Large Gauge Transformations and Black Hole Entropy

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

This thesis does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

Diffeomorphisms in general relativity can act non-trivially on the boundary of spacetimes. Of particular importance are the *BMS group* of transformations, originally found by Bondi, Metzner, van der Burg and Sachs. They describe the symmetries of asymptotically flat spacetimes.

We first consider symmetries such as these that act non-trivially at null infinity. We look in detail at the BMS group and then describe the conformal symmetries of asymptotically flat spacetime. These represent an extension of the BMS group that we call the *conformal BMS group*.

Second, we explore the action of these large diffeomorphisms in the context of black holes. The emergence of Virasoro algebras as the asymptotic symmetry algebras of black hole spacetimes suggests a fundamental link to conformal field theory. For the case of the generic Kerr black hole, we hypothesize that the black hole is itself a thermal conformal field theory which transforms under a Virasoro action. A set of infinitesimal $\text{Virasoro}_{\text{L}} \otimes \text{Virasoro}_{\text{R}}$ diffeomorphisms are presented which act non-trivially on the horizon. Using the covariant phase space formalism, we can construct the corresponding surface charges on the black hole horizon and find the central terms in their algebras. Ambiguities in the construction of the charges allow for the addition of extra terms. Wald and Zoupas have provided a general framework for these counterterms, although the precise form is left undetermined. In computing the horizon charges, certain obstructions to the integrability and associativity of the charge algebra arise, calling for some counterterm to be used. A consistent counterterm is found that removes these obstructions and gives rise to central charges $c_L = c_R = 12J$. On the assumption that there exists a quantum Hilbert space on which these charges generate the symmetries, one can use the Cardy formula to compute the entropy of the conformal field theory. This Cardy entropy turns out to be exactly equal to the Bekenstein-Hawking entropy, providing a potential microscopic interpretation for this macroscopic area-entropy law. The results are generalised to the Kerr-Newman black hole with the addition of charge.

Research overview

The original contributions of this thesis are based upon the research described in the following publications, listed in chronological order:

- S. J. Haco, S. W. Hawking, M. J. Perry and J. L. Bourjaily, "The Conformal BMS Group," JHEP **1711**, 012 (2017)

- S. Haco, S. W. Hawking, M. J. Perry and A. Strominger, "Black Hole Entropy and Soft Hair," JHEP **1812**, 098 (2018)

- S. Haco, M. J. Perry and A. Strominger, "Kerr-Newman Black Hole Entropy and Soft Hair," arXiv:1902.02247 [hep-th].

Preface

The first chapter was done in collaboration with Stephen Hawking, Malcolm Perry and Jacob Bourjaily. The group algebra determined in section 3 was done in collaboration with Malcolm Perry, although I performed all calculations throughout the first chapter. Part II of the thesis begins with review material, except section 4.3 which is the result of my own work. The novel material of this chapter was largely done in collaboration with Malcolm Perry, Stephen Hawking and Andrew Strominger, including sections 7.3, 7.4 and 7.5. Sections 7.2, 7.6, the calculations of section 8 and Appendices A, C, D, E and F are the results of my own work.

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1 Introduction

One century ago, Albert Einstein published his ground-breaking General Theory of Relativity, a theory of space and time and of gravitation. For over two hundred years prior to this, the conception of space and time had followed the formulation of Newton, as absolute quantities, with gravity a force that acted at a distance. General Relativity altered the conventional Aristotelian view of the Universe on a large scale, fundamentally linking space and time and the action of gravity through geometry. This revolutionised understanding and gave enormous predictive power: the first major prediction, the bending of light by gravity, was verified by Eddington in 1919 with the observation of stars during a solar eclipse, and the second huge prediction of gravitational waves was confirmed only recently in September 2015 by the gravitational wave detectors of LIGO and Virgo [1]. The direct detection of gravitational waves is perhaps the most compelling evidence for the existence of black holes, which result from the death of massive stars. This first detection measured the gravitational waves that emanated from the collision of two black holes about 1.3 billion years ago. This cataclysmic event released about 5.3×10^{47} joules of energy as gravitational waves and radiated at a peak power of more than that of all the stars in the observable universe combined [1].

The black hole information problem

The black hole no hair theorems tell us that black holes are characterised only in terms of three quantities: the mass, M, charge, Q and angular momentum, J. Classically, it is possible to form a black hole with parameters M, J and Q, in an infinite number of ways. This suggests that the black hole has infinite entropy. This was the generally accepted picture until a striking resemblance between the laws of black hole mechanics and the laws of thermodynamics was uncovered by Bardeen, Bekenstein, Carter and Hawking [2–4]. For these laws to coincide, black holes must have a finite temperature and a finite entropy. In a seminal paper by Hawking titled *Black hole explosions?* [5], black holes were shown to be thermal objects that emit radiation, with a Hawking temperature given in terms of the surface gravity κ as [1],

$$T_H = \frac{\kappa}{2\pi}. \tag{1.1}$$

One result of this finite temperature is that black holes must also have a finite entropy, which is given in terms of its area, A by the famous Bekenstein-Hawking formula [4, 6],

$$S = \frac{A}{4}. \tag{1.2}$$

This discovery uncovered a new and very deep problem in theoretical physics. In the statistical, thermodynamic picture, the entropy is a measure of the number of different states of a system. But for black holes, there is no clear way to count the degrees of freedom, without a microscopic description. A central question therefore arises: *what are the quantum states of a black hole?* This question remains unresolved despite considerable progress, particularly for black holes in string theory [7] and will become an essential feature of any theory of quantum gravity.

The fact that black holes radiate energy means that they can evaporate. Once a black hole has completely evaporated, all that is left will be thermal Hawking radiation. It might seem reasonable to assume that the whole process from black hole formation to evaporation obeys the laws of quantum mechanics: that an initial state will evolve according to an S-matrix as,

$$|\Psi_{final}\rangle = S\,|\Psi_{initial}\rangle\,. \tag{1.3}$$

Since the S-matrix is unitary, evolution of $|\Psi_{final}\rangle$ from $|\Psi_{initial}\rangle$ will be deterministic, and one will be able to recover the initial state from knowledge of the final state [8]. However, if we assume that the black hole was formed from some matter in a pure quantum state, then the overall process will involve the evolution from a pure to a mixed state. This is not a unitary process and

[1] We use units such that $G = \hbar = 1$

thus would violate quantum mechanics. Consequently, one is forced to draw the uncomfortable conclusion that information is lost in the process of black hole formation and subsequent evaporation [9]. This is the content of the black hole information paradox.

