A Simple and Complete Model Theory for Intensional and Extensional Untyped λ-Equality

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Abstract

We present a sound and complete model theory for theories of β-reduction with or without η-expansion. The models of this paper derive from structures of modal logic: we use ternary accessibility relations on 'possible worlds' to model the action of intensional and extensional lambda-abstraction in much the same way binary accessibility relations are used to model the box operators of a normal multi-modal logic.

Keywords: Lambda Calculus, Reduction, Intensional Equality, Extensional Equality, Model Theory, Completeness, Kripke Frames, Possible World Semantics, Modal Logic.

1 Introduction

We extend the method of [6] by which we interpret λ-terms compositionally on 'possible world' structures. The simplicity of the structures is striking, moreover, they provide a surprising richness of interpretations of function abstraction and application.

Our primary goal is to show how the models can differentiate between *extensional* and *intensional* λ-equality, and provide semantic characterisation (i.e. completeness) theorems for both. We shall then hint at how richer λ-languages can be interpreted.

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The key idea in this paper is a class of models, presented in Section 2.2, although an important syntactic consideration is required first in Section 2.1. These ideas bear some similarity to the reduction models of [18] in that they get us as far as λ-reduction only. Then, in Section 4 we use the results of the earlier sections to provide a characterisation theorem for λ-equality (both with and without η-equality, i.e. extensional and intensional).

2 The models, computation, logic

2.1 The language and logic

Definition 2.1. Fix a countably infinite set of **variables**.

Define a language $\mathcal{L}_\lambda$ of **λ-terms** by: $\mathtt{t} \ ::= \ \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{t} \mid \mathtt{t}\cdot\mathtt{t}$

$\lambda\mathtt{x}$ binds in $\lambda\mathtt{x}.\mathtt{t}$. For example, $\mathtt{x}$ is bound (not free) in $\lambda\mathtt{x}.\mathtt{x}\cdot\mathtt{y}$.

We write $\mathtt{t}[\mathtt{x}/\mathtt{s}]$ for the usual capture-avoiding substitution. For example, $(\lambda\mathtt{z}.\mathtt{y})[\mathtt{y}/\mathtt{x}] = \lambda\mathtt{z}.\mathtt{x}$, and $(\lambda\mathtt{x}.\mathtt{y})[\mathtt{y}/\mathtt{x}] = \lambda\mathtt{z}.\mathtt{x}$ where $\mathtt{z}$ is an arbitrary fresh variable. If $\mathtt{x}_1 \dots \mathtt{x}_n$ is a sequence of variables and $\mathtt{t}_1 \dots \mathtt{t}_n$ is an equally long sequence of terms then we write $\mathtt{t}[\mathtt{x}_i/\mathtt{t}_i]$ for the simultaneous substitution in $\mathtt{t}$ of each $\mathtt{x}_i$ by its corresponding $\mathtt{t}_i$.

We write $\mathtt{t}[\mathtt{x}{:-}\mathtt{s}]$ for the (unusual) non-capture avoiding substitution. For example, $(\lambda\mathtt{x}.\mathtt{x})[\mathtt{x}{:-}\mathtt{y}] = \lambda\mathtt{x}.\mathtt{y}$, and $(\lambda\mathtt{x}.\mathtt{y})[\mathtt{y}{:-}\mathtt{x}] = \lambda\mathtt{x}.\mathtt{x}$

We now turn to λ-reduction. It is important for us to consider not merely the relation of λ-reduction, but a relation of λ-reduction *with assumptions*. We therefore need to define some basic, and familiar, rules of λ-reduction but allow for a set of assumed additional reductions.

Remark 2.2. We shall define a basic relation on terms that follows the familiar reduction rule of β-contraction (Definition 2.3). To help with the completeness theorem of Section 3 we will need to consider a conservative extension of the familiar λ-calculus (Definition 2.5). To facilitate the proof that this extension really is conservative (Theorem 2.8), we present the λ-calculus in the non-axiomatic style of [12, Def. 1.24].

Definition 2.3. Let Γ be a set of pairs of terms of $\mathcal{L}_\lambda$. We define a **reduction relation** $\longrightarrow_\Gamma$ on terms of $\mathcal{L}_\lambda$ using Figure 1. A **derivation** is a sequence of terms $\mathtt{t}_1, \dots, \mathtt{t}_n$ such that $\mathtt{t}_i \longrightarrow_\Gamma \mathtt{t}_{i+1}$ for each $1 \leq i < n$.

Remark 2.4. If $\Gamma = \varnothing$ then $\longrightarrow_\Gamma$ is the familiar relation of untyped β-reduction.

Definition 2.5. Define $\mathcal{L}^*_\lambda$ by: $\mathtt{t} \ ::= \ \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{t} \mid \mathtt{t}\cdot\mathtt{t} \mid \mathtt{t} * \mathtt{t}$

Let $\mathtt{x}$ occur free *only once* in $\mathtt{t}$. Let $\mathtt{x}_1 \dots \mathtt{x}_n$ be any sequence of variables and $\mathtt{t}_1 \dots \mathtt{t}_n$ be any (equally long) sequence of terms.

$$\begin{array}{lrcll} (\beta) & \mathtt{t}[\mathtt{x}{:-}(\lambda\mathtt{x}.\mathtt{s})\cdot\mathtt{r}] & \longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{x}/\mathtt{r}]] & \\ (ass) & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{x}_i/\mathtt{t}_i]] & \longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{r}[\mathtt{x}_i/\mathtt{t}_i]] & (\langle \mathtt{s},\mathtt{r}\rangle \in \Gamma) \\ (\alpha) & \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.\mathtt{s}] & \longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{z}.\mathtt{s}[\mathtt{y}/\mathtt{z}]] & \end{array}$$

The rule (ass) says that any (capture avoiding) substitution instance of $\mathtt{s}$ may be replaced in any $\mathtt{t}$, without worrying about variable capture, by its matching substitution instance of $\mathtt{r}$.[1]

Figure 1: λ-reduction for $\mathcal{L}_\lambda$

Let $\mathtt{x}$ occur free *only once* in $\mathtt{t}$. Let $\mathtt{x}_1 \dots \mathtt{x}_n$ be any sequence of variables and $\mathtt{t}_1 \dots \mathtt{t}_n$ be any (equally long) sequence of terms.

$$\begin{array}{lrcll} (\beta) & \mathtt{t}[\mathtt{x}{:-}(\lambda\mathtt{x}.\mathtt{s})\cdot\mathtt{r}] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{x}/\mathtt{r}]] & \\ (ass) & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{x}_i/\mathtt{t}_i]] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{r}[\mathtt{x}_i/\mathtt{t}_i]] & \langle \mathtt{s},\mathtt{r}\rangle \in \Gamma \\ (\alpha) & \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.\mathtt{s}] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{z}.\mathtt{s}[\mathtt{y}/\mathtt{z}]] & \\ (\beta^*) & \mathtt{t}[\mathtt{x}{:-}(\lambda\mathtt{x}.\mathtt{s}) * \mathtt{r}] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{x}/\mathtt{r}]] & \\ (sub) & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}\cdot\mathtt{r}] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\mathtt{s} * \mathtt{r}] & \\ (\eta^*) & \mathtt{t}[\mathtt{x}{:-}\mathtt{s}] & \Longrightarrow_\Gamma & \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})] & (\mathtt{y} \text{ not free in } \mathtt{t}) \end{array}$$

Figure 2: λ-reduction for $\mathcal{L}^*_\lambda$

Definition 2.6. Let Γ be a set of pairs of terms of $\mathcal{L}_\lambda$ (*not* $\mathcal{L}^*_\lambda$). Define a reduction relation $\Longrightarrow_\Gamma$ on terms of $\mathcal{L}^*_\lambda$ using Figure 2. Again, a **derivation** is a sequence of terms $\mathtt{t}_1, \dots, \mathtt{t}_n$ such that $\mathtt{t}_i \Longrightarrow_\Gamma \mathtt{t}_{i+1}$ for each $1 \le i < n$.

Remark 2.7. Notice that we do not allow terms unique to $\mathcal{L}^*_\lambda$ to be assumptions in derivations. This is because the paper is concerned with characterising reduction and equality only in the more familiar language $\mathcal{L}_\lambda$, and $\mathcal{L}^*_\lambda$ is merely a means to that end. Allowing assumed reductions for $\mathcal{L}^*_\lambda$ causes problems for the Theorem 2.8.

Theorem 2.8. *If* $\mathtt{t}_1$ *and* $\mathtt{t}_2$ *are terms of* $\mathcal{L}_\lambda$ *then* $\mathtt{t}_1 \Longrightarrow_\Gamma \mathtt{t}_2$ *implies* $\mathtt{t}_1 \longrightarrow_\Gamma \mathtt{t}_1$. *In other words* $\Longrightarrow_\Gamma$ *is* **conservative** *over* $\longrightarrow_\Gamma$.

[1]So for example if $\langle \mathtt{x}, \lambda\mathtt{y}.(\mathtt{x}\cdot\mathtt{y})\rangle \in \Gamma$ then $\longrightarrow_\Gamma$ is a reduction relation allowing η-expansion: $\mathtt{s}$ may be rewritten, inside any term $\mathtt{t}$, to $\lambda\mathtt{y}.(\mathtt{s}\cdot\mathtt{y})$ provided the lambda-operator $\lambda\mathtt{y}$ does not bind in $\mathtt{s}$ (although variables in $\mathtt{s}$ may be bound by abstractions in the wider context $\mathtt{t}$.

Proof. Suppose that $\mathtt{t}_1 \Longrightarrow_\Gamma \mathtt{t}_2$, where $\mathtt{t}_1, \mathtt{t}_2 \in \mathcal{L}_\lambda$. We argue that any such derivation can be converted into a derivation that $\mathtt{t}_1 \longrightarrow_\Gamma \mathtt{t}_2$.

We first argue that any application of (η) or (sub) can be pushed after an application of any other rule or eliminated entirely.

