Slow Relaxation and Diffusion in Holographic Quantum Critical Phases

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The dissipative dynamics of strongly interacting systems are often characterized by the timescale set by the inverse temperature $\tau_P \sim \hbar/(k_B T)$. We show that near a class of strongly interacting quantum critical points that arise in the infrared limit of translationally invariant holographic theories, there is a collective excitation (a quasinormal mode of the dual black hole spacetime) whose lifetime τ_{eq} is parametrically longer than τ_P: $\tau_{\mathrm{eq}} \gg T^{-1}$. The lifetime is enhanced due to its dependence on a dangerously irrelevant coupling that breaks the particle-hole symmetry and the invariance under Lorentz boosts of the quantum critical point. The thermal diffusivity (in units of the butterfly velocity) is anomalously large near the quantum critical point and is governed by τ_{eq} rather than τ_P. We conjecture that there exists a long-lived, propagating collective mode with velocity v_s, and in this case the relation $D = v_s^2 \tau_{\mathrm{eq}}$ holds exactly in the limit $T\tau_{\mathrm{eq}} \gg 1$. While scale invariance is broken, a generalized scaling theory still holds provided that the dependence of observables on the dangerously irrelevant coupling is incorporated. Our work further underlines the connection between dangerously irrelevant deformations and slow equilibration.

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In many-body quantum systems with strong interactions, the characteristic timescales relevant for a variety of dynamical processes are short, and are set by the inverse temperature $\tau_P = \hbar/(k_B T)$ [1]. For example, τ_P has been shown to control the onset of hydrodynamics in holographic plasmas, the postquench equilibration of the Sachdev-Ye-Kitaev model, as well as the Lyapunov exponent characterizing the growth rate of chaos in both of the aforementioned kinds of theories [2–7]. Transport measurements in the strange metallic phase of high-T_c superconductors (HTSCs) [8,9] further support the conjecture that τ_P fundamentally bounds the dynamics of strongly correlated phases [10–14].

Indeed, in the vicinity of a quantum critical point (QCP), T is the only energy scale and so the importance of τ_P is manifest [15]. However, there are circumstances in which nonuniversal effects are important and lead to dynamics that survive on timescales much longer than τ_P. The most familiar example is near a QCP where translational symmetry is broken by an irrelevant coupling g [16–20], leading to the slow relaxation of momentum and a parametrically small resistivity. More generally, whenever the dynamics near a QCP is sensitive to a dangerously irrelevant coupling, τ_P is no longer privileged since the irrelevant coupling provides an additional energy scale [21]. In such situations, it is not obvious what the relevant timescales for dynamical processes are.

We study a class of strongly interacting, $(d+1)$-dimensional, translationally invariant systems whose infrared (IR) physics are governed by hyperscaling violating QCPs with dynamical exponent $z = 1$. The particle-hole symmetry and the invariance under Lorentz boosts of the $T = 0$ IR QCP are broken by an irrelevant deformation with coupling $g \propto \rho$ the density of the state. We show that in these systems the incoherent current (i.e., the part of the electric current without momentum drag [22]) acquires a long lifetime τ_{eq}

$$\tau_{\mathrm{eq}} \sim \tau_P \left(\frac{T^{\Delta_g}}{g}\right)^2, \tag{1}$$

which is parametrically longer than τ_P, $\tau_{\rm eq} \gg T^{-1}$, and is controlled by the dimension of the coupling $\Delta_g < 0$. While we expect typical excitations to have a lifetime τ_P, it is only after a time $\tau_{\rm eq}$ that local equilibration will be achieved and the expected hydrodynamic behavior will take over. The slowly relaxing mode produces a narrow peak in the optical conductivity

$$\sigma(\omega) = \frac{\rho^2}{sT + \mu\rho}\frac{i}{\omega} + \frac{\sigma_o}{(1 - i\omega\tau_{\rm eq})}, \tag{2}$$

where σ_o and $\tau_{\rm eq}$ are given by Eq. (13), ρ is the charge density, μ the chemical potential, and s the entropy density. We expect that adding slow momentum relaxation to our theories (as in, e.g., Refs. [23–25]) will broaden the divergent $\omega \to 0$ contribution to the conductivity, Eq. (2), into a Drude-like peak. The interplay between multiple irrelevant deformations can be subtle but important for transport near QCPs [19,20,26–28].

