STRONGLY ANISOTROPIC TYPE II BLOW UP AT AN ISOLATED POINT

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Abstract. We consider the energy super critical $d+1$ dimensional semilinear heat equation

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^{d+1}, \quad p \geq 3, \ d \geq 14.$$

A fundamental open problem on this canonical nonlinear model is to understand the possible blow up profiles appearing after renormalization of a singularity. We exhibit in this paper a new scenario corresponding to the first example of strongly anisotropic blow up bubble: the solution displays a completely different behaviour depending on the considered direction in space. A fundamental step of the analysis is to solve the *reconnection problem* in order to produce finite energy solutions which is the heart of the matter. The corresponding anistropic mechanism is expected to be of fundamental importance in other settings in particular in fluid mechanics. The proof relies on a new functional framework for the construction and stabilization of type II bubbles in the parabolic setting using energy estimates only, and allows us to exhibit new unexpected blow up speeds.

1. Introduction

1.1. **A new type of singularity.** We deal here with the semi-linear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, \\ u_{|t=0} = u_0 \end{cases} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad p>1, \quad n \geq 1, \tag{1.1}$$

for which solutions dissipate the total energy of the flow

$$\frac{d}{dt}E(u) \leq 0, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u(t,x)|^{p+1} dx.$$

It admits a scaling invariance: if $u(t,x)$ is a solution then so is $\lambda^{-2/(p-1)}u(t/\lambda^2, x/\lambda)$ for any $\lambda > 0$. Smooth well localised initial data u_0 yield unique local in time solutions [2, 66]. Their maximal time of existence is finite $0 < T < \infty$ if and only if

$$\lim_{t \uparrow T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = \infty.$$

We then say that x_0 is a blow-up point if $\limsup_{t\uparrow T} \|u(t)\|_{L^\infty(|x-x_0|\leq \delta)} = \infty$ for any $\delta > 0$. Singularity formation for Equation (1.1) is part of the larger problem of the description of singular dynamics for general semi-linear equations such as the semi-linear wave and Schrödinger equations:

$$u_{tt} = \Delta u + |u|^{p-1}u \quad \text{and} \quad iu_t = \Delta u + |u|^{p-1}u. \tag{1.2}$$

Two fundamental problems are the determination of the blow-up rates and blow-up profiles. For Equation (1.1) the blow-up rate refers to the obtention of an equivalent of $\|u(t)\|_{L^\infty}$ as $t \to T$. Blow-up profiles are nonlinear structures emerging after renormalisation, here for *finite energy* singularities. A large part of the literature on these equations concerns either radially symmetric cases or energy critical and subcritical cases $p \leq 1+4/(n-2) = p_c$, while recent progress deal with the nonradial supercritical case (see below). The outcome is a robust analysis that could address more problems including geometrical flows and fluid dynamics. We construct in this paper, in particular, a *truly anisotropic blow-up* in the energy supercritical case above a critical exponent, the Joseph-Lundgren one $p > p_{JL} > p_c$:

$$p_{JL} = \begin{cases} +\infty & \text{for } n \leq 10, \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{for } n \geq 11, \end{cases} \tag{1.3}$$

We believe the result can be adapted to other equations, including the two above.

1.2. **On singularities of the semi-linear heat equation.** For singular solutions of (1.1) there holds the lower bound $\liminf_{t\uparrow T}(T-t)^{1/(p-1)}\|u(t)\|_{L^\infty} > 0$ by a scaling argument. Blow-up rates are then divided in two categories.

$$\left| \begin{array}{ll} \text{The blow-up is Type I if} & \limsup_{t\uparrow T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} < \infty. \\ \text{The blow-up is Type II if} & \limsup_{t\uparrow T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty} = \infty. \end{array} \right.$$

All solutions are believed to display some kind of self-similarity at the singularity. A further refinement is then the obtention of the blow-up profiles and scaling laws, giving a leading order expansion of the solution near the singularity. Three dynamics are known for $p > 1$, corresponding to three natural regimes: for the ODE blow-up the diffusion is lower order, for Type I non ODE all terms in (1.1) scale the same way, and for type II blow-up the time variation ∂_t is lower order. Let us first go over the existence results, and then the classification ones.

ODE blow-up. This is the first scenario to have been studied, since the pioneering results of Giga and Kohn [26, 27, 28, 29, 30]. For all $p > 1$ the equation $u' = |u|^{p-1}u$ admits the solution $\kappa(T - t + b|x|^2)^{-1/(p-1)}$ where $\kappa = (p-1)^{-1/(p-1)}$ and $b > 0$. It persists in the full equation (1.1). For all $p > 1$ a stable dynamics exists where $u \sim \kappa(T - t + b(t)|x|^2)^{-1/(p-1)}$, the diffusion becoming negligible but affecting the scale through the law $b \sim b^*|\log(T-t)|$, see [1, 32, 3, 50]. After the self similar renormalisation

$$u(t,x) = \frac{1}{(T-t)^{\frac{1}{p-1}}}v(\tau,y), \quad \tau = -\log(T-t), \quad y = \frac{x}{\sqrt{T-t}}, \tag{1.4}$$

which maps (1.1) onto the renomalized flow

$$\partial_\tau v = \Delta v - \frac{1}{2}\Lambda v + v^p, \quad \Lambda v = \frac{2}{p-1}v + y\cdot\nabla v, \tag{1.5}$$

such solutions converge to the constant κ that is a stationary state of (1.5).

Type I non ODE. New phenomena appear in the energy critical and supercritical range $p \geq p_c$. A third critical exponent exists, that of Lepin $p_L > p_{JL}$ [38]. In the range $p_c < p < p_L$, other radially symmetric stationary solutions Ψ to (1.5) which are localised have been constructed using ODE techniques [38, 63, 4, 5] or a direct bifurcation argument [6, 15]. They produce a truly self-similar blow-up $u \sim (T-t)^{-1/(p-1)}\Psi(x/\sqrt{T-t})$ with parabolic scale. They have been shown recently to be nonlinearly finite codimensionally stable within a class of finite energy initial data in the nonradial case [15]. In the radial case, they appear as threshold dynamics between global existence and the above mentioned ODE blow up [41].

Type II blow-up. For $p > p_{JL}$, new type II blow-up solutions appear, again as threshold between global existence and ODE blow-up [41]. The dynamical proofs were proposed in [33, 40, 53]. Their construction has been revisited recently in the nonradial case [11]. The structure of the singularity is deeply related to the *singular* self similar solution of (1.5)

$$\Phi^*(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}}, \quad c_\infty(p,n) = \left[\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right]^{\frac{2}{p-1}} \tag{1.6}$$

and its smooth regularisation at the origin given by the soliton profile

$$\begin{cases} Q'' + \frac{n-1}{r}Q' + Q^p = 0, \\ Q(0) = 1, \quad Q'(0) = 0. \end{cases} \tag{1.7}$$

These type II solutions concentrate a stationary state $u \sim \lambda^{-2/(p-1)}Q(x/\lambda)$ to leading order, at a smaller scale than the parabolic one $\lambda \ll \sqrt{T-t}$. The blow up rate is given by a countable sequence

$$\|u(t)\|_{L^\infty} \sim \frac{1}{(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha(d)}}}, \quad \ell \in \mathbb{N}, \quad \ell > \frac{\alpha}{2},$$

where $\alpha = \alpha(n,p) > 2$ given by (1.19) is one of the key phenomenological number of the super critical numerology. Type II blow-up solutions also exist in the critical cases $p = p_c$ and $p = p_{JL}$ [25, 60, 61].

Classification. While ODE blow-up exists and is stable for all $p > 1$, the existence of type I non ODE and and type II blow-up is not always guaranteed.

- In the energy subcritical case $1 < p < p_c$, the ODE dynamics is the only one: all singular solutions are type I and converge to κ after the renormalisation (1.4) [26, 27, 28, 29, 30, 51, 13, 64, 65, 67, 68].
- For $p_c < p < p_{JL}$ there is no Type II blow-up in the radial case [40].
- For $p > p_{JL}$ the above mentioned type II blow-up rates are the only possibility in the radial case, [53]. This leaves the nonradial case open.

1.3. **Other semi-linear equations.** Singularity formation for semi-linear equations is a major nonlinear phenomenon. Due to the availability of maximum principle and parabolic regularity, the heat equation (1.1) has been extensively studied. The results mentioned above however possess analogues for other equations, such as the wave and Schrödinger equations (1.2), yielding a *unified picture*, in particular via our approach [10, 11, 15, 34, 44, 46, 48, 49, 50, 55]. This is why we believe the new dynamics described in the present paper can also appear in such other contexts.

ODE blow-up exists for the wave equation. It is known to be the only dynamics in the subconformal range $p < 1 + 3/(n-1)$ [52]. It still generates a stable dynamics in the critical and supercritical cases (see [21] and references therein), where other additional self-similar solutions were found [8]. Type II blow-up was showed in the energy critical case [37, 34, 35] and only recently in the supercritical one $p > p_{JL}$ [10]. The supercritical or nonradial cases remain mostly open.

For the semi-linear Schrödinger equation the situation is even less understood. There is no analogue of ODE blow-up, and in the mass subcritical case $p < 1+4/n$ solutions are global. In the mass critical case $p = 1+4/n$ type II solutions exist: the conformal blow-up [9] and the log-log blow-up [44]. True self-similar behaviour is expected in the mass supercritical case [22] and has been proved in the slightly mass supercritical case [47]. Between the mass and energy critical cases $1+4/n < p < p_c$ type II dynamics still exist [48]. Nothing is rigorously known in the energy supercritical case, apart from the existence of type II blow-up for $p > p_{JL}$ [46].

1.4. **Anisotropic blow up bubbles.** Despite substantial efforts, the above known nonlinear structures all correspond to isotropic profiles, concentrating at an isolated blow-up point. The only other known geometry for the singular set is the sphere [55]. Few examples concern interacting bubbles [16, 42, 20] which produce additional dynamics. In various classical nonlinear models, *anisotropy* and the possiblity of completly different behaviours in the various directions in space is expected to be an essential feature of the problem. This is typically the case for the anisotropic nonlinear Schrödinger equation

$$i\partial_t u + \partial_{xx} u - \partial_{yy} u + u|u|^{p-1} = 0, \quad (x,y) \in \mathbb{R}^2$$

which arises from classical fluid mechanics models [62], and for which the blow up problem is completely open. More generally, the question here is to understand how dramatically unbounded solutions to the *time dependent* renormalized flow (1.5) may correspond to finite energy singularities. This problem is the *reconnection problem* and is the heart of the forthcoming analysis. Let us stress that this is mostly an unexplored domain, and that even formal predictions are to our knowledge unclear.

We solve in this paper the anisotropic reconnection problem for $p > p_{JL}$. We consider a canonical situation: for a space dimension

$$n = d + 1,$$

our goal is to lift a lower d-dimensional blow-up ensuring the solution remains of finite energy. Namely, any radially symmetric solution $U(t,r)$, $r \in \mathbb{R}^d$, to the d dimensional nonlinear heat equation provides a solution to the $d+1$ model with cylindrical symmetry, $x = (r,z) \in \mathbb{R}^d \times \mathbb{R}$, by considering $u(t,r,z) = U(t,r)$. This solution corresponds to a *line* singularity along the additional e_z axis, but has infinite energy for the $\mathbb{R}^{d+1}$ model. We claim that given $U(t,r)$ a type II blow up bubble for the d-dimensional radially symmetric problem, we may *solve the reconnection problem* for the additional z direction and produce *finite energy* $d+1$ dimensional initial data which after renormalization are very elongated along the e_z direction, and eventually reconnect $U(t,r)$ to a decreasing profile in the z direction. The associated reconnection profile is *universal and rigid.* This eventually produces a *blow-up at an isolated point and not a line singularity* but with an elongated pancake like profile in self similar variables and new blow up rates.

Theorem 1.1 (Existence and finite codimensional stability of anistropic blow up bubbles)**.** *Let* $\alpha = \alpha(d,p)$, $\Delta = \Delta(d,p)$ *be the super critical numbers given by* (1.17), (1.19)*, and assume:*

$$d \geq 11, \quad p \geq \max\{3, p_{JL}(d)\}, \quad \sqrt{\Delta} > 2, \quad \alpha \notin 2\mathbb{N}. \tag{1.8}$$

Pick

$$\ell \in \mathbb{N}^* \quad \textit{with} \quad \ell > \frac{\alpha}{2}.$$

Then there exists a finite codimensional set of initial data $u_0 \in \mathcal{C}^\infty_c(\mathbb{R}^{d+1}, \mathbb{R})$ *with cylindrical symmetry such that the corresponding solution* $u(t,x)$ *to* (1.1) *with* $n = d+1$ *blows up in finite time* $0 < T < +\infty$ *with the following asymptotics. The solution admits on* $[0,T)$ *in self similar variables* (1.4) *a decomposition*

$$v(t,r,z) = \frac{1}{D(t,z)^{\frac{2}{p-1}}} Q\left(\frac{r}{D(t,z)}\right) + V(t,r,z)$$

where Q *denotes the d-dimensional smooth radially symmetric soliton profile* (1.7)*, and with the following sharp description:*

1. Computation of the reconnection*: there holds*

$$D(t,z)=\sqrt{b(t)}(1+a(t)P_{2\ell}(z))^{\frac{1}{\alpha}} \tag{1.9}$$

where $P_{2\ell}(z)$ *is the* 2ℓ*-th one dimensional Legendre polynomial given by* (1.27)*, and* $(a,b)\in\mathcal{C}^1([0,T),\mathbb{R}_+^*)$ *with the sharp asymptotics near blow up time:*

$$b(t)=b^*(1+o_{t\to T}(1))(T-t)^{\frac{2\ell-\alpha}{\alpha}},\quad 0<b^*(u_0), \tag{1.10}$$

$$a(t)=a^*(1+o_{t\to T}(1)),\quad 0<a^*(u_0)\ll 1. \tag{1.11}$$

2. Soliton profile and blow up speed*:*

$$\lim_{t\to T}(\sqrt{b})^{\frac{2}{p-1}}\|V(t,\cdot)\|_{L^\infty}=0 \tag{1.12}$$

and

$$\|u(t,\cdot)\|_{L^\infty}=\frac{c(u_0)(1+o(1))}{(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha}}},\quad c(u_0)>0. \tag{1.13}$$

3. Isolatedness of the blow-up point*: The origin is the only blow-up point.*

Comments on the result.

1. New $d+1$*-dimensional blow up rates.* The blow up speed (1.13) is new and unexpected, and shows that all the type II blow up rates of the d-dimensional problem are admissible blow up speeds for the $d+1$ dimensional problem. This also shows that the classification of all type II blow up speeds for *radially symmetric data* obtained in [54] using maximum principle like arguments *no longer holds* for non radially symmetric data. The method clearly designs an iteration process for this dimensional reduction procedure. The extension of this result to the energy critical case and the construction of new non radial type II blow up bubbles below the Joseph Lundgren exponent is work in progress.

2. Structure of the reconnection profile. In the companion paper [49], we address a similar result in the context of type I blow up with decreasing at infinity self similar profile. The analysis of type I blow up is simpler, and the reconnection profile is universal

$$D(t,z)\sim\sqrt{1+b(t)z^2},\quad b(t)\sim\sqrt{\log(T-t)} \tag{1.14}$$

which is reminiscent to the stability of the ODE type I blow up [3, 50]. The structure of the reconnection profile (1.9) of type II blow up is more complicated and the associated moving free boundary $r=D(t,z)$ locating the region where the singularity is large has a non trivial geometrical description. This shapes follows from an *all order* algebraic cancellation, Lemma 3.1, related to the fact that the parameter a is nearly constant in time according to (1.11), and this was very much unexpected.

3. On technical assumptions. The first one is the restriction (1.8). The optimal range of exponents for the existence of radially symmetric type II blow up in $\mathbb{R}^d$ is $p>p_{JL}(d)$ which is equivalent to $\Delta>0$. However $p_{JL}(d)\to 1$ as $d\to+\infty$ and this causes lack of differentiability of the nonlinearity. From direct check, the assumption (1.8) is automatically satisfied for $p\geq 3$, $d\geq 14$ and it will avoid additional technicalities when Δ is small. For $\alpha\in 2\mathbb{N}$, some logarithmic corrections appear. The second technical assumption is the restriction to cylindrical data. We expect that the solutions of the Theorem persist in the non-cylindrical case, up to controlling an additional finite number of instability directions (see [11] for a related issue).

4. Stability of the dynamics. We expect that the solutions described by Theorem 1.1 correspond to a finite codimensional C^1 manifold of initial data in $H^3(\mathbb{R}^n)$, for T fixed and p large enough. In [10, 43] the Lipschitz and C^1 regularity respectively are obtained for related problems, and we expect that a similar strategy could be applied here that is the following. The heart of the proof of Theorem 1.1 is Proposition 3.4, giving the existence of a true solution staying close to an approximate one. This proposition provides, given an initial choice for a stable part of the perturbation, with the existence of an initial choice for the finitely many unstable modes so that the perturbation remains forward in time in the bootstrap regime. The regularity of the aforementioned set then corresponds to that of the mapping which to the initial stable part associates the finite dimensional initial unstable one. It could be obtained in a second step, using the a priori estimates on the solutions that have been provided by Proposition 3.4.

We are now ready to state some open problems about the blow-up rates for equation (1.1) in the energy supercritical case $p>p_c$. Given any $p>1$ we introduce:

$$n^*(p)=\min\{\tilde{n}\in\mathbb{N},\ p_{JL}(\tilde{n})<p\}.$$

Note $11\leq n^*(p)<\infty$ and $p_{JL}(n)<p$ for all $n\geq n^*$. We recall that the number $\alpha(n,p)$ is defined in (1.19).

Problem 1 (Persistence of all lower-dimensional supercritical type II blow-up)**.** *Consider Equation* (1.1) *for some* $p>0$ *and* $n\in\mathbb{N}$ *such that* $p>p_{JL}(n)$*. Then for any* $n^*(p)\leq n'\leq n$ *and any* $\ell\in\mathbb{N}$ *with* $2\ell>\alpha(n',p)$*, there exists a smooth and localised solution* u *to* (1.1) *that blows up at time* $T>0$ *with, for some* $C>0$*:*

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)}\underset{t\uparrow T}{\sim}C(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha(n',p)}}.$$

We believe the blow-up rates mentioned in the above Problem (1) could be the only possible ones for isolated singularities.

Problem 2 (Classification of blow-up rates in the case of an isolated singularity)**.** *Prove or disprove the following. Assume* $p>1$*,* $n\in\mathbb{N}$*, and* u *is a smooth, localised, and singular solution to* (1.1) *that has only one blow-up point. Assume further the non-degeneracy condition.*

$$\text{For all } n'\leq n: \quad p\neq 1+\frac{4}{n'-2},\ p\neq p_{JL}(n'),\quad \text{and } \alpha(n',p)\notin 2\mathbb{N}\ \text{ if } p>p_{JL}(n'). \tag{1.15}$$

- *If* $p<p_{JL}(n)$ *then the blow-up is of type I, that is, there exists* $C>0$ *with:*

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)}\underset{t\uparrow T}{\sim}C(T-t)^{\frac{1}{p-1}}.$$

- *If the blow-up is of type II, then there exists* $n^*(p)\leq n'\leq n$*, an* $\ell\in\mathbb{N}$ *with* $2\ell>\alpha(n',p)$*, and a constant* $C>0$ *with:*

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)}\underset{t\uparrow T}{\sim}C(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha(n',p)}}.$$

Problem 1 has been solved in the case $n'=n$, with references given in the introduction. Our present result solves it for $n'=n-1$, under the technical assumption (1.8), and gives a roadmap to investigate the remaining cases $n'\leq n-2$. Problem 2 is a hard and interesting open problem. We mention that the two above conjectures should be supplemented by considering the degenerate cases related to (1.15).

Namely if $\alpha \in 2\mathbb{N}$ logarithmic corrections could appear. In the case where there exists $n' \leq n$ with $p = p_{JL}(n')$, and the case where there exists $n' \leq n$ such that $p = 1 + 4/(n' - 2)$. The case $p = p_{JL}(n)$ has been investigated recently [61], while for $p = 1 + 4/(n - 2)$ we refer to [25, 60, 12]

The heart of the proof of Theorem 1.1 is the computation of the free boundary (1.9) which is the zone where the soliton Q profile dominates, and the development of a suitable functional setting to control the flow for solutions which after renormalization are nearly constant in z for $|z| \leq z^*(t)$, $z^*(t) \to +\infty$ as $t \to T$. A similar issue occurs for the study of the type I ODE blow up where the blow up profile is given by the constant profile κ. The analysis developed in [3, 50] amounts to first the formal derivation of the reconnection profile, and then the control of suitable L^∞ bounds for the perturbation. This last step is essential and sees the reconnection procedure, and is performed in [3, 50] using propagator estimates for the linearized flow close to κ *which is explicit*, and in [50] Liouville type classification theorems to rule out some possible growth scenario.

This approach seems hardly applicable in the setting of type II blow up bubbles, and the analysis of the proof of Theorem 1.1 requires two main new inputs.

1. New approach to type II blow up. First we propose a *new functional framework* for the study of type II blow up bubbles which relies on a Lyapounov type bifurcation argument to study the flow near the soliton Q seen as a *non compact* perturbation of the singular profile Φ^*. Here we clearly adopt a point view which is already central in the pioneering breakthrough work [33], but the main novelty is that the bifurcation argument allows us to construct the *exact time dependent* spectral basis needed for the analysis which avoids complicated matching procedures, and trivializes the computation of modulation equations and the derivation of the type II blow up speeds. Such an approach was implemented for the first time in [34] for the study of the Stefan melting problem which formally corresponds to a zero soliton case. The computation of the eigenvectors, Proposition 2.2, involves a universal sequence of functions, see (2.13), which is already central in the *tail computation* developed in [46, 56, 45, 57] and makes the link between the approaches developped in [33] and [46].

2. L^∞ bounds. Once the flow is controlled in suitable weighted norms, it remains to close the nonlinear term using L^∞ bounds. For the d-dimensional parabolic problem, the difficulty is at the origin $r = 0$ and this can be done in various ways using for example an elementary maximum principle like argument [7], or a direct brute force energy method. Treating the full cylindrical problem is more complicated. We first need to compute the free boundary $D(t, z)$, and here we rely on the spectacular algebra (3.7). Once the reconnection is computed, we derive L^∞ bounds from $W^{1,q}$ energy estimates for the linearized flow which are particularly efficient both at the origin and infinity in space. Here we use the parabolic structure again and the repulsive nature of the linearized operator close to Q *when measured in suitable weighted norms*, section 4.

3. Comment on other techniques. Recently, different gluing methods were developed by del Pino, Musso, Wei and co-authors [14, 17, 19, 20] in the context of critical parabolic equations. These methods improve on former matched asymptotic techniques [33, 24] by bringing a powerful treatment of the inner and outer zones

at the blow-up location. We believe these techniques are complementary to ours, as in particular the use of maximum principle allow for somewhat shorter proofs. The adaptation to truly energy supercritical cases where nonlinear object have slow decay at infinity or hyperbolic and dispersive equations where time oscillations are present remains however unclear. We believe in contrast the strategy of the present paper to be adaptable to other dispersive settings, such as the semi-linear wave and Schrödinger equations.

Hence the proof designs a new route map for the study type II blow up bubbles which uses in an optimal way the parabolic structure of the problem with respect to the pioneering works [57, 11]. The parabolic structure allows us to work with self adjoint operators and energy estimates which considerably simplify the analysis, but the essence of the argument which relies on energy estimates only may in principle be propagated to more complicated dispersive Schrödinger or wave like problems.

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Notations. We let

$$Y=(y,z)\in\mathbb{R}^{d+1},\quad r=|y|=\left(\sum_1^d y_i^2\right)^{\frac12}$$

and define the japanese bracket:

$$\langle z\rangle=\sqrt{1+z^2}.$$

We say that $u(Y)$ has even cylindrical symmetry if

$$u(Y)=u(r,z)=u(r,-z).$$

We note the generator of scaling

$$\Lambda=\Lambda_r+z\partial_z,\quad \Lambda_r=\frac{2}{p-1}+r\partial_r.$$

We let the roots of the quadratic equation

$$\gamma(d-2-\gamma)=pc_\infty^{p-1}. \tag{1.16}$$

be

$$\gamma=\frac{d-2-\sqrt{\Delta}}{2},\quad \Delta=\left(\frac{d-2}{2}\right)^2-pc_\infty^{p-1}>0\ \text{ for }\ p>p_{JL}(d) \tag{1.17}$$

and

$$\gamma_2=d-2-\gamma>\gamma. \tag{1.18}$$

We let for an arbitrary constant $\epsilon>0$

$$\alpha=\gamma-\frac{2}{p-1}>2,\quad g=\min\{\alpha,\sqrt{\Delta},2\}-\epsilon=2-\epsilon \tag{1.19}$$

because of the assumption (1.8), meaning that g is arbitrarily close to 2. We let $Q(r)$ be the unique radially symmetric solution to

$$\left|\begin{array}{l} Q''+\frac{d-1}{r}Q'+Q^p=0\\ Q(0)=1,\quad Q'(0)=0\end{array}\right.$$

and recall the following properties [39, 36]. $p > p_{JL}$ implies

$$0 < Q(r) < \Phi^*(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}}, \quad \Lambda Q(r) > 0 \tag{1.20}$$

where Φ^*, c_∞ are given by (1.6). At infinity,

$$Q(r) = \Phi^*(r) - \frac{c}{r^\gamma} + O\left(\frac{1}{r^{\gamma+\delta}}\right), \quad \text{as } r \to +\infty, \; c > 0, \; \delta \geq g, \tag{1.21}$$

which propagates for derivatives, where $0 < \delta < \min(\sqrt{\Delta}, \alpha)$ is a constant arbitrarily close to $\min(\sqrt{\Delta}, \alpha)$, and there holds in particular the fundamental cancellation:

$$\Lambda Q(r) = \frac{\tilde{c}}{r^\gamma} + O\left(\frac{1}{r^{\gamma+\delta}}\right) = \frac{\tilde{c}}{r^\gamma} + O\left(\frac{1}{r^{\gamma+g}}\right) \quad \text{as } r \to +\infty, \tag{1.22}$$

with $\tilde{c} = (\gamma - 2/(p-1))c \neq 0$, c defined above in (1.21). Given $b > 0$, we define

$$Q_b(r) = \frac{1}{(\sqrt{b})^{\frac{2}{p-1}}} Q\left(\frac{r}{\sqrt{b}}\right), \tag{1.23}$$

the linearized operators

$$\mathcal{L}_b = -\Delta_Y + \frac{1}{2}\Lambda - pQ_b^{p-1}, \quad H_b = -\Delta_r + \frac{1}{2}\Lambda_r - pQ_b^{p-1} \tag{1.24}$$

and the weights

$$\rho_Y(Y) = \frac{1}{2^d \pi^{d-\frac{1}{2}}} e^{-\frac{|Y|^2}{4}}, \quad \rho_r(r) = e^{-\frac{r^2}{4}}, \quad \rho_z = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} \tag{1.25}$$

so that for $f : \mathbb{R}^{d+1} \to \mathbb{R}$ with cylindrical symmetry:

$$\int_{Y \in \mathbb{R}^{d+1}} f(Y) \rho_Y dY = \int_{r=0}^{+\infty} \int_{z \in \mathbb{R}} f(r,z) r^{d-1} \rho_r \rho_z dr dz.$$

We denote by $L^2(\rho_Y)$, $L^2(\rho_r)$ the associated spaces of square integrable functions with scalar products $\langle \cdot \rangle_{L^2_{\rho_Y}}$ and $\langle \cdot \rangle_{L^2_{\rho_r}}$. We define

$$H = -\Delta_r - pQ^{p-1} \tag{1.26}$$

the linearized operator close to Q. We let

$$P_m(z) = c_m \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!} (-1)^k z^{m-2k} \tag{1.27}$$

be the m-th one dimensional Hermite polynomial which solves

$$-\partial_z^2 P_m + \frac{1}{2} z \partial_z P_m = \frac{m}{2} P_m, \quad m \in \mathbb{N} \tag{1.28}$$

and the normalization (implying $P_0 = 1$)

$$(P_m, P_{m'})_{L^2_{\rho_z}} = \delta_{mm'}. \tag{1.29}$$

All along the paper, we fix once and for all an integer

$$\ell > \frac{\alpha}{2}$$

and the associated set of indices

$$\mathcal{I} := \left\{ (j,k) \in \mathbb{N}^2, \quad 0 \leq j \leq \ell, \quad \middle| \begin{array}{ll} 1 \leq k \leq \ell - 1 & \text{for } \; j = 0 \\ 0 \leq k \leq \ell - j & \text{for } \; 1 \leq j \leq \ell \end{array} \right\} \tag{1.30}$$

A quantity b^δ denotes a small gain with a constant $\delta = \delta(\ell) > 0$ universal and small enough, independent of any of the constants that will appear in a central bootstrap argument. In the whole paper, we use the notation

$$\eta(a) = o_{a\to 0}(1).$$

For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ we use the notation

$$(x)_0 = 1, \quad (x)_k = x \times (x+1) \times ... \times (x+k-1) \text{ for } k \geq 1. \tag{1.31}$$

2. **A perturbative spectral theorem**

We start the analysis of the renormalized flow (1.5) by the diagonalization of the linearized operator closed to a concentrated soliton given by (1.24). This approach, initiated in [34] for the Stefan problem, is the conceptual link between the pioneering approach Herrero and Velazquez [33] and the soliton approach of Merle, Raphaël, Rodnianski [46]. This will produce an elementary functional framework to compute the blow up speed, and will allow us to control the full flow in the anisotropic geometry.

2.1. **Reduction of the problem.**

We first recall that the spectrum of the linearized operator H_∞ close to Φ^* is explicit.

Proposition 2.1 (Diagonalization of H_∞, [33])**.** *Assume $p > p_{JL}$. There exists a domain $\mathcal{D} \subset H^1_{\rho_r}$ with $H^2_{\rho_r} \subset \mathcal{D}$ such that $H_\infty : \mathcal{D} \to L^2(\rho_r)$ given by*

$$H_\infty = -\Delta + \frac{1}{2}\Lambda - \frac{pc_\infty^{p-1}}{r^2}$$

is self adjoint with compact resolvant. The spectrum in the radial sector is given by

$$\lambda_{i,\infty} = i - \frac{\alpha}{2}, \quad i \in \mathbb{N}$$

with eigenfunctions

$$\phi_{i,\infty}(r) = \frac{n!}{\left(\frac{d}{2}-\gamma\right)_n} \frac{L_i^{(\frac{d}{2}-\gamma-1)}\left(\frac{r^2}{2}\right)}{r^\gamma} = \sum_{j=0}^{i} c_{i,j} C_j r^{2j-\gamma}$$

where L is the modified Laguerre polynomial, and $c_{i,j}$ and C_j are defined by (2.19) *and* (2.15)*. Moreover, there holds the spectral gap estimate: $\forall u \in H^1_{\rho_r}$ with radial symmetry,*

$$\langle u, \phi_{i,\infty}\rangle_{L^2_{\rho_r}} = 0, \quad 0 \leq i \leq \ell \;\Rightarrow\; \langle H_\infty u, u\rangle_{L^2_{\rho_r}} \geq \lambda_{\ell+1,\infty}\|u\|^2_{L^2_{\rho_r}}. \tag{2.1}$$

.

We claim that this allows us to diagonalize the operator H_b given by (1.24) in the radial sector of R^d in the perturbative regime $0 < b \ll 1$.

Proposition 2.2 (Partial diagonalisation of H_b)**.** *Assume* (1.8)*. There exists a universal domain $\mathcal{D} \subset H^1_{\rho_r}$ with $H^2_{\rho_r} \subset \mathcal{D}$ such that for $b > 0$, $H_b : \mathcal{D} \to L^2(\rho_r)$ is self adjoint with compact resolvant. Moreover, for all $\ell \in \mathbb{N}$, there exists $C(\ell), c(\ell) > 0$ and $0 < b^*(\ell) \ll 1$ such that for all $0 < b \leq b^*(\ell)$, the following holds:*
1. Eigenvalue computation*: the i-th eigenvalue of H_b, $0 \leq i \leq \ell$, is:*

$$\lambda_{i,b} = i - \frac{\alpha}{2} + \tilde\lambda_{i,b}, \quad \textit{with } |\tilde\lambda_i| \leq C(\ell) b^{\frac{g}{2}}, \tag{2.2}$$

and it is associated to the eigenfunction

$$\phi_{i,b}=\sum_{j=0}^{i}c_{i,j}\sqrt{b}^{2j-\gamma}T_j\left(\frac{r}{\sqrt{b}}\right)+\tilde{\phi}_{i,b},\quad with\ \|\tilde{\phi}_{i,b}\|_{H^1_{\rho_r}}\leq C(\ell)b^{\frac{g}{2}} \tag{2.3}$$

where the coefficients $c_{i,j}$ *are defined by* (2.19) *and* $(T_i)_{i\in\mathbb{N}}$ *is defined by* (2.13).
2. Estimates*: there holds*

$$\|\phi_{i,b}-\phi_{i,\infty}\|_{H^1_{\rho_r}}\leq C(\ell)b^{\frac{g}{2}}. \tag{2.4}$$

and the more precise pointwise bounds: for $k=0,1,2$,

$$|\partial_r^k\phi_{i,b}|\lesssim\frac{\langle r\rangle^{2i+4}}{(\sqrt{b}+r)^{\gamma+k}},\quad |\partial_r^k b\partial_b\phi_{i,b}|\lesssim\frac{b^{\frac{g}{2}}}{(\sqrt{b}+r)^{g}}\frac{\langle r\rangle^{2i+4}}{(\sqrt{b}+r)^{\gamma+k}}. \tag{2.5}$$

$$|\partial_r^k\tilde{\phi}_{i,b}|+|\partial_r^k b\partial_b\tilde{\phi}_{i,b}|\lesssim\frac{b^{\frac{g}{2}}(1+r)^{2i+4}}{(\sqrt{b}+r)^{\gamma+k}}. \tag{2.6}$$

Moreover:

$$|b\partial_b\tilde{\lambda}_{i,b}|\leq C(\ell)b^{\frac{g}{2}},\quad \|\ b\partial_b\tilde{\phi}_{i,b}\ \|_{H^1_\rho}\leq C(\ell)b^{\frac{g}{2}}. \tag{2.7}$$

3. Spectal gap estimate*: there exists* $c(\ell)>0$*,* $\forall u\in H^1_{\rho_r}$ *with radial symmetry,*

$$(u,\phi_{i,b})_{L^2_{\rho_r}}=0,\quad 0\leq i\leq\ell\ \Rightarrow\ (H_bu,u)_{L^2_{\rho_r}}\geq\lambda_{\ell,b}\ \|\ u\ \|^2_{L^2_{\rho_r}}+c(\ell)\ \|\ u\ \|^2_{H^1_{\rho_r}}. \tag{2.8}$$

Since the potential term $Q_b(r)$ is independent of z, the diagonalization of the full linearized operator $\mathcal{L}_b$ for even in z cylindrical functions directly follows from Proposition 2.2 and an elementary tensorial claim.