- Suppose we have the following derivation segment:

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})] \Longrightarrow_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s}' * \mathtt{y})]$$

  where $\mathtt{s}'$ is derived from $\mathtt{s}$ by an application of any rule, then we may easily swap the rule applications:

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}] \Longrightarrow_\Gamma \mathtt{t}[\mathtt{x}{:-}\mathtt{s}'] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s}' * \mathtt{y})]$$

- The cases where $\mathtt{t}'[\mathtt{x}{:-}\mathtt{s}]$ is derived from $\mathtt{t}[\mathtt{x}{:-}\mathtt{s}]$ is similar. For example:

$$\mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}[\mathtt{x}{:-}\mathtt{s}]\big] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})]\big] \overset{(ass)}{\Longrightarrow}_\Gamma \mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}'[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})]\big]$$

  may be replaced, given that Γ contains only terms from $\mathcal{L}_\lambda$,[2] by:

$$\mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}[\mathtt{x}{:-}\mathtt{s}]\big] \overset{(ass)}{\Longrightarrow}_\Gamma \mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}'[\mathtt{x}{:-}\mathtt{s}]\big] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}\big[\mathtt{z}{:-}\mathtt{r}'[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})]\big]$$

  There are also the following three special cases:

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}{\cdot}\mathtt{r}] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y}){\cdot}\mathtt{r}] \overset{(\beta)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\mathtt{s}{\cdot}\mathtt{r}]$$

  becomes

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}{\cdot}\mathtt{r}]$$

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s} * \mathtt{r}] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y}) * \mathtt{r}] \overset{(\beta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\mathtt{s} * \mathtt{r}]$$

  becomes

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s} * \mathtt{r}]$$

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.(\mathtt{s} * \mathtt{y})] \overset{(\alpha)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{z}.(\mathtt{s} * \mathtt{z})]$$

  becomes

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s}] \overset{(\eta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{z}.(\mathtt{s} * \mathtt{z})]$$

- By a similar reasoning it follows that (sub) can be pushed in front of any other rule, with the exception of the special case of the derivation segment

$$\mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.\mathtt{s}{\cdot}\mathtt{r}] \overset{(sub)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}.\mathtt{s} * \mathtt{r}] \overset{(\beta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\mathtt{s}[\mathtt{y}/\mathtt{r}]]$$

[2] And so the application of (ass) cannot depend on the preceding application of (η^*).

which may be replaced by:

$$\texttt{t}[\texttt{x}{:-}\lambda\texttt{y}.\texttt{s}{\cdot}\texttt{r}] \overset{(\beta)}{\Longrightarrow}_{\Gamma} \texttt{t}[\texttt{x}{:-}\texttt{s}[\texttt{y}/\texttt{r}]]$$

It follows that any derivation may be replaced by a derivation where the last application is an instance of (η) or (sub), if either appears in the derivation at all. Since (η) and (sub) introduce an instance of $*$, if $\texttt{t}_2 \in \mathcal{L}_\lambda$ then no instances of (η) or (sub) occur in the derivation. Furthermore, since $\texttt{t}_1 \in \mathcal{L}_\lambda$ it follows that the derivation contains no instances of (β^*) or occurrences of $*$. This implies that $\texttt{t}_1 \longrightarrow_\Gamma \texttt{t}_2$. $\square$

2.2 Frames and interpreting λ-terms

Given Theorem 2.8 we will work with $\mathcal{L}^*_\lambda$.

Definition 2.9. If W is a set, write $\mathcal{P}(W)$ for the set of subsets of W.

An **intensional frame** F is a 4-tuple $(W, \bullet, \circ, H)$ where:

- W a set of **worlds**,
- $\bullet$ and $\circ$ are functions from $W \times W$ to $\mathcal{P}(W)$ such that $\bullet \subseteq \circ$.
- $H \subseteq \mathcal{P}(W)$.

Remark 2.10. Subsets of W will serve as denotations of λ-terms (Definition 2.13) and $H \subseteq \mathcal{P}(W)$ ('H' for 'Henkin') plays a similar role to the structure of Henkin models for higher-order logic [2, 11, 19]. This makes our completeness results possible and is a famous issue for second- and higher-order logics: powersets are too large and for completeness results to be possible we must cut them down — at least when we quantify. This is why in Definition 2.13, the binders restrict quantification from $\mathcal{P}(W)$ down to H.

The reader familiar with modal logic can think of $\bullet$ and $\circ$ as ternary 'accessibility relations' $R_\bullet$ and $R_\circ$ such that $R_\bullet w_1 w_2 w_3$ if and only if $w_3 \in w_1 \bullet w_2$ (and similarly for $R_\circ$). We can also think of $\bullet$ and $\circ$ as non-deterministic 'application' operations, but note that intensional frames are not applicative structures — an applicative structure would map $W \times W$ to W, whereas in the case of intensional frames, $W \times W$ maps to $\mathcal{P}(W)$.

Definition 2.11. Let $F = (W, \bullet, \circ, H)$ be an intensional frame and $S_1, S_2 \subseteq W$ and $w \in W$. Then the functions $\bullet$ and $\circ$ induce functions from $W \times \mathcal{P}(W)$ and $\mathcal{P}(W) \times \mathcal{P}(W)$ to $\mathcal{P}(W)$ by: $w \bullet S = \bigcup\{w \bullet w' \mid w' \in S\}$ and $S_1 \bullet S_2 = \bigcup\{w_1 \bullet w_2 \mid w_1 \in S_1,\ w_2 \in S_2\}$ (and similarly for $\circ$).

Definition 2.12. Suppose $F = (W, \bullet, \circ, H)$ is a frame. A **valuation** (to F) is a map from variables to sets of worlds (elements of $\mathcal{P}(W)$) that are in H. v will range over valuations.

If $\mathtt{x}$ is a variable, $h \in H$, and v is a valuation, then write $v[\mathtt{x} \mapsto h]$ for the valuation mapping $\mathtt{x}$ to h and mapping $\mathtt{y}$ to $v(\mathtt{y})$ for any other $\mathtt{y}$.

Definition 2.13. Define an **denotation** of $\mathtt{t}$ inductively by:

$$[\![\mathtt{x}]\!]^v = v(\mathtt{x}) \qquad [\![\mathtt{t}\cdot\mathtt{s}]\!]^v = [\![\mathtt{t}]\!]^v \bullet [\![\mathtt{s}]\!]^v \qquad [\![\mathtt{t} * \mathtt{s}]\!]^v = [\![\mathtt{t}]\!]^v \circ [\![\mathtt{s}]\!]^v$$

$$[\![\lambda\mathtt{x}.\mathtt{t}]\!]^v = \{w \mid w \circ h \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto h]} \text{ for all } h \in H\}$$

Remark 2.14. By elementary set theory: $[\![\lambda\mathtt{x}.\mathtt{t}]\!]^v = \bigcap_{h\in H}\{w \mid w \circ h \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto h]}\}$

We are particularly interested in frames where the denotation of every λ-term is a member of H. This is because Definition 2.13 interprets λ as a kind of quantifier over all members of H. β-reduction is then valid analogously to universal instantiation in first order logic ($\forall x.Fx \vDash Ft$),[3] and so requires that every possible instantiation (i.e. every term denotation) is a member of H.

Remark 2.15. Consider the definition of application and abstraction in a graph model with carrier set $\mathcal{P}(A)$ (Scott semantics), where $\mapsto$: $\mathcal{P}_{fin}(A) \times A \mapsto A$ is an arbitrary injective map and $v : Var \mapsto \mathcal{P}(A)$ is an arbitrary environment:

$$\begin{aligned} X \bullet Y &= \{\alpha \mid (\exists a \subseteq_{fin} A) \text{ s.t. } a \mapsto \alpha \in X \text{ and } a \subseteq Y\} && (X, Y \subseteq A) \\ [\![\lambda\mathtt{x}.\mathtt{t}]\!]^v &= \{a \mapsto \alpha \mid \alpha \in [\![\mathtt{t}]\!]^{v[x\mapsto a]}\} && (\mathtt{t} \text{ a lambda term}) \\ &= \{a \mapsto \alpha \mid (\forall h \in \mathcal{P}(A))\{a \mapsto \alpha\} \bullet h \subseteq [\![\mathtt{t}]\!]^{v[x\mapsto h]}\} \end{aligned}$$

Very roughly speaking (our construction is more general), the above definitions of application and abstraction in graph models are abstracted in this paper as follows, where H is a fixed subset of $\mathcal{P}(A)$:

$$\begin{aligned} \bullet : H \times H \mapsto H &= \text{any linear map in both arguments} \\ [\![\lambda\mathtt{x}.\mathtt{t}]\!]^v &= \{\alpha \mid (\forall h \in H)\{\alpha\} \bullet h \subseteq [\![\mathtt{t}]\!]^{v[x\mapsto h]}\} \end{aligned}$$

where $v : Var \mapsto H$ is an H-environment. This semantics is of perhaps of additional interest because it does not codify the step functions of continuous semantics.

Lemma 2.16. *1. If* $\mathtt{x}$ *is not free in* $\mathtt{t}$*, then for any* $h \in H$*,* $[\![\mathtt{t}]\!]^v = [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto h]}$.
2. $[\![\mathtt{t}[\mathtt{x}/\mathtt{s}]]\!]^v = [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto[\![\mathtt{s}]\!]^v]}$

[3] Perhaps a better analogy would be $\forall x.(Fx \to Gx) \wedge Ft \vDash Gt$, where conjunctions corresponds to $\cdot$ and the quantified expression corresponds to a λ-term.

Proof. Both parts follow by easy inductions on t. ☐

Definition 2.17. A frame is **faithful** when for every v and every $\mathtt{t} \in \mathcal{L}_\lambda$, $[\![\mathtt{t}]\!]^v \in H$. That is, a frame is faithful when H contains the interpretation of every λ term in $\mathcal{L}_\lambda$ independently of v.

Remark 2.18. Definition 2.17 is not ideal. A semantic characterisation of faithfulness — as a condition on H, $\bullet$ and $\circ$ — is desirable. We cannot present such a charactersiation in this paper except in the special cases of theories of β-equality which we defer until 5.1 of Section 5. In spite of this, the structures characterised by 2.17 are informative because they allow us to break down λ-abstraction into a quantification over worlds with a ternary accessibility relation; the denotations of λ-terms then become simply sets of worlds. On this analysis the domain of quantification for λ-abstraction — the actual set of denotations of λ-terms — is H, a subset of the set of all possible denotations $\mathcal{P}(W)$. This is a common pattern, for example in topological semantics for modal logics and for intuitionistic logic the set of denotations, the analogue of H, is the set of all open subsets of the domain rather than the powerset itself. It is then no accident that H will reveal further significance: it will be useful later in characterising the difference between extensional and intensional λ-calculus (see Remark 3.18). Unlike in the modal case, we offer no general way in this paper of characterising H beyond simply saying that it must contain $[\![\mathtt{t}]\!]^v$ for each valuation v and term t. For a considerably more complex characterisation of H in topological terms see [8].

Lemma 2.19. *If we interpret* $\Longrightarrow_\Gamma$ *as subset inclusion then all the rules of Figure 2 are sound for faithful intensional frames.*

Proof. By routine calculations from the definitions. We show only (β) and (η^*) here, the others are equally straightforward.

$$
\begin{array}{lll}
[\![\lambda\mathtt{x}.\mathtt{t}\cdot\mathtt{s}]\!]^v & & \\
\quad = [\![\lambda\mathtt{x}.\mathtt{t}]\!]^v \bullet [\![\mathtt{s}]\!]^v & & \text{Definition 2.13} \\
\quad = \bigcap_{h\in H}\{w \mid w \circ h \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto h]}\} \bullet [\![\mathtt{s}]\!]^v & & \text{Definition 2.13} \\
\quad \subseteq \{w \mid w \circ [\![\mathtt{s}]\!]^v \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto [\![\mathtt{s}]\!]^v]}\} \bullet [\![\mathtt{t}]\!]^v & & [\![\mathtt{s}]\!]^v \in H \\
\quad \subseteq \{w \mid w \bullet [\![\mathtt{s}]\!]^v \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto [\![\mathtt{s}]\!]^v]}\} \bullet [\![\mathtt{t}]\!]^v & & \bullet \subseteq \circ \\
\quad \subseteq [\![\mathtt{t}]\!]^{v[\mathtt{x}\mapsto [\![\mathtt{s}]\!]^v]} & & \text{Definition 2.11} \\
\quad = [\![\mathtt{t}[\mathtt{x}/\mathtt{s}]]\!]^v & & \text{Lemma 2.16}
\end{array}
$$

$$
\begin{array}{lll}
[\![\mathtt{t}]\!]^v & \subseteq \bigcap_{h\in H}\{w \mid w \circ h \subseteq [\![\mathtt{t}]\!]^v \circ h\} & \text{Definition 2.11} \\
 & = \bigcap_{h\in H}\{w \mid w \circ h \subseteq [\![\mathtt{t} * \mathtt{x}]\!]^{v[\mathtt{x}\mapsto h]}\} & \mathtt{x} \text{ not free in } [\![\mathtt{t}]\!]^v \\
 & = [\![\lambda\mathtt{x}.(\mathtt{t} * \mathtt{x})]\!]^v & \text{Definition 2.13}
\end{array}
$$

☐

2.3 Soundness

Definition 2.20. – A **model** M is a pair $\langle F, v\rangle$ where F is a faithful intensional frame and v is a valuation on F such that $v(\mathtt{t}) \in H \in F$ for every $\mathtt{t}$.