We study these systems using gauge-gravity duality, where the IR QCP is captured by a spacetime metric that is conformal to AdS_{d+2} and is a solution of Einstein-Dilaton theories with an exponential potential [29,30]. It is important to note that our models do not capture competing phases on either side of a QCP, only the dynamics of the quantum critical region itself. The irrelevant deformation is realized by a Maxwell field, with exponential coupling to the dilaton, that backreacts on this spacetime and drives a renormalization group (RG) flow to a nonzero density ultraviolet (UV) fixed point. In gravitational language, we show that certain charged, translationally invariant, asymptotically AdS_{d+2} black branes have quasinormal modes with parametrically long lifetimes $\sim\tau_{\rm eq}$.

Near the QCP, we furthermore show that $\tau_{\rm eq}$ is the timescale relevant for transport processes that do not involve the dragging of momentum. Specifically, at times $t \gtrsim \tau_{\rm eq}$, these processes are diffusive. Near the QCP, they are characterized by a single diffusivity D_T (the thermal diffusivity) where

$$D_T = \frac{2}{d + 1 - \theta} v_B^2 \tau_{\rm eq}, \tag{3}$$

θ is a universal number quantifying the violation of hyperscaling at the QCP, and v_B is the "butterfly" velocity at which quantum chaos spreads. The large value of D_T resulting from its sensitivity to irrelevant deformations was established in Ref. [25], and was in potential tension with the upper bounds on diffusivities proposed to ensure the causality of diffusive hydrodynamics [31,32]. The result [Eq. (3)] elegantly resolves this potential tension: at precisely the timescales at which causality appears to be violated, the diffusive hydrodynamic description breaks down due to the existence of the slowly relaxing mode. This is a consequence of the nontrivial fact that both D_T and $\tau_{\rm eq}$ are governed by the same irrelevant deformation of the QCP.

A number of recent works have established relations similar to Eq. (3) between thermal diffusivities and the spreading of quantum chaos [12,25,33–40]. In holographic theories, these have always been of the form $D_T \sim v_B^2 \tau_P$. Our result, Eq. (3), lends further support to the claim that in general the timescale appearing in this relation should be $\tau_{\rm eq}$, and not τ_P or the Lyapunov time τ_L (which governs the growth rate of quantum chaos) [31,32]. These timescales could not be distinguished in previous examples, which had $\tau_{\rm eq} \sim \tau_L \sim \tau_P$ [41]. Our results are also nontrivially consistent with the quantum hydrodynamic theory for maximally chaotic systems proposed in Ref. [40] and explored in Ref. [42]. The result $D_T \sim v_B^2 \tau_P$ follows from this theory provided that diffusive hydrodynamics applies at timescales $t \sim \tau_P$. Assuming the validity of this theory for the holographic QCPs we study, the parametrically large value of D_T therefore implies that hydrodynamics must break down at timescales $t \sim \tau_{\rm eq} \gg \tau_P$, as we explicitly show.

Another consequence of the additional energy scale g in the IR theory is the violation of naive ω/T scaling in response functions near the QCP. We close by illustrating this explicitly, and by showing that if one carefully takes into account the g dependence of the critical contribution to the conductivity, a generalized scaling theory [23,43–45], which has been applied to dc transport in cuprate strange metals [46], continues to hold. Nontrivial scaling theories near QCPs are attractive from a phenomenological point of view: we know that if the strange metallic phase of high T_c superconductors does originate from a QCP, then it cannot be governed by a simple, scale invariant theory, as such a theory is inconsistent with the observed T-linear resistivity [47].

In the remainder of this Letter, we describe our setup and outline the calculations leading to the results mentioned above. We have also found analogous results to Eqs. (1) and (3) in a closely related class of systems that are particle-hole symmetric and flow to QCPs with dangerously irrelevant translational symmetry-breaking deformations. The results for these systems, along with a number of technical details, are presented in Ref. [45].

Holographic quantum criticality.—In holographic theories, quantum critical states with dynamical exponent $z = 1$ can be described by the $(d+2)$-dimensional Einstein-Dilaton action [29]

$$S_{\rm crit} = \int d^{d+2}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - V_0 e^{-\delta\phi}\right). \tag{4}$$

We study theories where Eq. (4) is the effective action capturing the low temperature dynamics far from the boundary of an asymptotically AdS_{d+2} spacetime. By identifying the extra spatial dimension with the energy scale of a dual quantum field theory, Eq. (4) describes the

IR dynamics that arise at the endpoint of a RG flow generated by deforming an UV CFT. The details of the RG flow will determine the constants V_0 and δ, but are otherwise not important for our analysis.

