Networks, Clubs and Matching

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To Mom and Dad, who have always supported me unconditionally.
To Tai-tai, Po-ah, my grandpa and my aunt, who raised me in my childhood with lots of love.

Declaration

I hereby declare that this dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Abstract and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Abstract and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

This PhD dissertation is a study of how social networks and clubs form in different contexts.

Chapter 1 investigates the incentives of individuals to make introductions (the act of creating a link for two neighbours) in a social network. The chapter assumes that players are endowed with different ability levels and have a network among them. Given an ability endowment and a network, players undergo a matching process where one can only be matched with one of his neighbours or stay alone, and one always prefers a more capable matching partner to a less capable one to staying alone. A strict ability ranking would yield a unique stable matching for all network structures. Our research question is: If a player can create a link for a pair of his neighbours, when would he want to do so? Two results are derived to address this question. First, the matching of a player would be unchanged if he makes an introduction for two neighbours, at least one of whom is less capable than him. Second, an introduction could benefit the introducer when both neighbours involved are more capable than him, and there exists an even-length alternating path from one of the neighbours to him. The chapter also examines the stability of networks based on no profitable introductions and characterizes Pareto efficient networks.

Chapter 2 studies a general model of investment in relationships. Existing research on network formation proceeds under strong assumptions on how a link between two agents can be produced: typically link investments are assumed to be unweighted and links are formed either reciprocally or unilaterally. This chapter proposes a more general approach by allowing weighted link investment and employing a constant elasticity of substitution (CES) link formation function. This formulation has two advantages other than permitting a more flexible sponsorship of links. First, it nests the two commonly employed bilateral and unilateral link formation assumptions as special cases and thus enables robustness checks on existing works. Second, it introduces a variation in link investment substitutability and hence enables the analysis of how different link formation technologies affect network formation. We illustrate this approach through two applications: a game of pure network formation and a game of network formation with assorted activities.

Chapter 3, which is co-authored with Prof Sanjeev Goyal and Dr Marcin Dziubinski, explores club joining activities of individuals and member admission activities of clubs. We assume that links between clubs are formed when they share common members. The productivity of a club is determined by its number of members and how connected it is to other clubs. Individuals wish to join clubs with high productivity and clubs admit members with the aim to raise productivity. We study the efficient and the stable club membership structures and find that both efficiency and stability implies the segregation of individuals (and clubs) into two groups with very different levels of club joining (and member admission) activeness and welfare. Our results provide a simple explanation for the phenomena of the "power elite" and interlocking board of directors.

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Chapter 1

When to Make an Introduction

1.1 Introduction

Introductions can be a major channel of social network formation. It is common for two strangers to build a link as a result of someone who knows both of them making an introduction. For example, a scholar can refer one young researcher to another for a joint project, and a couple may come to know each other thanks to the introduction from a mutual friend.

In certain circumstances, individuals can rely heavily on introductions for the formation of a new relationship. One such case is when a person is looking for a collaborator to perform a task together that calls for trust. Introductions from a common friend would serve as a reliable source of new connections for this purpose.

Note also that a number of studies on network formation assume that some proportion of a player's links are built through meeting friends of friends.[1] This meeting process would not happen if the friend is not willing to introduce the player to his friends.

Despite the importance of introductions, as far as we know, there have not been any studies on the strategic reasoning behind this behaviour. Most existing literature on network formation analyses the creation, maintaining, and deletion of a link by examining the incentives of players sitting on the two sides of the link.[2] The influence a third party can have on a relationship (facilitating a link with an introduction, for example) is not taken into account.

1 For example, see Granovetter (1973) and Jackson and Rogers (2007).

2 For an overview of network formation research, see Goyal (2007), Jackson (2010), and more recently Mauleon and Vannetelbosch (2016).

The aim of this chapter is to provide a first understanding of how agents choose when and for whom to make an introduction, and how introductions affect network stability and efficiency.

We study the following model. Consider a set of players who are endowed with different ability levels and are connected in a network. The players undergo a one-to-one matching process where everyone tries to find a partner from his neighbours. A high-ability partner is preferred to a low-ability one. Having a partner is preferred to not having one. It can be shown that given any strict ability ranking and network structure, the above process admits a unique stable matching. We now ask ourselves: If a player can alter the network structure by making an introduction for a pair of his unlinked neighbours, would he do so?

We answer the question with two findings. First, an introducer's matching outcome would not change if at least one of the players being introduced has lower ability than the introducer. Second, when both players are more capable than the introducer, the matching of the introducer will be affected only if there is an alternating path (a path with pairs of players on it who are alternatively matched with each other and not matched with each other) from one of the players being introduced to the introducer. The parity of the alternating path length determines the direction of the influence: an even one leads to an improvement while an odd one leads to an impairment.

With the above understanding, we characterize networks that are introduction robust (networks where no player wants to introduce any pairs of his neighbours). We also characterize Pareto efficient networks and show that introduction robustness and Pareto efficiency do not imply each other.

Our model of introduction has two key ingredients. First, players are born with heterogeneous characteristics and have aligned preferences over them. Second, players do not interact with all neighbours in the network, but just one. There is thus, in some sense, a competition for quality neighbours in our model.

There are a few economic and social scenarios that feature this kind of competition. We provide two examples. The first is one where starting entrepreneurs are looking for co-founders of business. Some of the entrepreneurs must be viewed as better than others due to stronger technical or business background. And they cannot engage in multiple projects at one time because establishing a new firm requires devotion. A capable partner is a scarce resource. The entrepreneurs will want to decide whether or not to make an introduction wisely. The second example concerns the pursuits for attention and popularity that are common among adolescents. Youngsters can vary in their attractiveness and often want to have an exclusive best friend relationship with someone who is popular. An introduction for two friends can affect the best friend relationships in the network. The adolescents thus will want to make introductions in some cases but not others. Our results shed light on the understanding of introductions in these contexts.

The remainder of this chapter is structured as follows. Section 2 explains the model setup. Section 3 illustrates how we obtain the unique stable matching in a network. Section 4 shows the effects of a new link and explains players' incentives to make an introduction. Section 5 characterizes introduction robust networks and Pareto efficient networks. Section 6 concludes with a discussion of far-sighted introduction decisions. Proofs that are not given in the main text can be found in the Appendix.

1.2 The Model

Consider a set of players $N = \{1,2,...,n\}$. Each player $i \in N$ is endowed with an ability level $a_i \in \mathbb{R}_+$. Th endowments for all players are represented with a vector $a = \{a_1, a_2, ..., a_n\}$. We assume that there is a strict ordering of ability levels: $a_i \neq a_j$ for all $i \in N, j \neq i$.

The players are situated in a network $g = \{g_{ij}\}_{i,j \in N}$. The variable $g_{ij} \in \{0,1\}$ represents an undirected relationship between i and j: $g_{ij} = 1$ indicates a link between i and j while $g_{ij} = 0$ indicates no link between them. Since relationships are undirected, we have $g_{ij} = g_{ji}$ for all $i, j \in N$. Let G denote the set of all possible networks for the n players.

We now define a few notions based on a network g. First, player i and player j are neighbours to each other if $g_{ij} = 1$. We use $N_i(g) = \{j \in N : g_{ij} = 1\}$ to denote the set of neighbours i has in g. Second, a path in a network g is a sequence of distinct players $\{i_0, i_2, ..., i_k\}$ where every two consecutive players have a link between them: $g_{i_{m-1}i_m} = 1$ for $m = 1,2,...,k$. Finally, if $g_{ij} = 0$, we use $g + ij$ to represent the network that is network g with an additional link between i and j.

Given an ability endowment a and a network structure g, players undergo a one-to-one matching process where each player partners with one of his neighbours or remains single. The outcome of the process is a matching: $\mu_g : N \to N$ where $\mu_g(i) = j$ only if $j \in N_i(g) \cup \{i\}$ and $\mu_g(j) = i$. If $\mu_g(i) = i$, player i remains single in the matching μ_g.

We assume that players have the following aligned preference for matching. First, they always prefer to be matched to a partner with higher ability. Second, they always prefer to be matched than to remain single. In other words, let $w_i(j)$ be the payoff for player i if he is matched with player j. We assume that for all players $m \in N$, $i \neq m$ and $j \neq m$:

$$w_m(i) > w_m(j), \text{ if and only if } a_i > a_j \tag{1.1}$$

and

$$w_m(i) > w_m(m) \tag{1.2}$$

A matching under a network is stable if (i) no player wants to abandon his current partner and stay single, and (ii) no pair of linked players want to form a new match (and leave their current partners if they have one). Formally,

Definition 1.1. *A matching μ_g is stable under a network g if*

(i) it is not blocked by any player i: $\nexists i \in N$ with

$$w_i(i) > w_i(\mu_g(i))$$

(ii) it is not blocked by any pair of linked players (i,j): $\nexists (i,j) \in N^2$ with

$$\begin{gathered} w_i(j) > w_i(\mu_g(i)) \\ w_j(i) > w_j(\mu_g(j)) \\ j \in N_i(g) \end{gathered}$$

Note that with our assumption that players always prefer to be matched, the first condition of a stable matching is satisfied for all possible matchings that can happen in a network.

We show in Section 1.3 that for any network structure g, the matching process admits a unique stable matching μ_g^*. We use the payoffs from this matching to define the utility of players from a network: $u_i(g) = w_i(\mu_g^*(i))$ for all $i \in N$.

We then consider the question of whether a player wants to make an introduction for a pair of his unlinked neighbours. Given a network g, a player $m \in N$ who has two unlined neighbours $i, j \in N_m(g)$ with $g_{ij} = 0$ can make an introduction for i and j and change the network structure from g to $g + ij$. If $u_m(g+ij) > u_m(g)$, player m wants to make the introduction. If $u_m(g+ij) < u_m(g)$, player m does not want to make the introduction. In the tie-breaking case, we assume that m would make the introduction if he is *generous*.

A network is introduction robust if no player wants to make an introduction for any pairs of his unlinked neighbours.

Definition 1.2. *A network $g \in G$ is introduction robust if*

(i) there does not exist $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $u_m(g+ij) > u_m(g)$, and

(ii) if players are generous, there does not exist $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $u_m(g+ij) = u_m(g)$.

We also examine Pareto efficiency of networks. A network g is Pareto efficient if there does not exist another network g' where the utilities of players under g' are at least as large as those under g, with some strictly greater.

Definition 1.3. *A network $g \in G$ is Pareto efficient if there does not exists $g' \in G$ such that $u_i(g') \geq u_i(g)$ for all $i \in N$ and there exists $i \in N$ such that $u_i(g') > u_i(g)$.*

1.3 The Unique Stable Matching

We derive the unique stable matching μ_g^* under network g in this section.

First, note that if g is the complete network where all pairs of players are linked: $g_{ij} = 1$ for all $i \in N, j \neq i$, then the network structure imposes no restrictions on the matching of players. A unique stable matching under the complete network can be easily obtained.

Remark 1.1. *When players are situated in a complete network, the matching process admits a unique stable matching where the player with highest ability is matched to the player with the second-highest ability, the player with the third-highest ability is matched to the player with the forth-highest ability,.... If n is an even number, all players get matched. If n is an odd number, the player with the lowest ability is the only one unmatched.*

For a network g that is not necessarily complete, there is also a unique stable matching that can be derived with a simple algorithm.

Lemma 1.1. *For all $g \in G$, the matching process admits a unique stable matching μ_g^* which can be derived with the following algorithm:*

(i) With a given network, let the player with the highest ability be matched with his most capable neighbour.

(ii) Delete the pair of players who get matched in step (i) and their links from the original network and arrive at a reduced network.

(iii) Go back to step (i) until the remaining network is empty or consists only of players with no links.

(iv) All those get matched in step (i) will be matched in μ_g^*, and all other players will remain single in μ_g^*.

1.4 The Influence of a New Link

This section discusses how a new link between two players, say i and j, affects the matching of other players. We ask ourselves the following questions and answer them one by one. First, if there is a new link between i and j, will μ^*_{g+ij} and μ^*_g be different? Second, if different, whose matching will be affected? Finally, for a player whose matching is changed, will he obtain a higher or lower payoff?

We begin with the first question. We define a link between i and j as *useful* under g if i and j are matched in the unique stable matching μ^*_g.

We show that μ^*_{g+ij} is different from μ^*_g if and only if the new link between i and j is useful under $g+ij$. It is easy to see that μ^*_{g+ij} and μ^*_g are different if the link between i and j is useful because the two players who cannot be matched in μ^*_g are now matched in μ^*_{g+ij}. We verify that μ^*_{g+ij} and μ^*_g are different only if the link is useful by going through the algorithm used to derive μ^*_{g+ij} and finding that all steps lead to the same actions as those taken to derive μ^*_g if the new link is not useful.

Also, note that for a link between i and j to be useful under $g+ij$, the two players that are not matched under g get matched under $g+ij$. This requires that they prefer each other to their original match in μ^*_g:

$$a_i > a_{\mu^*_g(j)} \text{ if } \mu^*_g(j) \neq j$$
$$a_j > a_{\mu^*_g(i)} \text{ if } \mu^*_g(i) \neq i$$

We thus arrive at Lemma 1.2 that characterizes when a new link between i and j has an effect on the matching outcomes of players.

Lemma 1.2. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. There exists a player $m \in N$ such that $\mu^*_{g+ij}(m) \neq \mu^*_g(m)$ if and only if the link between i and j is useful under $g+ij$:*

$$a_i > a_{\mu^*_g(j)} \textit{ if } \mu^*_g(j) \neq j$$
$$a_j > a_{\mu^*_g(i)} \textit{ if } \mu^*_g(i) \neq i$$

We now discuss whose matching will be affected by a useful new link. Recall the algorithm we employ to derive the unique stable matching under a network. When it is executed, a sequence of steps are taken. And we can observe that given any network structure, the sequence is unique. Based on this unique sequence, we define player i as a *leading player* of player j under network g if i exists from the algorithm used to derive μ^*_g earlier than j.

We show that with a new link between i and j, leading players of i and j under g will not be affected by the link.

Lemma 1.3. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. If player $m \in N$ is a leading player of i and j under g, then $\mu^*_{g+ij}(m) = \mu^*_g(m)$.*

With the help of Lemma 1.2 and Lemma 1.3, we show that an introducer's matching will not be affected if at least one of the players he makes an introduction for is not as capable as him.

Proposition 1.1. *For all $m \in N$ and $g \in G$, if $i, j \in N_m(g)$ and $a_m > \min\{a_i, a_j\}$, then $\mu^*_{g+ij}(m) = \mu^*_g(m)$.*

The proof for this proposition follows from Lemmas 1.2 and 1.3. When there is a new link between i and j, it either does not facilitate a matching for i and j under $g+ij$ (the link is not useful under $g+ij$) so it does not influence the matching of any players, or it makes i and j match under $g+ij$. In the second case, player m must have been matched to a player who has higher ability than both i and j and have exited from the matching algorithm earlier than i and j (m is a leading player of i and j under g). If not, then m and the more capable player between i and j would match, contradicting the matching between i and j under $g+ij$.

This proposition shows that if a player is generous, he would always make introductions for neighbours that are not as capable as him.

Going back to the effects of a new link on players, we make a further observation on who will be affected by a new link with the notion of alternating paths. An *alternating path* under network g is a path where pairs of players on the path are alternative matched to each other and not matched to each other in μ^*_g: a path $\{i_0, i_1, ..., i_k\}$ in g is an alternating path if $\mu^*_g(i_0) = i_1$, $\mu^*_g(i_1) \neq i_2$, $\mu^*_g(i_2) = i_3$,$\mu^*_g(i_3) \neq i_4$....

We show that all the players that are affected by a new link between i and j must sit on an alternating path that starts from i or j.

Lemma 1.4. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. If $\mu^*_{g+ij}(m) \neq \mu^*_g(m)$, then there is an alternating path $\{i_0, i_1, ..., i_k = m\}$ in network g where $i_0 \in \{i, j\}$.*

The intuition behind Lemma 1.4 is the following. If the matching of a player m is affected by a new link between i and j, ignoring the case where m is not matched under g or $g+ij$, m must have broken his original match under g and formed a new match under $g+ij$. The player whom m breaks a match with is originally matched with m under g, and the player whom m forms a new match with is not matched with m under g. We can already see that m sits on a short alternating path in g that consists of his original match under g, himself, and his new partner under $g+ij$.

The two players that match with m under g and $g+ij$ respectively are also affected by the new link between i and j. So they are also parts of an alternating path that contains themselves, m and another player. By repeating this reasoning, we can extend the alternating path with affected players on it. This extension must end with a source of affections, which is the matching between i and j facilitated by their new link. So players affected by a new link between i and j must sit on an alternating path that starts from i or j.

We illustrate the alternating path pattern of affected players with an example in Figure 1.1. In Figure 1.1, a label on a node represents the ability ranking of a player. Figure 1.1 (a) shows the unique stable matching under a network g, where the matchings are indicated with red lines. Figure 1.1 (b) shows the new unique stable matching under the network with an additional link between players 1 and 2. Compare Figure 1.1 (a) and 1.1 (b), we can easily find the set of players whose matchings are influenced by the new link, which is illustrated in Figure 1.1 (c) where the affected players are painted red. We can see that there are two red paths starting from player 1 and player 2 respectively that connect the affected players. The solid red links are useful under the original network g and the dashed red links are not useful under g. These two paths are the alternating paths mentioned in Lemma 1.4.

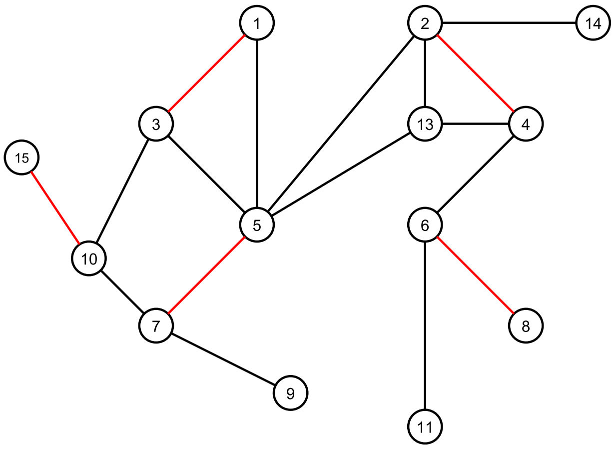
Finally, we illustrate the direction of influence a new link between i and j can have on an affected player. This depends on the parity of the alternating path length from i or j to the affected player.

Proposition 1.2. *Consider a network $g \in G$ and two players $i, j \in N$ where $g_{ij} = 0$. If $u_m(g+ij) > u_m(g+ij)$, then then there is an alternating path $\{i_0, i_1, ..., i_k = m\}$ in network g where $i_0 \in \{i, j\}$ and k is an even number. If $u_m(g+ij) < u_m(g+ij)$, then then there is an alternating path $\{i_0, i_1, ..., i_k = m\}$ in network g where $i_0 \in \{i, j\}$ and k is an odd number.*

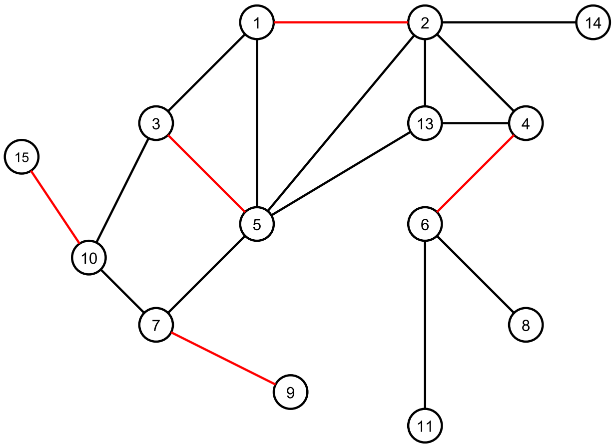
Proposition 1.2 shows how the parity of the alternating path length determines if an affected player experience positive or negative influence from a new link. An even-length path leads to an improvement and an odd-length path leads to an impairment.

To see why this happens, first note that the original partners of i and j under g must be worse off given the new link between i and j. They lose their original partners and need to match with some other neighbours. They can only be worse off because otherwise they would deviate from the matching with i and j under g. The length of the alternating paths from i and j to these two players are odd. Then observe that the players who are newly matched to the original partners of i and j must be better off, because otherwise they would stay with their original partners. The length of the alternating paths from i and j to these two players are even. We can extend this reasoning further and see that affected players on an alternating path take turns to lose an original preferable match and gain a new preferable match. So the parity of the alternating path pins down the direction of the influence for a new link.

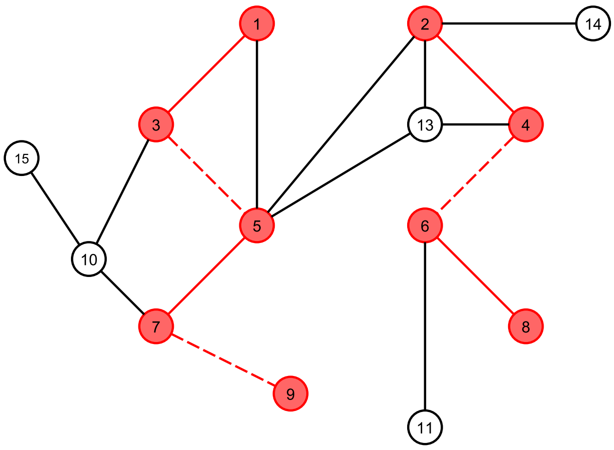
Proposition 1.2 also informs us about a player's incentive to make an introduction for two players i and j that are both more capable than him. If the player is generous, he will make an introduction as long as there does not exist an alternating path from i or j to him



(a) The unique stable matching μ_g^*



(b) The unique stable matching μ_{g+12}^*



(c) Players affected by a new link between 1 and 2

Figure 1.1 Alternating paths and players affected by a new link that is of odd-length. If the player is not generous, he will make the introduction only if there is an alternating path from i or j to him that is of even-length.

This characterization based on alternating paths and their lengths has two implications. First, it shows that players can benefit from an introduction for other players. This would not be expected at first sight of the model. Since players in our model are competing for good partners, it seems an introduction for two neighbours can only intensify two competitions a player is involved with and reduce the utility of the player. Nonetheless, if the two neighbours are matched with the new link, they withdraw from competitions for other players who the introducer might want to partner with. In this way, an introducer can benefit from an introduction.

Second, since affected players are alternatively better off and worse off given their positions on the two alternating paths, around half of them would have their matchings improved with an introduction and half of them would have their matchings impaired. This shows that the chances of an introducer benefiting from an introduction is not low.

1.5 Introduction Robustness and Pareto Efficiency

With the understanding developed in the previous section on how an introduction affects the matchings of players, we now characterize conditions for a network to be introduction robust and to be Pareto efficient.

The conditions for a network to be introduction robust follow directly from Propositions 1.1 and 1.2.

Proposition 1.3. *When players are generous, if a network $g \in G$ is introduction robust, then for all $m \in N$ and $i, j \in N_m(g)$ with $g_{ij} = 0$: $a_i > a_m$, $a_j > a_m$, and there is an alternating path $\{i_0, i_1, ..., i_k = m\}$ in network g where $i_0 \in \{i, j\}$ and k is an odd number.*

When players are not generous, a network $g \in G$ is introduction robust if for all $m \in N$, there does not exist $i, j \in N_m(g)$ with $g_{ij} = 0$ such that $a_i > a_m$, $a_j > a_m$, and there is an alternating path $\{i_0, i_1, ..., i_k = m\}$ in network g where $i_0 \in \{i, j\}$ and k is an even number.

When players are generous, all introductions that do not reduce the utility of an introducer will be made. So a network is introduction robust only if all possible introductions that can be made lead to a worse matching for the introducer which we characterize with the odd-length alternating path. When players are not generous, a player will make an introduction when it enhances his utility. So a network is introduction robust if there is no introduction that can be made and leads to a better matching for the introducer, which we characterize with the even-length alternating path.

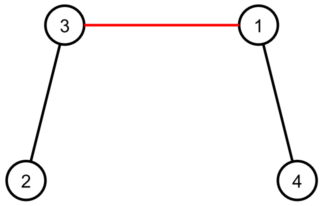


Figure 1.2 An introduction robust network that is not Pareto efficient

Regarding Pareto efficiency, we show that a network g is Pareto efficient if and only if at most one player is not matched in the unique stable matching under g.

Proposition 1.4. *A network $g \in G$ is Pareto efficient if and only if*

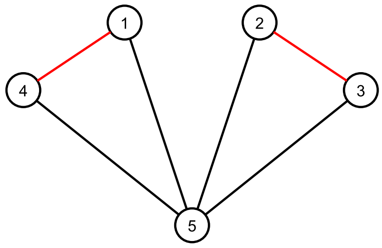
$$|\{m \in N : \mu_g^*(m) = m\}| \leq 1$$

The proof for this proposition is simple. If there are two players i and j that are not matched in μ_g^*, then it must be that $g_{ij} = 0$. Also, all players that are matched in μ_g^* are leading players of i and j in g. Adding a link between i and j would not affect the matching of other players (since they are leading players of i and j) and would facilitate a matching between i and j and increase their utilities. When no more than one player remains single in μ_g^*, it is can be shown that there is no Pareto improvements that can be made for the matching of players. Improving the matching of one player must lead to the impairment of matching of another player.

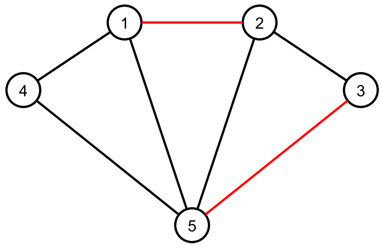
We now show that introduction robustness and Pareto efficiency do not imply each other.

Proposition 1.5. *Regardless of whether players are generous or not, there exists a network $g \in G$ that is introduction robust but not Pareto efficient and a network $g' \in G$ that is Pareto efficient but not introduction robust.*

The proposition can be shown with examples. First, an empty network where $g_{ij} = 0$ for all $i, j \in N$ is obviously introduction robust but not Pareto efficient. When players are not generous, we also show that a connected network can be introduction robust but not Pareto efficient. See Figure 1.2 for an example. There are four players in the network with labels indicating their ability ranks. The unique stable matching in the network features a single match between players 1 and 3. When players are not generous, this network is introduction robust since player 3 would not benefit from introducing players 1 and 2, and player 1 would not benefit from introducing players 3 and 4. This network is not Pareto efficient since an additional link between players 3 and 4 would lead to a Pareto improvement.



(a) the network



(b) player 5 benefits from an introduction

Figure 1.3 A Pareto efficient network that is not introduction robust

Now we show a Pareto efficient network is not necessarily introduction robust. We illustrate this with the example in Figure 1.3. In the five-player network in Figure 1.3 (a), the unique stable matching features a match between players 1 and 4 and a match between players 2 and 3. This network is Pareto efficient as only one player is not matched. However, it is not introduction robust since player 5 has an incentive to introduce player 1 and player 2. The network with the introduction is depicted in Figure 1.3 (b). We can see now player 5 is matched with player 3 which is utility enhancing for him.

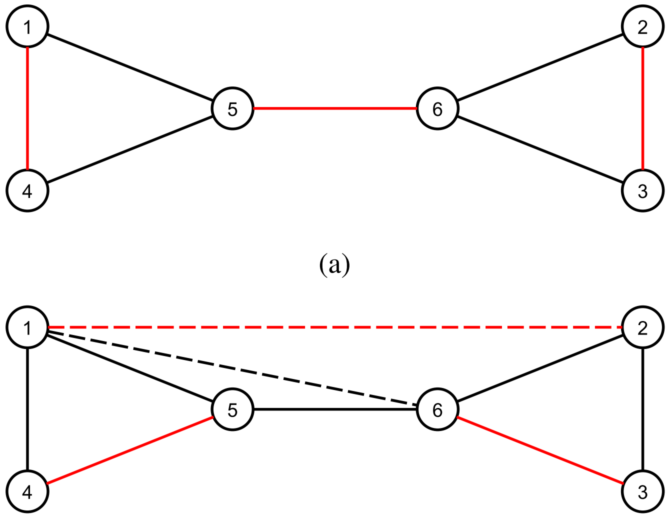
We have illustrated an absence of alignment between introduction robustness and Pareto efficiency.

1.6 Conclusion

This chapter studies introductions, which is a major way to bridge people.

In our model, links in the network serve as a contact book for players who wish to find a partner. Obviously, all players will want to amplify their contact books so that their chances of getting a better partner will increase. However, if links can only be formed via introductions, will a player do a favour for his neighbours by making the introduction?

First, we show that a player's own matching will not be influenced when he makes an introduction where at least one of the two being introduced is weaker than him. We next



(a)

(b)

Figure 1.4 Far-sightedness and introductions

characterize when a player will get a better match or worse match by making an introduction for a pair of players both more capable than him. Given this, we understand when a (myopic) player wants to make an introduction for a particular pair of players.

We then move to define a network as introduction robust when no (myopic) player wants to make an introduction.

Nonetheless, the introduction robustness notion defined in this chapter is short-sighted. We know that players choose between staying in the current network g and making an introduction by comparing their utility before and after the introduction. In this chapter, when we specify a player's utility after an introduction for i and j, we define it as the payoff from the unique stable matching under network $g+ij$. However, this may not be the most appropriate evaluation. For a far-sighted player, he knows that following his introduction for i and j, the network changes and other players can make further introductions. It is wiser to evaluate his utility after the introduction as the utility from a network with all introductions that will emerge after his introduction.

We illustrate this with an example in Figure 1.4. Look at the network depicted in Figure 1.4 (a). When players are not generous, no one has an incentive to make an introduction. However, we can see in Figure 1.4 (b) that if player 5 makes an introduction for player 1 and player 6, then player 6 will want to introduce player 1 and player 2 as this enables him to match with player 3 instead of player 5. With this introduction made by player 6, player 5 is now matched with player 4 which makes him better off. Hence, with future possible introductions taken into account, player 5 will want to make an introduction for player 1 and

6. This consideration may lead us to more sophisticated stability notions like far-sighted introduction robustness.

Appendix 1.A Proofs

Proof for Remark 1.1

The remark is a special case of Lemma 1.1.

Proof for Lemma 1.1

We prove that the matching μ_g^* is the unique stable matching under g.

To begin with, we show μ_g^* is stable. We already know that μ_g^* will not be blocked by any single player since players prefer to be matched than to stay alone. So we only need to show that μ_g^* is not blocked by any pairs of players.

Suppose the pair (i,j) blocks the matching μ_g^*, then according to Condition 1.1, it must be that:

$$
\begin{aligned}
a_j &> a_{\mu_g^*(i)} \\
a_i &> a_{\mu_g^*(j)} \\
j &\in N_i(g)
\end{aligned}
$$

Suppose $a_i > a_j$, in the algorithm that leads to μ_g^*, i may get picked by someone, picks someone, or stays alone. If i is picked up, then it must be:

$$a_{\mu_g^*(i)} > a_i > a_j$$

A contradiction.

If i picks someone, since j is not picked by i, it indicates that either j is not as capable as i's match in μ_g^*:

$$a_{\mu_g^*(i)} > a_j$$

or j is already picked by a more capable player:

$$a_{\mu_g^*(j)} > a_i$$

A contradiction.

Suppose i is left alone in μ_g^* with j being his neighbour, it must be j is already picked by a more capable player:

$$a_{\mu_g^*(j)} > a_i$$

A contradiction. So μ_g^* can not be dominated by any pair of players. Therefore, μ_g^* is a stable matching.

Now, we prove that μ_g^* is the unique stable matching. Suppose there is new matching μ_g' that is stable, then define the set $N' = \{i \in N \mid \mu_g'(i) \neq \mu_g(i)\} \neq \emptyset$ as the set of players who get a different matching partner under μ_g^* and μ_g'. Let player j be the most capable players in this set, i.e.

$$j = arg\max_{i \in N'} a_i$$

Also, suppose that:

$$\mu_g^*(j) = m$$
$$\mu_g'(j) = n$$
$$\mu_g'(m) = k$$

i.e. let j's original match be m, new match be n, and let m's original match be k. It is obvious that m, n and k belong to N' as well. I will prove that the pair (j, m) blocks the matching μ_g', hence proving that there are no other stable matching other then μ_g^*.

Since $j = arg\max_{i \in N'} a_i$, it is obvious that $a_j > a_k$, which makes m want to deviate from his match under μ_g' to j. Also, it can be shown that $a_m > a_n$. Suppose not, then in the matching algorithm, it must be that n has been picked by another player, say l, before it can be picked by j. Hence, $a_l > a_j$, and $l \in N'$ since player l is not matched with n at μ_g'. This contradicts with $j = arg\max_{i \in N'} a_i$.

Given $a_j > a_k$ and $a_m > a_n$, (j,m) blocks the matching μ_g'. Therefore, μ_g' is not stable. A contradiction.

Hence μ_g^* is the unique stable matching.

Proof for Lemma1.3

Without loss of generality, assume $a_i > a_j$. We first show that i is a leading player of j under g. Suppose not, then since $a_j < a_i$, for j to be the leading player instead, it must be that

$$a_{\mu_g^*(j)} > a_i$$

But since the link between i and j is useful, we need that

$$a_i > a_{\mu_g^*(j)}$$

A contradiction. So i is a leading player of j under g.

Now we apply the unique stable matching algorithm to g and $g+ij$, the steps before i's matching will be identical for both cases, hence the leading players of i under g will also be leading players of i under $g+ij$, and their matchings are not influenced.

Proof for Proposition 1.1

Suppose $a_i > a_j$, since at least one of i and j is less capable than m, $a_j < a_m$. Also, i, j and m are connected to each other under $g+ij$.

If the introduction is not useful, then obviously $\mu_g^*(m) = \mu_{g+ij}^*(m)$.

If the introduction is useful, since $a_j < a_m$, it indicates that i will prefer m to j, but i and j are matched, indicating m has exited from the algorithm earlier than i. Hence m is a leading player of i under $g+ij$. According to Lemma 1.3, m's matching is not affected.

Hence, a player's matching will not be affected by making an introduction when a less capable player is involved.

Proof for Lemma 1.4

If there is a useful new link between i and j, the set of affected players $A = \{m \in N : \mu_g^*(m) \neq \mu_{g+ij}^*(m)\}$ can be characterized with the following process.

Define $A_{k+1} = A_k \cup \{m \in N : \mu_g^*(m) \in A_k\} \cup \{m \in N : \mu_{g+ij}^*(m) \in A_k\}$. This function specifies an augmentation process for a player set.

We starts with the set $A_0 = \{i, j\}$ and apply the above augmentation iteratively to the set A_0 and obtain $A_1, A_2, \ldots$. There will be a point where the augmentation stops where $A_{j+1} = A_j$. We prove that $A_j = A$.

First, we show that $A_j \subset A$. Since the link between i and j is useful, we know $A_0 \subset A$. Moreover, observe that if $A_k \subset A$, then $A_{k+1} \subset A$, because if a player is affected, then his original and new partners are affected as well. Deduce iteratively, we have $A_j \subset A$.

Then, we show that $A \subset A_j$. Suppose there exists $m_0 \in A$ such that $m_0 \notin A_j$. Then since $m_0 \in A$, he is either better off or worse off as $a_i \neq a_j, \forall i, j \in N$. Suppose m_0 is better off, m_0 must have a new partner $m_1 = \mu_{g+ij}^*(m_0)$ that he prefer to his original match. There are two possible situations that make $m_1 \neq \mu_g^*(m_0)$ match to m_0 under $g+ij$.

First, both m_0 and m_1 are better off because there is a new link between them. Second, m_1 is worse off because $m_2 = \mu_g^*(m_1)$ finds a better match under $g+ij$ and left m_1.

If the second situation is the case, then m_2 is better off and we can do the same induction to m_2 again. This reasoning will not stop unless m_{2k} is better off under the first situation. The reasoning cannot proceed endlessly because we have a finite set of players. So it must be that there is a new link between m_{2k} and m_{2k+1} for some k, indicating m_{2k} and m_{2k+1} are i and j

respectively. But then we have $m_{2k} \in A_0$, $m_{2k-1} \in A_1$, $m_{2k-2} \in A_2$, ... Hence $m_0 \in A_{2k}$. And since $A_k \subset A_j$ for k=1, 2, ... We have $m_0 \in A_j$, a contradiction.

For m_0 to be worse off, it can be proved in a similar fashion that $m_0 \in A_j$ and hence a contradiction. So $A \subset A_j$.

As $A_j \subset A$ and $A \subset A_j$, we get $A_j = A$.

We now explore the pattern for the set of players generated by the above process. It is immediate for us to see that $A_1 \backslash A_0$ consists of the original partners of i and j, $A_2 \backslash A_1$ consists of the new partners of players in $A_1 \backslash A_0$, $A_3 \backslash A_2$ consists of the original partners of players in $A_2 \backslash A_1$,.... The set A_k develop itself is by tracing alternatively the original and new partners of players newly added to the set. Since the links between original partners are useful under g and the links between new partners are not useful under g, this means that the process provides us with the set of affected players by drawing two alternating paths under g starting from i and j with a useful link.

Hence the set of affected players $A = \{m \in N : \mu_g^*(m) \neq \mu_{g+ij}^*(m)\}$ lie on two alternating paths under g, starting from i and j respectively with a useful link.

Proof for Proposition 1.2

We know that the augmenting process we use to obtain the set of affected players can end in two ways. We show that Proposition 1.2 is true under both circumstances.

First, consider the case where the process stops because the newly added players do not have an original match under g or a new match under $g+ij$. Suppose player m_0 lies on the alternating path starting from i and m_0 does not have an original match, indicating that m_0 is added to the set of affected players because the new partner of m_0 is in the set of affected players. So, m_0 is better off because he moves from no partner to having a partner. Obviously, the length of the augmenting path from i to m_0 is an even number as analysed in the proof of Lemma 1.4. Hence the case for m_0 is aligned with Proposition 1.2.

Now, for player $m_1 = \mu_{g+ij}^*(m_0)$, we know the length of the alternating path from i to him is an odd number.We know for m_0 and m_1, at least one of them is worse off. Since m_0 is better off, m_1 must be worse off. Hence the case for m_1 also agrees with Proposition 1.2. Similarly, for $m_2 = \mu_g^*(m_1)$, from i to whom the length of the augmenting is an odd number, he must be better off.

Keep reasoning in this manner, we can see that Proposition 1.2 is true for all the affected members standing on the alternating path from i (j) to m_0.

Suppose player n_0 lies on the alternating path starting from j and n_0 does not have a new partner, indicating that n_0 is added to the set of affected players because the original partner of n_0 is in the set of affected players. So, n_0 is worse off because he moves from having a partner to no partner. Obviously, the length of the augmenting path from j to n_0 is an odd number as analysed. Hence the case for n_0 agrees with Proposition 1.2. Again, for $n_1 = \mu_g(n_0)$, from j to whom the length of the augmenting is an even number, he must be better off, because at least one of n_0 and n_1 is better off. Do the inductions again for $n_2, n_3, \ldots$, we can see that Proposition 1.2 is true for all the affected members standing on the alternating path from $i(j)$ to n_0.

So we have shown that Proposition 1.2 is true when the augmenting process ends because the newly added players do not have an original or new partner.

For the other circumstance, we can see that the two alternating paths starting from i and j actually coincide with each other. Also, it is immediate for us to see that the length of the path from i to j will be an odd number. As a result, for an affected player that stands on this path, if the length of the path from him to i is an even number, then the length of the path from him to j is an odd number. So no matter he is better off or worse off, Proposition 1.2 is true.

Chapter 2

Link Investment Substitutability and Network Formation

2.1 Introduction

The study of network formation has been an active research area for the past two decades. Economists are interested in this topic because we do observe individuals, firms, and nations making an effort to manage their relationships, and the resulting network can have substantial social and economic implications.[1]

Theoretical investigations on the topic build upon assumptions on how two agents can form a link between them, and how a network affects the utility of players. Existing works, while making various modelling choices on network utility specifications, typically adopt one of the following two link formation assumptions: bilateral link formation or unilateral link formation.