– A frame F is Γ-**sensitive** if $[[\mathtt{t}]]^v \subseteq [[\mathtt{s}]]^v$ for every v and every $\langle \mathtt{t}, \mathtt{s}\rangle \in \Gamma$.

– A model $\langle F, v\rangle$ is Γ-**sensitive** if F is Γ-sensitive.

Remark 2.21. We could have defined a model as a pair $\langle F, v\rangle$ where F is a (possibly not faithful) frame and v is a valuation on F such that $[[\mathtt{t}]]^v \in H \in F$ for every $\mathtt{t}$. But since the completeness theorem 3.12 holds for the stronger notion of a model we shall use that. Intuitively a Γ-sensitive frame or model can be thought of as giving $\langle \mathtt{t}, \mathtt{s}\rangle \in \Gamma$ the meaning that however the variables of $\mathtt{t}$ and $\mathtt{s}$ are interpreted, $\mathtt{t}$'s denotation is a subset of $\mathtt{s}$'s.

Remark 2.22. This paper approaches lambda calculus from the angle of modal logic and so we retain the 'normal' practice of describing the model theory in terms of frames and models: a frame is sufficient to fix the interpretation of the closed terms, and a model interprets the open terms (e.g. as in [10]. This also matches the 'normal' practice in the model theory of first order logic of distinguishing a structure from a model – a structure together with a variable assignment – as in [3]. This differs from the 'normal' terminology for lambda calculus which would use the term 'model' to refer to our frames (e.g. in [1]).

Lemma 2.23. $\bullet$ *and* $\circ$ *are* **monotone**. *That is,* $h_1 \subseteq h_2$ *implies* $h \bullet h_1 \subseteq h \bullet h_2$ *and* $h_1 \bullet h \subseteq h_2 \bullet h$ *for any* h*, and similarly for* $\circ$.

Proof. By the pointwise definitions of $\bullet$ and $\circ$. For example:

$$\begin{array}{rcll} h \bullet h_1 & = & \bigcup\{w \bullet w_1 \mid w \in h \text{ and } w_1 \in h_1\} & \text{Def. 2.11} \\ & \subseteq & \bigcup\{w \bullet w_1 \mid w \in h \text{ and } w_1 \in h_2\} & \text{if } h_1 \subseteq h_2 \end{array}$$

□

Lemma 2.24. *If* $[[\mathtt{s}]]^v \subseteq [[\mathtt{r}]]^v$ *for all* v *on some faithful* F*, then for any* v *on* F $[[\mathtt{t}[\mathtt{x}{:}{-}\mathtt{s}]]]^v \subseteq [[\mathtt{t}[\mathtt{x}{:}{-}\mathtt{r}]]]^v$

Proof. By induction on $\mathtt{t}$.

– If $\mathtt{t}$ is a variable the result is easy.

– If $\mathtt{t}$ is $\mathtt{t}_1{\cdot}\mathtt{t}_2$ or $\mathtt{t}_1 * \mathtt{t}_2$ then the result follows from the induction hypothesis and the monotonicity of $\bullet$ and $\circ$ (Lemma 2.23).

– If $\mathtt{t}$ is $\lambda \mathtt{x}\mathtt{t}'$, then $\mathtt{t}[\mathtt{x}{:-}\mathtt{s}]$ is $\lambda \mathtt{x}.\mathtt{t}'[\mathtt{x}{:-}\mathtt{s}]$. And so:

$$\begin{array}{rcll} [[\lambda \mathtt{x}.\mathtt{t}'[\mathtt{x}{:-}\mathtt{s}]]]^v & = & \bigcap_{h\in H}\{w \mid w\circ h \subseteq [[\mathtt{t}'[\mathtt{x}{:-}\mathtt{s}]]]^{v[\mathtt{x}\mapsto h]}\} & \text{Def. 2.13} \\ & \subseteq & \bigcap_{h\in H}\{w \mid w\circ h \subseteq [[\mathtt{t}'[\mathtt{x}{:-}\mathtt{r}]]]^{v[\mathtt{x}\mapsto h]}\} & \text{Ind. Hyp} \\ & = & [[\lambda \mathtt{x}.\mathtt{t}'[\mathtt{x}{:-}\mathtt{r}]]]^v & \text{Def. 2.13} \end{array}$$

And the argument is similar if $\mathtt{t}$ is $\lambda \mathtt{y}.\mathtt{t}'$ for $\mathtt{y} \neq \mathtt{x}$

□

Theorem 2.25. $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ *implies* $[[\mathtt{t}]]^v \subseteq [[\mathtt{s}]]^v$ *in all* Γ*-sensitive (faithful) models* M*.*

Proof. Theorem 2.19 entails that each rule of Figure 2 holds in all models, and by definition, if $\langle \mathtt{t}, \mathtt{s}\rangle \in \Gamma$ then $[[\mathtt{t}]]^v \subseteq [[\mathtt{s}]]^v$ for all v in any Γ-sensitive model. The result then follows by Lemma 2.24. □

3 Completeness for λ-reduction

Ultimately, we wish to show that if $\mathtt{t} \not\longrightarrow_\Gamma \mathtt{s}$ then there is a Γ-sensitive model M (Def. 2.20) where $[[\mathtt{t}]]^v \nsubseteq [[\mathtt{s}]]^v$. We first show that $\mathtt{t} \not\Longrightarrow_\Gamma \mathtt{s}$ implies such an M exists if $\mathtt{t}, \mathtt{s} \in \mathcal{L}_\lambda$, and then we appeal to Theorem 2.8.

First we form the languages $\mathcal{L}_{\lambda_c}, \mathcal{L}^*_{\lambda_c}$ by adding infinitely many new constant symbols $\mathtt{c}_1, \mathtt{c}_2 \ldots$ to $\mathcal{L}_\lambda$ and $\mathcal{L}^*_{\lambda_c}$. Since the language is countable we can enumerate its terms $\mathtt{t}_1, \mathtt{t}_2 \ldots$, which may contain the new constants, and the new constants alone $\mathtt{c}_1, \mathtt{c}_2 \ldots$. We describe a one-one function f from terms to constants.

> $f(\mathtt{t}_i) = \mathtt{c}_j$ where j is the least number such that $j > i$ and $\mathtt{c}_j$ does not occur in $\mathtt{t}_i$ nor is the value under f of any $\mathtt{t}_k$ for $k < i$.

Thus f is a one-one function that assigns a distinct 'fresh' constant to each term of the language, so $f(\mathtt{t})$ is a constant that 'names' $\mathtt{t}$. These play the role of witness constants in the construction of the canonical frame in Theorem 3.8. The $f(\mathtt{t})$ also help us carry out inductions on the size of λ-terms, as $\mathtt{t}[\mathtt{x}/f(\mathtt{s})]$ is smaller than $\lambda \mathtt{x}.\mathtt{t}$ even if $\mathtt{t}[\mathtt{x}/\mathtt{s}]$ might not be.

Definition 3.1. Define a reduction relation $\Longrightarrow^f_\Gamma$ on terms of of $\mathcal{L}^*_{\lambda_c}$ by setting $\mathtt{t} \Longrightarrow^f_\Gamma \mathtt{s}$ if $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ and using the rule:

$$(con) \quad \begin{array}{c} \mathtt{t}[\mathtt{x}/\mathtt{s}] \Longrightarrow^f_\Gamma \mathtt{t}[\mathtt{x}/f(\mathtt{s})] \\ \mathtt{t}[\mathtt{x}/f(\mathtt{s})] \Longrightarrow^f_\Gamma \mathtt{t}[\mathtt{x}/\mathtt{s}] \end{array}$$

In other words, $\Longrightarrow^f_\Gamma$ extends $\Longrightarrow_\Gamma$ with the rule (*con*), which makes $\mathtt{t}$ and its corresponding $f(\mathtt{t})$ inter-reducible.

Remark 3.2. Simply extending $\Longrightarrow_\Gamma$ by insisting that $\{\langle \mathtt{t}, f(\mathtt{t})\rangle, \langle f(\mathtt{t}), \mathtt{t}\rangle\} \subseteq \Gamma$ for every $\mathtt{t}$ is not equivalent to defining $\Longrightarrow^f_\Gamma$ as we have done above. For example, consider the individual variable $\mathtt{x}$: if $f(\mathtt{x})$ is $\mathtt{c}$ and $\langle \mathtt{x}, \mathtt{c}\rangle \in \Gamma$ then by (*ass*), $\mathtt{t} \Longrightarrow_\Gamma \mathtt{c}$ for any $\mathtt{t}$.

Lemma 3.3. *If* $\mathtt{t} \Longrightarrow^f_\Gamma \mathtt{s}$ *and neither* $\mathtt{s}$ *nor* $\mathtt{t}$ *contain any of the new constants* $\mathtt{c}_1, \mathtt{c}_2 \ldots$*, then* $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$*.*

Proof. f is defined in terms of an enumeration such that $\mathtt{r}$ always precedes $f(\mathtt{r})$. Thus if we repeatedly substituting each instance of $f(\mathtt{r})$ with $\mathtt{r}$ in a derivation, eventually all will be eliminated. But then instances of (*con*) depending on become trivial reductions $\mathtt{r} \Longrightarrow^f_\Gamma \mathtt{r}$ which can be removed without affecting the rest of the derivation. Certainly the first and final terms $\mathtt{t}$ and $\mathtt{s}$ are unaffected as they never contained any $f(\mathtt{r})$ in the first place. ☐

Definition 3.4. If $\mathtt{t}$ is a term let $w_\mathtt{t} = \{\mathtt{s} \mid \mathtt{t} \Longrightarrow^f_\Gamma \mathtt{s}\}$. Thus $w_\mathtt{t}$ is the closure of $\mathtt{t}$ under $\Longrightarrow^f_\Gamma$.

Definition 3.5. Define the **canonical** λ**-frame** $F_\lambda = \langle W_\lambda, \bullet_\lambda, \circ_\lambda, H_\lambda\rangle$:

$$W_\lambda = \{w_\mathtt{t} \mid \mathtt{t} \in \mathcal{L}^*_{\lambda_c}\} \qquad H_\lambda = \big\{\{w \mid \mathtt{t} \in w\} \mid w \in W_\lambda \text{ and } \mathtt{t} \in \mathcal{L}_{\lambda_c}\big\}$$

$$w_\mathtt{t} \bullet_\lambda w_\mathtt{s} = \{w \in W_\lambda \mid \mathtt{t}{\cdot}\mathtt{s} \in w\} \qquad w_\mathtt{t} \circ_\lambda w_\mathtt{s} = \{w \in W_\lambda \mid \mathtt{t} * \mathtt{s} \in w\}$$

Definition 3.6. Given $F_\lambda = \langle W_\lambda, \bullet_\lambda, \circ_\lambda, H_\lambda\rangle$, and a term $\mathtt{t}$ of $\mathcal{L}^*_\lambda$, let $\|\mathtt{t}\| = \{w \in W_\lambda \mid \mathtt{t} \in w\}$. Note that $H_\lambda = \big\{\|\mathtt{t}\| \mid \mathtt{t} \in \mathcal{L}_{\lambda_c}\big\}$

Remark 3.7. Given (*sub*) it is easy to see that $\bullet_\lambda \subseteq \circ_\lambda$. Frames where the converse does not hold are easy to construct (for example, Figure 3).