The quantum critical dynamics are captured by the following classical solutions of the action [29,30], in which the metric transforms covariantly under the $z = 1$ rescaling $(t, \vec{x}) \mapsto \lambda(t, \vec{x})$

$$ds^2 = \left(\frac{u}{L}\right)^{2(\theta/d)-2}(-L_t^2 dt^2 + \tilde{L}^2 du^2 + L_x^2 d\vec{x}^2),$$
$$\tilde{L}^2 = \frac{(d+1-\theta)(d-\theta)}{-V_0},$$
$$\phi = \kappa \ln\left(\frac{u}{L}\right), \qquad \kappa^2 = \frac{2}{d}\theta(\theta - d), \qquad \kappa\delta = 2\frac{\theta}{d}. \quad (5)$$

u is the radial coordinate in the IR region of the spacetime $u \gg L$. The running dilaton leads to violation of hyperscaling, parametrized by $\theta < 0$ (consistent with the null energy condition). At small temperatures, the entropy density $s \sim T^{d-\theta}$ [48] and so the critical state can be thought of as a "CFT" in $(d-\theta)$ spatial dimensions [49,50]. L_t, L_x, and L are functions of the deformations of the UV fixed point, and depend on the details of the RG flow. These length scales typically depend smoothly on the scalar source at the boundary (the deformation of the UV CFT) as it is varied over a continuous range of real values. Each such value allows us to represent a distinct QCP. From a gravitational perspective these are perhaps better thought of as quantum critical lines [14,51–53].

The RG flow away from the IR critical point produces corrections to the solution Eq. (5) in inverse powers of u/L. For our purposes, the most important correction comes from the Maxwell action

$$\Delta S_{\text{irr}} = \int d^{d+2}x\sqrt{-g}\frac{Z_0}{4}e^{\gamma\phi}F_{\mu\nu}F^{\mu\nu}, \quad (6)$$

where the constants Z_0 and γ depend on the details of the flow to the UV fixed point. γ encodes the dimension of an irrelevant deformation, as we will shortly illustrate.

Solving the Maxwell equations in the spacetime [Eq. (5)] gives the profile of the gauge field at leading order in large u/L

$$A = A_0\left(\frac{u}{L}\right)^{\zeta-1}L_t dt, \qquad \zeta = d - \kappa\gamma - (d-2)\frac{\theta}{d}. \quad (7)$$

The density $\rho = -ZC^{d/2}A'/\sqrt{BD} \propto A_0$ at $T = 0$, so while the gauge field does not backreact on the metric at the QCP, particle-hole symmetry is broken at all temperatures. A_0 is the bulk quantity corresponding to the dangerously irrelevant coupling g we referred to in the introduction. Indeed, the gauge field sources corrections to the solution Eq. (5) for the metric and dilaton, which at leading order in A_0 are $\sim 1 + \#A_0^2 u^{2\Delta_{A_0}}$ with $\Delta_{A_0} = (d - \theta + \zeta)/2$. This is an irrelevant deformation if $\Delta_{A_0} < 0$ (so the corrections vanish as $u/L \to \infty$), which we demand from now on. Treating u as an energy scale in the usual way indeed determines the dimension of the irrelevant coupling A_0 to be Δ_{A_0} and that of the corresponding irrelevant operator to be $\Delta_{\text{irr}} = d + 1 - \theta - \Delta_{A_0}$ [45]. Therefore, $\Delta_g = \Delta_{A_0}$ in Eq. (1).

Charge response near the QCP.—In order to compute the optical conductivity, we embed the preceding IR theory into a complete holographic RG flow described by the action

$$S = \int d^{d+2}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{Z(\phi)}{4}F^2 - V(\phi)\right), \quad (8)$$

where $V(\phi)$ and $Z(\phi)$ are chosen to reproduce the IR action (4)+(6) as $\phi \to \infty$. The states we are interested in are captured by the ansatz for the metric $ds^2 = -D(r)dt^2 + B(r)dr^2 + C(r)d\vec{x}^2$, gauge field $A = A(r)dt$, and scalar $\phi = \phi(r)$. r is a radial coordinate that goes to zero at the boundary, where the metric is asymptotically AdS and $A(0) = \mu \neq 0$ defines the chemical potential of the state. We are interested in thermal states, and so we assume there is a regular black brane horizon at $r = r_h > 0$, where $D(r \to r_h) = 4\pi T(r_h - r) + \cdots$, $B(r \to r_h) = 1/(4\pi T(r_h - r)) + \cdots$, $C(r \to r_h) = C_h + \cdots$, $\phi(r \to r_h) = \phi_h + \cdots$, $A(r \to r_h) = A_h(r_h - r) + \cdots$ The charge and entropy densities are given by the r-independent expressions $\rho = -ZC^{d/2}A'/\sqrt{BD} = Z_h A_h C_h^{d/2}$ and $s = -[\rho A - C^{1+d/2}(D/C)'/\sqrt{BD}]/T = 4\pi C_h^{d/2}$, where $Z_h \equiv Z(\phi(r_h))$. We are mainly interested in the low T solutions that reduce to Eq. (5) in the IR as $T \to 0$.