The two link formation setups both assume that agents make binary decisions on whether to invest in a relationship, and they differ in the requirement for the creation of a link: the bilateral one calls for mutual investments whereas the unilateral one demands effort from only one side. More specifically, let $a_{ij} \in \{0,1\}$ be the choice i makes on whether to invest in a link with j and g_{ij} be the relationship status between i and j, the bilateral link formation assumption assumes that $g_{ij} = \min\{a_{ij}, a_{ji}\}$ whereas the unilateral link formation assumption assumes $g_{ij} = \max\{a_{ij}, a_{ji}\}$.

The above assumptions are restrictive in two senses. First, they do not allow different intensities of link investment and link strength. Second, they imply either an extremely reciprocal way of link formation or a completely unilateral one. In practice, agents can

[1] See Goyal (2007) and Jackson (2010) for a review.

decide on how much-time or money-to devote to a relationship and links can be sponsored in a bilateral but unequal manner. For example, Kovanen et al. (2010) show with a mobile phone dataset that communication networks are weighted and efforts to maintain contacts can be unbalanced, Vaquera and Kao (2008) show with the Add Health data that friendships among adolescents have different intensities and are not always reciprocated, and we can observe from our own research experience that contributions towards a joint project are sometimes unequal.[2]

We relax the above restrictions by allowing continuous link investment choice $a_{ij} \in \mathbb{R}_+$ and modelling link formation with the constant elasticity of substitution (CES) function:

$$g_{ij} = \begin{cases} h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} & \text{for weighted link formation} \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{for unweighted link formation} \end{cases}$$

where $\mathbb{1}$ is the indicator function and β is a parameter that can take any real value.

This specification permits a more flexible way for players to form links and allows us to analyze equality of link investments, which is a feature of network structures that receives little attention from existing literature.

Our link formation assumption has two other related advantages.

First, the CES function nests the bilateral and unilateral link formation function as its special cases when β approaches negative and positive infinity respectively. This generalization enables us to analyze to what extent results from existing works are robust to deviations from the bilateral or unilateral link formation assumption they build upon.[3]

Second, the parameter β captures the substitutability of link investments in forming a connection. For a greater β, there is a smaller drop in the marginal enhancement of $(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$ as a_{ij} rises alone, which indicates that it is less costly to have disproportionate a_{ij} and a_{ji}, i.e. we have a higher degree of link investment substitutability. So, by imposing different values of β, we can examine the role of this substitutability in network formation processes.[4]

[2] See the following papers for a further reference on the presence and implications of weighted and unequally sponsored links: Granovetter (1973), Newman (2001), Yook et al. (2001), Garlaschelli and Loffredo (2004), and Squartini et al. (2013).

[3] Olaizola and Valenciano (2015) offer another approach of unifying bilateral and unilateral link formation. They assume binary link investment and set different link strength for the three possible cases of no investment, one-sided investment, and mutual investment. By imposing the same link strength to the cases of no investment and one-sided investment (one-sided investment and mutual investment), their specification nests the bilateral (unilateral) link formation assumption.

[4] Elliott (2015) provides a reference on how link investment substitutability can affect the outcome and efficiency of network formation activities. He analyzes and compares how players form trade networks

We believe that variation in link investment substitutability is notable. It results from a difference in the nature of activities that accompany a link. For example, in the case of communication networks, the degree of substitutability rises as we move from face-to-face contacts to phone calls and then to email correspondences since they demand less and less joint devotion. For a friendship network, link investment substitutability can be higher for those who frequent expensive restaurants together than those who chat in cafes, because the former involves a monetary cost that can be paid flexibly. For the research network, theoretical collaborations may require mutual understanding on all details of a model and hence feature a low degree of link investment substitutability while empirical or experimental projects can work the other way round.

We apply the CES formulation to well-known models that lie in two broad classes of problems: pure network formation games and network formation with assorted activities games.[5] We select the *connections model* analyzed by Jackson and Wolinsky (1996) that features pure network formation and extend its bilateral link formation assumption with the unweighted CES link formation specification.[6] We pick the *law of the few* model studied by Galeotti and Goyal (2010) that models a simultaneous choice of network formation and another activity and replace its unilateral link formation assumption with the weighted CES link formation specification.

We now summarize the modelling and results from the two applications.

In the application to the connections model, each player is endowed with one unit of intrinsic value that can be shared through a network. Direct and indirect connections facilitate exchanges of the benefit, but there is a loss in exchange efficiency for any additional player that stands in between the exchange. Players form links in an unweighted CES manner with costly relationship investment to maximize the gathering of worth.

We first characterize the efficient network structure under the setup: Proposition 2.1 shows that it must be the complete network, the star, or the empty network depending on the value of β and other parameters of the model. We then propose the solution concept of weighted pairwise stability as an adaptation of the notion of pairwise stability for a wider range of link investment and link formation possibilities. Proposition 2.2 illustrates that the position (measured by closeness to other nodes) of two connected players in a stable

under the perfect complements link formation protocol ($\beta \to -\infty$) and the perfect substitute link formation protocol ($\beta = 1$).

[5] See Mauleon and Vannetelbosch (2016) for a recent survey on network formation games. Papers that study network formation with assorted activities include Baetz (2015), Hiller (2017), and König et al. (2014).

[6] The connections model is also discussed in Bala and Goyal (2000) as one of the cases they study. They analyze the setup with the unilateral link formation assumption which is a limiting case of our CES extension when $\beta \to +\infty$. Feri (2007), Hojman and Szeidl (2008) and Bloch and Dutta (2009) work on extensions of the connections model.

network can be more unbalanced with a higher level of β. This implies a greater likelihood for the star to be weighted pairwise stable for higher β. Finally, we look at the relationship between efficiency and stability by comparing the set of parameter values that make a network efficient with the set of parameter values that make it weighted pairwise stable. Proposition 2.3 demonstrates that the two sets are generally different but there is a tendency for the two sets to overlap more as we move closer to $\beta = 1$ (when investments are perfect substitutes). We infer from this that there is a tension between efficiency and stability but it is alleviated as link investments become more substitutable.

The above results show that the efficient network characterization in Jackson and Wolinsky (1996) (which features the complete network, the star, and the empty network) can be generalized to setups with weighted link investment and CES link formation technology. More importantly, we point out that although the tension between efficiency and stability illustrated in Jackson and Wolinsky (1996) is common, it is less prominent when we move away from the bilateral link formation assumption ($\beta \to -\infty$).[7]

In the application to the law of the few model, players aim to collect information and there are two ways for them to do so: to search on their own or to invest in relationships that are created with the weighted CES technology and get a proportion of information searched by those they relate to.[8] Players get utility from information and pay costs for searching and link investments.[9]

We provide the Nash equilibrium and strict Nash equilibrium solution of the game in Proposition 2.4 and 2.5 for different ranges of β. We find that when β is smaller than a cutoff value that is between 0 and 1, all players search for some information and links are sponsored bilaterally. The equilibrium network structure can be combinations of isolated, regular and bipartite components which do not put connected players in very different network positions. When β is greater than 1, players specialize in searching or in building connections and all links are formed unilaterally. Equilibrium network structures, e.g. the star, feature distinct network position for connected players. When β is between the cutoff value and 1, equilibrium can be a mixture of the two forms summarized above.

The above characterization points to a general trend of moving towards a dispersion of searching, inequality in link investment, and a less regular network structure when β rises.

[7] The change in tension we characterize here is consistent with the finding in Bala and Goyal (2000) which shows that with the unilateral link formation assumption ($\beta \to +\infty$), efficient networks are often in equilibrium.

[8] The object players try to obtain need not be restricted to information, it can be any public good that is shared with neighbours, e.g. influence.

[9] Kinateder and Merlino (2017) and Sethi and Yildiz (2016) study similar models where agents form networks and acquire information from neighbours. The local public good provision setup of the law of the few model was first analyzed by Bramoulle and Kranton (2007) in a fixed network.

Proposition 2.6 provides further support for this finding by showing that there is a smooth growth in cross-player searching and link investment difference as β increases.

Proposition 2.7 looks at the welfare distribution of players and shows that a larger β indicates an advantage of being a searcher because it makes it easier for the searcher to attract link investments from others.

Note that our characterization of equilibrium when $\beta > 1$ is analogous to that obtained in Galeotti and Goyal (2010), so it suggests that their results are robust to weighted link formation and a wider range of link formation technology. When link investments are more complementary ($\beta < 1$), we make predictions different from Galeotti and Goyal (2010).

Our work falls into the strand of literature on weighted network formation. In that field, Rogers (2008) studies a directed weighted network formation problem where agents allocate their link investment budget strategically to maximize their Bonacich centrality. Our paper studies undirected network formation is closer to the papers by Bloch and Dutta (2009) and Baumann (2017).

Bloch and Dutta (2009) analyze a weighted version of the connections model. They assume that agents have a fixed budget and link strength is an additively separable and convex function of individual investments (when $\beta = 1$, the CES formulation satisfies their link formation assumption). They show that both the stable network and the efficient network are the star network. In our application of the CES formulation to the connections model, we maintain the model's original assumption of unweighted link and constant link formation cost. Our characterization when $\beta = 1$ shows that the star is stable and efficient for a wide range of situations, which is in accordance with the result of Bloch and Dutta (2009). We differ from Bloch and Dutta (2009) as we are interested in understanding the network formation process with a broader scope of link formation technology along the dimension of link investment substitutability and how variation in that dimension affects the choice of players. This is not discussed in their paper.

Baumann (2017) looks at a weighted network formation model where players maximize their aggregate link strength (including self-loops) with a fixed budget. She assumes a link formation function that features constant returns to scale, zero production with unilateral effort, and complementarity of link investments (the CES specification when $\beta \leq 0$ satisfy these conditions). She shows that in equilibrium, links can be sponsored in equal or unequal manners and the network can consist of regular and bipartite components. We study a different weighted network formation model, which is the law of the few model, in our second application. Yet, we have a similar characterization of equilibrium network that features equal or unequal links and regular or bipartite component when $\beta \leq 0$. Other than allowing substitutability in investments ($\beta > 0$) and modelling a different scenario, we differ from Baumann (2017) in the motivation of research. First, the application to the law of the few allows us to study not only network formation but also activities alongside. Second, we study how the degree of unequalness in link investment, the distribution of action intensity, and the network structure change as link investment substitutability change. This is not discussed in Baumann (2017).

The rest of the paper is organized as follows. In the next section, we give a detailed discussion of the CES link formation assumption. We explain the model and results for the connections model and law of the few model with the CES specification in Section 3 and 4 respectively. Section 5 concludes. All proofs can be found in the Appendix.

2.2 The CES Link Formation Assumption

For a set of players $N = \{1, ..., n\}$, let $a_{ij} \in \mathbb{R}_+$ be the link investment from player i to j. We assume that a link g_{ij} between i and j is produced with inputs a_{ij} and a_{ji} and a symmetric CES link formation technology:

$$g_{ij} = \begin{cases} h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} & \text{for weighted link formation} \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{for unweighted link formation} \end{cases}$$

where $\mathbb{1}$ is the indicator function and β is a parameter that can take any real value.

The specification can be employed to model both weighted and unweighted link formation. The weighted formulation assumes a CES production function directly. For the unweighted version, we assume that the sustainment of a link takes a level of overall effort aggregated by the CES function and two players can contribute in different ways to satisfy the requirement.

The CES function was first introduced in Arrow et al. (1961) as a way to generalize the Leontiff production function (of which the elasticity of substitution is 0) and the Cobb Douglas production function (of which the elasticity of substitution is 1) and is widely employed in macroeconomic literatures. The standard two factor CES function is:

$$\hat{h}(a_{ij}, a_{ji}) = \gamma(\alpha a_{ij}^{\beta} + (1-\alpha)a_{ji}^{\beta})^{\frac{1}{\beta}}$$

where $\gamma > 0$, $\alpha \in (0,1)$, and $\beta \leq 1$. Here we normalize γ to 1 since we will later introduce a link formation cost variable. We assume links are undirected, i.e. $\hat{h}(a_{ij}, a_{ji}) = \hat{h}(a_{ji}, a_{ij})$,

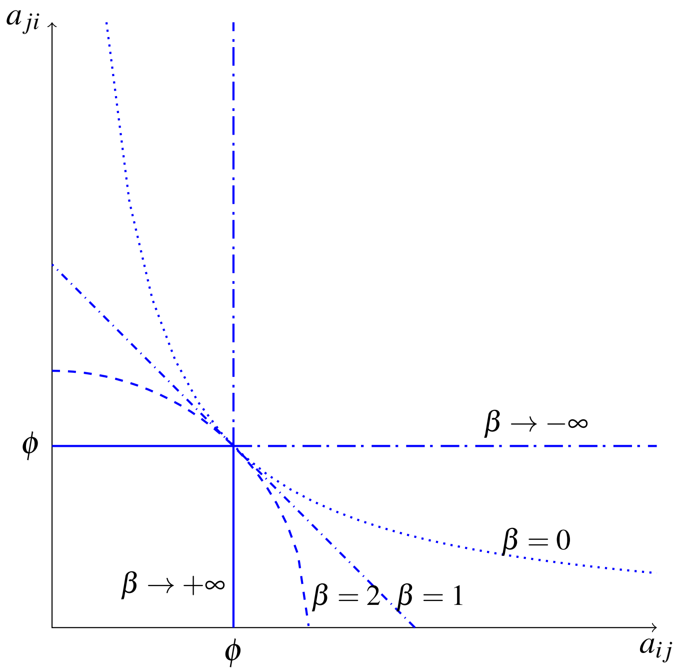


Figure 2.1 Isoquants for $h(a_{ij}, a_{ji}) = \phi$ under different values of β

which requires $\alpha = (1-\alpha) = 1/2$. So we arrive at:

$$h(a_{ij}, a_{ji}) = (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$$

that only has one parameter β, which we allow to take values greater than 1 to achieve the goal of including the unilateral link formation function as a special case of our formulation.

The CES assumption encompasses a wide range of specific link formation protocols. We have:

(i) $\lim_{\beta \to -\infty} h(a_{ij}, a_{ji}) = \min\{a_{ij}, a_{ji}\}$: the bilateral link formation assumption is a special case of our formulation.

(ii) $h(a_{ij}, a_{ji}) \mid_{\beta=1} = \frac{1}{2}(a_{ij} + a_{ji})$: this depicts the case where link investments are perfect substitutes and can be used to model scenarios where only the sum of investments matter for the formation of a link, e.g. rail and bridge construction. It can also be adopted to incorporate the possibility of side payments.

(iii) $\lim_{\beta \to 0} h(a_{ij}, a_{ji}) = \sqrt{a_{ij}a_{ji}}$: this Cobb-Douglas case can be a convenient functional form to work with when link investments are strategic complements.

(iv) $\lim_{\beta \to +\infty} h(a_{ij}, a_{ji}) = \max\{a_{ij}, a_{ji}\}$: the unilateral link formation assumption is a special case of our formulation.

The parameter β captures the substitutability of link investments in forming a connection. When $\beta < 1$, the second-order mixed derivative of h is positive:

$$\frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij} \partial a_{ji}}|_{\beta<1} > 0$$

so as one player increases his investments in a link, the other player will find it easier to create or strengthen their link with her investments. Link investments are strategic complements when $\beta < 1$.

When $\beta > 1$, the second-order mixed derivative of h is negative:

$$\frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij} \partial a_{ji}}|_{\beta>1} < 0$$

so as one player increases his investments in a link, the link investments from another player has less effect on the creation or strengthening of their link. Link investments are strategic substitutes when $\beta > 1$.

More specifically, as illustrated in Arrow et al. (1961), $\frac{1}{1-\beta}$ is the elasticity of substitution between a_{ij} and a_{ji}, so a greater β indicates a greater link investment substitutability.

Figure 2.1 pictures the isoquants of h for five different values of β and provides a graphical illustration of how β is related to link investment substitutability.

2.3 Application I: The Connections Model

The connections model was first studied in the seminal work of Jackson and Wolinsky (1996). In that paper, they introduce the solution concept of pairwise stability to predict what kinds of networks will be stable, and then examine the relationship between efficient networks-the networks that maximize the aggregate utility of players-and stable networks. The connections model is one of the two stylized models Jackson and Wolinsky (1996) carry out the above analysis on. They show that there is a strong tension between efficiency and stability.

Their findings are based on the bilateral link formation assumption. In this section, we extend their analysis by allowing weighted link investments and adopting a more flexible CES link formation specification. We propose and employ the solution concept of weighted pairwise stability and show that the efficiency-stability tension is alleviated as link investments become perfect substitutes.

2.3.1 The Model

There is a set of players $N = \{1, ..., n\}$. Each player $i \in N$ chooses how much investment to make to other players, which can be summarized by a vector $a_i = \{a_{i1}, ..., a_{in}\}$ where $a_{ij} \in A_{ij}$. The investment decisions of all agents $a = \{a_{ij}\}_{i,j\in N}$ is an investment profile, and we denote the space of investment profiles with $A = \prod_{i,j\in N} A_{ij}$.

An unweighted and undirected network $g = \{g_{ij}\}_{i,j\in N}$, where $g_{ij} = g_{ji} \in \{0,1\}$ for all $i, j \in N$, specifies whether there is a link between any two players. We use $G = \{g \mid g_{ij} = g_{ji} \in \{0,1\}\}$ to denote the set of networks n players can have.

Any investment profile creates an unweighted and undirected network. We use $g(a)$ to denote the network created with investment profile a.

The utility of a player is determined by the investment profile (and the network created with it). Before specifying the form of the utility function $(u_i(a))_{i\in N}$, we first define the efficiency and stability concept we adopt.

A network is *efficient* if it is created with an investment profile that maximizes the aggregate utility of players:

Definition 2.1. *A network $g \in G$ is efficient if $g = g(a^*)$ where*

$$a^* = \arg\max_{a\in A} \sum_{i\in N} u_i(a)$$

We propose the solution concept of *weighted pairwise stability* which adapts the notion of pairwise stability for weighted link investments. A network is weighted pairwise stable if it can be created with an investment profile where (i) no player wants to adjust his level of investment for a link he has, and (ii) no pair of unlinked players can coordinate their investments to each other for a mutually beneficial link. Formally, let $a_{-ij,kl,...}$ denote an investment profile a excluding the investment choices from i to j, k to l,....

Definition 2.2. *A network $g \in G$ is weighted pariwise stable if there exists an investment profile $a \in A$ such that $g = g(a)$ and for all pairs of players $(i, j) \in N^2$:*

(i) if $g_{ij} = 1$, then for all $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$,

$$u_i(a) \geq u_i(a'_{ij}, a_{-ij})$$
$$u_j(a) \geq u_j(a'_{ji}, a_{-ji})$$

(ii) if $g_{ij} = 0$, then there does not exist $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$ such that

$$u_i(a'_{ij}, a'_{ji}, a_{-ij,ji}) \geq u_i(a)$$
$$u_j(a'_{ij}, a'_{ji}, a_{-ij,ji}) \geq u_j(a)$$

with at least one inequality being strict.

We now discuss the relationship between the notion of weighted pairwise stability and the notion of pairwise stability. The solution concept of pairwise stability can be employed for network formation models with the bilateral link formation assumptions. In our terminology, a network formation model with the bilateral link formation assumption is one where link investment choices are binary: $\forall i, j \in N : A_{ij} = \{0, 1\}$, a link is created with mutual investments: $\forall i, j \in N : g_{ij} = \min\{a_{ij}, a_{ji}\}$, and an investment profile affects the utility of players only via the network it creates and the cost it incurs: $\forall a, a' \in A$ with $g(a) = g(a')$: $u_i(a') \geq u_i(a)$ if and only if $a_i \geq a'_i$. Note that a network $g \in G$ and an investment profile $a \in A$ are both $n \times n$ matrices and $G \subset A$, so $(u_i(g))_{i \in N}$ is well defined. Let $g - ij$ denote the network with the deletion of a link between i and j from g and $g + ij$ denote the network with the addition of a link between i and j to g.

Definition 2.3 (Jackson and Wolinsky (1996))**.** *A network $g \in G$ is pariwise stable if for all pairs of players $(i, j) \in N^2$:*

(i) if $g_{ij} = 1$, then

$$u_i(g) \geq u_i(g - ij)$$
$$u_j(g) \geq u_j(g - ji)$$

(ii) if $g_{ij} = 0$, then it cannot be

$$u_i(g + ij) \geq u_i(g)$$
$$u_j(g + ij) \geq u_j(g)$$

with at least one inequality being strict.

We show that for a network formation model with the bilateral link formation assumption, the notions of weighted pairwise stability and pairwise stability are equivalent.

Remark 2.1. *Consider a network formation model with the bilateral link formation assumption. A network $g \in G$ is weighted pairwise stable if and only if it is pairwise stable.*

We now specify the assumptions for the *weighted connections model* we examine in this section. For all $i, j \in N$, we set $A_{ij} = \mathbb{R}_+$ so that players can choose the level of an link investment,

$$g_{ij}(a) = \begin{cases} 0 & \text{when } i = j \\ \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} = \mathbb{1}\{(\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} \geq 1\} & \text{otherwise} \end{cases}$$

so links are formed according to the CES link formation specification, and

$$u_i(a) = \sum_{j \in N} \delta^{d_{ij}(g(a))} - c \sum_{j \in N} a_{ij}$$

where $\delta \in (0,1)$ is a discount factor, $c > 0$ is the cost of link investment, and $d_{ij}(g)$ is the distance between i and j in network g. To be specific, the distance between i and j in g is the number of links in the shortest path between i and j in g. A path between i and j in g is a sequence of distinct players $i = v_0, v_1, ..., v_m = j$ such that $g_{v_{k-1}v_k} = 1$ for all $k = 1, ..., m$. We set $d_{ii}(g) = 0$ for all $i \in N$ and $d_{ij}(g) = \infty$ when there is no path between i and j under g.

The rationale behind the utility function is the following. We assume that each player possesses a unit of non-rivalrous intrinsic value that is shared through the network. There is a decay in the sharing efficiency if two players are far away from each other. This decay is captured by parameter δ. The utility a network provides to a player is the sum of value he obtains, and he pays a constant cost c for each unit of link investments he makes.

In our analysis, we will encounter three specific networks: the *complete network*, the *star network* and the *empty network*. The complete network is a network where all players are linked to each other: $\forall i \in N, j \neq i : g_{ij} = 1$. The star network is a network where all links are between one centre player and the rest of the players whom we call periphery players: $\exists m \in N$ such that $g_{mj} = 1$ for all $j \neq m$ and $g_{ij} = 0$ for all $i \neq m, j \neq m$. The empty network is a network with no links: $\forall i, j \in N : g_{ij} = 0$.

2.3.2 Results

We first solve for efficient networks.

Proposition 2.1. *Consider the weighted connections model. The efficient network is:*

(i) the complete network when $0 \leq c \leq \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2)$,

(ii) the star network when $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta - \delta^2) \leq c \leq \frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2)$, *and*

(iii) the empty network when $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1}(\delta + \frac{n-2}{2}\delta^2) \leq c$.

The above efficient network characterization is similar to the one obtained in Jackson and Wolinsky (1996). When $\beta < 1$ so that link investments are strategic complements, the term $\frac{1}{2}^{\frac{1}{\max\{\beta,1\}}-1} = 1$ and the two characterizations are identical. This is because when $\beta < 1$, the most efficient way for two players to sponsor a link would be for them to both make one unit of link investment to each other, making the cost of a link $2c$, which is the same as the cost of a link assumed in Jackson and Wolinsky (1996).

When $\beta > 1$ so that link investments are strategic substitutes, the value of β plays a role in the efficient characterization. Now, the most efficient way for two players to sponsor a link would be for one of them to bear all the cost while the other does not invest, making the cost of a link $2^{\frac{1}{\beta}}c$ which drops in β. A greater β essentially reduces the cost of a link because it supports the unilateral sponsorship of links more. A denser network is hence more likely to be efficient. We can see from our characterization that given the discount factor, when β rises, the complete network is preferred to the star network for a greater range of link investment cost c and the star network is preferred over the empty network for a greater range of cost c.

We now move to characterize the weighted stable networks.

We start by analysing the incentive for a player to build, maintain, or sever a link. To do so, we define and derive the *marginal connection benefit* and the *marginal investment cost* of a link for a player.

The *marginal benefit from connection* player i obtains by linking with player j in network g is the difference in the amount of intrinsic value i obtains through the network with and without a link to j:

$$MB(i \leftarrow j, g) = \begin{cases} \sum_{k \in N} \delta^{d_{ik}(g+ij)} - \sum_{k \in N} \delta^{d_{ik}(g)} & \text{if } g_{ij} = 0 \\ \sum_{k \in N} \delta^{d_{ik}(g)} - \sum_{k \in N} \delta^{d_{ik}(g-ij)} & \text{if } g_{ij} = 1 \end{cases}$$

The *marginal cost of investment* player i pays for a link with player j under an investment profile a is simply the cost of investment a_{ij}:

$$MC(i \to j, a) = ca_{ij}$$

We can see that the evaluation of marginal connection benefit $MB(i \leftarrow j, g)$ is not as straight forward as that of the marginal investment cost $MC(i \to j, a)$. To assess $MB(i \leftarrow j, g)$, we need to know how i's distances to others change with an addition or deletion of link with j.

When a link with j is added (deleted), i not only reduces (increases) his distance to j to 1 (at least 2) but also reduces (increases) distances to other players who j is relatively closer to. This effect can be captured with information on i's, j's and i's neighbours' distances to other players in network g. Let $d_i(g) = \{d_{i1}(g), ..., d_{in}(g)\}$ be the *distance vector* of player i in network g. We make the following observation on how adding or severing a link between two players affect their distance vectors.

Lemma 2.1. *For players $i \in N$, $j \neq i$ and $k \neq i$:*

(i)

$$d_{ik}(g+ij) = \begin{cases} d_{jk}(g)+1 & \textit{when } d_{ik}(g) > d_{jk}(g) \\ d_{ik}(g) & \textit{otherwise} \end{cases}$$

(ii)

$$d_{ik}(g-ij) = \begin{cases} \min\limits_{l:l\neq j, g_{il}=1} d_{lk}(g)+1 & \textit{when } d_{ik}(g) > d_{jk}(g) \\ d_{ik}(g) & \textit{otherwise} \end{cases}$$

The intuition behind Lemma 2.1 is simple. If i gets a new link with j, i's distance to other nodes may shorten as there is now a new path from i to other players via j. Whether the new path is at least as short as all the original ones depends on if $d_{ik}(g) > d_{jk}(g)$. If it is, the new distance will be $d_{jk}(g)+1$; if not, the distance remains $d_{ik}(g)$. Similarily, if i cuts his link with j, i's distance to other nodes may lengthen as he looses a path to other players via j. Whether this is the case depends on if the original shortest path between i and another node passes through j, indicated by whether $d_{ik}(g) < d_{jk}(g)$. If it is, the new distance will be the shortest path from i to the other node without passing j, which is $\min\limits_{l:l\neq j, g_{il}=1} d_{lk}(g)+1$; if not, the distance remains $d_{ik}(g)$.

With Lemma 2.1, we can rewrite the marginal connection benefit i obtains from a link with j in network g as:

$$MB(i \leftarrow j, g) = \begin{cases} \sum\limits_{k\in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{jk}(g)+1} - \delta^{d_{ik}(g)}\right) & \text{if } g_{ij} = 0 \\ \sum\limits_{k\in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{ik}(g)} - \delta^{\min\limits_{l\in N: l\neq j, g_{il}(g)=1} d_{lk}(g)+1}\right) & \text{if } g_{ij} = 1 \end{cases}$$

We can see that $MB(i \leftarrow j, g)$ is determined by how many players j has relatively closer access to than i does, and how much closer j is compared to i.

Going back to the question of whether i wants to build or sever a link with j, it can be answered by comparing $MB(i \leftarrow j, g)$ and $MC(i \rightarrow j, a)$. Since $MC(i \rightarrow j, a) = ca_{ij}$, the comparison can be made between $MB(i \leftarrow j,g)/c$ and a_{ij}. If $MB(i \leftarrow j,g)/c \geq a_{ij}$, i would want to build or sustain the link with j. If $MB(i \leftarrow j,g)/c < a_{ij}$, i would not want to have a link with j.

We know that a link between i and j is created and sustained if $h(a_{ij}, a_{ji}) \geq 1$. So, there is a way of link sponsorship where both i and j are willing to maintain the link if

$$h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) \geq 1$$

and there is no way of link sponsorship where both i and j find creating a link beneficial (one finds if strictly beneficial) if

$$h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) \leq 1$$

The conditions for a network g to be weighted pairwise stable can thus be reformulated as:

Proposition 2.2. *Consider the weighted connections model. A network $g \in G$ is weighted pairwise stable if and only if for all pairs of players $(i,j) \in N^2$:*

(i) if $g_{ij} = 1$, then

$$h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) \geq 1$$

(ii) if $g_{ij} = 0$, then

$$h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) \leq 1$$

where

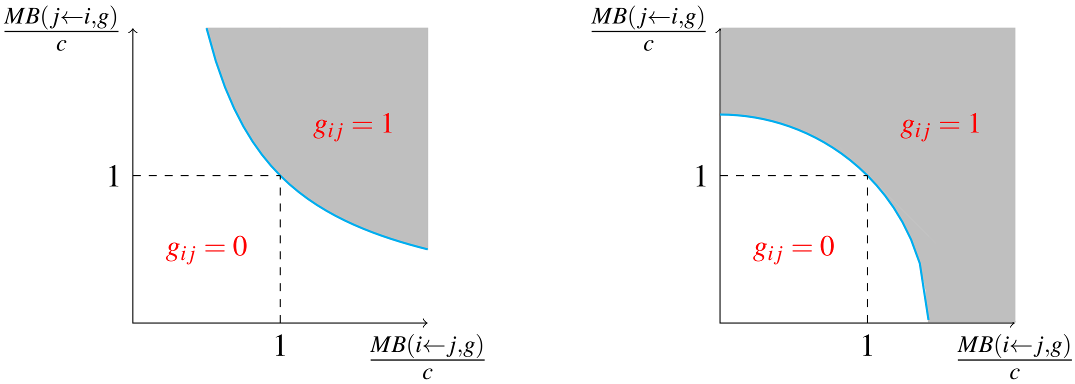
$$MB(i \leftarrow j,g) = \begin{cases} \sum\limits_{k \in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{jk}(g)+1} - \delta^{d_{ik}(g)} \right) & \text{if } g_{ij} = 0 \\ \sum\limits_{k \in N: d_{ik}(g) > d_{jk}(g)} \left(\delta^{d_{ik}(g)} - \delta^{\min\limits_{l \in N: l \neq j, g_{il}(g)=1} d_{lk}(g)+1} \right) & \text{if } g_{ij} = 1 \end{cases}$$

Note that with different levels of β, the shape of the h function changes. Therefore, the level of link investment substitutability influences players' decision on when to keep and form links and hence affect the stability of networks. We illustrate this below.

It can be shown the for the CES function h, when $\beta_1 \geq \beta_2$,

$$h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c})|_{\beta=\beta_1} \geq h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c})|_{\beta=\beta_2}$$

So in a network g, if player i and player j want to maintain or form a link with $\beta = \beta_2$, they would also want to do so with $\beta = \beta_1$. A greater link investment substitutability increases the chances for two players to have a link between them. Figure 2.2 provides a simple graphic illustration for this effect.



(a) when $\beta = 0$ (b) when $\beta = 2$

Figure 2.2 Link investment substitutability and g_{ij}

The blue curves in Figure 2.2(a) and Figure 2.2(b) plot the combinations of $\frac{MB(i \leftarrow j,g)}{c}$ and $\frac{MB(j \leftarrow i,g)}{c}$ such that $h(\frac{MB(i \leftarrow j,g)}{c}, \frac{MB(j \leftarrow i,g)}{c}) = 1$ when $\beta = 0$ and $\beta = 2$ respectively. When the marginal connection benefits i and j provide to each other lie in the grey area, i and j would maintain their link if $g_{ij} = 1$ and create a link if $g_{ij} = 0$. By comparing Figure 2.2(a) and Figure 2.2(b), we can see a greater β leads to a larger space of marginal connection benefit pair that facilitates a link. In particular, two players, one of whom enjoys a large marginal connection benefit from a link while the other gets a small marginal connection benefit from the link, can only be linked when link investment substitutability is large.

We now combine the above observation with our understanding on how $MB(i \leftarrow j,g)$ is affected by the network positions of i and j to explain how link investment substitutability affects the structure of weighted pairwise stable networks.

When β is small, for two players i and j to have a link, the marginal connection benefits they obtain from the link must both be beyond a certain level. Hence, i needs to have a shorter distance to some players than j does and j needs to have a shorter distance to some other players than i does. It is not likely that two players are linked and one is much better connected than the other.

When β is large, two players can have a link as long as one enjoys a large marginal connection benefit from the link. A link between a well-connected player and a poorly-connected player is now common.

We illustrate the above influence of β on network structures with an examination of the stability of the star network. Since links in the star network are all between a very well connected centre player and a relatively poorly connected periphery player, we would expect the star network to be weighted pairwise stable in rare cases when β is small but much more frequently when β is large. We now verify this conjecture.

For the star network to be weighted pairwise stable, we require that the centre player m and a periphery player p want to maintain their link:

$$h(\frac{MB(m \leftarrow p, star)}{c}, \frac{MB(p \leftarrow m, star)}{c}) \geq 1 \tag{2.1}$$

and that two periphery players p and p' do not want to form a new link:

$$h(\frac{MB(p \leftarrow p', star)}{c}, \frac{MB(p' \leftarrow p, star)}{c}) \leq 1 \tag{2.2}$$

For a star network with n players, if the centre player cuts his link with a periphery player, his distances to other players are not changed and he only forgoes the exchange of intrinsic value with the periphery player that worth δ, so

$$MB(m \leftarrow p, star) = \delta \tag{2.3}$$

If a periphery player loses his link with the centre player, he loses connections to all other players. Since his distance to the centre player was 1 and his distances to the other $(n-2)$ periphery players were 2,

$$MB(p \leftarrow m, star) = \delta + (n-2)\delta^2 \tag{2.4}$$

Finally, if two periphery players form a link, they shorten the distance to each other. But their distances to other players are not changed, hence

$$MB(p \leftarrow p', star) = MB(p' \leftarrow p, star) = \delta - \delta^2 \tag{2.5}$$

Combine Requirements 2.1 and 2.2 with Equations 2.3, 2.4 and 2.5, the star network is weighted pairwise stable if and only if

$$\begin{cases} \left(\frac{1}{2}(\frac{\delta}{c})^\beta + \frac{1}{2}(\frac{\delta+(n-2)\delta^2}{c})^\beta\right)^{\frac{1}{\beta}} \geq 1 \\ \left(\frac{1}{2}(\frac{\delta-\delta^2}{c})^\beta + \frac{1}{2}(\frac{\delta-\delta^2}{c})^\beta\right)^{\frac{1}{\beta}} \leq 1 \end{cases}$$

which can be simplified to

$$\delta-\delta^2 \le c \le \left(\frac{1}{2}\delta^\beta+\frac{1}{2}(\delta+(n-2)\delta^2)^\beta\right)^{\frac{1}{\beta}}$$

So the range of link investment cost c that makes the star network weighted pairwise stable changes with the value of β. We demonstrate this with Figure 2.3 that draws the range of c for the star to be weighted pairwise stable under $\delta=0.8, n=12$ and different values of β. We can see that the range significantly broadens as β gets larger. Greater link investment substitutability supports the star structure.

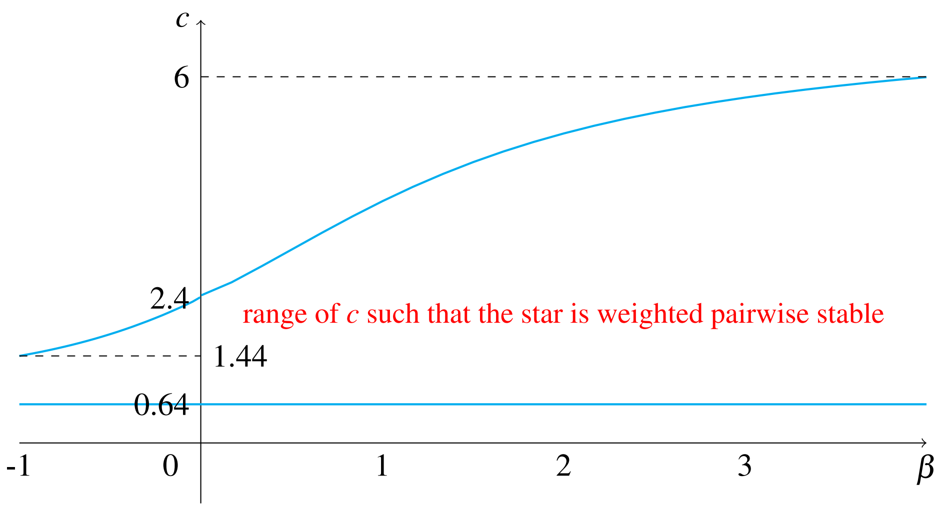


Figure 2.3 Link investment substitutability and the stability of the star ($\delta=0.8, n=12$)

We finish our analysis for the weighted connections model with an investigation on the tension between efficiency and stability.

We measure the tension in the following way. For a network g that can be efficient, we compare the range of situations in which it is efficient but not stable with the range of situations in which it is efficient and obtain a ratio that is between 0 and 1. The larger the ratio, the stronger the tension. Formally, we restrict the cost of link investment to be between 0 and n: $c \in [0,n]$. Note that if $c>n$, the empty network is efficient and is the unique weighted pairwise stable network. Let $G^e(c)$ be the set of networks that are efficient when the cost of link investment is c and $G^{wps}(c)$ be the set of networks that are weighted pairwise stable when the cost of link investment is c. We define

$$T(g)=\frac{|\{x\in[0,n]: g\in G^e(x), g\notin G^{wps}(x)\}|}{|\{x\in[0,n]: g\in G^e(x)\}|}$$

We show that

Proposition 2.3.

(i) $T(complete)$ is 0 when $\beta \leq 1$ and increases with β when $\beta > 1$,

(ii) $T(star)$ decreases with β when $\beta \leq 1$, reaches 0 when $\beta = 1$, and increases with β when $\beta > 1$,

(iii) $T(empty)$ is 0 for all $\beta \in \mathbb{R}$

Proposition 2.3 shows that when $\beta = 1$ (so that link investments are perfect substitutes: $h(a_{ij}, a_{ji}) = \frac{1}{2}(a_{ij} + a_{ji})$, whenever a network is efficient, it is also weighted pairwise stable. As the link formation technology move towards the bilateral link formation assumption ($\beta \to -\infty$) or the unilateral link formation assumption ($\beta \to +\infty$), a tension between efficiency and stability emerges for certain networks.

The intuition behind the result is the following. When an efficient network is not stable, there is a pair of players who do not find it mutually profitable to form a link that enhances aggregate welfare. The reason that they cannot sponsor the link in a mutually beneficial way might be the inflexibility of the link formation protocol. For example, under the bilateral link formation specification, if i benefits significantly from a link with j but j only benefits trivially from the link, the link cannot be formed since i and j need to make the same level of link investments. When the link formation technology becomes more flexible, the above restriction is relaxed. This makes it easier for i to compensate j with a greater amount of link investment. So the tension between efficiency and stability reduces as link investments approach perfect substitutes.

This finding shows the great impact link investment substitutability can have for the welfare of players.

2.4 Application II: Law of the Few

The second network formation scenario we examine is one where players produce goods themselves and simultaneously form links to enjoy the externalities from the production of others. Galeotti and Goyal (2010) model such interactions with the assumption that link investments are binary and links can be unilaterally formed. They solve for the Nash equilibrium and strict Nash equilibrium of the game and show that in equilibrium, only a small fraction of players engage in production activities while the majority of players focus on link formation. They use the results to explain the empirical finding that in many online information sharing networks, the number of users writing comments and reviews is small and most users only follow the content providers. Based on this finding, they title their paper *Law of the Few*.