Possible ways out of the paradox

Over the last few decades, several potential resolutions to the information paradox have been proposed, but to date, the problem remains unsolved. Previous options that have been considered include: information stored in a stable remnant which survives after evaporation [10–12], evaporation treated as quantum tunnelling [13, 14], information prevented from entering the black hole in the first place due to 'bleaching' [15, 16], information emerging at the final stages of evaporation [17, 18], information encoded in 'quantum hair' [19], information escaping to 'baby universes' [20, 21], information stored both outside and inside the black hole via complementarity [22, 23] and numerous other scenarios. For information about each of these proposals, as well as a discussion of their various merits and flaws, see [16], [24] and [25].

Recent developments: soft particles

Diffeomorphisms (diffeos) in general relativity can act non-trivially - on both the classical phase space and the physical quantum states - whenever the spacetime has a boundary. In string theory, 'large' diffeomorphisms which act nontrivially at the horizon have been used to count black hole microstates and microscopically reproduce the macroscopic area law [7, 26]. This, however, is restricted to supersymmetric or near-supersymmetric black holes, not the astrophysical black holes that we see in the sky.

More recently, the effects of large diffeos on physically realistic black holes have been studied from a different point of view [27–63], beginning from the observation of Bondi, Metzner, van der Burg and Sachs [64, 65] that they can act nontrivially on the boundary of spacetime at infinity. They discovered an infinite group of symmetries, containing so-called ‘supertranslations’, which change the physical state. These supertranslations act to shift the individual light rays of null infinity forward or backward in retarded or advanced time. This *BMS group* of symmetries of so-called ‘asymptotically flat spacetime’ has been the subject of much scrutiny over recent years, with additional symmetries identified and referred to as ‘superrotations’ [66–68], but which are still only partially understood.

However, new research into the infrared behaviour of quantum gravity has put the subject on firmer footing, for example [69–76]. It turns out [69] that certain antipodally-matched combinations of supertranslations at the future endpoint of past null infinity and the past endpoint of future null infinity correspond to exact symmetries of gravitational scattering. Each of these supertranslations can be associated with a corresponding ‘supertranslation charge’ and the antipodal combinations give rise to an infinite set of conservation laws for all gravitational theories in asymptotically Minkowskian spacetimes. These conservation laws can be interpreted as identifying net incoming energy at a given angle with the net outgoing energy at the opposing angle. It has been shown [69–71] that these relations are actually equivalent to the soft (i.e. zero energy) graviton theorems, dating back to the work of Weinberg in 1965 [77].

There are two major insights that can be inferred from these developments, as set out in [27], which call into question the assumptions that underpin the information paradox. The first is that the vacuum in quantum gravity is not unique. Supertranslations act to transform the vacuum into a physically distinct vacuum, also with zero energy. These two vacuum states are then related to each other by the creation or annihilation of soft gravitons. The infinite number of supertranslations means that there is an infinity of degenerate vacua, which differ from one another by their numbers of soft gravitons. After a black hole evaporates, one can no longer assume that the quantum state settles down to a unique vacuum.

The second assumption concerns the no hair theorems. In [27, 28], it was shown that the BMS transformations act on the horizon of a generic 4D Kerr black hole and create distinguishing features referred to as soft hair. Thus black holes are not 'bald', or with just the few 'hairs' corresponding to mass, charge and angular momentum, but perhaps they have an infinite head of soft hair, corresponding to the infinite number of BMS symmetries.

This shift in the interpretation of two key principles, driven by new understanding of the symmetries of asymptotically flat spacetimes and the infrared structure of quantum gravity, has hinted at a possible way forward in understanding the information paradox. We remain, however, far from any resolution.

One key hurdle that must be overcome in reaching a resolution is deriving the Bekenstein-Hawking entropy law (1.2) for generic black holes in terms of a microscopic description, without reliance on string theory. Although an actual derivation of the entropy has not yet been found, in [78], following the earlier work [79] on the hidden conformal symmetry of a spin J Kerr black hole, a set of diffeos were found which act non-trivially on the horizon. These diffeos have the following properties:

(i) The Lie bracket algebra acts as a left-right Virasoro pair on the horizon,

(ii) The corresponding Iyer-Wald-Zoupas charges [80–86] with a judicious choice of counterterm, have central terms with $c_L = c_R = 12J$.

Assuming that there exists a unitary Hilbert space (including horizon edge states) transforming under these Virasoro algebras, along with the Cardy formula, the area law then follows.

Outline of the thesis

The structure of this thesis is as follows. In the first chapter we will look in general at large gauge transformations: those which are not simply coordinate transformations, but act nontrivially. In section 2 we will discuss the known symmetries of flat space: the Poincaré group of translations, rotations and boosts. We will further constrain the spacetime to be conformally flat and hence find the conformal group. We will then define asymptotically flat spacetimes and explore the corresponding symmetries here, resulting in the BMS group. Again, we will add in conformal symmetry and find the analogue of the conformal group for asymptotically conformally flat space. We call this the conformal BMS group and we will consider it in detail in section 3. In particular, we will find the group algebra. This involves a careful analysis of the action of the group generators and the use of a modified Lie bracket. The relation of the conformal BMS group to the BMS group is considered, along with it applications in a wider context.

The second chapter considers the problem of black hole information, with the ultimate goal being to recover the entropy of the Kerr black hole using microstates on the horizon. We begin in section 4 by recapping the covariant phase space techniques used to compute surface charges associated to diffeomorphisms. We will consider the algebra of these charges and the possibility that there might be central terms. We will explicitly compute the various forms of the charges and central terms for the gravitational case, expressions which will be used repeatedly in later sections. There may be certain issues in defining these charges, which result from lack of integrability or associativity. We will discuss how these issues arise and how they may be avoided. In addition, several ambiguities arise in constructing these charges. We will unpick these ambiguities in detail and consider how they might be used to add extra terms to the charges or to manipulate the expressions to find more convenient versions.

In section 5 we consider the AdS/CFT correspondence, to understand what techniques we can learn from this duality to take forward into computing the entropy of the Kerr black hole. We start in section 5.1 by reviewing the main features and properties of conformal field theory (CFT) and show how the Virasoro algebra emerges. We will then look at three-dimensional gravity in general, before focusing on the specific case of Anti de Sitter (AdS) spacetime. We will examine the symmetry algebra of AdS_3 and then consider in detail the BTZ black hole, which arises from asymptotically AdS_3 spacetimes. In section 5.2.3, we will explore the class of metrics that asymptote to AdS_3, as well as the vector fields which preserve this class of metrics. We will calculate the charges corresponding to these vector fields, using the covariant phase space methods. In section 5.3 we will compute the algebra of these charges and find the central terms. These central terms can be used to calculate the entropy of the CFT via the Cardy formula, which will be derived in section 5.3.1. We will see that this Cardy entropy precisely matches the black hole entropy in this case.