Theorem 3.8. *Let* F_λ *be the canonical intensional* λ*-frame (Definition 3.5), let* $v(\mathtt{x}) = \|\mathtt{x}\|$ *for any variable* $\mathtt{x}$*, and extend* v *so that* $v(\mathtt{c}) = \mathtt{c}$ *for any constant* $\mathtt{c}$*. Then for any term* $\mathtt{t} \in \mathcal{L}_{\lambda_c}$*,* $[\![\mathtt{t}]\!]^v = \|\mathtt{t}\|$*.*

Proof. By induction on $\mathtt{t}$ we show that $w \in \|\mathtt{t}\|$ (i.e. $\mathtt{t} \in w$) iff $w \in [\![\mathtt{t}]\!]^v$.

– $\mathtt{t}$ is a variable $\mathtt{x}$. Then $\|\mathtt{x}\| = v(\mathtt{x}) = [\![\mathtt{x}]\!]^v$ by the definition of v.

– $\mathtt{t}$ is $\mathtt{t}_1{\cdot}\mathtt{t}_2$. Then $\mathtt{t}_1, \mathtt{t}_2 \in \mathcal{L}_{\lambda_c}$.

Suppose $\mathtt{t}_1{\cdot}\mathtt{t}_2 \in w$, and consider the worlds $w_{\mathtt{t}_1}$ and $w_{\mathtt{t}_2}$ in W_λ. If $\mathtt{s}_1 \in w_{\mathtt{t}_1}$ and $\mathtt{s}_2 \in w_{\mathtt{t}_2}$ then by Definition 3.4, $\mathtt{t}_1 \Longrightarrow^f_\Gamma \mathtt{s}_1$ and $\mathtt{t}_2 \Longrightarrow^f_\Gamma \mathtt{s}_2$. Thus $\mathtt{t}_1{\cdot}\mathtt{t}_2 \Longrightarrow^f_\Gamma \mathtt{s}_1{\cdot}\mathtt{s}_2$

and $\mathtt{s}_1{\cdot}\mathtt{s}_2 \in w$. Then by the definition of $\bullet_\lambda$ we have that $w \in w_{\mathtt{t}_1} \bullet_\lambda w_{\mathtt{t}_2}$. Furthermore, $w_{\mathtt{t}_1} \in \|\mathtt{t}_1\|$ and so by the induction hypothesis, $w_{\mathtt{t}_1} \in [\![\mathtt{t}_1]\!]^v$. Similarly $w_{\mathtt{t}_2} \in [\![\mathtt{t}_2]\!]^v$. Hence $w \in [\![\mathtt{t}_1{\cdot}\mathtt{t}_2]\!]^v$ by Definition 2.13.

Conversely, suppose that $w \in [\![\mathtt{t}_1{\cdot}\mathtt{t}_2]\!]^v$. Then there are $w_{\mathtt{s}_1}, w_{\mathtt{s}_2}$ such that $w_{\mathtt{s}_1} \in [\![\mathtt{t}_1]\!]^v$ and $w_{\mathtt{s}_2} \in [\![\mathtt{t}_2]\!]^v$ and $w \in w_{\mathtt{s}_1} \bullet_\lambda w_{\mathtt{s}_2}$. By the induction hypothesis $w_{\mathtt{s}_1} \in \|\mathtt{t}_1\|$ and $w_{\mathtt{s}_2} \in \|\mathtt{t}_2\|$. Then $\mathtt{s}_1 \Longrightarrow^f_\Gamma \mathtt{t}_1$ and $\mathtt{s}_2 \Longrightarrow^f_\Gamma \mathtt{t}_2$. Furthermore, by the construction of $\bullet_\lambda$, $\mathtt{s}_1{\cdot}\mathtt{s}_2 \in w$ and hence by (*cong*) $\mathtt{t}_1{\cdot}\mathtt{t}_2 \in w$.

– $\mathtt{t}$ is $\lambda\mathtt{x.s}$. $\mathtt{s} \in \mathcal{L}_{\lambda_c}$.

Suppose $\lambda\mathtt{x.s} \in w_1$. Suppose that $w_3 \in w_1 \circ_\lambda w_2$, and that $w_2 \in h$ for some $h \in H_\lambda$, then $h = \|\mathtt{r}\|$ for some term $\mathtt{r}$. By (ζ_f) we have that $\mathtt{r} \Longrightarrow^f_\Gamma \mathtt{c}$ and $\mathtt{c} \Longrightarrow^f_\Gamma \mathtt{r}$ for some $\mathtt{c} \in \mathcal{L}_{\lambda_c}$. So $h = \|\mathtt{c}\|$ and $\mathtt{c} \in w_2$. By the construction of $\circ_\lambda$, $\lambda\mathtt{x.s} * \mathtt{r} \in w_3$ and so $\mathtt{s[x/c]} \in w_3$ by (β^*), i.e. $w_3 \in \|\mathtt{s[x/c]}\|$. Since $\mathtt{s[x/c]} \in \mathcal{L}_{\lambda_c}$, it follows by the induction hypothesis that $\|\mathtt{s[x/c]}\| = [\![\mathtt{s[x/c]}]\!]^v$. Furthermore by Lemma 2.16 $[\![\mathtt{s[x/c]}]\!]^v = [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto[\![\mathtt{c}]\!]^v]}$. But by the definition of v, $[\![\mathtt{c}]\!]^v = \|\mathtt{c}\|$, and so $w_3 \in [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto\|\mathtt{c}\|]}$. But $h = \|\mathtt{c}\|$ so $w_3 \in [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto h]}$. Thus $w_1 \in \{w \mid \forall h \in H_\lambda . w \circ_\lambda h \subseteq [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto h]}\} = [\![(\lambda\mathtt{x.s})]\!]^v$. Hence, $\|\lambda\mathtt{x.s}\| \subseteq [\![(\lambda\mathtt{x.s})]\!]^v$

Conversely, suppose that $\lambda\mathtt{x.s} \notin w_\mathtt{r}$ for some $\mathtt{r}$. Let $\mathtt{y}$ be a variable not free in $\mathtt{r}$ or $\mathtt{s}$ and consider the worlds $w_\mathtt{y}$ and $w_{\mathtt{r}*\mathtt{y}}$. If $\mathtt{s[x/y]} \in w_{\mathtt{r}*\mathtt{y}}$ then $\mathtt{r} * \mathtt{y} \Longrightarrow^f_\Gamma \mathtt{s[x/y]}$, so $\lambda\mathtt{y}.(\mathtt{r} * \mathtt{y}) \Longrightarrow^f_\Gamma \lambda\mathtt{y}(\mathtt{s[x/y]})$ by (ξ). But by our choice of $\mathtt{y}$, (η) entails that $\mathtt{r} \Longrightarrow^f_\Gamma \lambda\mathtt{y}.(\mathtt{r} * \mathtt{y})$. So $\mathtt{r} \Longrightarrow^f_\Gamma \lambda\mathtt{y.s[x/y]}$, which contradicts our initial supposition that $\lambda\mathtt{x.s} \notin w_\mathtt{r}$, therefore $\mathtt{s[x/y]} \notin w_{\mathtt{r}*\mathtt{y}}$. In other words $w_{\mathtt{r}*\mathtt{y}} \notin \|\mathtt{s[x/y]}\|$. But $\mathtt{s[x/y]} \in \mathcal{L}_{\lambda_c}$, so by the induction hypothesis $w_{\mathtt{r}\cdot\mathtt{y}} \notin [\![\mathtt{s[x/y]}]\!]^v$. Since $[\![\mathtt{y}]\!]^v = \|\mathtt{y}\|$, it follows by Lemma 2.16 that $w_{\mathtt{r}\cdot\mathtt{y}} \notin [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto\|\mathtt{y}\|]}$. But clearly $w_{\mathtt{r}*\mathtt{y}} \in w_\mathtt{r} \circ_\lambda w_\mathtt{y}$, so it follows that $w_\mathtt{r} \notin \{w \mid \forall h \in H_\lambda . w \circ_\lambda h \subseteq [\![\mathtt{s}]\!]^{v[\mathtt{x}\mapsto h]}\}$. By the semantics of $\lambda\mathtt{y.s}$ this means that $w_\mathtt{r} \notin [\![(\lambda\mathtt{y.s})]\!]^v$. Hence, since every $w \in W_\lambda$ is $w_\mathtt{r}$ for some $\mathtt{r}$, $[\![(\lambda\mathtt{x.s})]\!]^v \subseteq \|\lambda\mathtt{x.s}\|$. □

Lemma 3.9. *If v_1, v_2 are any valuations on a frame F that such that*

1. *$v_1(\mathtt{x}) = v_2(\mathtt{x})$ for any variable $\mathtt{x}$ that occurs free in $\mathtt{t}$,*
2. *v_1, v_2 are extended so that $v_1(\mathtt{c}) = v_2(\mathtt{c})$ for any constant $\mathtt{c}$ that occurs in $\mathtt{t}$,*

then $[\![\mathtt{t}]\!]^{v_1} = [\![\mathtt{t}]\!]^{v_2}$.

Proof. By an easy induction on $\mathtt{t}$. □

Lemma 3.10. *If there is a valuation v on a frame F such that $\{[\![\mathtt{t}]\!]^v \mid \mathtt{t} \in \mathcal{L}_\lambda\} = H$, then F is faithful. Hence the canonical frame F_λ is faithful.*

Proof. Suppose there is such a v, then we must show that for any valuation v' and any term $\mathtt{t} \in \mathcal{L}_\lambda$ that $[\![\mathtt{t}]\!]^{v'} \in H$. By the definition of a valuation, $[\![\mathtt{x}]\!]^{v'} \in H$ for any variable $\mathtt{x}$. So if $[\![\mathtt{t}]\!]^{v'} \notin H$ then by Lemma 3.9

$$[\![\mathtt{t}]\!]^{v[\mathtt{x}_1 \mapsto [\![\mathtt{x}_1]\!]^{v'} \ldots \mathtt{x}_n \mapsto [\![\mathtt{x}_n]\!]^{v'}]} \notin H$$

where $\mathtt{x}_1 \ldots \mathtt{x}_n$ are the free variables of $\mathtt{t}$. Now, by assumption, v is such that every $h \in H$ is $[\![\mathtt{s}]\!]^v$ for some $\mathtt{s}$. It follows then that we can choose $\mathtt{s}_1 \ldots \mathtt{s}_n$ such that $[\![\mathtt{s}_i]\!]^v = v[\mathtt{x}_i \mapsto [\![\mathtt{x}_i]\!]^{v'}]$, and so:

$$[\![\mathtt{t}]\!]^{v[\mathtt{x}_1 \mapsto [\![\mathtt{s}_1]\!]^{v} \ldots \mathtt{x}_n \mapsto [\![\mathtt{s}_n]\!]^{v}]} \notin H$$

This entails, by Theorem 2.16 that $[\![\mathtt{t}[\mathtt{x}_i/\mathtt{s}_i]]\!]^v \notin H$. But this contradicts the assumption that $\{[\![\mathtt{t}]\!]^v \mid \mathtt{t} \in \mathcal{L}_\lambda\} = H$. $\square$

Lemma 3.11. *F_λ is Γ-sensitive.*

Proof. We must argue that for $\langle \mathtt{t}_1, \mathtt{t}_2 \rangle \in \Gamma$ and any v, $[\![\mathtt{t}_1]\!]^v \subseteq [\![\mathtt{t}_2]\!]^v$. Let $\mathtt{x}_1 \ldots \mathtt{x}_n$ be the free variables of $\mathtt{t}_1$ and $\mathtt{t}_2$. Then $v(\mathtt{x}_i)$ is some $\|\mathtt{s}_i\| \in H_\lambda$.