We extend the model in Galeotti and Goyal (2010) by allowing weighted link investments and assume that links can be formed with the CES link formation technology. We carry out this exercise because we believe in certain contexts, our assumption better describes the situations players face. For example, when information sharing is achieved with email exchanges, players can choose the frequency of correspondence (so we allow weighted link investments) and they enjoy better communication efficiency when the two sides put in a similar amount of effort (we can assume link investments to be complementary at a certain level by setting the value of β). When information sharing is achieved with face to face contacts, players can again choose how enthusiastic to be and the complementarity between link investments can be even greater than in the case of email exchanges (we can capture this by setting a smaller β).

We show that the law of the few prediction in Galeotti and Goyal (2010) is robust when link investments are strategic substitutes (when $\beta > 1$), but equilibrium characterization becomes quite different when link investments get more complementary. In that case, all players make some production effort on their own, links are sponsored in relatively reciprocal ways, and there is less disparity in the network positions of players. We also compare the welfare of players under different levels of link investment substitutability.

In the following parts of the analysis, we describe the model and results with an information searching and sharing story. However, the understanding obtained should not be restricted to that specific context.

2.4.1 The Model

Consider the following game played by a set of players $N = \{1, ..., n\}$. Each player $i \in N$ chooses $s_i = \{x_i, a_i\}$, where $x_i \in X$ represents player i's effort in searching and $a_i = \{a_{i1}, ..., a_{ii-1}, a_{ii+1}, ...a_{in}\} \in A_i$ represents i's investments in links with other players. We assume the space of strategies player i can choose from to be $S_i = X \times A_i = \mathbb{R}^n$. A strategy profile is $s = \{s_1, ..., s_n\}$ and the space of strategy profiles is $S = S_1 \times S_2 \times ... \times S_n$.

The link investments of all players $a = \{a_1, ..., a_n\}$ pin down a network structure (N, g) where $g = \{g_{ij}\}_{i,j \in N}$ is an $n \times n$ matrix and g_{ij} measures the strength of link between i and j. We assume that g_{ij} is determined by a_{ij} and a_{ji} according to the CES link formation specification:

$$g_{ij}(a) = h(a_{ij}, a_{ji}) == (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}}$$

where $\beta \in \mathbb{R}$ captures the degree of substitutability between link investments a_{ij} and a_{ji}.

We define the utility of player i under strategy profile $s \in S$ as:

$$u_i(s) = f(x_i + \sum_{j \neq i} g_{ij}(a)x_j) - cx_i - k\sum_{j \neq i} a_{ij}$$

where f is twice continuously differentiable, increasing, and strictly concave, $c > 0$ and $k > 0$. The utility specification is based on the following story. Players get utility from information that comes from searching or communication with other players they have links with. The proportion of information one can get from another player is proportional to the strength of link they have. There is a cost $c > 0$ for effort and a cost $k > 0$ for link investment. The f function captures how players value information: the value is increasing with information at a decreasing rate. Assume that $f'(0) > c$ and $\lim_{y \to +\infty} f'(y) < c$ so there exists a $\hat{y} > 0$ such that $f'(\hat{y}) = c$. Consider a player who does not have links, $\hat{y}$ would be his optimal effort level and the resulting amount of information he obtains.

We solve for the Nash equilibrium and strict Nash equilibrium of the game. A strategy profile s^* is a Nash equilibrium if for all $i \in N$ and all $s_i \in S_i$:

$$u_i(s^*) \geq u_i(s_i, s^*_{-i}).$$

A strategy profile s^* is a strict Nash equilibrium if for all $i \in N$ and all $s'_i \in S_i \backslash \{s^*_i\}$:

$$u_i(s^*) > u_i(s'_i, s^*_{-i}).$$

2.4.2 Equilibrium Characterization

We first characterize the Nash equilibrium of the game when link investments are strategic complements (when $\beta < 1$). In this case, the equilibrium network can consist of several components where a pair of players are connected if and only if they belong to the same component.

We now define the concept of a component. For the purpose of analysing our model, we also include information on efforts and link investments into our definition.

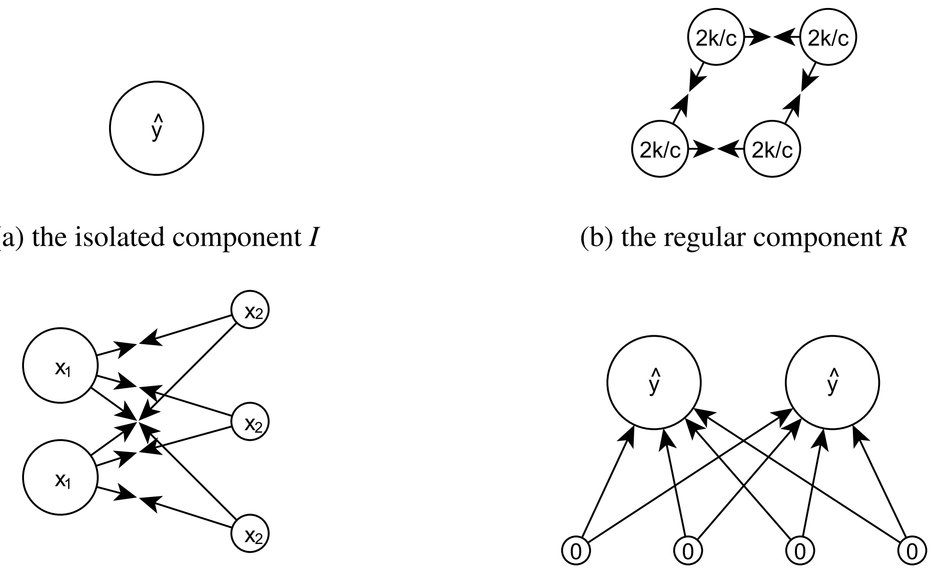
Definition 2.4. *Two players i and j are connected in a network g if there exists a sequence of nodes $i = v_0, v_1, v_2, ..., v_m = j$ such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$.*

A component $C = (N^C, s^C)$ of a strategy profile s is a subset of players and their strategies such that $\emptyset \neq N^C \subset N$, $s^C = \{s_i\}_{i \in N^C}$, and

(i) $\forall i, j \in N^C$: i and j are connected in $g(s)$,

(ii) $\forall i \in N^C, j \notin N^C$: i and j are not connected in $g(s)$.

Our equilibrium characterization involves four kinds of components: the *isolated component*, the *regular component*, the *bipartite component* and the *multi-centre star component*.



(a) the isolated component I

(b) the regular component R

(c) the bipartite component B

(d) the multi-centre star component S

Figure 2.4 Four kinds of equilibrium components

To be specific, the isolated component $I = (N^I, s^I)$ has a single player who searches for $\hat{y}$ of information and does not invest in relationships with others:

$$\begin{cases} \mid N^I \mid = 1 \\ \forall i \in N^I : x_i = \hat{y} \text{ and } a_{ij} = 0 \text{ for all} j \neq i \end{cases}$$

The regular component $R = (N^R, s^R)$ has a set of players who all search for $2k/c$ amount of information, make $c\hat{y}/2k - 1$ amount of link investments to other players in the regular component, and sponsor links to each other in a completely reciprocal manner:

$$\begin{cases} \mid N^R \mid > 1 \\ \forall i \in N^R : x_i = \frac{2k}{c} \text{ and } \sum a_{ij} = \frac{c\hat{y}}{2k} - 1 \\ \forall i, j \in N^R : a_{ij} = a_{ji} \end{cases}$$

The bipartite component $B = (N^B, s^B)$ has two groups of players searching for different amounts (x_1 and x_2) of information and investing in intergroup links. The way a link between two players is sponsored depends on the relative amount of efforts they make. Let x_1 and x_2 be two scalers where $x_1, x_2 \in (0, \hat{y})$, $x_1 \neq x_2$ and

$$x_1{}^{\frac{\beta}{1-\beta}} + x_2{}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} (\frac{k}{c})^{\frac{\beta}{1-\beta}}$$

The bipartite component features:

$$\begin{cases} N^B = U \cup V \text{ where } U \neq \emptyset, V \neq \emptyset, \text{ and } U \cap V = \emptyset \\ \forall i \in U : x_i = x_1, \sum a_{ij} = (\frac{c x_2^{\beta}}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_1), \text{ and } a_{ij} = 0 \text{ for all} j \notin V \\ \forall j \in V : x_j = x_2, \sum a_{ji} = (\frac{c x_1^{\beta}}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_2), \text{ and } a_{ji} = 0 \text{ for all} i \notin U \\ \forall i \in U, j \in V : \frac{a_{ij}}{a_{ji}} = (\frac{x_2}{x_1})^{\frac{1}{1-\beta}} \text{ or } a_{ij} = a_{ji} = 0 \end{cases}$$

The multi-centre star component $S = (N^S, s^S)$ has two groups of players where players in one group search for $\hat{y}$ amount of information while players in the other group do not search and unilaterally sponsor links to searchers:

$$\begin{cases} N^S = U' \cup V' \text{ where } U' \neq \emptyset, V' \neq \emptyset, \text{ and } U' \cap V' = \emptyset \\ \forall i \in U' : x_i = \hat{y} \text{ and } a_{ij} = 0 \text{ for all} j \neq i \\ \forall j \in V' : x_j = 0, \sum a_{ji} = 2^{\frac{1}{\beta}} \frac{\hat{z}}{\hat{y}} \text{ where } \hat{z} \text{ satisfies } f'(\hat{z}) = 2^{\frac{1}{\beta}} \frac{k}{\hat{y}} \end{cases}$$

We have the following Nash equilibrium characterization of the game when link investments are strategic complements.

Proposition 2.4. *Consider the game with $\beta < 1$. If $\hat{y} \leq 2k/c$, the game has a unique Nash equilibrium where each player forms an I component. If $\hat{y} > 2k/c$, then s^* is an equilibrium if and only if the components players form are*

(i) a combination of I, R and B components when $\beta < \frac{ln2}{ln(c\hat{y}/k)}$,

(ii) a combination of I, R, B and S components when $\beta = \frac{ln2}{ln(c\hat{y}/k)}$,

(iii) a combination of R and B components, or a single S component with $\mid V' \mid = 1$ when $\frac{ln2}{ln(c\hat{y}/k)} < \beta < 1$.

We can see that when $\hat{y} \leq 2k/c$, there will not be any linking activity in equilibrium because the amount information one can get from a neighbour is too small compared to the cost of a link. We are more interested in understanding the equilibrium when $\hat{y} > 2k/c$ such that there can be some linking activity. We make two general observations for this case. First, there can always be a very symmetric equilibrium structure that consists only of regular components. Second, an asymmetric equilibrium structure featuring the multi-centre star can emerge as β grows over a certain threshold (link investments are sufficiently substitutable).

Another thing to note is that link investment substitutability influences the structure of bipartite components that can be formed in equilibrium. From our previous definition of the bipartite component, we can see that β affects the relationship between the levels of efforts made by linked players from the two groups and how relative efforts influence link investments.

We illustrate the influence of β on efforts with Figure 2.5 that plots possible values of x_1 and x_2 under two different βs.

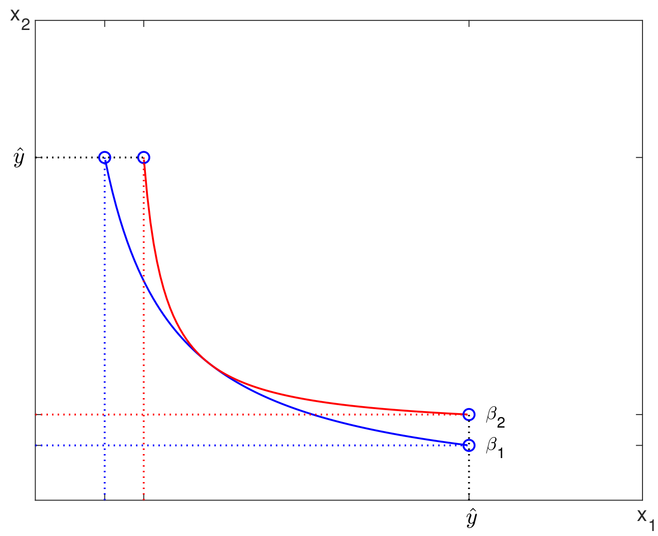


Figure 2.5 The efforts of two linked players when $\beta = \beta_1$ and when $\beta = \beta_2$ ($\beta_1 > \beta_2$)

Note that with both βs, when a player from one group of the bipartite component makes more efforts, his linked player from the other group searches less. Link investment substitutability plays a role in affecting how much the efforts players in one group make drop as players in the other group search more. We can see that a greater β (β_1) leads to a greater change and hence a larger possible difference in the efforts of two different players.

Regarding the effect of β on link investments, we know that

$$\frac{a_{ij}}{a_{ji}} = (\frac{x_2}{x_1})^{\frac{1}{1-\beta}} = (\frac{x_j}{x_i})^{\frac{1}{1-\beta}}$$

and $\frac{1}{\beta} > 0$ when $\beta < 1$. So a player i invests relatively more to a link with j if j searches relatively more. The level of link investment substitutability determines the scale of the influence relative efforts have in link investments. Since $\frac{1}{\beta}$ rises with β, a grater link investment substitutability leads to a more elastic response in link investments to differences in efforts.

We close our equilibrium analysis for strategic complements with a strict Nash equilibrium characterization. This can be easily obtained by ruling out the strategy profiles in Proposition 2.4 where some players are indifferent between the equilibrium strategy and another strategy.

Corollary 2.1. *Consider the game with $\beta < 1$. If $\hat{y} \leq 2k/c$, then there is a unique strict Nash equilibrium where each player forms an I component. If $\hat{y} > 2k/c$, then there is no strict Nash equilibrium when $\beta \leq \frac{ln2}{ln(c\hat{y}/k)}$, and there is a unique strict Nash equilibrium where a single S component with $\mid V' \mid = 1$ is formed when $\frac{ln2}{ln(c\hat{y}/k)} < \beta < 1$.*

We now move to investigate the Nash equilibrium of the game when link investments are strategic substitutes (when $\beta > 1$).[10] We show that a Nash equilibrium must have the following features.

Proposition 2.5. *Consider the game with $\beta > 1$. If $\hat{y} \leq 2^{\frac{1}{\beta}} k/c$, the game has a unique Nash equilibrium where each player forms an I component. If $\hat{y} > 2^{\frac{1}{\beta}} k/c$, then a Nash equilibrium s^* must satisfy:*

(i) the effort level of all players lies between $[0, \hat{y}]$ and there can be at most one player who searches for more than $2^{\frac{1}{\beta}} k/c$:

$$\begin{cases} x_i^* \in [0, \hat{y}] \text{ for all } i \in N \\ |\{i \in N : x_i^* > 2^{\frac{1}{\beta}} \frac{k}{c}\}| \leq 1 \end{cases}$$

(ii) all links are sponsored unilaterally and only players who search for at least $2^{\frac{1}{\beta}} k/c$ receive link investments:

$$\begin{cases} a_{ij}^* a_{ji}^* = 0 \text{ for all } i \in N, j \neq i \\ a_{ij}^* > 0 \text{ only if } x_j^* \geq 2^{\frac{1}{\beta}} \frac{k}{c} \end{cases}$$

Again, when $\hat{y}$ is small, players do not invest in links in equilibrium because the amount of information to obtain is too small to incentivize linking. When $\hat{y}$ exceeds the $2^{\frac{1}{\beta}} k/c$ threshold, players start to make link investments. Note that an equilibrium structure here is very different from those that can emerge when link investments are strategic complements.

[10] We provide the solution for Nash equilibrium when $\beta = 1$ in the appendix. The characterization is similar to the case when $\beta > 1$.

The most obvious difference is that all links are sponsored unilaterally when link investments are strategic substitutes. This contradicts the reciprocal link investments in the regular components and the proportional link investments in the bipartite components when link investments are strategic complements.

Also, note that now the equilibrium network has a core-periphery structure where players who search for at least $2^{\frac{1}{\beta}}k/c$ are in the core and can have links to each other while players who search for less than $2^{\frac{1}{\beta}}k/c$ are in the periphery and issue links to core players. We also show that there can only be one player who searches for a large amount of (more than $2^{\frac{1}{\beta}}k/c$) information.

Our Nash equilibrium characterization with $\beta > 1$ resonates with the equilibrium characterization in Galeotti and Goyal (2010). They show that under the unilateral link investment assumption ($A_{ij} \in \{0,1\}$ and $g_{ij} = \max\{a_{ij}, a_{ji}\}$ for all $i \in N, j \neq i$), the Nash equilibrium network has a core-periphery structure where core players search for more information than periphery players.

We also characterize the strict Nash equilibrium of the game when $\beta > 1$.

Remark 2.2. *Consider the game with $\beta > 1$. If $\hat{y} \leq 2^{\frac{1}{\beta}}k/c$, then there is a unique strict Nash equilibrium where each player forms an I component. If $\hat{y} > 2^{\frac{1}{\beta}}k/c$, then there is a unique strict Nash equilibrium where a single S component with $\mid V' \mid = 1$ is formed.*

2.4.3 Comparative Statics

We provide comparative statics analysis on β in this section. We ask ourselves two questions. First, as analysed in the previous section, how does β influence players' searching and linking behaviours. Second, in an asymmetric equilibrium where players choose different strategies, how does β affect the utility distribution.

We start with the first question. We define four measures of equilibrium disparity for our analysis. They focus on different aspects of equilibrium structures as explained below. Let S^* be the set of strategy profiles that are Nash equilibria.

The *supremum difference in equilibrium efforts SD_x* measures how different two players' efforts can be in an equilibrium:

$$SD_x = \sup_{s^* \in S^*, i, j \in N, j \neq i} \left| \frac{x_i^* - x_j^*}{x_i^* + x_j^*} \right|$$

The *infimum difference in equilibrium efforts ID_x* measures how close two players' efforts can be in an equilibrium:

$$ID_x = \inf_{s^* \in S^*, i,j \in N, j \neq i} \left| \frac{x_i^* - x_j^*}{x_i^* + x_j^*} \right|$$

The *supremum difference in link investments SD_a* measures how different link investments from two sides of a link can be for its formation:

$$SD_a = \sup_{s^* \in S^*, i,j \in N, g_{ij} * > 0} \left| \frac{a_{ij}^* - a_{ji}^*}{a_{ij}^* + a_{ji}^*} \right|$$

The *infimum difference in link investments ID_a* measures how close link investments from two sides of a link can be for its formation:

$$ID_a = \inf_{s^* \in S^*, i,j \in N, g_{ij} * > 0} \left| \frac{a_{ij}^* - a_{ji}^*}{a_{ij}^* + a_{ji}^*} \right|$$

We show that all four measures of disparity above are non-decreasing in β.

Proposition 2.6. *Consider the game with $\hat{y} > 2k/c$ so that there can always be linking activities in equilibrium:*

(i) SD_x rises with β when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and is 1 when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$,

(ii) ID_x is 0 for all $\beta \in \mathbb{R}$,

(iii) SD_a rises with β when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and is 1 when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$,

(iv) ID_a is 0 for when $\beta \leq 1$ and is 1 when $\beta > 1$.

From Proposition 2.6, we can see that SD_x and SD_a are affected by β in a similar pattern. They both rise with β and reach 1 when β gets greater than $\frac{ln(2)}{ln(c\hat{y}/k)}$. Recall that when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$, players can always form a multi-centre star in equilibrium where periphery players do not search and unilaterally make link investments to core players., so there can be a huge difference in how players obtain information when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$. When link investments are less substitutable ($\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$), the scale of disparity in equilibrium strategies is restricted. There is a smooth increase in how different equilibrium efforts can be and how difference equilibrium link investments can be as link investment suggestibility rises.

For the infimum difference measures, while ID_x remains 0 for all values of β, ID_a takes a huge jump from 0 to 1 when link investments change from strategic complements ($\beta < 1$) to strategic substitutes ($\beta > 1$). This is because when link investments are strategic complements, links are always reciprocally sponsored in a regular component that can emerge in equilibrium. But when link investments are strategic substitutes, links are always unilaterally sponsored.

We have learnt from Proposition 2.6 how different characteristics of equilibrium structure change smoothly or discretely with β. We now move to the second question on how β affects the utility distribution of players.

In an asymmetric equilibrium, players adopt different strategies to obtain information. Some can specialize in searching while others devote more to linking. The utility of players can also be different given different equilibrium strategies. We analyse the relationship between players' behaviour (their level of effort) and their utility and how β affects this relationship.

Proposition 2.7. *Consider the game with $\beta < 1$. In a Nash equilibrium s^*,*
if $0 < x_i^* < x_j^* < \beta\hat{y}$ or $\hat{y} > x_i^* > x_j^* > \beta\hat{y}$, then $u_i(s^*) < u_j(s^*)$.
Consider the game with $\beta > 1$. In a Nash equilibrium s^*,
(i) if $x_i^*, x_j^* \in [0, 2^{\frac{1}{\beta}}k/c)$, then $u_i(s^*) = u_j(s^*)$,
(ii) if $x_i^* \in [0, 2^{\frac{1}{\beta}}k/c)$ and $x_j^* = 2^{\frac{1}{\beta}}k/c$, then $u_i(s^*) \le u_j(s^*)$,
(iii) if $x_i^* \in [0, 2^{\frac{1}{\beta}}k/c]$ and $x_j^* > 2^{\frac{1}{\beta}}k/c$, then $u_i(s^*) > u_j(s^*)$.

Proposition 2.7 tells us that when link investments are strategic complements, there is an optimal level of effort $\max\{0, \beta\hat{y}\}$ that leads to the greatest equilibrium utility: when two players both search for less than $\beta\hat{y}$ or more than $\beta\hat{y}$, the one whose effort is closer to $\beta\hat{y}$ obtains a larger utility. Since this optimal level of effort rises with β, greater link investment substitutability favours those who search more. The reason for this favour is that with greater link investment substitutability, those who search more can attract more link investments from other players and then access their information at a lower cost.

When link investments are strategic substitutes, Proposition 2.7 shows an advantage for a player who searches for $2^{\frac{1}{\beta}}k/c$ but a disadvantage for a player who searches for more than $2^{\frac{1}{\beta}}k/c$. The advantage for those who make $2^{\frac{1}{\beta}}k/c$ amount of effort can be explained with the same reasoning above on why greater β favours searchers. When a player searches for $2^{\frac{1}{\beta}}k/c$, he will receive unilateral link investments from other players who might be searching. He hence accesses some information for free. However, if a player searches for more than $2^{\frac{1}{\beta}}k/c$, he will also receive unilateral link investments but those who invest will not search at all since acquiring information from the player is chapter than searching.

2.5 Discussion

This chapter suggests a research agenda on link formation technology and network formation. We have proposed an approach that utilizes the CES function and illustrated the applicability and usefulness of it with practices on two well-known models. We believe that the approach can be used in various other network formation setups and deliver meaningful messages.

The chapter also offers a prospect for empirical research on network formation. The variation we introduce on link investment substitutability could be an omitted variable in works that adopt the bilateral or unilateral link formation assumption. Adding a degree of freedom along that dimension can reduce the bias of estimator significantly in certain cases. Given the wide adoption of the CES function in macro literature, we believe the practice should not be very challenging. We believe that it would be a promising research agenda to further explore the connection between link investment substitutability and network formation, especially for more general network utility functions, instead of the two particular games analysed in this paper. Such an analysis might provide us with a network formation explanation of welfare disparity among agents.

Appendix 2.A Proofs

Proof for Remark 2.1

We first prove the "if" part, i.e. with the restrictions on the network formation game, when a network g satisfies the conditions specified in Remark 2.1, then it also satisfy the conditions specified in Definition 2.2 for it to be weighted pairwise stable.

Let g be a network that satisfy conditions specified in Remark 2.1, and let a be an investment profile where $a = g$. So $a_{ij} = a_{ji} = 1$ when $g_{ij} = 1$ and $a_{ij} = a_{ji} = 0$ when $g_{ij} = 0$. Given the specification on A and H in Remark 2.1, we can see $a \in A$ and $g = H(a)$. To show that g to be weighted pairwise stable, we only need that for this a:

(i) for all $i, j \in N$ and all $a'_{ij} \in A_{ij}$,

$$v_i(a;H) \geq v_i(a'_{ij}, a_{-ij};H)$$

(ii) there does not exists $i, j \in N$ such that for some $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$,

$$v_i(a'_{ij}, a'_{ji}, a_{-ij,ji};H) \geq v_j(a;H)$$
$$v_j(a'_{ij}, a'_{ji}, a_{-ij,ji};H) \geq v_j(a;H)$$

with at least one inequality being strict.

For (i), suppose there exists an i and $a'_{ij} \in A_{ij}$ such that $v_i(a;H) < v_i(a'_{ij}, a_{-ij};H)$. It cannot be $a'_{ij} > a_{ij}$, because given the specification on H, $h(a'_{ij}, a_{ji}) = (a_{ij}, a_{ji})$ when $a'_{ij} > a_{ij}$, and since $u_i(g,a)$ is decreasing in a_{ij}, an increase in a_{ij} can only lead to a decrease in utility. It also cannot be $a'_{ij} < a_{ij}$. When $a_{ij} = 0$, this is obvious from the specification on A; when $a_{ij} = 1$, having $a'_{ij} < a_{ij}$ means $h(a'_{ij}, a_{ji}) < 1$, indicating a cut of link between i and j. With condition (i) of Remark 2.1, we know that $v_i(g;H) \geq v_i(g - g_{ij};H)$. Since $a = g$, we have $v_i(a;H) = v_i(g;H)$. Since $u_i(g,a)$ is decreasing in a_{ij} and independent of a_k for $k \neq i$, $v_i(g - g_{ij};H) \geq v_i(a'_{ij}, a_{-ij};H)$. So, we have $v_i(a;H) = v_i(g;H) \geq v_i(g - g_{ij};H) \geq v_i(a'_{ij}, a_{-ij};H)$, which contradicts $v_i(a;H) < v_i(a'_{ij}, a_{-ij};H)$.

For (ii), suppose there exists $i, j \in N$ such that for some $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$, both players are better off (one strictly). Given that $u_i(g,a)$ is decreasing in a_{ij} and given our specification on H, it suffice to examine $a'_{ij} = a'_{ji} = 0$ when $a_{ij} = a_{ij} = 1$ and $a'_{ij} = a'_{ji} = 1$ when $a_{ij} = a_{ij} = 0$. For the former one, we have

$$v_i(a;H) = v_i(g;H) \geq v_i(g - g_{ij};H) = v_i(a'_{ij}, a'_{ji}, a_{-ij,ji};H)$$

so players cannot benefit from the deviation. For the latter one, we have that it cannot be

$$v_i(a'_{ij}, a'_{ji}, a_{-ij,ji}; H) = v_i(g + g_{ij}; H) \geq v_j(g; H) = v_i(a; H)$$
$$v_j(g - g_{ij}; H) = v_i(a'_{ij}, a'_{ji}, a_{-ij,ji}; H) = v_j(g + g_{ij}; H) \geq v_j(g; H) = v_j(a; H)$$

with at least one inequality being strict, so players cannot benefit (with one strictly) from the deviation.

We have finished our proof for the "if" part and move to the "only if" part now. We need to show that with the restrictions on the network formation game, when a network g is weighted pairwise stable, it satisfies the conditions specified in Remark 2.1.

When a network is weighted pairwise stable, we know the a that supports it need to satisfy $v_i(a; H) \geq v_i(a'_{ij}, a_{-ij}; H)$ for all $i, j \in N$ and all $a'_{ij} \in A_{ij}$. With our specification on H, and that $u_i(g, a)$ is decreasing in a_{ij}, it implies that $a = g$.

When $g_{ij} = 1$, we have $a_{ij} = a_{ji} = 1$, let $\hat{a_{ij}} = \hat{a_{ji}} = 0$. Since $v_i(a; H) \geq v_i(a'_{ij}, a_{-ij}; H)$ for all $a'_{ij} \in A_{ij}$, we have $v_i(g; H) \geq v_i(\hat{a_{ij}}, a_{-ij}; H)$. Since $u_i(g, a)$ is independent of a_k for all $k \neq i$, we have $v_i(\hat{a_{ij}}, a_{-ij}; H) = v_i(\hat{a_{ij}}, \hat{a_{ji}}, a_{-ij,ji}; H) = v_i(g - g_i j; H)$ given our specification for H. So we have

$$v_i(g; H) \geq v_i(g - g_{ij}; H)$$

when $g_{ij} = 1$.

When $g_{ij} = 0$, we have $a_{ij} = a_{ji} = 0$, since there does not exists $a'_{ij} \in A_{ij}$ and $a'_{ji} \in A_{ji}$ that make both players are better off (one strictly), $\hat{a_{ij}} = \hat{a_{ji}} = 1$ wouldn't. So it cannot be

$$v_i(g + g_{ij}; H) = v_i(\hat{a_{ij}}, \hat{a_{ji}}, a_{-ij,ji}; H) \geq v_j(a; H) = v_i(g; H)$$
$$v_j(g + g_{ij}; H) = v_j(\hat{a_{ij}}, \hat{a_{ji}}, a_{-ij,ji}; H) \geq v_j(a; H) = v_j(g; H)$$

with at least one inequality being strict.

We have finished our proof for the "only if" part.

Proof for Proposition 2.1

We show that the cost of a link is minimized when $a_{ij} = a_{ji} = 1$ for $\beta < 1$ and $a_{ij} = 2^{\frac{1}{\beta}}, a_{ji} = 0$ or $a_{ij} = 0, a_{ji} = 2^{\frac{1}{\beta}}$ when $\beta > 1$. Our goal is to:

$$\min(a_{ij} + a_{ji})c \quad \text{s.t.} \quad a_{ij} \geq 0, a_{ji} \geq 0 \text{ and } (\frac{1}{2}a_{ij}^{\beta} + \frac{1}{2}a_{ji}^{\beta})^{\frac{1}{\beta}} = 1 \tag{2.6}$$

i.e.

$$\min[a_{ij}+2^{\frac{1}{\beta}}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}}]c \quad \text{s.t.} \quad a_{ij}\geq 0 \text{ and } 1-\frac{1}{2}a_{ij}^{\beta}\geq 0 \tag{2.7}$$

First order derivative of the objective w.r.t a_{ij} is:

$$1+2^{\frac{1}{\beta}}\frac{1}{\beta}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}(-\frac{1}{2})\beta a_{ij}^{\beta-1} \tag{2.8}$$

which is equal to 0 when $a_{ij}=2^{\frac{1}{\beta}}(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}}=a_{ji}=1$, and the second order derivative of the objective w.r.t a_{ij} is:

$$\begin{aligned}&2^{\frac{1}{\beta}-1}[(\frac{1}{\beta}-1)(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-2}(-\frac{1}{2})\beta a_{ij}^{2\beta-2}+(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}(\beta-1)a_{ij}^{\beta-2}]\\ =&2^{\frac{1}{\beta}-1}(\beta-1)[(\frac{1}{2})(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-2}\beta a_{ij}^{2\beta-2}+(1-\frac{1}{2}a_{ij}^{\beta})^{\frac{1}{\beta}-1}a_{ij}^{\beta-2}]\end{aligned} \tag{2.9}$$

which is positive when $\beta>1$ and negative when $\beta<1$. So the minimizing problem has an interior solution of $a_{ij}=a_{ji}=1$ when $\beta<1$ and a boundary solution of $a_{ij}=2^{\frac{1}{\beta}}, a_{ji}=0$ or $a_{ij}=0, a_{ji}=2^{\frac{1}{\beta}}$ when $\beta>1$. This proves that in the efficient case, a link is reciprocally sponsored when $\beta<1$ and unilaterally sponsored when $\beta>1$. When $\beta=1$, we can see the cost of link is always *2c* whatever is the way two players share the cost of a link. And hence we can see the minimized cost of a link is *2c* when $\beta\leq 1$ and $2^{\frac{1}{\beta}}c$ when $\beta>1$.

The efficient network structure characterization then follows directly from Proposition 1 of Jackson and Wolinsky (1996).

Proof for Lemma 2.1

For point (i), first it's obvious that for all $k\in N$, either $d_{ik}(g+ij)<d_{ik}(g)$ or $d_{ik}(g+ij)=d_{ik}(g)$ since the addition of a link would not make the distance from *i* to other nodes longer. If $d_{ik}(g+ij)<d_{ik}(g)$, it must be that there is a new path from *i* to *k* that is shorter than all original paths. Since there is only one such new path that travels through *j*, the condition for $d_{ik}(g+ij)<d_{ik}(g)$ to be the case is $d_{jk}(g)+1<d_{ik}(g)$ and the new distance is $d_{jk}(g)+1$.

For point (ii), we have for all $k\in N$, either $d_{ik}(g-ij)>d_{ik}(g)$ or $d_{ik}(g-ij)=d_{ik}(g)$ since the deduction of a link would not make the distance from *i* to other nodes shorter. If $d_{ik}(g-ij)>d_{ik}(g)$, it must be that the original shortest path from *i* to *k* that travels through *j*. The condition for that to be the case is $d_{ik}(g)=d_{jk}(g)+1$ and the new distance is $\min_{l:l\neq j,d_{il}(g=1} d_{lk}(g)+1$.

Proof for Proposition 2.2

For the "if" part, we need to show that if a distance matrix d satisfies condition (i) and (ii) of Proposition 2.2, then the underlying network g is pairwise stable. To see that, let investment profile a be such that:

$$a_{ij} = \begin{cases} WS_i^{ij}(d) & \text{if } d_{ij} = 1 \\ 0 & \text{if } d_{ij} > 1 \end{cases}$$

Then we can see, first, if $d_{ij} = 1$:

$$\begin{aligned} g_{ij}(a) &= \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} \\ &= \mathbb{1}\{h(WS_i^{ij}(d), WS_j^{ij}(d)) \geq 1\} \\ &= 1 \end{aligned}$$

and if $d_{ij} > 1$:

$$\begin{aligned} g_{ij}(a) &= \mathbb{1}\{h(a_{ij}, a_{ji}) \geq 1\} \\ &= \mathbb{1}\{h(0,0) \geq 1\} \\ &= 0 \end{aligned}$$

so a constructs the network g associated with d. Condition (i) for the definition of a pairwise stable network is satisfied.

Also, for all $i, j \in N$ where $g_{ij} = 1$, since we set $a_{ij} = WS_i^{ij}(d)$, we have:

$$\begin{gathered} (WS_i^{ij}(d) - a_{ij})c = 0 \\ \sum_{k:d_{ik}=d_{jk}+1} \delta^{d_{ik}}(1 - \delta^{\min_{l:l\neq j, d_{il}=1} d_{lk}+1-d_{ik}}) - c \cdot a_{ij} = 0 \\ \sum_{k\in N} \delta^{d_{ik}(g)} - c \cdot \sum_{k\neq i} a_{ik} - (\sum_{k\in N} \delta^{d_{ik}(g-ij)} - c \cdot \sum_{k\neq i,j} a_{ik}) = 0 \\ u_i(g, a_i) = u_i(g - ij, a_i - a_{ij}^{\langle ij \rangle}) \end{gathered}$$

and similarily

$$u_j(g, a_j) = u_j(g - ij, a_j - a_{ji}^{\langle ji \rangle})$$

fulfilling condition (ii) for the definition of a pairwise stable network.

Finally, for all $i, j \in N$ where $j \neq i$ and $g_{ij} = 0$, if there exists $a'_{ij}, a'_{ji} \geq 0$ such that $h(a'_{ij}, a'_{ji}) \geq 1$ and that $u_i(g + ij, a_i + {a'_{ij}}^{\langle ij \rangle}) \geq u_i(g, a_i)$ and $u_j(g + ij, a_j + {a'_{ji}}^{\langle ji \rangle}) \geq u_j(g, a_j)$, with at least one inequality being strict, then we have:

$$\sum_{k\in N}\delta^{d_{ik}(g+ij)}-c\cdot\sum_{k\neq i}a_{ik}-c\cdot a'_{ij}-(\sum_{k\in N}\delta^{d_{ik}(g)}-c\cdot\sum_{k\neq i}a_{ik})\geq 0$$
$$\sum_{k:d_{ik}>d_{jk}+1}\delta^{d_{jk}+1}(1-\delta^{d_{ik}-d_{jk}-1})-c\cdot a'_{ij}\geq 0$$
$$(WF_i^{ij}(d)-a'_{ij})c\geq 0$$
$$WF_i^{ij}(d)\geq a'_{ij}$$

and similarily

$$WF_j^{ij}(d)\geq a'_{ji}$$

with at least one inequality being strict. But then since

$$h(a'_{ij},a'_{ji})\geq 1$$

we have

$$h(WF_i^{ij}(d),WF_j^{ij}(d))\geq 1$$

violating condition (ii) in Proposition 2.2. So d also satisfies condition (iii) for the definition of a pairwise stable network.

Since the g underlying d satisfies all three conditions for a network to be pairwise stable, we proved the "if" part of the proposition.

For the "only if" part, we need to show that if network g is pairwise stable, then the associated distance matrix d satisfies condition (i) and (ii) in Proposition 2.2.

If g is pairwise stable, then there exists an investment profile a such that: when $g_{ij}=1$ (i.e. $d_{ij}=1$), $h(a_{ij},a_{ji})\geq 1$ and

$$u_i(g,a_i)=\sum_{k\in N}\delta^{d_{ik}(g)}-c\cdot\sum_{k\neq i}a_{ik}\geq\sum_{k\in N}\delta^{d_{ik}(g-ij)}-c\cdot\sum_{k\neq i,j}a_{ik}=u_i(g-ij,a_i-a_{ij}^{\langle ij\rangle})$$
$$\sum_{k\in N}(\delta^{d_{ik}(g)}-\delta^{d_{ik}(g-ij)})-c\cdot a_{ij}\geq 0$$
$$\sum_{k:d_{ik}=d_{jk}+1}\delta^{d_{ik}}(1-\delta^{\min\limits_{l:l\neq j,d_{il}=1}d_{lk}+1-d_{ik}})-c\cdot a_{ij}\geq 0$$
$$WS_i^{ij}(d)\geq a_{ij}$$

and similarily

$$WS_j^{ij}(d) \geq a_{ji}$$

Since $h(a_{ij}, a_{ji}) \geq 1$, we have $h(WS_i^{ij}(d), WS_j^{ij}(d)) \geq 1$ which is condition (i) in Proposition 2.2.

When $g_{ij} = 0$ and $j \neq i$ (i.e. $d_{ij} > 1$), according to the definition, there cannot exists $a'_{ij}, a'_{ji} \geq 0$ such that $h(a'_{ij}, a'_{ji}) \geq 1$ and that

$$u_i(g+ij, a_i + a'_{ij}{}^{\langle ij \rangle}) = \sum_{k \in N} \delta^{d_{ik}(g+ij)} - c \cdot \sum_{k \neq i} a_{ik} - c \cdot a'_{ij} \geq \sum_{k \in N} \delta^{d_{ik}(g)} - c \cdot \sum_{k \neq i} a_{ik} = u_i(g, a_i) \tag{2.10}$$

$$u_j(g+ij, a_j + a'_{ji}{}^{\langle ji \rangle}) = \sum_{k \in N} \delta^{d_{jk}(g+ji)} - c \cdot \sum_{k \neq j} a_{ik} - c \cdot a'_{ji} \geq \sum_{k \in N} \delta^{d_{jk}(g)} - c \cdot \sum_{k \neq j} a_{jk} = u_i(g, a_j) \tag{2.11}$$

with one inequality being strict.