In section 6 we will move on to look at four dimensional black holes, beginning with the case of extreme Kerr, as an intermediate step before studying the case of the generic Kerr black hole. We will again examine the symmetries of the spacetime and calculate the corresponding charges, then use the Cardy formula to compute the entropy.

From here we consider the case of the generic Kerr black hole. This makes up the bulk of the novel material of this thesis. This involves the consideration of a different viewpoint, in which the boundary of the spacetime becomes the black hole horizon. We will discuss this boundary for the Kerr black hole in section 7 but also show in Appendices D and E how this alternative viewpoint will affect the discussions for the two examples already considered, BTZ and extreme Kerr.

We review the hidden conformal symmetry found in Kerr spacetimes in section 7.1, by analysing the wave equation and the near region contribution to the soft absorption cross-sections.

Part of the reason that computing charges for the Kerr black hole is so much more challenging than the cases already mentioned is that it is hard to write the Kerr metric in a simple way that is also well-adapted for this context. In section 7.2 we introduce different coordinate systems for the Kerr spacetime and outline the various merits of each. This involves the development of Kruskal-like coordinates, the details of which may be found in Appendix F.

In section 7.3 we will provide a heuristic derivation for the entropy of the Kerr black hole, before proceeding to calculate it in detail, again using covariant phase space methods. We start by finding suitable vector fields in section 7.4 and then proceeding to calculate the charges in section 7.5. This involves the development of a counterterm to avoid problems with constructing charges with well-defined associative Dirac brackets. A suitable counterterm is proposed and a consistent charge is found. In a similar manner to the previous cases of BTZ and extreme Kerr, the central charge is used to compute the entropy of the CFT by use of the Cardy formula. This calculation is performed, and its implications are discussed, in section 7.7.

In section 8 we add charge and generalize the previous work to the case of the Kerr-Newman black hole. This again involves understanding the hidden conformal symmetry, choosing appropriate vector fields and computing the charges to find the entropy.

In section 9 there is a discussion of the further work needed to stand a chance of resolving the information paradox, before we conclude in section 10.

In this work we will use units such that $c = G = \hbar = 1$.

Part I

Large Gauge Transformations

2 Spacetime symmetries

In four-dimensional Minkowski space, the isometries of the spacetime are given by the ten independent solutions to Killing's equation. These solutions allow one to form the Poincaré algebra, made up of four translations in each of the spacetime directions, plus three boosts and three spatial rotations. These are the symmetries of special relativity.

As soon as gravitational fields are included via general relativity, the standard isometry transformations of flat space must be revised. In the 1960s Bondi, van der Burg, Metzner and Sachs (BMS) postulated that there must be some way in which the full Poincaré group represent 'approximate' symmetry transformations [64, 65, 87]. They studied these approximate symmetries of curved spacetime by investigating the asymptotic symmetries of asymptotically flat spacetimes at null infinity - if the spacetime were asymptotically flat, then infinitely far away from any gravitational fields we must in some sense be able to reproduce the Poincaré group as the symmetry group. This group of asymptotic symmetries is known as the BMS group [64, 87], a larger group than the Poincaré group of flat space, that consists of the ordinary Lorentz transformations plus an infinite number of 'supertranslations'.

This BMS group has been extensively studied over the years. Penrose investigated the BMS group as a symmetry group on null infinity [88] and later with Newman, he looked into possible subgroups of BMS that might arise when considering scattering problems and the emission of radiation out to infinity [89].

More recently, the BMS group has received renewed attention. An extension to the BMS group has been proposed to include 'superrotations' [66–68] and work has been done on the conserved quantities that would be associated to the asymptotic symmetries of the BMS group [29, 62, 90–92]. In the quantum picture, these conservation laws amount to relations between ingoing and outgoing scattering states [69, 74] and have been shown to be equivalent to so-called soft theorems [70, 71] and subleading soft-theorems [93], originally formulated by Weinberg and Low [77, 94]. Within the last few years, the effect of these symmetries on black hole spacetimes has been investigated along with the potential for these conservation laws to provide answers to the black hole information paradox [9, 27].

While the Poincaré and BMS groups describe the symmetries of special and general relativity, for any theory that also admits a conformal symmetry, the necessary group of isometries must be larger. In flat space, the Poincaré group gets extended to the conformal group at spacelike infinity and at null infinity, one needs not the BMS group but a conformal version of it, which is developed here.

Conformal symmetry is at the heart of many important physical theories. For example, Maxwell's free field equations are conformally invariant, as is the massless Dirac equation. In terms of gravity, the situation is less clear, but for empty space, the Weyl tensor is unchanged by conformal transformations to the metric [95]. Another hint at conformal symmetry in gravity is through the connection with Yang-Mills theory: some aspects of gravity, particularly scattering amplitudes, can be regarded as the product of two Yang-Mills theories [96] - and we know Yang-Mills to be a classically conformally invariant theory in Minkowski space.

Given that $\mathcal{N} = 4$ Yang-Mills theory exhibits conformal symmetry, an obvious next step will be to study the action of the *conformal BMS group* in this context. The BMS group has previously been shown to be a conformal extension of the Carroll group [97, 98]. A generalization of the BMS group for supergravity has also been studied [99], although without investigation into asymptotically conformal transformations. Recently, work on classifying the asymptotic symmetry algebras of theories in different dimensions has been studied in the context of holography [100].

2.1 Conformal symmetries of flat space: Poincaré and conformal groups

In four-dimensional flat Minkowski spacetime, it is possible to identify certain symmetries of the metric - transformations that leave the spacetime invariant. These are the ten isometries which form the well-known Poincaré group of the symmetries of special relativity. These symmetries are found by asking for which vector fields ξ does the Lie derivative of the metric vanish, in other words, solutions to Killing's equation,

$$(\mathcal{L}_\xi g)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0. \tag{2.1}$$

$\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field ξ. In (3+1)-dimensional Minkowski space, we get ten independent solutions (Killing vectors (KV)) that make up the Poincaré group. This Poincaré group consists of the Lorentz group, a subgroup made up of three boosts and three spatial rotations, as well as an abelian normal subgroup of four translations in each of the spacetime directions.