The optical conductivity is given by

$$\sigma(\omega) \equiv -\frac{i}{\omega}\lim_{r\to 0}\left(r^{2-d}\frac{a_x'(r)}{a_x(r)}\right), \quad (9)$$

where a_x is the ingoing linear perturbation of the spatial component of the gauge field and obeys the equation [22]

$$\frac{d}{dr}[FG\tilde{a}_x'] + \omega^2\frac{G}{F}\tilde{a}_x = 0, \quad (10)$$

with $\tilde{a}_x \equiv a_x/(sT + \rho A)$, $F \equiv \sqrt{D/B}$, $G \equiv ZC^{(d/2)-1} \times (sT + \rho A)^2$.

To calculate the low frequency optical conductivity, we use the usual perturbative ansatz [54]

$$\tilde{a}_x = c\left(\frac{r_h - r}{r_h}\right)^{-i\omega/4\pi T}\left(1 + \sum_{n=1}^{\infty}\left(\frac{i\omega}{4\pi T}\right)^n \mathcal{A}_n(r)\right), \quad (11)$$

where $\mathcal{A}_n(r_h) = 0$. Substituting this into Eq. (10) and solving at $O(\omega)$ gives

$$\mathcal{A}_1(r) = \int_{r_h}^{r} d\tilde{r}\left[\alpha\frac{C}{D}\frac{d}{d\tilde{r}}\left(\frac{1}{sT + \rho A}\right) - \frac{1}{r_h - \tilde{r}}\right], \quad (12)$$

where $\alpha = s^3T^3Z_h\rho^{-2}(s/4\pi)^{-2/d}$. This results in an optical conductivity [Eq. (2)] where

$$\sigma_o = \frac{s^2T^2Z_h}{(sT+\rho\mu)^2}\left(\frac{s}{4\pi}\right)^{1-2/d}, \qquad \tau_{\rm eq} = -\frac{\mathcal{A}_1(0)}{4\pi T}. \tag{13}$$

The first term in the optical conductivity [Eq. (2)] is the usual small ω divergence due to momentum conservation, while the second term arises from charge-carrying processes in which no momentum flows [22]. The pole in the second term at $\omega = -i\tau_{\rm eq}^{-1}$ indicates the existence of a collective excitation with lifetime $\tau_{\rm eq}$. The result [Eq. (13)] for $\tau_{\rm eq}$ can only be trusted if $\tau_{\rm eq}T \gg 1$, as the perturbative expansion is reliable for $\omega \ll T$.

For a low T state that is sufficiently close to the QCP described by Eqs. (5) and (7), we will now verify that indeed $\tau_{\rm eq}$ is parametrically longer than T^{-1}. The deep IR geometry of such a state will have an event horizon at a large value of $u = u_h$, but will still be described by Eqs. (5) and (7) over the range $u_{\rm IR} > u > u_{\rm UV}$, with $u_h \gg u_{\rm IR}$ and $u_{\rm UV} \gg L$. Integrating over this part of the spacetime yields a contribution to $\tau_{\rm eq}$ that is independent of the cutoffs [45]:

$$\tau_{\rm eq} = \frac{\tilde{L}(d+1-\theta)}{L_tZ_0(1-\zeta)^2}\frac{1}{A_0^2}\left(\frac{u_h}{L}\right)^{1-2\Delta_{A_0}} \sim \frac{1}{T}\frac{T^{2\Delta_{A_0}}}{A_0^2}. \tag{14}$$

Recalling that $\Delta_{A_0} < 0$, this contribution to $\tau_{\rm eq}$ is parametrically larger than T^{-1} and should dominate the full integral in the limit $T \to 0$. It is manifest that the irrelevant deformation sourced by A_0 is responsible for the slow relaxation of the mode, and indeed $\tau_{\rm eq}$ is of the form given in Eq. (1) with $g \sim A_0$ and $\Delta_g = \Delta_{A_0}$.