Inequality 2.10 can be simplified to:

$$u_i(g+ij, a_i + a'_{ij}{}^{\langle ij \rangle}) = \sum_{k \in N} \delta^{d_{ik}(g+ij)} - c \cdot \sum_{k \neq i} a_{ik} - c \cdot a'_{ij} \geq \sum_{k \in N} \delta^{d_{ik}(g)} - c \cdot \sum_{k \neq i} a_{ik} = u_i(g, a_i)$$

$$\sum_{k \in N} (\delta^{d_{ik}(g+ij)} - \delta^{d_{ik}(g)}) - c \cdot a'_{ij} \geq 0$$

$$\sum_{k: d_{ik} > d_{jk}+1} \delta^{d_{jk}+1} (1 - \delta^{d_{ik} - d_{jk} - 1}) - c \cdot a'_{ij} \geq 0$$

$$WF_i^{ij}(d) \geq a'_{ij}$$

and similarily inequality 2.11 can be simplified to:

$$WF_j^{ij}(d) \geq a'_{ji}$$

So for g to be pairwise stable, for any $a'_{ij}, a'_{ji} \geq 0$ such that $h(a'_{ij}, a'_{ji}) = 1$, it cannot be $WF_i^{ij}(d) \geq a'_{ij}$ and $WF_j^{ij}(d) \geq a'_{ji}$, with one inequality being strict. Since $h(a_{ij}, a_{ji})$ is increasing in both a_{ij} and a_{ji}, we have

$$h(WF_j^{ij}(d), WF_j^{ij}(d)) \leq 1$$

for all $i, j \in N$ with $d_{ij} > 1$, which is condition (ii) in Proposition 2.2.

Proof for Proposition 2.3

(i) The complete network is weighted pairwise stable when $c \in [0, \delta - \delta^2]$. We now compare this parameter range with that that makes the complete network efficient. When $\beta \leq 1$, the two ranges coincide with each other. Yet when $\beta > 1$, we can see that $[0, \delta - \delta^2] \subset [0, 2^{1-\frac{1}{\beta}}(\delta - \delta^2)]$ and since $2^{1-\frac{1}{\beta}}$ is increasing in β, there is a growing tendency for the complete network to be efficient yet not MI-pairwise stable as β rises.

(ii) For the star network to be weighted pairwise stable, we need the distance advantage pair for any two periphery players to be in the infeasible production range, or on the production frontier such that they do not have strict incentive to form a link, and the distance advantage pair for the center player and a periphery player to be in the feasible production range such that they can coordinate in some way to maintain the link. This holds when $c \in [\delta - \delta^2, (\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}]$. Note that when $\beta \leq 1$, with the power mean inequality, we can show that $[\delta - \delta^2, (\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}] \subset [\delta - \delta^2, 2^{1-\frac{1}{\beta}}(\delta + \frac{n-2}{2}\delta^2)]$ and $(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}$ is increasing in β. So the range of link investment cost c that makes the star weighted pairwise stable is a subset of the range of it that makes the star efficient, but the difference in the two ranges are shrinking in β, i.e. in some cases, when star is the efficient network, it is not supported as weighted pairwise stable, but the likelihood of such cases is declining as β grows when $\beta \leq 1$.

When $\beta > 1$, we will compare the parameter range $[2^{1-\frac{1}{\beta}}(\delta - \delta^2), (\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}}]$ such that the star is efficient and weighted pairwise stable when c lies in it, with $[2^{1-\frac{1}{\beta}}(\delta - \delta^2), 2^{1-\frac{1}{\beta}}(\delta + \frac{n-2}{2}\delta^2)$ such that the star is efficient when c lies in it.[11] We can show that as $n \to \infty$, the two ranges tend to be identical for all $\beta > 1$. Combining our analysis on the weighted pairwise stability of the star when $\beta \leq 1$, we can infer that the asymmetric structure is supported under larger β and this leads to a higher level of coordination between efficient and weighted pairwise network structure when β rises, given that the efficient structure is asymmetric.

(iii) Finally, for the empty network structure to be weighted pairwise stable, we need that no pairs of players can coordinate in any way to form a link, i.e. $\delta \leq c$ which is satisfied for all cases when the empty network is efficient.

Proof of Propsition 2.4

First, we do some best response analysis for players.

[11] We arrive at the first range because we can show that $\delta - \delta^2 \leq 2^{1-\frac{1}{\beta}}(\delta - \delta^2)$ and $(\frac{1}{2}\delta^\beta + \frac{1}{2}(\delta + (n-2)\delta^2)^\beta)^{\frac{1}{\beta}} \leq 2^{1-\frac{1}{\beta}}(\delta + \frac{n-2}{2}\delta^2)$ with the help of properties of p-norms.

Let $BR_i(s_{-i}) = \{s_i \mid u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \; \forall s'_i \in S_i\}$ be the set of i's best responses given s_{-i}. We know that a strategy profile s^* is a Nash equilibrium if and only if for all player $i \in N$:

$$s^*_i \in BR_i(s^*_{-i})$$

We first list some basic conditions on s^*_i for it to be a best response:

Lemma 2.2. *If $s^*_i = \{x^*_i, a^*_{i1}, ..., a^*_{ii-1}, a^*_{ii+1}, ...a^*_{in}\} \in BR_i(s_{-i})$, then:*
(i) $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j \geq \hat{y}$, and $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j = \hat{y}$ if $x^*_i > 0$,
(ii) $x^*_i \leq \hat{y}$, and
(iii) $x^*_i > 0$ or $a^*_{ij} > 0$ for some j, in any equilibrium s^*.

Proof for Lemma 2.2:

Proof. For the first point, suppose $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j < \hat{y}$, then since $f'(\hat{y}) = c$ and $f''() < 0$, we have $f'(x^*_i + \sum_{t \neq i} h(a^*_{it}, a_{ti}) x_t) - c > 0$, player i will want to put in more effort and s^*_i cannot be a best response.

When $x^*_i > 0$, suppose $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j > \hat{y}$, we can infer that $f'(x^*_i + \sum_{t \neq i} h(a^*_{it}, a_{ti}) x_t) - c < 0$, player i will want to put in less effort and s^*_i cannot be a best response. And knowing that $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j \geq \hat{y}$, we get $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a_{ji}) x_j = \hat{y}$ when $x^*_i > 0$.

Moreover, since $h(a^*_{ij}, a_{ji}) x_j \geq 0 \; \forall j \neq i$, combine the first point, we know $x^*_i \leq \hat{y}$.

Finally, in equilibrium it cannot be that $x^*_i = 0$ and $a^*_{ij} = 0 \; \forall j \neq i$. This is because if $x^*_i = 0$, then $a^*_{ji} = 0 \; \forall j \neq i$, as linking to i incurs no benefit but cost for j. Thus $h(a^*_{ij}, a^*_{ji}) = h(0,0) = 0 \; \forall j \neq i$. This leads to $x^*_i + \sum_{j \neq i} h(a^*_{ij}, a^*_{ji}) x^*_j = 0$, contradicting the first condition. $\square$

From Lemma 2.2, we know that there is a minimum amount of information to obtain and a maximum amount of effort to exert, and that every player has to pay something in equilibrium, in the form of searching or link investment.

We now make some further notes on s^*. Given other players' strategy s_{-i}, player i best respond by solving a nonlinear optimization problem with the constraints that $x_i \geq 0$ and $a_{ij} \geq 0 \; \forall j \neq i$. Therefore, if $s^*_i = \{x^*_i, a^*_{i1}, ..., a^*_{ii-1}, a^*_{ii+1}, ...a^*_{in}\} \in BR_i(s_{-i})$, we have the following Karush-Kuhn-Tucker necessary conditions:

$$\begin{cases} f'(x^*_i + \sum_{l \neq i} h(a^*_{il}, a_{li}) x_l) - c = 0, & \text{if } x^*_i > 0 \\ f'(x^*_i + \sum_{l \neq i} h(a^*_{il}, a_{li}) x_l) - c \leq 0, & \text{if } x^*_i = 0 \end{cases} \tag{2.12}$$

and for all $j \neq i$:

$$\begin{cases} f'(x_i^* + \sum_{l\neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j - k = 0, & \text{if } a_{ij}^* > 0 \\ f'(x_i^* + \sum_{l\neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j - k \leq 0, & \text{if } a_{ij}^* = 0 \end{cases} \tag{2.13}$$

i.e. if x_i^* or a_{ij}^* is an interior solution, then the marginal benefit from it must be zero; if it is a corner solution, then the marginal benefit must be less than or equal to zero.

We know that f is a concave function and $h(a_{ij}, a_{ji})$ taken a_{ji} as given is a concave function of a_{ij} when $\beta \leq 1$. So the utility function of player i is a concave function of x_i and $\{a_{ij}\}_{j\neq i}$ when $\beta \leq 1$. Moreover we know that the inequality constraints are convex, so the KKT necessary conditions are also sufficient for optimazation when $\beta \leq 1$.

Regardless of whether the KKT conditions are sufficient, we know the following must be true for the best response:

Lemma 2.3. *If $s_i^* = \{x_i^*, a_{i1}^*, ..., a_{ii-1}^*, a_{ii+1}^*, ...a_{in}^*\} \in BR_i(s_{-i})$, then*

$$h_x(a_{ij}^*, a_{ji})x_j \leq k/c \qquad \forall j \neq i \tag{2.14}$$

when $x_i^* > 0$,

$$h_x(a_{ij}^*, a_{ji})x_j \geq k/c \tag{2.15}$$

when $a_{ij}^* > 0$, and

$$h_x(a_{ij}^*, a_{ji})x_j = k/c \tag{2.16}$$

when $x_i^* > 0$ and $a_{ij}^* > 0$.

Proof for Lemma 2.3:

Proof. First, if $x_i^* > 0$, then we know from (2.12):

$$f'(x_i^* + \sum_{l\neq i} h(a_{il}^*, a_{li})x_l) = c \tag{2.17}$$

and from (2.13) we know for all $j \neq i$:

$$f'(x_i^* + \sum_{l\neq i} h(a_{il}^*, a_{li})x_l)h_x(a_{ij}^*, a_{ji})x_j \leq k \tag{2.18}$$

Combining (2.17) and (2.18), we have for all $j \neq i$:

$$h_x(a_{ij}^*, a_{ji})x_j \leq k/c \tag{2.19}$$

Then, if $a^*_{ij} > 0$, then we know from (2.13):

$$f'(x^*_i + \sum_{l \neq i} h(a^*_{il}, a_{li})x_l)h_x(a^*_{ij}, a_{ji})x_j = k \tag{2.20}$$

and from (2.12) we have:

$$f'(x^*_i + \sum_{l \neq i} h(a^*_{il}, a_{li})x_l) \leq c \tag{2.21}$$

Combining (2.20) and (2.21), we have:

$$h_x(a^*_{ij}, a_{ji})x_j \geq k/c \tag{2.22}$$

Finally, if $x^*_i > 0$ and $a^*_{ij} > 0$, we know (2.19) and (2.22) are both true, so:

$$h_x(a^*_{ij}, a_{ji})x_j = k/c \tag{2.23}$$

$\square$

Intuitively, we can interpret the relationships specified by (2.14), (2.15) and (2.16) as comparing the relative marginal revenue between investing in relationship with j and searching, which is $h_x(a^*_{ij}, a_{ji})x_j$, with the relative cost of the two options, which is k/c. For example, if $x^*_i > 0$ and $a^*_{ij} > 0$, it indicates that the relative marginal revenue must be equal to the relative marginal cost such that player i is indifferent between the two ways of acquiring information.

Now we start to prove Proposition 2.4.

Proposition 2.4 gives the necessary and sufficient conditions for s^* to be an equilibrium for different parameter values when $\beta < 1$. It is straightforward to prove that the conditions are sufficient: we can verify that all players are best responding. So here we only prove that the conditions are necessary, i.e. we have characterized all possible equilibrium.

Suppose there is a Nash equilibrium s' such that there exists a resulting component $C' = (N^{C'}, g^{C'})$ that is not characterized by I, R, B, or S.

It cannot be that $\mid N^{C'} \mid = 1$. Because when $\mid N^{C'} \mid = 1$, from Lemma 2.2 it must be $x'_i = \hat{y}$ in equilibrium, which is the same as I.

If $\mid N^{C'} \mid > 1$, then there exisits $i, j \in N^{C'}$ such that $g'_{ij} > 0$. First look at the case when $a'_{ij} > 0$ and $a'_{ji} > 0$ for all $i, j \in N^{C'}$ where $g'_{ij} > 0$.

Since $a'_{ij} > 0$ and $a'_{ji} > 0$, we know that $x'_i > 0$ and $x'_j > 0$, otherwise at least on player would not want to make investment in the link. Then from Lemma 2.3, we get:

$$\begin{cases} h_x(a'_{ij}, a'_{ji}) = \frac{k}{cx'_j} \\ h_y(a'_{ij}, a'_{ji}) = \frac{k}{cx'_i} \end{cases} \tag{2.24}$$

We also know that for CES link formation function, the following is true:

$$h_x(a_{ij}, a_{ji})^{\frac{\beta}{\beta-1}} + h_y(a_{ij}, a_{ji})^{\frac{\beta}{\beta-1}} = 2^{\frac{1}{1-\beta}} \tag{2.25}$$

Substitute $h_x(a_{ij}, a_{ji})$ and $h_y(a_{ij}, a_{ji})$ with $\frac{k}{cx'_j}$ and $\frac{k}{cx'_i}$ respectively, we get:

$$x'^{\frac{\beta}{1-\beta}}_i + x'^{\frac{\beta}{1-\beta}}_j = 2^{\frac{1}{1-\beta}}(k/c)^{\frac{\beta}{1-\beta}} \tag{2.26}$$

With Implict Function Theorem, We can see that $\frac{dx'_j}{dx'_i} = -(x'_i/x'_j)^{\frac{2\beta-1}{1-\beta}} < 0$ for positive x'_i and x'_j. This means that if i and j are connected, then there is a one-to-one relationship between their effort.

If there exists $i, j \in N^{C'}$ s.t. $a'_{ij} > 0$ and $x'_i = x'_j$, then for all $l \in N^{C'}$, we have $x'_l = x'_i = x'_j$ given the one-to-one relationship for linked players. Then with equation (2.26), we get $x'_l = 2k/c$, and from equation (2.24), we get $h_x(a'_{ij}, a'_{ji}) = 1/2$, and hence $a'_{ij} = a'_{ji}$. Finally, from Lemma 2.2, since $x'_i + \sum_{j \neq i} h(a'_{ij}, a'_{ji}) x'_j = \hat{y}$, we get $\sum_{j \neq i} a'_{ij} = c\hat{y}/2k - 1$. In this case C' belongs to R type of component.

If there exists $i, j \in N^{C'}$ s.t. $a'_{ij} > 0$ and $x'_i \neq x'_j$, let $x'_i = x_1$ and $x'_j = x_2$. Since $a'_{ij} > 0$ and $a'_{ji} > 0$, from Lemma 2.2 we know $0 < x'_i < \hat{y}$ and $0 < x'_j < \hat{y}$, so we have $0 < x_1 < \hat{y}$ and $0 < x_2 < \hat{y}$. It is immediate to see that if there exists a sequence of nodes $i = v_0, v_1, ..., v_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an even number, then $x'_l = x'_i = x_1$. If there exists a sequence of nodes $i = i_0, i_1, i_2, ..., i_m = l$, such that $g_{i_k i_{k+1}} > 0$ for $k = 0, 1, ..., m-1$ and m is an odd number, then $x'_l = x'_j = x_2$. This follows from the one-to-one relationship between two connected nodes. And from this we can see for all $l \in N^{C'}$, $x'_l = x_1$ or $x'_l = x_2$.

Define $U = \{i \in N^{C'} \mid x'_i = x_1\}$ and $V = \{j \in N^{C'} \mid x'_j = x_2\}$, we can see that in equilibrium, a player from U can only be connected to a player from V and vice versa, i.e. $a'_{ij} = 0$ if $i, j \in U$ or $i, j \in V$.

Also, an examination of the CES function gives us:

$$h_x(a_{ij}, a_{ji}) = \frac{\partial h(a_{ij}, a_{ji})}{\partial a_{ij}} = (1/2)^{\frac{1}{\beta}}(1 + (a_{ji}/a_{ij})^{\beta})^{\frac{1-\beta}{\beta}} \tag{2.27}$$

So for $i \in U$, $j \in V$ and $g'_{ij} > 0$, from equation (2.27) and (2.26), we can infer that $a'_{ij}/a'_{ji} = (x_2/x_1)^{\frac{1}{1-\beta}}$. Finally, from Lemma 2.2, we can get $\sum a'_{ij} = (\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_1)\ \forall i \in U$, and $\sum a'_{ji} = (\frac{cx_1^\beta}{2k})^{\frac{1}{1-\beta}}(\hat{y} - x_2)\ \forall j \in V$. In this case C' belongs to B type of component.

If there exists $i, j \in N^{C'}$ such that $a'_{ij} > 0$ and $a'_{ji} = 0$, then we know that $\beta > 0$ and there exist $j \in N^{C'}$ such that $x'_j \geq 2^{\frac{1}{\beta}} k/c$. Because we have $a'_{ij} > 0$ and $a'_{ji} = 0$, if $x'_j < 2^{\frac{1}{\beta}} k/c$, then:

$$h_x(a'_{ij}, a'_{ji})x'_j = (1/2)^{\frac{1}{\beta}} x'_j < k/c \tag{2.28}$$

violating the condition for $a'_{ij} > 0$ in Lemma 2.3.

Since $x'_j \geq 2^{\frac{1}{\beta}} k/c$, we know that $a'_{jl} = 0\ \forall l \neq j$. If there is a l such that $a'_{jl} > 0$, then

$$h_x(a'_{lj}, a'_{jl})x'_j > (1/2)^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} k/c = k/c. \tag{2.29}$$

violating the condition for $x'_l > 0$ in Lemma 2.3, so $x'_l = 0$. But as $x'_l > 0$, we have $a'_{jl} = 0$.

Now, for all l such that $g'_{jl} > 0$, we know that $a'_{jl} = 0$, hence $a'_{lj} > 0$. It must be that $x'_l = 0$, because otherwise

$$h_x(a'_{jl}, a'_{lj})x'_l \to +\infty > k/c \tag{2.30}$$

violating the condition for $x'_j > 0$ in Lemma 2.3.

So for all l such that $g'_{jl} > 0$, $x'_l = 0$. From Lemma 2.2, we can infer $x'_j = \hat{y}$.

Returning to the link $g'_{ij} > 0$ we looked at in the start, since $g'_{ij} > 0$, we know that $x'_i = 0$. This implies that $a'_{li} = 0\ \forall l \neq i$ since linking to i brings no benefit for l. So for all l such that $g'_{il} > 0$, we know that $a'_{li} = 0$, hence $a'_{il} > 0$. We can show $x'_l = \hat{y}$ using the same logic for showing $x'_j = \hat{y}$.

In brief, we have now proved that if there exists $i, j \in N^{C'}$ such that $a'_{ij} > 0$ and $a'_{ji} = 0$, then $x'_i = 0, x'_j = \hat{y}$. And for all players l that is connected to j, we have $a'_{jl} = 0$, $a'_{lj} > 0$, and $x'_l = 0$. For all players l connected to i, we have $a'_{li} = 0$, $a'_{il} > 0\ x'_l = \hat{y}$.

With the above result, using iterative reasoning, it's immediate to see that for $a'_{ij} > 0$ and $a'_{ji} = 0$, if there exists a sequence of nodes $i = v_0, v_1, ..., i_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an even number, then $x'_l = x'_i = 0$. If there exists a sequence of nodes $i = v_0, v_1, ..., i_m = l$, such that $g_{v_{k-1}v_k} > 0$ for $k = 1, ..., m$ and m is an odd number, then $x'_l = x'_j = \hat{y}$. And from this we can see for all $l \in (N^{C'}, g^{C'})$, $x'_l = 0$ or $x'_l = \hat{y}$.

Define $U' = \{i \in N^{C'} \mid x'_i = 0\}$ and $V' = \{j \in N^{C'} \mid x'_j = \hat{y}\}$, we can see that for $i \in U'$: $a'_{ij} > 0$ only if $j \in V'$, and for $a'_{ij} > 0$, we know the following from the best response analysis:

$$f'(x'_i + \sum_{j \in V'} h(a'_{ij}, a'_{ji})x'_j)h_x(a'_{ij}, a'_{ji})x'_j - k = 0$$

Substitute the variables with our results, get:

$$f'(\sum_{j\in V'}(1/2)^{\frac{1}{\beta}}a'_{ij}\hat{y})(1/2)^{\frac{1}{\beta}}\hat{y}-k=0$$

Let $\hat{z}$ be a scaler such that $f'(\hat{z})=2^{\frac{1}{\beta}}k/\hat{y}$, we can see $\sum_{j\in V'}a'_{ij}=2^{\frac{1}{\beta}}\hat{z}/\hat{y}$.

For $j\in V'$, we have already shown that $a'_{jl}=0\ \forall l\neq j$. We can see that C' belong to S type of components now.

We have looked at all possibilities for $C'=(N^{C'},g^{C'})$ under Nash equilibrium, and show that it can always be characterized by one of the I,R,B or S components.

Now we show that equilibrium network can only be the specific kind of combinations of I,R,B and S components for different parameter ranges. When $\hat{y}\leq 2k/c$, we need to show that there does not exist $s'\neq s^*$ such that s' is a Nash equilibrium. Suppose there is a s' where $\exists i\in N$ such that $x'_i\neq\hat{y}$ or $\exists i,j\in N$ such that $a'_{ij}>0$.

If $x'_i\neq\hat{y}$, from Lemma 2.2 we know that it must be $\sum_{j\neq i}h(a'_{ij},a'_{ji})>0$, indicating it cannot be $a'_{ij}=a'_{ji}=0$. So for $s'\neq s^*$ to be an equilibrium, there exists $i,j\in N$ such that $a'_{ij}>0$.

When $\hat{y}<2k/c$, from Lemma 2.2 we know:

$$x'_j\leq\hat{y}<2k/c \tag{2.31}$$

and from Lemma 2.3 we know:

$$h_x(a'_{ij},a'_{ji})x'_j\geq k/c \tag{2.32}$$

Combining the above two conditions, we need:

$$h_x(a'_{ij},a'_{ji})>1/2 \tag{2.33}$$

An examination of how $h_x(a_{ij},a_{ji})$ varies with a_{ji}/a_{ij} tells that when $\beta<0$, (2.33) implies:

$$a'_{ji}/a'_{ij}>1 \tag{2.34}$$

So $a'_{ji}>0$ as well. Then we can do the same analysis for a'_{ji} and get:

$$a'_{ij}/a'_{ji}>1 \tag{2.35}$$

A contradiction.

When $\hat{y} = 2k/c$, if both x'_i and x'_j are less than $\hat{y}$, we can again get $x'_i < 2k/c$ and $x'_j < 2k/c$ like in equation (2.31). Using the same logic we arrive at a contradiction. If x'_i or x'_j equal to $\hat{y}$, then from Lemma 2.2, we have $h(a'_{ij}, a'_{ji}) = 0$, indicating $a'_{ij} = a'_{ji} = 0$. A contradiction.

So when $\hat{y} \leq 2k/c$, the unique equilibrium is s^* where $a^*_{ij} = 0$ for all $i, j \in N$, and $x^*_i = \hat{y}$ for all $i \in N$.

When $\beta < \frac{ln2}{ln(c\hat{y}/k)}$, we only need to show that there cannot be any S component in equilibrium when. We prove this by showing that if s^* is a Nash equilibrium, then $\nexists i, j \in N$ such that $a^*_{ij} > 0$ and $a^*_{ji} = 0$. If $a^*_{ji} = 0$, then for $s^*_i \in BR_i(s^*_{-i})$, it must be $a^*_{ij} = 0$, otherwise:

$$h_x(a^*_{ij}, a^*_{ji})x^*_j < (1/2)^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} k/c = k/c \tag{2.36}$$

violating the condition for $a^*_{ij} > 0$ in Lemma 2.3. So there cannot be unilaterally sponsored link in equilibrium when $\hat{y} < 2^{\frac{1}{\beta}} k/c$. But all links in S are unilarterally sponsored, so there cannot be any S component in this case.

When $\beta = \frac{ln2}{ln(c\hat{y}/k)}$, nothing to be further proved.

When $\beta > \frac{ln2}{ln(c\hat{y}/k)}$, we first show that there cannot be any I component in equilibrium when $0 < \beta < 1$ and $\hat{y} > 2^{\frac{1}{\beta}} k/c$. If there is an equilibrium s' with a I component, then there exist $j \in N$ such that $j \in N^I$, indicating $x'_j = \hat{y}$ and $g'_{ij} = 0 \; \forall i \neq j$.

Then $\forall i \neq j$ we have $x'_i = 0$, because it is always true that:

$$h_x(a'_{ij}, a'_{ji})x'_j > (1/2)^{\frac{1}{\beta}} 2^{\frac{1}{\beta}} k/c = k/c \tag{2.37}$$

violating the condition for $x'_i > 0$ in Lemma 2.3. Moreover, from Lemma 2.2 we know $x'_i + \sum_{l \neq i} g'_{il} a'_l \geq \hat{y} \; \forall i \neq j$. But as $x'_i = 0$ and $x'_l = 0 \; \forall l \neq j$, we can infer that $g'_{ij} > 0$, violating the assumption.

So when $\beta > \frac{ln2}{ln(c\hat{y}/k)}$, we can only have R, B, and S components in equilibrium. There are seven possible combinations for the three elements: (1) all components are R; (2) all components are B; (3) all components are S; (4) there are R and B components; (5) there are R and S components; (6) there are B and S components; and (7) all three kinds of components coexists.

We show that if there is an S component in equilibrium, then $| V' | = 1$ and there cannot be any other component. This is because for $j \in V'$, we have $x^*_j = \hat{y}$ and $a^*_{ji} = 0 \; \forall i \neq j$, then we can follow the same argument above to show that $x^*_i = 0$ and $g^*_{ij} > 0 \; \forall i \neq j$. This eliminates the possibility that an S coexists with other components or $| V' | > 1$.

Hence we know that combination (5) (6) and (7) cannot be equilibrium, and for situation (3), we know there can only be one component S with $\mid V' \mid = 1$.

We have finished the proof.

Proof for Proposition 2.5

Suppose there exisits a Nash equilibrium s' with $a'_{ij} > 0$ and $a'_{ji} > 0$. This implies $x'_i > 0$ and $x'_j > 0$ as otherwise linking incurs no benefit and $a'_{ij} > 0$ and $a'_{ji} > 0$ can not be part of best response for i and j.

Since $x'_i > 0$ and $a'_{ij} > 0$, from Lemma 2.3, we have:

$$h_x(a'_{ij}, a'_{ji})x'_j = \frac{k}{c} \tag{2.38}$$

Let s^*_i be a strategy for player i where:

- $x^*_i = 0$
- $a^*_{ij} = a'_{ij} + \frac{x'_i}{h_x(a'_{ij}, a'_{ji})x'_j}$
- $a^*_{il} = a'_{ij}$ for all $l \neq j$

We can compare s^*_i and s'_i for player i facing s'_{-i}:

$$\begin{aligned}
u_i(x^*_i, s'_{-i}) - u_i(s') &= f(x^*_i + \sum_{l \neq i} h(a^*_{il}, a'_{li})x'_l) - f(x'_i + \sum_{l \neq i} h(a'_{il}, a'_{li})x'_l) \\
&\quad - cx^*_i - k\sum_{l \neq i} a^*_{il} + cx'_i + k\sum_{l \neq i} a'_{il} \\
&= f(h(a^*_{ij}, a'_{ji})x'_j + \sum_{l \neq i,j} h(a'_{il}, a'_{li})x'_l) \\
&\quad - f(x'_i + h(a'_{ij}, a'_{ji})x'_j + \sum_{l \neq i} h(a'_{il}, a'_{li})x'_l) \\
&\quad - k(a'_{ij} + \frac{x'_i}{h_x(a'_{ij}, a'_{ji})x'_j}) - k\sum_{l \neq i,j} a'_{il} + cx'_i + ka'_{ij} + k\sum_{l \neq i,j} a'_{il} \\
&= f(h(a^*_{ij}, a'_{ji})x'_j + \sum_{l \neq i,j} h(a'_{il}, a'_{li})x'_l) \\
&\quad - f(x'_i + h(a'_{ij}, a'_{ji})x'_j + \sum_{l \neq i} h(a'_{il}, a'_{li})x'_l)
\end{aligned} \tag{2.39}$$

Since $f'(y) > 0$, the sign of $u_i(x_i^*, s'_{-i}) - u_i(s')$ is the same as the sign of:

$$\begin{aligned}
&h(a_{ij}^*, a'_{ji})x'_j + \sum_{l \neq i,j} h(a'_{il}, a'_{li})x'_l - x'_i - h(a'_{ij}, a'_{ji})x'_j - \sum_{l \neq i} h(a'_{il}, a'_{li})x'_l \\
=&[h(a'_{ij} + \frac{x'_i}{h_x(a'_{ij}, a'_{ji})x'_j}, a'_{ji}) - h(a'_{ij}, a'_{ji})]x'_j - x'_i \\
>&h_x(a'_{ij}, a'_{ji})x'_j \frac{x'_i}{h_x(a'_{ij}, a'_{ji})x'_j} x'_j - x'_i = 0
\end{aligned} \tag{2.40}$$

since $h_{xx}(a_{ij}, a_{ji}) > 0$ for all $a_{ij} > 0, a_{ji} > 0$ under $\beta > 1$.

So we can see that $s'_i \notin BR_i(s'_{-i})$, s' cannot be a Nash equilibrium.

We first prove that when $\hat{y} < 2^{\frac{1}{\beta}} k/c$, the strategy profile s^* where $a_{ij}^* = 0 \; \forall i \neq j$ and $x_i^* = \hat{y} \; \forall i \in N$ is an equililbrium.

For all player $i \in N$, he is facing $x_j^* = \hat{y}$ and $a_{ji}^* = 0 \; \forall j \neq i$. Hence:

$$h_x(a_{ij}, a_{ji}^*)x_j^* = (1/2)^{\frac{1}{\beta}} x_j^* < k/c \tag{2.41}$$

for all $a_{ij} > 0$. From Lemma 2.3, we know that if $s'_i \in BR_i(s^*_{-i})$, then $a'_{ij} = 0 = a^*_{ij} \; \forall j \neq i$. Then, from Lemma 2.2, we get $x'_i = \hat{y} = x^*_i$. Since $BR_i(s_{-i}) \neq \emptyset$, we know $s^*_i \in BR_i(s^*_{-i})$ $\forall i \in N$. So s^* is a Nash equililbrium.

We can also prove that there is no other Nash equilibrium. Suppose there is a strategy profile $s' \neq s^*$ that is also an equilibrium. It can either be that there exists $i \in N$ where $x'_i \neq \hat{y}$ or there exists $i, j \in N$ where $a'_{ij} > 0$. If $x'_i \neq \hat{y}$, from Lemma 2.2, we need $h(a'_{ij}, a'_{ji})x'_j > 0$, so we have at least $a'_{ij} > 0$ or $a'_{ji} > 0$. Thus if there is a equilibrium $s' \neq s^*$, then there exists $i, j \in N$ where $a'_{ij} > 0$.

Since $a'_{ij} > 0$, from Lemma 2.3 we know:

$$h_x(a'_{ij}, a'_{ji})x'_j \geq k/c \tag{2.42}$$

We also know that $h_x(a'_{ij}, a'_{ji}) \leq (1/2)^{\frac{1}{\beta}}$ when $\beta > 1$, so we need $x'_j \geq 2^{\frac{1}{\beta}} k/c > \hat{y}$, violating Lemma 2.2. So s' cannot be an equilibrium.

Now we have proved that when $\hat{y} < 2^{\frac{1}{\beta}} k/c$, the strategy profile s^* where he strategy profile s^* where $a^*_{ij} = 0 \; \forall i \neq j$ and $x^*_i = \hat{y} \; \forall i \in N$ is the unique Nash equilibrium.

When $\hat{y} \geq 2^{\frac{1}{\beta}} k/c$, we seperate our analysis of equilibrium s^* to the case where there exist $m \in N$ such that $x^*_m > 2^{\frac{1}{\beta}} k/c$ and the case where $x^*_i \leq 2^{\frac{1}{\beta}} k/c \; \forall i \in N$.

When there exist $m \in N$ such that $x^*_m > 2^{\frac{1}{\beta}} k/c$, we can show $x^*_i \leq 2^{\frac{1}{\beta}} k/c \; \forall i \neq m$. Suppose this is not the case and there exisit $i \neq m$ such that $x^*_i > 2^{\frac{1}{\beta}} k/c$, then it cannot be that

$a^*_{im} > 0$ and $a^*_{mi} > 0$ as shown in Lemma 2.2 . It also cannot be $a^*_{im} = 0$ or $a^*_{mi} = 0$. Suppose $a^*_{mi} = 0$, this leads to

$$h_x(a^*_{im}, a^*_{mi})x^*_m > k/c \tag{2.43}$$

and according to Lemma 2.3, it is only true when $x^*_i = 0$, contradicting our assumption. Using the same logic, we can show it cannot be $a^*_{im} = 0$ as well. We cannot find an equilibrium relationship for i and m when they are both greater than $2^{\frac{1}{\beta}}k/c$, so there can only be one player m with $x^*_m > 2^{\frac{1}{\beta}}k/c$.

We also have $g^*_{im} > 0\ \forall i \neq m$. Suppose this is not the case and there exists $i \in N$ such that $a^*_{im} = a^*_{mi} = 0$, then again

$$h_x(a^*_{im}, a^*_{mi})x^*_m > k/c \tag{2.44}$$

implying $x^*_i = 0$. From Lemma 2.2, we know that there must exists $j \in N$ such that $a^*_{ij} > 0$. Since $x^*_m > x^*_j\ \forall j \neq m$, it must be $a^*_{im} > 0$, violating the assumption.

Since $g^*_{im} > 0\ \forall i \neq m$ and links are sponsored unilaterally in equilibrium, we can either have $a^*_{im} > 0, a^*_{mi} = 0$ or $a^*_{mi} > 0, a^*_{im} = 0$.

Let $U = \{i \in N \mid a^*_{im} > 0, a^*_{mi} = 0\}$. We can see that for $i \in U$:

(1) $x^*_i = 0$. Because as we always have:

$$h_x(a^*_{im}, a^*_{mi})x^*_m > k/c \tag{2.45}$$

and from Lemma 2.3 we know $x^*_i = 0$.

(2) $a^*_{il} = 0\ \forall l \neq m$. As links are unilaterally sponsored, the marginal production from investments is the same. So when agents decide who to link to, they only care about the level of effort. Since m puts in most effort, players will only want to link with m.

(3) $a^*_{im} = 2^{\frac{1}{\beta}}\hat{z}/x^*_m$ where $f'(\hat{z}) = 2^{\frac{1}{\beta}}k/x^*_m$. With the previous two points, we only need to solve the optimization problem for i to figure out the value of a^*_{im}. At the optimal point, we have the following first order condition:

$$f'(h(a^*_{im}, 0)x^*_m)h_x(a^*_{im}, 0)x^*_m - k = 0 \tag{2.46}$$

and we can get obtain from it $a^*_{im} = 2^{\frac{1}{\beta}}\hat{z}/x^*_m$ where $f'(\hat{z}) = 2^{\frac{1}{\beta}}k/x^*_m$.

Let $V = \{j \in N \mid a^*_{jm} = 0, a^*_{mj} > 0\}$. We can see that for $j \in V$:

(1) $x^*_j = 2^{\frac{1}{\beta}}k/c$. Since $a^*_{mj} > 0$, from Lemma 2.3 we know that:

$$h_x(a^*_{mj}, 0)x^*_j \geq k/c \tag{2.47}$$

Hence $x^*_j \geq 2^{\frac{1}{\beta}}k/c$. We have shown already that $x^*_i \leq 2^{\frac{1}{\beta}}k/c\ \forall i \neq n$, so $x^*_j = 2^{\frac{1}{\beta}}k/c$.

(2) $a^*_{jl} = 0\ \forall l \notin V$. This is obvious as in definition we know $a^*_{jm} = 0$ and since $x^*_i = 0$ $\forall i \in U$, so $a^*_{ji} = 0\ \forall i \in U$.

(3) $\sum_{l \in V} a^*_{jl} = (\hat{y} - (1/2)^{\frac{1}{\beta}} a^*_{mj} x^*_m)k/c - 2^{\frac{1}{\beta}} - \sum_{l \neq j} a^*_{lj}$. From Lemma 2.2, we know that:

$$x^*_j + \sum_{l \neq j} h(a^*_{jl}, a^*_{lj}) x^*_l = \hat{y} \tag{2.48}$$

and can then derive the result.

This is the (a) kind of characterization in the Lemma.

Now we look at the case when $x^*_i \leq 2^{\frac{1}{\beta}} k/c\ \forall i \in N$.

First we show that there exists $j \in N$ such that $x^*_j = 2^{\frac{1}{\beta}} k/c$. Otherwise $x^*_j < 2^{\frac{1}{\beta}} k/c$ $\forall j \in N$, then for all $i \neq j$ we have:

$$h_x(a^*_{ij}, a^*_{ji}) x^*_j < k/c \tag{2.49}$$

and according to Lemma 2.3, we have $a^*_{ij} = 0\ \forall j \neq i$. Then $x^*_i + h(a^*_{ij}, a^*_{ji}) x^*_j = x^*_i < 2^{\frac{1}{\beta}} k/c \leq \hat{y}$, violating Lemma 2.2. So we know there exists $j \in N$ such that $x^*_j = 2^{\frac{1}{\beta}} k/c$.

Let $U' = \{i \in N \mid x^*_i < 2^{\frac{1}{\beta}} k/c\}$ and $V' = \{j \in N \mid x^*_j = 2^{\frac{1}{\beta}} k/c\}$, we then know $V' \neq \emptyset$. And for $i \in U'$, we know $a^*_{il} = 0\ \forall l \in U'$ according to Lemma 2.3, since

$$h_x(a^*_{il}, a^*_{li}) x^*_l < k/c \tag{2.50}$$

and $\sum_{l \in V'} a^*_{il} = (\hat{y} - x^*_i)c/k$ from Lemma 2.2.

For $j \in V'$, we need $a^*_{jl} = 0\ \forall l \in U'$ according to Lemma 2.3, since

$$h_x(a^*_{jl}, a^*_{lj}) x^*_l < k/c \tag{2.51}$$

and $\sum_{l \in B'} a^*_{jl} = (\hat{y} - (1/2)^{\frac{1}{\beta}} \sum_{i \in A'} a^*_{ij} x^*_i) - 2^{\frac{1}{\beta}} - \sum_{l \in B'} a^*_{lj}$ from Lemma 2.2.

This is the (b) kind of characterization in the Lemma.

We have done an exhaustive analysis on Nash equilibrium for the case when $\hat{y} \geq 2^{\frac{1}{\beta}} k/c$, and can see that it can only be the two kinds of characterization.

First we show that there exist a Nash equilibrium s^* that satisfies conditions in (a) of Lemma **??**. Let s^* be a strategy profile where $N = \{m, U\}$ and:

- $x^*_m = \hat{y}$; $x^*_i = 0\ \forall i \in U$
- $a^*_{mi} = 0\ \forall i \in U$
- $\forall i \in U$: $a^*_{il} = 0\ \forall l \neq m$; $a^*_{im} = 2^{\frac{1}{\beta}} \hat{z}/\hat{y}$ where $f'(\hat{z}) = 2^{\frac{1}{\beta}} k/\hat{y}$

We can see that s^* belong to the set of strategy profiles characterized by (a) of Lemma **??** when $V = \emptyset$. We can easily verify that this is a Nash equilibrium by showing that all players are best responding. For m, his best response is to exert $\hat{y}$ effort and issue links to no one. For $\forall i \in U$, he is best responding by getting all information from m. And from the KKT condtion, the optimal investment level will be $a^*_{im} = 2^{\frac{1}{\beta}} \hat{z}/\hat{y}$ where $f'(\hat{z}) = 2^{\frac{1}{\beta}} k/\hat{y}$. So s^* is a Nash equilibrim.