The generators of these symmetry transformations may be written,

$$M_{ab} \equiv (x_a \partial_b - x_b \partial_a), \qquad P_a \equiv \partial_a \,, \tag{2.2}$$

where the M_{ab} give the Lorentz transformations and P_a the translations. The commutation relations are,

$$\begin{aligned}
[P_a, P_b] &= 0, \\
[M_{ab}, P_c] &= \eta_{bc}\, P_a - \eta_{ac}\, P_b, \\
[M_{ab}, M_{cd}] &= \eta_{ad}\, M_{bc} + \eta_{bc}\, M_{ad} - \eta_{bd}\, M_{ac} - \eta_{ac}\, M_{bd},
\end{aligned} \tag{2.3}$$

where η_{ab} is the Minkowski metric of signature $(-,+,+,+)$. These are the generators of the group $O(3,1)$.

We may also look at transformations which preserve the metric up to a conformal factor,

$$\mathcal{L}_\xi \, g = \Omega^2 \, g \,. \tag{2.4}$$

By taking the trace, we can solve for Ω^2 and find that the transformations ξ correspond to solutions to the *conformal Killing equation*, which in four dimensions is:

$$\nabla_a \, \xi_b + \nabla_b \, \xi_a - \frac{1}{2} \, g_{ab} \, \nabla_c \, \xi^c = 0 \,. \tag{2.5}$$

The solutions are *conformal Killing vectors* (CKV).

In flat space, the conformal Killing vectors consist of the Poincaré group, along with an extension to include special conformal transformations generated by K_μ and dilatations (scalings) generated by D:

$$\begin{aligned} D &\equiv x^a \, \partial_a \,, \\ K_a &\equiv x^2 \, \partial_a - 2 \, x_a x^b \, \partial_b \,. \end{aligned} \tag{2.6}$$

The commutation relations are given by:

$$\begin{aligned} [D, K_a] &= K_a, \\ [D, P_a] &= -P_a, \\ [K_a, P_b] &= 2\eta_{ab} D + 2M_{ab}, \\ [K_a, M_{bc}] &= \eta_{ab} K_c - \eta_{ac} K_b \,. \end{aligned} \tag{2.7}$$

These are the generators of the group $O(4, 2)$.

2.2 Symmetries of asymptotically flat spacetimes

In a curved spacetime the above transformations no longer hold as exact symmetries. However, in any asymptotically flat spacetime one can define 'asymptotic symmetries' which correspond to those transformations that are consistent with the boundary conditions of asymptotic flatness. This amounts to the consideration of an 'asymptotic Killing equation' - the solutions to which are known to form a larger group of symmetries, known as the BMS group [64].

This consists of the ordinary Lorentz transformations, plus an infinite number of 'supertranslations' and 'superrotations'. Let us briefly review how these symmetries arise in some detail, before extending this algebra to include also the asymptotic manifestations of conformal symmetry.

Using retarded Bondi coordinates (u, r, x^A), the flat space Minkowski metric is given by,

$$ds^2 = -du^2 - 2du\,dr + r^2\,\gamma_{AB}\,dx^A dx^B\,, \tag{2.8}$$

where γ_{AB} is the unit metric on the two-sphere at infinity. In the Bondi gauge,

$$g_{rr} = g_{rA} = 0, \qquad \partial_r\left(\det\left(\frac{g_{AB}}{r^2}\right)\right) = 0\,. \tag{2.9}$$

In order to maintain this metric asymptotically, any allowed transformations are constrained by a set of boundary conditions. These ensure that any non-zero components of the resulting Riemann tensor have suitable r-dependence as $r \to \infty$, so that the curvature falls off sufficiently fast. The corresponding changes to the metric must therefore obey certain fall-off conditions, given by

$$\begin{aligned}
\delta g_{uA} &\sim \mathcal{O}(r^0)\,,\\
\delta g_{ur} &\sim \mathcal{O}(r^{-2})\,,\\
\delta g_{uu} &\sim \mathcal{O}(r^{-1})\,,\\
\delta g_{AB} &\sim \mathcal{O}(r)\,.
\end{aligned} \tag{2.10}$$

In order to satisfy the Bondi gauge, we also require,

$$\delta g_{rr} = \delta g_{rA} = 0, \qquad \partial_r\left(\det\left(\frac{g_{AB} + \delta g_{AB}}{r^2}\right)\right) = 0\,. \tag{2.11}$$

If peeling holds [101], any asymptotically flat metric can be written as an expansion in powers of $1/r$. In Bondi coordinates near null infinity, this is,

$$\begin{aligned}
ds^2 =& - du^2 - 2du\,dr + r^2\,\gamma_{AB}\,dx^A dx^B\\
&+ 2\frac{m_b}{r}du^2 + r\,C_{AB}dx^A dx^B + D_A\,C^A_B\,du\,dx^B + \dots\,,
\end{aligned} \tag{2.12}$$

where D_A is the covariant derivative with respect to the metric on the two-sphere, and m_b and C_{AB} are functions of (u, x^A) and denote first order corrections to flat space. m_b is the 'Bondi mass aspect' and $\partial_u C_{AB} = N_{AB}$ where N_{AB} is the 'Bondi news'. Capital letters $A, B, ...$ can be raised and lowered with respect to γ_{AB}.

Transformations that preserve these conditions and therefore maintain the structure of the metric correspond to asymptotic solutions to the Killing equation. These are generated by the vector fields,

$$\begin{aligned} \xi_T &\equiv f\,\partial_u + \frac{1}{2}D^2 f\,\partial_r - \frac{1}{r}D^A f\,\partial_A\,, \\ \xi_R &\equiv \frac{1}{2}u\,\psi\;\partial_u - (\frac{1}{2}r\,\psi - \frac{1}{4}u\,D^2\psi)\,\partial_r + (Y^A - \frac{u}{2r}D^A\psi)\partial_A\,, \end{aligned} \tag{2.13}$$

where f is any scalar spherical harmonic, Y^A are conformal Killing vectors on the 2-sphere and $\psi \equiv D_A Y^A$. Further terms that are subleading in r have been neglected. The vectors ξ_T generate infinitesimal 'supertranslations' and the ξ_R give the 'superrotations'. The supertranslations act to shift individual light rays of null infinity forwards or backwards in retarded time. The standard BMS group of infinitesimal transformations preserving the asymptotically flat metric contains only the superrotations that are globally well defined on the sphere. These correspond to supertranslations ξ_T and superrotations ξ_R for which $Y^z = 1, z, z^2$ and its conjugates, when expressed in stereographic coordinates on the two-sphere [87]. More recently, an 'extended BMS' group has been proposed to include all vector fields ξ_R with $Y^z = z^{n+1}$ (and conjugates) for any n [31, 66–68, 92].