Proposition 2.3 (Partial diagonalization of $\mathcal{L}_b$)**.** *Assume* (1.8). *There exists a universal domain* $\mathcal{D}\subset H^1_{\rho_Y}$ *with* $H^2_{\rho_Y}\subset\mathcal{D}$ *such that for all* $b>0$*, the operator* $\mathcal{L}_b:\mathcal{D}\to L^2(\rho_Y)$ *is self adjoint with compact resolvant. Moreover, for all* $\ell\in\mathbb{N}$*, there exists* $C(\ell),c(\ell)>0$ *and* $b^*(\ell)>0$ *such that for all* $0<b<b^*(\ell)$ *and* $0\leq i\leq\ell$*, the following holds. The function*

$$\psi_{i,k}(r,z)=\phi_{i,b}(r)P_{2k}(z),\quad k\in\mathbb{N}$$

with P_{2k} *given by* (1.27) *is an eigenfunction*

$$\mathcal{L}_b\psi_{i,k}=\lambda_{i,k}\psi_{i,k},\quad \lambda_{i,k}=i+k-\frac{\alpha}{2}+\tilde{\lambda}_{j,b}, \tag{2.9}$$

and for all $u\in H^1_{\rho_Y}$ *with even cylindrical symmetry satisfying*

$$(u,\psi_{i,k})_{L^2_{\rho_Y}}=0,\quad 0\leq i\leq\ell,\quad 0\leq k\leq\ell-i,$$

there holds:

$$(\mathcal{L}_bu,u)_{L^2_{\rho_Y}}\geq\lambda_{\ell,0}\|u\|^2_{L^2_{\rho_Y}}+c(\ell)\|u\|^2_{H^1_{\rho_Y}}. \tag{2.10}$$

Proof. Let

$$\mathcal{L}_\infty=-\Delta_Y+\frac{1}{2}\Lambda-p(\Phi^*)^{p-1},$$

let $k\in\mathbb{N}$ and P_{2k} be the k-th even Hermite polynomial (1.27). Let

$$\psi_{i,k,\infty}(r,z)=\phi_{i,\infty}(r)P_{2k}(z),$$

then since $Q(r)$ is independent of z, we compute from Proposition 2.1 and (1.28):

$$\mathcal{L}_\infty\psi_{i,k,\infty}=P_{2k}(z)H_b(\phi_{i,\infty})+\phi_{i,\infty}\left[-\partial_z^2+\frac{1}{2}z\partial_z\right]P_{2k}=\left(i+k-\frac{\alpha}{2}\right)\psi_{i,k,\infty}.$$

Since from standard tensorial claim the family $(\psi_{i,k,\infty})_{i,k\geq 0}$ is total in $H^1_{\rho_Y}$ restricted to functions with even cylindrical symmetry, we obtain the spectral gap estimate:

$$(u,\psi_{i,k,\infty})_{L^2_{\rho_Y}}=0,\quad 0\leq i\leq \ell,\quad 0\leq k\leq \ell-i,$$

with even cylindrical symmetry implies :

$$(\mathcal{L}_\infty u,u)_{L^2_{\rho_Y}}\geq \left(\ell+k+1-\frac{\alpha}{2}\right)\|u\|^2_{L^2_{\rho_Y}}. \tag{2.11}$$

Similarly from Proposition 2.2,

$$\mathcal{L}_b\psi_{i,k}=(\lambda_{i,b}+k)\,\psi_{i,k,\infty}$$

and the uniform closeness estimate (2.4) injected into (2.11) now implies (2.10). □

The rest of this section is devoted to a sketch of proof of Proposition 2.2 using a direct brute force matching ODE approach, see for example [18, 15] for related approaches.

2.2. **Interior problem.** We construct in this subsection eigenfunctions of H_b in the zone $0\leq r\leq r_0\ll 1$. We zoom on the soliton core by introducing

$$y=\frac{r}{\sqrt{b}},\quad v(r)=u(y),\quad H_bv(r)=\frac{1}{b}(H+\frac{b}{2}\Lambda)u(y)$$

with H given by (1.26), and study the Schrödinger operator $H+\frac{b}{2}\Lambda$ in the zone $0\leq y\leq y_0$ for

$$1\ll y_0:=\frac{r_0}{\sqrt{b}}\ll\frac{1}{\sqrt{b}} \tag{2.12}$$

restricted to radially symmetric functions. Let us recall from Lemmas 2.3 and 2.10 in [11], see also [46], the description of the iterates of the kernel of H:

Lemma 2.4 (Generators of the generalized kernel of H)**.** *There exists a family of smooth radial functions $(T_i)_{i\in\mathbb{N}}$ satisfying the identities*

$$T_0=c'\Lambda Q,\quad HT_0=0,\quad HT_{i+1}=-T_i \tag{2.13}$$

where $c'=\tilde{c}^{-1}\neq 0$, $\tilde{c}$ being defined in (1.22)*, with the expansion*

$$\left|\begin{array}{l} T_i \underset{y\to 0}{=} \sum_{l=0}^q d_{i,l}y^{2i+2l}+O(y^{2+2q}),\quad \text{for all } q\in\mathbb{N},\\ T_i \underset{y\to+\infty}{=} C_iy^{-\gamma+2i}+O(y^{-\gamma+2i-g}).\end{array}\right. \tag{2.14}$$

where (with the convention $C_0=1$)

$$C_i=\frac{2^{-2i}}{i!\left(\frac{d}{2}-\gamma\right)_i}, \tag{2.15}$$

and the additional cancellation at infinity:

$$\Lambda T_i=(2i-\alpha)T_i+\Theta_i,\quad \Theta_i=O(\langle y\rangle^{-\gamma+2i-g}). \tag{2.16}$$

These profiles provide the *non perturbative* leading order term of inner eigenfunctions which is the heart of the proof of Proposition 2.2. Let for $a\in\mathbb{R}$ the norm

$$\|f\|_{X^a_{y_0}}=\sup_{0\leq y\leq y_0}\sum_{i=0}^2\frac{|(\langle y\rangle\partial_y)^if|}{\langle y\rangle^a}.$$

Lemma 2.5 (Inner eigenfunctions). *Let $i \in \mathbb{N}$, $\tilde{\lambda} = O(1)$, $0 < r_0 \ll 1$ small enough and $0 < b < b^*(r_0)$ small enough. Then there exists a smooth profile $\phi_{int} \in \mathcal{C}^\infty([0,y_0],\mathbb{R})$ satisfying*

$$\left(H + \frac{b}{2}\Lambda\right)\phi_{int} = b\left(i - \frac{\alpha}{2} + \tilde{\lambda}\right)\phi_{int}, \tag{2.17}$$

with the decomposition:

$$\phi_{int} = \sum_{j=0}^{i} c_{i,j} b^j T_j + \tilde{\lambda}\sum_{j=0}^{i} b^{j+1}(-c_{i,j}T_{j+1} + S_j) + bR_i \tag{2.18}$$

where the constants $(c_{i,j})_{0\leq j\leq i}$ are given by

$$c_{i,j} = (-1)^j \frac{i!}{(i-j)!}, \quad c_{i,j+1} = -c_{i,j}(i-j), \tag{2.19}$$

and the correction functions $R_i, (S_j)_{0\leq j \leq i} : [0,y_0] \to \mathbb{R}$ satisfy

$$\|S_j\|_{X_{y_0}^{2j+2-\gamma}} \lesssim r_0^2, \quad \|\partial_b S_j\|_{X_{y_0}^{2j+4-\gamma}} \lesssim 1, \quad \|\partial_{\tilde{\lambda}} S_j\|_{X_{y_0}^{2j+2-\gamma}} \lesssim r_0^2. \tag{2.20}$$

$$\|R_i\|_{X_{y_0}^{2-\gamma-g}} \lesssim 1, \quad \|\partial_b R_i\|_{X_{y_0}^{4-\gamma-g}} \lesssim 1, \quad \|\partial_{\tilde{\lambda}} R_i\|_{X_{y_0}^{4-\gamma-g}} \lesssim b. \tag{2.21}$$

Proof of Lemma 2.5. Injecting (2.18) into (2.17) and using (2.13), (2.15), (2.19), we are left with solving

$$HS_j = b\left[\left(i - \frac{\alpha}{2} + \tilde{\lambda}\right)(-c_{i,j}T_{j+1} + S_j) - \frac{1}{2}\Lambda(-c_{i,j}T_{j+1} + S_j)\right]$$

$$HR_i = b\left[\left(i - \frac{\alpha}{2} + \tilde{\lambda}\right)R_i - \frac{1}{2}\Lambda R_i\right] + \sum_{j=0}^{i} \frac{c_{i,j}}{2} b^j \Theta_j.$$

For this, we use the basis of fundamental solutions to $Hu = 0$ given by

$$\Lambda Q, \quad \tilde{\Gamma} = -\Lambda Q \int_1^y \frac{d\tau}{(\Lambda Q)^2 \tau^{d-1}}$$

which yields the explicit inverse

$$H^{-1}f(y) := -\Lambda Q(y)\int_0^y \widetilde{\Gamma}(s) f(s) s^{d-1} ds + \widetilde{\Gamma}(y)\int_0^y \Lambda Q(s) f(s) s^{d-1} ds. \tag{2.22}$$

The behaviour of $\Lambda Q, \tilde{\Gamma}$ at both the origin and $+\infty$ can be computed from (1.21) and yield from direct check the continuity estimate: for $a > -\gamma$,

$$\|H^{-1}f\|_{X_{y_0}^a} \lesssim \sup_{0\leq y\leq y_0} \langle y\rangle^{-a+2} |f(r)|. \tag{2.23}$$

This draws a route map for the construction of S_j, R_j using the Banach fixed point theorem with the bounds (2.20), (2.21), and the details are left to the reader. □

2.3. **Exterior problem.** We construct in this subsection eigenfunctions of H_b in the zone $r \in [r_0, +\infty)$ where $0 < r_0 \ll 1$. Since $pQ_b^{p-1} \sim \Phi^*(r) = pc_\infty^{p-1} r^{-2}$ as $b \to 0$ from (1.21), the problem can be written to leading order as

$$\mathcal{L}_{\text{ext}} u = \left(i - \frac{\alpha}{2}\right) u, \quad \mathcal{L}_{\text{ext}} := -\Delta + \frac{1}{2}\Lambda - \frac{pc_\infty^{p-1}}{r^2} \tag{2.24}$$

We first construct the outer homogeneous basis.

Lemma 2.6 (Fundamental solutions of the exterior problem). *Let $i \in \mathbb{N}$. Then the solutions of* (2.24) *are spanned by two functions ψ_1, ψ_2 with*

$$\psi_1 = \sum_{j=0}^{i} c_{i,j} C_j r^{2j-\gamma} = \begin{cases} r^{-\gamma} + O(r^{-\gamma+2}) & \text{as } r \to 0, \\ c_{i,i} C_i r^{2i-\gamma} + O(r^{2i-\gamma-2}) & \text{as } r \to +\infty, \end{cases} \tag{2.25}$$

where $c_{i,j}$ and C_j are defined by (2.15) *and* (2.19)*, and*

$$\psi_2 = \begin{cases} -\frac{2}{c_{i,i}C_i} r^{-2i+\gamma-d} e^{\frac{r^2}{4}} \left[1 + O(r^{-2})\right] & \text{as } r \to +\infty, \\ \frac{r^{-\gamma_2}}{d-2\gamma-2} + O(r^{-\gamma_2+2}) & \text{as } r \to 0. \end{cases} \tag{2.26}$$

Moreover, there exists a unique solution to $[\mathcal{L}_{ext} - (i - \alpha/2)]\tilde{\psi}_1 = \psi_1$ satisfying

$$\tilde{\psi}_1 = r^{2i-\gamma} \left[2c_{i,i} C_i log(r) + O(1)\right] \quad \text{as } r \to +\infty, \tag{2.27}$$

$$\tilde{\psi}_1 = r^{-\gamma_2} \left[K + O(r^2)\right] \quad \text{as } r \to 0, \ K \neq 0, \tag{2.28}$$

The proof of Lemma 2.6 follows either by reducing the problem to known Laguerre type polynomials, or proceeding to a brute force expansion, the details are left to the reader. For $a, a' \in \mathbb{R}$, $0 < r_0 \ll 1$, we let the norms:

$$\|f\|_{X^{a,a'}_{r_0}} = \sup_{r_0 \leq r \leq 1} r^{-a}|f| + r^{-a+1}|\partial_r f| + r^{-a+2}|\partial_{rr} f| + \sup_{r \geq 1} r^{-a'}(|f| + |\partial_{rr} f|) + r^{-a'+1}|\partial_r f|.$$

Lemma 2.7 (Eigenfunctions of the exterior problem). *For any $i \in \mathbb{N}$, $0 < r_0 < 1$, there exists $\tilde{\lambda}^* > 0$ and $b^* > 0$ such that for all $0 < b < b^*$ and $|\tilde{\lambda}| \leq \tilde{\lambda}^*$, there exists a solution ϕ_{ext} on $[r_0, +\infty)$ of*

$$H_b \phi_{ext} = \left(i - \frac{\alpha}{2} + \tilde{\lambda}\right) \phi_{ext}, \quad \phi_{ext} = \psi_1 + \tilde{\lambda}(\tilde{\psi}_1 + \tilde{R}_1) + \tilde{R}_2 \tag{2.29}$$

satisfying the estimates:

$$\|\tilde{R}_1\|_{X^{-\gamma_2, 2i+2-\gamma}_{r_0}} \lesssim |\tilde{\lambda}|, \quad \partial_b \tilde{R}_1 = 0, \quad \|\partial_{\tilde{\lambda}} \tilde{R}_1\|_{X^{-\gamma_2, 2i+2-\gamma}_{r_0}} \lesssim 1, \tag{2.30}$$

and for $a = -\gamma_2 - 2 - \alpha$, $a' = 2i + 2 - \gamma$

$$\|\tilde{R}_2\|_{X^{a,a'}_{r_0}} \lesssim b^{\frac{\alpha}{2}}, \quad \|\partial_{\tilde{\lambda}} \tilde{R}_2\|_{X^{a,a'}_{r_0}} \lesssim b^{\frac{\alpha}{2}}, \qquad \|\partial_b \tilde{R}_2\|_{X^{a,a'}_{r_0}} \lesssim b^{\frac{\alpha}{2}-1}. \tag{2.31}$$

Proof of Lemma 2.7. We ensure (2.29) by solving

$$\left(\mathcal{L}_{\text{ext}} - \left(i - \frac{\alpha}{2}\right)\right) \tilde{R}_1 - \tilde{\lambda}\tilde{\psi}_1 - \tilde{\lambda}\tilde{R}_1 = 0,$$

$$\left(\mathcal{L}_{\text{ext}} - \left(i - \frac{\alpha}{2}\right)\right) \tilde{R}_2 + \left(\frac{pc_\infty^{p-1}}{r^2} - pQ_b^{p-1}\right) (\psi_1 + \tilde{\lambda}(\tilde{\psi}_1 + \tilde{R}_1) + \tilde{R}_2) - \tilde{\lambda}\tilde{R}_2 = 0.$$

For this, we consider the explicit inverse:

$$\left[\mathcal{L}_{\text{ext}} - (i - \frac{\alpha}{2})\right]^{-1} f(r) = -\psi_1(r) \int_1^r f\psi_2 s^{d-1} e^{-\frac{s^2}{4}} ds - \psi_2(r) \int_r^{+\infty} f\psi_1 s^{d-1} e^{-\frac{s^2}{4}} ds$$

which satisfies from direct check the continuity estimate: for $a \leq -\gamma_2$, $a \neq \gamma - d$ and $a' > 2i - \gamma$,

$$\|\mathcal{L}_{\text{ext}}^{-1}(f)\|_{X^{a,a'}_{r_0}} \lesssim \sup_{r_0 \leq r \leq 1} r^{-a}|f| + \sup_{r \geq 1} r^{-a'}|f|.$$

This designs again a route map for the construction of $\tilde{R}_1, \tilde{R}_2$ with the Banach fixed point argument which is left to the reader. One additional difficulty is the dependance in $\tilde{\lambda}$ which is in fact explicit to leading order. ☐

2.4. **Matching.** We now match the inner and outer solutions provided by Lemma 2.5 and Lemma 2.7. The following Lemma directly implies Proposition 2.2.

Lemma 2.8 (Matching)**.** *Fix* $\ell \in \mathbb{N}$*. There exists* $0 < r_0 \ll 1$ *and* $b^*(r_0) > 0$ *small enough, such that for* $0 < b \le b^*$ *and* $j \in \mathbb{N}$*,* $0 \le i \le \ell$*, there exists a unique* $\tilde{\lambda}_i = \tilde{\lambda}_i(b) \in \mathbb{R}$ *with*

$$|\tilde{\lambda}_i| \lesssim b^{\frac{g}{2}}, \quad |\partial_b \tilde{\lambda}_i| \lesssim b^{\frac{g}{2}-1} \tag{2.32}$$

such that ϕ_i *defined by*

$$\phi_i(r) = \begin{cases} b^{-\frac{\gamma}{2}} \phi_{int}\left(\frac{r}{\sqrt{b}}\right) & \text{if } 0 \le r \le r_0, \\ \frac{b^{-\frac{\gamma}{2}} \phi_{int}\left(\frac{r_0}{\sqrt{b}}\right)}{\phi_{ext(r_0)}} \phi_{ext}(r) & \text{if } r \ge r_0, \end{cases} \tag{2.33}$$

where ϕ_{int} *and* ϕ_{ext} *are given by Lemma 2.5 and Lemma 2.7, is a smooth solution of* $H_b \phi_i = \lambda_i \phi_i$*,* $\lambda_i = i - \alpha/2 + \tilde{\lambda}_i$*. Moreover, all the bounds in Proposition 2.2 hold true.*

Proof of Lemma 2.8. From standard argument, the gluing (2.33) gives a global smooth solution if the two parts of the solutions, on $[0, r_0]$ and $[r_0, \infty)$ respectively, agree at r_0, as well as their first order derivatives:

$$\begin{cases} b^{-\frac{\gamma}{2}} \phi_{\text{int}}\left(\frac{r_0}{\sqrt{b}}\right) = \frac{b^{-\frac{\gamma}{2}} \phi_{\text{int}}\left(\frac{r_0}{\sqrt{b}}\right)}{\phi_{\text{ext}(r_0)}} \phi_{\text{ext}}(r_0) \\ b^{-\frac{\gamma}{2}-\frac{1}{2}} \phi'_{\text{int}}\left(\frac{r_0}{\sqrt{b}}\right) = \frac{b^{-\frac{\gamma}{2}} \phi_{\text{int}}\left(\frac{r_0}{\sqrt{b}}\right)}{\phi_{\text{ext}(r_0)}} \phi'_{\text{ext}}(r_0). \end{cases} \tag{2.34}$$

Note that the first condition in (2.34) is always met. Indeed, if ϕ is an eigenfunction of the interior or exterior problem, then so is $a\phi$ for any $a \in \mathbb{R}$; so that we already renormalised the exterior solution in the formula (2.33) to deal with this natural invariance. Hence only the second condition needs to be checked and is equivalent to the cancellation:

$$\Phi[r_0](b, \tilde{\lambda}) = 0, \quad \Phi[r_0](b, \tilde{\lambda}) := \frac{b^{-\frac{1}{2}} \phi'_{\text{int}}(y_0)}{\phi_{\text{int}}(y_0)} - \frac{\phi'_{\text{ext}}(r_0)}{\phi_{\text{ext}(r_0)}}.$$

A tedius sequence of computations using the correction bounds of Lemma 2.5 and Lemma 2.7 yields the expansion:

$$\Phi[r_0](b, \tilde{\lambda}) = \tilde{\lambda}(\gamma_2 - \gamma) K r_0^{\gamma - \gamma_2 - 1}[1 + O(|\tilde{\lambda}|)] + O(b^{\frac{g}{2}})$$

$$\partial_b \Phi[r_0](b, \tilde{\lambda}) = O(b^{-1+\frac{g}{2}}) + O(|\tilde{\lambda}| b^{-1} r_0^3) - O(b^{\frac{\alpha}{2}-1}) = O(b^{-1+\frac{g}{2}}) + O(|\tilde{\lambda}| b^{-1} r_0^3)$$

$$\partial_{\tilde{\lambda}} \Phi[r_0](b, \tilde{\lambda}) = (\gamma_2 - \gamma) K r_0^{\gamma - \gamma_2 - 1}(1 + O(|\tilde{\lambda}|)).$$

Above, as $(\gamma_2 - \gamma)K \neq 0$, the partial derivative $\partial_{\tilde{\lambda}} \Phi$ is non-degenerate. Therefore, using the intermediate value theorem, for r_0 small enough, then for $0 < b < b^*$ small enough, there exists at least one $\tilde{\lambda}_i = \tilde{\lambda}_i(b) = O(b^{\frac{g}{2}})$ such that the matching condition (2.34) is satisfied. The uniqueness of the corresponding eigenvalue is now a simple consequence of the spectral gap (2.1) and the smallness (2.3). □

3. **Setting up the bootstrap argument**

This section is devoted to the set up of the bootstrap argument at the heart of the proof of Theorem 1.1. The initial datum is constructed by following the flow in a suitable regime, and we avoid the instability directions of type II blow up and the additional transversal directions using a nowadays standard outgoing flux argument. In this section, we prepare the analysis and show in particular how the spectral Proposition 2.3 provides an elementary setting to compute the type II blow up speeds (1.10) and control the flow in the associated $H^1_{\rho_Y}$ weighted norms.

3.1. **The reconnection function.** In order to build a solution to (1.1), we pick $0 < T \ll 1$ and consider the self similar change of variables

$$u(t,x) = \frac{1}{(T-t)^{\frac{1}{p-1}}} U(\tau, Y), \quad Y = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t),$$

which maps (1.1) onto the self similar equation

$$\partial_\tau U - \Delta U + \frac{1}{2}\Lambda U - U|U|^{p-1} = 0 \tag{3.1}$$

on a global time interval

$$\tau \in [\tau_0, +\infty), \quad \tau_0 = -\log T \gg 1. \tag{3.2}$$

In order to produce a suitable approximate solution to (3.1), we let $P_{2\ell}(z)$ be the even Legendre polynomial (1.27) so that

$$\lim_{|z|\to+\infty} P_{2\ell}(z) = +\infty. \tag{3.3}$$

Given $0 < a < a^*$ small enough, this implies $1 + aP_{2\ell}(z) \geq \frac{1}{2}$ for all $z \in \mathbb{R}$ and we may therefore consider

$$\mu = \mu(a, z) = (1 + \nu(z))^{\frac{1}{\alpha}}, \quad \nu(z) = aP_{2\ell}(z). \tag{3.4}$$

Given another parameter $0 < b < b^*$ small enough, we introduce the reconnection function

$$D = D(a, b, z) = \sqrt{b}\mu = \sqrt{b}(1 + aP_{2\ell}(z))^{\frac{1}{\alpha}}$$

and the even cylindrical blow up profile:

$$\Phi_{a,b}(r, z) = \frac{1}{D^{\frac{2}{p-1}}} Q\left(\frac{r}{D}\right) = \frac{1}{\mu(z)^{\frac{2}{p-1}}} Q_b\left(\frac{r}{\mu(z)}\right) \tag{3.5}$$

where we recall the notation (1.23). We claim that $\Phi_{a,b}$ is an approximate solution to (3.1) in the following sense:

Lemma 3.1 (All order cancellation in a)**.** *Assume that* $(a, b) \in \mathcal{C}^1([\tau_0, \tau^*], (0, a^*] \times (0, b^*])$*, then*

$$\partial_\tau \Phi_{a,b} - \Delta\Phi_{a,b} + \frac{1}{2}\Lambda\Phi_{a,b} - \Phi_{a,b}|\Phi_{a,b}|^{p-1} = \Psi_1 \tag{3.6}$$

with

$$\Psi_1 = \left[\frac{1}{2}\left(-\frac{b_\tau}{b} + 1\right) - \frac{\ell}{\alpha} - \frac{1}{2\alpha}\frac{\partial_\tau \nu}{1+\nu}\right]\Lambda_r \Phi_{a,b} + \frac{\ell}{\alpha}\Lambda Q_b(r) + \tilde{\Psi}_1 \tag{3.7}$$

and

$$\tilde{\Psi}_1 = \frac{\ell}{\alpha}\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right) - \Lambda Q_b(r)\right] - \frac{1}{\alpha^2}\left(\frac{\partial_z \nu}{1+\nu}\right)^2 \frac{1}{\mu^{\frac{2}{p-1}}}(\Lambda^2 Q_b + \alpha\Lambda Q_b)\left(\frac{r}{\mu}\right). \tag{3.8}$$

Remark 3.2. We claim that (3.7), (3.8) correspond to an all order cancellation in the parameter a. Indeed, replace Q by its far away asymptotic expansion (1.21), then from (1.19), both terms in $\tilde{\Psi}_1$ vanish. The first term in (3.7) will vanish for the laws

$$\frac{1}{2}\left(-\frac{b_\tau}{b} + 1\right) - \frac{\ell}{\alpha} = 0, \quad a_\tau = 0$$

which yield (1.10), (1.11), and hence to leading order

$$\Psi_1 \sim \frac{\ell}{\alpha}\Lambda Q_b(r),$$

see (3.50) and Appendix C for quantitative statements. This external force will be adjusted *exactly* using an excitation of the eigenmode $\psi_{\ell,0}$.

Proof. This is a brute force computation. First:

$$\begin{aligned}
\partial_\tau\Phi_{a,b} + \frac{1}{2}\Lambda_r\Phi_{a,b} &= \left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right) - \frac{\partial_\tau\mu}{\mu}\right]\Lambda_r\Phi_{a,b}\\
&= \left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right) - \frac{\ell}{\alpha} - \frac{\partial_\tau\mu}{\mu}\right]\Lambda_r\Phi_{a,b} + \frac{\ell}{\alpha}\Lambda_r\Phi_{a,b}.
\end{aligned}$$

We now let

$$\mu^\alpha = \tilde{\mu} = 1 + aP_{2\ell} = 1 + \nu$$

and compute:

$$\begin{aligned}
&\frac{\ell}{\alpha}\Lambda_r\Phi_{a,b} - \partial_z^2\Phi_{a,b} + \frac{1}{2}z\partial_z\Phi_{a,b}\\
= \;& \left[\frac{\ell}{\alpha} - \frac{1}{2}z\frac{\partial_z\mu}{\mu}\right]\frac{1}{\mu^{\frac{2}{p-1}}}\Lambda Q_b\left(\frac{r}{\mu}\right) - \frac{1}{\mu^{\frac{2}{p-1}}}\left[\left(\frac{\partial_z\mu}{\mu}\right)^2\Lambda^2 Q_b - \partial_{zz}(\log\mu)\Lambda Q_b\right]\left(\frac{r}{\mu}\right)\\
= \;& \left[\frac{\ell}{\alpha} - \frac{1}{2}z\frac{\partial_z\mu}{\mu} + \alpha\left(\frac{\partial_z\mu}{\mu}\right)^2 + \partial_{zz}\log\mu\right]\frac{1}{\mu^{\frac{2}{p-1}}}\Lambda Q_b\left(\frac{r}{\mu}\right) - \frac{1}{\mu^{\frac{2}{p-1}}}\left(\frac{\partial_z\mu}{\mu}\right)^2(\Lambda^2Q_b + \alpha\Lambda Q_b)\left(\frac{r}{\mu}\right)\\
= \;& \frac{1}{\alpha\widetilde{\mu}}\left[\ell\widetilde{\mu} - \frac{1}{2}z\partial_z\widetilde{\mu} + \partial_{zz}\widetilde{\mu}\right]\frac{1}{\mu^{\frac{2}{p-1}}}\Lambda Q_b\left(\frac{r}{\mu}\right) - \frac{1}{\alpha^2}\left(\frac{\partial_z\widetilde{\mu}}{\widetilde{\mu}}\right)^2\frac{1}{\mu^{\frac{2}{p-1}}}(\Lambda^2Q_b + \alpha\Lambda Q_b)\left(\frac{r}{\mu}\right)\\
= \;& \frac{\ell}{\alpha}\frac{1}{1+\nu}\Lambda_r\Phi_{a,b} - \frac{1}{\alpha^2}\left(\frac{\partial_z\nu}{1+\nu}\right)^2\frac{1}{\mu^{\frac{2}{p-1}}}(\Lambda^2Q_b + \alpha\Lambda Q_b)\left(\frac{r}{\mu}\right)
\end{aligned}$$

where we used in the last step (1.28) which ensures:

$$\ell\widetilde{\mu} - \frac{1}{2}z\partial_z\widetilde{\mu} + \partial_{zz}\widetilde{\mu} = \ell - a\left(-\partial_z^2 + \frac{1}{2}z\partial_z - \ell\right)P_{2\ell} = \ell.$$

We now observe from (1.19):

$$\frac{\ell}{\alpha}\frac{1}{1+\nu}\Lambda_r\Phi_{a,b} = \frac{\ell}{\alpha}\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)$$

and, since $-\Delta_r Q + |Q|^{p-1}Q = 0$, (3.7) and (3.8) follow. □

3.2. **Geometrical decomposition of the flow.** We now describe our set of initial data and the associated bootstrap bounds required to control their time evolution through the renormalized flow (3.1).

Let $(\psi_{j,k})_{0\le j\le\ell, 0\le k\le \ell-j}$ be the eigenmodes of Proposition 2.3, then given parameters

$$0 < b < b^*, \quad 0 < a < a^*$$

small enough, we first claim the non degeneracy of the scalar products generated by the profile $\Phi_{a,b}$ given by (3.5) with the eigenbasis.

Lemma 3.3 (Computation of the scalar products)**.** *There holds for some universal constants $I\neq 0$ and $\delta>0$ for $0\leq j\leq \ell$, $0\leq k\leq \ell-j$:*

$$(b\partial_b\Phi_{a,b},\psi_{j,k})_{L^2_{\rho_Y}} = -\frac{I}{2}(\sqrt{b})^\alpha\left[\delta_{j0}\delta_{k0}+O(b^\delta+a)\right] \tag{3.9}$$

$$(\partial_a\Phi_{a,b},\psi_{j,k})_{L^2_{\rho_Y}} = -\frac{I}{\alpha}(\sqrt{b})^\alpha\left[\delta_{j0}\delta_{k\ell}+O(b^\delta+a)\right]. \tag{3.10}$$

Proof. The proof follows from the asymptotics (1.21) and the relation (1.17) to provide integrability at the origin.

step 1 Norm computation. Let $\zeta\in\mathbb{R}$, $\mu\geq\frac12$, $k\in\{0,1\}$, $(i,j)\in\mathbb{N}^2$, we claim:

$$\begin{aligned}&\left\|\frac{1}{(\sqrt{b})^\alpha}\partial_r^k\left(\frac{1}{\mu^{\frac{2}{p-1}}}\Lambda^iQ_b+\zeta\frac{1}{\mu^{\frac{2}{p-1}}}\Lambda^jQ_b\right)\left(\frac{r}{\mu}\right)\right\|^2_{L^2_{\rho_r}} \\ =\ & c^2(\gamma)_k^2\mu^{2\left(\frac d2-\frac{2}{p-1}-k\right)}\left[1+O(\mu^2-1)\right]\left[(-\alpha)^i+\zeta(-\alpha)^j)+O(b^\delta)\right]^2\int_0^{+\infty}\frac{1}{r^{2\gamma+2k}}\rho_r r^{d-1}dr.\end{aligned} \tag{3.11}$$

for some universal constant $c>0$ and $\delta>0$ smal enough. Indeed, recall the asymptotics (1.21) which implies for $c_k=c(-1)^k(\gamma)_k\neq 0$

$$\partial_y^k\Lambda^iQ(y)=c_k\frac{(-\alpha)^i}{y^{\gamma+k}}+O\left(\frac{1}{y^{\gamma+\delta+k}}\right)\quad\text{for }\ |y|\gg 1,\quad k\in\{0,1\}. \tag{3.12}$$

Let $A=\frac{1}{\sqrt{b}^\delta}$ for δ universal small enough, then for $k\in\{0,1\}$ and changing variables:

$$\begin{aligned}&\frac{1}{\mu^{2\left(\frac d2-\frac{2}{p-1}-k\right)}}\left\|\frac{1}{(\sqrt{b})^\alpha}\partial_r^k\left[\frac{1}{\mu^{\frac{2}{p-1}}}(\Lambda^iQ_b+\zeta\Lambda^jQ_b)\left(\frac r\mu\right)\right]\right\|^2_{L^2_{\rho_r}}\\
=\ &\int_0^{+\infty}\left|\frac{1}{(\sqrt{b})^\alpha}\partial_r^k(\Lambda^iQ_b+\zeta\Lambda^jQ_b)(r)\right|e^{-\frac{\mu^2r^2}{4}}r^{d-1}dr\\
=\ &\int_{r\leq A\sqrt{b}}\left|\frac{1}{(\sqrt{b})^{\alpha+\frac{2}{p-1}+k}}\left[\partial_y^k(\Lambda^iQ+\zeta\Lambda^jQ)\right]\left(\frac{r}{\sqrt{b}}\right)\right|^2e^{-\frac{\mu^2r^2}{4}}r^{d-1}dr\\
+\ &\int_{r\geq A\sqrt{b}}\left|\frac{c_k}{(\sqrt{b})^{\alpha+\frac{2}{p-1}+k}}\frac{(\sqrt{b})^{\gamma+k}}{r^{\gamma+k}}\left((-\alpha)^i+\zeta(-\alpha)^j+O\left(\frac{1}{A^\delta}\right)\right)\right|^2e^{-\frac{\mu^2r^2}{4}}r^{d-1}dr\\
=\ &O\left(\frac{(A\sqrt{b})^d}{(\sqrt{b})^{2\gamma+2k}}\right)+\left[c_k^2((-\alpha)^i+\zeta(-\alpha)^j))^2+O\left(\frac{1}{A^\delta}\right)\right]\int_{r\geq A\sqrt{b}}\frac{1}{r^{2\gamma+2k}}e^{-\frac{\mu^2r^2}{4}}r^{d-1}dr\\
=\ &O\left(b^\delta\right)+c_k^2\left[(-\alpha)^i+\zeta(-\alpha)^j)+O\left(b^\delta\right)\right]^2\int_0^{+\infty}\frac{1}{r^{2\gamma+2k}}e^{-\frac{\mu^2r^2}{4}}r^{d-1}dr.\end{aligned} \tag{3.13}$$

where we used

$$d-1-(2\gamma+2)=-1+\sqrt{\Delta}>-1. \tag{3.14}$$

We now write

$$\left|e^{-\frac{\mu^2r^2}{4}}-e^{-\frac{r^2}{4}}\right|=\left|\int_1^\mu\mu'\frac{r^2}{2}e^{-\frac{(\mu')^2r^2}{4}}d\mu'\right|\lesssim|\mu^2-1|e^{-\frac{r^2}{16}}$$

from $\mu\geq\frac12$ and (3.11) follows.

step 2 a correction. We now estimate the a correction and claim:

$$\|\Lambda_r\Phi_{a,b}-\Lambda Q_b(r)\|_{H^1_{\rho_Y}}\lesssim|a|\sqrt{b}^\alpha. \tag{3.15}$$

Indeed,

$$\frac{d}{d\mu}\Lambda_r\Phi_{a,b}=-\frac{1}{\mu^{1+\frac{2}{p-1}}}\Lambda^2Q_b\left(\frac{r}{\mu}\right).$$

The global bound

$$|\partial_i\Lambda^jQ|\lesssim\frac{1}{1+|y|^{\gamma+i}},\quad i=0,1 \tag{3.16}$$

implies the global control

$$|\partial_\mu\partial_r^i\Lambda_r\Phi_{a,b}|\lesssim\frac{1}{\mu}\frac{1}{(\mu\sqrt{b})^i}\frac{1}{(\mu\sqrt{b})^{\frac{2}{p-1}}}\frac{1}{1+\left(\frac{r}{\mu\sqrt{b}}\right)^{\gamma+i}}\lesssim\frac{\mu^{\alpha-1}(\sqrt{b})^\alpha}{r^{\gamma+i}}$$

which together with the Taylor Lagrange formula yields:

$$\partial_r^i\Lambda_r\Phi_{a,b}=\partial_r^i\Lambda Q_b(r)+O\left(\int_1^\mu(\sqrt{b})^\alpha\frac{(\mu')^{\alpha-1}}{r^{\gamma+i}}d\mu'\right)=\partial_r^i\Lambda Q_b(r)+O\left((\sqrt{b})^\alpha\frac{|\mu^\alpha-1|}{r^{\gamma+i}}\right)$$

and hence, since $d-2\gamma-2=\sqrt{\Delta}>0$ and $\mu^\alpha-1=aP_{2\ell}$,

$$\|\partial_r^i\left(\Lambda_r\Phi_{a,b}-\Lambda Q_b(r)\right)\|_{L^2_{\rho_Y}}\lesssim|a|\sqrt{b}^\alpha\|\frac{P_{2\ell}}{r^{\gamma+i}}\|_{L^2_{\rho_Y}}\lesssim|a|\sqrt{b}^\alpha,\quad i=0,1$$

and (3.15) is proved.