Now we show all strategy profile s^* characterized by (b) of Lemma 2.4 are Nash equilibrium. We can verify this by showing that all players are best responding in s^*. For $\forall i \in U'$, we can see that i is indifferent between exerting effort and unilaterally issuing links to players in set V'. And there are no better options for i. So $s^*_i \in BR_i(s^*_{-i})$. Similarily, for $j \in V'$, we can see that j is indifferent between exerting effort and unilaterally issuing links to players in set V'. And there are no better options for j. So $s^*_j \in BR_j(s^*_{-j})$. Thus s^* is a Nash equilibrim.

Proof for Proposition 2.6

Follow our equilibrium characterization, we can see that DS and DR are less than one when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and are equal to one when $\beta \geq \frac{ln(2)}{ln(c\hat{y}/k)}$. So we only need to show how DS and DR change with β when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ in the proof.

First we show that when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and $\hat{y} > 2k/c$, then:

$$x^*_{inf}{}^{\frac{\beta}{1-\beta}} + \hat{y}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} (k/c)^{\frac{\beta}{1-\beta}} \tag{2.52}$$

We know that when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$ and $\hat{y} > 2k/c$, in equilibrium we have:

- If $i \in N^I$: $x^*_i = \hat{y}$
- If $i \in N^R$: $x^*_i = 2k/c$
- If $i \in N^B$: $x^*_i = x_1$ or $x^*_i = x_2$, where $0 < x_1 < \hat{y}$, $0 < x_2 < \hat{y}$, $x_1 \neq x_2$, and:

$$x_1{}^{\frac{\beta}{1-\beta}} + x_2{}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} (k/c)^{\frac{\beta}{1-\beta}} \tag{2.53}$$

Since $\hat{y} > 2k/c$, we know $x^*_{inf} \neq \hat{y}$, and $2k/c$ solves equation (2.53) at $0 < x_1 = x_2 < \hat{y}$, so we have $x^*_{inf} = \min\{\inf x_1, \inf x_2\}$. As x_1 and x_2 is symmetric in equation (2.53), we have $x^*_{inf} = \inf x_1 = \inf x_2$

Returning to equation (2.53), take k, c as constants, we can see x_1 is determined by x_2 and β, so let $x_1 = \phi(x_2, \beta)$. With Implicit Function Theorem we can get:

$$\frac{dx_1}{dx_2} = -(x_2/x_1)^{\frac{2\beta-1}{1-\beta}} < 0 \tag{2.54}$$

Hence we have:

$$(\inf x_1)^{\frac{\beta}{1-\beta}} + (\sup x_2)^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}}(k/c)^{\frac{\beta}{1-\beta}} \tag{2.55}$$

And since $x^*_{inf} = \inf x_1$ and $\sup x_2 = \hat{y}$, we have:

$$x^{*\frac{\beta}{1-\beta}}_{inf} + \hat{y}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}}(k/c)^{\frac{\beta}{1-\beta}} \tag{2.56}$$

So we have $x^*_{inf} = \phi(\hat{y}, \beta)$, we now show x^*_{inf} is decreasing in β.

From equation (2.54), we can see that there is a negative relationship between x_1 and x_2, we can also check how fast x_1 decrease with x_2:

$$\frac{d\frac{dx_1}{dx_2}}{d\beta} = -(x_2/x_1)^{\frac{2\beta-1}{1-\beta}} \frac{1}{(1-\beta)^2} ln\frac{x_2}{x_1} \tag{2.57}$$

which is negative when $x_1 < x_2$. Since when $x_2 = 2k/c$, we have $x_1 = x_2$, so $\frac{d\frac{dx_1}{dx_2}}{d\beta}$ is negative when $x_2 > 2k/c$. Hence we know that for a greater β, x_1 is decreasing at a faster rate as x_2 increases when $x_2 > 2k/c$. This indicates that if $\beta_1 > \beta_2$, we have:

$$\phi(\hat{y}, \beta_1) - \phi(2k/c, \beta_1) < \phi(\hat{y}, \beta_2) - \phi(2k/c, \beta_2) \tag{2.58}$$

Since $\phi(2k/c, \beta_1) = \phi(2k/c, \beta_2) = 2k/c$, we have:

$$\phi(\hat{y}, \beta_1) < \phi(\hat{y}, \beta_2) \tag{2.59}$$

for $\beta_1 > \beta_2$, i.e. x^*_{inf} is smaller when β is greater.

First we show $\frac{dDS}{d\beta} > 0$ when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$. We have:

$$\begin{aligned} DS &= \sup_{s^* \in S^*} \max_{i,j \in N} \left| \frac{x^*_i - x^*_j}{x^*_i + x^*_j} \right| \\ &= \sup_{s^* \in S^*} \frac{\max_{i \in N} x^*_i - \min_{i \in N} x^*_i}{\max_{i \in N} x^*_i + \min_{i \in N} x^*_i} \\ &= \frac{\hat{y} - x^*_{inf}}{\hat{y} + x^*_{inf}} \end{aligned} \tag{2.60}$$

since $x_i^* > 0$ for all $i \in N$ and $s^* \in S^*$.

We know $x_{inf}^* > 0$ and we have shown that $\frac{dx_{inf}^*}{d\beta} < 0$ when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$, so we get:

$$\begin{aligned} \frac{dDS}{d\beta} &= \frac{dDS}{dx_{inf}^*}\frac{dx_{inf}^*}{d\beta} \\ &= -\frac{2\hat{y}}{(\hat{y}+x_{inf}^*)^2}\frac{dx_{inf}^*}{d\beta} > 0 \end{aligned} \tag{2.61}$$

Now we show $\frac{dDR}{d\beta} > 0$ when $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$, and $\lim_{\beta \to -\infty} DR = 0$. Since:

$$a_{ij}^*/a_{ji}^* = (x_j^*/x_i^*)^{\frac{1}{1-\beta}} \tag{2.62}$$

We have:

$$\begin{aligned} DR &= \sup_{s^* \in S^*} \max_{i,j \in N, g_{ij}^* > 0} \mid \frac{a_{ij}^* - a_{ji}^*}{a_{ij}^* + a_{ji}^*} \mid \\ &= \sup_{s^* \in S^*} \max_{i,j \in N, g_{ij}^* > 0} \mid \frac{(x_j^*/x_i^*)^{\frac{1}{1-\beta}} - 1}{(x_j^*/x_i^*)^{\frac{1}{1-\beta}} + 1} \mid \\ &= 1 - \frac{2}{(\hat{y}/x_{inf}^*)^{\frac{1}{1-\beta}} + 1} \end{aligned} \tag{2.63}$$

since the player exerting most effort can always be connected to the player exerting least effort. We already proved $\frac{d(\hat{y}/x_{inf}^*)}{d\beta} > 0$ just now. Also we have $\hat{y}/x_{inf}^* > 1$ and $\frac{d\frac{1}{1-\beta}}{d\beta} > 0$, hence we have $\frac{dDR}{d\beta} > 0$.

Moreover, we know $\lim_{\beta \to -\infty} \frac{1}{1-\beta} = 0$ and $\lim_{\beta \to -\infty} x_{inf}^* > 0$, so $\lim_{\beta \to -\infty} DR = 1 - 1 = 0$.

Proof for Proposition 2.7

In Proposition 2.7, we categorize all equilibria into three kinds, and show how a player's utility is related to his effort level respectively. We now prove the results for the three situations one by one.

First, if we exclude the case that $\beta = \frac{ln(2)}{ln(c\hat{y}/k)}$, then we know from analysis in Section 3.3 that the resulting network can always consists of R and B components. When $\beta < \frac{ln(2)}{ln(c\hat{y}/k)}$, the equilibrium network may also encompass I components. When $\beta = \frac{ln(2)}{ln(c\hat{y}/k)}$, the equilibrium can consists of I, R, B and a particular kind of S components where players exerting 0 effort get $\hat{y}$ information. We can also show that all players in this kind of equilibrium will obtain exactly $\hat{y}$ information. So the difference in utility results from different costs players need to pay.

If $i \in N^I \cup N^S$, we can see that the aggregate cost player i needs to pay is $c\hat{y}$.

If $i \in N^R$, we can view it as a special case of $i \in N^B$ with $x_i^* = 2k/c$. So for $i \in N^R \cup N^B$, let $x_i^* = x_1$ hence all players connected to i exert effort level x_2 where:

$$x_1{}^{\frac{\beta}{1-\beta}} + x_2{}^{\frac{\beta}{1-\beta}} = 2^{\frac{1}{1-\beta}} (k/c)^{\frac{\beta}{1-\beta}}$$

We can show that aggregate cost player i needs to pay in this case is:

$$\begin{aligned}
c_i &= cx_1 + k(\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_1) \\
&= c\hat{y} - [c\hat{y} - cx_1 - k(\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}} (\hat{y} - x_1)] \\
&= c\hat{y} - (\hat{y} - x_1)(c - k(\frac{cx_2^\beta}{2k})^{\frac{1}{1-\beta}}) \\
&= c\hat{y} - (\hat{y} - x_1)(c - k(\frac{c}{2k})^{\frac{1}{1-\beta}} (2^{\frac{1}{1-\beta}} (\frac{k}{c})^{\frac{1}{1-\beta}} - x_1^{\frac{\beta}{1-\beta}})) \\
&= c\hat{y} - (\hat{y} - x_1)(c - c + k(\frac{c}{2k})^{\frac{1}{1-\beta}} x_1^{\frac{\beta}{1-\beta}}) \\
&= c\hat{y} - (\hat{y} - x_1) k(\frac{c}{2k})^{\frac{1}{1-\beta}} x_1^{\frac{\beta}{1-\beta}} \\
&< c\hat{y}
\end{aligned} \tag{2.64}$$

We can differenciate the cost c_i with $x_i = x_1$, and get:

$$\begin{aligned}
\frac{dc_i}{dx_1} &= -k(\frac{c}{2k})^{\frac{1}{1-\beta}} [-x_1^{\frac{\beta}{1-\beta}} + (\hat{y} - x_1) \frac{\beta}{1-\beta} x_1^{\frac{2\beta-1}{1-\beta}}] \\
&= -k(\frac{c}{2k})^{\frac{1}{1-\beta}} [\frac{\hat{y} - x_1}{x_1} \frac{\beta}{1-\beta} - 1]
\end{aligned} \tag{2.65}$$

And we can see when $\beta \leq 0$, $\frac{dc_i}{dx_1}$ is positive for all $x_1 \in (0, \hat{y})$, so utility is always decreasing in effort level and the statement is true for $\beta \leq 0$. When $0 < \beta < 1$, $\frac{dc_i}{dx_1}$ is negative when $x_1 < \beta\hat{y}$ and positive when $x_1 > \beta\hat{y}$. So utility is decreasing in effort level when $\beta \leq 0$ and when x_i exceeds $\beta\hat{y}$ when $0 < \beta < 1$, and increasing in effort level when it is less than $\beta\hat{y}$ when $0 < \beta < 1$, so the statement is true for $0 < \beta < 1$. This is the proof for the first statement in Proposition 2.7.

For the second statement in Proposition 2.7, from analysis in Section 3.3, we know the kind of equilibrium we are looking at are the (b) kind of equilibrium when $\beta = 1$ and (b) kind of equilibrium when $\beta > 1$. For both cases, we can see that players can be seperated into two groups, $U' = \{i \in N \mid x_i^* < 2^{\frac{1}{\beta}} k/c\}$ and $V' = \{j \in N \mid x_j^* = 2^{\frac{1}{\beta}} k/c\}$. We can see that all players will get $\hat{y}$ information, so the difference in utility is related to different costs players pay. For $i \in U'$, we have:

$$\begin{aligned} c_i &= cx_i + k(\hat{y} - x_i)c/k \\ &= c\hat{y} \end{aligned} \tag{2.66}$$

For $j \in V'$, we have:

$$\begin{aligned} c_j &= c2^{\frac{1}{\beta}} k/c + k[(\hat{y} - (1/2)^{\frac{1}{\beta}} \sum_{i \in U'} a_{ij}^* x_i^*)c/k - 2^{\frac{1}{\beta}} - \sum_{l \in V'} a_{lj}^*] \\ &= c\hat{y} - c(1/2)^{\frac{1}{\beta}} \sum_{i \in U'} a_{ij}^* x_i^* - k \sum_{l \in V'} a_{lj}^* \\ &\leq c\hat{y} \end{aligned} \tag{2.67}$$

And we know $\sum_{j \in V'} a_{lj}^* > 0$ so there must exist a $j \in V'$ such that $\sum_{l \in V'} a_{lj}^* > 0$. Hence inequality (2.67) must be strict for some $j \in V'$. We have shown the second statement of Proposition 2.7 now.

For the third statement in Proposition 2.7, based on the analysis from Section 3, we know the kind of equilibrium that satisfies the description are unique S component when $\frac{ln(2)}{ln(c\hat{y}/k)} < \beta < 1$ and (b) kind of equilibrium when $\beta = 1$ and $\beta > 1$.

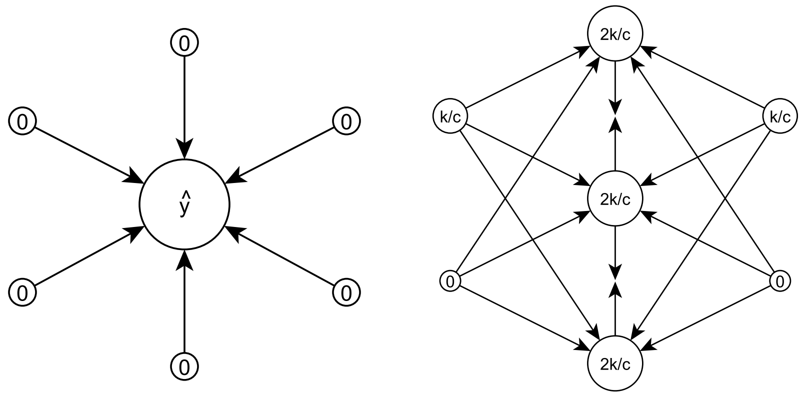
For the first two cases, player m gets information at cost c, other players get information at cost $2^{\frac{1}{\beta}} k/\hat{y} < c$, so other players are better off.

For the last case, player m gets information at cost c, player $i \in A$ get information at cost $2^{\frac{1}{\beta}} k/x_m^* < c$, and player $j \in B$ gets information at cost c but receives some free information from player m, so m is worst off.

So player m is always worst off. We have shown the third statement of Proposition 2.7 now.

Appendix 2.B Nash Equilibrium When Link Investments are Perfect Substitutes

We quickly examine the case when $\beta = 1$ so that a_{ij} and a_{ji} are perfect substitutes.



(a) (b)

Figure 2.6 Examples of Nash equilibrium when $\beta = 1$, $\hat{y} \geq 2k/c$

Lemma 2.4. *When $\beta = 1$, if $\hat{y} < 2k/c$, then there is a unique Nash equilibrium s^* where $a^*_{ij} = 0\ \forall i, j \in N$, and $x^*_i = \hat{y}\ \forall i \in N$. If $\hat{y} \geq 2k/c$, then s^* is an equilibrium if and only if it belongs to one of the following:*

(a) $N = \{m, U\}$ where $m \notin U$ and:

- $x^*_m = \hat{y}$, *and* $x^*_i = 0\ \forall i \in U$
- $a^*_{mi} = 0\ \forall i \neq m$
- $\forall i \in U$: $a^*_{ij} = 0\ \forall j \neq m$, *and* $a^*_{im} = 2\hat{z}/\hat{y}$ *where* $f'(\hat{z}) = 2k/\hat{y}\ \forall i \in U$

(b) $N = \{U', V'\}$ with $V' \neq \emptyset$, and:

- $x^*_i < 2k/c\ \forall i \in U'$, *and* $x^*_j = 2k/c\ \forall j \in V'$
- $\forall i \in U'$: $a^*_{ij} > 0$ *only if* $j \in V'$, *and* $\sum a^*_{ij} = (\hat{y} - x^*_i)c/k$
- $\forall j \in V'$: $a^*_{jl} > 0$ *only if* $l \in V'$, *and* $\sum a^*_{jl} = (\hat{y} - \frac{1}{2}\sum_{i \in U'} a^*_{ij} x^*_i)c/k - 2 - \sum_{l \in V'} a^*_{lj}$

Figure 2.6 gives two examples of Nash equilibrium that match the description of (a) and (b) in Lemma 2.4 respectively. The equilibrium characterization when $\beta = 1$ differs from that for the case when $\beta > 1$ in two ways. First, in the (a) kinds of equilibrium network, the possibility of V group is eliminated. Second, in the (b) kind of characterization, links can now be sponsored bilaterally between core players when $\beta = 1$.

Proof for Lemma 2.4:

When $\beta = 1$ we know that still the KKT conditions are necessary and sufficient for optimization, so it is easy to show that s^* characterized in Lemma 2.4 is Nash equilibrium. Here we focus on showing that all Nash equilibria have been characterized by conditions in Lemma 2.4.

When $\hat{y} < 2k/c$, we need to show that there does not exist an equililbrium $s' \neq s^*$. Suppose there is such a s' then we have $x'_i \neq \hat{y}$ or there exists $i, j \in N$ such that $a'_{ij} > 0$.

If $x'_i \neq \hat{y}$, from Lemma 2.2 we know that it must be $\sum_{j \neq i} h(a'_{ij}, a'_{ji}) > 0$, indicating there exists $a'_{ij} > 0$. So for $s' \neq s^*$ to be an equilibrium, there exists $i, j \in N$ such that $a'_{ij} > 0$.

When $\hat{y} < 2k/c$, from Lemma 2.2 we know:

$$x'_j \leq \hat{y} < 2k/c \tag{2.68}$$

and we know that $h_x(a'_{ij}, a'_{ji})x'_j = 1/2$ for all $a'_{ij} \geq 0$, $a'_{ji} \geq 0$, so

$$h_x(a'_{ij}, a'_{ji})x'_j = \frac{1}{2}x'_j < k/c \tag{2.69}$$

violating the condition for $a'_{ij} > 0$ in Lemma 2.3. A contradiction.

So when $\hat{y} < 2k/c$, the unique equilibrium is s^* where $a^*_{ij} = 0$ for all $i, j \in N$, and $x^*_i = \hat{y}$ for all $i \in N$.

When $\hat{y} \geq 2k/c$, suppose there is a strategy profile s' that is a Nash equilibrium is neither characterized by (a) nor (b).

If there exist $m \in N$ such that $x'_m > 2k/c$, then $\forall i \neq m$ we have $x'_i = 0$, because it's always true that:

$$h_x(a'_{im}, a'_{mi})x'_m > (1/2)2k/c = k/c \tag{2.70}$$

violating the condition for $x'_i > 0$ in Lemma 2.3.

From Lemma 2.2, we know that $x'_m = \hat{y}$. Also we know $a'_{mi} = 0\ \forall i \neq m$, as linking to a player other than m incurs no benefit but cost. Then from Lemma 2.2 we know that for all $i \neq m$: $x'_i + \sum_{l \neq i} h(a'_{il}, a'_{li})a'_l \geq \hat{y}$. But as $x'_i = 0$, $x'_l = 0\ \forall l \neq m$ and $a'_{mi} = 0$, we can infer that $a'_{im} > 0$. And with the KKT condition, we know that when $a'_{im} > 0$, we have:

$$f'(x'_i + \sum_{l \neq i} h(a'_{il}, a'_{li})x'_l)h_x(a'_{im}, a'_{mi})x'_m = k \tag{2.71}$$

so $a'_{im} = 2\hat{z}/\hat{y}$ where $f'(\hat{z}) = 2k/\hat{y}$. We can see that s' is the same as s^* characterized in (a) of Lemma 2.4.

If $x'_i \leq 2k/c\ \forall i \in N$. First we show that there exists $j \in N$ such that $x'_j = 2k/c$. If not, we have $x'_j < 2k/c\ \forall j \in N$, then for all $i \neq j$ we have:

$$h_x(a'_{ij}, a'_{ji})x'_j < k/c \tag{2.72}$$

and according to Lemma 2.3, we have $a'_{ij} = 0\ \forall j \neq i$. Then $x'_i + h(a'_{ij}, a'_{ji})x'_j = x'_i < 2k/c \leq \hat{y}$, violating Lemma 2.2. So we know there exists $j \in N$ such that $x'_j = 2^{\frac{1}{\beta}}k/c$.

Let $U' = \{i \in N \mid x'_i < 2k/c\}$ and $V' = \{j \in N \mid x'_j = 2k/c\}$, we then know $V' \neq \emptyset$. And for $i \in U'$, we know $a'_{il} = 0\ \forall l \in U'$ according to Lemma 2.3, since:

$$h_x(a'_{il}, a'_{li})x'_l < k/c \tag{2.73}$$

and $\sum_{l \in B'} a'_{il} = (\hat{y} - x'_i)c/k$ from Lemma 2.2.

For $j \in V'$, we need $a'_{jl} = 0\ \forall l \in U'$ according to Lemma 2.3, since:

$$h_x(a'_{jl}, a'_{lj})x'_l < k/c \tag{2.74}$$

and $\sum_{l \in V'} a'_{jl} = (\hat{y} - (1/)2^{\frac{1}{\beta}} \sum_{i \in V'} a'_{ij} x'_i) - \sum_{l \in B'} a'_{lj}$ from Lemma 2.2

We can see that s' is the same as s^* characterized in (b) of Lemma 2.4.

We have done an exhaustive analysis on Nash equilibrium for the case when $\hat{y} \geq 2k/c$, and can see that it leads to the two kinds of characterization in Lemma 2.4.

Chapter 3

A Simple Model of the Power Elite

3.1 Introduction

In *The Republic*, the choice of those who will rule and the relation between the rulers and the ruled is a central question (Plato, 1992). Closer to our times, in the 19^{th} century, the Italian school of sociology proposed a theory of elites defined in terms of the membership of the top echelons of different – government and non-government – organizations. Building on this tradition, in his well-known study of mid-twentieth-century American society, Wright Mills (1956) argued that the power to make major decisions was highly concentrated: a very small group of individuals moved between the top levels of the Federal government, a few hundred largest corporations, and the military. He referred to these individuals as the *The Power Elite*. Similar claims have been made about the concentration of power and influence in other societies.[1]

In a more empirical spirit, a number of scholars have studied the relations between the boards of directors of firms (banks and corporations). If a director sits on multiple boards, the different boards are 'linked' to each other. Studies have documented the role of prominent individuals who hold multiple directorships in creating interlocking boards (Baker et al., 2001, Useem, 1984, Kogut, 2012). While there is broad agreement on these empirical patterns, opinion is divided on the impact of such small worlds. On the one hand, there is the view that such cohesion facilitates the spread of information and best practices of cooperative behaviour. On the other hand, others have argued that such personal connections create and sustain major lobbies and perpetuate inequality.[2]

[0] This chapter is co-authored with Prof Sanjeev Goyal and Dr Marcin Dziubinski.

[1] For an overview of the theory of elites, see Bottomore (1993), and for a critique of theories of elite power and control, see Dahl (1958).

[2] For an overview of the issues, see Granovetter (2017).

We are led to ask: what are the determinants of exclusive structures and the emergence of dominant groups? Are such structures socially desirable?

To explore these questions, we propose a model of club memberships. Individuals seek to belong to clubs. The attractiveness of clubs depends on their productivity, which is a function of their membership and their connections with other clubs. Clubs seek to admit members who enhance their appeal. More specifically, we assume that a link between two clubs is formed when they share common members and the strength of the link is rising in the membership overlap. The productivity of a club is increasing in its membership size and its aggregate link strength with other clubs. In the tradition of the theory of clubs, we assume that there is congestion in the provision of club services: this is captured in a capacity constraint for every club. As individuals have budget and time constraints we also assume that there is a maximum number of clubs they can join.

The model has two types of active agents: individuals seeking to join clubs and club owners. We study efficient and stable club memberships. A club membership profile is efficient if it maximizes the utility of individuals. It is stable if no individual wishes to quit clubs, no club wishes to expel any of its members, and there is no individual-club pair that can jointly deviate and do better.

Our first result provides a characterization of efficient club memberships. It is useful to separate the result in two parts, one, where the membership availability exceeds club capacity and two, where the converse holds. We will focus on the former case as it probably is the more natural setting. Efficiency dictates that some individuals join all the clubs they can while others join no clubs at all. This concentration of membership is a consequence of the benefit a club receives by admitting someone who is a member of other clubs. Turning to the pattern of ties between clubs, the outcome relies on whether club productivity returns in overlapping common members are convex (marginal returns to membership overlap are increasing) or concave (marginal returns to membership overlap are decreasing). When the returns are convex, efficiency requires that clubs have maximal overlap: this in turn means that strongly connected clubs may get partitioned into distinct cliques. By contrast, when returns from overlapping members are concave the network of clubs will be evenly connected with relatively weaker links between clubs.

Our second result provides a characterization of stable club memberships. We will focus again on the case where membership availability exceeds club capacity. A stable club membership structure features the separation of most individuals into two distinct groups: either they join multiple most productive clubs or they are entirely excluded from all clubs. Convex/concave productivity returns from overlapping members affect the stable network of clubs similarly as they work for efficiency.

To summarize, in the natural setting where individual availability is larger than club capacity, both efficiency and stability dictate that a subset of individuals fully exhaust their club membership capacity while all others are entirely excluded from all membership. Thus efficiency and stability both imply significant payoff inequality among otherwise symmetric individuals.

The returns from overlapping members is key to understanding our model: in the baseline setting with zero spillovers across clubs, efficiency and stability have very different implications: in a setting where the number of individuals exceeds club capacity, a club membership profile in which a large number of individuals are members of one club each and there are no links between clubs is both efficient and stable.

Our model draws on the theory of clubs and the theory of networks to explain phenomena such as power elites and interlocking boards of directors. Specifically, we combine the ideas of congestion and capacity constraints from club theory (Buchanan, 1965, Cornes, 1996, Demange and Wooders, 2005) with the ideas of multiple memberships and returns from links from the theory of networks (Bala and Goyal, 2000, Jackson and Wolinsky, 1996, Bloch and Dutta, 2012). Our analysis identifies returns from club connections and the key role of bridging individuals as decisive factors in leading to exclusivity in memberships. Over and above this, we show that changes in marginal returns form the strength of connections – convex vs concave – determines whether the network of clubs is 'cliquish' (and possibly disjoint) or sparse and connected.

In a recent paper, Fershtman and Persitz (2018) propose a model of clubs and networks. At a general level, there are similarities – both papers study a club memberships model – but the motivation of the two papers is different and as a result, the models and main insights are very different. Their interest is in studying social networks among individuals, while our interest is in understanding the forces that lead to power elites. In their model, a link between two individuals arises out of common membership of a club and its strength is inversely related to the size of the club. They allow for benefits that flow through paths in the network of individuals. By contrast, in our model, individuals care about returns from a club: these returns depend on the size of the club and its connection with other clubs. These differences in modelling, in turn, lead to very different results. Their primary insight is that there is a trade-off between the size of clubs, the depreciation of indirect connections and the membership fee. By contrast, our main result is about how positive returns from overlapping membership leads to the partition of individuals into an excluded set and a group of insiders. And we derive a mapping between the curvature of these returns and the architecture of the club networks. These results have no parallel in their work.

The rest of the paper is organized as follows. Section 2 specifies the model. We explain our main results in Section 3 and we examine extensions of our model in Section 4. All proofs can be found in the Appendix.

3.2 The Model

There is a set of individuals $I = \{i_1, ..., i_n\}$ and a set of clubs $C = \{c_1, ..., c_m\}$. Use i to denote a typical individual and c to denote a typical club. Individuals join clubs to become members. The *club membership structure* is represented by a matrix $a = (a_{ic})_{i \in I, c \in C}$ where $a_{ic} \in \{0, 1\}$ shows whether individual i is a member of club c. Let A be the set of feasible club membership structures, we know $A \subset \{0, 1\}^{n \times m}$.

We define a few notions based on a club membership structure a.

The *degree* of individual i given a club membership structure a measures the number of clubs individual i joins:

$$d_i(a) = \sum_{c \in C} a_{ic}$$

The *membership size* of club c given a club membership structure a captures the number of individuals that join c:

$$s_c(a) = \sum_{i \in I} a_{ic}$$

The *membership overlap* between a club c and a club c' given a club membership structure a represents the number of individuals that join both c and c':

$$o_{cc'}(a) = \sum_{i \in I} a_{ic} a_{ic'}$$

We make the following assumptions based on a club membership structure a.

Assumption 1. *A link between two clubs c and c' is created if they have membership overlap, and the link strength is non-decreasing in the membership overlap. We assume a link formation function $g_{cc'} : A \mapsto \mathbb{R}$ for club c and c' that takes the form:*

$$g_{cc'}(a) = h(o_{cc'}(a)) \tag{3.1}$$

where function $h(x)$ satisfies $h(0) = 0$ and $h'(x) \geq 0$.

Assumption 2. *A club provides goods and services and its productivity depends on its membership size and its aggregate link strength with other clubs. We assume a production* *function $\pi_c : A \mapsto \mathbb{R}$ for club c that takes the form:*

$$\pi_c(a) = F(s_c(a)) + \sum_{c' \neq c} g_{cc'}(a) \tag{3.2}$$

where for function $F(x)$, we assume that $F(0) = 0$ and that there is a positive integer S such that the marginal benefit of membership size on productivity $\Delta F(x) = F(x+1) - F(x)$ is positive when $x \leq S-1$ and negative when $x \geq S$, so S is the membership size that would maximize the productivity of a club if the aggregate link strength factor is not present. The assumption that club productivity is first rising and then decreasing in membership size is standard in club theory literature.

Note that under Assumption 2, admitting an individual as a new member affects the productivity of a club in two ways. First, the membership size grows. Second, depending on what other clubs the individual joins, the club creates or strengthens links with other clubs. We call the latter effect *connection externality*.

Assumption 3. *An individual enjoys benefits from clubs she joins and pays costs for club memberships. We assume a utility function $u_i : A \mapsto \mathbb{R}$ for individual i that takes the form:*

$$u_i(a) = \sum_{c \in C} a_{ic} \pi_c(a) - \kappa(d_i(a)) \tag{3.3}$$

where for function $\kappa(x)$, we assume that $\kappa(0) = 0$ and that the marginal club membership cost $\Delta\kappa(x) = \kappa(x+1) - \kappa(x)$ is non-negative and non-decreasing in x.

A *club membership model* can be represented by a tuple (I, C, A, g, π, u) where I and C specifies the individuals and clubs in the model respectively, A specifies the set of feasible club membership structures, $g = (g_{cc'})_{c,c' \in C}$ summarizes the link formation functions for pairs of clubs, $\pi = (\pi_c)_{c \in C}$ summarizes the production functions for clubs, and $u = (u_i)_{i \in I}$ summarizes the utility functions for individuals.

We define the efficient club membership structure of a club membership model as a club membership structure a that maximizes the aggregate utility of individuals. Formally,

Definition 3.1. *A club membership structure $a \in A$ is efficient if there is no club membership structure $a' \in A$ such that*

$$\sum_{i \in I} u_i(a') > \sum_{i \in I} u_i(a)$$

We define the stable club membership structure of a club membership model as a club membership structure a that satisfies (i) no individual i can increase her utility by quitting any clubs she is currently in, (ii) no club c can enhance its productivity by exiling any members it currently has, and (iii) no pair of individual i and club c can both benefit from a joint deviation where i is allowed to quit any clubs she is in, c is allowed to exile any members it has and i can join c. Formally,

Definition 3.2. *A club membership structure $a \in A$ is stable if*

(i) $\forall i \in I$: there is no $a' \in A$ with

$$\begin{aligned} a'_{ic} &\leq a_{ic} && \forall c \in C \\ a'_{i'c} &= a_{i'c} && \forall i' \neq i, c \in C \end{aligned}$$

such that

$$u_i(a') > u_i(a)$$

(ii) $\forall c \in C$: there is no $a' \in A$ with

$$\begin{aligned} a'_{ic} &\leq a_{ic} && \forall i \in I \\ a'_{ic'} &= a_{ic'} && \forall c' \neq c, i \in I \end{aligned}$$

such that

$$\pi_c(a') > \pi_c(a)$$

(iii) $\forall (i,c) \in I \times C$: there is no $a' \in A$ with

$$\begin{aligned} a'_{ic'} &\leq a_{ic'} && \forall c' \neq c \\ a'_{i'c} &\leq a_{i'c} && \forall i' \neq i \\ a'_{i'c'} &= a_{i'c'} && \forall i' \neq i, c' \neq c \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

We now give a few specifications for a club membership model.

The ***capacity constraints model*** refers to a club membership model (I, C, A, g, π, u) where $A = \{0,1\}^{n \times m}$, Assumptions 2 and 3 are satisfied, and

(i)

$$F(x) = \begin{cases} f(x) & \text{when } x \leq S \\ f(S) - M_1(x - S) & \text{when } x > S \end{cases}$$

where $f(0) = 0$, $f'(x) > 0$, $1 < S < n$, and M_1 is a number that is sufficiently large such that when a club has at least S members, any rise in its membership size reduces its productivity sufficiently so that no club should admit more than S members, and

(ii)

$$\kappa(x) = \begin{cases} 0 & \text{when } x \leq D \\ M_2(x - D) & \text{when } x > D \end{cases}$$

where $1 < D < m$ and M_2 is a number sufficiently large such that when an individual joins at least D clubs, any rise in her degree reduces her utility sufficiently so that no individual should join more than D clubs.

By focusing on the capacity constraints model, we essentially impose a constraint S on the number of members a club can admit and a constraint D on the number of clubs an individual can join. Note that since there are n individuals in the model, nD measures the aggregate membership availability (from the side of individuals), and since there are m clubs in the model, mS measures the aggregate club capacity (from the side of clubs).

The model with ***constant returns from membership overlap*** is one in which Assumption 1 is satisfied and $h'(x) = \alpha$ with $\alpha > 0$.

The model with ***zero returns from membership overlap*** is one in which Assumption 1 is satisfied and $h'(x) = 0$. This is equivalent to assuming $g_{cc'}(a) = 0$ for all $c, c' \in C$ and for all $a \in A$ and hence is a model where there is no connection externality.

In our main results section, we characterize the efficient and the stable club membership structures for the capacity constraints model with constant returns from membership overlap and with zero returns from membership overlap respectively. By comparing the results under the two settings, we explore the role of connection externality in a club membership structure.

After that, we extend our analysis to richer setups. First, instead of assuming constant or zero returns from membership overlap, we discuss how *increasing returns from membership overlap* and *decreasing returns from membership overlap* would affect club membership structures. Second, moving away from the capacity constraints model, we investigate a *smooth club membership model* where individuals and clubs can make flexible decisions on how many clubs to join and how many members to admit.

3.3 Main Results

In this section, we first characterize the efficient and the stable club membership structures for the capacity constraints model with constant returns from membership overlap.

We show that when aggregate membership availability exceeds aggregate club capacity ($nD > mS$), a club membership structure is efficient when all clubs are full with S members and all but one individuals are separated into two groups, those from one join D clubs while those from another join no clubs.[3] And when aggregate club capacity exceeds aggregate membership availability ($mS > nD$), a club membership structure is efficient when all individuals are fully affiliated with D clubs and all but one clubs are separated into two groups, those from one admit S members while those from another admit no clubs.[4] So, in an efficient club membership structure, club memberships concentrate either in a group of individuals or in a group of clubs.

For stability, we have a similar finding. We partition individuals into four groups based on their degree and the productivity of the clubs they join, and we partition clubs into three groups based on their productivity which is determined by their membership size and the degree of their members. We find that in a stable club membership structure, most individuals belong either to the first individual group where they join D most productive clubs or to the last individual group where they join no clubs, and most clubs belong either to the first club group where they admit S highest-degree members or to the last club group where they admit no members. Since the individuals and clubs in the first group obtain the highest possible utility and productivity in the model while individuals and clubs in the last group obtain zero utility and productivity, there can be large welfare gaps in a stable club membership structure.

We then characterize the efficient and the stable club membership structures for the capacity constraints model with zero returns from membership overlap. We show that any club membership structure that is efficient or stable under the constant returns from membership overlap assumption is also efficient or stable under the zero returns from membership overlap assumption, but not vice versa. In particular, consider the case when aggregate membership availability exceeds aggregate club capacity, if there is zero returns from membership overlap, then a club membership structure where all clubs are full and club memberships are equally allocated to all individuals is both efficient and stable. This even structure is not efficient nor stable when there is constant returns from membership overlap.

So, our finding that efficiency and stability imply concentrated club memberships and welfare gaps is due to the connection externality.

We now formally state our results.

[3] There can be one individual who joins some but less than D clubs due to the discrete nature of our model.

[4] Again, there can be one club that admits some but less than S members due to the discreteness of our set-up.

We use the standard notation of $\lfloor \frac{x}{y} \rfloor$ to represent the floor of x divided by y and the notation of $x \bmod y$ to represent the reminder of x divided by y.

Proposition 3.1. *Consider the capacity constraint model with constant returns from membership overlap. If $nD \geq mS$, then a club membership structure $a \in A$ is efficient if and only if all clubs have S members, and there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs. If $mS \geq nD$, then a club membership structure $a \in A$ is efficient if and only if all individuals join D clubs, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs that admit D members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.*

Proposition 3.1 tells us that when $nD \geq mS$, we maximize aggregate utility by fully filling all clubs and concentrating club memberships to a group of individuals, and when $mS \geq nD$, we maximize aggregate utility by exhausting membership availability of all individuals and concentrating club memberships to a group of clubs.

The intuition behind Proposition 3.1 is simple. First, we know that within the capacity constraints, the productivity of a club always rises with its membership size and the utility of an individual always rises with its degree, so an efficient club membership structure must use up all club capacity by filling all clubs with S members when $nD \geq mS$ and use up all membership availability by letting all individuals join D clubs when $mS \geq nD$. The next question is what individuals should clubs admit and what clubs should individuals join. With constant returns from membership overlap, if an individual joins more clubs, she creates greater membership overlap and raises the productivity of clubs and hence the utility of individuals more. Therefore, efficiency is achieved when club memberships concentrate on a group of individuals so that they have large degrees. Given that clubs produce local public goods whose value and range of beneficiary both rise with its membership size, efficiency is achieved when club memberships concentrate on a group of clubs so that they have large membership size.

We depict three efficient structures when $nD \geq mS$ in Example 3.1 and one efficient structure when $mS \geq nD$ in Example 3.2.