There is a similar construction at past null infinity.

3 The conformal BMS symmetry groups

For the conformal case, we look for asymptotic solutions to the conformal Killing equation and ask that the infinitesimal changes in the metric satisfy the same fall-off conditions as above.

The group of solutions involves the ordinary BMS supertranslations (T) and superrotations (R), plus a dilatation (D), another sort of conformal dilatation, a 'BMS dilatation' (E) and a new type of extended special conformal transformation, a 'BMS special conformal transformation' (C). In our coordinates, at leading order, these are given by,

$$
\begin{aligned}
T &\equiv f\,\partial_u + \frac{1}{2}D^2 f\,\partial_r - \frac{1}{r}D^A f\,\partial_A\,,\\
R &\equiv \frac{1}{2}u\psi\,\partial_u - (\frac{1}{2}r\,\psi - \frac{1}{4}u\,D^2\psi)\,\partial_r + (Y^A - \frac{u}{2r}D^A\psi)\,\partial_A\,,\\
D &\equiv u\,\partial_u + r\,\partial_r\,,\\
E &\equiv \frac{u^2}{2}\,\partial_u + r(u+r)\,\partial_r\,,\\
C &\equiv \frac{u^2}{4}\zeta\,\partial_u - \left(\frac{u^2}{4} + \frac{r^2}{2} + \frac{u\,r}{2}\right)\zeta\,\partial_r - \frac{u}{2}\left(1 + \frac{u}{2r}\right)D^A\zeta\,\partial_A\,,
\end{aligned}
\qquad (3.1)
$$

where $\psi \equiv D_A Y^A$, $\zeta \equiv D_A Z^A$ and Y^A and Z^A are conformal Killing vectors on the 2-sphere. Note that while the superrotations may be formed from any conformal Killing vectors, the special conformal transformations however vanish if Z^A is a Killing vector. Therefore C is only formed from the divergence of 'strictly conformal Killing vectors'.

Thus the conformal BMS group is larger than both the conformal group and the BMS group. As well as the infinite number of supertranslations and superrotations, the new special conformal transformation also gives an infinite number of symmetries - generated by the infinity of strictly conformal Killing vectors Z^A. Just as for the superrotations we can define both global and local special conformal transformations. The conformal BMS group described above is the group $CBMS^+$, as it is defined on future null infinity, $\mathscr{J}^+$. Performing a similar calculation on past null infinity, $\mathscr{J}^-$, we can obtain the corresponding (although different) group $CBMS^-$.

It is also worthwhile considering how the original (*i.e.* flat space) conformal group fits into this larger asymptotic group. In flat space, there are four special conformal transformations, given by equation (2.6). When written in (u, r, x^A) coordinates, these are,

$$\begin{aligned} K_u &\equiv u^2\,\partial_u + 2r(u+r)\partial_r\,, \\ K_r &\equiv 2u^2\,\partial_u - u^2\partial_r\,, \\ K_A &\equiv -u(u+2r)\partial_A\,; \end{aligned} \tag{3.2}$$

we can thus identify,

$$K_u = 2E\,, \tag{3.3}$$

and the other components are contained within the superrotation and the new special conformal transformation C, for suitable choice of ψ and ζ.

3.1 The modified bracket

In order to compute the algebra, there is an important subtlety that must be taken into account: it is not the Lie bracket that is required, but a modified version of it (see *e.g.* [92]). This is because the vector fields generate perturbations in the metric and these vector fields are themselves metric-dependent. Thus, in calculating the commutator an extra piece must be added or subtracted from the usual bracket in order to take into account how each vector field varies as the metric changes.

Consider the action of a vector field, ξ_1 on the metric, followed by another vector, ξ_2. We allow metric variations $g_{ab}\to g_{ab}+\hat{h}_{ab}$ which satisfy the fall-off conditions given above and then calculate the possible vector fields, ξ, which can give rise to such variations. Thus these vector fields are defined through,

$$\hat{\mathcal{L}}_{\xi_1} g = \hat{h}\,, \tag{3.4}$$

where the 'conformal' Lie derivative is defined by,

$$(\hat{\mathcal{L}}_\xi g)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - \frac{1}{2} g_{ab} \nabla_c \xi^c \,. \tag{3.5}$$

When the second vector field ξ_2 acts on the metric we allow for additional perturbations:

$$\begin{aligned} \xi_2 &\rightarrow \xi_2 + \mu_2 \,, \\ g + \hat{h} &\rightarrow g + \hat{h} + \hat{K} \,, \end{aligned} \tag{3.6}$$

where μ_2 is a first order perturbation to the vector field and $\hat{K}$ is a second order variation of the metric. We then find the action of $\hat{\mathcal{L}}_{\xi_2} g$ to second order. Explicitly,

$$\begin{aligned} \hat{K}_{ab} = \;& \mu_2^c \, \partial_c g_{ab} + \xi^c \, \partial_c \hat{h}_{ab} + \partial_a \xi^c \hat{h}_{bc} + \partial_a \mu_2^c g_{bc} + \partial_b \mu_2^c g_{ac} + \partial_b \xi^c \hat{h}_{ac} \\ & - \frac{1}{2} g_{ab} \, \partial_c \mu_2^c - \frac{1}{2} \hat{h}_{ab} \, \partial_c \xi^c - \frac{1}{2} g_{ab} \Gamma^c{}_{cd} \mu^d - \frac{1}{2} \hat{h}_{ab} \Gamma^c{}_{cd} \xi^d - \frac{1}{2} g_{ab} \delta \Gamma^c{}_{cd} \xi^d \,, \end{aligned} \tag{3.7}$$

where $\delta\Gamma^c{}_{cd}$ is the perturbation of the connection $\Gamma^c{}_{cd}$ due to the change $g \rightarrow g + \hat{h}$. Asking that the corresponding changes to the metric still satisfy the boundary conditions and the Bondi gauge as above, we may solve for μ_2.

In order to find the commutator, $[\xi_1, \xi_2]$ of two generators we must repeat the process - acting first with ξ_2 and then with ξ_1 and find the corresponding values of μ_1. We can then compute,

$$\delta\mu = \mu_2 - \mu_1 \,, \tag{3.8}$$

which gives the necessary piece that must be subtracted from the ordinary commutator to account for changes to the metric from the vector fields being themselves metric-dependent.

It turns out that the only commutators for which this modification is important are those involving T. In Appendix A we illustrate this modified bracket in the most subtle case - showing that the commutator of two supertranslations, $[T,T]$, vanishes.