Let v' be a valuation extended such that $v'(\mathtt{r}) = \|\mathtt{r}\|$ for for any variable or constant $\mathtt{r}$ (i.e. v' meets the condition of Theorem 3.8). Then:

$$\begin{aligned}
[\![\mathtt{t}_1]\!]^v &= [\![\mathtt{t}_1]\!]^{v'[\mathtt{x}_1 \mapsto [\![\mathtt{x}_1]\!]^v \ldots \mathtt{x}_n \mapsto [\![\mathtt{x}_n]\!]^v]} && \text{Lemma 3.9} \\
&= [\![\mathtt{t}_1]\!]^{v'[\mathtt{x}_1 \mapsto \|\mathtt{s}_1\| \ldots \mathtt{x}_n \mapsto \|\mathtt{s}_n\|]} && \\
&= [\![\mathtt{t}_1]\!]^{v'[\mathtt{x}_1 \mapsto [\![\mathtt{s}_1]\!]^{v'} \ldots \mathtt{x}_n \mapsto [\![\mathtt{s}_n]\!]^{v'}]} && \text{Theorem 3.8} \\
&= [\![\mathtt{t}_1[\mathtt{x}_i/\mathtt{s}_i]]\!]^{v'} && \text{Lemma 2.16} \\
&= \|\mathtt{t}_1[\mathtt{x}_i/\mathtt{s}_i]\| && \text{Theorem 3.8}
\end{aligned}$$

and similarly for $\mathtt{t}_2$. But $\mathtt{t}_1[\mathtt{x}_i/\mathtt{s}_i] \Longrightarrow_\Gamma \mathtt{t}_2[\mathtt{x}_i/\mathtt{s}_i]$ by (ass), and so $\|\mathtt{t}_1[\mathtt{x}_i/\mathtt{s}_i]\| \subseteq \|\mathtt{t}_2[\mathtt{x}_i/\mathtt{s}_i]\|$ and so $[\![\mathtt{t}_1]\!]^v \subseteq [\![\mathtt{t}_2]\!]^v$. $\square$

Theorem 3.12. $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ *if and only if* $[\![\mathtt{t}]\!]^v \subseteq [\![\mathtt{s}]\!]^v$ *for all* Γ*-sensitive models.*[4]

Proof. The left-right direction is Theorem 2.25.

If $\mathtt{t} \not\Longrightarrow_\Gamma \mathtt{s}$ then $\mathtt{s} \notin w_\mathtt{t}$ in F_λ. Therefore $\|\mathtt{t}\| \not\subseteq \|\mathtt{s}\|$ and so by Theorem 3.8 there is a valuation v such that $[\![\mathtt{t}]\!]^v \not\subseteq [\![\mathtt{s}]\!]^v$ on the canonical frame F_λ. Furthermore, by Lemmas 3.10 and 3.11, F_λ is faithful and Γ-sensitive. $\square$

Corollary 3.13. *If* $\mathtt{t}$ *and* $\mathtt{s}$ *are terms of* $\mathcal{L}_\lambda$ *then* $\mathtt{t} \longrightarrow_\Gamma \mathtt{s}$ *if and only if* $[\![\mathtt{t}]\!]^v \subseteq [\![\mathtt{s}]\!]^v$ *for all* Γ*-sensitive models.*

[4] Equivalently: $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ if and only if $[\![\mathtt{t}]\!]^v \subseteq [\![\mathtt{s}]\!]^v$ for any valuation v on any Γ-sensitive frame.

Proof. Using Theorem 2.8 and the assumption that $\mathtt{t}$ and $\mathtt{s}$ are terms of $\mathcal{L}_\lambda$ we get that $\mathtt{t} \longrightarrow_\Gamma \mathtt{s}$ if and only if $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ □

Definition 3.14. An **extensional frame** is an intensional frame where $\bullet = \circ$, we may define them simply as a triple $\langle W, \bullet, H\rangle$. Similarly an **extensional model** is a pair $\langle F, v\rangle$ where F is an extensional frame.

Corollary 3.15. *Let* $\Gamma = \{\langle \mathtt{x}, \lambda\mathtt{y}.(\mathtt{x}\cdot\mathtt{y})\rangle\}$. *Then* $\mathtt{t} \Longrightarrow_\Gamma \mathtt{s}$ *if and only if* $[\![\mathtt{t}]\!]^v \subseteq [\![\mathtt{s}]\!]^v$ *for any faithful extensional model.*

Proof. For the left-right direction it is a simple matter to apply the reasoning of Theorem 2.25. For the right-left direction it is enough to note that:

$$\mathtt{t}[\mathtt{x}{:-}\mathtt{s} * \mathtt{r}] \overset{(ass)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\lambda\mathtt{y}(\mathtt{s}\cdot\mathtt{y}) * \mathtt{r}] \overset{(\beta^*)}{\Longrightarrow}_\Gamma \mathtt{t}[\mathtt{x}{:-}\mathtt{s}\cdot\mathtt{r}]$$

so in the construction of the canonical frame F_λ of Theorem 3.8, $\bullet_\lambda = \circ_\lambda$. □

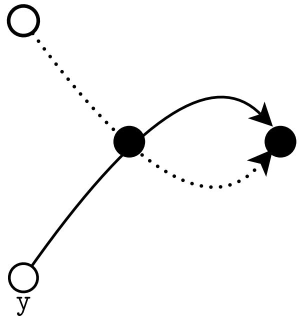
Remark 3.16. An extensional frame satisfies η-expansion. An intensional frame is like an extensional frame except with an additional 'outer' application function $\circ$. We interpret λ in terms of the outer function and application in terms of the inner function $\bullet$ to block η-expansion (Definition 2.13). η-expansion will prove useful in constructing models of λ-equality in Section 4. Other authors have also noted reasons to include η-expansion in models [13].

Remark 3.17. Given 3.15, we can say that λ-reduction with η-expansion is *complete* for extensional frames.

Remark 3.18. Notice also a crucial purpose served by H in the completeness proof. Any subset of a frame is a potential denotation of a λ-term, and H may be seen as listing the subsets that actually are denotations of λ-terms. We have used this distinction to characterise intensional λ-reduction.

We took an (intensional) set of λ-reductions Γ (in $\mathcal{L}_\lambda$) and we extended it using $*$ to help us interpret λ (Definition 2.5 and Theorem 2.8). Then, when we constructed the frame (Definition 3.5) for Γ we left out of H the denotations depending explicitly on $*$. We obtained a frame which is sensitive to all the reductions of Γ, in the original language $\mathcal{L}_\lambda$, but where the interpretation of λ still depends on $*$ which is not mentioned in Γ (Theorem 3.8).

As we shall see, this provides a simple characterisation of intensional and extensional λ-abstraction. Abusing notation somewhat: extensional $\lambda\mathtt{x}.\mathtt{t}$ is something that maps objects h in the domain to $\mathtt{t}(h)$; intensional $\lambda\mathtt{x}.\mathtt{t}$ is something maps objects h in the domain *and also some in a hidden domain* to $\mathtt{t}(h)$. Furthermore, the 'hidden' objects are the denotations of terms in $\mathcal{L}^*_\lambda$ that require $*$.



A solid arrow passing from w_1 through w_2 to w_3 represents that $w_3 \in w_1 \bullet w_2$, and a dotted arrow represents that $w_3 \in w_1 \circ w_2$. H can be set as $\mathcal{P}(W)$ and $v(\mathtt{y})$ is as indicated. Then the two worlds on the left (unfilled) both are in $[\![\lambda\mathtt{x}.(\mathtt{y}{\cdot}\mathtt{x})]\!]^v$ and $[\![\lambda\mathtt{x}.(\mathtt{y} * \mathtt{x})]\!]^v$, but only one is in $[\![\mathtt{y}]\!]^v$.

Figure 3: A counterexample to η-reduction in an intensional model where W contains 4 worlds.

3.1 η-reduction

As already noted, if $\mathtt{x}$ is not free in $\mathtt{t}$, then $[\![\mathtt{t}]\!]^v \subseteq [\![\lambda\mathtt{x}.(\mathtt{t} * \mathtt{x})]\!]^v$ in any intensional frame. That is, η-expansion is satisfied by any frame, but what about η-reduction? Figure 3 gives an example of a simple frame where $[\![\mathtt{y}]\!]^v \nsubseteq [\![\lambda\mathtt{x}.(\mathtt{y}{\cdot}\mathtt{x})]\!]^v$ (and since $\bullet \subseteq \circ$, also $[\![\mathtt{y}]\!]^v \nsubseteq [\![\lambda\mathtt{x}.(\mathtt{y} * \mathtt{x})]\!]^v$).

We can characterise η-reduction syntactically easily enough:

Definition 3.19. Let $\boldsymbol{\eta}^- = \{\langle \lambda\mathtt{x}.(\mathtt{y}{\cdot}\mathtt{x}), \mathtt{y}\rangle\}$.

Then $\mathtt{t} \Longrightarrow_{\boldsymbol{\eta}^-} \mathtt{s}$ is the relation we want. Furthermore, we can use the completeness theorem 3.12 to describe a class of models for which this relation is complete:

Definition 3.20. A frame is η-**reductive** when $\bigcap_{h' \in H}\{w \mid w \circ h' \subseteq h \bullet h'\} \subseteq h$ for any h.

Theorem 3.21. $\mathtt{t} \Longrightarrow_{\boldsymbol{\eta}^-} \mathtt{s}$ *iff* $\mathtt{t} \subseteq \mathtt{s}$ *in all η-reductive models.*

Proof. It is straightforward to verify that $\mathtt{t} \Longrightarrow_{\boldsymbol{\eta}^-} \mathtt{s}$ implies that $[\![\mathtt{t}]\!]^v \subseteq [\![\mathtt{s}]\!]^v$ in all η-reductive models. Conversely, if $\mathtt{t} \not\Longrightarrow_{\boldsymbol{\eta}^-} \mathtt{s}$ then $[\![\mathtt{t}]\!]^v \nsubseteq [\![\mathtt{s}]\!]^v$ in the canonical model for $\boldsymbol{\eta}^-$:

$$\bigcap_{h' \in H_\lambda} \{w \mid w \circ h' \subseteq \|\mathtt{t}\| \bullet h'\} = \|\lambda\mathtt{x}.(\mathtt{t}{\cdot}\mathtt{x})\| \subseteq \|\mathtt{t}\|$$

since each $h \in H$ is $\|\mathtt{t}\|$ for some $\mathtt{t}$. ☐

4 Equality

Definition 4.1. Let $\boldsymbol{\beta} = \{\langle \mathtt{t}, \lambda\mathtt{x.t}\cdot\mathtt{x}]\rangle \mid \mathtt{t} \in \mathcal{L}_\lambda\}$.