Example 3.1. *Consider the capacity constraints model with 12 individuals and 6 clubs where individuals can join up to 3 clubs and clubs can admit up to 4 members ($n = 12$, $D = 3$, $m = 6$ and $S = 4$). Assume constant returns from membership overlap, the three club membership structures depicted in Figure 3.1 are all efficient:*

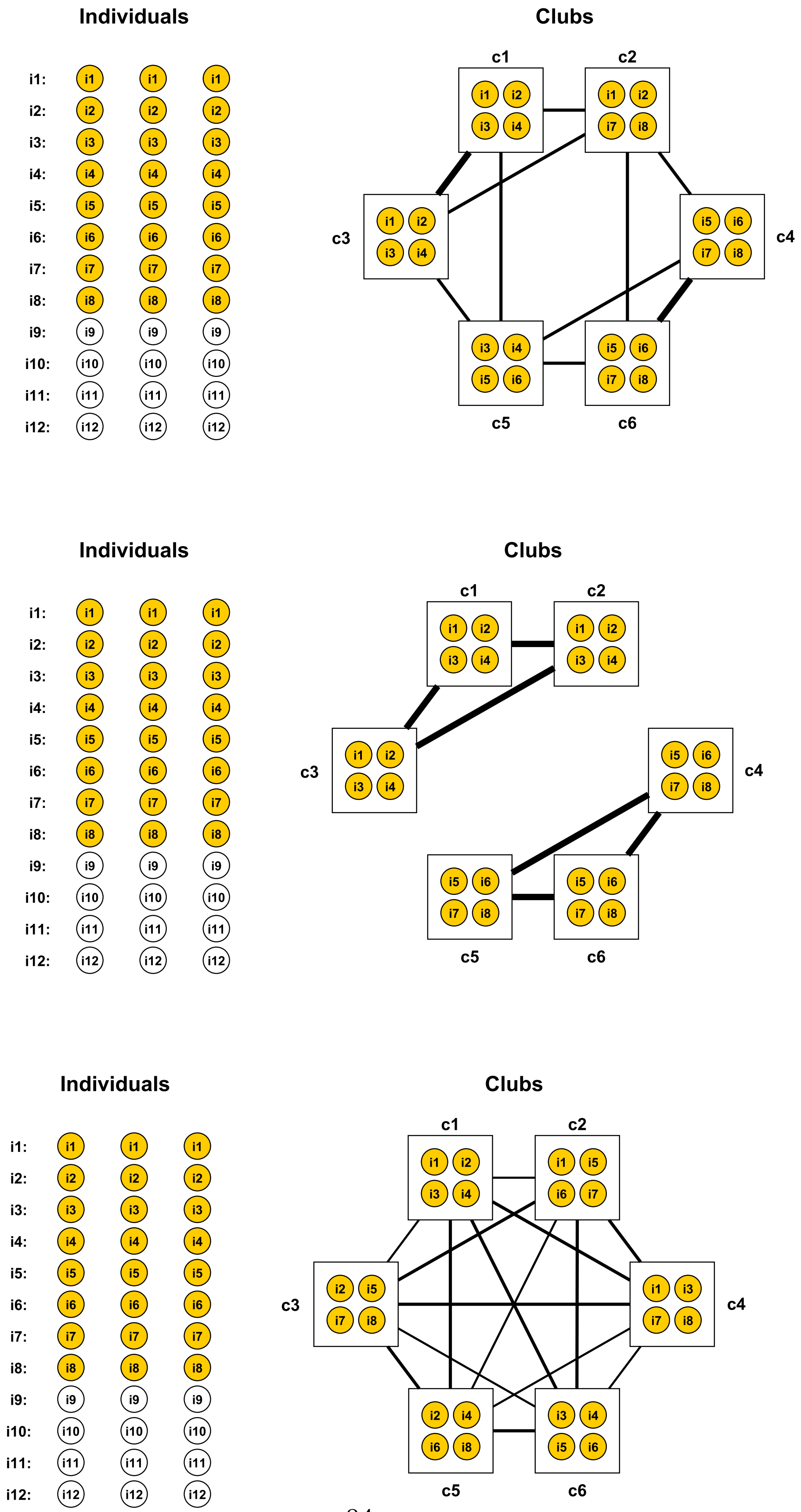


Figure 3.1 Three efficient club membership structures with constant returns from membership overlap ($nD \geq mS$)

The three graphs in Figure 3.1 can be read in the following way. On the left-hand side of each graph, we have the 12 individuals. Each individual has three nodes representing her membership availability. If the odd is solid, it indicates a club membership for the individual. If the node is empty, it indicates an unused membership availability. On the right-hand side of the graph, we have the 6 clubs. Each club has four nodes representing its club capacity. Each solid node with the name of an individual indicates that the club admits that individual and each empty node (which we do not see in this example) indicates an unused club capacity. There are links between clubs, the thickness of which captures the link strength induced by the club membership structure.

In all three club membership structures in Figure 3.1, we can see all clubs are full with 4 members, and 8 out of the 12 individuals use up their membership availability while the other 4 individuals join no clubs. The difference between the three club membership structures lies in their induced club network. The middle structure leads to a club network with strongly connected cliques, the bottom structure leads to a club network with weakly connected complete graph, and the top structure leads to something in between. We will discuss this difference in our extension section.

Example 3.2. *Consider the capacity constraints model with 5 individuals and 6 clubs where individuals can join up to 3 clubs and clubs can admit up to 4 members ($n = 5$, $D = 3$, $m = 6$ and $S = 4$). Assume constant returns from membership overlap, the club membership structure depicted in Figure 3.2 is efficient:*

In this club membership structure, we can see all individuals are fully affiliated with 3 club memberships, the first 3 clubs are full with 4 members, club c_4 has 3 members, and the last 2 clubs have no members. Club memberships are concentrated to clubs c_1 to c_4.

For the stability characterization when there is constant returns from membership overlap, we first derive the highest productivity a club can achieve in the model and define a way to partition individuals and clubs based on it. We then characterize the stable club membership structure with that partition.

Let π^* be the highest productivity a club can achieve. In the capacity constraint model with constant returns from membership overlap, the productivity of a club c under a club

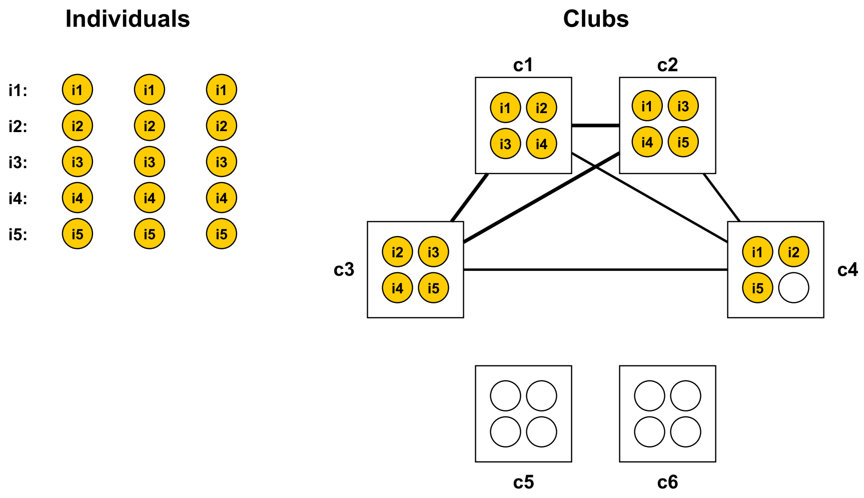


Figure 3.2 An efficient club membership structures with constant returns from membership overlap and ($mS \geq nD$)

membership structure a is

$$\begin{aligned}
\pi_c(a) =& F(s_c(a)) + \sum_{c' \neq c} g_{cc'}(a) \\
=& \alpha \sum_{c' \neq c} \sum_{i \in I} a_{ic} a_{ic'} + F(s_c(a)) \\
=& \begin{cases} \alpha \sum_{i \in I} a_{ic}(d_i(a) - 1) + f(s_c(a)) & \text{when } s_c(a) \leq S \\ \alpha \sum_{i \in I} a_{ic}(d_i(a) - 1) + f(S) - M_1(s_c(a) - S) & \text{when } s_c(a) > S \end{cases}
\end{aligned} \tag{3.4}$$

From expression (3.4), it is easy to see that here

$$\pi^* = \alpha S(D-1) + f(S)$$

and a club achieves π^* when it admits S members, all of whom join D clubs.

We define the following *individual degree-club productivity* partition for a club membership structure a. Individuals are categorized into four groups, where the first group $I_1(a)$ consists of individuals who join D clubs and all clubs they join achieve productivity π^*, the second group $I_2(a)$ consists of individuals who join D clubs and the productivity of some clubs they join is less than π^*, the third group $I_3(a)$ consists of individuals who join some but less than D clubs, and the fourth group $I_4(a)$ consists of individuals who join no clubs:

$$I_1(a) = \{i \in I : d_i(a) = D \text{ and } \forall c \in C \text{ with } a_{ic} = 1, \pi_c(a) = \pi^*\}$$
$$I_2(a) = \{i \in I : d_i(a) = D \text{ and } \exists c \in C \text{ with } a_{ic} = 1 \text{ s.t. } \pi_c(a) < \pi^*\}$$
$$I_3(a) = \{i \in I : 0 < d_i(a) < D\}$$
$$I_4(a) = \{i \in I : d_i(a) = 0\}$$

Observe that in a club membership structure a where all individuals join at most D clubs, the set of individuals I can be partitioned into the four groups $I_1(a)$, $I_2(a)$, $I_3(a)$, and $I_4(a)$. Also note that if $i \in I_1(a)$, then $u_i(a) = D\pi^*$ which is the highest utility an individual can obtain. And if $i \in I_4(a)$, then $u_i(a) = 0$.

Clubs are categorized into three groups, where the first group $C_1(a)$ consists of clubs with productivity π^*, the second group $C_2(a)$ consists of clubs with positive productivity that is less than π^*, and the third group $C_3(a)$ consists of clubs with zero productivity:

$$C_1(a) = \{c \in C : \pi_c(a) = \pi^*\}$$
$$C_2(a) = \{c \in C : 1 < \pi_c(a) < \pi^*\}$$
$$C_3(a) = \{c \in C : \pi_c(a) = 0\}$$

Observe that in a club membership structure a where all clubs admits at most S members, the set of clubs C can be partitioned into the three groups $C_1(a)$, $C_2(a)$, and $C_3(a)$.

By definition of the individual degree-club productivity partition, we can see if an individual is in $I_1(a)$, she joins D clubs in $C_1(a)$; if a club is in $C_1(a)$, it admits S members that belong to $I_1(a)$ or $I_2(a)$; and if a club is in $C_3(a)$, it admits no members.

We are now ready to characterize the stable club membership structures.

Proposition 3.2. *Consider the capacity constraints model with constant returns from membership overlap, a club membership structure $a \in A$ is stable if and only if:*
(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,
(ii) $\forall i \in I_2(a), c, c' \in C_2(a)$: if $a_{ic} > a_{ic'}$, then $\pi_c(a) > \pi_{c'}(a)$,
(iii) $\forall i, i' \in I_3(a)$: if $d_i(a) \geq d_{i'}(a)$, then $a_{ic} \geq a_{i'c}$ $\forall c \in C$,
(iv) $\forall c, c' \in C_2(a)$: if $\pi_c(a) \geq \pi_{c'}(a)$, then $a_{ic} \geq a_{ic'}$ $\forall i \in I_2(a)$ and $a_{ic} \leq a_{ic'}$ $\forall i \in I_3(a)$.

Proposition 3.2 tells us that a club membership structure is stable if and only if the following conditions are satisfied. First, since we are considering the capacity constraint model, all individuals join at most D clubs and all clubs admit at most S members. Also, there cannot exist a pair of individual and club where the individual is not fully affiliated with D clubs and the club is not full with S members, because otherwise they would both benefit if the individual joins the club. Then, we have some club membership requirements for individuals and clubs that belong to different groups. By definition of the individual degree-club productivity partition, we already know the club joining pattern of individuals in $I_1(a)$ and $I_4(a)$ and the member admission pattern of clubs in $C_1(a)$ and $C_3(a)$. The proposition specifies how individuals in $I_2(a)$ and $I_3(a)$ join clubs and how clubs in $C_2(a)$ admit members in a stable club membership structure.

First, for any individual i in $I_2(a)$, we know she joins D clubs and some clubs she joins do not achieve productivity π^*. So she must join some clubs from $C_2(a)$. Our proposition says that the clubs i joins in $C_2(a)$ must be the most productive clubs in $C_2(a)$, because otherwise i would want to join a club in $C_2(a)$ with higher productivity instead and the club would also be willing to substitute another member with i given the high degree of i.

Second, for any two individuals i and i' in $I_3(a)$, we know they join some but less than D clubs. Our proposition says that if i joins at least as many clubs as i', then i joins all the clubs i' is in. Suppose this is not the case and there is a club c that admits i' but not i, then i would want to join this club since she has not used up her membership availability, and c would want to substitute i' with i since i would have a higher degree than i' with her new membership in c and creates larger connection externality for c.

Finally, for any two clubs c and c' in $C_2(a)$, since they do not achieve productivity π^*, they do not have members from $I_1(a)$ and can only have members from $I_2(a)$ and $I_3(a)$. Our proposition says that if c is at least as productive as c', then c has all the members c' has from $I_2(a)$ and c' has all the members c has from $I_3(a)$. The reason that c should have all the members c' has from $I_2(a)$ is that if not, the individual from $I_2(a)$ that is in c' but not c would want to quit c' and join c which would have a higher productivity with her join. The reason why c' should have all the members c has from $I_3(a)$ is that individuals in $I_3(a)$ want to join any clubs that are willing to admit then since they have not used up their membership availability and that c must have a stricter selection criteria for members from $I_3(a)$ than c' since c has more members from $I_2(a)$ and thus fewer spots for members from $I_3(a)$.

The above reasoning explains how we prove the necessity of the four conditions (i), (ii), (iii), and (iv). We illustrate in the proof that the four conditions also guarantee a stable club membership structure.

Example 3.3 depicts a stable club membership structure when there is constant returns from membership overlap.

Example 3.3. *Consider the same capacity constraints model as in Example 3.1 where there are 12 individuals and 6 clubs where individuals can join up to 3 clubs and clubs can* *admit up to 4 members ($n = 12$, $D = 3$, $m = 6$ and $S = 4$). Assume constant returns from membership overlap, the club membership structure depicted in Figure 3.3 is stable:*

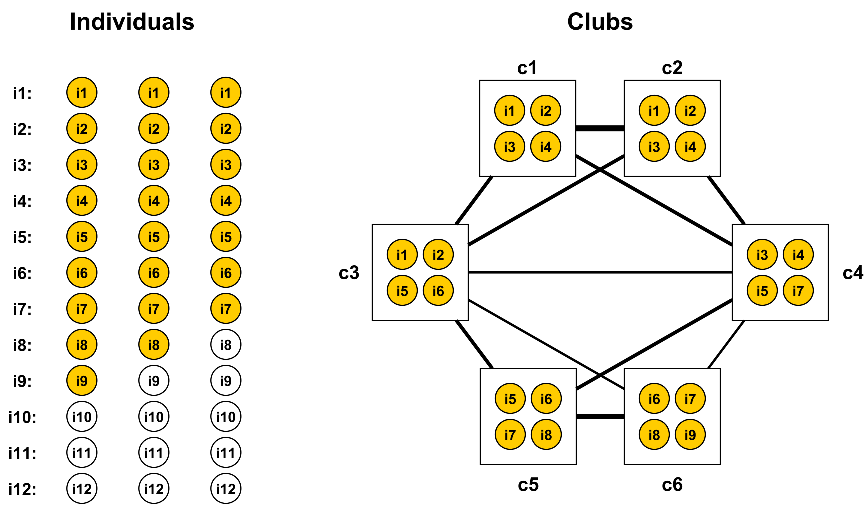


Figure 3.3 A stable club membership structure with constant returns from membership overlap ($nD \geq mS$)

In this example, we have $I_1(a) = \{i_1, i_2, i_3, i_4\}$, $I_2(a) = \{i_5, i_6, i_7\}$, $I_3(a) = \{i_8, i_9\}$, $I_4(a) = \{i_{10}, i_{11}, i_{12}\}$, $C_1(a) = \{c_1, c_2, c_3, c_4\}$, $C_2(a) = \{c_5, c_6\}$, and $C_3(a) = \emptyset$.

An important implication of Proposition 3.2 is that we can derive lower bounds for the size of the first and last individual group $I_1(a) \cup I_4(a)$ and the size of the first and last club group $C_1(a) \cup C_3(a)$ for a stable club membership structure a.

Corollary 3.1. *Consider the capacity constraint model with constant returns from membership overlap, if a club membership structure $a \in A$ is stable, then*

$$|I_1(a) \cup I_4(a)| \geq n - 2S + 1$$
$$|C_1(a) \cup C_3(a)| \geq m - D$$

We derive the corollary by restricting the cardinality of $I_2(a)$, $I_3(a)$ and $C_2(a)$. First, since all individuals in $I_2(a)$ join some clubs in $C_2(a)$ and the clubs they join always have higher utility than the clubs they do not join according to condition (ii) of Proposition 3.2, the most productive club in $C_2(a)$ admits all individuals in $I_2(a)$. This most productive club cannot admit D members from $I_2(a)$ since it would achieve productivity π^* and belong to $C_1(a)$ if it does. So $|I_2(a)| \leq S - 1$. Second, we know from the definition of $C_1(a)$ and

$C_3(a)$ that individuals in $I_3(a)$ can only join clubs in $C_2(a)$. According to condition (iv) of Proposition 3.2, if we find the least productive club in $C_2(a)$, the set of its members from $I_3(a)$ is a superset of members other clubs have from $I_3(a)$. Given the capacity constraint, we know $|I_2(a)| \leq S$. Finally, we derive the upper bound for $C_2(a)$. When set $I_3(a)$ is not empty, according to condition (i) of Proposition 3.2, all clubs in $C_2(a)$ have S members. Since clubs in $C_2(a)$ do not achieve π^*, they must admit members from $I_3(a)$. Then, according to condition (iii) of Proposition 3.2, the individual with highest degree in $I_3(a)$ joins all clubs in $C_2(a)$, so $C_2(a) \leq D-1$. When set $I_3(a)$ is empty, all members clubs in $C_2(a)$ have are from $I_2(a)$. According to condition (iv) of Proposition 3.2, a member the least productive club in $C_2(a)$ has is also a member in other clubs in $C_2(a)$. Since an individual cannot join more than D clubs, $C_2(a) \leq D$. With the upper bound for $I_2(a)$, $I_3(a)$ and $C_2(a)$, it is obvious how we arrive at the lower bound for $|I_1(a) \cup I_4(a)|$ and $|C_1(a) \cup C_3(a)|$.

We know that individuals from $I_1(a)$ join D clubs and earn highest utility possible in the model while individuals from $I_4(a)$ join no clubs and earns zero utility and that clubs from $C_1(a)$ admit S members and achieve highest utility possible in the model while clubs from $C_3(a)$ admit no members and have zero productivity. So, in a large society where $n >> 2S$ and $m >> D$, the corollary tells us that individuals and clubs can be divided to two extreme groups with distinct activeness and welfare level.

Now we turn to the model with zero returns from membership overlap. Proposition 3.3 characterizes the efficient club membership structure.

Proposition 3.3. *Consider the capacity constraints model with zero returns from membership overlap. If $nD \geq mS$, then a club membership structure $a \in A$ is efficient if and only if all clubs have S members. If $mS \geq nD$, then a club membership structure $a \in A$ is efficient if and only if all individuals join D clubs, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs that admit D members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.*

Compare Proposition 3.3 with Proposition 3.1. We can see that when $mS \geq nD$, the efficiency characterization is the same under constant returns from membership overlap and under zero returns from membership overlap. This is because the intuition we explained for Proposition 3.1 that individuals should use up their membership availability and club memberships should concentrate on a group of clubs since the productivity and the range of beneficiary of a club both rise with its membership size still holds when there is zero returns from membership overlap. However, when $nD \geq mS$, the condition for efficiency when there is zero returns from membership overlap is weaker than the corresponding condition when there is constant returns from membership overlap. This is because we still want to use up club capacity but the force that drives clubs to admit high-degree members for greater connection externality is absent.

Moving to stability, our characterization for the model with zero returns from membership overlap still builds upon the individual degree-club productivity partition. Note that now for a club to achieve π^*, it only needs to admit S members. So if a club c is in $C_1(a)$, it admits S members from $I_1(a)$, $I_2(a)$ or $I_3(a)$. This is a relaxation of our previous requirement that a club in $C_1(a)$ should admit S members from $I_1(a)$ or $I_2(a)$ for the case when there is constant returns from membership overlap.

Proposition 3.4 characterizes the stable club membership structures and further illustrates that stability under zero returns from membership overlap is less demanding than stability under constant returns from membership overlap.

Proposition 3.4. *Consider the capacity constraints model with zero returns from membership overlap, a club membership structure $a \in A$ is stable if and only if:*

(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I_2(a), c, c' \in C_2(a)$: if $a_{ic} > a_{ic'}$, then $\pi_c(a) > \pi_{c'}(a)$.

Compare Proposition 3.4 with Proposition 3.2, we can see that the conditions for stability when there is zero returns from membership overlap is a subset of the conditions for stability when there is constant returns from membership overlap. So if a club membership structure is stable under constant returns from membership overlap, it must also be stable under zero returns from membership overlap.

We have shown that for both efficiency and stability, the model with zero returns from membership overlap admits a wider range of club membership structures than the model with constant returns from membership overlap. In fact, there can be club membership structures that are very different from any efficient or stable club membership structure under constant returns from membership overlap but are efficient and stable under zero returns from membership overlap. We show this with Example 3.4.

Example 3.4. *Consider the same capacity constraints model as in Example 3.1 where there are 12 individuals and 6 clubs where individuals can join up to 3 clubs and clubs can admit up to 4 members ($n = 12$, $D = 3$, $m = 6$ and $S = 4$). Assume zero returns from membership overlap, the club membership structure depicted in Figure 3.4 is efficient and stable:*

In this club membership structure, all clubs are full with 4 members. This is the same as what we have in Example 3.1 and Example 3.3 where we demonstrate three efficient club membership structures and one stable club membership structure when there is constant returns from membership overlap. However, all club memberships are allocated equally to all individuals and all individuals have the same utility. This departs from the exclusiveness and the welfare gap we show for efficient and stability when there is constant returns from membership overlap.

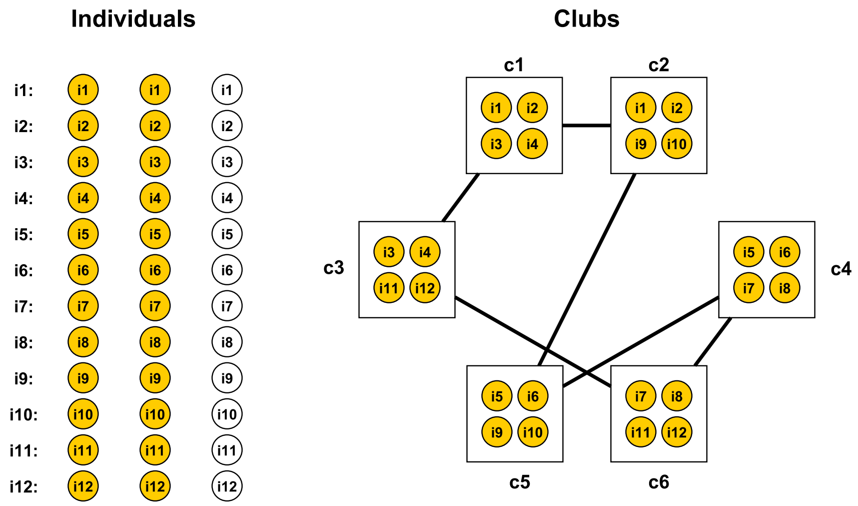


Figure 3.4 An efficient and stable club membership structure with zero returns from membership overlap and $(nD \geq mD)$

3.4 Extensions

3.4.1 Increasing/Decreasing Returns from Membership Overlap

Proposition 3.1-3.2 separates individuals into well connected and unconnected but are relatively permissive in terms of club networks (see Figure 3.1-3.2). In this section, we show with an example how the curvature of return from membership overlap would have prominent effects on the efficient and stable club memberships.

We say a club membership model (I, C, A, g, π, u) has ***increasing returns from membership overlap*** if Assumption 1 is satisfied and $h''(x) > 0$. We say a club membership model (I, C, A, g, π, u) has ***decreasing returns from membership overlap*** if Assumption 1 is satisfied and $h''(x) < 0$.

We use the following example to discuss the impact of having increasing or decreasing returns from membership overlap.

Example 3.5. *Consider the same capacity constraints model as in Example 3.1 where there are 12 individuals and 6 clubs where where individuals can join up to 3 clubs and clubs can admit up to 4 members ($n = 12$, $D = 3$, $m = 6$ and $S = 4$).*

When there is increasing returns from membership overlap, the club membership structure depicted in Figure 3.5 is efficient, and it is the unique efficient club membership structure in the sense that an efficient club membership structure must have clubs admit members in

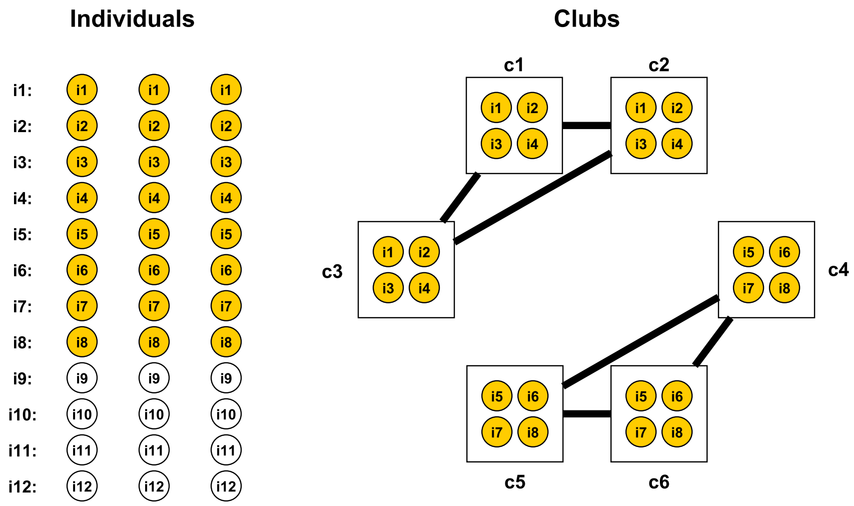


Figure 3.5 The unique efficient club membership structure with increasing returns from membership overlap ($nD \geq mS$)

a way so that each three clubs for a clique with the strongest links possible. Observe that the club membership structure in Figure 3.5 is the same as the club membership structure depicted in the middle graph of Figure 3.1, so this structure is also efficient under constant returns from membership overlap. But the other two club membership structures in Figure 3.1 are not efficient when there is increasing returns from membership overlap. We can see that the increasing returns from membership overlap encourage clubs to build stronger links with fewer clubs and refines the set of efficient club membership structures in this direction.

When there is decreasing returns from membership overlap, the club membership structure depicted in Figure 3.6 is efficient, and it is the unique efficient club membership structure in the sense that an efficient club membership structure must have clubs admit members in a way so that all clubs are linked to each other and the strength of links are almost the same (the difference in the membership overlap between any pairs of clubs is not greater than one). Observe that the club membership structure in Figure 3.5 is the same as the club membership structure depicted in the bottom graph of Figure 3.5, so this structure is also efficient under constant returns from membership overlap. But the other two club membership structures in Figure 3.1 are not efficient when there is decreasing returns from membership overlap. We can see that the decreasing returns from membership overlap encourage clubs to build weaker links with more clubs and refines the set of efficient club membership structures in this direction.

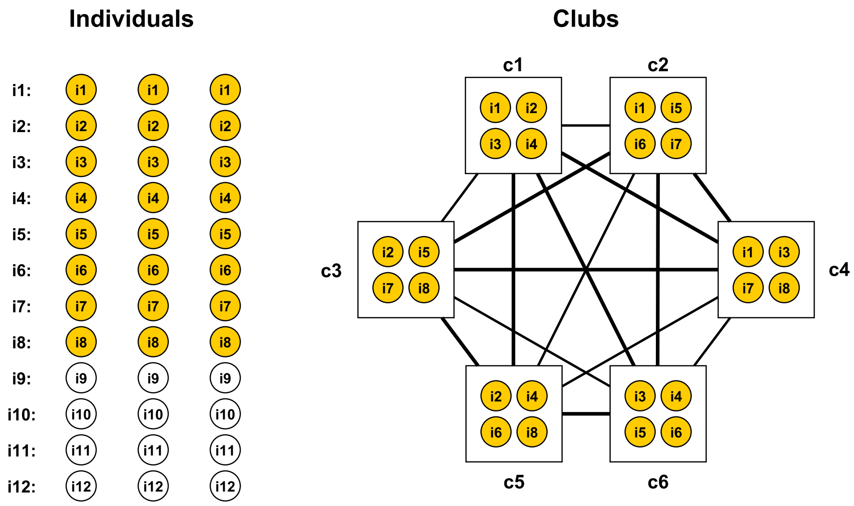


Figure 3.6 The unique efficient club membership structure with decreasing returns from membership overlap ($nD \geq mS$)

We have a similar finding for stability. While all the three club membership structures in Figure 3.1 are also stable under constant returns from membership overlap, when there is increasing returns from membership overlap, the structure in Figure 3.5 is the only stable one out of the three, and when there is decreasing returns from membership overlap, the structure in Figure 3.6 is the only stable one out of the three.

3.4.2 The Smooth Club Membership Model

In the capacity constraints model, an individual wants to join as many clubs as possible but faces a constraint on her degree, and a club wants to admit as many members as possible but has a constraint on its membership size. In a more general setting, individuals and clubs make flexible club joining and member admission decisions by comparing the marginal benefit and the marginal cost that change smoothly with their degree and membership size. We show that Propositions 3.1-3.2 on efficiency and stability of club membership structures are robust to this generalization

Consider a ***smooth club membership model*** that refers to a model (I, C, A, g, π, u) where Assumptions 2 and 3 are satisfied, $A = \{0, 1\}^{n \times m}$, and

(i) the marginal effect of club size on productivity $\Delta F(x) = F(x+1) - F(x)$ is dropping in club size:

$$\frac{d\Delta F(x)}{dx} < 0$$

and that $\Delta F(n-1)$ is sufficiently negative such that a club should never admit all n individuals.

(ii) the marginal club joining cost $\Delta\kappa(x)$ is rising at an increasing rate:

$$\frac{d\Delta\kappa(x)}{dx} > 0$$
$$\frac{d^2\Delta\kappa(x)}{dx^2} > 0$$

and that $\Delta\kappa(1)$ is sufficiently small such that an individual should always join multiple clubs and $\Delta\kappa(m-1)$ is sufficiently large such that an individual should never join all m clubs.

We have the following results for this model with constant returns from membership overlap.

For efficiency, let s_1^*, d_1^* be the solution to the maximization problem (3.5) and s_2^*, d_2^* be the solution to the maximization problem (3.6):

$$\max_{d\in\{1,2,\dots,m\},s\in\{1,2,\dots,n\}} (\alpha sd - \alpha s + F(s) - \frac{\kappa(d)}{d})s \tag{3.5}$$

$$\max_{d\in\{1,2,\dots,m\},s\in\{1,2,\dots,n\}} (\alpha sd - \alpha s + F(s) - \frac{\kappa(d)}{d})d \tag{3.6}$$

Remark 3.1. *Consider the smooth club membership model with constant returns from membership overlap. If $nd_1^* \geq ms_1^*$ and d_1^* divides ms_1^*, then a club membership structure a is efficient if and only if all clubs admits s_1^* members, and there are $\frac{ms_1^*}{d_1^*}$ individuals that join d_1^* clubs and the remaining individuals join no clubs. If $ms_2^* \geq nd_2^*$ and s_2^* divides nd_2^*, then a club membership structure a is efficient if and only if all individuals join d_2^* clubs, and there are $\frac{nd_2^*}{s_2^*}$ clubs that admit s_2^* members and the remaining clubs admit no members.*

The above remark demonstrates that in certain non-trivial cases, efficiency indicates the concentration of club membership to one group of individuals/clubs and the exclusion to club activities of the other group. This is consistent with our finding for the capacity constraints model.

For stability, define two functions $\hat{s}(d)$ and $\hat{\pi}(d)$ where $\hat{s}(d)$ is the solution to and $\hat{\pi}(d)$ is the outcome of the following maximizing problem:

$$\max_{s\in\mathbb{R}} F(s) + \alpha s(d-1)$$

In the smooth club membership model with constant returns from membership overlap, the productivity of a club is

$$\begin{aligned}\pi_c(a) =& F(s_c(a)) + \alpha \sum_{c' \neq c} o_{cc'}(a) \\ =& F(s_c(a)) + \alpha \sum_{i \in I} a_{ic}(d_i(a) - 1)\end{aligned}$$

So, if all members of a club c have degree d, then

$$\pi_c(a) = F(s_c(a)) + \alpha s_c(a)(d-1)$$

Therefore, $\hat{s}(d)$ captures the optimal membership size for a club when all its members have degree d, and $\hat{\pi}(d)$ captures the highest productivity of a club when all its members have degree d. For simplicity, we assume $\hat{s}(d) \in \mathbb{Z}$ when $d \in \mathbb{Z}$.

Consider the following two conditions on d:

$$\begin{cases} \Delta\kappa(d) > \hat{\pi}(d) + \alpha(d+1) \\ \Delta\kappa(d-1) < \hat{\pi}(d) + \alpha(d-1) \end{cases} \tag{3.7}$$

Note that for a club, if all its members have degree d and it achieves productivity $\hat{\pi}(d)$, then it has no incentive to admit more members with degree d and it has no incentive to exile any existing members.

For an individual with degree d, we show in the appendix that $\hat{\pi}(d) + \alpha(d+1)$ is the maximum additional benefit she can obtain by joining another club that has all its members with degree d and achieves productivity $\hat{\pi}(d)$, and $\hat{\pi}(d) + \alpha(d-1)$ is the reduction in the benefit she suffers by quitting a club that has all its members with degree d and achieves productivity $\hat{\pi}(d)$. We also know for an individual with degree d, $\Delta\kappa(d)$ and $\Delta\kappa(d-1)$ are the marginal cost she pays by joining another club and the marginal cost she saves by quitting a club respectively. So, if the two conditions in (3.7) hold, for an individual with degree d, she has no incentive to join or quit a club that has $\hat{s}(\hat{d})$ members with degree $\hat{d}$.

Therefore, an integer solution $\hat{d}$ to the two inequities in (3.7) and the associated $\hat{s} = \hat{s}(\hat{d})$ suggest a stable club membership structure for a group of individuals and clubs where all individuals in the group join $\hat{d}$ clubs in the group and all clubs in the group admit $\hat{s}$ individuals in the group.

Let $\{\hat{d}^1, \hat{d}^2, ..., \hat{d}^k\}$ with $\hat{d}^1 > \hat{d}^2 > ... > \hat{d}^k$ be the set of positive integers that satisfy the inequities in (3.7) and let $\{\hat{s}^1, \hat{s}^2, ..., \hat{s}^k\} = \{\hat{s}(\hat{d}^1), \hat{s}(\hat{d}^2), ..., \hat{s}(\hat{d}^k)\}$. For a club membership structure a, we define the following $k+1$ layers of individuals and clubs.

For $j \in \{1,2,...,k\}$, layer j contains the following subset of individuals and clubs:

$$\hat{I}_j(a) = \{i \in I : d_i(a) = \hat{d}^j\}$$
$$\hat{C}_j(a) = \{c \in C : s_c(a) = \hat{s}^j\}$$

For $j = k+1$, layer j contains the following subset of individuals and clubs:

$$\hat{I}_j(a) = \{i \in I : d_i(a) = 0\}$$
$$\hat{C}_j(a) = \{c \in C : s_c(a) = 0\}$$

Remark 3.2. *Consider the smooth club membership model with constant returns from membership overlap, a club membership structure a is stable if*
(i) $\bigcup_{j=1}^{k+1} \hat{I}_j(a) = I$ *and* $\bigcup_{j=1}^{k+1} \hat{C}_j(a) = C$,
(ii) $|\hat{I}_{k+1}(a)| \times |\hat{C}_{k+1}(a)| = 0$,
(iii) $\forall i \in \hat{I}_j(a), c \in \hat{C}_{j'}(a)$: *if* $j \neq j'$, *then* $a_{ic} = 0$.

The above remark illustrates a type of stable club membership structures where individuals and clubs are separated into several layers. In each layer, individuals join the same number of clubs and clubs admit the same number of members, and all club memberships are between individuals and clubs within the layer.

Note that in a stable club membership structure demonstrated above, individuals/clubs in the same layer will obtain the same utility/productivity and individuals/clubs in different layers will generically obtain different utility/productivity. So, in a stable club membership structure characterized by Remark 3.2, individuals and clubs are separated into a few groups with distinct club membership pattern and welfare level. This is consistent with our finding for the capacity constraints model.

3.5 Discussion

In Section 3.4, we examine two extensions of our model. The analysis provides a sense of how our results would remain robust or change, but is partial. We are working on getting complete analytical solutions. There are also other ways to extend the model. For example, we assume that the productivity of a club rise in its aggregate link strength with other clubs without taking into account the characteristics of clubs it has links with. We can extend our analysis by assuming that the benefit from a link depends on the membership size or the productivity of the club on the other side of the link. We aim to explore this issue in the future.

Appendix 3.A Proofs

Proof for Proposition 3.1

In the capacity constraints model, we know that if $a \in A$ is efficient, we must have

$$\forall i \in I, c \in C : d_i(a) \leq D, s_c(a) \leq S$$

The aggregate utility of individuals given a club membership structure a is

$$\begin{aligned}
U(a) =& \sum_{i \in I} u_i(a) \\
=& \sum_{i \in I} \sum_{c \in C} a_{ic} \left(f(s_c(a)) + \sum_{c' \neq c} g_{cc'}(a) \right) \\
=& \sum_{i \in I} \sum_{c \in C} a_{ic} \left(f(s_c(a)) + \alpha \sum_{c' \neq c} o_{cc'}(a) \right) \\
=& \sum_{i \in I} \sum_{c \in C} a_{ic} \left(f(s_c(a)) + \alpha \sum_{c' \neq c} \sum_{i' \in I} a_{i'c} a_{i'c'} \right) \\
=& \sum_{i \in I, c \in C} a_{ic} f(s_c(a)) + \alpha \sum_{i \in I, c \in C, c' \neq c, i' \in I} a_{ic} a_{i'c} a_{i'c'} \\
=& \sum_{c \in C} f(s_c(a)) \sum_{i \in I} a_{ic} + \alpha \sum_{c \in C, c' \neq c, i' \in I} a_{i'c} a_{i'c'} \sum_{i \in I} a_{ic} \\
=& \sum_{c \in C} f(s_c(a)) s_c(a) + \alpha \sum_{c \in C, c' \neq c, i' \in I} a_{i'c} a_{i'c'} s_c(a) \\
=& \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i' \in I} s_c(a) a_{i'c} \sum_{c' \neq c} a_{i'c'} \\
=& \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i' \in I} s_c(a) a_{i'c} (d_{i'}(a) - a_{i'c}) \\
=& \begin{cases} \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i' \in I} s_c(a) a_{i'c} (d_{i'}(a) - 1) & \text{when } a_{i'c} = 0 \\ \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i' \in I} s_c(a) a_{i'c} (d_{i'}(a) - 1) & \text{when } a_{i'c} = 1 \end{cases} \\
=& \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i' \in I} s_c(a) a_{i'c} (d_{i'}(a) - 1) \\
=& \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i \in I} a_{ic} s_c(a) (d_i(a) - 1) \qquad (3.8)
\end{aligned}$$

We now prove our efficiency characterization for the situation when $nD \geq mS$. From equation (3.8), we can get

$$\frac{dU(a)}{ds_c(a)} = f(s_c(a)) + s_c(a)f'(s_c(a)) + \alpha \sum_{i \in I} a_{ic}(d_i(a) - 1)$$

We know $d_i(a) \in \{0, 1, ..., m\}$. When $d_i(a) \geq 1$, we have $a_{ic}(d_i(a) - 1) \geq 0$. When $d_i(a) = 0$, it must be $a_{ic} = 0$ and therefore $a_{ic}(d_i(a) - 1) = 0$. So we have

$$\alpha \sum_{i \in I} a_{ic}(d_i(a) - 1) \geq 0$$

Also, when $s_c(a) \geq 0$, we have $f(s_c(a)) \geq 0$ and $s_c(a)f'(s_c(a)) \geq 0$. When $s_c(a) > 0$, we have $f(s_c(a)) > 0$ and $s_c(a)f'(s_c(a)) > 0$. So

$$\begin{aligned} &\frac{dU(a)}{ds_c(a)} \geq 0 && \text{when } s_c(a) \geq 0 \\ &\frac{dU(a)}{ds_c(a)} > 0 && \text{when } s_c(a) > 0 \end{aligned}$$

Since $0 \leq s_c(a) \leq S$,

$$U(a) \leq \sum_{c \in C} Sf(S) + \alpha \sum_{c \in C, i \in I} a_{ic}S(d_i(a) - 1)$$

where the equality can be reached if and only if $\forall c \in C : s_c(a) = S$.

