \cup EX$) and the free group generated by letters s_y (for $y \in EX$), subject to the following relations:

- $g = s_y^{1-\epsilon(y)} f_y(g) s_y^{\epsilon(y)-1}$ for $y \in EX$ and $g \in G_y$;
- $s_y = 1$ for all $y \in ET$, and $s_{\bar{y}} = s_y$ for all $y \in EX$.

All the groups G_x inject into G under this construction, and we identify them with their images in G. Note that for an edge y of X, the group G_y is not necessarily contained in $G_{t(y)}$, but in a conjugate of it; and note that the map f_y is equal to the composition

$$f_y = \left(G_y \xrightarrow{\subseteq} s_y^{1-\epsilon(y)} G_{t(y)} s_y^{\epsilon(y)-1} \longrightarrow G_{t(y)}\right) \qquad (*)$$

where the final map is left conjugation by $s_y^{\epsilon(y)-1}$.

The Bass–Serre tree of G dual to $(X, G_\bullet)$ is the tree $\widetilde{X}$ with vertex and edge sets

$$V\widetilde{X} = \bigsqcup_{x \in VX} G/G_x, \quad E\widetilde{X} = \bigsqcup_{y \in EX} G/G_y$$

For $x \in X$ define $\tilde{x}$ to be the coset $1 \cdot G_x$ viewed as an element of $\widetilde{X}$. The adjacency maps in $\widetilde{X}$ are

$$o(g\tilde{y}) = gs_y^{-\epsilon(y)}\widetilde{o(y)}, \quad t(g\tilde{y}) = gs_y^{1-\epsilon(y)}\widetilde{t(y)}$$

There is a natural (left-)action of G on $\widetilde{X}$ with quotient graph X and with point stabilisers

$$G(g\tilde{x}) \coloneqq \mathrm{stab}_G(g\tilde{x}) = gG_xg^{-1}$$

Conversely [5, Section I.5.4], an action of G on a tree $\widetilde{X}$ gives rise to a graph of groups $(X, G_\bullet)$ whose Bass–Serre tree is G-isomorphic to $\widetilde{X}$.

Results

For the proof of the main theorem we will make use of the following standard fact. We include a proof for completeness. Note that there is no requirement that G be finitely generated. The notation '$H \lhd_p G$' means 'H is a normal subgroup of G with index a power of p'.

Lemma 2. *Let $H \lhd_p G$. If H is residually p-finite then G is residually p-finite.*

Proof. Let $g \in G$. If $g \notin H$ then there is nothing to prove. If $g \in H$ then by assumption there is $U \lhd_p H$ such that $g \notin U$. Consider

$$V = \bigcap_{g \in G} gUg^{-1}.$$

Since U is normal in H and H has finite index in G, there are only finitely many subgroups in this intersection. All are normal in H of p-power index, so the intersection V also has p-power index in H, and hence in G. By construction V is normal in G and $g \notin V$. This completes the proof. □

We proceed now to the main theorem. The criterion for residual p-finiteness is stated in terms of chief series.

Definition 3. A *chief series* for a finite p-group P is a sequence

$$P = P^{(0)} \geq P^{(1)} \geq \cdots \geq P^{(k)} \geq \cdots$$

of normal subgroups of P such that each successive quotient

$$\gamma^{(k)}(P) = P^{(k)}/P^{(k+1)}$$

is either trivial or of order p and such that $P^{(n)} = 1$ for some n. The *length* of the chief series is the smallest n such that $P^{(n)} = 1$.

Remark. This differs slightly from the usual definition of a chief series in that the sequence does not terminate. This is a purely formal difference enabling us to state the next theorem in its greatest generality.

Theorem 4. *Let $(X, G_\bullet)$ be a graph of finite p-groups with fundamental group $G = \pi_1(X, G_\bullet)$. Suppose there is a chief series $(G_x^{(k)})_{k\geq 0}$ of G_x for each $x \in X$, such that the following two properties hold.*

(I) For all $k \in \mathbb{N}$ and all $y \in EX$, we have $f_y(G_y^{(k)}) = f_y(G_y) \cap G_{t(y)}^{(k)}$.

(II) For each k there exists a family of injections $\phi_x^{(k)} \colon \gamma^{(k)}(G_x) \hookrightarrow \mathbb{F}_p$ (for $x \in X$) such that the diagrams

$$\begin{array}{ccc} \gamma^{(k)}(G_y) & \xrightarrow{\;f_y\;} & \gamma^{(k)}(G_{t(y)}) \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_p & \end{array}$$

commute for all $y \in EX$.

Then G is residually p-finite.

Remark. The converse to Theorem 4 holds if the graph X is finite. In this case, if G is residually p-finite then there is a finite p-group P and a map $\Phi \colon G \to P$ restricting to an injection on all the (finitely many) G_x. Taking intersections of $\Phi(G_x)$ with a chief series $(P^{(k)})_{0\leq k\leq n}$ of P yields chief series of the G_x satisfying the conditions of the theorem.