Corollary 4.2. *When restricted to $\mathcal{L}_\lambda$, $\Longrightarrow_{\boldsymbol{\beta}}$ is the familiar relation of (intensional) λ-equality, and by Theorem 3.12 is complete for $\boldsymbol{\beta}$-sensitive models.*

Remark 4.3. Corollary 4.2 is itself not so significant as it only tells us half the story about what these models look like, and does not tell us if there are any non-trivial ones. Of course, given independent nontriviality proofs for λ-equality,[5] we can use Theorem 3.12 to conclude that there are nontrivial $\boldsymbol{\beta}$-sensitive models. This section is concerned with producing a purely semantic characterisation of $\boldsymbol{\beta}$-sensitivity. In fact, in characterising $\boldsymbol{\beta}$-sensitivity, we can complete Definition 2.17 and provide a semantic characterisation of faithfulness.

The strategy we shall employ is as follows. First we introduce some shorthands to stand in for constructions involving λ-expressions, so for example $\mathbf{K}\cdot\mathtt{z}$ will stand in for $(\lambda\mathtt{x}\lambda\mathtt{y.x})\cdot\mathtt{z}$. This will allow us to work with certain complex λ-expressions as if they are free of the symbol λ. Then, for each $\mathtt{t}$ we describe a new term $[\mathtt{x}]\mathtt{t}$, constructible only out of application and the new shorthands (effectively the familiar combinator abstraction of [12, p.26], but extended to a language that includes the λ-operator). Then, with the help of the completeness theorem 3.12, we describe conditions in which a model (or frame) entails that $[\![\mathtt{t}[\mathtt{x}/\mathtt{s}]]\!]^v = [\![[\mathtt{x}]\mathtt{t}\cdot\mathtt{s}]\!]^v$. It then turns out that $[\![[\mathtt{x}]\mathtt{t}]\!]^v \subseteq [\![\lambda\mathtt{x.t}]\!]^v$ and we thereby obtain models of β-expansion.

Definition 4.4. Define the following shorthands:

$$
\begin{aligned}
\mathbf{K} &= \lambda\mathtt{xy.x}\\
\mathbf{C} &= \lambda\mathtt{xyz.}((\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y}))\\
\mathbf{S} &= \lambda\mathtt{xyz.}((\mathtt{x}\cdot\mathtt{z})\cdot(\mathtt{y}\cdot\mathtt{z}))
\end{aligned}
$$

Definition 4.5. – Say that an instance of λ in a term $\mathtt{t}$ is **free** if it is not part of an occurrence of **K**, **C**, **S** in $\mathtt{t}$.

– When defining or proving a property of a term $\mathtt{t}$, we write '**by induction on (l, d)**' to describe an induction on the pair (l, d), lexicographically ordered, where d is the number of occurrences of $\cdot$ in $\mathtt{t}$ and l is the number of occurrences of λ in $\mathtt{t}$ that are free.[6]

[5] Nontriviality follows syntactically from the Church-Rosser property [12, Ch. A2], the cut-elimination theorem of [5]; and it follows semantically from Scott's famous model D_ω [12, Ch. 16], among others.

[6] More loosely, if we were to treat **K**, **C**, **S** as constants in $\mathtt{t}$ (i.e. not containing λ at all), then l would be the number of occurrences of λ in $\mathtt{t}$.

Definition 4.6. For any $\mathtt{t} \in \mathcal{L}_\lambda$, define $[\mathtt{x}]\mathtt{t}$ by induction on (l, d):

1. $[\mathtt{x}]\mathtt{x}$ is $(\mathbf{S}{\cdot}\mathbf{K}){\cdot}\mathbf{K}$
2. $[\mathtt{x}]\mathtt{r}$ is $\mathbf{K}{\cdot}\mathtt{r}$ if $\mathtt{x}$ is not free in $\mathtt{r}$.
3. $[\mathtt{x}](\mathtt{s}{\cdot}\mathtt{r})$ is $(\mathbf{S}{\cdot}[\mathtt{x}]\mathtt{s}){\cdot}[\mathtt{x}]\mathtt{r}$
4. $[\mathtt{x}]\lambda\mathtt{z}.\mathtt{s}$ is $\mathbf{C}{\cdot}[\mathtt{z}][\mathtt{x}]\mathtt{s}$

Without loss of generality we assume that $\mathtt{x}$ and $\mathtt{z}$ are distinct in (4).

Lemma 4.7. *If* $\mathtt{t} \in \mathcal{L}_\lambda$ *then* $[\mathtt{x}]\mathtt{t} \in \mathcal{L}^*_\lambda$ *and (1) is well defined, (2) contains no free instances of* λ*, and (3) contains no free occurrences of* $\mathtt{x}$*.*

Proof. By induction on (l, d).

– If $\mathtt{t}$ is atomic or of the form $\mathtt{s}{\cdot}\mathtt{r}$ or $\mathtt{s} * \mathtt{r}$ then the result is easily proved.

– If $\mathtt{t}$ is $\lambda\mathtt{z}.\mathtt{s}$ then by the induction hypothesis $[\mathtt{x}]\mathtt{s}$ is well defined and contains no free occurrences of λ. So the induction hypothesis applies again and the same may be said of $[\mathtt{z}][\mathtt{x}]\mathtt{s}$. It then follows easily that the properties (1), (2) and (3) hold for $[\mathtt{x}]\lambda\mathtt{z}.\mathtt{s}$. □

Lemma 4.8. *Suppose* $\mathtt{t} \in \mathcal{L}_\lambda$ *and let variable* $\mathtt{v}$ *not occur in* $\mathtt{t}$*, then* $[\mathtt{x}](\mathtt{t}[\mathtt{y}/\mathtt{v}]) = ([\mathtt{x}]\mathtt{t})[\mathtt{y}/\mathtt{v}]$

Proof. By induction on (l, d).

– If $\mathtt{t}$ is $\mathtt{x}$ then $\mathtt{x}[\mathtt{y}/\mathtt{v}] = \mathtt{x}$ and so

$$\begin{array}{rcll} [\mathtt{x}]\mathtt{x} & = & (\mathbf{S}{\cdot}\mathbf{K}{\cdot})\mathbf{K} & \text{Def. 4.6} \\ & = & ([\mathtt{x}]\mathtt{x})[\mathtt{y}/\mathtt{v}] & \mathtt{y} \notin (\mathbf{S}{\cdot}\mathbf{K}{\cdot})\mathbf{K} \end{array}$$

– If $\mathtt{t}$ is $\mathtt{y}$ then $\mathtt{y}[\mathtt{y}/\mathtt{v}] = \mathtt{v}$ and $[\mathtt{x}]\mathtt{y} = \mathbf{K}{\cdot}\mathtt{y}$. So:

$$\begin{array}{rcll} [\mathtt{x}]\mathtt{v} & = & \mathbf{K}{\cdot}\mathtt{v} & \text{Def. 4.6} \\ & = & (\mathbf{K}{\cdot}\mathtt{y})[\mathtt{y}/\mathtt{v}] & \mathtt{y} \notin \mathbf{K} \\ & = & ([\mathtt{x}]\mathtt{y})[\mathtt{y}/\mathtt{v}] & \end{array}$$

– The case where $\mathtt{t}$ is $\mathbf{K}$, $\mathbf{C}$, $\mathbf{S}$ or some variable other than $\mathtt{x}$ or $\mathtt{y}$ is similar.

– If $\mathtt{t}$ is $\mathtt{s}{\cdot}\mathtt{r}$ or $\mathtt{s} * \mathtt{r}$ then the result follows easily by the induction hypothesis.

– If $\mathtt{t}$ is $\lambda\mathtt{z}.\mathtt{s}$, then we may assume that $\mathtt{z}$ is not $\mathtt{x}$, then

$$\begin{array}{rcll} [\mathtt{x}](\mathtt{t}[\mathtt{y}/\mathtt{v}]) & = & \mathbf{C}{\cdot}[\mathtt{z}][\mathtt{x}](\mathtt{s}[\mathtt{y}/\mathtt{v}]) & \text{Def. 4.6} \\ & = & \mathbf{C}{\cdot}([\mathtt{z}][\mathtt{x}]\mathtt{s})[\mathtt{y}/\mathtt{v}] & \text{ind. hyp, Lemma 4.7} \\ & = & (\mathbf{C}{\cdot}[\mathtt{z}][\mathtt{x}]\mathtt{s})[\mathtt{y}/\mathtt{v}] & \mathtt{y} \notin \mathbf{C} \\ & = & ([\mathtt{x}]\mathtt{t})[\mathtt{y}/\mathtt{v}] & \end{array}$$

□

Theorem 4.9. *If* $\texttt{t} \in \mathcal{L}_\lambda$*, then for any* $M = \langle F, v\rangle$*,* $[\![[\texttt{x}]\texttt{t} * \texttt{s}]\!]^v \subseteq [\![\texttt{t}[\texttt{x}/\texttt{s}]]\!]^v$

Proof. By induction on (l, d).

We appeal to known facts about β-reduction and Theorem 3.12 (completeness).

– $\texttt{t} = \texttt{x}$. Then $[\texttt{x}]\texttt{t} * \texttt{s}$ is $((\mathbf{S}{\cdot}\mathbf{K}{\cdot})\mathbf{K}){\cdot}\texttt{s}$, and it is easy to show that that

$$((\mathbf{S}{\cdot}\mathbf{K}{\cdot})\mathbf{K}) * \texttt{s} \Longrightarrow_\varnothing \texttt{s}$$

So the result follows by Theorem 3.12.

– The argument is similar for the case where $\texttt{t}$ is a variable $\texttt{y} \neq \texttt{x}$ or $\mathbf{K}$, $\mathbf{C}$, $\mathbf{S}$. We appeal to the easily shown fact that:

$$(\mathbf{K}{\cdot}\texttt{t}) * \texttt{s} \Longrightarrow_\varnothing \texttt{t}$$

– $\texttt{t} = \texttt{t}_1{\cdot}\texttt{t}_2$. Then

$$\begin{array}{rcl} ((\mathbf{S}{\cdot}[\texttt{x}]\texttt{t}_1) * [\texttt{x}]\texttt{t}_2){\cdot}\texttt{s} & \Longrightarrow_\varnothing & \texttt{t}_1[\texttt{x}/\texttt{s}]{\cdot}\texttt{t}_2[\texttt{x}/\texttt{s}] \\ & = & (\texttt{t}_1{\cdot}\texttt{t}_2)[\texttt{x}/\texttt{s}] \end{array}$$

and the result follows as above.