Since there are m clubs, we have

$$\begin{aligned} U(a) \leq & mSf(S) + \alpha S \sum_{i \in I} (d_i(a) - 1) \sum_{c \in C} a_{ic} \\ = & mSf(S) + \alpha S \sum_{i \in I} (d_i(a) - 1)d_i(a) \end{aligned}$$

We now look st the problem of

$$\max_{(d_i(a))_{i \in I}} \sum_{i \in I} (d_i(a) - 1)d_i(a)$$

Note that when $nD \geq mS$, the aggregate number of club membership is constrained by mS, i.e. $\sum_{i \in I} d_i(a) \leq mS \leq nD$. Also, in the capacity constraints model, we have $\forall i \in I : 0 \leq$

$d_i(a) \leq D$. So we have the following constraints for the maximizing problem above:

$$\sum_{i \in I} d_i(a) \leq mS \leq nD$$
$$\forall i \in I : 0 \leq d_i(a) \leq D$$

We show that this constrained maximizing problem has the following solution $(d_i(a)^*)_{i \in I}$. When $(mS) \bmod D \neq 1$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = (mS) \bmod D$, and the remaining individuals with $d_i(a)^* = 0$. When $(mS) \bmod D = 1$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = 1$ or $d_i(a)^* = 0$, and the remaining individuals with $d_i(a)^* = 0$. We also show that the outcome of the maximization problem is

$$\lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D)$$

The proof of the above statement can be found at the end of this subsection.

Hence

$$\sum_{i \in I} (d_i(a) - 1) d_i(a) \leq \lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D)$$

where the equality can be reached if and only if $(d_i(a))_{i \in I}$=$(d_i(a)^*)_{i \in I}$.

Note that when $(mS) \bmod D = 1$, if there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = 0$, and the remaining individuals with $d_i(a)^* = 0$,

$$\begin{aligned} \sum_{i \in I} d_i(a) &= \lfloor \frac{mS}{D} \rfloor D \\ &= mS - (mS) \bmod D \\ &= mS - 1 \end{aligned}$$

and we can not have $\forall c \in C : s_c(a) = S$.

So

$$\begin{aligned} U(a) \leq & mSf(S) + \alpha S \sum_{i \in I} (d_i(a) - 1) \sum_{c \in C} a_{ic} \\ = & mSf(S) + \alpha S \lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D) \end{aligned}$$

where the equality can be reached if and only if $\forall c \in C : s_c(a) = S$, and there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a) = D$, one individual with $d_i(a) = (mS) \bmod D$, and the remaining individuals with $d_i(a) = 0$.

When $nD \geq mS$, there exists a club membership structure a where $\forall c \in C : s_c(a) = S$, and there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a) = D$, one individual with $d_i(a) = (mS) \bmod D$, and the remaining individuals with $d_i(a) = 0$. This can be obtained by letting individuals join clubs in sequence. Make each individual join D clubs that has the smallest membership size at her turn before moving to the next individual. Stop when all clubs have S members.

We have proved that in the capacity constraint model with constant returns from membership overlap. If $nD \geq mS$, then a club membership structure $a \in A$ is efficient if and only if all clubs have S members, and there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs.

We now prove our efficiency characterization for the situation when $mS \geq nD$. From equation (3.8), we can get

$$\frac{dU(a)}{dd_i(a)} = \alpha \sum_{c \in C} a_{ic} s_c(a) \geq 0$$

When $d_i(a) > 0$, there must exist $i \in I, c \in C$ with $a_{ic} > 0$ and $s_c(a) > 0$. So

$$\begin{aligned} \frac{dU(a)}{dd_i(a)} &\geq 0 && \text{when } d_i(a) \geq 0 \\ \frac{dU(a)}{dd_i(a)} &> 0 && \text{when } d_i(a) > 0 \end{aligned}$$

Since $0 \leq d_i(a) \leq S$,

$$\begin{aligned} U(a) &\leq \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C, i \in I} a_{ic} s_c(a)(D-1) \\ &= \sum_{c \in C} s_c(a) f(s_c(a)) + \alpha \sum_{c \in C} s_c(a)(D-1) \sum_{i \in I} a_{ic} \\ &= \sum_{c \in C} s_c(a) \left(f(s_c(a)) + \alpha(D-1) s_c(a) \right) \end{aligned}$$

where the equality can be reached if and only if $\forall i \in I : d_i(a) = S$.

We now look st the problem of

$$\max_{(s_c(a))_{c \in C}} \sum_{c \in C} s_c(a) \left(f(s_c(a)) + \alpha(D-1) s_c(a) \right)$$

Note that when $mS \geq nD$, the aggregate number of club membership is constrained by nD, i.e. $\sum_{i \in I} s_c(a) \leq nD \leq Ms$. Also, in the capacity constraints model, we have $\forall c \in C : 0 \leq s_c(a) \leq S$. So we have the following constraints for the maximizing problem above:

$$\sum_{c \in C} s_c(a) \leq nD \leq mS$$
$$\forall c \in C : 0 \leq s_c(a) \leq S$$

We show that this constrained maximizing problem has the following solution $(s_c(a)^*)_{c \in C}$. There are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a)^* = S$, one club with $s_c(a)^* = (nD) \bmod S$, and the remaining clubs with $s_c(a)^* = 0$. We also show that the outcome of the maximization problem is

$$\lfloor \frac{nD}{S} \rfloor S[f(S) + \alpha(D-1)S] + (nD) \bmod S[f((nD) \bmod S) + \alpha(D-1)(nD) \bmod S]$$

The proof of the above statement can be found at the end of this subsection.

Hence

$$\begin{aligned} &\sum_{c \in C} s_c(a) \left(f(s_c(a)) + \alpha(D-1)s_c(a)\right) \\ \leq &\lfloor \frac{nD}{S} \rfloor S[f(S) + \alpha(D-1)S] + (nD) \bmod S[f((nD) \bmod S) + \alpha(D-1)(nD) \bmod S] \end{aligned}$$

where the equality can be reached if and only if $(s_c(a))_{c \in C}$=$(s_c(a)^*)_{c \in C}$.

So

$$U(a) \leq \lfloor \frac{nD}{S} \rfloor S[f(S) + \alpha(D-1)S] + (nD) \bmod S[f((nD) \bmod S) + \alpha(D-1)(nD) \bmod S]$$

where the equality can be reached if and only if $\forall i \in I : d_i(a) = D$, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a)^* = S$, one club with $s_c(a)^* = (nD) \bmod S$, and the remaining clubs with $s_c(a)^* = 0$.

When $mS \geq nD$, there exists a club membership structure a where $\forall i \in I : d_i(a) = D$, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a)^* = S$, one club with $s_c(a)^* = (nD) \bmod S$, and the remaining clubs with $s_c(a)^* = 0$. This can be obtained by letting clubs admit individuals in sequence. Make each club admit S individuals that has the smallest degree its turn before moving to the next club. Stop when all individuals have degree D.

We have proved that in the capacity constraint model with constant returns from membership overlap. If $mS \geq nD$, then a club membership structure $a \in A$ is efficient if and only if all individuals join D clubs, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs that admit D members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.

Now, we prove our previous statement that for the problem of

$$\begin{aligned} &\max_{(d_i(a))_{i\in I}} \sum_{i\in I}(d_i(a)-1)d_i(a) \\ &\text{s.t. } \sum_{i\in I} d_i(a) \le mS \le nD \\ &\text{and } \forall i \in I : 0 \le d_i(a) \le D \end{aligned}$$

its solution takes $(d_i(a)^*)_{i\in I}$ takes the following form. When $(mS) \bmod D \neq 1$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = (mS) \bmod D$, and the remaining individuals with $d_i(a)^* = 0$. When $(mS) \bmod D = 1$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = 1$ or $d_i(a)^* = 0$, and the remaining individuals with $d_i(a)^* = 0$. And its outcome is

$$\lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D)$$

We prove in three steps. First, we show in a solution $(d_i(a)^*)_{i\in I}$ to the problem, there can only be one individual i with $d_i(a)^* \notin \{0, D\}$. Second, we show that a solution $(d_i(a)^*)_{i\in I}$ to the problem must satisfy $\sum_{i\in I} d_i(a)^* \ge mS - 1$. Finally, if $\sum_{i\in I} d_i(a) = mS$, combine our requirement for $(d_i(a)^*)_{i\in I}$ in step 1, if $(d_i(a))_{i\in I}$ is the solution to the problem, we know there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a) = D$, one individual with $d_i(a) = (mS) \bmod D$, and the remaining individuals with $d_i(a) = 0$ and we have

$$\sum_{i\in I}(d_i(a)-1)d_i(a) = \lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D)$$

if $\sum_{i\in I} d_i(a) = mS - 1$, combine our requirement for $(d_i(a)^*)_{i\in I}$ in step 1, if $(d_i(a))_{i\in I}$ is the solution to the problem, we know there are $\lfloor \frac{mS-1}{D} \rfloor$ individuals with $d_i(a) = D$, one individual with $d_i(a) = (mS-1) \bmod D$, and the remaining individuals with $d_i(a) = 0$ and we have

$$\sum_{i\in I}(d_i(a)-1)d_i(a) = \lfloor \frac{mS-1}{D} \rfloor (D-1)D + ((mS-1) \bmod D - 1)((mS-1) \bmod D)$$

Since when $(mS) \bmod D \neq 1$,

$$\lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D) > \lfloor \frac{mS-1}{D} \rfloor (D-1)D + ((mS-1) \bmod D - 1)((mS-1) \text{ mo}$$

So $\sum\limits_{i \in I} d_i(a)^* = mS$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = (mS) \bmod D$, and the remaining individuals with $d_i(a)^* = 0$.

When $(mS) \bmod D \neq 1$, we have

$$\lfloor \frac{mS}{D} \rfloor (D-1)D + ((mS) \bmod D - 1)((mS) \bmod D) = \lfloor \frac{mS-1}{D} \rfloor (D-1)D + ((mS-1) \bmod D - 1)((mS-1)$$

So $\sum\limits_{i \in I} d_i(a)^* = mS$ or $\sum\limits_{i \in I} d_i(a)^* = mS - 1$, there are $\lfloor \frac{mS}{D} \rfloor$ individuals with $d_i(a)^* = D$, one individual with $d_i(a)^* = 1$ or $d_i(a)^* = 0$, and the remaining individuals with $d_i(a)^* = 0$.

We now elaborate on the first two steps.

Step 1: Suppose $\exists i' \in I, i'' \neq i$ with $0 < d_{i'}(a)^* < D$ and $0 < d_{i''}(a)^* < D$. Let $d_{i'}(a)^* > d_{i''}(a)^*$. Construct a $(d_i(a)')_{i \in I}$ where $d_{i'}(a)' = d_{i'}(a)^* + 1 \leq D$, $d_{i''}(a)' = d_{i''}(a)^* - 1 \geq 0$, and $\forall i \neq i', i''$: $d_i(a)' = d_i(a)^*$. $(d_i(a)')_{i \in I}$ satisfies all the constraints of the maximization problem. We have

$$\begin{aligned}
\sum_{i \in I}(d_i'(a) - 1)d_i(a) =& (d_{i'}(a)' - 1)d_{i'}(a)' + (d_{i''}(a)' - 1)d_{i''}(a)' + \sum_{i \neq i', i''}(d_i^*(a) - 1)d_i(a)^* \\
=& (d_{i'}(a)^*)(d_{i'}(a)^* + 1) + (d_{i''}(a)^* - 2)(d_{i''}(a)^* - 1) + \sum_{i \neq i', i''}(d_i^*(a) - 1)d_i(a)^* \\
=& (d_{i'}(a)^* - 1)d_{i'}(a)^* + 2d_{i'}(a)^* + (d_{i''}(a)^* - 1)d_{i''}(a)^* - 2(d_{i''}(a)^* - 1) \\
& + \sum_{i \neq i', i''}(d_i^*(a) - 1)d_i(a)^* \\
=& (d_{i'}(a)^* - 1)d_{i'}(a)^* + (d_{i''}(a)^* - 1)d_{i''}(a)^* + \sum_{i \neq i', i''}(d_i^*(a) - 1)d_i(a)^* \\
& + 2(d_{i'}(a)^* - d_{i''}(a)^* + 1) \\
>& (d_{i'}(a)^* - 1)d_{i'}(a)^* + (d_{i''}(a)^* - 1)d_{i''}(a)^* + \sum_{i \neq i', i''}(d_i^*(a) - 1)d_i(a)^* \\
=& \sum_{i \in I}(d_i^*(a) - 1)d_i(a)^*
\end{aligned}$$

so $(d_i(a)^*)_{i \in I}$ cannot be the solution to the optimization problem.

Step 2: Suppose $\sum\limits_{i \in I} d_i(a)^* < mS - 1$, we know $\exists i' \in I$ with $d_{i'}(a)^* < D$.

If $d_{i'}(a)^* < D-1$, construct a $(d_i(a)')_{i\in I}$ where $d_{i'}(a)' = d_{i'}(a)^* + 2 \le D$ and $\forall i \ne i'$: $d_i(a)' = d_i(a)^*$. $(d_i(a)')_{i\in I}$ satisfies all the constraints of the maximization problem. We have

$$\begin{aligned}
\sum_{i\in I}(d_i'(a)-1)d_i(a) =&(d_{i'}(a)'-1)d_{i'}(a)' + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
=&(d_{i'}(a)^*+1)(d_{i'}(a)^*+2) + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
>&(d_{i'}(a)^*-1)d_{i'}(a)^* + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
=&\sum_{i\in I}(d_i^*(a)-1)d_i(a)^*
\end{aligned}$$

so $(d_i(a)^*)_{i\in I}$ cannot be the solution to the optimization problem.

If $d_{i'}(a)^* = D-1$, construct a $(d_i(a)')_{i\in I}$ where $d_{i'}(a)' = d_{i'}(a)^* + 1 \le D$ and $\forall i \ne i'$: $d_i(a)' = d_i(a)^*$. $(d_i(a)')_{i\in I}$ satisfies all the constraints of the maximization problem. We have

$$\begin{aligned}
\sum_{i\in I}(d_i'(a)-1)d_i(a) =&(d_{i'}(a)'-1)d_{i'}(a)' + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
=&(d_{i'}(a)^*)(d_{i'}(a)^*+1) + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
>&(d_{i'}(a)^*-1)d_{i'}(a)^* + \sum_{i\ne i'}(d_i^*(a)-1)d_i(a)^* \\
=&\sum_{i\in I}(d_i^*(a)-1)d_i(a)^*
\end{aligned}$$

so $(d_i(a)^*)_{i\in I}$ cannot be the solution to the optimization problem.

We can follow the same argument to prove our previous statement that for the problem of

$$\begin{aligned}
&\max_{(s_c(a))_{c\in C}} \sum_{c\in C} s_c(a)\left(f(s_c(a)) + \alpha(D-1)s_c(a)\right) \\
&\text{s.t. } \sum_{c\in C} s_c(a) \le nD \le mS \\
&\text{and } \forall c \in C: 0 \le s_c(a) \le S
\end{aligned}$$

its solution takes $(s_c(a)^*)_{c\in C}$ takes the following form. There are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a)^* = S$, one club with $s_c(a)^* = (nD) \bmod S$, and the remaining clubs with $s_c(a)^* = 0$. And its outcome is

$$\lfloor \frac{nD}{S} \rfloor S[f(S) + \alpha(D-1)S] + (nD) \bmod S[f((nD) \bmod S) + \alpha(D-1)(nD) \bmod S]$$

Proof for Proposition 3.2

We start by making note of two Lemmas which we will frequently use later.

First, for the capacity constraints model with constant returns from membership overlap, we know all individuals join at most D clubs and all clubs admit at most S members. As individuals and clubs always prefer larger degree and larger membership size before they meet their constraints, there are no incentives for individuals to quit clubs and for clubs to exile members, unless they want to make a substitution. So,

Lemma 3.1. *Consider the capacity constraints model with constant returns from membership overlap, a club membership structure $a \in A$ is stable if and only if:*

(i) $\forall i \in I, c \in C$: $d_i(a) \le D$, $s_c(a) \le S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I, c \in C$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\le a_{ic'} \\ a'_{i'c} &\le a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

The first condition says that all individuals join at most D clubs and all clubs admit at most S member and that there cannot exist a pair of individual and club where the individual is not fully affiliated with D memberships and the club is not full with S members, because otherwise they would both benefit if the individual joins the club. The second conditions say that there is no pair of individual i and club c that would both benefit when i joins c. When i joins c, i would want to quit at most one other club c' and c would want to exile at most one existing member i'.

Second, we show how the utility of an individual i change and how the productivity of a club c change when they move from a club membership structure a to a' described condition (ii) of Lemma 3.1.

Lemma 3.2. *In the capacity constraints model with constant returns from membership overlap, consider a club membership structure a and a'. If $a_{ic} = 0$ and*

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

then:

If $a'_{i'c} = a_{i'c}$, $\pi_c(a') > \pi_c(a)$ if and only if $s_c(a) < S$,
If $a'_{i'c} < a_{i'c}$, $\pi_c(a') > \pi_c(a)$ if and only if $d_i(a') > d_{i'}(a)$,
If $a'_{ic'} = a_{ic'}$, $u_i(a') > u_i(a)$ if and only if $d_i(a) < D$,
If $a'_{ic'} < a_{ic'}$, $u_i(a') > u_i(a)$ if and only if $\pi_c(a') < \pi_{c'}(a)$.

Lemma 3.2 can be proved in the following way.

In the capacity constraint model with constant returns from membership overlap, the productivity of a club c is:

$$\begin{aligned} \pi_c(a) =& F(s_c(a)) + \sum_{c' \neq c} g_{cc'}(a) \\ =& \alpha \sum_{c' \neq c} \sum_{i \in I} a_{ic} a_{ic'} + F(s_c(a)) \\ =& \alpha \sum_{i \in I} a_{ic}(d_i(a) - 1) + F(s_c(a)) \end{aligned}$$

So

$$\pi_c(a) = F(s_c(a)) + \alpha a_{ic}(d_i(a) - 1) + \alpha a_{i'c}(d_{i'}(a) - 1) + \alpha \sum_{i'' \neq i, i'} a_{i''c}(d_{i''}(a) - 1)$$

$$\pi_c(a') = F(s_c(a')) + \alpha a'_{ic}(d_i(a') - 1) + \alpha a'_{i'c}(d_{i'}(a') - 1) + \alpha \sum_{i'' \neq i, i'} a'_{i''c}(d_{i''}(a') - 1)$$

Since for individuals $i'' \neq i, i'$, their club memberships are the same under a and a', we have

$$\alpha \sum_{i'' \neq i, i'} a_{i''c}(d_{i''}(a) - 1) = \alpha \sum_{i'' \neq i, i'} a'_{i''c}(d_{i''}(a') - 1)$$

and so

$$\begin{aligned}\pi_c(a') - \pi_c(a) =& F(s_c(a')) - F(s_c(a)) \\ &+ \alpha \left[a'_{ic}(d_i(a') - 1) + a'_{i'c}(d_{i'}(a') - 1) - a_{ic}(d_i(a) - 1) - a_{i'c}(d_{i'}(a) - 1)\right]\end{aligned}$$

If $a'_{i'c} = a_{i'c}$, i.e. club c does not exile i', then $s_c(a') = s_c(a) + 1$ and $d_{i'}(a') = d_{i'}(a)$,

$$\pi_c(a') - \pi_c(a) = F(s_c(a) + 1) - F(s_c(a)) + \alpha a'_{ic}(d_i(a') - 1) \begin{cases} > 0 & \text{when } s_c(a) < S \\ < 0 & \text{when } s_c(a) \geq S \end{cases}$$

So $\pi_c(a') > \pi_c(a)$ if and only if $s_c(a) < S$.

If $a'_{i'c} < a_{i'c}$, i.e. club c exiles i', then $s_c(a') = s_c(a)$,

$$\pi_c(a') - \pi_c(a) = \alpha \left[d_i(a') - d_{i'}(a)\right] \begin{cases} > 0 & \text{when } d_i(a') > d_{i'}(a) \\ \leq 0 & \text{when } d_i(a') \leq d_{i'}(a) \end{cases}$$

So $\pi_c(a') > \pi_c(a)$ if and only if $d_i(a') > d_{i'}(a)$,

In the capacity constraint model, the utility of individual i is:

$$\begin{aligned} u_i(a) &= a_{ic}\pi_c(a) + a_{ic'}\pi_{c'}(a) + \sum_{c'' \neq c,c'} a_{ic''}\pi_{c''}(a) - \kappa(d_i(a)) \\ u_i(a') &= a'_{ic}\pi_c(a) + a'_{ic'}\pi_{c'}(a) + \sum_{c'' \neq c,c'} a'_{ic''}\pi_{c''}(a') - \kappa(d_i(a')) \end{aligned}$$

Since for clubs $c'' \neq c, c'$, their membership structure are the same under a and a', we have

$$\sum_{c'' \neq c,c'} a_{ic''}\pi_{c''}(a) = \sum_{c'' \neq c,c'} a'_{ic''}\pi_{c''}(a')$$

and so

$$\begin{aligned} u_i(a') - u_i(a) =& - \left[\kappa(d_i(a')) - \kappa(d_i(a))\right] \\ &+ a'_{ic}\pi_c(a) + a'_{ic'}\pi_{c'}(a) - a_{ic}\pi_c(a) - a_{ic'}\pi_{c'}(a)\end{aligned}$$

If $a'_{ic'} = a_{ic'}$, i.e. individual i does not quit c', then $d_i(a') = d_i(a)+1$ and $\pi_{c'}(a') = \pi_{c'}(a)$,

$$u_i(a') - u_i(a) = -\left[\kappa(d_i(a)+1) - \kappa(d_i(a))\right] + \pi_c(a) \begin{cases} > 0 & \text{when } d_i(a) < D \\ < 0 & \text{when } d_i(a) \geq D \end{cases}$$

So $u_i(a') > u_i(a)$ if and only if $d_i(a) < D$.

If $a'_{ic'} < a_{ic'}$, i.e. individual i quits c', then $d_i(a') = d_i(a)1$,

$$u_i(a') - u_i(a) = \pi_c(a) - \pi_{c'}(a') \begin{cases} > 0 & \text{when } \pi_c(a) > \pi_{c'}(a') \\ \leq 0 & \text{when } \pi_c(a) \leq \pi_{c'}(a') \end{cases}$$

So $u_i(a') > u_i(a)$ if and only if $\pi_c(a) > \pi_{c'}(a')$.

We have now finished our preparation and start to prove Proposition 3.2.

With Lemma 3.1, we can prove Proposition 3.2 by showing that in the capacity constrains model with constant returns from membership overlap, a club membership structure $a \in A$ satisfies

(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I, c \in C$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

if and only if a satisfies

(a) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(b) $\forall i \in I_2(a), c, c' \in C_2(a)$: if $a_{ic} > a_{ic'}$, then $\pi_c(a) > \pi_{c'}(a)$,

(c) $\forall i, i' \in I_3(a)$: if $d_i(a) \geq d_{i'}(a)$, then $a_{ic} \geq a_{i'c}$ $\forall c \in C$,

(d) $\forall c, c' \in C_2(a)$: if $\pi_c(a) \geq \pi_{c'}(a)$, then $a_{ic} \geq a_{ic'}$ $\forall i \in I_2(a)$ and $a_{ic} \leq a_{ic'}$ $\forall i \in I_3(a)$.

We start with the 'if' part. If (a) holds, (i) obviously also holds. So we show (ii) holds when (a), (b), (c) and (d) hold.

With (a), we have $\forall i \in I$: $d_i(a) \leq D$ and we know $\forall i \in I$: $d_i(a) \geq D$ by definition. So

$$I_1(a) \cup I_2(a) \cup I_3(a) \cup I_4(a) = I$$

Again, with (a), we have $\forall c \in C$: $0 \leq s_c(a) \leq S$, and hence $\forall c \in C$: $0 \leq \pi_c(a) \leq \pi^*$. So

$$C_1(a) \cup C_2(a) \cup C_3(a) = C$$

We show that if (a), (b), (c) and (d) hold, then $\forall x \in \{1,2,3,4\}, y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

When $i \in I_1(a)$, since $d_i(a) = D$ and $\pi_c(a) = \pi^*$ for all $c \in C$ with $a_{ic} = 1$.

$$u_i(a) = D\pi^*$$

We know that $\forall a' \in A$, $i \in I$:

$$u_i(a') \leq D\pi^*$$

since when $d_i(a') \leq D$,

$$u_i(a') = \sum_{c \in C} a'_{ic} \pi_c(a') \leq D\pi^*$$

and when $d_i(a') > D$,

$$u_i(a') = \sum_{c \in C} a'_{ic}\pi_c(a') - M_2(d_i(a') - D) < 0$$

given M_2 is sufficiently large.

So $\forall a' \in A$: $u_i(a) = D\pi^* \geq u_i(a')$. Therefore, when $x = 1, \forall y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

When $i \in I_4(a)$, we know $d_i(a) = 0$. With condition (a) that says $(d_i(a) - D)(s_c(a) - S) = 0 \; \forall i \in I, c \in C$, we know $\forall c \in C$: $s_c(a) = S$ if there exists $i \in I_4(a)$.

For $i \in I_4(a), c \in C$, when $a_{ic} = 0$, suppose there is an $a' \in A$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{i'c} &\leq a_{i'c} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \end{aligned}$$

such that

$$\pi_c(a') > \pi_c(a)$$

Since $a'_{ic} = 1$, $a_{ic} = 0$, and $\forall i'' \neq i, i' : a'_{i''c} = a_{i''c}$, if $a'_{i'c} = a_{i'c}$, given $s_c(a) = S$, we know from Lemma 3.2 that $\pi_c(a') \leq \pi_c(a)$. If $a'_{i'c} < a_{i'c}$, we know from Lemma 3.2 that $\pi_c(a') > \pi_c(a)$ only if $d_i(a') > d_{i'}(a)$, which cannot be true since $d_i(a') \leq 1 \leq d_{i'}(a)$.

Therefore, when $x = 4, \forall y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

When $c \in C_1(a)$, since $\pi_c(a) = \pi^*$, we know by definition of π^* that $\forall a' \in A$: $\pi_c(a) = \pi^* \geq \pi_c(a')$. Therefore, when $y = 1, \forall x \in \{1,2,3,4\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

When $c \in C_3(a)$, we know $\pi_c(a) = 0$ and $s_c(a) = 0$. With condition (a) that says $(d_i(a) - D)(s_c(a) - S) = 0 \; \forall i \in I, c \in C$, we know $\forall i \in I$: $d_i(a) = D$ if there exists $c \in C_3(a)$.

For $i \in I, c \in C_3(a)$, when $a_{ic} = 0$, suppose there is an $a' \in A$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \end{aligned}$$

such that

$$u_i(a') > u_i(a)$$

Since $a'_{ic} = 1$, $a_{ic} = 0$, and $\forall c'' \neq c, c' : a'_{ic''} = a_{ic''}$, if $a'_{ic'} = a_{ic'}$, given $d_i(a) = D$, we know from Lemma 3.2 that $u_i(a') \leq u_i(a)$. If $a'_{ic'} < a_{ic'}$, we know from Lemma 3.2 that $u_i(a') > u_i(a)$ only if $\pi_c(a') > \pi_{c'}(a)$, which cannot be true since $d_i(a') \leq f(1) \leq \pi_{c'}(a)$.

Therefore, when $y = 3, \forall x \in \{1,2,3,4\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

Now we only need to show that $\forall x \in \{2,3\}, y \in \{2\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

When $i \in I_2(a), c \in C_2(a)$ and $a_{ic} = 0$. Since $i \in I_2(a)$, $d_i(a) = D$. If $a'_{ic'} = a_{ic'}$, we know from Lemma 3.2 that $u_i(a') \leq u_i(a)$.

If $a'_{ic'} < a_{ic'}$, then $a_{ic'} = 1$. We know $c' \notin C_3(a)$ since $C_3(a)$ consists of clubs that have no members in a. If $c' \in C_1(a)$, then $\pi_c(a') \leq \pi_{c'}(a) = \pi^*$. We know from Lemma 3.2 that $u_i(a') \leq u_i(a)$.

If $c' \in C_2(a)$, since $a_{ic} = 0$, $a_{ic'} = 1$, we know from condition (b) that $\pi_{c'}(a) > \pi_c(a)$. Since $\pi_{c'}(a) > \pi_c(a)$, we know from condition (d) that $\forall i \in I_2(a)$: $a_{ic'} \geq a_{ic}$ and $\forall i \in I_3(a)$: $a_{ic'} \leq a_{ic}$.

We know that $\forall c, c' \in C_2(a)$: $a_{ic} = 1$ only if $i \in I_2(a) \cup I_3(a)$. Given that $\pi_{c'}(a) > \pi_c(a)$, there must exists $i \in I_2(a)$ where $a_{ic'} = 1$ and $a_{ic} = 0$, otherwise $\forall i \in I_2(a) \cup I_3(a)$: $a_{ic'} \leq a_{ic}$, violating $\pi_{c'}(a) > \pi_c(a)$.

So now we know

$$\forall i \in I_2(a) : a_{ic'} \geq a_{ic} \tag{3.9}$$

$$\exists i \in I_2(a) \text{ s.t. } a_{ic'} > a_{ic} \tag{3.10}$$

$$\forall i \in I_3(a) : a_{ic'} \leq a_{ic} \tag{3.11}$$

If $\forall i \in I_3(a) : a_{ic'} = a_{ic}$, then $s_c(a) < s_{c'}(a) \leq S$. We know $forall c \in C_2(a)$:

$$\pi_c(a) = f(s_c(a)) + \alpha(D-1) \sum_{i \in I_2(a)} a_{ic} + \sum_{i \in I_3(a)} a_{ic}(d_i(a) - 1)$$

so

$$\begin{aligned}
\pi_c(a') =& f(s_c(a')) + \alpha(D-1)\sum_{i\in I_2(a)} a'_{ic} + \sum_{i\in I_3(a)} a'_{ic}(d_i(a')-1)\\
\leq& f(s_{c'}(a)) + \alpha(D-1)\sum_{i\in I_2(a)} a_{ic'} + \sum_{i\in I_3(a)} a_{ic'}(d_i(a)-1)\\
=& \pi_{c'}(a)
\end{aligned}$$

We know from Lemma 3.2 that $u_i(a') \leq u_i(a)$.

If $\exists i'' \in I_3(a) : a_{i''c'} < a_{i''c}$, so $a_{i''c'} = 0$. We must have $s_{c'} = S$, otherwise $(d_{i''}(a) - D)(s_{c'} - S) \neq 0$, violating condition (a). Also, we mush have

$$\forall i''' \in I \text{ with } a_{i'''c} = 1 : d_{i'''}(a) > d_{i''}(a)$$

otherwise condition (c) is violated.

Hence we have

$$\pi_{c'}(a) = f(S) + \alpha(D-1)S - \sum_{i\in I_3(a)} a_{ic'}(D - d_i(a))$$

and

$$\begin{aligned}
\pi_c(a') &\leq f(S) + \alpha(D-1)S - \sum_{i\in I_3(a)} a'_{ic}(D - d_i(a'))\\
&\leq f(S) + \alpha(D-1)S - \sum_{i\in I_3(a), i\neq i''} a_{ic}(D - d_i(a))\\
&\leq \pi_{c'}(a)
\end{aligned}$$

We know from Lemma 3.2 that $u_i(a') \leq u_i(a)$.

Hence when $x = 2$, $y = 2$, $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1\\
a'_{ic'} &\leq a_{ic'}\\
a'_{i'c} &\leq a_{i'c}\\
\forall c'' \neq c, c' : & a'_{ic''} = a_{ic''}\\
\forall i'' \neq i, i' : & a'_{i''c} = a_{i''c}\\
\forall i'' \neq i, c'' \neq c : & a'_{i''c''} = a_{i''c''}
\end{aligned}$$

such that

$$
\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}
$$

When $i \in I_3(a), c \in C_2(a)$ and $a_{ic} = 0$. We know $d_i(a) < D$, so we must have $\forall c \in C$: $s_c = S$, otherwise $(d_i(a) - D)(s_c - S) \neq 0$, violating condition (a). If $a'_{i'c} = a_{i'c}$, we know from Lemma 3.2 that $\pi_c(a') \leq \pi_c(a)$.

If $a'_{i'c} < a_{i'c}$, so $a_{i'c} = 1$, we have $d_i(a) < d_{i'}(a)$, otherwise we should have $a_{ic} \geq a_{i'c}$ according to condition (c), violating $a_{ic} = 0 < a_{i'c} = 1$. So $d_i(a') \leq d_i(a) + 1 \leq d_{i'}(a)$. We know from Lemma 3.2 that $\pi_c(a') \leq \pi_c(a)$.

Hence when $x = 3$, $y = 2$, $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$
\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}
$$

such that

$$
\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}
$$

We have finished the 'if' part. For the 'only if' part. If (i) holds, (a) obviously also hold. So we show (b), (c), (d) holds when (i) and (ii) hold.

Suppose (i) and (ii) hold, but (b) does not hold, then there exists $i \in I_2(a), c, c' \in C_2(a)$ with $a_{ic} > a_{ic'}$ and $\pi_c(a) \leq \pi_{c'}(a)$. Note that since $c' \in C_2(a)$, either $s_{c'}(a) < S$ or $\exists i' \in I_3(A)$ with $a_{i'c'} = 1$.

If $s_{c'}(a) < S$, consider the following deviation from a to a' for i and c'.

$$\begin{aligned} a'_{ic'} &= 1 \\ a'_{ic} &< a_{ic} \\ a'_{i'c'} &= a_{i'c'} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c'} &= a_{i''c'} \\ \forall i'' \neq i, c'' \neq c' : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

According to Lemma 3.2, $pi_{c'}(a') > pi_{c'}(a)$. Also according to Lemma 3.2, $u_i(a) > u_i(a')$ since $\pi_{c'}(a') > \pi_{c'}(a) \geq \pi_c(a)$. Condition (ii) is violated.

If $\exists i' \in I_3(A)$ with $a_{i'c'} = 1$, consider the following deviation from a to a' for i and c'.

$$\begin{aligned} a'_{ic'} &= 1 \\ a'_{ic} &< a_{ic} \\ a'_{i'c} &< a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

Since $d_i(a') = D > d_{i'}(a)$, according to Lemma 3.2, $pi_{c'}(a') > pi_{c'}(a)$. Also according to Lemma 3.2, $u_i(a) > u_i(a')$ since $pi_{c'}(a') > pi_{c'}(a) \geq \pi_c(a)$. Condition (ii) is violated.

So, if (i) and (ii) hold, (b) holds.

Suppose (i) and (ii) hold, but (c) does not hold, then there exists $i, i' \in I_3(a), c \in C$ with $d_i(a) \geq d_{i'}(a)$ and $a_{ic} < a_{i'c}$. Consider the following deviation from a to a' for i and c.

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &= a_{ic'} \\ a'_{i'c} &< a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

Since $i \in I_3(a)$, we know $d_i(a) < D$. According to Lemma 3.2, we have $u_i(a) > u_i(a')$. Also, we know $d_i(a') = d_i(a) + 1 > d_{i'}(a)$, according to Lemma 3.2, we have $pi_c(a') > pi_c(a)$. Condition (ii) is violated.

So, if (i) and (ii) hold, (c) holds.

Suppose (i) and (ii) hold, but (d) does not hold, then there exists $c, c' \in C_2(a), i \in I_2(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} < a_{ic'}$ or there exists $c, c' \in C_2(a), i \in I_3(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} > a_{ic'}$.

If there exists $c, c' \in C_2(a), i \in I_2(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} < a_{ic'}$. Note that since $c \in C_2(a)$, either $s_c < S$ or $\exists i' \in I_3(A)$ with $a_{i'c} = 1$.

When $s_c < S$, consider the following deviation from a to a' for i and c.

$$\begin{aligned}
a'_{ic'} &= 1 \\
a'_{ic} &< a_{ic} \\
a'_{i'c} &= a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

According to Lemma 3.2, $pi_c(a') > pi_c(a)$. Also according to Lemma 3.2, $u_i(a) > u_i(a')$ since $pi_c(a') > pi_c(a) \geq \pi_{c'}(a)$. Condition (ii) is violated.

When $\exists i' \in I_3(A)$ with $a_{i'c} = 1$, consider the following deviation from a to a' for i and c.

$$\begin{aligned}
a'_{ic'} &= 1 \\
a'_{ic} &< a_{ic} \\
a'_{i'c} &< a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

Since $d_i(a') = D > d_{i'}(a)$, according to Lemma 3.2, $pi_c(a') > pi_c(a)$. Also according to Lemma 3.2, $u_i(a) > u_i(a')$ since $pi_c(a') > pi_c(a) \geq \pi_{c'}(a)$. Condition (ii) is violated.

So there does not exist $c, c' \in C_2(a), i \in I_2(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} < a_{ic'}$.

If there exists $c, c' \in C_2(a), i \in I_3(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} > a_{ic'}$. Since $d_i(a) < D$, we know $s_c(a) = s_{c'}(a) = S$, otherwise $(d_i(a) - D)(s_c(a) - S) \neq 0$ or $(d_i(a) - D)(s_{c'}(a) - S) \neq 0$, condition (i) is violated. Then since $c' \in C_2(a)$, $\exists i' \in I_3(a)$ with $a_{i'c'} = 1$.

Suppose $\exists i' \in I_3(a)$ with $a_{i'c'} = 1$ and $d_{i'}(a) \leq d_i(a)$, according to condition (c), which we have shown must hold when (i) and (ii) hold, we have $a_{ic'} \geq a_{i'c'}$. This contradicts with $a_{ic'} = 0$ and $a_{i'c'} = 1$.

Suppose $\forall i' \in I_3(a)$ with $a_{i'c'} = 1$, we have $d_{i'}(a) > d_i(a)$, according to condition (c), which we have shown must hold when (i) and (ii) hold, we have $a_{i'c} \geq a_{ic} = 1$. Combine this with our previous finding that $\forall i' \in I_2(a)$: $a_{i'c} \geq a_{i'c'}$ and our assumption that $a_{ic} = 1$, $a_{ic'} = 0$. The membership size of club c is greater than the membership size of c', violating $s_c(a) = s_{c'}(a) = S$.

So there does not exist $c, c' \in C_2(a), i \in I_3(a)$ with $\pi_c(a) \geq \pi_{c'}(a)$ and $a_{ic} > a_{ic'}$.

Hence, if (i) and (ii) hold, (d) holds.

We have finished the 'only if' part.

Proof for Corollary 3.1

From Proposition 3.2, we know that in the capacity constraints model with constant returns from membership overlap, if a club membership structure a is stable, then

(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I_2(a), c, c' \in C_2(a)$: if $a_{ic} > a_{ic'}$, then $\pi_c(a) > \pi_{c'}(a)$,

(iii) $\forall i, i' \in I_3(a)$: if $d_i(a) \geq d_{i'}(a)$, then $a_{ic} \geq a_{i'c}$ $\forall c \in C$,

(iv) $\forall c, c' \in C_2(a)$: if $\pi_c(a) \geq \pi_{c'}(a)$, then $a_{ic} \geq a_{ic'}$ $\forall i \in I_2(a)$ and $a_{ic} \leq a_{ic'}$ $\forall i \in I_3(a)$.

We first show that $|I_2(a)| \leq S - 1$.

We know $\forall i \in I_2(a)$: there exists $c' \in C_2(a)$ such that $a_{ic'} = 1$, otherwise $i \in I_1(a)$ since $a_{ic} = 1$ only if $c \in C_1(a)$.

Let $c \in C_2$ be the club with the highest productivity in $C_2(a)$:

$$\forall c' \in C_2(a) : \pi_c(a) \geq \pi_{c'}(a)$$

Given (iv), we know

$$\forall i \in I_2(a), c' \in C_2(a) : a_{ic} \geq a_{ic'}$$

so

$$\forall i \in I_2(a) : a_{ic} = 1$$

and therefore

$$|I_2(a)| = \sum_{i \in I_2(a)} 1 = \sum_{i \in I_2(a)} a_{ic} < s_c(a) \leq S$$
$$|I_2(a)| \leq S - 1$$

We then show that $|I_3(a)| \leq S$.