Counterparts of the QCPs [Eq. (5)] with $z \neq 1$ are well known [29,30,48]. For these solutions, the deformation parametrized by A_0 is marginal ($\Delta_{A_0} = 0$), and the integral for $\tau_{\rm eq}$ is no longer dominated by the IR spacetime. In these cases we expect $\tau_{\rm eq} \sim 1/T$, as has been observed numerically in a variety of holographic theories [55–61].

Diffusivity and hydrodynamics.—As mentioned above, there are two distinct contributions to the small ω optical conductivity [Eq. (2)]. The divergence at $\omega \to 0$ is due to current (J) flow that drags (conserved) momentum (P), while the remainder is due to current flow that does not. The latter processes can be conveniently isolated by examining the dynamics of the "incoherent" current $J_{\rm inc} \equiv \chi_{PP}J - \chi_{JP}P$, where χ denote static susceptibilities [22]. We will concentrate on $J_{\rm inc}$: its small ω conductivity $\sigma_{\rm inc}(\omega)$ is proportional to the second term of Eq. (2), and is sensitive to the slowly relaxing mode [62].

Over sufficiently long timescales, we expect relativistic hydrodynamics to govern the system and thus the conductivity of $J_{\rm inc}$ to be ω independent [22]. From Eq. (2), it is apparent that this is the case at times $t \gg \tau_{\rm eq}$. In this regime, long wavelength perturbations of $J_{\rm inc}$ and its associated charge $\delta\rho_{\rm inc} \equiv s^2T\delta(\rho/s)$ diffuse with the usual diffusivity D of relativistic hydrodynamics (see, e.g., Ref. [63]). D obeys the Einstein relation $D = \sigma_{\rm inc}^{\rm dc}/\chi_{\rm inc}$ where $\sigma_{\rm inc}^{\rm dc} = (sT+\mu\rho)^2\sigma_o$ and $\chi_{\rm inc}$ is the static susceptibility of $\delta\rho_{\rm inc}$. While in general $\chi_{\rm inc}$ depends in a complicated way on the thermodynamic properties of the state, near a QCP it simplifies to $\chi_{\rm inc} = \rho^2T^2(\partial s/\partial T)_\rho$ [45]. Furthermore, as $\sigma_{\rm inc}^{\rm dc}$ is related to the open-circuit thermal conductivity κ by $\sigma_{\rm inc}^{\rm dc} = T\rho^2\kappa$ in a relativistic hydrodynamic system [45], near the QCP D is equal to the thermal diffusivity $D_T \equiv \kappa/(T\partial s/\partial T)_\rho$. Using our explicit results [Eq. (13)] for holographic theories, in addition to the temperature scaling of s, both diffusivities near the QCP can be written simply as Eq. (3).

The relation [Eq. (3)] is possible because D_T, v_B, and $\tau_{\rm eq}$ are all related to near-horizon properties of the dual black hole [64]. This fact also lies behind the existence of a relation analogous to Eq. (3) for $z \neq 1$ QCPs, with $\tau_{\rm eq}$ replaced by τ_P [25,34]. But unlike in those cases, where D_T and v_B are both properties of the QCP, for the $z = 1$ cases at hand the relation Eq. (3) relies crucially on the fact that both D_T and $\tau_{\rm eq}$ depend in the same way on the irrelevant deformation away from the QCP sourced by A_0. This is also different to the case of $z = \infty$, $\theta = 0$ QCPs, where a relation similar to Eq. (3) with $\tau_{\rm eq}$ replaced by τ_P arises due to the fact that both D_T and v_B^2 are determined by the same irrelevant coupling [35–37].