– The argument is similar for the case where $\texttt{t} = \texttt{t}_1 * \texttt{t}_2$

– If $\texttt{t}$ is $\lambda\texttt{y}.\texttt{r}$, then choose a variable $\texttt{z}$ that does not occur in $\texttt{r}$ or $\texttt{s}$. Now, $\texttt{s}$ and $[\texttt{x}]\texttt{s}$ contain fewer free instances of λ than $\texttt{t}$ (Lemma 4.7), so given Theorem 3.12 we may apply the induction hypothesis as follows:

$$\begin{array}{rcll} [\![(\mathbf{C}{\cdot}[\texttt{y}][\texttt{x}]\texttt{r}){\cdot}\texttt{s}]\!]^v & \subseteq & [\![\lambda\texttt{z}.\big(((\mathbf{C}{\cdot}[\texttt{y}][\texttt{x}]\texttt{r}){\cdot}\texttt{s}) * \texttt{z}\big)]\!]^v & \text{Thrm 2.25} \\ & \subseteq & [\![\lambda\texttt{z}.\big(([\texttt{y}][\texttt{x}]\texttt{r}{\cdot}\texttt{z}){\cdot}\texttt{s}\big)]\!]^v & \text{Thrm 3.12} \\ & \subseteq & [\![\lambda\texttt{z}.([\texttt{x}]\texttt{r}[\texttt{y}/\texttt{z}]{\cdot}\texttt{s})]\!]^v & \text{Ind. Hyp.} \\ & \subseteq & [\![\lambda\texttt{z}.(\texttt{t}[\texttt{y}/\texttt{z}, \texttt{x}/\texttt{s}])]\!]^v & \text{Ind. Hyp.} \\ & = & [\![\lambda\texttt{y}.(\texttt{t}[\texttt{x}/\texttt{s}])]\!]^v & \end{array}$$

□

Definition 4.10. A frame is **λ-complete** when for any $h_1, h_2, h_3 \in H$

1. $h_1 = ([\![\mathbf{K}]\!]^v \bullet h_1) \bullet h_2$
2. $(h_1 \bullet h_3) \bullet (h_2 \bullet h_3) = (([\![\mathbf{S}]\!]^v \bullet h_1) \bullet h_2) \bullet h_3$
3. $\bigcap_{h \in H}\{w \mid (h_1 \bullet h) \bullet h_2\} = ([\![\mathbf{C}]\!]^v \bullet h_1) \bullet h_2$

A model $\langle F, v\rangle$ is λ-complete if F is.

Remark 4.11. Notice that no $h \in H$ can be empty if F is a non-trivial λ-complete frame. For if $\varnothing \in H$ then for any $h \in H$, $h = ([\![\mathbf{K}]\!]^v \bullet h) \bullet \varnothing = \varnothing$, so then $H = \{\varnothing\}$ and $[\![\mathtt{t}]\!]^v = [\![\mathtt{s}]\!]^v = \varnothing$ for any $\mathtt{t}, \mathtt{s}$ and v.

We could have equivalently defined λ-complete frames by requiring that, for any v $[\![\mathtt{x}]\!]^v = [\![(\mathbf{K}\cdot\mathtt{x})\cdot\mathtt{y}]\!]^v$, $[\![(\mathtt{x}\cdot\mathtt{z})\cdot(\mathtt{y}\cdot\mathtt{z})]\!]^v = [\![((\mathbf{S}\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v$, $[\![\lambda\mathtt{z}.((\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y})]\!]^v = [\![(\mathbf{C}\cdot\mathtt{x})\cdot\mathtt{y}]\!]^v$ and so on. But 4.10 is preferable as its form is less dependent on the syntax. Since $\mathbf{K}$, $\mathbf{S}$ and $\mathbf{C}$ are closed terms, we could even go further and replace $[\![\mathbf{K}]\!]^v$, $[\![\mathbf{S}]\!]^v$ and $[\![\mathbf{C}]\!]^v$ with purely semantic expressions using Definition 2.13.

Theorem 4.12. *For any λ-complete $\langle F, v\rangle$, if $\mathtt{t} \in \mathcal{L}_\lambda$ then $[\![\mathtt{t}[\mathtt{x}/\mathtt{s}]]\!]^v \subseteq [\![[\mathtt{x}]\mathtt{t}\cdot\mathtt{s}]\!]^v$.*

Proof. Again, we proceed by induction on (l, d).

– $\mathtt{t} = \mathtt{x}$. Then $\mathtt{x}[\mathtt{x}/\mathtt{s}] = \mathtt{s}$ and it is not hard to see that the definition of lambda completeness (4.10) implies that $[\![\mathtt{s}]\!]^v = [\![((\mathbf{S}\cdot\mathbf{K})\cdot\mathbf{K})\cdot\mathtt{s}]\!]^v$.

– The argument is similar for the case where $\mathtt{t} = \mathtt{y} \neq \mathtt{x}$ or $\mathtt{t}$ is $\mathbf{K}$, $\mathbf{C}$, $\mathbf{S}$.

– $\mathtt{t} = \mathtt{t}_1\cdot\mathtt{t}_2$. Then:

$$\begin{array}{rcll} [\![(\mathtt{t}_1\cdot\mathtt{t}_2)[\mathtt{x}/\mathtt{s}]]\!]^v & = & [\![\mathtt{t}_1[\mathtt{x}/\mathtt{s}]]\!]^v \bullet [\![\mathtt{t}_2[\mathtt{x}/\mathtt{s}]]\!]^v & \\ & \subseteq & [\![([\mathtt{x}]\mathtt{t}_1\cdot\mathtt{s})]\!]^v \bullet [\![([\mathtt{x}]\mathtt{t}_2\cdot\mathtt{s})]\!]^v & \text{Ind. Hyp.} \\ & = & ([\![[\mathtt{x}]\mathtt{t}_1]\!]^v \bullet [\![\mathtt{s}]\!]^v) \bullet ([\![[\mathtt{x}]\mathtt{t}_2]\!]^v \bullet [\![\mathtt{s}]\!]^v) & \\ & = & (([\![\mathbf{S}]\!]^v \bullet [\![[\mathtt{x}]\mathtt{t}_1]\!]^v) \bullet [\![[\mathtt{x}]\mathtt{t}_2]\!]^v) \bullet [\![\mathtt{s}]\!]^v & \text{Def 4.10} \\ & = & [\![((\mathbf{S}\cdot[\mathtt{x}]\mathtt{t}_1)\cdot[\mathtt{x}]\mathtt{t}_2)\cdot[\![\mathtt{s}]\!]^v]\!]^v & \text{Def. 2.13} \\ & = & [\![([\mathtt{x}]\mathtt{t}_1\cdot\mathtt{t}_2)\cdot\mathtt{s}]\!]^v & \text{Def. 4.6} \end{array}$$

and the result follows as above.

– Suppose $\mathtt{t}$ is $\lambda\mathtt{y}.\mathtt{r}$. Let $\mathtt{z}$ be chosen so that it does not occur in $\mathtt{t}$ or $\mathtt{s}$. Then using Lemma 2.24:

$$\begin{array}{rcll} [\![\lambda\mathtt{y}.(\mathtt{r}[\mathtt{x}/\mathtt{s}])]\!]^v & = & [\![\lambda\mathtt{z}.(\mathtt{r}[\mathtt{y}/\mathtt{z}, \mathtt{x}/\mathtt{s}])]\!]^v & \\ & \subseteq & [\![\lambda\mathtt{z}.\big([\mathtt{x}](\mathtt{r}[\mathtt{y}/\mathtt{z}])\cdot\mathtt{s}\big)]\!]^v & \text{Ind. Hyp.} \\ & = & [\![\lambda\mathtt{z}.\big(([\mathtt{x}]\mathtt{r})[\mathtt{y}/\mathtt{z}]\cdot\mathtt{s}\big)]\!]^v & \text{Lemma 4.8} \\ & \subseteq & [\![\lambda\mathtt{z}.\big(([\mathtt{y}][\mathtt{x}]\mathtt{r}\cdot\mathtt{z})\cdot\mathtt{s}\big)]\!]^v & \text{Ind. Hyp.} \\ & \subseteq & [\![(\mathbf{C}\cdot[\mathtt{y}][\mathtt{x}]\mathtt{r})\cdot\mathtt{s}]\!]^v & \text{Def. 4.10} \\ & = & [\![([\mathtt{x}]\lambda\mathtt{y}.\mathtt{r})\cdot\mathtt{s}]\!]^v & \text{Def. 4.6} \end{array}$$

□

Corollary 4.13. *If $\mathtt{t} \in \mathcal{L}_\lambda$ then $[\![\mathtt{t}[\mathtt{x}/\mathtt{s}]]\!]^v = [\![\lambda\mathtt{x}.\mathtt{t}\cdot\mathtt{s}]\!]^v$ for any λ-complete model $\langle F, v\rangle$.*

Proof. Since $\bullet \subseteq \circ$ we have that $[[\lambda\texttt{x}.\texttt{t}\cdot\texttt{s}]]^v \subseteq [[\lambda\texttt{x}.\texttt{t} * \texttt{s}]]^v$, and so Theorem 4.9 entails that $[[\lambda\texttt{x}.\texttt{t}\cdot\texttt{s}]]^v \subseteq [[\texttt{t}[\texttt{x}/\texttt{s}]]]^v$.

And conversely:

$$\begin{array}{lll} [[\texttt{t}[\texttt{x}/\texttt{s}]]]^v & \subseteq & [[[\texttt{x}]\texttt{t}\cdot\texttt{s}]]^v \quad \text{Thrm 4.12} \\ & \subseteq & [[\lambda\texttt{x}.([\texttt{x}]\texttt{t} * \texttt{x})\cdot\texttt{s}]]^v \quad \text{Thrm 3.12, Lemma 2.24, Lemma 4.7} \\ & \subseteq & [[\lambda\texttt{x}.\texttt{t}\cdot\texttt{s}]]^v \quad \text{Thrm 4.9} \end{array}$$

□

We now have a means of characterising β-equality semantically.

Corollary 4.14. *If* $\texttt{t},\texttt{s} \in \mathcal{L}_\lambda$ *then,* $\texttt{t} \Longrightarrow_{\boldsymbol{\beta}} \texttt{s}$ *iff* $[[\texttt{t}]]^v \subseteq [[\texttt{s}]]^v$ *for all λ-complete models* $\langle F, v\rangle$*.*

Proof. By 4.13 if $\langle \texttt{t},\texttt{s}\rangle \in \boldsymbol{\beta}$ then $[[\texttt{t}]]^v \subseteq [[\texttt{s}]]^v$ in all λ-complete frames. Furthermore if $\texttt{t} \not\Longrightarrow_{\boldsymbol{\beta}} \texttt{s}$ then $[[\texttt{t}]]^v \nsubseteq [[\texttt{s}]]^v$ in the canonical model for $\boldsymbol{\beta}$. It is not hard to verify that the canonical frame is λ-complete. □

Corollary 4.15. *A frame is λ-complete iff it is* $\boldsymbol{\beta}$*-sensitive*

Definition 4.16. A frame is **fully extensional** when $h = \bigcap_{h' \in H}\{w \mid w \circ h' \subseteq h \bullet h'\}$ for all $h \in H$. A model $\langle F, v\rangle$ is fully extensional when F is.

Remark 4.17. Looking at Definition 2.13 $h = [[\texttt{t}]]^v$ implies $\bigcap_{h' \in H}\{w \mid w \circ h' \subseteq h \bullet h'\} = [[\lambda\texttt{x}.(\texttt{t}\cdot\texttt{x})]]^v$ for $\texttt{x} \notin \texttt{t}$. So if a frame is fully extensional then, for any $\texttt{t}$ and any v, $[[\texttt{t}]]^v = [[\lambda\texttt{x}(\texttt{t}\cdot\texttt{x})]]^v$ for $\texttt{x}$ not free in $\texttt{t}$. This implies, by reasoning similar to Corollary 3.15, that $\bullet = \circ$.

Definition 4.18. Let $\boldsymbol{\eta} = \{\langle \lambda\texttt{x}.(\texttt{y}\cdot\texttt{x}), \texttt{y}\rangle, \langle \texttt{y}, \lambda\texttt{x}.(\texttt{y}\cdot\texttt{x})\rangle\}$ and let $\boldsymbol{\beta\eta} = \boldsymbol{\beta} \cup \boldsymbol{\eta}$.