We know $\forall i \in I_3(a)$: there exists $c' \in C_2(a)$ such that $a_{ic'} = 1$, since there does not exist $c \in C_1(a) \cup C_3(a)$ such that $a_{ic} = 1$ by definition.

Let $c \in C_2$ be the club with the lowest productivity in $C_2(a)$:

$$\forall c' \in C_2(a) : \pi_c(a) \leq \pi_{c'}(a)$$

Given (iv), we know

$$\forall i \in I_3(a), c' \in C_2(a) : a_{ic} \geq a_{ic'}$$

so

$$\forall i \in I_3(a) : a_{ic} = 1$$

and therefore

$$|I_2(a)| = \sum_{i \in I_2(a)} 1 = \sum_{i \in I_3(a)} a_{ic} \leq s_c(a) \leq S$$

Finally, we show that $|C_2(a)| \leq D$.

If $I_3(a) = \emptyset$, let $c \in C_2$ be the club with the lowest productivity in $C_2(a)$. Given (iv), we know

$$\forall i \in I_2(a), c' \in C_2(a) : a_{ic} \leq a_{ic'}$$

We know there exists $i \in I_2$ such that $a_{ic} = 1$, otherwise c has no members and belong to $C_3(a)$ since $I_3(a) = \emptyset$. So there exists $i \in I_2$ such that

$$\forall c' \in C_2(a) : a_{ic'} \geq a_{ic} = 1$$

therefore

$$|C_2(a)| = \sum_{c \in C_2(a)} 1 = \sum_{c \in C_2(a)} a_{ic} \leq d_i(a) \leq D$$

If $I_3(a) \neq \emptyset$, then $\forall c \in C_2(a)$: $s_c(a) = S$, otherwise there exists $i \in I_3(a), c \in C_2(a)$ such that $(d_i(a) - D)(s_c(a) - S) \neq 0$.

Hence $\forall c \in C_2(a)$: there exists $i' \in I_3(a)$ such that $a_{i'c} = 1$, otherwise $c \in C_1(a)$ since it has S members from $I_1(a) \cup I_2(a)$.

Let $i \in I_3(a)$ be the individual with the highest degree in $I_3(a)$:

$$\forall i' \in I_3(a) : d_i(a) \geq d_{i'}(a)$$

Given (iii), we know

$$\forall c \in C_2(a) : a_{ic} \geq a_{i'c}$$

so

$$\forall c \in C_2(a) : a_{ic} = 1$$

and therefore

$$|C_2(a)| = \sum_{c \in C_2(a)} 1 = \sum_{c \in C_2(a)} a_{ic} \leq d_i(a) < D$$

So $|C_2(a)| \leq D$.

Proof for Proposition 3.3

In the capacity constraint model with no returns from membership overlap, we know that if $a \in A$ is efficient, we must have

$$\forall i \in I, c \in C : d_i(a) \leq D, s_c(a) \leq S$$

and the aggregate utility of individuals given a club membership structure a is

$$\begin{aligned} U(a) &= \sum_{i \in I} u_i(a) \\ &= \sum_{i \in I} \sum_{c \in C} a_{ic} f(s_c(a)) \\ &= \sum_{c \in C} f(s_c(a)) \sum_{i \in I} a_{ic} \\ &= \sum_{c \in C} f(s_c(a)) s_c(a) \end{aligned}$$

When $nD \geq mS$, we can achieve $\forall c \in C$: $s_c(a) = S$ by letting individuals join clubs in any sequence. In each turn, let an individual join the club with the lowest membership size.

Stop when all clubs have S members. So

$$U(a) \leq mf(S)S$$

where the equality can be reached if and only if $\forall c \in C : s_c(a) = S$.

We have proved that in the capacity constraint model with no returns from membership overlap. If $nD \geq mS$, then a club membership structure $a \in A$ is efficient if and only if all clubs have S members.

When $mS \geq nD$, we look at the constrained maximizing problem of

$$\max_{(s_c(a))_{c\in C}} \sum_{c\in C} f(s_c(a))s_c(a) \qquad \text{s.t.} \sum_{c\in C} s_c(a) \leq nD \leq mS$$
$$\text{and } \forall c \in C : 0 \leq s_c(a) \leq S$$

It is easy to see that the solution $(s_c(a)^*)_{c\in C}$ takes the following form. There are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a)^* = S$, one club with $s_c(a)^* = (nD) \bmod S$, and the remaining clubs with $s_c(a)^* = 0$. And its outcome is

$$\lfloor \frac{nD}{S} \rfloor Sf(S) + (nD) \bmod Sf((nD) \bmod S)$$

So

$$U(a) \leq \lfloor \frac{nD}{S} \rfloor Sf(S) + (nD) \bmod Sf((nD) \bmod S)$$

where the equality can be reached if and only if there are $\lfloor \frac{nD}{S} \rfloor$ clubs with $s_c(a) = S$, one club with $s_c(a) = (nD) \bmod S$, and the remaining clubs with $s_c(a) = 0$. Also, note to achieve the above $(s_c(a))_{c\in C}$, we must have $\forall i \in I : d_i(a) = D$.

We have proved that in the capacity constraint model with no returns from membership overlap. If $mS \geq nD$, then a club membership structure $a \in A$ is efficient if and only if all individuals join D clubs, and there are $\lfloor \frac{nD}{S} \rfloor$ clubs that admit D members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.

Proof for Proposition 3.4

In the same spirit of Lemma 3.1, for the capacity constraints model with zero returns from membership overlap, we know all individuals join at most D clubs and all clubs admit at most S members. As individuals and clubs always prefer larger degree and larger membership size before they meet their constraints, there is no incentives for individuals to quit clubs and for clubs to exile members, unless they want to make an substitution. So, a club membership structure $a \in A$ is stable if and only if:

(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I, c \in C$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

We prove Proposition 3.4 by showing that in the capacity constrains model with zero returns from membership overlap, a club membership structure $a \in A$ satisfies

(i) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(ii) $\forall i \in I, c \in C$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

if and only if a satisfies

(a) $\forall i \in I, c \in C$: $d_i(a) \leq D$, $s_c(a) \leq S$, and $(d_i(a) - D)(s_c(a) - S) = 0$,

(b) $\forall i \in I_2(a), c, c' \in C_2(a)$: if $a_{ic} > a_{ic'}$, then $\pi_c(a) > \pi_{c'}(a)$,

We start with the 'if' part. If (a) holds, (i) obviously also hold. So we show (ii) holds when (a) and (b) hold.

With (a), we have $\forall i \in I$: $d_i(a) \leq D$ and we know $\forall i \in I$: $d_i(a) \geq D$ by definition. So

$$I_1(a) \cup I_2(a) \cup I_3(a) \cup I_4(a) = I$$

Again, with (a), we have $\forall c \in C$: $0 \leq s_c(a) \leq S$, and hence $\forall c \in C$: $0 \leq \pi_c(a) \leq \pi^*$. So

$$C_1(a) \cup C_2(a) \cup C_3(a) = C$$

We show that if (a) and (b) hold, then $\forall x \in \{1,2,3,4\}, y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

When $i \in I_1(a)$, since $d_i(a) = D$ and $\pi_c(a) = \pi^*$ for all $c \in C$ with $a_{ic} = 1$.

$$u_i(a) = D\pi^*$$

We know that $\forall a' \in A$, $i \in I$:

$$u_i(a') \leq D\pi^*$$

since when $d_i(a') \leq D$,

$$u_i(a') = \sum_{c \in C} a'_{ic} \pi_c(a') \leq D\pi^*$$

and when $d_i(a') > D$,

$$u_i(a') = \sum_{c \in C} a'_{ic}\pi_c(a') - M_2(d_i(a') - D) < 0$$

given M_2 is sufficiently large.

So $\forall a' \in A$: $u_i(a) = D\pi^* \geq u_i(a')$. Therefore, when $x = 1, \forall y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

When $i \in I_3(a) \cup I_4(a)$, we know $d_i(a) < D$. With condition (a) that says $(d_i(a) - D)(s_c(a) - S) = 0 \; \forall i \in I, c \in C$, we know $\forall c \in C$: $s_c(a) = S$ if there exists $i \in I_3 \cup I_4(a)$.

In the capacity constraints model with zero returns from membership overlap, we know

$$\pi_c(a) = F(s_c(a)) = \begin{cases} f(s_c(a)) & \text{if } s_c(a) \leq S \\ f(S) - M_1(s_c(a) - S) & \text{if } s_c(a) > S \end{cases}$$

Since $\forall c \in C$: $s_c(a) = S$, we have $\forall c \in C, a' \in A$: $\pi_c(a) = f(S) = \pi^* \geq \pi_c(a')$.

Therefore, $\forall x \in \{3,4\}, y \in \{1,2,3\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

When $c \in C_1(a)$, since $\pi_c(a) = \pi^*$, we know by definition of π^* that $\forall a' \in A$: $\pi_c(a) = \pi^* \geq \pi_c(a')$. Therefore, when $y = 1, \forall x \in \{1,2,3,4\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

When $c \in C_3(a)$, we know $\pi_c(a) = 0$ and $s_c(a) = 0$. With condition (a) that says $(d_i(a) - D)(s_c(a) - S) = 0 \; \forall i \in I, c \in C$, we know $\forall i \in I$: $d_i(a) = D$ if there exists $c \in C_3(a)$.

For $i \in I, c \in C_3(a)$, when $a_{ic} = 0$, suppose there is an $a' \in A$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \end{aligned}$$

such that

$$u_i(a') > u_i(a)$$

We know $a'_{ic} = 1$, $a_{ic} = 0$, and $\forall c'' \neq c, c' : a'_{ic''} = a_{ic''}$. So, if $a'_{ic'} = a_{ic'}$, $d_i(a') = d_i(a) + 1 = D + 1$,

$$u_i(a') = \sum_{c \in C} a'_{ic} \pi_c(a') - M_2 < \sum_{c \in C} a_{ic} \pi_c(a) = u_i(a)$$

given M_2 is sufficiently large, violating $u_i(a') > u_i(a)$.

If $a'_{ic'} < a_{ic'}$, $d_i(a') = d_i(a) = D$,

$$\begin{aligned} u_i(a') =& \pi_c(a') + \sum_{c'' \neq c, c'} a'_{ic''} \pi_c(a') \\ =& f(1) + \sum_{c'' \neq c, c'} a'_{ic''} \pi_c(a') \end{aligned}$$

Since the membership structure for $c'' \neq c, c'$ is the same under a and a', ,

$$\begin{aligned} u_i(a') =& f(1) + \sum_{c'' \neq c, c'} a_{ic''} \pi_c(a) \\ \leq& \pi_{c'}(a) + \sum_{c'' \neq c, c'} a_{ic''} \pi_c(a) = u_i(a) \end{aligned}$$

violating $u_i(a') > u_i(a)$.

Therefore, when $y = 3, \forall x \in \{1,2,3,4\}$: $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

Now we only need to show that when $x = 2, y = 2$, $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

When $i \in I_2(a), c \in C_2(a)$ and $a_{ic} = 0$. Since $i \in I_2(a)$, $d_i(a) = D$. If $a'_{ic'} = a_{ic'}$, $d_i(a') = d_i(a) + 1 = D + 1$,

$$u_i(a') = \sum_{c \in C} a'_{ic} \pi_c(a') - M_2 < \sum_{c \in C} a_{ic} \pi_c(a) = u_i(a)$$

given M_2 is sufficiently large, violating $u_i(a') > u_i(a)$.

If $a'_{ic'} < a_{ic'}$, we know $c' \notin C_3(a)$ since $C_3(a)$ consists of clubs that have no members in a.

Suppose $c' \in C_1(a)$,

$$\begin{aligned}
u_i(a') =& \pi_c(a') + \sum_{c'' \neq c,c'} a_{ic''}\pi_c(a) \\
\leq& \pi^* + \sum_{c'' \neq c,c'} a_{ic''}\pi_c(a) \\
=& \pi_{c'}(a) + \sum_{c'' \neq c,c'} a_{ic''}\pi_c(a) = u_i(a)
\end{aligned}$$

violating $u_i(a') > u_i(a)$.

Suppose $c' \in C_2(a)$, since $a_{ic} = 0$, $a_{ic'} = 1$, we know from condition (b) that $\pi_{c'}(a) > \pi_c(a)$, so

$$\begin{aligned}
f(s_c(a)) = \pi_c(a) < pi_{c'}(a) = f(s_{c'}(a)) \\
s_c(a) < s_{c'}(a) \\
s_c(a') \leq s_{c'}(a) \\
pi_{c'}(a) \geq \pi_{c'}(a)
\end{aligned}$$

and hence

$$\begin{aligned}
u_i(a') =& \pi_c(a') + \sum_{c'' \neq c,c'} a'_{ic''}\pi_c(a) \\
\leq& \pi_{c'}(a) + \sum_{c'' \neq c,c'} a_{ic''}\pi_c(a) = u_i(a)
\end{aligned}$$

given the membership structure for $c'' \neq c, c'$ is the same under a and a'. $u_i(a') \leq u_i(a)$ violates $u_i(a') > u_i(a)$.

So, when $x = 2, y = 2$, $\forall i \in I_x(a), c \in C_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

We have finished the 'if' part. For the 'only if' part. If (i) holds, (a) obviously also holds. So we show (b) holds when (i) and (ii) hold.

Suppose (i) and (ii) hold, but (b) does not hold, then there exists $i \in I_2(a), c, c' \in C_2(a)$ with $a_{ic} > a_{ic'}$ and $\pi_c(a) \leq \pi_{c'}(a)$. Note that since $c' \in C_2(a)$, $s_{c'}(a) < S$. Consider the following deviation from a to a' for i and c'.

$$\begin{aligned} a'_{ic'} &= 1 \\ a'_{ic} &< a_{ic} \\ a'_{i'c'} &= a_{i'c'} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c'} &= a_{i''c'} \\ \forall i'' \neq i, c'' \neq c' : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

So $s_{c'}(a') = s_{c'}(a) + 1 \leq S$, and hence $\pi_{c'}(a') > \pi_{c'}(a) \geq \pi_c(a)$.

$$
\begin{aligned}
u_i(a') =& \pi_c(a') + \sum_{c'' \neq c, c'} a'_{ic''} \pi_c(a') \\
>& \pi_c(a) + \sum_{c'' \neq c, c'} a'_{ic''} \pi_c(a') \\
=& \pi_c(a) + \sum_{c'' \neq c, c'} a_{ic''} \pi_c(a') = u_i(a)
\end{aligned}
$$

since the membership structure for $c'' \neq c, c'$ is the same under a and a'.

So, if (i) and (ii) hold, (b) holds.

We have finished the 'only if' part.

Proof for Remark 3.1

In the smooth club membership model with constant returns from membership overlap, the aggregate utility of individuals given a club membership structure a is

$$
\begin{aligned}
U(a) =& \sum_{i \in I} u_i(a) \\
=& \sum_{i \in I} \left[\sum_{c \in C} a_{ic} \left(F(s_c(a)) + \sum_{c' \neq c} g_{cc'}(a) \right) - \kappa(d_i(a)) \right] \\
=& \sum_{i \in I, c \in C} a_{ic} f(s_c(a)) + \alpha \sum_{i \in I, c \in C, c' \neq c} o_{cc'}(a) - \sum_{i \in I} \kappa(d_i(a)) \\
=& \sum_{i \in I, c \in C} a_{ic} f(s_c(a)) + \alpha \sum_{c \in C, i \in I} a_{ic} s_c(a)(d_i(a) - 1) - \sum_{i \in I} \kappa(d_i(a)) \\
=& \sum_{i \in I, c \in C} a_{ic} f(s_c(a)) + \alpha \sum_{c \in C, i \in I} a_{ic} s_c(a)(d_i(a) - 1) - \sum_{i \in I} d_i(a) \frac{\kappa(d_i(a))}{d_i(a)} \\
=& \sum_{i \in I, c \in C} a_{ic} f(s_c(a)) + \alpha \sum_{c \in C, i \in I} a_{ic} s_c(a)(d_i(a) - 1) - \sum_{i \in I, c \in C} a_{ic} \frac{\kappa(d_i(a))}{d_i(a)} \\
=& \sum_{i \in I, c \in C} a_{ic} \left[f(s_c(a)) + \alpha s_c(a)(d_i(a) - 1) - \frac{\kappa(d_i(a))}{d_i(a)} \right] \\
=& \sum_{i \in I, c \in C} a_{ic} \left[\alpha s_c(a) d_i(a) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_i(a))}{d_i(a)} \right]
\end{aligned}
$$

We first prove our efficiency characterization for the situation when $nd_1^* \geq ms_1^*$ and d_1^* divides ms_1^*.

Let $d_1^*(s)$ be the solution to

$$\max_{d\in\{1,2,\dots,m\}} \alpha sd - \alpha s + f(s) - \frac{\kappa(d)}{d}$$

we have

$$\begin{aligned}
&a_{ic}\left[\alpha s_c(a)d_i(a) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_i(a))}{d_i(a)}\right]\\
\le &a_{ic}\left[\alpha s_c(a)d_1^*(s_c(a)) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_1^*(s_c(a)))}{d_1^*(s_c(a))}\right]
\end{aligned}$$

where the equality is reached if and only $\forall i \in I, c \in C$ with $a_{ic} = 1$: $d_i(a) = d_1^*(s_c(a))$.

So,

$$\begin{aligned}
U(a) \le &\sum_{i\in I, c\in C} a_{ic}\left[\alpha s_c(a)d_1^*(s_c(a)) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_1^*(s_c(a)))}{d_1^*(s_c(a))}\right]\\
= &\sum_{c\in C}\left[\alpha s_c(a)d_1^*(s_c(a)) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_1^*(s_c(a)))}{d_1^*(s_c(a))}\right]\sum_{i\in I} a_{ic}\\
= &\sum_{c\in C}\left[\alpha s_c(a)d_1^*(s_c(a)) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_1^*(s_c(a)))}{d_1^*(s_c(a))}\right] s_c(a)
\end{aligned}$$

Let s_1^* be the solution to

$$\max_{s\in\{1,2,\dots,n\}} \left[\alpha s d_1^*(s) - \alpha s + f(s) - \frac{\kappa(d_1^*(s))}{d_1^*(s)}\right] s$$

we have

$$\left[\alpha s_c(a)d_1^*(s_c(a)) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_1^*(s_c(a)))}{d_1^*(s_c(a))}\right] s_c(a) \le \left[\alpha s_1^* d_1^*(s_1^*) - \alpha s_1^* + f(s_1^*) - \frac{\kappa(d_1^*(s_1^*))}{d_1^*(s_1^*)}\right] s_1^*$$

where the equality is reached if and only $\forall c \in C$: $s_c(a) = s_1^*$.

So,

$$U(a) \le \left[\alpha s_1^* d_1^*(s_1^*) - \alpha s_1^* + f(s_1^*) - \frac{\kappa(d_1^*(s_1^*))}{d_1^*(s_1^*)}\right] s_1^*$$

where the equality is reached if and only $\forall i \in I, c \in C$: $s_c(a) = s_1^*$ and $d_i(a) = d_1^* = d_1^*(s_1^*)$ when $a_{ic} = 1$.

Note that by definition of $d_1^*(s)$ and s_1^*, $d_1^* = d_1^*(s_1^*)$ and s_1^* are the solution to

$$\max_{d\in\{1,2,\ldots,m\},s\in\{1,2,\ldots,n\}} (\alpha sd - \alpha s + F(s) - \frac{\kappa(d)}{d})s \tag{3.12}$$

When $nd_1^* \geq ms_1^*$ and d_1^* divides ms_1^*, there exists a club membership structure $a \in A$ where $\forall i \in I, c \in C$: $s_c(a) = s_1^*$ and $d_i(a) = d_1^*$ when $a_{ic} = 1$. This can be obtained by letting individuals join clubs in sequence. Make each individual join d_1^* clubs that has the smallest membership size at her turn before moving to the next individual. Stop when all clubs have s_1^* members. There will be $\frac{ms_1^*}{d_1^*}$ individuals who join clubs.

We have proved that in the smooth club membership model with constant returns from membership overlap. If $nd_1^* \geq ms_1^*$ and d_1^* divides ms_1^*, then a club membership structure a is efficient if and only if all clubs admits s_1^* members, and there are $\frac{ms_1^*}{d_1^*}$ individuals that join d_1^* clubs and the remaining individuals join no clubs.

We now prove our efficiency characterization for the situation when $ms_2^* \geq nd_2^*$ and s_2^* divides nd_2^*.

Let $s_2^*(d)$ be the solution to

$$\max_{s\in\{1,2,\ldots,n\}} \alpha sd - \alpha s + f(s) - \frac{\kappa(d)}{d}$$

we have

$$\begin{aligned}
&a_{ic}\left[\alpha s_c(a)d_i(a) - \alpha s_c(a) + f(s_c(a)) - \frac{\kappa(d_i(a))}{d_i(a)}\right] \\
\leq &a_{ic}\left[\alpha s_2^*(d_i(a))d_i(a) - \alpha s_2^*(d) + f(s_2^*(d_i(a))) - \frac{\kappa(d_i(a))}{d_i(a)}\right]
\end{aligned}$$

where the equality is reached if and only $\forall i \in I, c \in C$ with $a_{ic} = 1$: $s_c(a) = s_2^*(d_i(a))$.

So,

$$\begin{aligned}
U(a) \leq &\sum_{i\in I,c\in C} a_{ic}\left[\alpha s_2^*(d_i(a))d_i(a) - \alpha s_2^*(d_i(a)) + f(s_2^*(d_i(a))) - \frac{\kappa(d_i(a))}{d_i(a)}\right] \\
= &\sum_{i\in I}\left[\alpha s_2^*(d_i(a))d_i(a) - \alpha s_2^*(d_i(a)) + f(s_2^*(d_i(a))) - \frac{\kappa(d_i(a))}{d_i(a)}\right]\sum_{c\in C} a_{ic} \\
= &\sum_{i\in I}\left[\alpha s_2^*(d_i(a))d_i(a) - \alpha s_2^*(d_i(a)) + f(s_2^*(d_i(a))) - \frac{\kappa(d_i(a))}{d_i(a)}\right] d_i(a)
\end{aligned}$$

Let d_2^* be the solution to

$$\max_{d\in\{1,2,\ldots,m\}}\left[\alpha s_2^*(d)d-\alpha s_2^*(d)+f(s_2^*(d))-\frac{\kappa(d)}{d}\right]d$$

we have

$$\begin{aligned}&\left[\alpha s_2^*(d_i(a))d_i(a)-\alpha s_2^*(d_i(a))+f(s_2^*(d_i(a)))-\frac{\kappa(d_i(a))}{d_i(a)}\right]d_i(a)\\ \leq&\left[\alpha s_2^*(d_2^*)d_2^*-\alpha s_2^*(d_2^*)+f(s_2^*(d_2^*))-\frac{\kappa(d_2^*)}{d_2^*}\right]d_2^*\end{aligned}$$

where the equality is reached if and only $\forall i\in I$: $d_i(a)=d_2^*$.

So,

$$U(a)\leq\left[\alpha s_2^*(d_2^*)d_2^*-\alpha s_2^*(d_2^*)+f(s_2^*(d_2^*))-\frac{\kappa(d_2^*)}{d_2^*}\right]d_2^*$$

where the equality is reached if and only $\forall i\in I,c\in C$: $d_i(a)=d_2^*$ and $s_c(a)=s_2^*=s_2^*(d_2^*)$ when $a_{ic}=1$.

Note that by definition of d_2^* and $s_2^*(d)$, d_2^* and $s_2^*=s_2^*(d_2^*)$ are the solution to

$$\max_{d\in\{1,2,\ldots,m\},s\in\{1,2,\ldots,n\}}(\alpha sd-\alpha s+F(s)-\frac{\kappa(d)}{d})d \tag{3.13}$$

When $ms_2^*\geq nd_2^*$ and s_2^* divides nd_2^*, there exists a club membership structure $a\in A$ where $\forall i\in I,c\in C$: $d_i(a)=d_2^*$ and $s_c(a)=s_2^*$ when $a_{ic}=1$. This can be obtained by letting clubs admit individuals in sequence. Make each club admit s_2^* individuals that has the smallest degree its turn before moving to the next club. Stop when all individuals have degree d_2^*. There will be $\frac{nd_2^*}{s_2^*}$ clubs that have members.

We have proved that in the smooth club membership model with constant returns from membership overlap. If $ms_2^*\geq nd_2^*$ and s_2^* divides nd_2^*, then a club membership structure a is efficient if and only if all individuals join d_2^* clubs, and there are $\frac{nd_2^*}{s_2^*}$ clubs that admit s_2^* members and the remaining clubs admit no members.

Proof for Remark 3.2

To prove remark 2, we know that when a club membership structure a satisfies

(a) $\bigcup\limits_{j=1}^{k+1}\hat{I}_j(a)=I$ and $\bigcup\limits_{j=1}^{k+1}\hat{C}_j(a)=C$,

(b) $|\hat{I}_{k+1}(a)|\times|\hat{C}_{k+1}(a)|=0$,

(c) $\forall i\in\hat{I}_j(a),c\in\hat{C}_{j'}(a)$: if $j\neq j'$, then $a_{ic}=0$.

then

(i) $\forall i \in I$: there is no $a' \in A$ with

$$a'_{ic} \leq a_{ic} \qquad \forall c \in C$$
$$a'_{i'c} = a_{i'c} \qquad \forall i' \neq i, c \in C$$

such that

$$u_i(a') > u_i(a)$$

(ii) $\forall c \in C$: there is no $a' \in A$ with

$$a'_{ic} \leq a_{ic} \qquad \forall i \in I$$
$$a'_{ic'} = a_{ic'} \qquad \forall c' \neq c, i \in I$$

such that

$$\pi_c(a') > \pi_c(a)$$

(iii) $\forall i \in I, c \in C$: there is no $a' \in A$ with

$$a'_{ic'} \leq a_{ic'} \qquad \forall c' \neq c$$
$$a'_{i'c} \leq a_{i'c} \qquad \forall i' \neq i$$
$$a'_{i'c'} = a_{i'c'} \qquad \forall i' \neq i, c' \neq c$$

such that

$$u_i(a') > u_i(a)$$
$$\pi_c(a') > \pi_c(a)$$

First we show that if (a), (b) and (c) hold, then (i) is true, i.e., no individual has an incentive to quit any set of clubs. Note that since $\bigcup_{j=1}^{k+1} \hat{I}_j(a) = I$, and $d_i(a) = 0$ when $i \in \hat{I}_{k+1}(a)$, we only need to show $\forall i \in \hat{I}_j(a)$, $j \in \{1, 2, ..., k\}$, i has no incentive to quit any clubs.

If $i \in \hat{I}_j(a)$, we know

$$u_i(a) = \hat{d}^j \hat{\pi}(\hat{d}^j) - \kappa(\hat{d}^j)$$

Consider $a' \in A$ with

$$\begin{aligned} a'_{ic} &\leq a_{ic} && \forall c \in C \\ a'_{i'c} &= a_{i'c} && \forall i' \neq i, c \in C \end{aligned}$$

where $\sum_{c \in C}(a_{ic} - a'_{ic}) = x$, so x captures the number of clubs i quits by moving from a to a'. So $d_i(a') = d_i(a) - x$.

We know

$$\begin{aligned} \pi_c(a') &= F(s_c(a')) + \alpha a'_{ic}(d_i(a') - 1) + \alpha \sum_{i' \neq i} a'_{i'c}(d_{i'}(a') - 1) \\ &= F(s_c(a')) + \alpha a'_{ic}(d_i(a) - x - 1) + \alpha \sum_{i' \neq i} a'_{i'c}(d_{i'}(a') - 1) \end{aligned}$$

since the membership structure for $i' \neq i$ is the same under a and a',

$$\pi_c(a) = F(s_c(a')) + \alpha a'_{ic}(d_i(a) - x - 1) + \alpha \sum_{i' \neq i} a_{i'c}(d_{i'}(a) - 1)$$

If $a_{ic} = a'_{ic} = 1$, then $s_c(a') = s_c(a)$ and

$$\begin{aligned} \pi_c(a) &= F(s_c(a)) + \alpha a_{ic}(d_i(a) - 1) + \alpha \sum_{i' \neq i} a_{i'c}(d_{i'}(a) - 1) - \alpha x \\ &= \pi_c(a) - \alpha x \end{aligned}$$

So, in a', the utility of i is

$$\begin{aligned} u_i(a') &= \sum_{c \in C} a'_{ic}\pi_c(a') - \kappa(\hat{d}^j - x) \\ &= \sum_{c \in C} a'_{ic}(\hat{\pi}(\hat{d}^j) - \alpha x) - \kappa(\hat{d}^j - x) \\ &= (\hat{d}^j - x)(\hat{\pi}(\hat{d}^j) - \alpha x) - \kappa(\hat{d}^j - x) \end{aligned}$$

We know that the utility of i in a is

$$u_i(a) = \hat{d}^j \hat{\pi}(\hat{d}^j) - \kappa(\hat{d}^j)$$

So

$$
\begin{aligned}
u_i(a)-u_i(a') =& \hat{d}^j\hat{\pi}(\hat{d}^j)-\kappa(\hat{d}^j)-\left[(\hat{d}^j-x)(\hat{\pi}(\hat{d}^j)-\alpha x)-\kappa(\hat{d}^j-x)\right] \\
=& \hat{d}^j\hat{\pi}(\hat{d}^j)-\hat{d}^j\hat{\pi}(\hat{d}^j)+x\hat{d}^j\hat{\pi}(\hat{d}^j)+x\alpha\hat{d}^j-x\alpha-\left[\kappa(\hat{d}^j)-\kappa(\hat{d}^j-x)\right] \\
=& x\left[\hat{d}^j\hat{\pi}(\hat{d}^j)+\alpha(\hat{d}^j-1)\right]-\left[\kappa(\hat{d}^j)-\kappa(\hat{d}^j-1)+\ldots+\kappa(\hat{d}^j-(x-1))-\kappa(\hat{d}^j-x)\right] \\
=& \hat{d}^j\hat{\pi}(\hat{d}^j)+\alpha(\hat{d}^j-1)-\Delta\kappa(\hat{d}^j-1) \\
& +\hat{d}^j\hat{\pi}(\hat{d}^j)+\alpha(\hat{d}^j-1)-\Delta\kappa(\hat{d}^j-2) \\
& +\ldots \\
& +\hat{d}^j\hat{\pi}(\hat{d}^j)+\alpha(\hat{d}^j-1)-\Delta\kappa(\hat{d}^j-x)
\end{aligned}
$$

Since $\hat{d}^j\hat{\pi}(\hat{d}^j)+\alpha(\hat{d}^j-1)>\Delta\kappa(\hat{d}^j-1)>\Delta\kappa(\hat{d}^j-2)>\ldots>\Delta\kappa(\hat{d}^j-x)$,

$$u_i(a)-u_i(a')>0$$

when $x>0$. So, $\forall i\in I$: there is no $a'\in A$ with

$$
\begin{aligned}
&a'_{ic}\le a_{ic} && \forall c\in C \\
&a'_{i'c}=a_{i'c} && \forall i'\neq i, c\in C
\end{aligned}
$$

such that

$$u_i(a')>u_i(a)$$

Then we show that if (a), (b) and (c) hold, then (ii) is true, i.e., no club has an incentive to exile any set of members.

Since $\bigcup_{j=1}^{k+1}\hat{C}_j(a)=C$ and and $s_c(a)=0$ when $c\in\hat{C}_{k+1}(a)$, we only need to show $\forall c\in \hat{c}_j(a)$, $j\in\{1,2,\ldots,k\}$, c has no incentive to exile any members.

If $c\in\hat{c}_j(a)$, c admits $\hat{s}^j$ members with degree $\hat{d}^j$. Since

$$\hat{s}^j=\hat{s}^j(\hat{d}^j)=\arg\max_{s\in\mathbb{R}}F(s)+\alpha s(\hat{d}^j-1)$$

c has the optimal membership size and would not want to exile any member.

Finally, we show that if (a), (b) and (c) hold, then (iii) is true.

Since $\bigcup_{j=1}^{k+1}\hat{I}_j(a)=I$ and $\bigcup_{j=1}^{k+1}\hat{C}_j(a)=C$ and given that $\Delta\kappa(x)$ is increasing and $\Delta F(x)$ is decreasing, we only need to show that $\forall x\in\{1,2,\ldots,k+1\}, y\in\{1,2,\ldots,k+1\}$: $\forall i\in$

$\hat{I}_x(a), c \in \hat{C}_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned} a'_{ic} &= 1 \\ a'_{ic'} &\leq a_{ic'} \\ a'_{i'c} &\leq a_{i'c} \\ \forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\ \forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\ \forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''} \end{aligned}$$

such that

$$\begin{aligned} u_i(a') &> u_i(a) \\ \pi_c(a') &> \pi_c(a) \end{aligned}$$

First, consider the incentive of a joint deviation from i and c in the same layer. Since $|\hat{I}_{k+1}(a)| \times |\hat{C}_{k+1}(a)| = 0$, we know there does not exist a pair of i and c where $i \in \hat{I}_{k+1}(a)$ and $c \in \hat{C}_{k+1}(a)$. So consider $i \in \hat{I}_j(a), c \in \hat{C}_j(a)$, $j \in \{1, 2, ..., k\}$.

If $a'_{ic'} < a_{ic'}$, we have $d_i(a') = d_i(a) = \hat{d}^j$. Since $c \in \hat{C}_j(a)$, we know c admits $\hat{s}^j$ members with degree $\hat{d}^j$. Since

$$\hat{s}^j = \hat{s}^j(\hat{d}^j) = \arg\max_{s \in \mathbb{R}} F(s) + \alpha s(\hat{d}^j - 1)$$

c would not benefit by admitting individual i or replacing a member i' with i.

If $a'_{ic'} = a_{ic'}$, we have $d_i(a') = d_i(a) + 1 = \hat{d}^j + 1$. Under this case, we first look at how the productivity of c changes.

When $a'_{i'c} = a_{i'c}$, $s_c(a') = s_c(a) + 1 = \hat{s}^j + 1$,

$$\begin{aligned} \pi_c(a') =& F(s_c(a')) + \alpha a'_{ic}(d_i(a') - 1) + \alpha \sum_{i'' \neq i} a'_{i''c}(d_{i''}(a') - 1) \\ =& F(\hat{s}^j + 1) + \alpha \hat{d}^j + \alpha \hat{s}^j(\hat{d}^j - 1) \\ =& F(\hat{s}^j) + \alpha \hat{s}^j(\hat{d}^j - 1) + \Delta F(\hat{s}^j)\alpha(\hat{d}^j - 1) + \alpha \end{aligned}$$

When $a'_{i'c} < a_{i'c}$, $s_c(a') = s_c(a) = \hat{s}^j$,

$$\begin{aligned}\pi_c(a') =&F(s_c(a')) + \alpha a'_{ic}(d_i(a') - 1) + \alpha \sum_{i'' \neq i,i'} a'_{i''c}(d_{i''}(a') - 1)\\ =&F(\hat{s}^j) + \alpha \hat{d}^j + \alpha(\hat{s}^j - 1)(\hat{d}^j - 1)\\ =&F(\hat{s}^j) + \alpha \hat{s}^j(\hat{d}^j - 1) + \alpha\\ =&\pi_c(a) + \alpha\end{aligned}$$

Since

$$\hat{s}^j = \hat{s}^j(\hat{d}^j) = \arg\max_{s \in \mathbb{R}} F(s) + \alpha s(\hat{d}^j - 1)$$

we know

$$\Delta F(\hat{s}^j)\alpha(\hat{d}^j - 1) \leq 0$$

so $\pi_c(a') \leq \pi_c(a) + \alpha = \hat{\pi}(\hat{d}^j) + \alpha$ and the equality is reached when $a'_{i'c} < a_{i'c}$.

Then, still under the case of $a'_{ic'} = a_{ic'}$ which implies $d_i(a') = d_i(a) + 1 = \hat{d}^j + 1$, we look at how the productivity of $c'' \neq c$ changes.

$$\begin{aligned}\pi_{c''}(a') =&F(s_{c''}(a')) + \alpha a_{ic''}(d_i(a') - 1) + \alpha \sum_{i'' \neq i} a'_{i''c''}(d_{i''}(a') - 1)\\ =&F(s_{c''}(a)) + \alpha a_{ic''}(d_i(a) + 1 - 1) + \alpha \sum_{i'' \neq i} a_{i''c''}(d_{i''}(a) - 1)\\ =&F(s_{c''}(a)) + \alpha a_{ic''}(d_i(a) - 1) + \alpha \sum_{i'' \neq i} a_{i''c''}(d_{i''}(a) - 1) + \alpha a_{ic''}\\ =&\pi_{c''}(a) + \alpha a_{ic''}\end{aligned}$$

Now we look ar how the utility of i changes,

$$\begin{aligned}u_i(a') =&a'_{ic}\pi_c(a') + \sum_{c'' \neq c} a'_{ic''}\pi_{c''}(a') - \kappa(d_i(a'))\\ =&\pi_c(a') + \sum_{c'' \neq c} a'_{ic''}(\pi_{c''}(a) + \alpha a_{ic''}) - \kappa(hatd^j + 1)\\ =&\pi_c(a') + \hat{d}^j(\hat{\pi}(\hat{d}^j)) + \alpha \hat{d}^j - \kappa(hatd^j) - \Delta\kappa(hatd^j)\\ =&u_i(a) + \pi_c(a') + \alpha \hat{d}^j - \Delta\kappa(hatd^j)\\ \leq&u_i(a) + \hat{\pi}(\hat{d}^j) + \alpha + \alpha \hat{d}^j - \Delta\kappa(hatd^j)\end{aligned}$$

where the equality can be reached when $a'_{i'c} < a_{i'c}$.

Since

$$\Delta\kappa(hatd^j) > \hat{\pi}(\hat{d}^j) + \alpha(\hat{d}^j + 1)$$

we have

$$u_i(a') < u_i(a)$$

So i would not benefit by joining club c.

We have shown that $\forall x \in \{1,2,...,k+1\}, y = x$: $\forall i \in \hat{I}_x(a), c \in \hat{C}_y(a)$: if $a_{ic} = 0$, there is no $a' \in A$, $c' \neq c$, $i' \neq i$ with

$$\begin{aligned}
a'_{ic} &= 1 \\
a'_{ic'} &\leq a_{ic'} \\
a'_{i'c} &\leq a_{i'c} \\
\forall c'' \neq c, c' : a'_{ic''} &= a_{ic''} \\
\forall i'' \neq i, i' : a'_{i''c} &= a_{i''c} \\
\forall i'' \neq i, c'' \neq c : a'_{i''c''} &= a_{i''c''}
\end{aligned}$$

such that

$$\begin{aligned}
u_i(a') &> u_i(a) \\
\pi_c(a') &> \pi_c(a)
\end{aligned}$$

Now we consider the incentive of a joint deviation for i and c in different layers. Consider $i \in \hat{I}_x(a), c \in \hat{C}_y(a)$.

If $x > y$, so that the club is in a higher layer, then

$$d_i(a') \leq d_i(a) + 1 \leq \hat{d}^y$$

we know c has $\hat{s}^y$ members with degree $\hat{d}^y$. Since

$$\hat{s}^y = \hat{s}^y(\hat{d}^y) = \arg\max_{s \in \mathbb{R}} F(s) + \alpha s(\hat{d}^y - 1)$$

c would not benefit by admitting individual i or replacing a member i' with i.

If $y > x$, so that the individual is in a higher layer, since i cannot benefit by joining a club in the same layer, it cannot benefit by joining a club in a lower layer.

So if (a), (b) and (c) hold, then (iii) is true.

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