step 3 Proof of (3.9), (3.10). We compute:

$$\partial_b\Phi_{a,b}=-\frac{\partial_bD}{D}\Lambda_r\Phi_{a,b}=-\frac{1}{2b}\Lambda_r\Phi_{a,b}.$$

Now from (2.3):

$$\phi_{0,b}=\frac{1}{(\sqrt{b})^\alpha}\Lambda Q_b+\tilde{\phi}_{0,b},\quad\|\tilde{\phi}_{0,b}\|_{H^1_{\rho_r}}\lesssim b^\delta. \tag{3.17}$$

Moreover, (3.11) with $\mu=1$ yields the expansion:

$$\|\frac{1}{\sqrt{b}^\alpha}\Lambda Q_b\|^2_{L^2_{\rho_r}}=I+O(b^\delta),\quad I:=c^2\int_0^{+\infty}\frac{r^{d-1}}{r^{2\gamma}}e^{-\frac{r^2}{4}}r^{d-1}dr \tag{3.18}$$

and hence using the $L^2_{\rho_z}$ orthonormality of Hermite polynomials and the $L^2_{\rho_r}$ orthogonality of the $\phi_{i,b}$ and the fact that $P_0=1$:

$$\left(\frac{1}{(\sqrt{b})^\alpha}\Lambda Q_b,\psi_{j,k}\right)_{L^2_{\rho_Y}}=\left(\frac{1}{(\sqrt{b})^\alpha}\Lambda Q_b,\phi_{j,b}P_{2k}\right)_{L^2_{\rho_Y}}=I\delta_{j0}\delta_{k0}+O(b^\delta). \tag{3.19}$$

We conclude using (3.15):

$$\begin{aligned}(\partial_b\Phi_{a,b},\psi_{j,k})_{L^2_{\rho_Y}}&=-\frac{1}{2b}(\Lambda_r\Phi_{a,b},\phi_{j,b}P_{2k})_{L^2_{\rho_Y}}=-\frac{1}{2b}\left[(\Lambda Q_b(r),\phi_{j,b}P_{2k})_{L^2_{\rho_Y}}+O(a(\sqrt{b})^\alpha)\right]\\&=-\frac{(\sqrt{b})^\alpha}{2b}\left[I\delta_{j0}\delta_{k0}+O(a+b^\delta)\right]\end{aligned}$$

and (3.9) is proved. similarly,

$$\partial_a\Phi_{a,b}=-\frac{\partial_aD}{D}\Lambda_r\Phi_{a,b}=-\frac{1}{\alpha}\frac{P_{2\ell}}{1+aP_{2\ell}}\Lambda_r\Phi_{a,b}$$

and hence using the exponential localization of the $\rho_z dz$ measure and the $L^2_{\rho_z}$ orthogonality of Hermite polynomials:

$$(\partial_a \Phi_{a,b}, \psi_{j,k})_{L^2_{\rho_Y}} = -\frac{1}{\alpha}\left(\frac{P_{2\ell}}{1+aP_{2\ell}}\Lambda_r \Phi_{a,b}, \psi_{j,k}\right)_{L^2_{\rho_Y}}$$

$$= -\frac{1}{\alpha}\left[\left(\frac{P_{2\ell}}{1+aP_{2\ell}}\Lambda Q_b(r), \psi_{j,k}\right)_{L^2_{\rho_Y}} + O(a(\sqrt{b})^\alpha)\right]$$

$$= -\frac{(\sqrt{b})^\alpha}{\alpha}\left[\left(\frac{P_{2\ell}}{1+aP_{2\ell}}\phi_{0,b}, \phi_{j,b}P_{2k}\right)_{L^2_{\rho_Y}} + O(a+b^\delta)\right] = -\frac{(\sqrt{b})^\alpha}{\alpha}\left[I\delta_{j0}\delta_{k\ell} + O(a+b^\delta)\right]$$

and (3.10) is proved. □

A standard application of the implicit function theorem using (3.9), (3.10) and the smallness (2.7) now ensures the following. Given a constant $K>0$, there exist $0<a^*(K), b^*(K), c^*(K) \ll 1$ and a universal constant such that for all

$$|a_1| < a^*(K), \quad 0 < b_1 < b^*(K)$$

the following holds: recall the definition of the set of indices (1.30) and let parameters

$$\mathbf{b}_1 = (b_{j,k})_1, \quad (j,k) \in \mathcal{I}$$

with $|\mathbf{b}_1| < K(\sqrt{b_1})^\alpha$, and define the finite dimensional projection

$$\psi_{\mathbf{b}_1,b_1} = \sum_{j=1}^{\ell}(b_{j,0})_1\psi_{j,0,b_1} - \left(\sum_{j=1}^{\ell}(b_{j,0})_1\right)\psi_{0,0,b_1} + \sum_{k=1}^{\ell-1}(b_{0,k})_1\psi_{0,k,b_1} + \sum_{j=1}^{\ell-1}\sum_{k=1}^{\ell-j}(b_{j,k})_1\psi_{j,k,b_1}.$$

Let χ be a cut-off function, $\chi(Y)=1$ for $|Y|\le 1$ and $\chi(Y)=0$ for $|Y|\ge 2$ and $\kappa>0$ is universal. Then any function of the form

$$U = \Phi_{a_1,b_1} + \chi(b_1^\kappa Y)\psi_{\mathbf{b}_1,b_1} + \varepsilon_1 \tag{3.20}$$

with $\|\varepsilon_1\|_{L^2_{\rho_Y}} \le c^*(K)(\sqrt{b_1})^\alpha$ can be uniquely reparametrized as

$$U = \Phi_{a,b} + \psi_{\mathbf{b},b} + \varepsilon, \tag{3.21}$$

where ε satisfies the orthogonality conditions:

$$(\varepsilon, \psi_{j,k,b})_{L^2_{\rho_Y}} = 0, \quad 0\le j\le \ell, \quad 0 \le k \le \ell - j. \tag{3.22}$$

Moreover,

$$\left|\frac{b}{b_1} - 1\right| + |a_1 - a| < \sqrt{c^*(K)}$$

and

$$|\mathbf{b} - \mathbf{b}_1| + \|\varepsilon\|_{L^2_{\rho_Y}} < \sqrt{c^*(K)}(\sqrt{b_1})^\alpha.$$

We therefore pick an initial data of the form (3.20) ie equivalently

$$U(\tau_0) = \Phi_{a,b}(\tau_0) + \varepsilon(\tau_0) + \psi(\tau_0)$$

with $\varepsilon(\tau_0)$ satisfying (3.22) and

$$\|\varepsilon(\tau_0)\|_{H^1_{\rho_Y}} < c(\sqrt{b(\tau_0)})^\alpha, \quad |b_{j,k}(\tau_0)| < K(\sqrt{b(\tau_0)})^\alpha \tag{3.23}$$

and where we note to ease notations

$$\psi = \psi_{\mathbf{b},b} = \sum_{j=1}^{\ell} b_{j,0}\psi_{j,0} - \left(\sum_{j=1}^{\ell} b_{j,0}\right)\psi_{0,0} + \sum_{k=1}^{\ell-1} b_{0,k}\psi_{0,k} + \sum_{j=1}^{\ell-1}\sum_{k=1}^{\ell-j} b_{j,k}\psi_{j,k}. \tag{3.24}$$

Then the corresponding solution (3.1) admits a unique time dependent decomposition
$$U = \Phi_{a,b} + V, \quad V = \psi + \varepsilon \tag{3.25}$$
with $\varepsilon(\tau, \cdot)$ satisfying (3.22) on any time interval $[0, \tau^*]$ such that:
$$\forall \tau \in [0, \tau^*], \quad 0 < b(\tau) < b^*(K), \quad |a(\tau)| < a^*(K)$$
and
$$\|\varepsilon(\tau)\|_{H^1_{\rho_Y}} < c(\sqrt{b(\tau)})^\alpha, \quad |b_{j,k}(\tau)| < K(\sqrt{b(\tau)})^\alpha$$
For initial data satisfying additional regularity assumptions as the one that we will now consider, the regularity $(a, b, b_{j,k}) \in \mathcal{C}^1([\tau_0, \tau^*], \mathbb{R})$ is a standard consequence of the smoothness of the flow for (1.1). We refer to [34, 12, 46] for related statements.

3.3. **Bootstrap.** We now specify more carefully a set of bootstrap estimates in the variables of the decomposition (3.25). In particular the exponentially weighted norms *do not provide enough information* to control the full nonlinear flow. Let us define
$$b_{j,k} = b_{\ell,0}\tilde{b}_{j,k}, \quad (\sqrt{b})^\alpha = -\alpha b_{\ell,0}(1 + \tilde{b}). \tag{3.26}$$
We pick two small enough universal constants $0 < \tilde{\eta} \ll \eta \ll 1$, a large enough integer $q \gg 1$, a large enough universal constant $K \gg 1$, and assume the following bounds on the initial data:

- normalization of the dominant mode:
$$b_{\ell,0}(\tau_0) = -e^{-(\ell - \frac{\alpha}{2})\tau_0}; \tag{3.27}$$
- a_0 is very large compared to $b(\tau_0)$ and nonnegative:
$$0 < \frac{1}{|\log b(\tau_0)|^{\frac{1}{K}}} < a(\tau_0) < a^* \ll 1; \tag{3.28}$$
- initial exit condition:
$$|\tilde{b}(\tau_0)|^2 + \sum_{(j,k) \neq (\ell,0), j+k \leq \ell} |\tilde{b}_{j,k}(\tau_0)|^2 < (\sqrt{b_0})^{2\eta}; \tag{3.29}$$
- initial smallness of the weighted norm:
$$\|\varepsilon(\tau_0)\|_{L^2_{\rho_Y}} < (\sqrt{b_0})^{\alpha + 2\eta}, \tag{3.30}$$
- inital smallness of the global $W^{1,2q+2}$ like norm:
$$\mathcal{N}(\tau_0) < (\sqrt{b_0})^{2\tilde{\eta}} \tag{3.31}$$
where $\mathcal{N}$ is given (4.8) and is constructed to control in particular
$$\|\phi V\|_{L^\infty} \lesssim \mathcal{N}(V). \tag{3.32}$$
where ϕ is a smooth function satisfying
$$\phi(r) = \left| \begin{array}{l} r^{\frac{2}{p-1}} \text{ for } 0 \leq r \leq 1 \\ 1 \text{ for } r \geq 2. \end{array} \right. \tag{3.33}$$

The fact that our set of initial data is non empty and contains smooth profiles with finite energy follows from an elementary localization claim which is left to the reader, see for example [15] for related computations. We then define $\tau_0 \leq \tau^* \leq +\infty$ as the supremum of times such that the following bounds hold on $[\tau_0, \tau^*)$:

- control of the dominant modes:
$$-2 < b_{\ell,0}e^{(\ell - \frac{\alpha}{2})\tau} < -\frac{1}{2}; \tag{3.34}$$

- control of a:
$$\frac{a(\tau_0)}{2} < a < 2a(\tau_0), \tag{3.35}$$
- exit condition:
$$|\tilde{b}|^2 + \sum_{(j,k)\in\mathcal{I}} |\tilde{b}_{j,k}|^2 < (\sqrt{b})^{2\eta} \tag{3.36}$$
- smallness of the weighted norm:
$$\|\varepsilon\|_{L^2_{\rho_Y}} < (\sqrt{b})^{\alpha+\eta}, \tag{3.37}$$
- smallness of the global $W^{1,2q+2}$ like norm:
$$\mathcal{N}(V) < (\sqrt{b})^{\tilde{\eta}} \tag{3.38}$$

The heart of our analysis is the following bootstrap proposition:

Proposition 3.4 (Bootstrap)**.** *There exists $K, q, \tau_0^* \gg 1$ and $0 < \tilde{\eta} \ll \eta \ll 1$ such that the following holds. Let $\tau_0' \geq \tau_0^*$, $(b'_{\ell,0}, \tau_0', a_0', \varepsilon_0')$ satisfy* (3.27), (3.28), (3.22), (3.30), (3.31)*. Then there exists $(\tilde{b}_0', \tilde{b}'_{j,k,0})$ satisfying* (3.36) *such that the solution to* (3.1) *with data at τ_0 given by* (3.20), (3.21) *satisfies*
$$\tau^* = +\infty.$$

Following a now classical path, we prove Proposition 3.4 by contradiction and assume that $\tau^* < +\infty$ for all $(\tilde{b}_0', \tilde{b}_{j,k,0})'$ satisfying (3.36). We then study the flow on $[\tau_0, \tau^*)$ and prove that the finite dimensional exit condition (3.29) is necessarily saturated at τ^*. The control of the modulation equations will ensure that this condition corresponds to a strictly outgoing finite dimensional vector field, hence yielding a contradiction to Brouwer's theorem.

The application of Brouwer's fixed point theorem requires the study of all small enough initial perturbations along the instable modes. These latter being unbounded at infinity, we use a localisation and take initial data of the form (3.20). We however now always use the decomposition (3.21) in the analysis. We claim that passing from one decomposition to another preserves the bootstrap bounds and do not give the technical but straightforward proof here.

From now on, we therefore consider a time interval $[\tau_0, \tau^*)$ on which (3.34), (3.35), (3.36), (3.37), (3.38) hold.

We observe that the bootstrap regime implies the following bounds.

Lemma 3.5 (Direct bootstrap estimates)**.** *There exists universal constants $c = c(\ell) > 0$ and $C > 0$ such that on $[\tau_0, \tau^*]$:*
$$C^{-1} b^{\frac{\alpha}{2}} \leq |b_{\ell,0}| \leq C b^{\frac{\alpha}{2}} \quad \text{and} \quad |b_{j,k}| \lesssim b^{\frac{\alpha}{2}+\eta} \text{ for } (j,k) \in \mathcal{I},\ (j,k) \neq (\ell,0), \tag{3.39}$$
$$|\psi| \lesssim \frac{b^{\eta}}{(\sqrt{b}+r)^{\frac{2}{p-1}}} \langle r\rangle^c \langle z\rangle^c, \quad |\psi| \lesssim \frac{b^{\frac{\alpha}{2}}}{(\sqrt{b}+r)^{\gamma}} \langle r\rangle^c \langle z\rangle^c, \tag{3.40}$$
$$|V| \lesssim \frac{(\sqrt{b})^{\tilde{\eta}}}{r^{\frac{2}{p-1}}} \text{ for } r \leq 2, \quad |V| \lesssim (\sqrt{b})^{\tilde{\eta}} \text{ for } r \geq 2, \tag{3.41}$$
$$|\varepsilon| \lesssim \frac{(\sqrt{b})^{\tilde{\eta}}}{r^{\frac{2}{p-1}}} \langle r\rangle^c \langle z\rangle^c. \tag{3.42}$$

Proof. (3.39) is a direct consequence of (3.26) and (3.29). From (3.24)

$$\psi = b_{\ell,0}(\psi_{\ell,0}-\psi_{0,0}) + \sum_{j=1}^{\ell-1} b_{j,0}\psi_{j,0} - \left(\sum_{j=1}^{\ell-1} b_{j,0}\right)\psi_{0,0} + \sum_{k=1}^{\ell-1} b_{0,k}\psi_{0,k} + \sum_{j=1}^{\ell-1}\sum_{k=1}^{\ell-j} b_{j,k}\psi_{j,k}.$$

For the first term from (2.3), (2.14), (2.6) and (3.39), there holds since $c_{i,0}=1$:

$$|b_{\ell,0}(\psi_{\ell,0}-\psi_{0,0})| = |b_{\ell,0}|\left|\sum_{j=1}^{\ell} c_{i,j}\sqrt{b}^{2j-\gamma}T_j\left(\frac{r}{\sqrt{b}}\right)+\tilde\phi_\ell-\tilde\phi_0\right|$$

$$\lesssim \sqrt{b}^{\alpha}\left|\sum_{j=1}^{\ell}\sqrt{b}^{2j-\gamma}\left(1+\frac{r}{\sqrt{b}}\right)^{2j-\gamma}+\frac{b^{\frac{g}{2}}}{(\sqrt{b}+r)^{\gamma}}\right| \lesssim \frac{\sqrt{b}^{\alpha}}{(\sqrt{b}+r)^{\gamma}}\left|\left(\sqrt{b}+r\right)^2+b^{\frac{g}{2}}\right|\langle r\rangle^c.$$

For the second term from (2.5) and (3.39):

$$\left|\sum_{j=1}^{\ell-1} b_{j,0}\psi_{j,0} - \left(\sum_{j=1}^{\ell-1} b_{j,0}\right)\psi_{0,0} + \sum_{k=1}^{\ell-1} b_{0,k}\psi_{0,k} + \sum_{j=1}^{\ell-1}\sum_{k=1}^{\ell-j} b_{j,k}\psi_{j,k}\right|$$

$$\lesssim \sum_{\mathcal{I}\backslash\{(\ell,0)\}}|b_{j,k}|\frac{1}{(\sqrt{b}+r)^{\gamma}}\langle r\rangle^c\langle z\rangle^c \lesssim \frac{b^{\frac{\alpha}{2}+\eta}}{(\sqrt{b}+r)^{\gamma}}\langle r\rangle^c\langle z\rangle^c$$

and (3.40) follows from the two above identities. The estimate (3.41) is then a direct consequence of (3.38) and (4.9). Finally, (3.42) is a consequence of the relation $V=\psi+\varepsilon$, of the two estimates (3.40) and (3.41), and of the fact that $\tilde\eta \ll \eta$ (note that we can assume $c>2/(p-1)$ as no estimate on c is required in the analysis later on).

□

3.4. **Modulation equations.** Injecting the decomposition (3.25) and the identity (3.6) into (3.1) yields the ε equation:

$$\partial_\tau\varepsilon + \mathcal{L}_b\varepsilon = -\Psi + L(V) + \mathrm{NL}(V), \quad V=\psi+\varepsilon \tag{3.43}$$

with

$$\Psi = \Psi_1+\Psi_2, \quad \Psi_2 = \partial_\tau\psi + \mathcal{L}_b\psi \tag{3.44}$$

$$L(V) = p(\Phi_{a,b}^{p-1} - Q_b^{p-1})V \tag{3.45}$$

$$NL(V) = (\Phi_{a,b}+V)|\Phi_{a,b}+V|^{p-1} - \Phi_{a,b}|\Phi_{a,b}|^{p-1} - p\Phi_a^{p-1}V \tag{3.46}$$

We start the study of the flow on $[\tau_0,\tau^*)$ by deriving the modulation equations for the geometrical parameters $(a,b,b_{j,k})$ as a consequence of the orthogonality conditions (3.22) and the non degeneracies (3.9), (3.10). We let

$$B = \frac{1}{2}\left(-\frac{b_\tau}{b}+1\right) - \frac{\ell}{\alpha}. \tag{3.47}$$

In order to prepare the outgoing flux argument, we compute the modulation equations in the variables (3.26).

Lemma 3.6 (Modulation equations)**.** *Let*

$$\begin{aligned}\text{Mod} &= \sum_{(j,k)\in\mathcal{I}}\left|\partial_\tau\tilde b_{j,k}-(\ell-(k+j))\tilde b_{j,k}\right|+|a_\tau|+\left|\tilde b_\tau-\ell\tilde b+\sum_{j=1}^{\ell-1}\tilde b_{j,0}\right|\\ &\quad+\sum_{(j,k)\in\mathcal{I}}\frac{|\partial_\tau b_{j,k}+\lambda_{j,k}b_{j,k}|}{(\sqrt{b})^\alpha},\end{aligned}$$

then there holds for some small enough universal constant $\delta>0$:

$$\text{Mod}\lesssim\frac{(a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}}{(\sqrt{b})^\alpha}+(\sqrt{b})^\delta+|a|\left(|\tilde b|+\sum_{j=1}^{\ell-1}|\tilde b_{j,0}|\right)\tag{3.48}$$

Proof of Lemma 3.6. We take the $L^2_{\rho_Y}$ scalar product of (3.43) with $\psi_{j,k}$ for $0\leq j\leq\ell$ and $0\leq k\leq j-1$, and compute the resulting identity using (3.22):

$$(\Psi_1,\psi_{j,k})_{L^2_{\rho_Y}}+(\Psi_2,\psi_{j,k})_{L^2_{\rho_Y}}=(\varepsilon,\partial_\tau\psi_{j,k})_{L^2_{\rho_Y}}+(L(V)+N(V),\psi_{j,k})_{L^2_{\rho_Y}}.\tag{3.49}$$

We now estimate all terms in the above identity.

step 1 Scalar products on Ψ_1. We first compute from (3.7), (3.47):

$$E_{j,k}:=(\Psi_1,\psi_{j,k})_{L^2_{\rho_Y}}=\left(\left[B-\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right]\Lambda_r\Phi_{a,b}+\frac{\ell}{\alpha}\Lambda Q_b(r),\psi_{j,k}\right)_{L^2_{\rho_Y}}+(\tilde\Psi_1,\psi_{j,k})_{L^2_{\rho_Y}}.$$

The lower order term. We claim the degeneracy

$$\|\tilde\Psi_1\|_{H^1_{\rho_Y}}\lesssim(\sqrt{b})^\alpha b^\delta,\quad|(\tilde\Psi_1,\psi_{j,k})_{L^2_{\rho_Y}}|\lesssim(\sqrt{b})^\alpha b^\delta\tag{3.50}$$

The second estimate in the above identity is a direct consequence of the first one. To prove the first one, recalling (3.8), we estimate using (3.11) and $\mu\geq\frac12$:

$$\|\frac{1}{(\sqrt{b})^\alpha}\frac{1}{\alpha^2}\left(\frac{\partial_z\nu}{1+\nu}\right)^2\frac{1}{\mu^{\frac{2}{p-1}}}(\Lambda^2Q_b+\alpha\Lambda Q_b)\left(\frac{r}{\mu}\right)\|^2_{H^1_{\rho_Y}}\lesssim\sum_{k=0,1}\int b^\delta\frac{\mu^C}{r^{2\gamma+2k}}\rho_YdY\lesssim b^\delta,$$

and similarly:

$$\begin{aligned}&\frac{1}{\sqrt{b}^\alpha}\left\|\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right\|_{H^1_{\rho_r}}\\ =\;&\sum_{k=0,1}\frac{1}{\sqrt{b}^\alpha}\left\|\int_1^\mu\frac{1}{(\mu')^{\gamma+1}}\partial_r^k\left[-\gamma\Lambda Q_b-r\partial_r(\Lambda Q_b)\right]\left(\frac{r}{\mu'}\right)d\mu'\right\|_{L^2_{\rho_r}}\\ \lesssim\;&\sum_{k=0,1}\frac{1}{\sqrt{b}^\alpha}\int_1^\mu\frac{1}{(\mu')^{\gamma+1}}\left\|\partial_r^k(\alpha\Lambda Q_b+\Lambda^2Q_b)\left(\frac{r}{\mu'}\right)\right\|_{L^2_{\rho_r}}d\mu'\lesssim b^\delta\int_1^\mu(\mu')^C\lesssim b^\delta\mu^C\end{aligned}\tag{3.51}$$

from which using the exponential weight in z:

$$\frac{1}{\sqrt{b}^\alpha}\left\|\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right\|^2_{H^1_{\rho_Y}}\lesssim b^\delta\int\mu^C\rho_zdz\lesssim b^\delta,$$

and from (3.8) and the above identities, (3.50) is proved.

The main order term. By definition:

$$-\frac{\partial_\tau\nu}{\alpha(1+\nu)}\Lambda_r\Phi_{a,b}=a_\tau\partial_a\Phi_{a,b}$$

and hence from (3.10):

$$\left(-\frac{1}{2\alpha}\frac{\partial_\tau \nu}{1+\nu}\Lambda_r\Phi_{a,b},\psi_{j,k}\right)_{L^2_{\rho_Y}}=-\frac{a_\tau}{2\alpha}(\sqrt{b})^\alpha\left[I\delta_{j0}\delta_{k\ell}+O(b^\delta+a)\right].$$

Similarly,

$$\Lambda_r\Phi_{a,b}=-2b\partial_b\Phi_{a,b}$$

and hence from (3.9):

$$(B\Lambda_r\Phi_{a,b},\psi_{j,k})_{L^2_{\rho_Y}}=B(\sqrt{b})^\alpha\left[I\delta_{j0}\delta_{k0}+O(b^\delta+a)\right].$$

Finally from (3.19):

$$(\Lambda Q_b,\psi_{j,k})_{L^2_{\rho_r}}=(\sqrt{b})^\alpha\left[I\delta_{j0}\delta_{k0}+O(b^\delta)\right].$$

Using (3.50), this yields the values:

$$\frac{E_{0,0}}{(\sqrt{b})^\alpha}=B\left[I+O(b^\delta+a)\right]+\frac{\ell}{\alpha}\left(I+O(b^\delta)\right)+a_\tau O\left(b^\delta+a\right),\qquad(3.52)$$

and

$$\frac{E_{0,\ell}}{(\sqrt{b})^\alpha}=-\frac{a_\tau}{2\alpha}\left[I+O(b^\delta+a)\right]+BO(b^\delta+a)+O(b^\delta)\qquad(3.53)$$

and for $(j\geq 1, k\geq 0)$ and $(j=0, 1\leq k\leq \ell-1)$:

$$\frac{E_{j,k}}{(\sqrt{b})^\alpha}=BO(b^\delta+a)+a_\tau O(b^\delta+a)+O(b^\delta).\qquad(3.54)$$

step 2 Ψ_2 term. We compute from (3.24), (3.44):

$$\Psi_2=\sum_{(j,k)\in\mathcal{I}}(\partial_\tau b_{j,k}+\lambda_{j,k}b_{j,k})\psi_{j,k}-\left(\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{0,0}b_{j,0})\right)\psi_{0,0}+\tilde{\Psi}_2\qquad(3.55)$$

with

$$\tilde{\Psi}_2=\frac{b_\tau}{b}\left[\sum_{j=1}^{\ell}b_{j,0}(b\partial_b\psi_{j,0}-b\partial_b\psi_{0,0})+\sum_{k=1}^{\ell-1}b_{0,k}b\partial_b\psi_{0,k}+\sum_{j=1}^{\ell}\sum_{k=0}^{\ell-j}b_{j,k}b\partial_b\psi_{j,k}\right].(3.56)$$

From (2.7), (3.39):

$$\|\tilde{\Psi}_2\|_{H^1_{\rho_Y}}=\frac{b_\tau}{b}O\left(b^\delta\sum_{j,k}|b_{j,k}|\right)=\frac{b_\tau}{b}O\left(b^\delta(\sqrt{b})^\alpha\right)\qquad(3.57)$$

and hence using the $L^2_{\rho_Y}$ orthogonality of eigenfunctions:

$$(\Psi_2,\psi_{j,k})_{L^2_{\rho_Y}}=\left|\begin{array}{l}(\partial_\tau b_{j,k}+\lambda_{j,k}b_{j,k})\,\|\psi_{j,k}\|^2_{L^2_{\rho_Y}}\quad\text{for}\quad(j,k)\in\mathcal{I}\\ -\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{0,0}b_{j,0})\|\psi_{0,0}\|^2_{L^2_{\rho_Y}}\quad\text{for}\quad(j,k)=(0,0)\\ 0\quad\text{for }(j,k)=(0,\ell)\end{array}\right.$$
$$+\frac{b_\tau}{b}O\left(b^\delta(\sqrt{b})^\alpha\right).\qquad(3.58)$$

step 3 $L(V)$ terms. We now estimate the rhs of (3.49) which are lower order and start with the $L(V)$ term given by (3.45). We claim:

$$\left|(L(V),\psi_{j,k})_{L^2_{\rho_Y}}\right|\lesssim b^\delta(\sqrt{b})^\alpha+a\|\varepsilon\|_{L^2_{\rho_Y}}.\qquad(3.59)$$

We compute:

$$\frac{d}{d\mu}\Phi_{a,b}^{p-1} = -\frac{p-1}{\mu^{1+\frac{2}{p-1}}}\Phi_{a,b}^{p-2}\Lambda Q_b\left(\frac{r}{\mu\sqrt{b}}\right) \tag{3.60}$$

We now use the global bounds

$$|Q(y)| \lesssim \frac{1}{1+|y|^{\frac{2}{p-1}}}, \quad |\Lambda Q(y)| \lesssim \frac{1}{1+|y|^{\gamma}}$$

to estimate:

$$\begin{aligned}\left|\frac{d}{d\mu}\Phi_{a,b}^{p-1}\right| &\lesssim \frac{1}{\mu}\left[\frac{1}{(r+\mu\sqrt{b})^{\frac{2}{p-1}}}\right]^{p-2}\frac{(\mu\sqrt{b})^{\alpha}}{(r+\mu\sqrt{b})^{\gamma}}\\ &\lesssim \frac{(\mu\sqrt{b})^{\alpha}}{(r+\mu\sqrt{b})^{\alpha}}\frac{1}{\mu(r+\mu\sqrt{b})^2} &(3.61)\\ &\lesssim \min\left\{\frac{\mu^{\alpha-3}(\sqrt{b})^{\alpha-2}}{r^{\alpha}}, \frac{1}{\mu r^2}, \frac{(\mu\sqrt{b})^{\alpha}}{\mu}\frac{1}{(r+\sqrt{b})^{\alpha+2}}\right\} &(3.62)\end{aligned}$$

where we used $\mu \geq \frac{1}{2}$ for the last estimate, and hence the pointwise bound by integration in μ:

$$\left|\Phi_{a,b}^{p-1} - Q_b^{p-1}\right| \lesssim \min\left\{\frac{(\mu^{\alpha-2}-1)(\sqrt{b})^{\alpha-2}}{r^{\alpha}}, \frac{|\log\mu|}{r^2}, \frac{|\mu^{\alpha}-1|(\sqrt{b})^{\alpha}}{(r+\sqrt{b})^{\alpha+2}}\right\}. \tag{3.63}$$

This allows us to control the ψ term using the pointwise bound (3.40) and (2.5):

$$\begin{aligned}\left|(L(\psi),\psi_{j,k})_{L^2_{\rho_Y}}\right| &\lesssim \int_{r\leq\sqrt{b}}\frac{|\log\mu|}{r^2}\frac{b^{\eta}\langle z\rangle^c}{b^{\frac{1}{p+1}}}\frac{1}{b^{\frac{\gamma}{2}}}\rho_Y dY + \int_{r\geq\sqrt{b}}\frac{\mu^{\alpha-2}\sqrt{b}^{\alpha-2}}{r^{\alpha}}\frac{\sqrt{b}^{\alpha}}{r^{\gamma}}\frac{\langle r\rangle^c\langle z\rangle^c}{r^{\gamma}}\rho_Y dY\\ &\lesssim b^{\frac{d}{2}-1-\frac{1}{p-1}-\frac{\gamma}{2}+\eta} + b^{\alpha-1}\max(1, b^{\frac{d}{2}-\gamma-\frac{\alpha}{2}}) \lesssim b^{\frac{\alpha}{2}+\delta}\end{aligned}$$

since $\alpha > 2$ and $d-2\gamma-2>0$. We then control the ε term using the second estimate in (3.63), the bound $|\log(\mu)| \lesssim a|P_{2\ell}(z)|$ as $a>0$ and (2.5):

$$\begin{aligned}|\left((\Phi_{a,b}^{p-1}-Q_b^{p-1})\varepsilon, \psi_{j,k}\right)_{L^2_{\rho_Y}}| &\lesssim \int|\varepsilon|\frac{a}{r^2}\frac{\langle z\rangle^{c(\ell)}\langle r\rangle^{c(\ell)}}{r^{\gamma}}\rho_Y dY\\ &\lesssim a\left(\int\varepsilon^2\rho_Y dY\right)^{\frac{1}{2}}\left(\int\frac{\langle z\rangle^{c(\ell)}\langle r\rangle^{c(\ell)}}{r^{2\gamma+4}}\rho_Y dY\right)^{\frac{1}{2}} \lesssim a\|\varepsilon\|_{L^2_{\rho_Y}}\end{aligned}$$

where the constant $c(\ell)$ whose value does not matter changed from the first to the second line but only depends on ℓ, and where we used in the last step (1.8) which ensures:

$$d-(2\gamma-4) = \sqrt{\Delta}-2 > 0 \tag{3.64}$$

to ensure the convergence of the integrals at the origin. This concludes the proof of (3.59).

step 4 NL(V) terms. We claim the bound for some universal $\delta > 0$ small enough:

$$|(\mathrm{NL}(v),\psi_{j,k})_{L^2_{\rho_Y}}| \lesssim (\sqrt{b})^{\alpha+\delta} + (\sqrt{b})^{\tilde{\eta}}\|\varepsilon\|_{L^2_{\rho_Y}}. \tag{3.65}$$

First observe that (3.40) yields the two following bounds for $C, c > 0$ and $0<\delta<\delta^*$ universal small enough:

$$\int\frac{\psi^2}{r^{2+2\delta}}\rho_Y dY \lesssim b^{\alpha}\int\frac{\langle r\rangle^C\langle z\rangle^C}{r^{2\gamma+2+2\delta}}\rho_Y dY \lesssim b^{\alpha} \tag{3.66}$$

$$\|r^{\frac{2}{p-1}+\delta}\psi\|_{L^\infty(r\leq 1,|z|\leq K\sqrt{|\log b|})}\lesssim(\sqrt{b})^{\eta+c\delta}|K\log b|^C\lesssim(\sqrt{b})^{c\delta}. \quad (3.67)$$

and that from (3.41) and (1.20):

$$|\Phi_{a,b}|\lesssim\frac{1}{r^{\frac{2}{p-1}}},\quad |V|\lesssim\frac{(\sqrt{b})^{\tilde\eta}}{r^{\frac{2}{p-1}}}\text{ for } r\leq 2,\quad |V|\lesssim(\sqrt{b})^{\tilde\eta}\text{ for } r\geq 2. \quad (3.68)$$

Estimate for $|z|>M\sqrt{|\log b|}$: We estimate from the above identity, using the Gaussian weight and the fact that $d-2-2/(p-1)-\gamma>d-2\gamma>0$:

$$\int_{|z|\geq M\sqrt{|\log b|}}|\mathrm{NL}(V)||\psi_{j,k}|\rho_Y dY\lesssim\int_{|z|\geq M\sqrt{|\log b|}}(|V|^p+\Phi_{a,b}^{p-2}|V|^2)|\psi_{j,k}|\rho_Y dY$$
$$\lesssim\int_{|z|\geq M\sqrt{|\log b|}}\left(1+\frac{1}{r^{2+\frac{2}{p-1}}}\right)\frac{\langle r\rangle^c\langle z\rangle^c}{r^\gamma}\rho_Y dY\lesssim(\sqrt{b})^{cM}\leq(\sqrt{b})^{\alpha+\delta}$$

for M universal large enough.