Theorem 4.19. *If* $\texttt{t},\texttt{s} \in \mathcal{L}_\lambda$ *then,* $\texttt{t} \Longrightarrow_{\boldsymbol{\beta\eta}} \texttt{s}$ *iff* $[[\texttt{t}]]^v \subseteq [[\texttt{s}]]^v$ *for all fully extensional λ-complete models* $\langle F, v\rangle$*.*

Proof. It is straightforward to verify (see Remark 4.17) that $\texttt{t} \Longrightarrow_{\boldsymbol{\beta\eta}} \texttt{s}$ implies that $[[\texttt{t}]]^v \subseteq [[\texttt{s}]]^v$ in all fully extensional, λ-complete models. Conversely, if $\texttt{t} \not\Longrightarrow_{\boldsymbol{\beta\eta}} \texttt{s}$ then $[[\texttt{t}]]^v \nsubseteq [[\texttt{s}]]^v$ in the canonical model for $\boldsymbol{\beta\eta}$, it is not hard to show that it is λ-complete and fully extensional. □

Definition 4.20. Say that a frame is **combinatorially complete** when for any $h_1, h_2, h_3 \in H$

1. $h_1 = ([[\mathbf{K}]]^v \bullet h_1) \bullet h_2$
2. $(h_1 \bullet h_3) \bullet (h_2 \bullet h_3) = (([[\mathbf{S}]]^v \bullet h_1) \bullet h_2) \bullet h_3$

This is the familiar notion of combinatory completeness as used in characterisations of lambda models in terms of combinatory algebras (e.g. see [12, p.228]). We now get the following result.

Theorem 4.21. *A fully extensional frame (model) is λ-complete if it is combinatorially complete.*

Proof. Again, given Theorem 3.12 we argue partially syntactically. First note that the first two conditions of Definition 4.10 are met if F is combinatorially complete.

Now we argue that if a frame F is combinatorially complete, then for any v,

$$[\![(\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y}]\!]^v = [\![((\mathbf{C}\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v$$

Given soundness (Theorem 2.25) and known facts about combinators (e.g. [12, p.25]), if F is combinatorially complete then $[\![(\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y}]\!]^v = [\![((\mathbf{C}'\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v$ for some particular complex expression $\mathbf{C}'$ given in terms of $\mathbf{S}$ and $\mathbf{K}$.[7] Moreover, since F is fully extensional, i.e. $[\![\lambda\mathtt{x}(\mathtt{t}\cdot\mathtt{x})]\!]^v = [\![\mathtt{t}]\!]^v$ for any $\mathtt{t} \in \mathcal{L}_\lambda$, then

$$[\![\mathbf{C}']\!]^v = [\![\lambda\mathtt{xyz}.((\mathbf{C}'\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v = [\![\lambda\mathtt{xyz}.((\mathtt{x}\cdot\mathtt{r})\cdot\mathtt{y})]\!]^v = [\![\mathbf{C}]\!]^v$$

and so $[\![(\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y}]\!]^v = [\![((\mathbf{C}'\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v = [\![((\mathbf{C}\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}]\!]^v$.

So if F is combinatorially complete then:

$$\begin{array}{rcll} [\![\lambda\mathtt{z}.((\mathtt{x}\cdot\mathtt{z})\cdot\mathtt{y})]\!]^v & \subseteq & [\![\lambda\mathtt{z}.\big(((\mathbf{C}\cdot\mathtt{x})\cdot\mathtt{y})\cdot\mathtt{z}\big)]\!]^v & \text{by the above} \\ & \subseteq & [\![(\mathbf{C}\cdot\mathtt{x})\cdot\mathtt{y}]\!]^v & \text{as } F \text{ is fully extensional} \end{array}$$

and so the third condition of 4.10 is met. □

5 Faithfulness

The completeness result 4.15 relates λ theories to *faithful* λ-complete frames. Faithfulness was defined in 2.17 partially syntactically: a faithful frame is one that has a denotation $h \in H$ for every λ-term $\mathtt{t}$.

It is natural to seek a characterisation of faithfulness that does not require explicit reference to the syntax, i.e. a purely semantic one. Can we provide a description, only in terms of H and R, of structural properties a frame must have in order that there is an $h \in H$ to be the denotation of each λ-term? The difficulty lies in the denotation of λ-terms of the form $\lambda\mathtt{x.s}$. We might know what must hold of H for it to include the denotation of $\mathtt{s}$, but what of $\lambda\mathtt{x.s}$? $\lambda\mathtt{x}$ acts like a kind of quantifier

[7] $\mathbf{C}'$ is $\big(\mathbf{S}\cdot\big((\mathbf{B}\cdot\mathbf{B})\cdot\mathbf{S}\big)\big)\cdot(\mathbf{K}\cdot\mathbf{K})\big)$ where $\mathbf{B}$ is short for $(\mathbf{S}\cdot(\mathbf{K}\cdot\mathbf{S}))\cdot\mathbf{K}$.

which binds in $\mathtt{s}$. So the denotation of $\lambda\mathtt{x.s}$ depends not just on $\mathtt{s}$ but on the denotations of $\mathtt{s}$ for all possible interpretations of $\mathtt{x}$ (assuming it is free in $\mathtt{s}$).

What we have just described is an instance of the more general problem of syntax-free interpretations of quantification and binding (and substitution). We can solve the problem here for the the special case of a syntactic theory of λ-equality (i.e. a β-sensitive, or λ-complete, theory):[8]

Theorem 5.1. *If an intensional frame F is λ-complete and also for any $S_1, S_2 \subseteq \mathcal{P}(W)$:*

1. $[\![\mathbf{K}]\!]^v, [\![\mathbf{S}]\!]^v, [\![\mathbf{C}]\!]^v \in H$ *(for some/any v),*
2. *if $S_1, S_2 \in H$ then $S_1 \bullet S_2 \in H$.*
3. *if $S_1 \in H$ then $\bigcap_{h\in H}\{w \mid S \circ h \subseteq S_1 \circ h\} \in H$ (i.e. $[\![\lambda\mathtt{y}(\mathtt{x} * \mathtt{y})]\!]^{v[\mathtt{x}\mapsto S_1]} \in H$),*

then F is faithful.

Proof. First notice that condition (1) is independent of v as $\mathbf{K}$, $\mathbf{S}$ and $\mathbf{C}$ are all closed terms. Also note that by Definition 2.17, a frame is faithful when it guarantees an interpretation in H for every term of $\mathcal{L}_\lambda$ (i.e. terms not containing $*$). Condition (2) states that H is closed under $\bullet$. Condition (3) says that if $[\![\mathtt{x}]\!]^v \in H$ then so is $[\![\lambda\mathtt{y}(\mathtt{x} * \mathtt{y})]\!]^v$. Given closure under $\bullet$ this condition (3) could be replaced by the condition that $[\![\lambda\mathtt{xy}(\mathtt{x} * \mathtt{y})]\!]^v \in H$.

We must argue that for any valuation v, $[\![\mathtt{t}]\!]^v \in H$ for all $\mathtt{t} \in \mathcal{L}_\lambda$. We do so by induction on $\mathtt{t}$.

– $\mathtt{t}$ is a variable $\mathtt{x}$. Then $[\![\mathtt{t}]\!]^v \in H$ by the definition of valuations 2.12.

– $\mathtt{t}$ is $\mathtt{s}\cdot\mathtt{r}$.Then the result follows by condition (2) and the induction hypothesis.

– $\mathtt{t}$ is $\lambda\mathtt{x.s}$. Then by the induction hypothesis $[\![\mathtt{r}]\!]^v \in H$ for every subterm $\mathtt{r}$ of $\mathtt{s}$. Now $[\mathtt{x}]\mathtt{s}$ contains no free occurrences of $\mathtt{x}$ and is a concatenation, by the $\cdot$ symbol, only of instances of $\mathbf{K}$, $\mathbf{S}$, $\mathbf{C}$ and subterms of $\mathtt{s}$. So by conditions (1), (2) $[\![[\mathtt{x}]\mathtt{s}]\!]^v \in H$. But then by condition (3) and Lemma 2.16.2, $[\![\lambda\mathtt{x}.([\mathtt{x}]\mathtt{s} * \mathtt{x})]\!]^v \in H$. Finally, given λ-completeness we may conclude from Theorems 4.9 and 4.12.1 that $[\![[\mathtt{x}]\mathtt{s} * \mathtt{x}]\!]^v = [\![\mathtt{s}]\!]^v$ and so $[\![\lambda\mathtt{x.s}]\!]^v \in H$

□

Corollary 5.2. *If an extensional frame F is λ-complete and the three conditions of 5.1 hold, then F is faithful.*

[8]The mathematical designs of this paper, combined with those of [5, 7], give rise to a more general solution for λ-reduction in [8].

Remark 5.3. It is easy to verify that the converses of 5.1 and 5.2 hold. Moreover, for the case of a fully extensional frame, we know from 4.21 that combinator completeness implies λ-completeness. It is then not hard to see that the conditions of 5.1 become instances of the definition of a syntax-free model of the λ-calculus. For example [12, p.237], condition (3) corresponds to the so-called (although not by [12]) Meyer-Scott axiom.

6 Further work

The methods used here resemble those behind the models of λ-calculus constructed by Engeler, Meyer, Plotkin and Scott (e.g. in [4, 14] and in [1, §18-19]) which are the basis of *graph models*.

The frames presented here have the components W, $\bullet$ and H. Both $\bullet$ and W have an analogue in graph models, and the differences between them and their analogues are not of great significance: it is not hard to associate each graph model with an equivalent extensional frame (see Remark 2.15). The analogue of H in graph models is that denotations are drawn from the powerset of the domain (the analogue of W). The fact that in the models and frames of this paper H can be something other than $\mathcal{P}(W)$ is significant. The completeness theorem 3.12 shows this, for it implies that every consistent λ-theory can be associated with a frame, and yet as shown by Salibra [17] there are λ-theories for which there are no graph models.

The models of this paper separate *expansion* and *reduction* for both $\beta-$ and $\eta-$ as distinct semantic properties of a model. Interestingly, β-reduction and η-expansion are natural features of the models (there is independent evidence that this is natural [13]). η-expansion arises from λ-abstraction and application being defined over the same underlying function $\bullet$. If we use two underlying functions $\bullet$ and $\circ$ instead, where $\bullet \subseteq \circ$, so that λ abstracts over $\circ$ and application applies $\bullet$, then we obtain models free from η-expansion.

We could also reverse this 'trick' so that λ abstracts over the $\bullet$ and application applies $\circ$, and thus obtain models free from β-reduction. This may sound perverse but recall that meta-programming languages — languages that can suspend their own evaluation and/or quote their own syntax — are devoted to switching off β-reduction in a controlled manner, and the connections to modal logic have already been noted, where possible worlds correspond to deeper or shallower levels of suspension or quoting (see MetaML [15] and CMTT [16, 9]). This suggests the possibility of models for a variety of interacting λ-operators over a hierarchy of underlying application relations.

Finally, further work is needed in improving the semantic characterisations, in terms of $\bullet$ and $\circ$, of frames that are Γ-sensitive for interesting Γ. For example, can we provide a helpful semantic characterisation of theories that contain the schema of η-reduction? We can do it in terms of H by specifying that, for any $S \in H$, $\bigcap_{h\in H}\{w \mid w \circ h \subseteq S \bullet h\} \subseteq S$, but it would also be interesting to look for conditions on $\bullet$ and $\circ$ alone that correspond to this, independently of H.

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