At times $t \lesssim \tau_{\rm eq}$, relativistic hydrodynamics is not applicable to the system since it doesn't incorporate the dynamics of the slowly relaxing mode that appears at times $t \sim \tau_{\rm eq}$. Since we expect typical excitations near the QCP to have lifetimes $\sim T^{-1} \ll \tau_{\rm eq}$, then it may be possible to identify an effective theory valid to earlier times $t \gtrsim T^{-1}$ by supplementing the hydrodynamic equations to incorporate the existence of the slowly relaxing mode [66]. In the Supplemental Material [76], we compute holographically the other entries in the matrix of retarded Green's functions for J and P and show they match those of a hydrodynamic theory with a slowly decaying mode $J_{\rm inc}$: $\partial_t J_{\rm inc} = -J_{\rm inc}/\tau_{\rm eq}$, using standard techniques [63,77,78]. Such effective theories typically display pole collisions in the lower half frequency plane, whereby a diffusive mode acquires a real part and turns into a propagating mode at short distances. The velocity v_s of this propagating mode then determines the diffusivity $D = v_s^2\tau_{\rm eq}$ (see, eg, (2.17) of Ref. [67]). For Eq. (3) to take this form, we require a velocity $v_s^2 = 2v_B^2/(d+1-\theta) = 1/(d-\theta)$. It is known [49,50] that $z = 1$, $\theta \neq 0$ theories contain a mode with this velocity in their spectrum. We therefore conjecture that this mode of the IR spacetime is promoted to a mode of the full asymptotically AdS spacetime [79], and thus it is because $D = v_s^2\tau_{\rm eq}$ that Eq. (3) is realized. We plan to confirm this picture in Ref. [80], using the techniques developed in Ref. [68].

In light of this discussion, it would be interesting to identify for $z \neq 1$ QCPs (where the irrelevant deformation is unimportant) a lifetime $\tau_{\rm eq} \sim 1/T$ and velocity $v_s^2 \sim v_B^2$ of a collective mode such that $D_T = v_s^2 \tau_{\rm eq}$. Such a relation would indicate that it is not the butterfly velocity v_B that fundamentally sets the thermal diffusivity, but instead that Eq. (3) arises due to a relation between the velocities of collective modes and the butterfly velocity near quantum critical points.

Breakdown of ω/T scaling.—The existence of a collective mode with the parametrically long lifetime $\tau_{\rm eq}$ is the most striking consequence of the breakdown in quantum critical scaling caused by the dangerously irrelevant coupling A_0, but it is not the only one. It was previously shown that the conductivity $\sigma_{\rm inc}(\omega, T)$ does not exhibit ω/T scaling near the QCPs [Eq. (5)]: specifically, $\sigma_{\rm inc}(\omega, T = 0) \sim \omega^{-\zeta}$ [29,30,43] while $\sigma_{\rm inc}^{\rm dc} \sim T^{\zeta+2(d-\theta)}$ [22]. By carefully keeping track of the dependence on A_0 [45], we can explicitly attribute this breakdown in ω/T scaling to the presence of the irrelevant coupling in the IR theory:

$$\sigma_{\rm inc}^{\rm dc} \sim T^{d-\theta+2\Delta_{A_0}}, \qquad \sigma_{\rm inc}(T=0) \sim A_0^4 \omega^{d-\theta-2\Delta_{A_0}}. \tag{15}$$

Recalling that $\Delta_{A_0} = (d - \theta + \zeta)/2$, it is clear that when $z = 1$ we can consistently assign $\sigma_{\rm inc}$ the dimension $\zeta + 2(d - \theta)$, and that ω/T scaling fails because of the nontrivial dependence of $\sigma_{\rm inc}$ on the irrelevant coupling A_0.

In contrast, near the $z \neq 1$ counterparts of the QCPs [Eq. (5)] where A_0 sources a marginal deformation $\Delta_{A_0} = 0$, the incoherent conductivity obeys ω/T scaling: $\sigma_{\rm inc}(\omega, T = 0) \sim \omega^{2+(d-2-\theta)/z}$ [23,29,30,43] and $\sigma_{\rm inc}^{\rm dc} \sim T^{2+(d-2-\theta)/z}$ [22].

In both cases ($z = 1$ and $z \neq 1$), the scaling theory required to account for the total dimension of $\sigma_{\rm inc}$ is nontrivial. It involves anomalous dimensions for both the entropy density $\Delta_s = d - \theta$ (i.e., hyperscaling violation) and the charge density $\Delta_\rho = d - \theta + \Phi$ [23,43,44] and is explained in more detail in Ref. [45]. The anomalous dimension for charge density Φ is related to the profile of the Maxwell field [Eq. (7)] by $\Phi = (\zeta + \theta - d)/2$, and thus $\Delta_\rho = \Delta_{A_0}$ (consistent with our previous observation that $\rho \propto A_0$ at $T = 0$). Note that the close relation between ρ and A_0, supplemented by a matched asymptotics argument, is at the root of why the charge response near the QCP is sensitive to the irrelevant deformation sourced by A_0.

Both the anomalous dimensions, and the extra dimensionful coupling A_0, permit a much richer family of T dependence in the quantum critical contribution to the conductivity [Eq. (15)] than is allowed in a simple scale invariant theory [23,44], and may be necessary to explain the various scalings observed in strange metals [46,47].

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