Estimate for $|z|<M\sqrt{|\log b|}$. First one decomposes

$$\int_{|z|\leq M\sqrt{|\log b|}}|\mathrm{NL}(V)|\psi_{j,k}\rho_Y dY\lesssim\int_{|z|\leq M\sqrt{|\log b|}}\left(|V|^p+\Phi_{a,b}^{p-2}|V|^2\right)|\psi_{j,k}|\rho_Y dY$$
$$\lesssim\int_{|z|\leq M\sqrt{|\log b|}}\left(|\varepsilon|^p+|\psi|^p+\Phi_{a,b}^{p-2}|\varepsilon|^2+\Phi_{a,b}^{p-2}|\psi|^2\right)|\psi_{j,k}|\rho_Y dY$$

Near the origin, we estimate using (3.67) and (3.68), as $V=\varepsilon+\psi$

$$\int_{r\leq 1,|z|\leq K\sqrt{|\log b|}}(|\varepsilon|^p+\Phi_{a,b}^{p-2}|\varepsilon|^2)|\psi_{jk}|\rho_Y dY$$
$$\lesssim\int_{r\leq 1}\left[\|r^{\frac{2}{p-1}}V\|_{L^\infty(r\leq 1)}+\|r^{\frac{2}{p-1}}\psi\|_{L^\infty(r\leq 1,|z|\leq K\sqrt{|\log b|})}\right]\frac{|\varepsilon\psi_{j,k}|}{r^2}\rho_Y dY$$
$$\lesssim((\sqrt{b})^{\tilde\eta}+(\sqrt{b})^{\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\lesssim(\sqrt{b})^{\tilde\eta}\|\varepsilon\|_{L^2_{\rho_Y}}$$

where we used (3.64), and similarly:

$$\int_{r\leq 1,|z|\leq K\sqrt{|\log b|}}(|\psi|^p+\Phi_{a,b}^{p-2}|\psi|^2)|\psi_{j,k}|\rho_Y dY$$
$$\lesssim\|r^{\frac{2}{p-1}+\delta}\psi\|_{L^\infty(r\leq 1,|z|\leq K\sqrt{|\log b|})}\int_{r\leq 1,|z|\leq K\sqrt{|\log b|}}\frac{|\psi|}{r^\delta}|\psi_{j,k}|\frac{dY}{r^2}$$
$$\lesssim\|r^{\frac{2}{p-1}+\delta}\psi\|_{L^\infty(r\leq 1,|z|\leq K\sqrt{|\log b|})}\left(\int\frac{\psi^2}{r^{2+2\delta}}\rho_Y dY\right)^{\frac12}\left(\int\frac{\psi_{j,k}^2}{r^2}\rho_Y dY\right)^{\frac12}\leq(\sqrt{b})^{\alpha+c\delta}.$$

Away from the origin, we use the Gaussian weight in r, (3.41) and (3.40) to estimate:

$$\int_{r\geq 1,|z|\leq K\sqrt{|\log b|}}\left(|V|^p+\Phi_{a,b}^{p-2}|V|^2\right)|\psi_{j,k}|\rho_Y dY\lesssim\int_{r\geq 1,|z|\leq K\sqrt{|\log b|}}|V|^2\langle r\rangle^c\langle z\rangle^c\rho_Y dY$$
$$\lesssim\int_{r\geq 1}|\varepsilon|^2\langle r\rangle^c\langle z\rangle^c\rho_Y dY+\int_{r\geq 1}|\psi|^2\langle r\rangle^c\langle z\rangle^c\rho_Y dY\lesssim\int_{r\geq 1}|\varepsilon|(|V|+|\psi|)\langle r\rangle^c\langle z\rangle^c\rho_Y dY+b^\alpha$$
$$\lesssim\|\varepsilon\|_{L^2_{\rho_Y}}\left(\|V\|_{L^\infty_{\rho_Y}(r\geq 1)}+\|\langle r\rangle^c\langle z\rangle^c\psi\|_{L^2_{\rho_Y}}\right)+b^\alpha\lesssim(\sqrt{b})^{\tilde\eta}\|\varepsilon\|_{L^2_{\rho_Y}}+b^\alpha$$

This concludes the proof of (3.65).

step 6 Computation of the modulation equations. We estimate from (2.7), (3.47):

$$|(\varepsilon,\partial_\tau\psi_{j,k})_{L^2_{\rho_Y}}|=\frac{b_\tau}{b}O(b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})=BO(b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})+O(b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}).$$

Injecting this together with (3.59), (3.65) and (3.37) into (3.49) yields:

$$(\Psi,\psi_{j,k})_{L^2_{\rho_Y}}=BO(b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right)+O((\sqrt{b})^{\alpha+\delta}).$$

We then combine the estimates (3.58), (3.52), (3.53), (3.54) and obtain the following:

<u>*law for* $b_{j,k}$*,* $(j,k)\neq\{(0,0);(0,\ell)\}$</u>. We obtain:

$$\begin{aligned}(\partial_\tau b_{j,k}+\lambda_{j,k}b_{j,k})\|\psi_{j,k}\|^2_{L^2_{\rho_Y}} &= BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)\\ +\ & a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

which implies using (3.26), (3.39) and (2.9):

$$\begin{aligned}(\sqrt{b})^\alpha\left[\partial_\tau\tilde b_{j,k}-(\ell-(k+j))\tilde b_{j,k}\right] &= BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)\\ +\ & a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right)\end{aligned}\tag{3.69}$$

<u>*law for* a</u>. We obtain:

$$\begin{aligned}-\frac{a_\tau}{2\alpha}I(\sqrt{b})^\alpha &= BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))\\ &+ O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

which can be rewritten as, since $|a|,b\ll1$ and $I\neq0$:

$$\begin{aligned}-\frac{a_\tau}{2\alpha}I(\sqrt{b})^\alpha &= BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)\\ &+ O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}\tag{3.70}$$

<u>*law for* b</u>. Finally:

$$\begin{aligned}&(\sqrt{b})^\alpha BI+I\frac{\ell}{\alpha}(\sqrt{b})^\alpha-\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{0,0}b_{j,0})\|\psi_{0,0}\|^2_{L^2_{\rho_Y}}\\ =\ & BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))\\ +\ & O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left(a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

Since from (2.3) and (3.18), $\|\psi_{0,0}\|^2_{L^2_{\rho_Y}}=I+O(b^\delta)$, this last expression can be reformulated

$$\begin{aligned}&(\sqrt{b})^\alpha B+\frac{\ell}{\alpha}(\sqrt{b})^\alpha-\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{0,0}b_{j,0})\\ =\ & BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))\\ +\ & O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde\eta})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

We now reformulate these estimates using the renormalized variables (3.26). First recalling (3.47) and using (2.9):

$$B=\frac{1}{2}\left[-\frac{2}{\alpha}\left(\frac{\partial_\tau b_{\ell,0}}{b_{\ell,0}}+\frac{\partial_\tau\tilde{b}}{1+\tilde{b}}\right)+1\right]-\frac{\ell}{\alpha}=-\frac{1}{\alpha}\frac{\partial_\tau\tilde{b}}{1+\tilde{b}}-\frac{1}{\alpha}\frac{\partial_\tau b_{\ell,0}+[\lambda_{\ell,0}+O(b^\delta)])b_{\ell,0}}{b_{\ell,0}} \tag{3.71}$$

and hence from (3.39), (3.69):

$$\begin{aligned}(\sqrt{b})^\alpha\left(B+\frac{1}{\alpha}\frac{\partial_\tau\tilde{b}}{1+\tilde{b}}\right) &= BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))\\ &+ O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned} \tag{3.72}$$

Moreover from (3.69) again and (2.9), (3.39):

$$\begin{aligned}&\frac{\ell}{\alpha}(\sqrt{b})^\alpha-\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{0,0}b_{j,0})=\frac{\ell}{\alpha}(\sqrt{b})^\alpha-\sum_{j=1}^{\ell}(\partial_\tau b_{j,0}+\lambda_{j,0}b_{j,0})+\sum_{j=1}^{\ell}jb_{j,0}+O(b^\delta(\sqrt{b}^\alpha))\\ =\ &\frac{\ell}{\alpha}(\sqrt{b})^\alpha+\ell b_{\ell,0}+\sum_{j=1}^{\ell-1}jb_{j,0}+BO\left(a(\sqrt{b})^\alpha+(\sqrt{b})^{\alpha+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}}\right)\\ &+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

Moreover

$$\frac{\ell}{\alpha}(\sqrt{b})^\alpha+\ell b_{\ell,0}=\ell\left[\frac{(\sqrt{b})^\alpha}{\alpha}+b_{\ell,0}\right]=-\ell b_{\ell,0}\tilde{b}=\frac{\ell}{\alpha}(\sqrt{b})^\alpha\frac{\tilde{b}}{1+\tilde{b}}$$

and hence the bound:

$$\begin{aligned}&\frac{1}{\alpha}(\sqrt{b})^\alpha\left[\frac{-\partial_\tau\tilde{b}+\ell\tilde{b}}{1+\tilde{b}}\right]+\sum_{j=1}^{\ell-1}jb_{j,0}=\partial_\tau\tilde{b}O(a(\sqrt{b})^\alpha+b^{\frac{\alpha}{2}+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})\\ &+a_\tau(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

Now using (3.26):

$$b_{j,0}=\tilde{b}_{j,0}b_{\ell,0}=-\frac{(\sqrt{b})^\alpha}{\alpha}\frac{\tilde{b}_{j,0}}{1+\tilde{b}}$$

and hence the law:

$$\begin{aligned}&(\sqrt{b})^\alpha\left[\tilde{b}_\tau-\ell\tilde{b}+\sum_{j=1}^{\ell-1}j\tilde{b}_{j,0}\right]=\partial_\tau\tilde{b}O(a(\sqrt{b})^\alpha+b^{\frac{\alpha}{2}+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})+a_\tau(O(a(\sqrt{b})^\alpha)\\ +\ &O(\sqrt{b}^{\alpha+\delta}))+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned} \tag{3.73}$$

step 7 Conclusion. The estimates (3.69), (3.70), (3.73) together with (3.72) yield the system

$$\begin{aligned}&\sum_{(j,k)\in\mathcal{I}}(\sqrt{b})^\alpha\left|\partial_\tau\tilde{b}_{j,k}-(\ell-(k+j))\tilde{b}_{j,k}\right|+(\sqrt{b})^\alpha|a_\tau|+(\sqrt{b})^\alpha\left|\tilde{b}_\tau-\ell\tilde{b}+\sum_{j=1}^{\ell-1}j\tilde{b}_{j,0}\right|\\ \lesssim\ &O(|\partial_\tau\tilde{b}|)O(a(\sqrt{b})^\alpha+b^{\frac{\alpha}{2}+\delta}+b^\delta\|\varepsilon\|_{L^2_{\rho_Y}})+|a_\tau|(O(a(\sqrt{b})^\alpha)+O(\sqrt{b}^{\alpha+\delta}))\\ &+O\left((\sqrt{b})^{\alpha+\delta}\right)+O\left((a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}\right).\end{aligned}$$

which is invertible thanks to (3.35), (3.37) and implies:

$$\begin{array}{ll} & \sum_{(j,k)\in\mathcal{I}}\left|\partial_\tau\tilde{b}_{j,k}-(\ell-(k+j))\tilde{b}_{j,k}\right|+|a_\tau|+\left|\tilde{b}_\tau-\ell\tilde{b}+\sum_{j=1}^{\ell-1}j\tilde{b}_{j,0}\right| \\ \lesssim & \frac{(a+(\sqrt{b})^{\tilde{\eta}})\|\varepsilon\|_{L^2_{\rho_Y}}}{(\sqrt{b})^\alpha}+b^\delta+|a|\left(|\tilde{b}|+\sum_{j=1}^{\ell-1}|\tilde{b}_{j,0}|\right)\end{array}$$

this is (3.48). Injecting this into (3.69) with (3.72) yields (3.48) and concludes the proof of Lemma 3.6. □

3.5. **Local $L^2_{\rho_Y}$ bound.** The geometrical decomposition (3.25) build on the eigenbasis constructed in Proposition 2.3 yields an elementary setting to compute the modulation equations of Lemma 3.6 and the underlying outgoing vector field structure. A second elementary fruit is the control of the flow in the $L^2_{\rho_Y}$ topology.

Lemma 3.7 ($L^2_{\rho_Y}$ control)**.** *There holds the pointwise bound:*

$$\|\varepsilon\|_{L^2_{\rho_Y}}\lesssim\eta(a)(\sqrt{b})^{\alpha+\eta} \tag{3.74}$$

where $\eta(a)=o(1)$ as $a\to 0$.

Proof of Lemma 3.7. We claim that (3.74) follows from the differential inequality

$$\begin{array}{ll} & \frac{1}{2}\frac{d}{d\tau}\|\varepsilon\|^2_{L^2_{\rho_Y}}+\lambda_{\ell,0}\|\varepsilon\|^2_{L^2_{\rho_Y}}+\frac{c^*}{2}\|\varepsilon\|^2_{H^1_{\rho_Y}} \\ \lesssim & (\sqrt{b})^{2\alpha}\left[(\sqrt{b})^\delta+\eta(a)(\tilde{b}^2+\sum_{j=1}^{\ell-1}|\tilde{b}_{j,0}|^2)\right]\end{array} \tag{3.75}$$

for some universal constant $c^*>0$ and $\eta(a)=o_{a\to 0}(1)$. Indeed, assume (3.75), it implies from (3.39):

$$\begin{array}{ll} & \frac{d}{d\tau}\left(e^{2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})\tau}\|\varepsilon\|^2_{L^2_{\rho_Y}}\right)+c^*e^{2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})\tau}\|\varepsilon\|^2_{H^1_{\rho_Y}} \\ \lesssim & \eta(a)(\sqrt{b})^{2\alpha+2\eta}e^{2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})\tau}\lesssim\eta(a)e^{c^*\tau}(\sqrt{b})^{2\eta}\end{array}$$

whose time integration using (3.39), (3.30) yields for η universal small enough:

$$\begin{array}{lll}\|\varepsilon(\tau)\|^2_{L^2_{\rho_Y}} & \lesssim & e^{-2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})(\tau-\tau_0)}\|\varepsilon(\tau_0)\|^2_{L^2_{\rho_Y}}+\eta(a)e^{-2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})\tau}\int_{\tau_0}^{\tau}e^{c^*\tau'}(\sqrt{b})^{2\eta}d\tau' \\ & \lesssim & \left(\frac{\sqrt{b(\tau)}}{\sqrt{b(\tau_0)}}\right)^{2\alpha+c\delta}\|\varepsilon(\tau_0)\|^2_{L^2_{\rho_Y}}+\eta(a)e^{-2(\ell-\frac{\alpha}{2}+\frac{c^*}{2})\tau}e^{c^*\tau}(\sqrt{b(\tau)})^{2\eta} \\ & \lesssim & \eta(a)(\sqrt{b(\tau)})^{2\alpha+2\eta}\end{array}$$

and (3.74) is proved. We now turn to the proof of (3.75).

step 1 Energy identity. We compute from (3.43):

$$\frac{1}{2}\frac{d}{d\tau}\|\varepsilon\|^2_{L^2_{\rho_Y}}=(\partial_\tau\varepsilon,\varepsilon)_{L^2_{\rho_Y}}=(-\mathcal{L}_b\varepsilon-\Psi+L(V)+\mathrm{NL}(V),\varepsilon)_{L^2_{\rho_Y}}. \tag{3.76}$$

The linear term is coercive from the spectral gap estimate (2.10) and the choice of orthogonality conditions (3.22):

$$(\mathcal{L}_b\varepsilon,\varepsilon)_{L^2_{\rho_Y}}\geq\lambda_{\ell,0}\|\varepsilon\|^2_{L^2_{\rho_Y}}+c^*\|\varepsilon\|^2_{H^1_{\rho_Y}} \tag{3.77}$$

for some $c^* > 0$. We now estimate all remaining terms in (3.76).

step 2 Ψ terms. We claim:

$$|(\varepsilon, \Psi)_{L^2_{\rho_Y}}| \lesssim \|\varepsilon\|_{H^1_{\rho_Y}} (\sqrt{b})^\alpha \left[b^\delta + \eta(a)(|\tilde{b}| + \sum_{j=1}^{\ell-1} |\tilde{b}_{jk}|) \right] + [\eta(a) + b^{c\eta}] \|\varepsilon\|^2_{H^1_{\rho_Y}}. \tag{3.78}$$

We observe from (3.48), (3.39) the rough bounds:

$$|\partial_\tau b_{j,k}| + |b_{j,k}| + (\sqrt{b})^\alpha |a_\tau| + (\sqrt{b})^\alpha (|\tilde{b}_\tau| + |\tilde{b}|) \lesssim (\sqrt{b})^\alpha, \quad \left| \frac{b_\tau}{b} \right| \lesssim 1. \tag{3.79}$$

We now estimate the Ψ_2 term. Recall (3.55), then from (3.57), (3.22), (3.79):

$$|(\varepsilon, \Psi_2)_{L^2_{\rho_Y}}| = |(\varepsilon, \tilde{\Psi}_2)_{L^2_{\rho_Y}}| \lesssim \|\varepsilon\|_{H^1_{\rho_Y}} b^\delta (\sqrt{b})^\alpha.$$

We now estimate Ψ_1 and recall (3.7), (3.8). From (3.50):

$$|(\varepsilon, \tilde{\Psi}_1)_{L^2_{\rho_Y}}| \lesssim \|\varepsilon\|_{H^1_{\rho_Y}} b^\delta (\sqrt{b})^\alpha$$

and from (3.17), (3.22):

$$|(\varepsilon, \Lambda Q_b)_{L^2_{\rho_Y}}| = |(\varepsilon, (\sqrt{b})^\alpha \tilde{\phi}_{0,b})_{L^2_{\rho_Y}}| \lesssim b^\delta (\sqrt{b})^\alpha \|\varepsilon\|_{H^1_{\rho_Y}}.$$

Finally, from (3.72), (3.48):

$$(\sqrt{b})^\alpha (|B| + |\partial_\tau a|) \quad \lesssim \quad (\sqrt{b})^\alpha \left[b^\delta + |\tilde{b}| + \sum_{j=1}^{\ell-1} |\tilde{b}_{j,0}| \right] + \left[a + (\sqrt{b})^{\tilde{\eta}} \right] \|\varepsilon\|_{L^2_{\rho_Y}}$$

and from (3.15), (3.17), (3.22):

$$\begin{aligned} |(\varepsilon, \Lambda \Phi_{a,b})_{L^2_{\rho_Y}}| &\lesssim \|\varepsilon\|_{L^2_{\rho_Y}} \left[\|\Lambda \Phi_{a,b} - \Lambda Q_b\|_{L^2_{\rho_Y}} + \|\Lambda Q_b - (\sqrt{b})^\alpha \psi_{0,0}\|_{L^2_{\rho_Y}} \right] \\ &\lesssim (\sqrt{b})^\alpha (a + b^\delta) \|\varepsilon\|_{H^1_{\rho_Y}}. \end{aligned}$$

We conclude:

$$|(\varepsilon, \Psi_1)|_{L^2_{\rho_Y}} \lesssim \|\varepsilon\|_{H^1_{\rho_Y}} (\sqrt{b})^\alpha \left[b^\delta + |a|(|\tilde{b}| + \sum_{j=1}^{\ell-1} |\tilde{b}_{j,0}|) \right] + [\eta(a) + b^{c\eta}] \|\varepsilon\|^2_{H^1_{\rho_Y}}$$

and (3.78) is proved.

step 3 $L(V)$ term. We claim:

$$(L(V), \varepsilon)_{L^2_{\rho_Y}} \lesssim b^\delta (\sqrt{b})^\alpha \|\varepsilon\|_{H^1_{\rho_Y}} + a \|\varepsilon\|^2_{H^1_{\rho_Y}}. \tag{3.80}$$

Indeed, first compute:

$$(L(V), \varepsilon)_{L^2_{\rho_Y}} = p \left((\Phi_{a,b}^{p-1} - Q_b^{p-1}) \varepsilon, \varepsilon \right)_{L^2_{\rho_Y}} + p \left((\Phi_{a,b}^{p-1} - Q_b^{p-1}) \psi, \varepsilon \right)_{L^2_{\rho_Y}}.$$

For the linear term, we recall (3.61) which implies for $r \leq \sqrt{b}$ using $\mu \geq \frac{1}{2}$:

$$\left| \frac{d}{d\mu} \Phi_{a,b}^{p-1} \right| \lesssim \frac{(\mu\sqrt{b})^\alpha}{(r + \mu\sqrt{b})^\alpha} \frac{1}{\mu (r + \mu\sqrt{b})^2} \lesssim \frac{1}{b}.$$

Hence

$$|\Phi_{a,b}^{p-1} - Q_b^{p-1}| \lesssim \frac{1 + |\mu|}{b} \quad \text{for} \quad r \leq \sqrt{b}$$

and we estimate using (2.5), (3.39), (B.1):

$$\int_{r\leq\sqrt{b}}|\Phi_{a,b}^{p-1}-Q_b^{p-1}||\psi||\varepsilon|\rho_Y dY \lesssim \sum|b_{j,k}|\int_{r\leq\sqrt{b}}\frac{|\varepsilon|\langle z\rangle^c}{br^\gamma}\rho_Y dY$$

$$\lesssim (\sqrt{b})^\alpha\|\frac{\varepsilon}{r}\|_{L^2_{\rho_Y}}\left(\int_{r\leq\sqrt{b}}\frac{r^4}{b^2r^{2\gamma+2}}r^{d-1}dr\right)^{\frac12}\lesssim b^\delta(\sqrt{b})^\alpha\|\varepsilon\|_{H^1_{\rho_Y}}.$$

where we used (3.14) in the last step. For $r\geq\sqrt{b}$, we use (3.61) again which implies:

$$\left|\frac{d}{d\mu}\Phi_{a,b}^{p-1}\right|\lesssim\frac{(\mu\sqrt{b})^\alpha}{(r+\mu\sqrt{b})^\alpha}\frac{1}{\mu(r+\mu\sqrt{b})^2}\lesssim\frac{(\sqrt{b})^\alpha\mu^{\alpha-1}}{r^{\alpha+2}}.$$

Hence

$$|\Phi_{a,b}^{p-1}-Q_b^{p-1}|\lesssim\frac{(\mu\sqrt{b})^\alpha}{r^{\alpha+2}}\quad\text{for}\quad r\geq\sqrt{b}$$

and we estimate using (2.5), (3.39), (B.1):

$$\int_{r\geq\sqrt{b}}|\Phi_{a,b}^{p-1}-Q_b^{p-1}||\psi||\varepsilon|\rho_Y dY\lesssim\sum|b_{j,k}|\int_{r\geq\sqrt{b}}\frac{(\sqrt{b})^\alpha|\varepsilon|\langle z\rangle^c\langle r\rangle^c}{r^{2+\alpha+\gamma}}\rho_Y dY$$

$$\lesssim (\sqrt{b})^\alpha\|\frac{\varepsilon}{r}\|_{L^2_{\rho_Y}}\left(\int_{r\geq\sqrt{b}}\frac{(\sqrt{b})^{2\alpha}\langle r\rangle^c}{r^{2+2\alpha+2\gamma}}\rho_r r^{d-1}dr\right)^{\frac12}$$

$$\lesssim (\sqrt{b})^\alpha\|\varepsilon\|_{H^1_{\rho_Y}}\left(\int_{r\geq\sqrt{b}}\left(\frac{\sqrt{b}}{r}\right)^{2\alpha-\delta}\frac{(\sqrt{b})^\delta\langle r\rangle^c}{r^{2+2\gamma+\delta}}\rho_r r^{d-1}dr\right)^{\frac12}\lesssim b^\delta(\sqrt{b})^\alpha\|\varepsilon\|_{H^1_{\rho_Y}}$$

where we used (3.14) again in the last step. For the quadratic term, we recall (3.60), (1.20) which imply

$$\frac{d}{d\mu}\Phi_{a,b}^{p-1}<0$$

from which $\Phi_{a,b}^{p-1}-Q_b^{p-1}\leq 0$ for $\mu\geq 1$. From (3.3), (3.35), $\nu(z)=1+aP_{2\ell}(z)>1$ for $|z|\geq z^*$ universal large enough, and we thus conclude using the bound (3.63) and (B.1):

$$\begin{aligned}\left((\Phi_{a,b}^{p-1}-Q_b^{p-1})\varepsilon,\varepsilon\right)_{L^2_{\rho_Y}} &\leq \int_{|z|\leq z^*}|\Phi_{a,b}^{p-1}-Q_b^{p-1}|\varepsilon^2\rho_Y dY\lesssim\int_{|z|\leq z^*}\frac{|aP_{2\ell}(z)|}{r^2}\varepsilon^2\rho_Y dY\\ &\lesssim a\|\varepsilon\|^2_{H^1_{\rho_Y}}.\end{aligned}$$

This concludes the proof of (3.80).

step 4 Nonlinear term. We treat the nonlinear term carefully and claim for some δ universal:

$$|(\text{NL}(V),\varepsilon)_{L^2_{\rho_Y}}|\leq\eta(a)\|\varepsilon\|^2_{H^1_{\rho_Y}}+(\sqrt{b})^{2\alpha+\delta}. \tag{3.81}$$

Estimate for $|z|>M\sqrt{|\text{log}b|}$: We estimate from (3.41), using the Gaussian weight, (3.40) and $V=\psi+\varepsilon$:

$$\begin{aligned}\int_{|z|\geq M\sqrt{|\text{log}b|}}|\text{NL}(V)|\varepsilon\rho_Y dY &\lesssim \int_{|z|\geq M\sqrt{|\text{log}b|}}(|V|^p+\Phi_{a,b}^{p-2}|V|^2)(|V|+|\psi|)\rho_Y dY\\ &\lesssim \frac{(\sqrt{b})^{cM}}{(\sqrt{b})^c}\leq(\sqrt{b})^{2\alpha+\delta}\end{aligned}$$

for M universal large enough.

Estimate for $|z|<M\sqrt{|\log b|}$. By homogeneity:

$$\begin{aligned}\int_{|z|\le M\sqrt{|\log b|}}|\mathrm{NL}(V)|\varepsilon\rho_Y dY &\lesssim \int_{|z|\le M\sqrt{|\log b|}}\left(|V|^p+\Phi_{a,b}^{p-2}|V|^2\right)|\varepsilon|\rho_Y dY\\ &\lesssim \int_{|z|\le M\sqrt{|\log b|}}\left(|\varepsilon|^p+|\psi|^p+\Phi_{a,b}^{p-2}|\varepsilon|^2+\Phi_{a,b}^{p-2}|\psi|^2\right)|\varepsilon|\rho_Y dY\end{aligned}$$

Near the origin, we estimate using (B.1), (3.40) and (3.41):

$$\begin{aligned}&\int_{r\le 1,|z|\le M\sqrt{|\log b|}}(|\varepsilon|^p+\Phi_{a,b}^{p-2}|\varepsilon|^2)|\varepsilon|\rho_Y dY\\ \lesssim\ &\int_{r\le 1}\left[\|r^{\frac{2}{p-1}}V\|_{L^\infty(r\le 1)}+\|r^{\frac{2}{p-1}}\psi\|_{L^\infty(r\le 1,|z|\le M\sqrt{|\log b|})}\right]\frac{\varepsilon^2}{r^2}\rho_Y dY\\ \lesssim\ &((\sqrt{b})^{\tilde\eta}+\sqrt{b}^{\eta})\|\varepsilon\|^2_{H^1_{\rho_Y}}\end{aligned}$$

and similarly:

$$\begin{aligned}&\int_{r\le 1,|z|\le M\sqrt{|\log b|}}(|\psi|^p+\Phi_{a,b}^{p-2}|\psi|^2)|\varepsilon|\rho_Y dY\\ \lesssim\ &\|r^{\frac{2}{p-1}+\delta}\psi\|_{L^\infty(r\le 1,|z|\le M\sqrt{|\log b|})}\int_{r\le 1,|z|\le M\sqrt{|\log b|}}|\psi||\varepsilon|\frac{dY}{r^{2+\delta}}\\ \lesssim\ &\sqrt{b}^{\eta+c\delta}\left(\int\frac{\psi^2}{r^{2+2\delta}}\rho_Y dY\right)^{\frac12}\left(\int\frac{\varepsilon^2}{r^2}\rho_Y dY\right)^{\frac12}\lesssim\sqrt{b}^{\delta}\|\varepsilon\|^2_{H^1_{\rho_Y}}+(\sqrt{b})^{2\alpha+\delta}.\end{aligned}$$

Away from the origin, we use the Gaussian weight in r and (3.40) to estimate:

$$\begin{aligned}&\int_{r\ge 1,|z|\le K\sqrt{|\log b|}}(|\psi|^p+\Phi_{a,b}^{p-2}|\psi|^2)|\varepsilon|\rho_Y dY\lesssim(\sqrt{b})^{2\alpha}\int_{r\ge 1,|z|\le K\sqrt{|\log b|}}\langle r\rangle^c\langle z\rangle^c|\varepsilon|\rho_Y dY\\ \lesssim\ &\sqrt{b}^{\delta}\|\varepsilon\|^2_{H^1_{\rho_Y}}+(\sqrt{b})^{2\alpha+\delta}\end{aligned}$$

This concludes the proof of (3.81).

step 5 Conclusion. The collection of estimates (3.77), (3.78), (3.80) and (3.81), injected in (3.76), yields (3.75).

□

4. L^∞ bound through energy estimates

The energy estimate (3.75) easily closes the control of the modulation equations of Lemma 3.6 *provided the L^∞ control of $V=\psi+\varepsilon$*. It is a classical difficulty in the study of singularity formation that the description of the solution near the singularity involves growing in space profiles like ψ which are unbounded in L^∞, and exponentially localized norms which are too weak to control the nonlinear term both at the origin and infinity in space. In the setting of the radially symmetric type II blow up, the L^∞ bound for r large is obvious and relies on a scaling argument, section 4.6, see [46], [15] for related arguments. At the origin, the unconditional $L^2_{\rho_Y}$ control provides an outer estimate on the sphere $r=1$ which can easily be propagated inside using upper and lower solutions and the maximun principle, see for example [53, 7]. We propose a more energetic proof based on $W^{1,q}$ estimates which is well suited for the cylindrical geometry and handles both the difficulties of type II and the ode type I blow up as in [3, 50]. This provides a pure energy method for the control of the non linear flow.

4.1. **Definition of $\mathcal{N}$ and weighted Sobolev bound.** We prepare the analysis for the L^∞ bound by introducing two new decompositions of the flow. We let

$$0<r^*,\nu\ll 1,\quad A,q\gg 1$$

universal constants independent of a,b to be chosen later.

New decomposition away from the origin. We extract from the decomposition $V=\varepsilon+\psi$ the leading order term and consider

$$V=v+\zeta,\quad \zeta=b_{\ell,0}(\psi_{\ell,0}-\psi_{0,0})(r). \tag{4.1}$$

We estimate from (2.3), (2.14), (2.6), (3.39) for $r\le A$ using the *essential degeneracy* $c_{i,0}=1$, $\alpha>2$ and $g\le 2$:

$$\begin{aligned}
|\zeta(r)| &= |b_{\ell,0}|\left|\sum_{j=1}^{\ell}(\sqrt{b})^{2j-\gamma}T_j\left(\frac{r}{\sqrt{b}}\right)+\tilde\phi_\ell(r)-\tilde\phi_0(r)\right|\\
&\lesssim (\sqrt{b})^\alpha\left(\sum_{j=1}^{\ell}(\sqrt{b})^{2j-\gamma}\left(1+\frac{r}{\sqrt{b}}\right)^{2j-\gamma}+(\sqrt{b})^g\frac{(1+r)^{2\ell+4}}{(\sqrt{b}+r)^\gamma}\right)\\
&\lesssim (\sqrt{b})^\alpha\left|\begin{array}{l}(\sqrt{b})^{g-\gamma}\ \text{ for }\ r\le\sqrt{b}\\ r^{-\gamma}(r^2+(\sqrt{b})^g)\ \text{ for }\ \sqrt{b}\le r\le 1\\ r^{2\ell+2}\ \text{ for }\ r\ge 1\end{array}\right.\lesssim(\sqrt{b})^{g-\frac{2}{p-1}}(1+r^{2\ell+2})
\end{aligned}\tag{4.2}$$

which implies in particular

$$\|r^{\frac{2}{p-1}}\zeta\|_{L^\infty(r\le 1)}\lesssim(\sqrt{b})^g. \tag{4.3}$$

We moreover compute from (3.26)

$$\begin{aligned}
&\partial_\tau\zeta+\mathcal{L}_b\zeta=(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})\psi_{\ell,0}-(\partial_\tau b_{\ell,0}+\lambda_{0,0}b_{\ell,0})\psi_{0,0}+b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\\
=\ &(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})+(\lambda_{\ell,0}-\lambda_{0,0})b_{\ell,0}\psi_{0,0}+b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\\
=\ &(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{\ell+\tilde\lambda_{\ell,0}-\tilde\lambda_{0,0}}{\alpha}\frac{(\sqrt{b})^\alpha}{1+\tilde b}\psi_{0,0}+b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\\
=\ &(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})+\frac{\ell}{\alpha}(\sqrt{b})^\alpha\frac{\tilde b}{1+\tilde b}\psi_{0,0}-\frac{\ell}{\alpha}(\sqrt{b})^\alpha\psi_{0,0}\\
+\ &b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})-\frac{\tilde\lambda_{\ell,0}-\tilde\lambda_{0,0}}{\alpha}\frac{(\sqrt{b})^\alpha}{1+\tilde b}\psi_{0,0}\\
=\ &(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{(\sqrt{b})^\alpha}{\alpha(1+\tilde b)}\left[-\ell\tilde b+\tilde\lambda_{\ell,0}-\tilde\lambda_{0,0}\right]\psi_{0,0}\\
+\ &b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})-\frac{\ell}{\alpha}(\sqrt{b})^\alpha\left[\psi_{0,0}-\frac{1}{(\sqrt{b})^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right]-\frac{\ell}{\alpha}\Lambda_r Q_b(r).
\end{aligned}$$

This yields using Lemma 3.1 the v equation:

$$\partial_\tau v+\mathcal{L}_a v=F,\quad F=-\Psi_3+L(\zeta)+\text{NL}(V),\quad \mathcal{L}_a=-\Delta+\frac12\Lambda-p\Phi_{a,b}^{p-1}, \tag{4.4}$$

with

$$\Psi_3=\tilde\Psi_1+\tilde\Psi_3,\quad L(\zeta)=p(\Phi_{a,b}^{p-1}-Q_b^{p-1})\zeta \tag{4.5}$$

and

$$\tilde{\Psi}_3 = \left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right)-\frac{\ell}{\alpha}-\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right]\Lambda_r\Phi_{a,b} \tag{4.6}$$
$$+ (\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{(\sqrt{b})^\alpha}{\alpha(1+\tilde{b})}\left[-\ell\tilde{b}+(\tilde{\lambda}_{\ell,0}-\tilde{\lambda}_{0,0})\right]\psi_{0,0}$$
$$- \frac{\ell}{\alpha}(\sqrt{b})^\alpha\left[\psi_{0,0}-\frac{1}{(\sqrt{b})^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right]+b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})$$

Change of functions at the origin. For the derivation of L^∞ bounds near the origin $0<r\lesssim r^*\ll 1$, it is more convenient to change variables and define:

$$w=\frac{v}{T},\quad T=\Lambda\Phi_{a,b}. \tag{4.7}$$

Definition of $\mathcal{N}$. We now consider the norms

$$\|v\|^2_{\text{extloc}} = \sum_{0\leq i+j\leq 3}\int_{r\geq\frac{r^*}{2}}\frac{|\partial_r^i(\langle z\rangle\partial_z)^jv|^2}{D^{2\alpha(1+\nu)}\langle z\rangle}\rho_r dY$$
$$\|V\|^{2q+2}_{\text{extglobal},q} = \int_{\sqrt{r^2+D^2}\geq A}\frac{V^{2q+2}}{\langle z\rangle}dY$$
$$\|w\|^{2q+2}_{\text{int,q}} = \int_{r\leq r^*}\frac{w^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY$$

and the quantity:

$$\mathcal{N}=\|v\|_{\text{extloc}}+\|V\|_{\text{extglobal},q}+\|\langle z\rangle\partial_zV\|_{\text{extglobal},q} \tag{4.8}$$
$$+ (\|V\|_{\text{extglobal},q}+\|\langle z\rangle\partial_zV\|_{\text{extglobal},q})^{1-\frac{d}{2q+2}}\|\partial_rV\|^{\frac{d}{2q+2}}_{\text{extglobal},q}$$
$$+ \|w\|_{\text{int,q}}+\|\langle z\rangle w\|_{\text{int,q}}+(\|w\|_{\text{int,q}}+\|\langle z\rangle w\|_{\text{int,q}})^{1-\frac{d}{2q+2}}\|\partial_rw\|^{\frac{d}{2q+2}}_{\text{int,q}}$$

We claim the weighted Sobolev embedding:

Lemma 4.1 (Weighted Sobolev embedding)**.** *Recall* (3.33)*, then:*

$$\|\phi V\|_{L^\infty}+\|V\|_{L^\infty(D\geq 2A)}+\left\|\frac{w}{D^{\alpha\nu}}\right\|_{L^\infty(\frac{r^*}{2}\leq r\leq 2A)}+\|w\|_{L^\infty(r\leq 2A,D\leq 2A)}\lesssim_{r^*,A,q}\mathcal{N}+(\sqrt{b})^g. \tag{4.9}$$

and

$$\left\|\partial_r^i(\langle z\rangle\partial_z)^j\left(\frac{v}{D^{\alpha(1+\nu)}}\right)\right\|_{L^\infty(\frac{r^*}{2}\leq r\leq 2A)}\lesssim_{r^*,A}\|v\|_{\text{extloc}},\quad 0\leq i+j\leq 1. \tag{4.10}$$

Proof of Lemma 4.1. Let a smooth cut off function

$$\chi(x)=\left|\begin{array}{ll}1 & \text{for } |x|\geq 2,\\ 0 & \text{for } |x|\leq 1\end{array}\right. .$$

$\underline{r\geq 2A \textit{ or } D\geq 2A}$. Note that $|\partial_r(\chi(r/A))|=A^{-1}|\partial_r\chi(r/A)|\lesssim 1$ since $A\geq 1$. We apply (B.3) to $\chi(r/A)V$ and conclude for q large enough:

$$\begin{aligned}
&\|V\|^{2q+2}_{L^\infty(r\geq 2A)}\lesssim \|\chi\left(\frac{r}{A}\right)V\|^{2q+2}_{L^\infty}\\
\lesssim_q\ & \left(\int\frac{|\chi\left(\frac{r}{A}\right)V|^{2q+2}+|\langle z\rangle\partial_z\left(\chi\left(\frac{r}{A}\right)V\right)|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int\frac{|\partial_r\left(\chi\left(\frac{r}{A}\right)V\right)|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}\\
\lesssim_q\ & \left(\int_{r\geq A}\frac{|V|^{2q+2}+|\langle z\rangle\partial_z\left(V\right)|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int_{r\geq A}\frac{|\partial_rV|^{2q+2}+|V|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}\\
\lesssim_q\ & \int_{r\geq A}\frac{|V|^{2q+2}+|\langle z\rangle\partial_zV|^{2q+2}}{\langle z\rangle}dY\\
+\ & \left(\int_{r\geq A}\frac{|V|^{2q+2}+|\langle z\rangle\partial_zv|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int_{r\geq A}\frac{|\partial_rV|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}\\
\lesssim_q\ & [\|V\|_{\text{extglobal},q}+\|\langle z\rangle\partial_zV\|_{\text{extglobal},q}]^{2q+2}\\
+\ & \left[(\|V\|_{\text{extglobal},q}+\|\langle z\rangle\partial_zV\|_{\text{extglobal},q})^{1-\frac{d}{2q+2}}\|\partial_rV\|^{\frac{d}{2q+2}}_{\text{extglobal},q}\right]^{2q+2}\lesssim_q\mathcal{N}^{2q+2}
\end{aligned}\tag{4.11}$$

since if $r\geq A$ one has $\sqrt{r^2+D^2}\geq A$. similarly, consider $\chi_A(z)=\chi\left(\frac{D}{A}\right)$, then

$$D\sim A\ \text{ implies }\ z^{2\ell}\sim\frac{1}{a}\left(\frac{A}{(\sqrt{b})}\right)^\alpha\gg 1$$

and hence

$$\langle z\rangle\left|\partial_z\left(\chi\left(\frac{D}{A}\right)\right)\right|=\frac{\langle z\rangle|\partial_zD|\left|\chi'\left(\frac{D}{A}\right)\right|}{A}\lesssim\frac{|z|a|\partial_zP_{2\ell}|}{1+aP_{2\ell}(z)}\frac{\sqrt{b}(1+aP_{2\ell}(z))^{\frac{1}{\alpha}}}{A}\lesssim 1,\tag{4.12}$$

and hence applying (B.3) to $\chi_A(z)V$ ensures:

$$\begin{aligned}
\|V\|^{2q+2}_{L^\infty(D\geq 2A)}&\lesssim_q\left(\int_{D\geq A}\frac{|V|^{2q+2}+|\langle z\rangle\partial_zV|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int_{D\geq A}\frac{|\partial_rV|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}\\
&\lesssim_q\mathcal{N}^{2q+2}
\end{aligned}\tag{4.13}$$

since $\sqrt{r^2+A^2}\geq D\geq A$.