3.2 The conformal BMS algebra

In order to get a sense of the general structure of the group, it is useful to look at the elements involved in the commutation relations. The general results take the following overall form,

$$\begin{aligned} [T,R] &\sim \quad T\,, \\ [T,D] &\sim \quad T\,, \\ [R,R] &\sim \quad R\,, \\ [C,D] &\sim \quad C\,, \\ [D,E] &\sim \quad E\,, \\ [E,R] &\sim \quad C\,. \end{aligned} \tag{3.9}$$

We also have that

$$[R,C] \sim E\,, \tag{3.10}$$

except in the special case where the vector, Y^A that generates the superrotations is a Killing vector, *i.e.*, $\psi = 0$, in which case,

$$[R,C] \sim C\,. \tag{3.11}$$

All other commutators vanish:

$$\begin{aligned}
[T,T] &= \ 0\,, \\
[C,T] &= \ 0\,, \\
[C,C] &= \ 0\,, \\
[R,D] &= \ 0\,, \\
[T,E] &= \ 0\,, \\
[C,E] &= \ 0\,, \\
[E,E] &= \ 0\,, \\
[D,D] &= \ 0\,.
\end{aligned} \tag{3.12}$$

One can now compare this algebra with that of the flat space conformal group. The first thing to notice is that the structure is entirely different. In particular, no commutator ever produces a dilatation on the right hand side. In the case of flat space, a special conformal transformation commuted with a translation gives a combination of dilatations and rotations. In this conformal BMS group, the commutation of both C and E with a supertranslation give zero. In addition, when a BMS special conformal transformation is commuted with a superrotation that is generated by a Killing vector, the result is consistent with the flat space version: we get another BMS special conformal transformation. However, when the superrotation is generated by a conformal Killing vector then the commutator gives a different result, a BMS dilatation.

Both the flat space conformal group and the conformal BMS group have a subgroup involving the elements T, R, D and these subgroups have the same structure - as seen in the first three lines of (3.9). The superrotations form their own subgroup, just like the rotations in the flat space group.

Other subgroups of the conformal BMS group can be identified. There is one involving T, D, E, one with E, R, C and one with T, R. There is another involving all elements except for the supertranslations, R, C, D, E. A dilatation with any other element also generates a subgroup.

With this group structure in mind, we can now look at the commutation relations in more detail. The supertranslations are generated by the function f, so we write $T = T(f)$. Similarly, the superrotations and special conformal transformations are generated by vector fields, so we write $R = R(Y^A)$ and $C = C(Z^A)$. Then, more explicitly, the group algebra is given by,

$$
\begin{aligned}
[T(f), D] &= T(f'), \quad f' = f\,, \\
\big[T(f), R(Y^A)\big] &= T(f'), \quad f' = \frac{1}{2} f\,\psi - Y^A D_A f\,, \\
\big[D, C(Z^A)\big] &= C((Z')^A), \quad (Z')^A = Z^A\,, \\
\big[R(Y^A), E\big] &= C((Z')^A), \quad (Z')^A = Y^A\,, \\
\big[R(Y^A), R((Y')^A)\big] &= R((Y'')^A), \quad (Y'')^A = Y^B D_B (Y')^A - (Y')^B D_B Y^A\,.
\end{aligned}
\tag{3.13}
$$

When R is generated by a strict conformal Killing vector,

$$
\big[R(Y^A), C(Z^A)\big] = \frac{1}{4}(\zeta\psi + D^A\zeta D_A\psi)E\,, \tag{3.14}
$$

whereas when R is generated by a Killing vector,

$$
\big[R(Y^A), C(Z^A)\big] = C((Z')^A), \qquad (Z')^A = Y^A\zeta\,. \tag{3.15}
$$

At first sight, when R is generated by a strict CKV it does not look as though the commutator with C gives simply E. However, closer inspection of the prefactor reveals that it is indeed a constant. This requires the following identities that hold for a $2d$ strict CKV:

$$
\begin{aligned}
Y^A &= -\frac{1}{2} D^A \psi\,, \\
D_A D_B \psi &= -\gamma_{AB}\psi\,.
\end{aligned}
\tag{3.16}
$$

Note that since the generators of C must be strict conformal Killing vectors, equation (3.13) shows that if the superrotation involved is generated by a Killing vector, then the commutator vanishes. While equation (3.13) gives a general expression for the commutation of two superrotations, it is worthwhile examining the result for the different cases in which the superrotations are generated by two KVs, two strict CKVs, or one of each. For either two KVs or two strict CKVs, the resulting superrotation generator, $(Y'')^A$ is a KV, but for one KV and one strict CKV, one gets a strict CKV.

We have checked all the Jacobi identities and provide an illustrated example of how these commutation relations are computed according to the modified bracket in Appendix A.

3.3 Discussion

The symmetries of spacetime at asymptotic infinity - especially in the case of asymptotically flat geometry - are of particular interest to the physics of scattering processes. In particular, this is where the S-matrix should be measured. The fact that there are more symmetries at infinity than mere Poincaré is extremely suggestive and the connection between the holomorphically extended BMS group and the recently proposed infinite-dimensional symmetries of soft-particle scattering amplitudes [102] related to soft-theorems [73, 93, 103] may hint at a previously overlooked simplicity in the structure of four dimensional theories involving massless particles.

Because many of the most intriguing results along these lines have been found in the context of the scattering of massless particles, the extension of the BMS group to include spacetimes with conformal symmetry is both natural and important. Continuing this generalisation to the case of conformal theories with maximal supersymmetry is a natural road ahead - with exciting possibility of connecting the new symmetries proposed in [102] with those known to exist in the case of maximally supersymmetric Yang-Mills theory in the planar limit.

Part II

Black Hole Entropy

Part II of this thesis discusses how the Bekenstein-Hawking entropy of different black hole spacetimes may be reproduced by considering a microscopic description. The main goal of this chapter is to show that it is possible to reproduce the macroscopic area-entropy law for generic Kerr black holes.

Many supersymmetric or near-supersymmetric black holes in string theory admit a $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ action of nontrivial or 'large' diffeomorphisms [7,26] (diffeos) whose central charge can be determined by the analysis of Brown and Henneaux [104]. This fact, along with a few modest assumptions, allow one to determine the microscopic entropy of the black hole and reproduce [105] the macroscopic area law [4] without reliance on stringy microphysics.

In the following sections we will provide motivating evidence for the conjecture that the entropy of real-world Kerr black holes can be understood in a manner similar to their mathematically much better understood stringy counterparts.