Remark. If X is finite, then the conditions of the theorem are also sufficient for G to be conjugacy p-separable—this is equivalent to residual p-finiteness by [6, Theorem 4.2].

Remark. We note that in the case that X is a tree, condition II follows from condition I: one may choose $\phi_x^{(k)}$ arbitrarily at one point of each connected component of the graph $Y_k = \{x \in X \mid \gamma^{(k)}(G_x) \neq 1\}$, whereupon the maps $\phi_x^{(k)}$ for the remaining x in that component of Y_k may be uniquely defined by forcing condition II to hold.

If X is not a tree, one may still define the $\phi_x^{(k)}$ consistently on a maximal forest T of Y_k. For the remaining edges $y \in Y_k \cap E^+X$, one may again define $\phi_y^{(k)}$ so that condition II holds. The only remaining cases of condition II that must be satisfied are for the edges $\bar{y}_0$ for $y_0 \in E^+X \smallsetminus T$. Take an edge path $y_1, \ldots, y_m$ in Y_k from $o(y_0)$ to $t(y_0)$. Then condition II is easily seen to be satisfied if and only if the composite map

$$\gamma^{(k)}(G_{y_0}) \xrightarrow{f_{\bar{y}_0}} \gamma^{(k)}(G_{o(y_0)}) \xrightarrow{f_{\bar{y}_1}^{-1}} \gamma^{(k)}(G_{y_1}) \xrightarrow{f_{y_1}} \gamma^{(k)}(G_{t(y_1)}) \longrightarrow \cdots$$

$$\cdots \qquad \longrightarrow \gamma^{(k)}(G_{t(y_0)}) \xrightarrow{f_{y_0}^{-1}} \gamma^{(k)}(G_{y_0})$$

is the identity. This condition may be seen as the analogue in our context for the condition $(**)$ on HNN extensions given in [2].

The proof of Theorem 4 proceeds most smoothly if we translate conditions I and II of Theorem 4 into the language of the Bass–Serre tree dual to the graph of groups $(X, G_\bullet)$.

Lemma 5. *Let G be a group. Let G act on a Bass–Serre tree $\widetilde{X}$ dual to a graph of finite p-groups $(X, G_\bullet)$. Then $(X, G_\bullet)$ satisfies the conditions of Theorem 4 if and only if there exists a chief series $(G(z)^{(k)})_{k\geq 0}$ for each stabiliser $G(z)$ of $z \in \widetilde{X}$ such that the following conditions hold.*

(I′) For all $z \in E\widetilde{X}$, we have $G(z)^{(k)} = G(z) \cap G(t(z))^{(k)}$ and for each $z \in \widetilde{X}$ and each $g \in G$, we have $gG(z)^{(k)}g^{-1} = G(g \cdot z)^{(k)}$

(II′) For each k there exists a family of injections $\psi_z^{(k)} \colon \gamma^{(k)}(G(z)) \hookrightarrow \mathbb{F}_p$ for $z \in \widetilde{X}$ such that the diagrams

$$\begin{array}{ccc} \gamma^{(k)}(G(z)) & \longrightarrow & \gamma^{(k)}(G(t(z))) \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_p & \end{array}$$

commute for all $z \in EX$, and such that for all $z \in \widetilde{X}$ and all $g \in G$ the diagram

$$\begin{array}{ccc} \gamma^{(k)}(G(z)) & \xrightarrow{\zeta_g} & \gamma^{(k)}(G(g \cdot z)) \\ & \searrow \quad \swarrow & \\ & \mathbb{F}_p & \end{array}$$

commutes where ζ_g denotes left conjugation by g.

Proof. Suppose we have chief series $(G_x^{(k)})_{k\geq 0}$ for the graph of groups $(X, G_\bullet)$ satisfying conditions I and II of Theorem 4. For $g\widetilde{x} \in \widetilde{X}$ define a chief series

$$G(g\widetilde{x})^{(k)} = gG_x^{(k)}g^{-1}$$

of $G(g\tilde{x})$. This is well-defined (that is, it is invariant under replacing g by gh for $h \in G_x$) because $G_x^{(k)}$ is normal in G_x. Further define the map $\psi_{g\tilde{x}}^{(k)}$ to be the composition

$$\gamma^{(k)}(G(g\tilde{x})) \xrightarrow{\zeta_{g^{-1}}} \gamma^{(k)}(G(\tilde{x})) = \gamma^{(k)}(G_x) \xrightarrow{\phi_x^{(k)}} \mathbb{F}_p$$

This map is again well-defined under replacing g by gh for $h \in G_x$ because the conjugation action of G_x on itself induces the identity map on the $\gamma^{(k)}(G_x)$—a p-group cannot induce a non-trivial automorphism of either the trivial group or of a cyclic group of order p.