$\underline{r,D\leq 2A}$. For $\frac{r^*}{2}\leq r\leq 2A$, first note that

$$\langle z\rangle\frac{|\partial_zD|}{D}=\frac{\langle z\rangle a|\partial_zP_{2\ell}(z)|}{1+aP_{2\ell}(z)}\lesssim 1$$

we then apply (B.5) to $\frac{v}{D^{\alpha(1+\nu)}}$ and conclude that:

$$\begin{aligned}
&\left\|\frac{v}{D^{\alpha(1+\nu)}}\right\|^2_{L^\infty(\frac{r^*}{2}\leq r\leq 2A)}\lesssim_{r^*,A}\sum_{0\leq i+j\leq 2}\int_{\frac{r^*}{2}\leq r\leq 2A}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j(\frac{v}{D^{\alpha(1+\nu)}})|^2}{\langle z\rangle}dY\\
\lesssim_{r^*,A}\ &\sum_{0\leq i+j\leq 2}\int_{\frac{r^*}{2}\leq r}\frac{|\partial_r^i(\langle z\rangle\partial_z)^jv|^2}{D^{2\alpha(1+\nu)}}\frac{\rho_rdY}{\langle z\rangle}\lesssim_{r^*,A}\|v\|^2_{\text{extloc}}
\end{aligned}\tag{4.14}$$

where we used the lower bound $\rho_r\geq e^{-A^2}$ for $r\leq 2A$. The above estimate (4.14) implies from (4.2) the rough bound:

$$\|V\|_{L^\infty(\frac{r^*}{2}\leq r\leq 2A,D\leq 2A)}\lesssim\mathcal{N}+\|\zeta\|_{L^\infty(\frac{r^*}{2}\leq r\leq 2A,D\leq A)}\lesssim\mathcal{N}+(\sqrt{b})^\alpha.\tag{4.15}$$

Near the origin $0<r\le r^*$, we apply (B.3) to $(1-\chi(\frac{2r}{r^*}))(1-\chi(\frac{D}{2A}))w$, using (4.12), and obtain:

$$
\begin{aligned}
&\|w\|^{2q+2}_{L^\infty(r\le\frac{r^*}{2},D\le 2A)} \lesssim_{q,r^*} \int_{r\le r^*,D\le 4A} \frac{|w|^{2q+2}+|\langle z\rangle\partial_z w|^{2q+2}}{\langle z\rangle}dY\\
+&\left(\int_{r\le r^*,D\le 4A}\frac{|w|^{2q+2}+|\langle z\rangle\partial_z w|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int_{r\le r^*,D\le 4A}\frac{|\partial_r w|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}\\
\lesssim&\ \mathcal{N}^{2q+2}.
\end{aligned}\tag{4.16}
$$

Observing that the global rough bound $|\Lambda Q(y)|\lesssim |y|^{-2/(p-1)}$ implies that $r^{2/(p-1)}T=r^{2/(p-1)}D^{-2/(p-1)}\Lambda Q(r/D)\lesssim 1$, the above bound implies for $r\le\frac{r^*}{2}, D\le 2A$:

$$
\|r^{\frac{2}{p-1}}v(r)\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}=\|wr^{\frac{2}{p-1}}T\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}\lesssim\|w\|_{L^\infty(r\le\frac{r^*}{2},D\le A)}\lesssim\mathcal{N}. \tag{4.17}
$$

Finally, we observe that $T\gtrsim\frac{D^\alpha}{r^\gamma}$ for $r\ge D$ and hence from (4.14)

$$
\|\frac{w}{D^{\alpha\nu}}\|_{L^\infty(\frac{r^*}{2}\le r\le 2A)}\lesssim_A\left\|\frac{v}{D^{\alpha(1+\nu)}}\right\|_{L^\infty(\frac{r^*}{2}\le r\le 2A)}\lesssim_A\mathcal{N}. \tag{4.18}
$$

Conclusion. We conclude from (4.11), (4.13), (4.15), (4.17) and (4.3):

$$
\begin{aligned}
&\|\phi V\|_{L^\infty}\lesssim\|r^{\frac{2}{p-1}}V\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}+\|V\|_{L^\infty(D\ge 2A)}+\|V\|_{L^\infty(r\ge\frac{r^*}{2})}\\
\lesssim&\ \|r^{\frac{2}{p-1}}v\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}+\|r^{\frac{2}{p-1}}\zeta\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}+\|V\|_{L^\infty(r\ge 2A)}+\|V\|_{L^\infty(D\ge 2A)}\\
+&\ \|V\|_{L^\infty(\frac{r^*}{2}\le r\le 2A,D\le 2A)}\lesssim\mathcal{N}+\sqrt{b}^\alpha+\sqrt{b}^g\lesssim\mathcal{N}+\sqrt{b}^g
\end{aligned}
$$

as $\alpha\ge g$. We infer from (4.16) and (4.18) that:

$$
\|w\|_{L^\infty(r\le 2A,D\le 2A)}\le\|w\|_{L^\infty(r\le\frac{r^*}{2},D\le 2A)}+\|\frac{w}{D^{\alpha\nu}}\|_{L^\infty(\frac{r^*}{2}\le r\le 2A,D\le 2A)}\lesssim\mathcal{N}
$$

The two above inequalities, with (4.13) and (4.18) give (4.9). The bound (4.14) can be proven similarly for derivatives of v, this is (4.10).

□

The rest of this section is devoted to the control of the various compenents of $\mathcal{N}$ in (4.8) which each require a separate analysis.

4.2. **L^2 bound away from the origin.** We start with the outer L^2 bound $\|v\|_{\text{extloc}}$. It is a consequence of the spectral structure of the linearized operator near $\Phi_{a,b}$ in a polynomially weighted space, and from the fact that we know from the bootstrap assumptions that the first modes below $\ell-\alpha/2$ are not excited.

Lemma 4.2 (Weighted $H^1_{\rho_r}$ bound outside the origin)**.** *There holds the bound for all $0<\nu<\nu^*$ with ν^* depending on η but independent on $\tilde\eta$:*

$$
e^{c_1\nu\tau}\int\frac{v^2}{D^{2\alpha(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}+\int_{\tau_0}^{\tau}e^{c_1\nu\tau'}\int\frac{|\nabla v|^2+r^{-2}v^2}{D^{2\alpha(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}d\tau'\le 1 \tag{4.19}
$$

for some universal constant $c_1>0$.

Remark 4.3. The key feature of this lemma is the weighted D gain and $\langle z\rangle$ which are both sharp for the analysis.

Proof of Lemma 4.2. **step 1** General weighted L^2 energy identity. We compute from (4.4) for any function $\chi(\tau, r, z)$:

$$\frac{1}{2}\frac{d}{d\tau}\int v^2\chi\rho_r dY = \frac{1}{2}\int v^2\partial_\tau\chi\rho_r dY + \int v\chi\partial_\tau v\rho_r dY$$
$$= \frac{1}{2}\int v^2\partial_\tau\chi\rho_r dY + \int v\left[\Delta v + p\Phi_{a,b}^{p-1}v - \frac{1}{2}\Lambda v + F\right]\chi\rho_r dY.$$

We integrate by parts in r:

$$\int v\left[\Delta_r v - \frac{1}{2}r\partial_r v\right]\chi\rho_r dY = \int \chi v\partial_r(\rho_r r^{d-1}\partial_r v)drdz$$
$$= -\int \chi(|\partial_r v|^2\rho_r dY + \frac{1}{2}\int v^2\left[\Delta_r\chi - \frac{1}{2}r\partial_r\chi\right]\rho_r r^{d-1}drdz$$

and in z:

$$\int v\left[\partial_z^2 v - \frac{1}{2}z\partial_z v\right]\chi\rho_r dY = -\int \chi|\partial_z v|^2\rho_r dY + \frac{1}{2}\int v^2\left[\partial_z^2\chi + \frac{1}{2}\partial_z(z\chi)\right]\rho_r dY$$

to derive the algebraic relation:

$$\frac{1}{2}\frac{d}{d\tau}\int v^2\chi\rho_r dY = -\mathcal{Q}_{q,\chi}(v,v) + \int vF\chi\rho_r dY \tag{4.20}$$

where we introduced the quadratic form:

$$\mathcal{Q}_\chi(h,h) = \int \chi|\nabla h|^2\rho_r dY \tag{4.21}$$
$$- \frac{1}{2}\int h^2\left[2p\Phi_{a,b}^{p-1}\chi + \partial_\tau\chi + \partial_z^2\chi - \frac{2}{p-1}\chi + \frac{1}{2}\partial_z(z\chi) + \Delta_r\chi - \frac{1}{2}r\partial_r\chi\right]\rho_r dY$$

step 2 $L^2_{\rho_r}$ identity with polynomial weight. Let

$$\chi(z) = \frac{1}{\langle z\rangle^{4\ell(1+\nu)+1}}, \tag{4.22}$$

then for z large:

$$-\partial_z^2\chi - \frac{1}{2}\partial_z(z\chi) = \left[2\ell(1+\nu) + O\left(\frac{1}{\langle z\rangle}\right)\right]\chi.$$

We apply (4.21) and conclude:

$$Q_\chi(v,v) = \int \chi|\nabla v|^2\rho_r dY + \int\left(\ell(1+\nu) + \frac{1}{p-1} - p\Phi_{a,b}^{p-1}\right)\chi v^2\rho_r dY$$
$$+ O\left(\int_{|z|\leq z^*} v^2\rho_r dY + \nu^2\int \chi v^2\rho_r dY\right). \tag{4.23}$$

for some large enough $z^*(\nu)$. We recall from (1.20) and Proposition 2.1 the lower bound in terms of quadratic form on $H^1_{\rho_r}$:

$$-\Delta_r + \frac{1}{2}\Lambda_r - p\Phi_{a,b}^{p-1} \geq -\Delta_r + \frac{1}{2}\Lambda_r - \frac{pc_\infty^{p-1}}{r^2} \geq -\frac{\alpha}{2}$$

which implies from (B.1) that for some universal constant $c > 0$:

$$Q_\chi(v,v) \geq \int\left(\ell(1+\nu) - \frac{\alpha}{2} - c\nu^2\right)v^2\chi\rho_r dY + \nu^2\left[\int |\nabla v|^2\rho_r\chi dY + \int \frac{v^2}{r^2}\chi\rho_r dY\right]$$
$$+ O\left(\int_{|z|\leq z^*} v^2\rho_r dY\right).$$

We estimate the well localized quadratic term using (4.22), (3.28), (3.35):

$$\int_{|z|\leq z^*} v^2\rho_r dY \lesssim_\nu \int \left[\varepsilon^2+(\psi-\zeta)^2\right]\rho_Y dY \lesssim (\sqrt{b})^{2\alpha+2\eta} \tag{4.24}$$

where we used (3.74) and (3.29) in the last step, and the fact that ζ is the leading order term of ψ given by (3.24). We have therefore obtained from (4.20):

$$\begin{aligned} & \frac{d}{d\tau}\int \chi v^2\rho_r dY + 2\nu^2\int |\nabla v|^2\rho_r\chi dY \\ + & \left[(2\ell(1+\nu)-\alpha-c\nu^2)\int \chi v^2 + 2\nu^2\int \frac{v^2}{r^2}\chi dY\right] \lesssim |\int \chi F v\rho_r dY| + (\sqrt{b})^{2\alpha+2\eta}. \end{aligned} \tag{4.25}$$

step 4 Estimate for the forcing term. We now estimate the F term given by (4.4) and claim:

$$\left|\int \chi F v\rho_r dY\right| \lesssim \sqrt{b}^{2\alpha+2\eta} + \nu^2\int \chi v^2\rho_r dY. \tag{4.26}$$

To prove it, for the Ψ_3, $L(\zeta)$ terms, we estimate from (C.1), (C.3), Hölder and Young inequality:

$$\begin{aligned} \left|\int \chi(\Psi_3+L(\zeta)v\rho_r dY\right| & \lesssim \left(\int \frac{|\Psi_3|^2+|L(\zeta)|^2}{\langle z\rangle^{2(2\ell+2\nu)+1}}\rho_r dY\right)^{\frac12}\left(\int \chi v^2\rho_r dY\right)^{\frac12} \\ & \leq (\sqrt{b})^{2\alpha+2\eta} + \nu^2\int \chi v^2\rho_r dY. \end{aligned}$$

For the NL(V) term, we estimate by homogeneity with (3.41), (4.9):

$$\begin{aligned} & \int \chi|\mathrm{NL}(V)||v|\rho_r dY \lesssim \int |v|\left[|V|^p+|\Phi_{a,b}|^{p-2}V^2\right]\chi\rho_r dY \\ \lesssim & \int |v|\left(1+\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2} V^2\chi\rho_r dY \lesssim \int |v|\left(1+\frac{1}{r^2}\right)\phi V^2\chi\rho_r dY \end{aligned}$$

and split the integral in two parts for $M\gg 1$ large enough using the Gaussian weight in the dz integrability provided by χ. Indeed, using (4.2) and (3.41):

$$\begin{aligned} & \int_{r\leq M|\log b|} |v|\left(1+\frac{1}{r^2}\right)\phi V^2\chi\rho_r dY \lesssim \int_{r\leq K|\log b|} |v|\left(1+\frac{1}{r^2}\right)\phi(r)(v^2+\zeta^2)\chi\rho_r dY \\ \lesssim & \ (\|\phi V\|_{L^\infty}+\|\phi\zeta\|_{L^\infty(r\leq M|\log b|)})\int v^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY \\ + & \ \nu^2\int v^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY + c_\nu\int_{r\leq M|\log b|}\zeta^2(\phi\zeta)^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY \\ \leq & \ C\nu^2\int v^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY + (\sqrt{b})^{2\alpha+2g} \end{aligned}$$

as $\alpha > g$ and for some $c>0$

$$\begin{aligned} & \int_{r\geq M|\log b|} |v|\left(1+\frac{1}{r^2}\right)\phi V^2\chi\rho_r dY \lesssim \int_{r\geq M|\log b|} |\zeta|\left(1+\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2} V^2\chi\rho_r dY + b^{cM}\|\phi V\|^3_{L^\infty} \\ \lesssim & \ (\sqrt{b})^{cM} \leq (\sqrt{b})^{2\alpha+1} \end{aligned}$$

provided M has been chosen large enough. The collection of above bounds, injected in (4.4), gives (4.26).

step 5 Conclusion. From (4.25) and (4.26) we infer the pointwise differential inequality:

$$\begin{aligned} & \frac{d}{d\tau}\int \chi v^2\rho_r dY+\left[2\ell(1+\nu)-\alpha-C\nu^2\right]\int \chi v^2\rho_r dY \\ + & \frac{\nu^2}{2}\left[\int |\nabla v|^2\rho_r\chi dY+\int\frac{v^2}{r^2}\chi\rho_r dY\right]\lesssim(\sqrt{b})^{2\alpha+2\eta}. \end{aligned} \tag{4.27}$$

We now compute from (3.47), (3.71), (3.48) and (4.27):

$$\begin{aligned} & \frac{d}{d\tau}\left\{e^{c\nu\tau}\frac{\int \chi v^2\rho_r dY}{(\sqrt{b})^{2\alpha(1+\nu)}}\right\} \\ = & \frac{e^{c\nu\tau}}{(\sqrt{b})^{2\alpha(1+\nu)}}\left[\frac{d}{d\tau}\int\chi v^2\rho_r dY+c\nu\int \chi v^2\rho_r dY-\alpha(1+\nu)\frac{b_\tau}{b}\int\chi v^2\rho_r dY\right] \\ = & \frac{e^{c\nu\tau}}{(\sqrt{b})^{2\alpha(1+\nu)}}\left[\frac{d}{d\tau}\int\chi v^2\rho_r dY+[(2\ell-\alpha)(1+\nu)+c\nu+O((\sqrt{b})^\eta)]\int\chi v^2\rho_r dY\right\} \\ \leq & \frac{e^{c\nu\tau}}{(\sqrt{b})^{2\alpha(1+\nu)}}\left\{(-(\alpha-c)\nu+C\nu^2)\int\chi v^2\rho_r dY-\frac{\nu^2}{2}\int\left[|\nabla v|^2+\frac{v^2}{r^2}\right]\chi\rho_r dY\right\} \\ + & e^{c\nu\tau}\frac{(\sqrt{b})^{2\alpha+2\eta}}{(\sqrt{b})^{2\alpha(1+\nu)}}\leq-\nu^2\frac{e^{c\nu\tau}}{(\sqrt{b})^{2\alpha(1+\nu)}}\left[\int|\nabla v|^2\rho_r\chi dY+\int\frac{v^2}{r^2}\chi\rho_r dY\right]+O(\sqrt{b}^\eta) \end{aligned}$$

provided ν small enough and $c<\alpha$. Integration in time ensures:

$$e^{c\nu\tau}\frac{\int v^2\rho_r\chi dY}{(\sqrt{b})^{2\alpha(1+\nu)}}+\int_{\tau_0}^{\tau}\frac{e^{c\nu\tau'}}{(\sqrt{b})^{2\alpha(1+\nu)}}\left[\int|\nabla v|^2\rho_r\chi dY+\int\frac{v^2}{r^2}\chi\rho_r dY\right]d\tau'\lesssim 1.$$

We now observe for $|z|\geq z^*$:

$$\frac{1}{\langle z\rangle}\frac{1}{\mu^{2\alpha(1+\nu)}}\lesssim\frac{1}{a^C\langle z\rangle^{4\ell(1+\nu)+1}}=\frac{\chi}{a^C}$$

which together with (4.24) and (3.35) yields (4.19) for some $0<c=c_1\ll 1$ small enough. □

4.3. **Control of derivatives outside the origin.** We now propagate the L^2 bound (4.19) to higher derivatives.

Lemma 4.4 (Control of derivatives outside the origin)**.** *For $c_2<c_1$ independent of $\tilde\eta$ (c_1 is defined in Lemma 4.2), there holds the bounds for $0<r^*\ll 1$ for ν small enough depending on η but independent on $\tilde\eta$:*

$$\int_{r\geq\frac{r^*}{2}}\frac{(\partial_r^i(\langle z\rangle\partial_z)^j v)^2}{[(\mu\sqrt{b})^\alpha]^{2(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}\leq c(r^*)e^{-c_2\nu\tau},\quad 0\leq i+j\leq 3. \tag{4.28}$$

Proof. This is a standard parabolic regularity claim. We briefly sketch the proof of the main steps to take care of the $\langle z\rangle$ weight. We let

$$v_1=\partial_r v,\quad v_2=\partial_z v.$$

We let χ be given by (4.22).

step 1 Control of v_2. From (4.4):

$$\partial_\tau(zv_2)+\mathcal{L}_a(zv_2)+2\partial_z v_2=z\partial_z F+p(z\partial_z\Phi_{a,b}^{p-1})v.$$

The same chain of estimates like for the proof of (4.25) leads to:

$$\begin{aligned}
& \frac{d}{d\tau}\int \chi(zv_2)^2\rho_r dY + \nu^2\int\left[|\nabla(zv_2)|^2+\frac{(zv_2)^2}{r^2}\right]\rho_r\chi dY\\
+ & \ (2\ell(1+\nu)-\alpha-\nu^2)\int (zv_2)^2\rho_r\chi dY\\
\lesssim & \ |\int\left[z\partial_z F + p(z\partial_z\Phi_{a,b}^{p-1})v\right](zv_2)\chi\rho_r dY| + (z^*)^C\int_{|z|\leq z^*} v_2^2\rho_r dY
\end{aligned}$$

for some large enough $z^*(\nu)$. We now estimate all terms in the above identity. The crossed term is estimated using the rough bound:

$$|(z\partial_z\Phi_{a,b}^{p-1}|\lesssim \frac{z|\partial_z D|}{D}\frac{1}{D^2}\left|[2Q^{p-1}+(p-1)yQ^{p-2}Q'](\frac{r}{D})\right|\lesssim\frac{1}{D^2}\left(\frac{D}{r}\right)^2\lesssim\frac{1}{r^2}$$

and hence from Hölder:

$$\int\chi|z\partial_z\Phi_{a,b}^{p-1}v||zv_2|\rho_r dY\leq \nu^3\int\frac{(zv_2)^2}{r^2}\rho_r dY + c(\nu)\int\frac{v^2}{r^2}\rho_r dY.$$

The Ψ_3 and $L(\zeta)$ terms are estimated from (C.1), (C.3):

$$\begin{aligned}
& \int\chi(|z\partial_z\Psi_3|+|z\partial_z L(\zeta)|)|zv_2|\rho_r dY\\
\lesssim & \ \left(\int\frac{|z\partial_z\Psi_3|^2+|z\partial_z L(\zeta)|^2}{\langle z\rangle^{2(2\ell+2\eta)+1}}\chi\rho_r dY\right)^{\frac12}\left(\int\chi(zv_2)^2\rho_r dY\right)^{\frac12}\\
\lesssim & \ (\sqrt{b})^{2\alpha+2\eta}+\nu^3\int\chi(zv_2)^2\rho_r dY.
\end{aligned}$$

For the nonlinear term, we compute using $\partial_z\zeta=0$:

$$\partial_z\mathrm{NL}(V)=p\partial_z\Phi_{a,b}((\Phi_{a,b}+V)^{p-1}-\Phi_{a,b}^{p-1}-(p-1)\Phi_{a,b}^{p-2}V)+\partial_z v((\Phi_{a,b}+V)^{p-1}-\Phi_{a,b}^{p-1}).$$

We use

$$|z\partial_z\Phi_{a,b}|\lesssim\frac{|z\partial_z D|}{D}\frac{1}{D^{\frac{2}{p-1}}}\Lambda Q\left(\frac{r}{D}\right)\lesssim\Phi_{a,b}$$

to estimate by homogeneity

$$\begin{aligned}
& |z\partial_z\mathrm{NL}(V)|\lesssim\Phi_{a,b}^{p-2}V^2+\Phi_{a,b}|V|^{p-1}+|z\partial_z v|(|V|^{p-1}+|V|\Phi_{a,b}^{p-2})\\
\lesssim & \ \left(1+\frac{1}{r^2}\right)\left[|\phi V|+|\phi V|^{p-2}\right]|V|+|zv_2|\left(1+\frac{1}{r^2}\right)(|\phi V|^{p-1}+|\phi V|)
\end{aligned}$$

We therefore split the integral in $r\leq M|\mathrm{log}b|$, $r\geq M|\mathrm{log}b|$ for some $M\gg 1$ as before and estimate using (3.41), (4.2), (4.3):

$$\int_{r\geq M|\mathrm{log}b|}\chi|z\partial_z\mathrm{NL}(v)||zv_2|\rho_r dY\lesssim\nu^3\int(zv_2)^2\left(1+\frac{1}{r^2}\right)\rho_r dY+(\sqrt{b})^{2\alpha+4}$$

because of the exponential weight, and as $p\geq 3$

$$
\begin{aligned}
&\int_{r\leq M|\log b|}\chi|z\partial_z \mathrm{NL}(v)||zv_2|\rho_r dY \lesssim \int_{r\leq M|\log b|}\chi|zv_2|\left(1+\frac{1}{r^2}\right)\left[|\phi V|+|\phi V|^{p-2}\right]|V|\\
+&\int_{r\leq M|\log b|}\chi|zv_2|^2\left(1+\frac{1}{r^2}\right)(|\phi V|^{p-1}+|\phi V|)\\
\leq&\ \nu^3\int (zv_2)^2\left(1+\frac{1}{r^2}\right)\rho_r dY+c_\nu\int\left[|\phi V|+|\phi V|^{p-2}\right]^2|V|^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY\\
+&\ (\sqrt{b})^{\tilde\eta}\int_{r\leq M|\log b|}\chi|zv_2|^2\left(1+\frac{1}{r^2}\right)\\
\leq&\ 2\nu^3\int (zv_2)^2\left(1+\frac{1}{r^2}\right)\rho_r dY+c_\nu\int \phi^2(\zeta^4+v^4)\left(1+\frac{1}{r^2}\right)\chi\rho_r dY\\
\leq&\ 2\nu^3\int (zv_2)^2\left(1+\frac{1}{r^2}\right)\rho_r dY+c_\nu\int\left[|\phi\zeta|^2\zeta^2+(|\phi V|^2+|\phi\zeta|^2)v^2\right]\left(1+\frac{1}{r^2}\right)\chi\rho_r dY\\
\leq&\ 3\nu^3\int (zv_2)^2\left(1+\frac{1}{r^2}\right)\rho_r dY+\sqrt{b}^{2\alpha+2g}+(\sqrt{b})^{2\tilde\eta}\int v^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY
\end{aligned}
$$

The collection of above bounds yields the differential inequality:

$$
\begin{aligned}
&\frac{d}{d\tau}\int\chi(zv_2)^2\rho_r dY+\frac{\nu^2}{2}\int\left[|\nabla(zv_2)|^2+\frac{(zv_2)^2}{r^2}\right]\rho_r\chi dY\\
+&\ (2\ell(1+\eta)-\alpha-C\nu^2)\int(zv_2)^2\rho_r\chi dY\\
\lesssim&\ \int_{|z|\leq z^*}v_2^2\rho_r\chi dY+(\sqrt{b})^{2\alpha+2\eta}+\int v^2\left(1+\frac{1}{r^2}\right)\chi\rho_r dY.
\end{aligned}
$$

Integrating in time using the space time bound (4.19) yields the pointwise bound

$$
e^{c_2\nu\tau}\int\frac{(zv_2)^2}{D^{2(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}+\int_{\tau_0}^{\tau}e^{c_2\nu\tau'}\int\frac{|\nabla(zv_2)|^2}{D^{2(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}d\tau'\leq 1
$$

by possibly taking any smaller constant $c_2<c_1$ for ν and b small enough. Iterating $z\partial_z$ derivatives can be done along the same lines, using the estimates (C.1) and (C.3) for the source terms, and the nonlinearity being three times differentiable as $p\geq 3$, and is left to the reader.

step 2 Control of v_1. We now let one ∂_r derivative go through (4.4), and we run the energy identity (4.25) with a mulitplier χ localized strictly away from the origin. Since all terms without derivatives have been estimated at the previous step, the control of ∂_r derivatives and mixed derivatives follows. The elementary details are left to the reader.

□

4.4. $L^{2q+2}_{\rho_r}$ **bound at the origin.** We now study L^∞ bounds near the origin $0<r\lesssim r^*\ll 1$. For this, it is more convenient to use the variable w given by (4.7) and derive suitable L^{2q+2} monotonicity estimates in w, which equivalently correspond to suitably weighted estimates for v near the origin. The key point here is that we may derive such bounds for a uniform weight in z.

Lemma 4.5 (L^{2q+2} bound at the origin). *There exists $0<c_3<c_2$ such that the following holds. Pick K large enough, then there holds for all $q>q(d,p)$ large* *enough:*

$$e^{c_3\nu q\tau}\int_{r\leq 1}\frac{w^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY+\int_{\tau_0}^{\tau}\int_{r\leq 1}e^{c_3\nu q\tau'}\frac{w^{2q+2}}{r^2\langle z\rangle(1+D^{2Kq})}dYd\tau'\leq 1. \quad (4.29)$$

Proof. Observe

$$(\Delta_r+p\Phi_{a,b}^{p-1})T=0 \quad (4.30)$$

and hence (4.4) becomes:

$$\partial_t w=\Delta w+2\frac{\nabla T\cdot\nabla w}{T}+V_1 w-\frac{1}{2}Y\cdot\nabla w-\frac{1}{p-1}w+\widetilde{F},\quad \widetilde{F}=\frac{F}{T} \quad (4.31)$$

with

$$V_1=\frac{1}{T}\left[-\partial_t T+\partial_{zz}T-\frac{1}{2}Y\cdot\nabla T\right]. \quad (4.32)$$

step 1 L^{2q+2} energy identity. We compute from (4.31) for a given $\chi(t,r,z)$:

$$\begin{aligned}\frac{1}{2q+2}\frac{d}{dt}\int\chi w^{2q+2}dY &= \frac{1}{2q+2}\int\partial_t\chi w^{2q+2}dY\\ +&\int w^{2q+1}\left\{\Delta w+2\frac{\nabla T\cdot\nabla w}{T}+V_1 w-\frac{1}{2}Y\cdot\nabla w-\frac{1}{p-1}w+\widetilde{F}\right\}\chi dY\\ =&-Q_\chi(w^{q+1},w^{q+1})+\int w^{2q+1}\tilde{F}\chi dY\end{aligned}$$

with

$$\begin{aligned}Q_\chi(h,h)&=\frac{2q+1}{(q+1)^2}\int|\nabla h|^2\chi dY+\int h^2\left\{\frac{2}{2q+2}\Delta(\log T)-V_1-\frac{d+1}{2(2q+2)}-\frac{1}{p-1}\right\}\chi dY\\ &-\int h^2\left[\frac{1}{2q+2}\Delta\chi-\frac{2}{2q+2}\frac{\nabla T\cdot\nabla\chi}{T}+\frac{1}{2(2q+2)}Y\cdot\nabla\chi+\frac{\partial_t\chi}{2q+2}\right]dY.\end{aligned}$$

We choose

$$\chi(r,z)=\frac{\phi_1(r)}{\langle z\rangle(1+D^{2Kq})},\quad \phi_1(r)=\left\{\begin{array}{ll}1 & \text{for}\ \ r\leq 1,\\ 0 & \text{for}\ \ r\geq 2.\end{array}\right. . \quad (4.33)$$

step 2 Lower bound on Q_χ. We estimate:

$$\left|\frac{\partial_t\chi}{\chi}\right|\lesssim\frac{D}{1+D^{2Kq}}\frac{|\partial_t D|}{D}\lesssim\frac{|\partial_t\mu|}{\mu}+\frac{|b_t|}{b}\lesssim 1.$$

We then estimate in brute force using the definition of χ and the pointwise bounds (A.9), (A.10):

$$\begin{aligned}Q_\chi(h,h)&\gtrsim\frac{2q+1}{(q+1)^2}\int|\nabla h|^2\chi dY+\int h^2\left\{\frac{2}{2q+2}\Delta_r(\log T)\right\}\chi dY\\ &+O\left(\int_{r\leq 2}\frac{h^2}{\langle z\rangle(1+D^{2Kq})}dY\right).\end{aligned}$$

We now estimate from (A.1), (1.16), (4.30), (1.20):

$$\Delta_r(\log T)=-p\Phi_{a,b}^{p-1}-\left(\frac{\partial_r T}{T}\right)^2\geq\frac{-\gamma(d-2-\gamma)-\gamma^2}{r^2}=-\frac{\gamma(d-2)}{r^2}.$$

We then use the sharp Hardy inequality

$$\int\chi|\partial_r h|^2 r^{d-1}dr\geq\left(\frac{d-2}{2}\right)^2\int\frac{\chi h^2}{r^2}r^{d-1}dr+O\left(\int\frac{|\partial_r\chi|^2}{\chi}h^2r^{d-1}dr\right) \quad (4.34)$$

to lower bound:

$$\begin{aligned} &\frac{2q+1}{(q+1)^2}\int|\nabla h|^2\chi dY+\int h^2\left\{\frac{2}{2q+2}\Delta_r(\log T)\right\}\chi dY\\ \geq\ &\left[\frac{2q+1}{(q+1)^2}\left(\frac{d-2}{2}\right)^2-\frac{2(d-2)\gamma}{2q+2}\right]\int\frac{h^2}{r^2}\chi dY+O\left(\int_{r^*\leq r\leq 2}h^2\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY\right)\end{aligned}$$

Observe

$$\frac{2q+1}{(q+1)^2}\left(\frac{d-2}{2}\right)^2-\frac{2(d-2)\gamma}{2q+2}=\frac{d-2}{4(q+1)^2}\left[2q(d-2-2\gamma)+d-2-4\gamma\right]>0$$

for q large enough from (1.17), and hence for $q>q(d,p)\gg 1$ and a universal $c>0$:

$$\begin{aligned} &\frac{2q+1}{(q+1)^2}\int|\nabla h|^2\chi dY+\int h^2\left\{\frac{2}{2q+2}\Delta_r(\log T)\right\}\chi dY\\ \geq\ &\frac{c}{q}\int\left(|\nabla h|^2+\frac{h^2}{r^2}\right)\chi dY+O\left(\int_{r^*\leq r\leq 2}h^2\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY\right).\end{aligned}$$

and hence the lower bound

$$Q_\chi(h,h)\ \geq\ \frac{c}{q}\int\left(|\nabla h|^2+\frac{h^2}{r^2}\right)\chi dY+O\left(\int_{r^*\leq r\leq 2}h^2\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY\right)$$

provided $0<r^*(K,q)\ll 1$ universal has been chosen small enough.
step 2 L^{2q+2} bound. We conclude for q large enough:

$$\begin{aligned} &\frac{1}{2q+2}\frac{d}{dt}\int w^{2q+2}\chi dY+\frac{c}{q}\left[\int|\nabla(w^{q+1})|^2\chi dY+\int\frac{w^{2q+2}}{r^2}\chi dY\right]\\ \lesssim\ &\int_{r^*\leq r\leq 2}w^{2q+2}\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY+\left|\int_{r\leq 2}w^{2q+1}\widetilde{F}\chi dY\right|\end{aligned}$$

and we now estimate the rhs.