The large diffeomorphisms in the string theory examples are not ordinarily taken to act on the entire asymptotically flat spacetime. Roughly speaking, the spacetime is divided into two pieces. One piece contains the black hole and the other asymptotically flat piece has an inner boundary surrounding a hole. The large diffeos are taken to act on the black hole. The dividing surface is often taken to be the 'outer boundary' of a decoupled near-horizon AdS_3 region and the large diffeos are taken to act on this region. However, there is some ambiguity in the choice of dividing surface, and with a suitable extension inward, the large diffeos can alternately be viewed as acting on the horizon. Indeed, when the black hole is embedded in an asymptotically flat spacetime there is no clear location to place the outer boundary of the AdS_3 region, and the horizon itself provides a natural dividing surface. Using the covariant phase space formalism [80–86] (see also the recent review [106]) with a surface term reproduces the standard entropy results for BTZ black holes in AdS_3 from an intrinsically horizon viewpoint, albeit with a slight shift in interpretation (see Appendix D). Further comments on this division of the spacetime, and the corresponding split of the Hilbert space into two pieces, appear in section 7.7.

Using the horizon itself as the dividing surface permits the analysis of a more general class of black holes without near-horizon decoupling regions, such as most of those seen in the sky. It was recently shown [27, 28] that supertranslations act non-trivially on a generic black hole, changing both its classical charges and quantum state *i.e.* generating soft hair. However, supertranslations form an abelian group and are clearly inadequate for an inference of the entropy along the lines of the stringy analysis. As emphasized in [27, 28, 38, 47] a richer type of soft hair, as in the stringy examples, associated to nonabelian large diffeos, is needed.

In [78] and as will be explained in this chapter, we considered a more general class of $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ diffeos of a generic spin J Kerr black hole, inspired by the discovery some years ago [79] of a 'hidden conformal symmetry' which acts on solutions of the wave equation in a near-horizon region of *phase space* rather than *spacetime*. In [79] and subsequent work *e.g.* [107–121] the numerological observation was made that, if one assumes the black hole Hilbert space is a unitary two-dimensional CFT with $c_L = c_R = 12J$, the Cardy formula reproduces the entropy. In [78] we brought this enticing numerological observation two steps closer to an actual explanation of the entropy. First we gave precise meaning to the hidden conformal symmetry in the form of an explicit set of $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ vector fields which generate it and moreover act non-trivially on the horizon in the sense that their boundary charges are non-vanishing. Secondly, within the covariant formalism, we found a Wald-Zoupas boundary counterterm which removes certain obstructions to the existence of a well-defined charge and gives $c_L = c_R = 12J$. Assuming the existence of horizon edge states in a unitary Hilbert space transforming under these Virasoro algebras, the area law then follows (as expected) from the Cardy formula.

We do not, however, prove uniqueness of the counterterm, attempt to tackle the difficult problem of characterizing 'all' diffeos which act non-trivially on the black hole horizon, or show that the charges defined are integrable or actually generate the associated symmetries via Dirac brackets. These tasks are left to future investigations. For these reasons, the work of [78] might be regarded as incremental evidence for, but certainly not a demonstration of, the hypothesis that hidden conformal symmetry explains the leading black hole microstate degeneracy.

Previous potentially related attempts to obtain 4D black hole entropy from a Virasoro action at the horizon include [40, 63, 122–128].

The structure of this chapter is as follows. We will first recap the covariant phase space techniques that are used to compute charges, then go over the lessons that we have learnt from the AdS/CFT correspondence, including three-dimensional gravity and the well-known example of the BTZ black hole. We will then extend this approach to explore four dimensional spacetimes, considering the extreme Kerr black hole and then finally the Kerr black hole. In each case we will find the appropriate diffeomorphisms and compute the corresponding charges. We will then find the associated entropy of the field theory using the Cardy formula. The main challenge (and the novel material of this chapter) involves exploring how this may be done for the case of the generic Kerr black hole.

4 Covariant phase space

4.1 A generalised Noether theorem and covariant charges

The construction of covariant charges has a long history including [80–86] and has been reviewed in many places (e.g. [106]). A recent comprehensive discussion, including counterterm ambiguities and also adapted to black hole horizons, can be found in [58].

In a d-dimensional spacetime, consider a Lagrangian density $\mathcal{L}$ that depends on dynamical fields Φ, which include for example the metric and any matter fields.

The dynamical fields Φ are defined on the phase space. Phase space is a symplectic manifold, $\mathcal{M}$, with a symplectic 2-form, Ω, which has the following properties:

- It is skew-symmetric, $\Omega_{AB} = -\Omega_{BA}$,
- It is closed, $\delta\Omega = 0$, where δ is the exterior derivative on the phase space,
- It is non-degenerate, $\Omega_{AB}V^B = 0 \implies V^B = 0$.

A vector field X tangent to the phase space (rather than spacetime) is a symplectic symmetry over the phase space if,

$$\mathcal{L}_X\Omega = 0. \tag{4.1}$$

The dynamical fields obey

$$\delta_1\delta_2\Phi - \delta_2\delta_1\Phi = 0. \tag{4.2}$$

Again, δ is the exterior derivative on the space of field configurations, in contrast to d, the exterior derivative operator on the spacetime. The fields Φ^i solve the equations of motion, and $\delta\Phi^i$ solve the linearised equations of motion around the solution Φ^i.

The action of the theory is defined,

$$\mathcal{S} = \int d^d x \sqrt{-g}\mathcal{L} = \int \mathbf{L}, \tag{4.3}$$

where $\mathbf{L} = *\mathcal{L}$. The equations of motion are obtained by varying the action. In the language of forms, this can be written,

$$\delta\mathcal{S} = \int \delta\mathbf{L} = \int \mathbf{E}_\Phi \delta\Phi + d\boldsymbol{\Theta}, \tag{4.4}$$

where the first term on the right hand side involves summation over all fields Φ^i. The equations of motion for each dynamical field Φ is given by

$$\mathbf{E}_\Phi = 0, \tag{4.5}$$

subject to some boundary conditions.