The parts of conditions I′ and II′ concerning invariance under G-conjugation hold by construction. The conditions on edges follow from conditions I and II of Theorem 4 together with the G-conjugation invariance, by recalling from $(*)$ that the inclusion of an edge stabiliser into a vertex stabiliser is, up to a G-conjugacy, equal to the map f_y followed by a conjugation by $s_y^{\epsilon(y)-1}$.

Conversely, given chief series for the point stabilisers $G(z)$ satisfying I′ and II′, we may define

$$G_x^{(k)} = G(\tilde{x})^{(k)}, \quad \phi_x^{(k)} = \psi_{\tilde{x}}^{(k)} \colon \gamma^{(k)}(G_x) \to \mathbb{F}_p$$

for $x \in X$. Conditions I and II now follow from conditions I′ and II′ via the expression $(*)$ of the maps f_y as a composition of an inclusion of edge stabilisers and a conjugacy. □

Proof of Theorem 4. Suppose first that the chief series $(G^{(k)})_{k\geq 0}$ all have length at most N for some N—this is automatic if X is finite. We prove the theorem by induction on N. If $N = 0$ then G is free, hence is residually p-finite. Suppose $N > 0$. The maps $\phi_x^{(0)}$ in condition II define, by the universal property of the fundamental group of a graph of groups, a homomorphism $\Phi \colon G \to \mathbb{F}_p$ whose restriction to each G_x is the composite

$$G_x \longrightarrow G_x/G_x^{(1)} = \gamma^{(0)}(G_x) \xhookrightarrow{\phi_x^{(0)}} \mathbb{F}_p$$

Let $H = \ker \Phi$. The group G acts on its Bass–Serre tree $\widetilde{X}$ as in Lemma 5. Consider the action of H on $\widetilde{X}$. The point stabilisers $H(z)$ for $z \in \widetilde{X}$ are by construction the groups $G(z)^{(1)}$. The chief series $H(z)^{(k)} = G(z)^{(k+1)}$ now automatically satisfy conditions I′ and II′, and all have length at most $N - 1$. Hence by Lemma 5 the graph of groups decomposition of H dual to its action on $\widetilde{X}$ is equipped with chief series of length at most $N - 1$ satisfying conditions I and II. Therefore by induction H is residually p-finite. Since H is a normal subgroup of index p in G, it follows from Lemma 2 that G is also residually p-finite.

Now move to the general case. Let $g \in G \smallsetminus \{1\}$. In the graph of groups $(X, G_\bullet)$, the element g is equal to a reduced word [5, Section I.5.2] which is supported on some finite subgraph Z of X. Let N be such that the chief series

$(G_z^{(k)})_{k\geq 0}$ of G_z has length at most N for all $z \in Z$. We may take the quotient of each G_x by $G_x^{(N)}$ to obtain a new graph of groups $(X, G_\bullet/G_\bullet^{(N)})$. There is a natural map of fundamental groups

$$\Phi\colon G = \pi_1(X, G_\bullet) \to \pi_1(X, G_\bullet/G_\bullet^{(N)}) =: G'$$

Then $\Phi(g)$ is non-trivial in G', for it is given by a reduced word—the same reduced word as in G, since $G_z^{(N)} = 1$ for all $z \in Z$. But G' is residually p-finite by the first part of the theorem, since the chief series for all $G_\bullet/G_\bullet^{(N)}$ have length at most N. Therefore there is some map $\Psi\colon G' \to P$ for a finite p-group P such that $\Psi\Phi(g) \neq 1$. This attests that G is residually p-finite. $\square$

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References

[1] Matthias Aschenbrenner and Stefan Friedl. 3-manifold groups are virtually residually p. *Mem. Amer. Math. Soc.*, 225(1058), 2013.

[2] Zoé Maria Chatzidakis. Some remarks on profinite HNN extensions. *Israel J. Math.*, 85(1-3):11–18, 1994.

[3] John Hempel. Residual finiteness for 3-manifolds. *Combinatorial group theory and topology*, 111:379–396, 1987.

[4] Graham Higman. Amalgams of p-groups. *J. Algebra*, 1(3):301–305, 1964.

[5] Jean-Pierre Serre. *Trees. Translated from the French original by John Stillwell. Corrected 2nd printing of the 1980 English translation.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[6] Emmanuel Toinet. Conjugacy p-separability of right-angled Artin groups and applications. *Groups Geom. Dyn.*, 7(3):751–790, 2013.

[7] Gareth Wilkes. Virtual pro-p properties of 3-manifold groups. *J. Group Theory*, 20(5):999–1023, 2017.

[8] Henry Wilton and Pavel Zalesskii. Profinite properties of graph manifolds. *Geom. Dedicata*, 147(1):29–45, 2010.