Quadratic term. We first observe from (4.10) and (4.28) the pointwise bound:

$$\left\|\partial_r^i(\langle z\rangle\partial_z)^j\left(\frac{v}{D^{\alpha(1+\nu)}}\right)\right\|_{L^\infty(\frac{r^*}{2}\leq r\leq 2A)}\lesssim e^{-\frac{c_2}{2}\nu\tau},\quad 0\leq i,j\leq 1. \tag{4.35}$$

We then split the integral in two parts. For $\frac{r}{D}\geq 1$, we have

$$\Lambda\Phi_{a,b}\gtrsim\frac{c(r^*)D^\alpha}{r^\gamma}$$

and hence using (4.35), (4.19):

$$\begin{aligned} &\int_{r^*\leq r\leq 2,r\geq D}w^{2q+2}\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY\lesssim\int_{r^*\leq r\leq 2,r\geq D}\left(\frac{v}{D^\alpha}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\\ \lesssim\ &\left\|\frac{v}{D^{\alpha(1+\nu)}}\right\|^{2q}_{L^\infty(r^*\leq r\leq 2,D\leq 2)}\int\frac{v^2}{D^{2\alpha(1+\nu)}}\frac{\rho_r dY}{\langle z\rangle}\lesssim e^{-c_2\nu q\tau}\end{aligned}$$

with $0<c\ll 1$ independent of q. For $\frac{r}{D}\leq 1$, we use $\Lambda\Phi_{a,b}\gtrsim\frac{1}{D^{\frac{2}{p-1}}}$ and (4.35) to estimate:

$$\int_{r^*\leq r\leq 2,r\leq D}w^{2q+2}\frac{\rho_r}{\langle z\rangle(1+D^{2Kq})}dY\ \lesssim\ e^{-c_2\nu q\tau}\int\frac{D^{Cq}}{1+D^{2Kq}}\frac{dz}{\langle z\rangle}\lesssim e^{-\frac{c_2}{2}\nu q\tau}$$

for K large enough independent of q.

$\Psi_3, L(\zeta)$ *term.* We estimate from Hölder, (C.16) and (C.19):

$$\begin{aligned}\left|\int_{r\leq 2} w^{2q+1}\frac{\Psi_3+L(\zeta)}{T}\chi dY\right| &\leq \kappa\int\frac{w^{2q+2}}{r^2}\chi dY + c_\kappa\int\left(\frac{r^{2\frac{2q+1}{2q+2}}\Psi_3}{T}\right)^{2q+2}\chi dY\\ &\leq \kappa\int\frac{w^{2q+2}}{r^2}\chi dY + b^{c\eta q}\end{aligned}$$

Nonlinear term. We claim for a universal $c>0$:

$$\int_{r\leq 2}|\mathrm{NL}(V)|w^{2q+1}\chi dY \leq \kappa\int\frac{w^{2q+2}}{r^2}\chi dY + b^{cq}. \tag{4.36}$$

Indeed, we estimate by Hölder:

$$\int_{r\leq 2}\frac{|\mathrm{NL}(V)|}{T}w^{2q+1}\chi dY \leq \kappa\int\frac{w^{2q+2}}{r^2}\chi dY + c_\kappa\int\left(\frac{r^{\frac{2(2q+1)}{2q+2}}\mathrm{NL}(V)}{T}\right)^{2q+2}\chi dY.$$

We now estimate by homogeneity for $r\leq 2$ using (4.3), (3.38):

$$\begin{aligned}\frac{|\mathrm{NL}(V)|}{T} &\lesssim \frac{1}{T}\left[|v|^p+|\zeta|^p+\Phi_{a,b}^{p-2}(|v|^2+|\zeta|^2)\right]\lesssim \frac{1}{T}\|\phi v\|_{L^\infty(r\leq 2)}\frac{|v|}{r^2}+\frac{1}{T}\|\phi\zeta\|_{L^\infty(r\leq 2)}|\zeta|\\ &\lesssim \|\phi v\|_{L^\infty(r\leq 2)}\frac{|w|}{r^2}+(\sqrt{b})^g\frac{\zeta}{T}\lesssim(\sqrt{b})^{\tilde\eta}\frac{|w|}{r^2}+(\sqrt{b})^g\frac{\zeta}{T}.\end{aligned}$$

Moreover, we extract from (4.2) the rough bound for $r\leq 2$

$$|\zeta(r)|\lesssim\frac{(\sqrt{b})^\alpha}{r^\gamma}$$

which implies from (C.26) for $D\leq r\leq 2$:

$$\frac{|\zeta|}{T}\lesssim\frac{(\sqrt{b})^\alpha}{r^\gamma}\frac{D^\gamma+r^\gamma}{D^\alpha}\lesssim\frac{1}{\mu^\alpha}.$$

On the other hand for $r\leq D$ from (4.2):

$$\frac{|\zeta|}{T}\lesssim(\sqrt{b})^{g-\frac{2}{p-1}}D^{\frac{2}{p-1}}$$

and hence the bound for $r\leq 2$

$$\frac{|\mathrm{NL}(V)|}{T}\lesssim(\sqrt{b})^{\tilde\eta}\frac{|w|}{r^2}+\frac{b^{\frac{g}{2}}}{\mu^\alpha}\mathbf{1}_{D\leq r\leq 2}+D^{\frac{2}{p-1}}b^{\frac{g}{2}-\frac{1}{p-1}}\mathbf{1}_{r\leq\min\{2,D\}} \tag{4.37}$$

which implies since $\mu\geq 1/2$, $g\geq 1$ and $p\geq 3$:

$$\int\left(\frac{r^{2\frac{2q+1}{2q+2}}\mathrm{NL}(V)}{T}\right)^{2q+2}\chi dY\lesssim\|\phi v\|^{2q}_{L^\infty(r\leq 2)}\int\frac{w^{2q+2}}{r^2}\chi dY+b^{cq}$$

for K large enough independent of q and c universal. The collection of above bounds concludes the proof of (4.36).

step 3 Conclusion. We conclude for $r^*(q,K)$ small enough since $0<\nu\ll\eta\ll 1$:

$$\frac{1}{2q+2}\frac{d}{dt}\int w^{2q+2}\chi dY+\frac{c}{q}\left[\int|\nabla(w^{q+1})|^2\chi dY+\int\frac{w^{2q+2}}{r^2}\chi dY\right]\lesssim e^{-\frac{c_2}{2}\nu q\tau}$$

which implies provided $r^*(q)$ has been chosen small enough:

$$\frac{1}{2q+2}\frac{d}{dt}\int w^{2q+2}\chi dY+2cq\int w^{2q+2}\chi dY+\frac{c}{q}\left[\int|\nabla(w^{q+1})|^2\chi dY+\int\frac{w^{2q+2}}{r^2}\chi dY\right]\lesssim e^{-\frac{c_2}{2}\nu q\tau}$$

whose time integration ensures that there exists $0 < c_3 \leq c_2$ for which:

$$\int w^{2q+2}\chi dY + \int_{\tau_0}^{\tau} e^{c_3\nu q\sigma}\left[\int |\nabla(w^{q+1})|^2\chi dY + \int \frac{w^{2q+2}}{r^2}\chi dY\right]d\sigma \lesssim e^{-c_3\nu q\tau}.$$

The definition (4.33) of χ now yields (4.29). □

4.5. $W^{1,2q+2}$ **bound.** We now turn to the weigted control of derivatives in large L^{2q+2} norms. ∂_r derivatives yield singular term at the origin and we claim a lossy bound which for q large is sufficient thanks to (4.8) and the underlying weighted Sobolev estimate (B.3).

Lemma 4.6 (W_1^{2q+2} estimate)**.** *There holds:*

$$\int_{r\leq 1}\frac{(\partial_r w)^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY \lesssim \frac{1}{b^{Cq}} \tag{4.38}$$

and:

$$\int_{r\leq 1}\frac{(\langle z\rangle\partial_z w)^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY \lesssim e^{-c_4\nu q\tau} \tag{4.39}$$

for some universal constants $C = C(d,p)$ independent of q and $c_4 \lesssim c_3$.

Proof. Let

$$w_1 = \partial_r w, \quad w_2 = \partial_z w.$$

step 1 ∂_r bound. Taking ∂_r of (4.31):

$$\begin{aligned}
\partial_t w_1 &= \left[\Delta w_1 + 2\frac{\nabla T\cdot\nabla w_1}{T} + V_1 w_1 - \frac{1}{2}Y\cdot\nabla w_1\right]\\
&- \left[\frac{d-1}{r^2} - 2\partial_{rr}\text{log}T\right]w_1 + 2\partial_{rz}(\text{log}T)w_2 + (\partial_r V_1)w - \frac{1}{2}w_1\\
&+ \partial_r\widetilde{F}.
\end{aligned}$$

We let χ be given by (4.33) and hence arguing as above, we have for q large enough:

$$\begin{aligned}
&\frac{1}{2q+2}\frac{d}{dt}\int w_1^{2q+2}\chi dY\\
\leq\ & -\frac{c}{q}\int\left[|\nabla(w_1^{q+1})|^2 + \int\frac{w_1^{2q+2}}{r^2}\right]\chi dY - \int\left[\frac{d-1}{r^2} - 2\partial_{rr}\text{log}T\right]w_1^{2q+2}\chi dY\\
+\ & C(r^*)\int_{r^*\leq r\leq 2} w_1^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})} + \int\left[2\partial_{rz}(\text{log}T)w_2 + (\partial_r V_1)w + \partial_r\widetilde{F}\right]w_1^{2q+1}\chi dY.
\end{aligned}$$

We now observe from (A.2):

$$\frac{d-1}{r^2} - 2\partial_{rr}\text{log}T \geq \frac{d-1-2\gamma}{r^2} > 0$$

and hence using (A.9), (A.10):

$$\begin{aligned}
&\frac{1}{2q+2}\frac{d}{dt}\int w_1^{2q+2}\chi dY + \frac{c}{q}\int\left[|\nabla(w_1^{q+1})|^2 + \int\frac{w_1^{2q+2}}{r^2}\right]\chi dY\\
\lesssim\ & C(r^*)\int_{r^*\leq r\leq 2} w_1^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})} + \int\left(\frac{|w_2|}{r\langle z\rangle} + \frac{|w|}{r} + |\partial_r\widetilde{F}|\right)|w_1|^{2q+1}\chi dY.
\end{aligned}$$

We split the crossed term:

$$\int_{r\leq 2}(|w_2|+|w|)|w_1|^{2q+1}\frac{dY}{r\langle z\rangle(1+D^{2Kq})} = \int_{r\leq 2} r(|w_2|+|w|)|w_1|^{2q+1}\frac{dY}{r^2\langle z\rangle(1+D^{2Kq})}$$
$$\lesssim \ r^*\int_{r\leq 2}[w_2^{2q+2}+w^{2q+2}+w_1^{2q+2}]\frac{dY}{r^2\langle z\rangle(1+D^{2Kq})}$$
$$+ \ c(r^*)\int_{r^*\leq r\leq 2}\left[w_1^{2q+2}+w^{2q+2}+w_2^{2q+2}\right]\frac{dY}{\langle z\rangle(1+D^{2Kq})}$$

and therefore obtain:

$$\frac{1}{2q+2}\frac{d}{dt}\int w_1^{2q+2}\chi dY+\frac{c}{q}\int\left[|\nabla(w_1^{q+1})|^2+\int\frac{w_1^{2q+2}}{r^2}\right]\chi dY$$
$$\lesssim \ r^*\int_{r\leq 2}[w_2^{2q+2}+w^{2q+2}+w_1^{2q+2}]\frac{dY}{r^2\langle z\rangle(1+D^{2Kq})}+\int|\partial_r\widetilde{F}||w_1|^{2q+1}\chi dY$$
$$+ \ c(r^*)\int_{r^*\leq r\leq 2}\left[w_1^{2q+2}+w^{2q+2}+w_2^{2q+2}\right]\frac{dY}{\langle z\rangle(1+D^{2Kq})}.$$

We now estimate all terms in the above identity.

Exterior term. From (4.35), for K large enough

$$\int_{r^*\leq r\leq 2}\left[w_1^{2q+2}+w^{2q+2}+w_2^{2q+2}\right]\frac{dY}{\langle z\rangle(1+D^{2Kq})}.\lesssim e^{-c_2\nu q\tau}\int_{r\leq 2}\frac{1+D^{Cq}}{\langle z\rangle(1+D^{2Kq})}dY\lesssim e^{-\frac{c_2}{2}\nu q\tau}.$$

$\Psi_3, L(\zeta)$ *term.* We estimate from Hölder, (C.17), (C.20):

$$\int_{r\leq 2}|w_1|^{2q+1}\left|\partial_r\left(\frac{\Psi_3}{T}+\frac{L(\zeta)}{T}\right)\right|\chi dY \ \leq \ \kappa\int\frac{w_1^{2q+2}}{r^2}\chi dY+c_\kappa\int\left[r^{\frac{2(2q+1)}{2q+2}}\partial_r\left(\frac{\Psi_3}{T}\right)\right]^{2q+2}\chi dY$$
$$\leq \ \kappa\int\frac{w_1^{2q+2}}{r^2}\chi dY+\frac{1}{b^{cq}}$$

Nonlinear term. The control of the nonlinear term demands a loss due to the singularity at the origin which will however be manageable. We integrate by parts to estimate using (4.35)

$$\left|\int w_1^{2q+1}\partial_r\left(\frac{\mathrm{NL}}{T}\right)\chi dY\right|\lesssim 1+\int_{r\leq 2}\frac{|\mathrm{NL}|}{|T|}\left[|\partial_r(w_1^{2q+1})|+\frac{|w_1|^{2q+1}}{r}\right]\chi dY.$$

We now claim from (4.9) the rough bound:

$$\|v\|_{L^\infty(r\leq 2)}\lesssim\frac{\mathcal{N}}{(\sqrt{b})^{\frac{2}{p-1}}}+(\sqrt{b})^{\frac12}. \tag{4.40}$$

Indeed, let $r\leq 2$, then for $D\geq A$, since $g\geq 3/2$ and $p\geq 3$:

$$\|v\|_{L^\infty(r\leq 2,D\geq A)}\lesssim\|V\|_{L^\infty(r\leq 2,D\geq A)}+\|\zeta\|_{L^\infty(r\leq 2)}\lesssim\mathcal{N}+\sqrt{b}^{g-\frac{2}{p-1}}\lesssim\mathcal{N}+\sqrt{b}^{\frac12}, \tag{4.41}$$

and for $D\leq A$ we use $|T|\lesssim\frac{1}{D^{\frac{2}{p-1}}}\lesssim\frac{1}{(\sqrt{b})^{\frac{2}{p-1}}}$ to estimate from (4.9):

$$\|v\|_{L^\infty(r\leq 2,D\leq A)}\lesssim\|v\|_{L^\infty(r^*\leq r\leq 2,D\leq A)}+\frac{1}{(\sqrt{b})^{\frac{2}{p-1}}}\|w\|_{L^\infty(r\leq 2,D\leq A)}\lesssim\frac{\mathcal{N}}{(\sqrt{b})^{\frac{2}{p-1}}}+(\sqrt{b})^{\frac12},$$

and (4.41) is proved. This implies the rough bound

$$\|v\|_{L^\infty(r\leq 2)}\lesssim\frac{1}{b^{\frac{1}{p-1}}}.$$

We now use

$$T \gtrsim \left| \begin{array}{ll} \frac{D^\alpha}{r^\gamma} & \text{for} \quad r \geq D, \\ \frac{1}{D^{\frac{2}{p-1}}} & \text{for} \quad r \leq D \end{array} \right.$$

to derive the rough bound for some universal $C > 0$:

$$\frac{|\mathrm{NL}|}{T} \lesssim \frac{|v|}{T}\left[|v|^{p-1} + \frac{|v|}{D^{\frac{2(p-2)}{p-1}}}\right] \lesssim \frac{1 + D^C}{b^C}$$

which yields the lossy bound:

$$\int_{r\leq 2} \frac{|\mathrm{NL}|}{|T|}\left[|\partial_r(w_1^{2q+1})| + |\frac{|w_1|^{2q+1}}{r}\right]\chi dY \lesssim \frac{1}{b^C}\int \left[|\partial_r(w_1^{2q+1})| + |\frac{|w_1|^{2q+1}}{r}\right](1 + D^C)\chi dY.$$

Now from Hölder:

$$\begin{array}{lll} \frac{1}{b^C}\int_{r\leq 2} \frac{|w_1|^{2q+1}}{r}(1+D^C)\chi dY & \lesssim & \frac{1}{b^C}\left(\int_{r\leq 2}\frac{w_1^{2q+2}}{r^2}\chi dY\right)^{\frac{2q+1}{2q+2}}\left(\int_{r\leq 2}\frac{r^{2q}(1+D^{Cq})}{\langle z\rangle(1+D^{2Kq})}dY\right)^{2q+2} \\ & \leq & \kappa\int \frac{w_1^{2q+2}}{r^2}\chi dY + \frac{1}{b^{Cq}} \end{array}$$

for some universal constant $C = c(d,p)$ independent of q. similarly:

$$\frac{1}{b^C}\int_{r\leq 2}|\partial_r(w_1^{2q+1})|(1+D^C)\chi dY \lesssim \frac{1}{b^C}\left(\int_{r\leq 2}|\partial_r(w_1^{q+1})|^2\chi dY\right)^{\frac{1}{2}}\left(\int_{r\leq 2}w_1^{2q}(1+D^C)^2\chi dY\right)^{\frac{1}{2}}$$

$$\leq \kappa\int(|\partial_r(w_1^{q+1})|^2 + w_1^{2q+2})\chi dY + \frac{1}{b^{Cq}}\int_{r\leq 2}\frac{1+D^{Cq}}{1+D^{Kq}}dY \leq \kappa\int(|\partial_r(w_1^{q+1})|^2 + w_1^{2q+2})\chi dY + \frac{1}{b^{Cq}}$$

The collection of above bounds yields the pointwise differential inequation:

$$\begin{array}{l} \frac{1}{2q+2}\frac{d}{dt}\int w_1^{2q+2}\chi dY + \frac{c}{q}\int\left[|\nabla(w_1^{q+1})|^2 + \int\frac{w_1^{2q+2}}{r^2}\right]\chi dY \qquad (4.42) \\ \leq \quad \frac{1}{b^{Cq}} + r^*\int_{r\leq 2}\frac{w_2^{2q+2} + w^{2q+2}}{r^2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}. \end{array}$$

step 2 ∂_z bound. Taking ∂_z of (4.31):

$$\begin{array}{lll} \partial_t w_2 & = & \left[\Delta w_2 + 2\frac{\nabla T\cdot\nabla w_2}{T} + V_1 w_2 - \frac{1}{2}Y\cdot\nabla w_2\right] \\ & + & [2\partial_{zz}\mathrm{log}T]\, w_2 + 2\partial_{rz}(\mathrm{log}T)w_1 + (\partial_z V_1)w - \frac{1}{2}w_2 + \partial_z\widetilde{F}. \end{array}$$

We therefore let

$$\tilde{\chi} = \langle z\rangle^{2q+2}\chi,$$

and obtain arguing as above:

$$\begin{array}{ll} & \frac{1}{2q+2}\frac{d}{dt}\int w_2^{2q+2}\tilde{\chi}dY \\ \leq & -\frac{c}{q}\int\left[|\nabla(w_2^{q+1})|^2 + \int\frac{w_2^{2q+2}}{r^2}\right]\tilde{\chi}dY + C(r^*)\int_{r^*\leq r\leq 2}(\langle z\rangle w_2)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})} \\ + & \int\left[2\partial_{zz}(\mathrm{log}T)w_2 + 2\partial_{rz}\mathrm{log}Tw_1 + (\partial_z V_1)w + \partial_z\widetilde{F}\right]w_2^{2q+1}\tilde{\chi}dY. \end{array}$$

We estimate all terms in this identity.

Exterior term. From (4.35):

$$\int_{r^*\leq r\leq 2}(\langle z\rangle w_2)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim\|\langle z\rangle\partial_z w\|^{2q}_{L^\infty(r^*\leq r\leq 2)}\int_{r^*\leq r\leq 2}\frac{(\langle z\rangle\partial_z w)^2}{\langle z\rangle(1+D^{2Kq})}dY\lesssim e^{-c_2q\nu\tau}.$$

Crossed term. From (A.10), (4.35):

$$\left|\int\partial_{zz}(\log T)w_2\tilde{\chi}w_2^{2q+1}dY\right|\lesssim\int_{r\leq 2}w_2^{2q+2}\tilde{\chi}dY\leq r^*\int_{r\leq 2}\frac{w_2^{2q+2}}{r^2}\tilde{\chi}dY+e^{-c_2\nu q\tau}.$$

The crossed term in integrated by parts in r using (A.11) and the L^∞ bound (4.35):

$$\begin{aligned}&\left|\int_{r\leq 2}\partial_{rz}\log T w_1w_2^{2q+1}\tilde{\chi}dY\right|\lesssim e^{-c_2\nu q\tau}\\ +&\int|w|\left[|\partial_{rz}\log T|\partial_r(w_2^{2q+1})|+\frac{|\partial_{rz}\log T||w_2^{2q+1}|}{r}+|\partial_{rrz}\log T||w_2^{2q+1}|\right]\tilde{\chi}dY\\ \lesssim&\ e^{-c_2\nu q\tau}+\int|w|\left[\frac{|w_2^q||\partial_r(w_2^{q+1})|}{r\langle z\rangle}+\frac{|w_2|^{2q+1}}{r^2\langle z\rangle}\right]\tilde{\chi}dY\end{aligned}$$

and then from Hölder:

$$\begin{aligned}\int|w|\frac{|w_2^q||\partial_r(w_2^{q+1})|}{r\langle z\rangle}\tilde{\chi}dY&\leq\kappa\int|\partial_r(w_2^{q+1})|^2\tilde{\chi}dY+\frac{1}{\kappa}\int\frac{w^2w_2^{2q}}{r^2\langle z\rangle^2}\tilde{\chi}dY\\ &\leq\kappa\int\left[|\partial_r(w_2^{q+1})|^2+\frac{w_2^{2q+2}}{r^2}\right]\tilde{\chi}dY+c_\kappa\int\frac{w^{2q+2}}{r^2\langle z\rangle^{2q+2}}\tilde{\chi}dY\\ &\leq\kappa\int\left[|\partial_r(w_2^{q+1})|^2+\frac{w_2^{2q+2}}{r^2}\right]\tilde{\chi}dY+c_\kappa\int\frac{w^{2q+2}}{r^2}\chi dY.\end{aligned}$$

similarly for the second term:

$$\begin{aligned}\int|w|\frac{|w_2|^{2q+1}}{r^2\langle z\rangle}\tilde{\chi}dY&\leq\kappa\int\frac{w_2^{2q+2}}{r^2}\tilde{\chi}dY+c_\kappa\int\frac{w^{2q+2}}{r^2\langle z\rangle^{2q+2}}\tilde{\chi}dY\\ &\leq\kappa\int\left[|\partial_r(w_2^{q+1})|^2+\frac{w_2^{2q+2}}{r^2}\right]\tilde{\chi}dY+c_\kappa\int\frac{w^{2q+2}}{r^2}\chi dY.\end{aligned}$$

Next from (A.9), (4.35):

$$\begin{aligned}&\left|\int(\partial_zV_1)ww_2^{2q+1}\tilde{\chi}dY\right|\lesssim\int\frac{|w|}{\langle z\rangle}|w_2^{2q+1}|\tilde{\chi}dY\lesssim\int_{r\leq 2}|w||\langle z\rangle w_2|^{2q+1}\chi dY\\ \leq&\ r^*\int_{r\leq 2}[w^{2q+2}+(\langle z\rangle w_2)^{2q+2}]\frac{\chi dY}{r^2}+c(r^*)\int_{r^*\leq r\leq 2}[w^{2q+2}+(\langle z\rangle w_2)^{2q+2}]\chi dY\\ \leq&\ r^*\int_{r\leq 2}\frac{w^{2q+2}+(\langle z\rangle w_2)^{2q+2}}{r^2}\chi dY+e^{-c_2\nu q\tau}\end{aligned}$$

$\Psi_3, L(\zeta)$ term. From Hölder and (C.18), (C.21):

$$\begin{aligned}\left|\int_{r\leq 2}\partial_z\left(\frac{\psi_3+L(\zeta)}{T}\right)w_1^{2q+1}\tilde{\chi}dY\right|&\leq\kappa\int\frac{w_2^{2q+2}}{r^2}\tilde{\chi}dY+c_\kappa\int\left[r^{2\frac{2q+1}{2q+2}}\partial_z\left(\frac{\Psi_3+L(\zeta)}{T}\right)\right]^{2q+2}\tilde{\chi}dY\\ &\leq\kappa\int\frac{w_2^{2q+2}}{r^2}\chi dY+b^{c\eta q}\end{aligned}$$

Nonlinear term. We estimate by homogeneity using $\partial_z\zeta=0$

$$|\partial_z\mathrm{NL}|\lesssim|\partial_z\Phi_{a,b}|(|V|^{p-1}+|V|^2|\Phi_{a,b}|^{p-3})+|\partial_zv|(|V|^{p-1}+|\Phi_{a,b}|^{p-2}|V|).$$

Moreoever using (A.11):

$$|\partial_z v| \lesssim |\partial_z T||w| + |T\partial_z w| \lesssim T\left(\frac{|w|}{\langle z\rangle} + |w_2|\right)$$

and

$$|\partial_z \Phi_{a,b}| \lesssim \frac{T}{\langle z\rangle}, \quad |T| \lesssim \Phi_{a,b} \lesssim \frac{1}{r^{\frac{2}{p-1}}}.$$

$\underline{D \le A}$. This implies for $r \le 2$, $D \le A$ using (4.3):

$$\begin{aligned}\frac{|\partial_z \mathrm{NL}|}{T} &\lesssim \frac{1}{\langle z\rangle}\left[|\zeta|^{p-1} + \zeta^2\Phi_{a,b}^{p-3}\right]\\ &+ \frac{1}{\langle z\rangle}\left[T^{p-1}|w|^{p-1} + w^2T^2\Phi_{a,b}^{p-3}\right] + \left[T^{p-1}|w|^{p-1} + wT\Phi_{a,b}^{p-2}\right]\left(\frac{|w|}{\langle z\rangle} + |w_2|\right)\\ &\lesssim \|w\|_{L^\infty(r\le 2, D\le A)}\left(\frac{|w|}{\langle z\rangle} + |w_2|\right)\frac{1}{r^2} + \frac{b^g}{\langle z\rangle r^2}\end{aligned}$$

similarly

$$\left|\mathrm{NL}\frac{\partial_z T}{T^2}\right| \lesssim \frac{|\mathrm{NL}|}{\langle z\rangle T} \lesssim \|w\|_{L^\infty(r\le 2,,D\le A)}\frac{|w|}{r^2\langle z\rangle} + \frac{b^g}{\langle z\rangle r^2}$$

and hence the bound using Hölder:

$$\begin{aligned}\left|\int_{D\le A}\partial_z\left(\frac{\mathrm{NL}}{T}\right)w_2^{2q+1}\tilde\chi dY\right| &\lesssim \int\left[\|w\|_{L^\infty(r\le 2, D\le A)}\left(\frac{|w|}{\langle z\rangle} + |w_2|\right) + b^g\right]\frac{w_2^{2q+1}}{r^2}\tilde\chi dY\\ \le\ \kappa&\left[\int\frac{w_2^{2q+2}}{r^2}\tilde\chi dY + \int\frac{w^{2q+2}}{r^2}\chi dY\right] + b^{gq}\end{aligned}$$

where we used (4.9) in the last step.

$\underline{D \ge A}$. Since $r \le 2$ and $D \ge A \gg r$, we have

$$T \sim \Phi_{a,b} \sim \frac{1}{D^{\frac{2}{p-1}}} \le 1$$

and hence for $r \le 2$:

$$\begin{aligned}\frac{|\partial_z \mathrm{NL}|}{T} &\lesssim \frac{1}{\langle z\rangle}\left[|V|^{p-1} + \Phi_{a,b}^{p-3}V^2\right] + (|V|^{p-1} + |\Phi_{a,b}|^{p-2}|V|)\left(\frac{|w|}{\langle z\rangle} + |w_2|\right)\\ &\lesssim \frac{1}{\langle z\rangle}\frac{\|\phi V\|_{L^\infty}}{r^2}\left(\frac{|w|}{\langle z\rangle} + |w_2| + (\sqrt{b})^{\frac12}\right)\end{aligned}$$

and the same chain of estimates as above yields for c universal:

$$\left|\int_{D\ge A}\partial_z\left(\frac{\mathrm{NL}}{T}\right)w_2^{2q+1}\tilde\chi dY\right| \le \kappa\left[\int\frac{w_2^{2q+2}}{r^2}\tilde\chi dY + \int\frac{w^{2q+2}}{r^2}\chi dY\right] + b^{cq}.$$

The collection of above bounds yields the pointwise differential inequation since $0 < \nu \ll \eta \ll 1$:

$$\frac{1}{2q+2}\frac{d}{dt}\int w_2^{2q+2}\tilde\chi dY + cq\int\frac{w_2^{2q+2}}{r^2}\tilde\chi dY \lesssim \int\frac{w^{2q+2}}{r^2}\chi dY + e^{-c_2\nu q\tau} \tag{4.43}$$

step 3 Conclusion. The time integration of (4.43) using the space time bound (4.29) yields that there exists $0 < c_4 \lesssim c_3$ such that

$$e^{c_4\nu\tau q}\int_{r\le 1}\frac{(\langle z\rangle\partial_z w)^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY + \int_{\tau_0}^{\tau}e^{c_4\nu\tau' q}\int_{r\le 1}\frac{(\langle z\rangle\partial_z w)^{2q+2}}{\langle z\rangle(1+D^{2Kq})}dY d\tau' \le 1$$

which implies (4.39). Injecting this bound into (4.42) and integrating in time yields the lossy bound (4.38). $\square$

4.6. **Far away scaling bound.** Recall (4.4). We now claim the far away $W^{1,2q+2}$ bound which is a simple consequence of the fact that this norm is always above scaling for $q > q(d,p)$ large enough. We let $q \gg 1$ and

$$(z^*)^{2\ell} = \frac{1}{a(\sqrt{b})^\alpha} \gg 1. \tag{4.44}$$

Given a large enough constant A, we let a cut off function

$$\chi_{A,b}(r,z) = \chi\left(\frac{r}{A}, \frac{z}{Az^*}\right), \quad \chi(Y) = \left| \begin{array}{ll} 0 & \text{for} \quad |Y| \leq 1, \\ 1 & \text{for} \quad |Y| \geq 2. \end{array}\right.$$

Note that for $az^{2\ell} \gg 1$:

$$D = \mu\sqrt{b} = \left[az^{2\ell}\left(1 + O\left(\frac{1}{z^2}\right)\right)\right]^{\frac{1}{\alpha}}\sqrt{b} = \left(\frac{z}{z^*}\right)^{\frac{2\ell}{\alpha}}\left[1 + O\left(\frac{1}{z^2}\right)\right]$$

and hence

$$Y \in \mathrm{Supp}\chi_{\mathrm{A,b}} \quad \text{implies} \quad \mathrm{r} \gtrsim \mathrm{A} \quad \text{or} \quad \mathrm{D} \gtrsim \mathrm{A}. \tag{4.45}$$

Lemma 4.7 (Far away scaling bound)**.** *There holds for a constant c_5 depending on ν, c_2 and c_3:*

$$\int \chi_{A,b}\left[(\langle z\rangle\partial_z)^j V\right]^{2q+2} dY \lesssim b^{c_5 q}, \quad j = 0,1 \tag{4.46}$$

and for a universal constant C independent of q:

$$\int \chi_{A,b}\left[\partial_r V\right]^{2q+2} dY \lesssim \frac{1}{b^{Cq}}. \tag{4.47}$$

Proof. We compute the V equation:

$$\partial_\tau V - \Delta V + \frac{1}{2}\Lambda V - R(V) = 0, \quad R(V) = (\Phi_{a,b} + V)|\Phi_{a,b} + V|^{p-1} - \Phi_{a,b}^p.$$

step 1 L^{2q+2} estimate. We compute:

$$\begin{aligned}
& \frac{1}{2q+2}\frac{d}{dt}\int \chi_{A,b}V^{2q+2}dY \\
= & \int\left(\Delta V - \frac{1}{2}\Lambda V + R(V)\right)\chi_{A,b}V^{2q+1}dY + \frac{1}{2q+2}\int \partial_t\chi_{A,b}|V|^{2q+2}dY \\
= & -(2q+1)\int V^{2q}|\nabla V|^2\chi_{A,b}dY - \frac{1}{2}\left(\frac{2}{p-1} - \frac{d}{2q+2}\right)\int |V|^{2q+2}\chi_{A,b}dY \\
+ & \int R(V)V^{2q+1}\chi_{A,b}dY + \frac{1}{2q+2}\int V^{2q+2}\left[\partial_t\chi_{A,b} - \Delta\chi_{A,b} + \frac{1}{2}Y\cdot\nabla\chi_{A,b}\right]
\end{aligned}$$

From (4.45):

$$|\Phi_{a,b}| \lesssim \frac{1}{A^{\frac{2}{p-1}}} \quad \text{on} \quad \mathrm{Supp}\chi_{\mathrm{A,b}}$$

and hence using (4.9), (4.45):

$$|V| \lesssim \mathcal{N}, \quad |R(V)| \lesssim \frac{1}{A^c}|V| \quad \text{on} \quad \mathrm{Supp}(\chi_{A,b}).$$

Hence the bound:

$$\int |R(V)V^{2q+1}|\chi_{A,b}dY \leq (\sqrt{b})^{\tilde{\eta}}\int |V|^{2q+2}\chi_{A,b}dY.$$

We then estimate by definition of $\chi_{A,b}$ and (3.48):

$$|\partial_t\chi_{A,b}|+|\Delta\chi_{A,b}|+|Y\cdot\nabla\chi_{A,b}|\lesssim \mathbf{1}_{A\leq r\leq 2A,D\leq 2A}+\mathbf{1}_{r\leq 2A,cA\leq D\leq CA}.$$

From (4.2), (4.35):

$$\int_{A\leq r\leq 2A,D\leq 2A}|V|^{2q+2}dY\lesssim \int_{A\leq r\leq 2A,D\leq 2A}|\zeta|^{2q+2}dY+\int_{A\leq r\leq 2A,D\leq 2A}|v|^{2q+2}dY$$
$$\lesssim\ (\sqrt{b})^{\alpha(2q+2)}\,|\{z,\ D\leq 2A\}|+e^{-c_2\nu\tau(2q+2)}\,|\{z,\ D\leq 2A\}|$$
$$\lesssim\ (\sqrt{b})^{\alpha(2q+2)}(\sqrt{b})^{-\frac{\alpha}{2\ell}}+(\sqrt{b})^{\frac{c_2\alpha}{2\ell-\alpha}\nu(2q+2)}(\sqrt{b})^{-\frac{\alpha}{2\ell}}\lesssim b^{cq}$$

for q large enough and c depending on ν and c_2. Next from (4.2), (4.35), as $g>3/2$ and $p\geq 3$:

$$\int_{r\leq A}|\zeta|^{2q+2}r^{d-1}dr\lesssim(\sqrt{b})^{\alpha(2q+2)}+\int_{r\leq\sqrt{b}}(\sqrt{b})^{(2q+2)(g-\frac{2}{p-1})}r^{d-1}dr$$
$$+\ \int_{\sqrt{b}\leq r\leq 1}\left(\frac{(\sqrt{b})^{\alpha}(r^2+(\sqrt{b}^g))}{r^\gamma}\right)^{2q+2}r^{d-1}dr\lesssim(\sqrt{b})^q.$$

We then use

$$|v|=|Tw|\lesssim\frac{|w|}{D^{\frac{2}{p-1}}}$$

to estimate using (4.29):

$$\int_{D\sim A,r\leq 2A}|V|^{2q+2}dY\lesssim\int_{D\sim A,r\leq 2A}\frac{|w|^{2q+2}}{\langle z\rangle(1+D^{2qK})}dY+\frac{1}{b^c}\int_{r\leq A}|\zeta|^{2q+2}r^{d-1}dr\lesssim b^{cq}$$

with c depending on ν and c_3. The collection of above bounds yields the pointwise differential inequation:

$$\frac{d}{dt}\int\chi_{A,b}V^{2q+2}dY+cq\int\chi_{A,b}V^{2q+2}dY\lesssim b^{Cq}$$

with C depending on ν, c_2 and c_3, whose time integration yields

$$\int\chi_{A,b}V^{2q+2}dY\lesssim b^{Cq}.$$

step 2 Derivative estimate. The derivative bounds follow the exact same path using in particular $\partial_z\zeta=0$ and the inner bounds (4.38), (4.39), the details are left to the reader.

□

5. **Closing the bootstrap**

We are now in position to close the bootstrap and conclude the proof of Theorem 1.1.

5.1. **Proof of Proposition 3.4.** We argue by contradiction and conclude using a classical topological argument.

step 1 The (Exit) condition is saturated. From (3.74),

$$\|\varepsilon\|_{L^2_{\rho_Y}}\leq\eta(a)(\sqrt{b})^{\alpha+\eta}$$

and hence (3.37) is improved. Moreoever, we inject the bounds (4.19), (4.29), (4.38), (4.39), (4.46), (4.47) and conclude that there exists $\delta > 0$ depending on $\nu, c_1, c_2, c_3, c_4, c_5$ but independent of q and $\tilde{\eta}$, and a universal $C > 0$ such that:

$$\mathcal{N}^q \lesssim e^{-\delta q\tau} + \left(e^{-\delta q\tau}\right)^{1-\frac{d}{2q+2}} \left(\frac{1}{e^{Cq\tau}}\right)^{\frac{2}{2q+2}} \lesssim e^{-\delta q\tau}$$

provided q has been chosen universal large enough since δ, C are independent of q, and hence choosing $\tilde{\eta} \ll \delta$, (3.38) is improved: $\mathcal{N} \ll (\sqrt{b})^{\tilde{\eta}}$. We now oberve that (3.48) implies:

$$|a_\tau| + \left|\frac{d}{d\tau}\left(b_{\ell,0}e^{\left(\ell-\frac{\alpha}{2}\right)\tau}\right)\right| \lesssim (\sqrt{b})^{\eta}. \tag{5.1}$$

whose time integration ensures:

$$|a - a(\tau_0)| + \left|b_{\ell,0}(\tau)e^{\left(\ell-\frac{\alpha}{2}\right)\tau} - b_{\ell,0}(\tau_0)e^{\left(\ell-\frac{\alpha}{2}\right)\tau_0}\right| \leq b_0^{c\eta}$$

which using (3.27), (3.28) improves (3.34), (3.35). We conclude from a standard continuity argument that the (Exit) condition is saturated, meaning that a solution exits the trapped regime at time τ^* if and only if the instable modes have grown too big:

$$(\tilde{b}, \tilde{b}_{j,k}) \in \frac{(\sqrt{b}(\tau^*))^{\eta}}{(\sqrt{b(\tau_0)})^{\eta}}\mathcal{S}. \tag{5.2}$$

step 2 The topological argument. We now reformulate the (Exit) condition in diagonal form as required from (3.48). Indeed, we may diagonalize the associated matrix which has positive eigenvalues $\mu_{j,k} \geq 0$ such that letting

$$(\tilde{b}, (\tilde{b}_{jk})_{(j,k)\neq(\ell,0), j+k\leq\ell}) = P\tilde{B}_{jk}$$

for some suitable universal invertible matrix, (3.48) implies:

$$\sum_{j,k}\left|(\tilde{B}_{jk})_\tau + \mu_{j,k}\tilde{B}_{jk}\right| \lesssim \eta(a)(\sqrt{b})^{\eta}. \tag{5.3}$$

We now choose a small enough universal constant κ and consider initial values satisfying

$$\sum_{j,k}\left|\frac{\tilde{B}_{j,k}(\tau_0)}{(\sqrt{b})^{\eta}(\tau_0)}\right|^2 \leq \kappa^2$$

then from (5.2), (3.74), we may let $\tau^{**} < \tau^*$ the first time such that

$$\sum_{j,k}\left|\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}(\tau^{**})\right|^2 = \kappa^2,$$

we claim that the corresponding vector field is outgoing:

$$\frac{d}{d\tau}\left\{\sum_{j,k}\left|\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}\right|^2\right\}(\tau^{**}) > 0. \tag{5.4}$$

Assume (5.4) and that all solutions leave the trapped regime, then from standard argument, the map

$$\frac{1}{\kappa}\left(\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}(\tau_0)\right) \mapsto \frac{1}{\kappa}\left(\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}(\tau^{**})\right)$$

is continuous. It moreover sends by definition the unit sphere onto its boundary and is the identity when restricted to the boundary, a contradiction to Brouwer's

theorem.
Proof of (5.4): We compute:

$$I=\frac{d}{d\tau}\left\{\sum_{j,k}\left|\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}\right|^2\right\}=\frac{1}{(\sqrt{b})^{2\eta}}\left[\sum_{j,k}2\tilde{B}_{j,k}\left((\tilde{B}_{j,k})_\tau-\frac{\eta}{2}\frac{b_\tau}{b}\tilde{B}_{j,k}\right)\right].$$

We now observe from (3.47), (3.71), (3.48):

$$\frac{b_\tau}{b}=1-\frac{2\ell}{\alpha}+\eta(a)O\left((\sqrt{b})^{\eta}\right)$$

and hence from (5.3):

$$(\tilde{B}_{j,k})_\tau-\frac{\eta}{2}\frac{b_\tau}{b}\tilde{B}_{j,k}=\left[\mu_{j,k}+\frac{\eta}{2}\left(\frac{2\ell}{\alpha}-1\right)\right]\tilde{B}_{j,k}+\eta(a)O\left((\sqrt{b})^{\eta}\right).$$

Recalling $\mu_{j,k}\geq 0$
and the pointwise differential inequation (3.75) with $0<\eta\ll c_*$, we obtain:

$$\begin{aligned} I &\geq \frac{c\eta}{(\sqrt{b})^{2\eta}}\left[\sum_{j,k}\left|\frac{\tilde{B}_{j,k}}{(\sqrt{b})^{\eta}}(\tau^{**})\right|^2\right]+\frac{(\sqrt{b})^{\eta}}{(\sqrt{b})^{2\eta}}\eta(a)O\left((\sqrt{b})^{\eta}\right)\\ &\geq c\kappa-\eta(a)>0, \end{aligned}$$

and (5.4) is proved.

5.2. **Proof of Theorem 1.1.** Let $0<T\ll 1$ small enough and an initial data as in the assumptions of Proposition 3.4, then the corresponding solution to (1.1) generates a global in time to the self similar solution (3.1) which admits for all $t\in[0,T)$ a decomposition

$$u(t,x)=\frac{1}{(T-t)^{\frac{1}{p-1}}}U\left(t,\frac{x}{\sqrt{T-t}}\right),\quad U(t,r,z)=\Phi_{a,b}+V.$$

The law (1.11) follows from (5.1), (3.38) which yield the time integrability

$$\int_{\tau_0}^{+\infty}|a_\tau|d\tau<b_0^{c\eta}<\frac{a_0}{10},$$

and similarly for (1.10). It remains to prove the asymptotic stability (1.12). Recall (3.38) and (4.9) which first implies

$$\|V\|_{L^\infty(D\geq A)}+\|V\|_{L^\infty(r\geq 1)}\lesssim (T-t)^c\underset{t\uparrow T}{\longrightarrow}0,\quad c>0. \tag{5.5}$$

We then consider $D\leq A, r\leq 1$ and estimate in brute force:

$$\begin{aligned} (\sqrt{b})^{\frac{2}{p-1}}\|V\|_{L^\infty(D\leq A,r\leq 1)} &\lesssim (\sqrt{b})^{\frac{2}{p-1}}\|\zeta\|_{L^\infty(D\leq A,r\leq 1)}+\|(\sqrt{b})^{\frac{2}{p-1}}Tw\|_{L^\infty(D\leq A,r\leq 1)}\\ &\lesssim b^\delta+\|w\|_{L^\infty(D\leq A,r\leq 1)}\to 0\ \text{ as }\ t\to T \end{aligned}$$

where we used $D=\mu\sqrt{b}\geq\frac12\sqrt{b}$ and (4.2), (4.9). This concludes the proof of items 1. and 2. in Theorem 1.1. We now prove item 3. Recall the link between original and self-similar variables $x=(r\sqrt{T-t},z\sqrt{T-t})$. Let $\delta>0$. First, one has from (1.9), (1.11) and (1.10) that if $|z|\geq\delta(T-t)^{-1/2}$ then $D\gtrsim(T-t)^{-1/2}\gg 1$ for t close enough to T. Thus, (5.5) implies that:

$$\|V\|_{L^\infty(|(r,z)|\geq\delta(T-t)^{-1/2})}\lesssim (T-t)^c,\quad c>0.$$

It is direct to check from (1.20), (3.5), and the fact that $D \gtrsim (T-t)^{-1/2}$ for $|z| \geq \delta(T-t)^{-1/2}$ that $\|\Phi_{a,b}\|_{L^\infty(|(r,z)|\geq\delta(T-t)^{-1/2})} \lesssim (T-t)^{c'}$ for some $c'>0$ as well. Hence, up to changing the value of c, in original variables:

$$\|u\|_{L^\infty(|x|\geq\delta)} \leq C(\delta)(T-t)^{-\frac{1}{p-1}+c}.$$

The gain $c>0$ allows to apply Theorem 2.1 in [29]: the solution remains bounded for $|x|\geq\delta$. This prove item 3. an finishes the proof of Theorem 1.1.

Appendix A. Estimates on the soliton

This appendix is devoted to the derivation of sharp estimates on the soliton profile.

Lemma A.1 (Global control of the tails)**.** *Let* $V=\log\Lambda Q$, *then* $\forall r>0$,

$$-\frac{\gamma}{r} < \partial_r V \leq 0, \tag{A.1}$$

$$\partial_{rr} V \leq \frac{\gamma}{r^2}. \tag{A.2}$$

Proof. The lemma follows from the Q equation in the regime $p>p_{JL}(d)$ and a Sturm Liouville oscillation argument.

step 1 Proof of (A.1). Let from (1.20)

$$H=-\Delta_r - pQ^{p-1} > H_0 = -\Delta - \frac{\gamma(d-2-\gamma)}{r^2}. \tag{A.3}$$

Let $\psi_1=\Lambda Q$, $\psi_2=\frac{1}{r^\gamma}$, then

$$\psi_1>0, \quad H_0\psi_1 < H\psi_1 = 0, \quad H_0\psi_2=0. \tag{A.4}$$

This first implies

$$\frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r\psi_1) = -pQ^{p-1}\psi_1<0$$

from which with $\partial_r\psi_1(0)=0$:

$$r^{d-1}\partial_r\psi_1\leq 0, \quad \partial_r\psi_1\leq 0 \ \text{ and hence } \ \partial_r V=\frac{\partial_r\psi_1}{\psi_1}\leq 0.$$

Next from (A.4), the Wronskian $W=\psi_1'\psi_2-\psi_1\psi_2'$ satisfies

$$\frac{1}{r^{d-1}}\partial_r(r^{d-1}W) = (-H_0\psi_1)\psi_2>0.$$

At the origin,

$$r^{d-1}W(r) = O(r^{d-1-(\gamma+1)}) \to 0 \quad \text{as} \quad r\to 0$$

and hence $W(r)>0$, $\psi_1(r)>0$ implies

$$-\frac{\psi_1'}{\psi_1} < -\frac{\psi_2'}{\psi_2} = \frac{\gamma}{r}$$

and (A.1) is proved.

step 2 Proof of (A.2). We first compute from (A.4):

$$\Delta V = \nabla\cdot\left(\frac{\partial_r\Lambda Q}{\Lambda Q}\right) = \frac{\Delta\Lambda Q}{\Lambda Q} - \left(\frac{\partial_r\Lambda Q}{\Lambda Q}\right)^2 = -pQ^{p-1}-(\partial_r V)^2. \tag{A.5}$$

This implies:

$$\partial_{rr}V+\frac{\partial_r V}{r}=\Delta V-\frac{d-2}{r}\partial_r V=-pQ^{p-1}-\partial_r V\left[\frac{d-2}{r}+\partial_r V\right]. \tag{A.6}$$

Let $\Phi=-r\partial_r V$, then from (A.1): $0<\Phi<\gamma$ and Φ satisfies the order one nonlinear equation:

$$r\partial_r\Phi=pQ^{p-1}r^2-\Phi(d-2-\Phi). \tag{A.7}$$

We claim

$$\partial_r\Phi>0. \tag{A.8}$$

Indeed, at the origin, $\partial_r V(0)=0$ and hence from (A.5):

$$\Delta V(0)=d\partial_{rr}V(0)=-pQ^{p-1}(0)$$

from which near the origin:

$$\begin{aligned}\frac{\partial_r\Phi}{r} &= pQ^{p-1}+\frac{\partial_r V}{r}(d-2-\Phi)=pQ^{p-1}(0)+(d-2)\partial_r^2V(0)+o_{r\to 0}(1)\\ &= pQ^{p-1}(0)\left(1-\frac{d-2}{d}\right)+o_{r\to0}(1)>0.\end{aligned}$$

By contradiction, let $r_0>0$ be the first point where $\Phi'(r_0)=0$, then $\Phi''(r_0)\leq 0$ but deriving (A.7) at $r=r_0$ yields:

$$r_0\Phi''(r_0)=p(p-1)rQ^{p-2}\Lambda Q(r_0)>0$$

and a contradiction follows which concludes the proof of (A.8). We have therefore proved

$$0\leq\Phi\leq\gamma,\quad \partial_r\Phi>0.$$

Now

$$\partial_r\Phi=-r\partial_{rr}V-\partial_r V=-r\partial_{rrV}+\frac{\Phi}{r},\quad \partial_{rr}V=\frac{\Phi}{r^2}-\frac{\partial_r\Phi}{r}\leq\frac{\gamma}{r^2},$$

this is (A.2). □

Lemma A.2 (Estimates on log$T$ and V_1)**.** *Let T,V_1 be given by (4.7), (4.32), then for all $r\leq 2$:*

$$|V_1|+|r\partial_r V_1|+|\langle z\rangle\partial_z V_1|\lesssim 1, \tag{A.9}$$

$$|\partial_z^k\partial_r^j(\log T)|\lesssim\frac{1}{r^j\langle z\rangle^k},\quad k,j\geq 0,\quad k+j\geq 1 \tag{A.10}$$

Proof. Recall

$$T=\frac{1}{D^\gamma}\Lambda Q\left(\frac{r}{D}\right),\quad D=\mu(z)\sqrt{b}=\sqrt{b}\left(1+aP_{2\ell}(z)\right)^{\frac{1}{\alpha}}.$$

Hence

$$\frac{\partial_z T}{T}=-\frac{\partial_z D}{D}\zeta\left(\frac{r}{D}\right)\quad\text{where}\quad \zeta(y)=\gamma+\frac{y\partial_y\Lambda Q}{\Lambda Q}$$

satifies

$$\partial_y^k\zeta=O\left(\frac{1}{\langle y\rangle^k}\right),\quad k\in\mathbb{N}.$$

Moreover,

$$\frac{\partial_z D}{D}=\frac{\partial_z\mu}{\mu}=-\frac{1}{\alpha}\frac{aP'_{2\ell}}{1+aP_\ell}$$

so that

$$\partial_z^k\left(\frac{\partial_z D}{D}\right)=O\left(\frac{1}{\langle z\rangle^{k+1}}\right).$$

Similarly,

$$|\partial_r \log T| \lesssim \frac{1}{D}\frac{\partial_y \Lambda Q}{\Lambda Q}\left(\frac{r}{D}\right) \lesssim \frac{1}{r},$$

and (A.10) follows by induction. We now compute:

$$\partial_\tau \log T = -\frac{\partial_t D}{D}\zeta\left(\frac{r}{D}\right) = \left[-\frac{b_\tau}{2b} + \frac{1}{\alpha}\frac{a_\tau P_{2\ell}}{1+aP_{2\ell}}\right]\zeta\left(\frac{r}{D}\right)$$

and since

$$\left|\frac{b_\tau}{b}\right| + \left|\frac{a_\tau}{a}\right| \lesssim 1,$$

we obtain

$$\left|\partial_z^k \partial_r^j (\partial_\tau \log T)\right| = O\left(\frac{1}{r^j \langle z\rangle^k}\right), \quad k+j \geq 1. \tag{A.11}$$

□

Appendix B. Coercivity estimates

This Appendix is devoted to the proof of Hardy and Sobolev like inequalities that are used all along the proof. The proofs are standard and recalled for the convenience of the reader.

Lemma B.1 (Hardy inequality with Gaussian weight)**.** *There holds for ε with cylindrical symmetry:*

$$\int \left(|Y|^2 + \frac{1}{r^2}\right)\varepsilon^2 e^{-\frac{|Y|^2}{4}} dY \lesssim \int (|\nabla \varepsilon|^2 + \varepsilon^2) e^{-\frac{|Y|^2}{4}} dY \tag{B.1}$$

Proof. The proof follows a classical integration by parts and is left to the reader. □

Lemma B.2 (Weighted Hardy inequalities)**.** *There holds for all $\beta > -\frac{d-2}{2}$:*

$$\int r^{2\beta}|\partial_r u|^2 \rho_r dY \geq \left(\frac{d-2+2\beta}{2}\right)^2 \int \frac{r^{2\beta}}{r^2}u^2\rho_r dY - \frac{d-2+2\beta}{4}\int r^{2\beta}u^2\rho_r dY. \tag{B.2}$$

Proof. We integrate by parts and use Young inequality to compute:

$$\int \frac{u^2}{r^2}\chi\rho_r dY = \frac{1}{d-2}\int \chi u^2 \nabla\cdot\left(\frac{e_r}{r}\right)\rho_r dY = -\frac{1}{d-2}\int \frac{1}{r}\left[2\chi u\partial_r u + \chi' u^2 - \frac{r}{2}\chi u^2\right]\rho_r dY$$

$$\leq \frac{1}{d-2}\int \frac{u^2}{r^2}\left(-r\chi' + 2\beta\chi + \frac{r^2}{2}\chi\right)\rho_r dY - \frac{2\beta}{d-2}\int \frac{u^2}{r^2}\rho_r dY$$

$$+ \frac{1}{d-2}\left[\frac{1}{A}\int \chi|\partial_r u|^2\rho_r dY + A\int \frac{u^2}{r^2}\chi\rho_r dY\right]$$

and hence:

$$\int \chi|\partial_r u|^2 \rho_r dY \geq (d-2)A\int \frac{\chi}{r^2}u^2\left(1+\frac{2\beta - A}{d-2}\right)\rho_r dY + A\int \frac{u^2}{r^2}\left(r\chi' - 2\beta\chi - \frac{r^2}{2}\chi\right)\rho_r dY.$$

For $\beta > -(d-2)$, the optimal choice $A = \frac{d-2+2\beta}{2} > 0$ ensures:

$$\int \chi|\partial_r u|^2\rho_r dY \geq \left(\frac{d-2+2\beta}{2}\right)^2\int \frac{\chi}{r^2}u^2\rho_r dY + \frac{d-2+2\beta}{2}\int \frac{u^2}{r^2}\left(r\chi' - 2\beta\chi - \frac{r^2}{2}\chi\right)\rho_r dY.$$

Letting $\chi = r^{2\beta}$ yields the claim. □

Lemma B.3 (Weighted Sobolev bound). *Let $2q+2>d+1$ and $v(r,z)$ with cylindrical symmetry, then:*

$$\|v\|_{L^\infty}^{2q+2}\lesssim_q\left(\int\frac{|v|^{2q+2}+|\langle z\rangle\partial_z v|^{2q+2}}{\langle z\rangle}dY\right)^{1-\frac{d}{2q+2}}\left(\int\frac{|\partial_r v|^{2q+2}}{\langle z\rangle}dY\right)^{\frac{d}{2q+2}}. \quad (B.3)$$

Proof. This follows from Sobolev and a scaling argument. We first claim:

$$\|v\|_{L^\infty}^{2q+2}\lesssim\int\frac{|v|^{2q+2}+|\partial_r v|^{2q+2}+|\langle z\rangle\partial_z v|^{2q+2}}{\langle z\rangle}dY. \quad (B.4)$$

Indeed, we have from Sobolev for $2q+2>d+1$:

$$\begin{aligned}\|v\|_{L^\infty(|z|\le 1)}^{2q+2} &\lesssim \int_{|z|\le 1}\left(|v|^{2q+2}+|\partial_r v|^{2q+2}+|\partial_z v|^{2q+2}\right)dY\\ &\lesssim \int\frac{|v|^{2q+2}+|\partial_r v|^{2q+2}+|\langle z\rangle\partial_z v|^{2q+2}}{\langle z\rangle}dY.\end{aligned}$$

Let now $A\ge 1$ and the cylinder $\mathcal{C}_A=\{r\ge 0,\ \ \frac{A}{2}\le|z|\le A\}$ Let $V(r,z)=v(r,Az)$, then $2q+2>d+1$ and Sobolev in the cylinder $\mathcal{C}_1$ ensure:

$$\begin{aligned}\|v\|_{L^\infty(\mathcal{C}_A)}^{2q+2} &= \|V\|_{L^\infty(\mathcal{C}_1)}^{2q+2}\lesssim\|V\|_{L^{2q+2}(\mathcal{C}_1)}^{2q+2}+\|\nabla V\|_{L^{2q+2}(\mathcal{C}_1)}^{2q+2}\\ &\lesssim \frac{1}{A}\left[\|v\|_{L^{2q+2}(\mathcal{C}_A)}^{2q+2}+\|\partial_r v\|_{L^{2q+2}(\mathcal{C}_A)}^{2q+2}+\|A\partial_z v\|_{L^{2q+2}(\mathcal{C}_A)}^{2q+2}\right]\\ &\lesssim \int\frac{|v|^{2q+2}+|\partial_r v|^{2q+2}+|\langle z\rangle\partial_z v|^{2q+2}}{\langle z\rangle}dY\end{aligned}$$

and since the bound is independent of $A\ge 1$, (B.4) is proved. We now apply this estimate to

$$v_\lambda(r,z)=v\left(\frac{r}{\lambda},z\right)$$

which yields:

$$\|v\|_{L^\infty}^{2q+2}=\|v_\lambda\|_{L^\infty}^{2q+2}\lesssim\lambda^d\int\left[\frac{|v|^{2q+2}+|\langle z\rangle\partial_z v|^{2q+2}}{\langle z\rangle}+\frac{1}{\lambda^{2q+2}}\frac{|\partial_r v|^{2q+2}}{\langle z\rangle}\right]dY$$

and optimizing in λ yields (B.3). □

Lemma B.4 (Weighted outer Sobolev bound). *Let $v(r,z)$ with cylindrical symmetry, then for all $0<\delta<R$:*

$$\|v\|_{L^\infty(\delta\le r\le R)}^2\lesssim_{\delta,R}\sum_{0\le i+j\le 2}\int_{\delta\le r\le R}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j v|^2}{\langle z\rangle}dY. \quad (B.5)$$

Proof. Indeed, we have from the two dimensional Sobolev $H^2\subset L^\infty$:

$$\|v\|_{L^\infty(\delta\le r\le R,|z|\le 1)}^{2q+2}\lesssim_{q,\delta,R}\sum_{0\le i+j\le 2}\int_{\delta\le r\le R,|z|\le 1}|\partial_r^i\partial_z^j v|^2dY\lesssim\sum_{0\le i+j\le 2}\int_{\delta\le r\le R}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j v|^2}{\langle z\rangle}dY$$

Let now the cube

$$\mathcal{C}_A=\{\delta\le r\le R,\ \ \frac{A}{2}\le|z|\le A\},\quad A\ge 1.$$

Let $V(r,z)=v(r,Az)$, then the two dimensional Sobolev in the cylinder $\mathcal{C}_1$ ensures:

$$\begin{aligned}\|v\|^{2q+2}_{L^\infty(\mathcal{C}_A)} &= \|V\|^{2q+2}_{L^\infty(\mathcal{C}_1)}\lesssim_{\delta,R}\sum_{0\le i+j\le 2}\int_{\delta\le r\le R,\frac12\le|z|\le 1}|\partial_r^i\partial_z^jV|^2dY\\ &\lesssim_{\delta,R}\frac{1}{A}\sum_{0\le i+j\le 2}\int_{\delta\le r\le R,\frac{A}{2}\le|z|\le A}|\partial_r^iA^j\partial_z^jv|^2dY\\ &\lesssim_{\delta,R}\sum_{0\le i+j\le 2}\int_{\delta\le r\le R}\frac{|\partial_r^i(\langle z\rangle\partial_z)^jv|^2}{\langle z\rangle}dY\end{aligned}$$

and since the bound is independent of $A\ge 1$, (B.5) is proved. □

Appendix C. Nonlinear estimates on Ψ_3

This appendix is devoted to the control of various norms of the leading order driving term Ψ_3 and the error term $L(\zeta)$ given by (4.5). We start with $L^2_{\rho_r}$ bounds with sharp weight in z.

Lemma C.1 ($L^2_{\rho_r}$ bounds on Ψ_3 and $L(\zeta)$)**.** *We claim for $\nu>0$, $|b|<b^*(\nu)$:*

$$\int\frac{|(\langle z\rangle\partial_z)^j\Psi_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_rdY\lesssim(\sqrt{b})^{2\alpha+2\eta},\quad j=0,1,2,3,\tag{C.1}$$

and for any $r_0>0$, for $i\ge1$, $i+j\le3$, if $|b|<b^*(\nu,r_0)$:

$$\int_{r\ge r_0}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j\Psi_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_rdY\lesssim(\sqrt{b})^{2\alpha+2\eta},\quad i\ge1,\ i+j\le3,\tag{C.2}$$

and

$$\int\frac{|(\langle z\rangle\partial_z)^jL(\zeta)|^2}{1+z^{4(\ell+\nu)+1}}\rho_rdY\lesssim(\sqrt{b})^{2\alpha+2g},\quad j=0,1,2,3.\tag{C.3}$$

$$\int_{r\ge r_0}\frac{|\partial_r^i(\langle z\rangle\partial_z)^jL(\zeta)|^2}{1+z^{4(\ell+\nu)+1}}\rho_rdY\lesssim(\sqrt{b})^{4\alpha},\quad i\ge1,\ i+j\le3.\tag{C.4}$$

Proof. **step 1** Control of $\tilde\Psi_1$. Recall (3.8). Let $\delta<\alpha/\ell$. As $\delta<g$ one computes that for $i\in\mathbb{N}$:

$$\begin{aligned}&\partial_r^i\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\\ =&\int_1^\mu\frac{d\tilde\mu}{\tilde\mu}\frac{1}{(\sqrt{b})^{\frac{2}{p-1}}\tilde\mu^\gamma(\sqrt{b}\tilde\mu)^i}\left[\partial_r^i\left(-\Lambda^2Q-\alpha\Lambda Q\right)\right]\left(\frac{r}{\sqrt{b}\mu}\right)\\ =&\int_1^\mu\frac{d\tilde\mu}{\tilde\mu}\frac{1}{(\sqrt{b})^{\frac{2}{p-1}}\tilde\mu^\gamma(\sqrt{b}\tilde\mu)^i}O\left(\left(1+\frac{r}{\sqrt{b}\mu}\right)^{-\gamma-\delta-i}\right)=O\left(\mu^\delta\frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta+i}}\right),\end{aligned}$$

$$\begin{aligned}&\partial_r^i\langle z\rangle\partial_z\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\\ =&\frac{\langle z\rangle\partial_z\mu}{\mu}\frac{1}{\mu^\gamma(\sqrt{b})^{\frac{2}{p-1}}(\mu\sqrt{b})^i}\left[\partial_r^i\left(-\Lambda^2Q-\alpha\Lambda Q\right)\right]\left(\frac{r}{\sqrt{b}\mu}\right)\\ =&O\left(\mu^\delta\frac{(\sqrt{b})^{\alpha+\delta}}{(\mu\sqrt{b})^{\gamma+\delta+i}+r^{\gamma+\delta+i}}\right)=O\left(\mu^\delta\frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta+i}}\right),\end{aligned}\tag{C.5}$$

which can be easily generalized to show that for $j\in\mathbb{N}$:

$$\left|\partial_r^i(\langle z\rangle\partial_z)^j\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\right|\lesssim \mu^\delta\frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta+i}}\lesssim \mu^\alpha\frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta+i}}$$

as $g\leq\alpha$ and $\mu\geq 1/2$. Similarly, as for $j\in\mathbb{N}$

$$(\langle z\rangle\partial_z)^j\left(\frac{\partial_z\nu}{1+\nu}\right)^2=O(\langle z\rangle^{-2})$$

there holds the analogue estimate since $\mu^\delta\lesssim\langle z\rangle^{2\ell\delta/\alpha}\lesssim\langle z\rangle^2$:

$$\begin{aligned}&\left|\partial_r^i(\langle z\rangle\partial_z)^j\left[\left(\frac{\partial_z\nu}{1+\nu}\right)^2\frac{1}{\mu^{\frac{2}{p-1}}}\left(\Lambda^2Q_b+\alpha\Lambda Q_b\right)\left(\frac{r}{\mu}\right)\right]\right|\\ \lesssim&\ \frac{1}{(\sqrt{b})^{\frac{2}{p-1}}\langle z\rangle^2\mu^{\frac{2}{p-1}}(\sqrt{b}\mu)^i}O\left(\left(1+\frac{r}{\sqrt{b}\mu}\right)^{-\gamma-g-i}\right)\\ \lesssim&\ \frac{(\sqrt{b})^{\alpha+\delta}\mu^{\alpha+\delta}}{\langle z\rangle^2}\frac{1}{(\sqrt{b}\mu)^{\gamma+\delta+i}+r^{\gamma+\delta+i}}\lesssim\frac{\mu^{\alpha+\delta}}{\langle z\rangle^2}\frac{(\sqrt{b})^{\alpha+\delta}}{r^{-\gamma-\delta-i}}\lesssim\mu^\alpha\frac{(\sqrt{b})^{\alpha+\delta}}{r^{-\gamma-\delta-i}}.\end{aligned}\tag{C.6}$$

From (3.8) and the above bounds one infers that for $j\in\mathbb{N}$:

$$\begin{aligned}\int\frac{|(\langle z\rangle\partial_z)^j\tilde\Psi_1|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\ &\lesssim\ (\sqrt{b})^{2\alpha+2\delta}\int\frac{\mu^{2\alpha}}{1+z^{4(\ell+\nu)+1}}dz\int\frac{1}{r^{-2\gamma-2\delta}}r^{d-1}\rho_r dr\\ &\lesssim\ (\sqrt{b})^{2\alpha+2\delta}\int\frac{(1+aP_{2\ell}(z))^2}{1+z^{4(\ell+\nu)+1}}dz\lesssim(\sqrt{b})^{2\alpha+2\delta}\end{aligned}\tag{C.7}$$

and for $i\in\mathbb{N}$ for b small enough

$$\begin{aligned}&\int_{r\geq r_0}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j\tilde\Psi_1|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\\ \lesssim&\ (\sqrt{b})^{2\alpha+2\delta}\int\frac{\mu^{2\alpha}}{1+z^{4(\ell+\nu)+1}}dz\int\frac{1}{r^{-2\gamma-2\delta+2i}}r^{d-1}\rho_r dr\lesssim(\sqrt{b})^{2\alpha+2\delta}.\end{aligned}\tag{C.8}$$

step 2 Control of $\tilde\Psi_3$. One first computes for the first term in (4.5) that for $i\in\mathbb{N}$,

$$\begin{aligned}\partial_r^i\Lambda_r\Phi_{a,b}\ &=\ \frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}}\left(\partial_r^i\Lambda Q\right)\left(\frac{r}{\sqrt{b}\mu}\right)=\frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}}O\left(\left(1+\left(\frac{r}{\sqrt{b}\mu}\right)\right)^{-\gamma-i}\right)\\ &=\ O\left(\frac{(\sqrt{b})^\alpha\mu^\alpha}{r^{\gamma+i}}\right),\end{aligned}$$

$$\begin{aligned}\partial_r^i\langle z\rangle\partial_z\Lambda_r\Phi_{a,b}\ &=\ -\frac{\langle z\rangle\mu_z}{\mu}\frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}}\left(\partial_r^i\Lambda^2Q\right)\left(\frac{r}{\sqrt{b}\mu}\right)\\ &=\ \frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}}O\left(\left(1+\left(\frac{r}{\sqrt{b}\mu}\right)\right)^{-\gamma-i}\right)=O\left(\frac{(\sqrt{b})^\alpha\mu^\alpha}{r^{\gamma+i}}\right),\end{aligned}$$

which can easily be generalised to produce for $j\in\mathbb{N}$

$$\left|\partial_r^i(\langle z\rangle\partial_z)^j\Lambda_r\Phi_{a,b}\right|\lesssim\frac{(\sqrt{b})^\alpha\mu^\alpha}{r^{\gamma+i}}.$$

Also, from (3.48), (3.6), (3.74) and (3.39) one infers that for $j\in\mathbb{N}$

$$(\langle z\rangle\partial_z)^j\left(\frac{\partial_\tau\nu}{1+\nu}\right)=a_\tau(\langle z\rangle\partial_z)^j\left(\frac{P_{2\ell}(z)}{1+aP_{2\ell}(z)}\right)=O(|a_\tau|)=O((\sqrt{b})^\eta).\tag{C.9}$$

and from (3.47), (3.4):

$$\left|\frac{1}{2}\left(\frac{-b_\tau}{b}+1\right)-\frac{\ell}{\alpha}\right|=|B|\lesssim|\partial_\tau\tilde b|+\frac{|\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0}|}{b_{\ell,0}}\lesssim(\sqrt{b})^\eta. \tag{C.10}$$

We therefore infer from (4.6) that for $j\geq 0$:

$$\begin{aligned}&\int\frac{1}{1+z^{4(\ell+\nu)+1}}\left|(\langle z\rangle\partial_z)^j\left[\left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right)-\frac{\ell}{\alpha}+\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right]\Lambda_r\Phi_{a,b}\right]\right|^2\rho_r dY\\ &\lesssim(\sqrt{b})^{2\alpha+2\eta}\int\frac{\mu^{2\alpha}}{1+z^{4(\ell+\nu)+1}}dz\int r^{-2\gamma}r^{d-1}\rho_r dr\lesssim(\sqrt{b})^{2\alpha+2\eta}\end{aligned}$$

and for $i\geq 1$:

$$\begin{aligned}&\int_{r\geq r_0}\frac{1}{1+z^{4(\ell+\nu)+1}}\left|\partial_r^i(\langle z\rangle\partial_z)^j\left[\left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right)-\frac{\ell}{\alpha}+\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right]\Lambda_r\Phi_{a,b}\right]\right|^2\rho_r dY\\ &\lesssim(\sqrt{b})^{2\alpha+2\eta}\int\frac{\mu^{2\alpha}}{1+z^{4(\ell+\nu)+1}}dz\int_{r\geq r_0} r^{-2\gamma-2i}r^{d-1}\rho_r dr\lesssim(\sqrt{b})^{2\alpha+2\eta}.\end{aligned}$$

If $j\geq 1$ as this is the only term depending on z in $\tilde\Psi_3$ from (4.5) we conclude that:

$$\begin{aligned}&\int\frac{|(\langle z\rangle\partial_z)^j\tilde\Psi_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\\ =&\int\frac{\left|(\langle z\rangle\partial_z)^j\left[\left[\frac{1}{2}\left(-\frac{b_\tau}{b}+1\right)-\frac{\ell}{\alpha}+\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right]\Lambda_r\Phi_{a,b}\right]\right|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\lesssim(\sqrt{b})^{2\alpha+2\eta}\end{aligned} \tag{C.11}$$

and similarly for $i\geq 1$:

$$\int_{r\geq r_0}\frac{|\partial_r^i(\langle z\rangle\partial_z)^j\tilde\Psi_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\lesssim(\sqrt{b})^{2\alpha+2\eta}. \tag{C.12}$$

We turn to the other terms in (4.5). For $r\geq r_0$ and b small enough, $\partial_r^i\psi_{k,0}=O(r^{2k+4-\gamma})$ for $i=0,1,2$, and since $H_b\psi_{i,0}=\lambda_i\psi_{i,0}$ and $V_b=O(1)$ one deduces

$$\partial_r^3\psi_{i,0}=\partial_r\left[-\frac{d-1}{r}\partial_r+\frac{1}{p-1}+\frac{1}{2}r\partial_r+V_b-\lambda_i\right]\psi_{i,0}=O(r^{2i+5-\gamma}),\quad r\geq r_0. \tag{C.13}$$

Therefore, using (2.5) and (2.6) one obtains that for $r\in[0,+\infty)$:

$$|\psi_{\ell,0}|+|\psi_{0,0}|\lesssim\frac{1+r^{2\ell+5}}{r^\gamma},\quad\left|\psi_{0,0}-\frac{1}{(\sqrt{b})^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right|\lesssim\frac{(\sqrt{b})^g}{r^\gamma}(1+r^5)$$

and for $r\in[r_0,+\infty)$ and $i=1,2,3$:

$$|\partial_r^i\psi_{\ell,0}|+|\partial_r^i\psi_{0,0}|\leq r^{2\ell+5-\gamma},\quad\left|\partial_r^i\left(\psi_{0,0}-\frac{1}{(\sqrt{b})^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right)\right|\leq\sqrt{b}^g r^{5-\gamma}.$$

From (3.48), (3.6), (3.74), (2.2) and (3.39) there holds

$$|\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0}|\lesssim(\sqrt{b})^{\alpha+\eta},\quad|\tilde b|+|\tilde\lambda_\ell|+|\tilde\lambda_0|\lesssim(\sqrt{b})^\eta,\quad\left|\frac{b_\tau}{b}\right|\lesssim 1$$

since $\eta\ll g$. We then conclude from the three identities above, (2.5) and (2.7) that for

$$\begin{aligned}\overline{\Psi}_3\ :=\ &(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{(\sqrt{b})^\alpha}{\alpha(1+\tilde b)}(-\ell\tilde b+\tilde\lambda_\ell-\tilde\lambda_0)\psi_{0,0}\\ &-\frac{\ell}{\alpha}(\sqrt{b})^\alpha\left[\psi_{0,0}-\frac{1}{(\sqrt{b})^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right]+b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\end{aligned}$$

there holds

$$\begin{aligned}
&\int \frac{|\overline{\Psi}_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY \\
\lesssim\ & \sqrt{b}^{2\alpha}\int \frac{1}{1+z^{4(\ell+\nu)+1}}\left((\sqrt{b})^{2\eta}\frac{1+r^{4\ell+10}}{r^{2\gamma}}+|b\partial_b(\psi_{\ell,0})|^2+|b\partial_b(\psi_{0,0})|^2\right)\rho_r dY \\
\lesssim\ & \sqrt{b}^{2\alpha+2\eta}+\sqrt{b}^{2\alpha+2g}\lesssim \sqrt{b}^{2\alpha+2\eta}
\end{aligned}$$

and similarly for $i=1,2,3$:

$$\int_{r\geq r_0}\frac{|\partial_r^i\overline{\Psi}_3|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\lesssim \sqrt{b}^{2\alpha}\int \frac{(\sqrt{b}^{2\eta}r^{4\ell+10-2\gamma}+\sqrt{b}^{2g}r^{4\ell+10-2\gamma})}{1+z^{4(\ell+\nu)+1}}\rho_r dY\lesssim \sqrt{b}^{2\alpha+2\eta}.$$

step 3 Control of Ψ_3. From (4.5), the bounds (C.7), (C.8), (C.11), (C.12) and the two bounds above imply the desired bounds (C.1) and (C.2).

step 4 Proof of (C.3). Recall that $\zeta=b_\ell(\psi_{\ell,0}-\psi_{0,0})$ and $L(\zeta)=p(\Phi_{a,b}^{p-1}-Q_b^{p-1})\zeta$. We thus infer from (3.39) and (C.13):

$$\partial_z\zeta=0,\quad \text{and}\quad |\partial_r^i\zeta|\lesssim(\sqrt{b})^\alpha r^{2\ell+5}\quad \text{for } r\geq r_0\ \text{ and } i=0,...,3.$$

Also, one computes for $i\in\mathbb{N}$

$$\begin{aligned}
&\partial_r^i(\Phi_{a,b}^{p-1}-Q_b^{p-1})=(p-1)\int_1^\mu \frac{d\tilde\mu}{\tilde\mu}\frac{1}{(\tilde\mu\sqrt{b})^{2+i}}[\partial_r^i(Q^{p-2}\Lambda Q)]\left(\frac{r}{\tilde\mu\sqrt{b}}\right)\\
=\ &\int_1^\mu \frac{d\tilde\mu}{\tilde\mu}O\left(\frac{\tilde\mu^\alpha(\sqrt{b})^\alpha}{(\mu\sqrt{b})^{2+\alpha+i}+r^{2+\alpha+i}}\right)=O\left(\frac{\mu^\alpha\sqrt{b}^\alpha}{(\sqrt{b})^{2+\alpha+i}+r^{2+\alpha+i}}\right)=O\left(\frac{\mu^\alpha\sqrt{b}^\alpha}{r^{2+\alpha+i}}\right).
\end{aligned}$$

and

$$|\partial_r^i\langle z\rangle\partial_z\Phi_{a,b}|=\langle z\rangle\frac{|\mu_z|}{\mu}\frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}}(\partial_r^i\Lambda Q)\left(\frac{r}{\sqrt{b}\mu}\right)\lesssim\frac{\mu^\alpha(\sqrt{b})^\alpha}{(\sqrt{b}\mu)^{\gamma+i}+r^{\gamma+i}}$$

which can easily be generalized to prove that for $j\geq 1$:

$$|\partial_r^i(\langle z\rangle\partial_z)^j\Phi_{a,b}|\lesssim\frac{\mu^\alpha(\sqrt{b})^\alpha}{(\sqrt{b}\mu)^{\gamma+i}+r^{\gamma+i}}\lesssim\min\left(\frac{\sqrt{b}^\alpha\mu^\alpha}{r^{\gamma+i}},\frac{1}{(\sqrt{b}\mu)^{\frac{2}{p-1}+i}+r^{\frac{2}{p-1}+i}}\right)$$

implying that

$$|\partial_r^i(\langle z\rangle\partial_z)^j(\Phi_{a,b}^{p-1}-Q_b^{p-1})|=|\partial_r^i(\langle z\rangle\partial_z)^j(\Phi_{a,b}^{p-1})|\lesssim\frac{\mu^\alpha\sqrt{b}^\alpha}{(\sqrt{b})^{2+\alpha+i}+r^{2+\alpha+i}}\lesssim\frac{\sqrt{b}^\alpha\mu^\alpha}{r^{2+\alpha+i}}. \tag{C.14}$$

From the above estimates we infer that for $i\geq 1$ and $j\in\mathbb{N}$:

$$\int_{r\geq r_0}\frac{|\partial_r^i(\langle z\rangle\partial_z)^jL(\zeta)|^2}{1+z^{4(\ell+\nu)+1}}\rho_r dY\lesssim\int_{r\geq r_0}(\sqrt{b})^\alpha\frac{\mu^\alpha}{1+z^{4(\ell+\nu)+1}}r^{2\ell+5}\rho_r dY\lesssim(\sqrt{b})^{4\alpha}$$

which proves (C.4). We also infer that for $j\in\mathbb{N}$:

$$|(\langle z\rangle\partial_z)^jL(\zeta)|\lesssim\frac{(\mu\sqrt{b})^\alpha}{r^{\alpha+2}}|\zeta|. \tag{C.15}$$

Hence from (4.2), (3.14), (1.8) as $g \leq 2$:

$$
\begin{aligned}
&\int \frac{|(\langle z\rangle\partial_z)^j L(\zeta)|^2}{(1+z^{4\ell(1+\nu)+1})}dY \lesssim \int\frac{dz}{\langle z\rangle^{1+4\nu}}\int\frac{(\sqrt{b})^{2\alpha}\zeta^2}{r^{2(\alpha+2)}}r^{d-1}\rho_r dr\\
\lesssim\ & (\sqrt{b})^{2\alpha}\int_{r\leq\sqrt{b}}\frac{(\sqrt{b})^{2g}}{r^{\frac{4}{p-1}}}\frac{1}{r^{(2\alpha+4)}}r^{d-1}dr\\
+\ & (\sqrt{b})^{2\alpha}\int_{\sqrt{b}\leq r}(\sqrt{b})^{2\alpha}\frac{r^4+(\sqrt{b})^{2g}}{r^{2\gamma+2\alpha+4}}(1+r)^{2\gamma+4\ell+4}\rho_r r^{d-1}dr \lesssim (\sqrt{b})^{2\alpha+2g}
\end{aligned}
$$

and (C.3) is proved. □

We now turn to $W^{1,2q+2}$ near the origin.

Lemma C.2 ($W^{1,2q+2}$ bounds on Ψ_3)**.** *There hold the following estimates for some universal* $c, C>0$*:*

$$\int_{r\leq 2}\left(r^{2\frac{2q+1}{2q+2}}\frac{\Psi_3}{T}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim(\sqrt{b})^{c\eta q} \tag{C.16}$$

$$\int_{r\leq 2}\left[r^{2\frac{2q+1}{2q+2}}\partial_r\left(\frac{\Psi_3}{T}\right)\right]^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim\frac{1}{(\sqrt{b})^{Cq}} \tag{C.17}$$

$$\int_{r\leq 2}\left[r^{2\frac{2q+1}{2q+2}}\langle z\rangle\partial_z\left(\frac{\Psi_3}{T}\right)\right]^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim(\sqrt{b})^{c\eta q} \tag{C.18}$$

and similarly:

$$\int_{r\leq 2}\left(r^{2\frac{2q+1}{2q+2}}\frac{L(\zeta)}{T}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim(\sqrt{b})^{cq} \tag{C.19}$$

$$\int_{r\leq 2}\left[r^{2\frac{2q+1}{2q+2}}\partial_r\left(\frac{L(\zeta)}{T}\right)\right]^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim\frac{1}{(\sqrt{b})^{Cq}} \tag{C.20}$$

$$\int_{r\leq 2}\left[r^{2\frac{2q+1}{2q+2}}\langle z\rangle\partial_z\left(\frac{L(\zeta)}{T}\right)\right]^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim(\sqrt{b})^{cq} \tag{C.21}$$

Proof. **step 1** Pointwise bound. We claim the pointwise bound for any $0\leq\delta\leq\eta$ and $r\leq 2$:

$$\left|\frac{\Psi_3}{Tr}\right|+\left|\partial_r\left(\frac{\Psi_3}{T}\right)\right|\lesssim\frac{(\sqrt{b})^\delta}{r^{1+\delta}}+\frac{1}{r}\left|\begin{array}{ll}\frac{(\sqrt{b})^\delta}{r^\delta\mu^{\alpha-\delta}} & \text{for } r\geq D\\ D^{\frac{2}{p-1}}(1+D^\delta)\left[\frac{(\sqrt{b})^{\alpha+\delta}}{r^\gamma+(\sqrt{b})^\gamma}+\frac{(\sqrt{b})^\alpha}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\right] & \text{for } r\leq D\end{array}\right. \tag{C.22}$$

$\tilde\Psi_1$ *term.* Recall (4.5). We first prove the above bound (C.22) for $\tilde\Psi_1$. We decompose from (3.8):

$$\frac{\tilde\Psi_1}{T}=G_1+G_2$$

with

$$G_1=\frac{\ell}{\alpha T}\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right], \tag{C.23}$$

$$G_2=-\frac{1}{\alpha^2}\left(\frac{\partial_z\nu}{1+\nu}\right)^2\left(\alpha+\frac{\Lambda^2Q_b}{\Lambda Q_b}\right)\left(\frac{r}{D}\right). \tag{C.24}$$

G_2 *term.* We estimate in brute force using the asymptotics of Q:

$$\left|\alpha+\frac{\Lambda^2 Q}{\Lambda Q}(y)\right|\lesssim\frac{1}{1+|y|^g},\quad\left|\partial_y\left(\frac{\Lambda^2Q}{\Lambda Q}(y)\right)\right|\lesssim\frac{1}{1+|y|^{1+g}}$$

from which:

$$|G_2| \lesssim \frac{1}{\langle z\rangle^2}\frac{D^\delta}{D^\delta+r^\delta} \lesssim (\sqrt{b})^\delta \frac{1}{r^\delta\langle z\rangle}$$
$$|\partial_r G_2| \lesssim \frac{1}{\langle z\rangle^2}\frac{1}{D}\frac{1}{1+\left(\frac{r}{D}\right)^{\delta+1}} \lesssim \frac{1}{a^c\langle z\rangle^2}\frac{D^\delta}{r^{\delta+1}} \lesssim \frac{1}{r}(\sqrt{b})^\delta\frac{1}{r^\delta\langle z\rangle}.$$

G_1 *term.* Next we estimate using $\mu \geq \frac{1}{2}$:

$$\begin{aligned}\left|\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right| &\lesssim \int_1^\mu \frac{d\sigma}{\sigma^{\gamma+1}}\left|(\gamma+r\partial_r)\Lambda Q_b\right|\left(\frac{r}{\sigma}\right)\\ \lesssim & \int_1^\mu \frac{d\sigma}{\sigma^{\gamma+1}}\frac{1}{(\sqrt{b})^{\frac{2}{p-1}}}\frac{1}{1+\left(\frac{r}{\sigma\sqrt{b}}\right)^{\gamma+\delta}} \lesssim (\sqrt{b})^{\alpha+\delta}\int_1^\mu \frac{d\sigma}{\sigma^{1-\delta}}\frac{1}{(\sigma\sqrt{b})^{\gamma+\delta}+r^{\gamma+\delta}}\\ \lesssim & \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\int_1^\mu\frac{d\sigma}{\sigma^{1-\delta}} \lesssim \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\mu^\delta. \end{aligned} \tag{C.25}$$

Moreoever,

$$T=\Lambda\Phi_{a,b} \gtrsim \frac{D^\alpha}{D^\gamma+r^\gamma}. \tag{C.26}$$

and hence the pointwise bound for $r \geq D$:

$$\frac{1}{T}\left|\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right| \lesssim \frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta}}\mu^\delta\frac{D^\gamma+r^\gamma}{D^\alpha} \lesssim \frac{(\sqrt{b})^{\alpha+\delta}}{r^{\gamma+\delta}}\mu^\delta\frac{r^\gamma}{D^\alpha} \lesssim \frac{(\sqrt{b})^\delta}{r^\delta\mu^{\alpha-\delta}}$$

and for $r \leq D$:

$$\frac{1}{T}\left|\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right| \lesssim \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\mu^\delta D^{\frac{2}{p-1}} \lesssim \frac{(\sqrt{b})^{\alpha}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}D^{\frac{2}{p-1}+\delta}.$$

We now estimate the ∂_r derivative. First:

$$\begin{aligned}\left|\partial_r\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\right| &= \left|\frac{1}{\mu^{\gamma+1}}(\partial_r\Lambda Q_b)\left(\frac{r}{\mu}\right)-\partial_r\Lambda Q_b(r)\right|\\ \lesssim & \int_1^\mu\frac{d\sigma}{\sigma^{\gamma+2}}\left|(\gamma+1+r\partial_r)(\partial_r\Lambda Q_b)\right|\left(\frac{r}{\sigma}\right)\\ \lesssim & \int_1^\mu\frac{d\sigma}{\sigma^{\gamma+2}}\frac{1}{(\sqrt{b})^{\frac{2}{p-1}+1}}\frac{1}{1+\left(\frac{r}{\sigma\sqrt{b}}\right)^{\gamma+1+\delta}} \lesssim (\sqrt{b})^{\alpha+\delta}\int_1^\mu\frac{d\sigma}{\sigma^{1-\delta}}\frac{1}{(\sigma\sqrt{b})^{\gamma+1+\delta}+r^{\gamma+1+\delta}}\\ \lesssim & \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+1+\delta}+r^{\gamma+1+\delta}}\int_1^\mu\frac{d\sigma}{\sigma^{1-\delta}} \lesssim \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+1+\delta}+r^{\gamma+1+\delta}}\mu^\delta \lesssim \frac{1}{r}\frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\mu^\delta\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial_r\Lambda\Phi_{a,b}}{(\Lambda\Phi_{a,b})^2} &\lesssim \left(\frac{D^\gamma+r^\gamma}{D^\alpha}\right)^2\frac{1}{DD^{\frac{2}{p-1}}}\partial_r(\Lambda Q_b)\left(\frac{r}{D}\right) \lesssim \left(\frac{D^\gamma+r^\gamma}{D^\alpha}\right)^2\frac{1}{DD^{\frac{2}{p-1}}}\frac{1}{1+\left(\frac{r}{D}\right)^{\gamma+1}}\\ \lesssim & \left(\frac{D^\gamma+r^\gamma}{D^\alpha}\right)^2\frac{D^\alpha}{D^{\gamma+1}+r^{\gamma+1}} \lesssim \frac{1}{r}\frac{D^\gamma+r^\gamma}{D^\alpha}\end{aligned} \tag{C.27}$$

and hence:

$$\begin{aligned}\left|\partial_r\left(\frac{1}{T}\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\right)\right| &\lesssim \frac{1}{r}\frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\mu^\delta\frac{D^\gamma+r^\gamma}{D^\alpha}\\ + & \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\mu^\delta\frac{D^\gamma+r^\gamma}{rD^\alpha}.\end{aligned}$$

For $r\geq D$, this yields:

$$\left|\partial_r\left(\frac{1}{T}\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\right)\right|\lesssim\frac{1}{r}\frac{(\sqrt{b})^\delta}{r^\delta\mu^{\alpha-\delta}}$$

and for $r\leq D$:

$$\left|\partial_r\left(\frac{1}{T}\left[\frac{1}{\mu^\gamma}\Lambda Q_b\left(\frac{r}{\mu}\right)-\Lambda Q_b(r)\right]\right)\right|\lesssim\frac{1}{r}\frac{D^{\frac{2}{p-1}+\delta}(\sqrt{b})^\alpha}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}$$

Eigenvectors terms. Recall now (4.5), (4.6). We now prove the bound (C.22) for $\tilde\Psi_3$. We have from (3.48), (3.6), (3.74) and (3.39):

$$\left|\frac12\left(-\frac{b_\tau}{b}+1\right)-\frac{\ell}{\alpha}-\frac{1}{2\alpha}\frac{\partial_\tau\nu}{1+\nu}\right|\lesssim(\sqrt{b})^\eta,\ \ |\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0}|\lesssim(\sqrt{b})^{\alpha+\eta},\ \ |\tilde b|+|\tilde\lambda_\ell|+|\tilde\lambda_0|\lesssim(\sqrt{b})^\eta$$

from what we infer using (2.5), (C.26) and (C.27):

$$\begin{aligned}&\left|\frac{1}{T}(\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{(\sqrt{b})^\alpha[-\ell\tilde b+\tilde\lambda_\ell-\tilde\lambda_0]}{\alpha(1+\tilde b)}\psi_{0,0}\right|\\&+r\left|\partial_r\left[\frac{1}{T}\left((\partial_\tau b_{\ell,0}+\lambda_{\ell,0}b_{\ell,0})(\psi_{\ell,0}-\psi_{0,0})-\frac{(\sqrt{b})^\alpha[-\ell\tilde b+\tilde\lambda_\ell-\tilde\lambda_0]}{\alpha(1+\tilde b)}\psi_{0,0}\right)\right]\right|\\ \lesssim\ &\frac{D^\gamma+r^\gamma}{D^\alpha}\frac{\sqrt{b}^{\alpha+\eta}}{(\sqrt{b})^\gamma+r^\gamma}\lesssim\left|\begin{array}{ll}\frac{(\sqrt{b})^\eta}{\mu^\alpha} & \text{for } r\geq D\\ \frac{D^{\frac{2}{p-1}}(\sqrt{b})^{\alpha+\eta}}{r^\gamma+(\sqrt{b})^\gamma} & \text{for } r\leq D.\end{array}\right.\end{aligned}$$

Since Using (2.6), (C.26) and (C.27) one has:

$$\begin{aligned}&\frac{(\sqrt{b})^\alpha}{T}\left|\psi_{0,0}-\frac{1}{\sqrt{b}^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right|+r\left|\partial_r\left(\frac{(\sqrt{b})^\alpha}{T}\left(\psi_{0,0}-\frac{1}{\sqrt{b}^\gamma}\Lambda Q\left(\frac{r}{\sqrt{b}}\right)\right)\right)\right|\\=\ &\frac{(\sqrt{b})^\alpha}{T}\left|\tilde\phi_0\right|+r\left|\partial_r\left(\frac{(\sqrt{b})^\alpha}{T}\left(\tilde\phi_0\right)\right)\right|\\ \lesssim\ &\frac{D^\gamma+r^\gamma}{D^\alpha}\frac{\sqrt{b}^{\alpha+g}}{r^\gamma+(\sqrt{b})^\gamma}\lesssim\left|\begin{array}{ll}\frac{(\sqrt{b})^g}{\mu^\alpha} & \text{for } r\geq D,\\ \frac{D^{\frac{2}{p-1}}(\sqrt{b})^{\alpha+g}}{r^\gamma+(\sqrt{b})^\gamma} & \text{for } r\leq D.\end{array}\right.\end{aligned}$$

Finally, from (2.5), (C.26) and (C.27), since $(\sqrt{b})^\delta\lesssim D^\delta$:

$$\begin{aligned}&\frac{(\sqrt{b})^\alpha}{T}\left|b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\right|+r\left|\partial_r\left(\frac{1}{T}\left(b_{\ell,0}\frac{b_\tau}{b}b\partial_b(\psi_{\ell,0}-\psi_{0,0})\right)\right)\right|\\ \lesssim\ &\frac{D^\gamma+r^\gamma}{D^\alpha}\frac{\sqrt{b}^{\alpha+g}}{r^{\gamma+g}+(\sqrt{b})^{\gamma+g}}\lesssim\left|\begin{array}{ll}\frac{(\sqrt{b})^\delta}{r^\delta\mu^\alpha} & \text{for } r\geq D,\\ \frac{D^{\frac{2}{p-1}+\delta}(\sqrt{b})^\alpha}{r^{\gamma+\delta}+(\sqrt{b})^{\gamma+\delta}} & \text{for } r\leq D.\end{array}\right.\end{aligned}$$

step 2 Proof of (C.16), (C.17). We use the bound (C.22) to prove (C.16), noticing that $|\psi_3|/T\lesssim|\psi_3|/(Tr)$ as $r\leq 2$. Indeed, for any $0<\delta\leq\eta$ for q large enough:

$$\int_{r\leq 2}\left(r^{2\frac{2q+1}{2q+2}}\frac{(\sqrt{b})^\delta}{r^{1+\delta}}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim b^{cq\delta}.$$

For the second term, we split the integral. First:

$$\int_{D\leq r\leq 2}\left(r^{2\frac{2q+1}{2q+2}}\frac{(\sqrt{b})^\delta}{\mu^{\alpha-\delta}r^{1+\delta}}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\lesssim b^{cq\delta}$$

and then for the second term, we estimate for K universal large enough, since $r\leq 2$:

$$\begin{aligned}
&\int_{r\leq \min\{2,D\}}\left(r^{2\frac{2q+1}{2q+2}}\frac{D^{\frac{2}{p-1}}(1+D^\delta)(\sqrt{b})^{\alpha+\delta}}{r(r^\gamma+(\sqrt{b})^\gamma)}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\\
\lesssim\ &\int_{r\leq\sqrt{b}}\left((\sqrt{b})^{2\frac{2q+1}{2q+2}-1-\frac{2}{p-1}+\delta}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{Kq})}\\
+\ &\int_{\sqrt{b}\leq r\leq 2}\left(r^{2\frac{2q+1}{2q+2}-1-\frac{2}{p-1}}(\sqrt{b})^\delta\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{Kq})}\\
\lesssim\ &\int_{r\leq\sqrt{b}}(\sqrt{b})^{\delta(2q+2)-2}r^{d-1}\frac{dzdr}{\langle z\rangle(1+D^{Kq})}\\
+\ &\int_{\sqrt{b}\leq r\leq 2}r^{-2}(\sqrt{b})^{\delta(2q+2)}r^{d-1}\frac{dzdr}{\langle z\rangle(1+D^{Kq})}\lesssim b^{c\delta q},
\end{aligned}$$

where we used $p\geq 3$ and $q\gg 1$. Similarly for the last term in (C.22), for ψ_3/T:

$$\begin{aligned}
&\int_{r\leq \min\{2,D\}}\left(r^{2\frac{2q+1}{2q+2}}\frac{(\sqrt{b})^\alpha}{[\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}]}D^{\frac{2}{p-1}}(1+D^\delta)\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\\
\lesssim\ &\int_{r\leq\sqrt{b}}\left((\sqrt{b})^{2\frac{2q+1}{2q+2}-\frac{2}{p-1}-\delta}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{qK})}\\
+\ &\int_{\sqrt{b}\leq r\leq 2}\left(r^{2\frac{2q+1}{2q+2}-\frac{2}{p-1}-2\delta}(\sqrt{b})^\delta\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{qK})}\lesssim b^{c\delta q},
\end{aligned}$$

and for $\partial_r(\psi_3/T)$:

$$\begin{aligned}
&\int_{r\leq \min\{2,D\}}\left(r^{2\frac{2q+1}{2q+2}}\frac{(\sqrt{b})^\alpha}{r[\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}]}D^{\frac{2}{p-1}}(1+D^\delta)\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{2Kq})}\\
\lesssim\ &\int_{r\leq\sqrt{b}}\left((\sqrt{b})^{2\frac{2q+1}{2q+2}-1-\frac{2}{p-1}-\delta}\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{qK})}\\
+\ &\int_{\sqrt{b}\leq r\leq 2}\left(r^{2\frac{2q+1}{2q+2}-\frac{2}{p-1}-1-2\delta}(\sqrt{b})^\delta\right)^{2q+2}\frac{dY}{\langle z\rangle(1+D^{qK})}\lesssim\frac{1}{b^{Cq}},
\end{aligned}$$

This concludes the proof of (C.16), (C.17).

step 3 ∂_z derivative. We turn to the proof of (C.18). We first claim the pointwise bound for any $0\leq\delta\leq\eta$:

$$|\langle z\rangle\partial_z\Psi_3|\lesssim\frac{(\sqrt{b})^\delta}{r^\delta}+\left|\begin{array}{ll}\frac{(\sqrt{b})^\delta}{r^\delta\mu^{\alpha-\delta}} & \text{for}\ \ r\geq D\\ D^{\frac{2}{p-1}}(1+D^\delta)\left[\frac{(\sqrt{b})^{\alpha+\delta}}{r^\gamma+(\sqrt{b})^\gamma}+\frac{(\sqrt{b})^\alpha}{\sqrt{b}^{\gamma+\delta}+r^{\gamma+\delta}}\right] & \text{for}\ \ r\leq D\end{array}\right. \tag{C.28}$$

which implies (C.18) as above.
G_1 term. We first estimate:

$$|\langle z\rangle\partial_zT|=\left|\frac{\langle z\rangle\partial_z\mu}{\mu}\frac{1}{D^{\frac{2}{p-1}}}\Lambda^2Q\left(\frac{r}{D}\right)\right|\lesssim\frac{1}{D^{\frac{2}{p-1}}}\frac{1}{1+\left(\frac{r}{D}\right)^\gamma}\lesssim\frac{1}{D^{\frac{2}{p-1}}}\frac{D^\gamma}{D^\gamma+r^\gamma}$$

and hence using (C.26):

$$\left|\langle z\rangle\frac{\partial_zT}{T^2}\right|\lesssim\frac{1}{D^{\frac{2}{p-1}}}\frac{D^\gamma}{D^\gamma+r^\gamma}\left(\frac{D^\gamma+r^\gamma}{D^\alpha}\right)^2\lesssim\frac{D^\gamma+r^\gamma}{D^\alpha} \tag{C.29}$$

from what we infer using (C.5) and $\mu \geq 1/2$

$$|\langle z\rangle G_1| \lesssim \frac{D^\gamma + r^\gamma}{D^\alpha} \frac{(\sqrt{b})^{\alpha+\delta}}{(\mu\sqrt{b})^{\gamma+\delta} + r^{\gamma+\delta}} \mu^\delta$$

$$\lesssim \left| \begin{array}{l} \frac{(\sqrt{b})^\delta}{r^\delta \mu^{\alpha-\delta}} \quad \text{for} \quad r \geq D \\ D^{\frac{2}{p-1}} \frac{(\sqrt{b})^{\alpha+\delta}}{\sqrt{b}^{\gamma+\delta} + r^{\gamma+\delta}} \mu^\delta \lesssim \frac{(\sqrt{b})^\alpha}{(\sqrt{b})^{\gamma+\delta} + r^{\gamma+\delta}} D^{\frac{2}{p-1}+\delta} \quad \text{for} \quad r \leq D. \end{array} \right.$$

G_2 *term.* We estimate from (C.6), (C.26) and (C.29)

$$|\langle z\rangle \partial_z G_2| \lesssim \frac{D^\alpha + r^\gamma}{D^\alpha} \frac{(\sqrt{b})^{\alpha+\delta}\mu^{\alpha+\delta}}{\langle z\rangle^2} \frac{1}{(\sqrt{b}\mu)^{\gamma+\delta} + r^{\gamma+\delta}} \lesssim \frac{(\sqrt{b})^\delta}{r^\delta}$$

Eigenvectors term. Finally from (C.9), (C.10):

$$\left| \langle z\rangle \partial_z \left(\frac{\tilde{\Psi}_3}{T} \right) \right| \lesssim (\sqrt{b})^\eta$$

as the other terms do not depend on z. This concludes the proof of (C.28).

step 4 Control of $L(\zeta)$ terms. From the rough bound

$$|Q_b^{p-1}| + |\Phi_{a,b}|^{p-1} \lesssim \min\left(\frac{1}{r^2}, \frac{1}{b}\right),$$

the bounds (C.14), (4.2), (C.26), (C.29) we infer for $j = 0, 1$ and $r \leq 2$:

$$\left| (\langle z\rangle \partial_z)^j \left(\frac{L(\zeta)}{T} \right) \right| \lesssim \frac{D^\gamma + r^\gamma}{D^\alpha} \min\left(\frac{D^\alpha}{(\sqrt{b})^{2+\alpha} + r^{2+\alpha}}, \frac{1}{r^2}, \frac{1}{b} \right) \frac{(\sqrt{b})^\alpha (r^2 + (\sqrt{b})^g)}{(\sqrt{b})^\gamma + r^\gamma}$$

$$\lesssim \left| \begin{array}{l} D^{\frac{2}{p-1}} (\sqrt{b})^{g-2-\frac{2}{p-1}} \quad \text{for} \quad r \leq \sqrt{b} \\ D^{\frac{2}{p-1}} \left(\frac{(\sqrt{b})^\alpha}{r^\gamma} + \frac{(\sqrt{b})^{\alpha+g}}{r^{\gamma+2}} \right) \quad \text{for} \quad \sqrt{b} \leq r \leq D \\ \frac{(\sqrt{b})^\alpha}{r^\alpha} + \frac{\sqrt{b}^{\alpha+g}}{r^{2+\alpha}} \quad \text{for} \quad r \geq D \end{array} \right.$$

One then computes since $p \geq 3$ and $g \geq 3/2$ that

$$\int_{r \leq \sqrt{b}} \left(r^{2\frac{2q+1}{2q+2}} D^{\frac{2}{p-1}} (\sqrt{b})^{g-2-\frac{2}{p-1}} \right)^{2q+2} \frac{dY}{\langle z\rangle (1 + D^{2Kq})}$$
$$\lesssim (\sqrt{b})^{2(2q+1)+d+(g-2-\frac{2}{p-1})(2q+2)-1} \lesssim (\sqrt{b})^{q-2},$$

$$\int_{\sqrt{b} \leq r \leq 2} \left(r^{2\frac{2q+1}{2q+2}} D^{\frac{2}{p-1}} \left(\frac{(\sqrt{b})^\alpha}{r^\gamma} + \frac{(\sqrt{b})^{\alpha+g}}{r^{\gamma+2}} \right) \right)^{2q+2} \frac{dY}{\langle z\rangle (1 + D^{2Kq})}$$
$$\lesssim (\sqrt{b})^{2(2q+1)-\gamma(2q+2)+d+\alpha(2q+2)-1} + (\sqrt{b})^{2(2q+1)-(\gamma+2)(2q+2)+d+(\alpha+g)(2q+2)-1}$$
$$\lesssim (\sqrt{b})^{2(2q+1)-\frac{2}{p-1}(2q+2)+d-1} + (\sqrt{b})^{2(2q+1)-(\frac{2}{p-1}+2-g)(2q+2)+d-1} \lesssim (\sqrt{b})^{q-2}$$

and that

$$\int_{D \leq r \leq 2} \left(r^{2\frac{2q+1}{2q+2}} \left(\frac{(\sqrt{b})^\alpha}{r^\alpha} + \frac{\sqrt{b}^{\alpha+g}}{r^{2+\alpha}} \right) \right)^{2q+2} \frac{dY}{\langle z\rangle (1 + D^{2Kq})}$$
$$\lesssim (\sqrt{b})^{2(2q+1)-\alpha(2q+2)+d+\alpha(2q+2)-1} + (\sqrt{b})^{2(2q+1)-(\alpha+2)(2q+2)+d+(\alpha+g)(2q+2)-1} \lesssim (\sqrt{b})^q$$

yielding (C.19), (C.21). The estimate (C.20) can be proven with the same arguments and is left to the reader. □

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