The second term on the right hand side of (4.4) is the $(d-1)$-form, $\boldsymbol{\Theta}$, the surface term generated by the variation. It is known as the presymplectic potential. It is obtained by acting with Anderson's homotopy operator, $\mathbf{I}^d_{\delta\Phi}$ [129], on the Lagrangian. Anderson's homotopy operator is a fundamental operator which is defined for second order theories by [130],

$$\mathbf{I}^d_{\delta\Phi} = \left(\delta\Phi \frac{\partial}{\partial\Phi_{,\mu}} - \delta\Phi\partial_\nu \frac{\partial}{\Phi_{,\nu\mu}}\right) \frac{\partial}{\partial(dx^\mu)}. \tag{4.6}$$

We write $\boldsymbol{\Theta} = *\theta$, where the vector field $\theta^\mu[\delta\Phi^i, \Phi^i]$ depends on the fields and their variations, but not explicitly on the coordinates. The presymplectic potential gives rise to the presymplectic form, defined as the variation,

$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi] = \delta_1\boldsymbol{\Theta}[\delta_2\Phi, \Phi] - \delta_2\boldsymbol{\Theta}[\delta_1\Phi, \Phi]. \tag{4.7}$$

The Lee-Wald symplectic form is then given by,

$$\Omega[\delta_1\Phi, \delta_2\Phi, \Phi] = \int_\Sigma \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi, \Phi], \tag{4.8}$$

where Σ is a Cauchy surface in the spacetime.

Consider now that the variation of the Lagrangian, $\mathbf{L}$, is made with respect to a spacetime diffeomorphism ζ,

$$\begin{aligned}\delta_\zeta \mathbf{L} &= \mathcal{L}_\zeta \mathbf{L} \\ &= \mathbf{E}_\Phi \delta_\zeta \Phi + d\mathbf{\Theta}[\delta_\zeta \Phi, \Phi],\end{aligned} \tag{4.9}$$

where $\delta_\zeta \Phi = \mathcal{L}_\zeta \Phi$, is the field variation generated by ζ. According to Cartan's "magic formula", $\mathcal{L}_\zeta \mathbf{L} = d(\iota_\zeta \mathbf{L}) + \iota_\zeta d\mathbf{L}$, so $\delta_\zeta \mathbf{L} = \zeta \cdot d\mathbf{L} + d(\zeta \cdot \mathbf{L})$. Since $d\mathbf{L} = 0$, we have

$$\begin{aligned}-\mathbf{E}_\Phi \delta_\zeta \Phi &= d(\mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \zeta \cdot \mathbf{L}) \\ &= d\mathbf{J}_\zeta\end{aligned} \tag{4.10}$$

where $\mathbf{J}_\zeta = \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \zeta \cdot \mathbf{L}$ is the Noether current, a $(d-1)$-form.

When the equations of motion are satisfied, by (4.5) we have $d\mathbf{J} \approx 0$ (where $\approx$ is used to indicate that the equation holds when the equations of motion are satisfied) and thus $\mathbf{J}_\zeta$ is an exact form on the phase space,

$$\mathbf{J}_\zeta \approx d\mathbf{Q}^N_\zeta \tag{4.11}$$

and $\mathbf{Q}^N_\zeta$ is the Noether-Wald charge density, a $(d-2)$-form.

Given (4.9), we can see that,

$$(\delta_1\delta_2 - \delta_2\delta_1)\mathbf{L} = \delta_1 \mathbf{E}_\Phi \delta_2 \Phi + d\delta_1 \mathbf{\Theta}[\delta_2\Phi, \Phi] - \delta_2 \mathbf{E}_\Phi \delta_1 \Phi - d\delta_2 \mathbf{\Theta}[\delta_1 \Phi, \Phi]. \tag{4.12}$$

When the equations of motion are satisfied, $\delta \mathbf{E}_\Phi \approx 0$, and thus,

$$\begin{aligned}(\delta_1\delta_2 - \delta_2\delta_1)\mathbf{L} &\approx d\delta_1 \mathbf{\Theta}[\delta_2\Phi, \Phi] - d\delta_2 \mathbf{\Theta}[\delta_1\Phi, \Phi] \\ &= d\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi, \Phi],\end{aligned} \tag{4.13}$$

using the presymplectic form defined in (4.7). Therefore, since $(\delta_1\delta_2 - \delta_2\delta_1)\mathbf{L} = 0$

using (4.4), we have

$$d\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi, \Phi] \approx 0. \tag{4.14}$$

One consequence of this is that, with one variation generated by a gauge transformation ζ, one can find a $(d-2)$-form $\mathbf{k}_\zeta$ such that

$$\boldsymbol{\omega}[\delta\Phi, \delta_\zeta\Phi, \Phi] \approx d\mathbf{k}_\zeta[\delta\Phi, \Phi], \tag{4.15}$$

where again the fields solve the equations of motion and their variations solve the linearized equations of motion. In terms of the above quantities, the infinitesimal surface charge $\mathbf{k}_\zeta[\delta\Phi, \Phi]$ is given by,

$$\mathbf{k}_\zeta[\delta\Phi, \Phi] \;=\; \delta\mathbf{Q}^N_\zeta[\delta\Phi, \Phi] - \iota_\zeta\boldsymbol{\Theta}[\delta\Phi, \Phi]. \tag{4.16}$$

For a proof of this expression, see Appendix B.

The local variation of charge, $\delta\mathcal{Q}_\zeta$ is then given by the integral of $\boldsymbol{\omega}[\delta\Phi, \delta_\zeta\Phi, \Phi]$ over a Cauchy surface Σ, which, by Stokes theorem, reduces to a boundary integral of $\mathbf{k}_\zeta[\delta\Phi, \Phi]$ over $\partial\Sigma$,

$$\delta\mathcal{Q}_\zeta[\delta\Phi, \Phi] = \int_{\partial\Sigma} \mathbf{k}_\zeta[\delta\Phi, \Phi]. \tag{4.17}$$

One should interpret $\delta\mathcal{Q}_\zeta[\delta\Phi, \Phi]$ as the change in charge between the two solutions Φ^i and $\Phi^i + \delta\Phi^i$, conjugate to the vector field ζ.

4.1.1 Ambiguities

Several ambiguities arise in this formalism in going from the action to the final expression for the charge variation $\delta\mathcal{Q}_\zeta[\delta\Phi, \Phi]$. They have been catalogued by Wald and Zoupas [85], but we will here note how a few of these ambiguities arise.

- The first ambiguity arises from the Lagrangian, to which one may add an exact form, $\mathbf{L} \to \mathbf{L} + d\mathbf{W}$. The effect of this is to shift the Noether current by, $\mathbf{J}_\zeta \to \mathbf{J}_\zeta + d(\zeta \cdot \mathbf{W})$ [131].

- Given (4.4), we see that for a given Lagrangian, there is the freedom to add an exact $(d-1)$-form to the presymplectic potential, $\mathbf{\Theta}$, i.e. $\mathbf{\Theta}[\delta_\zeta\Phi,\Phi] \to \mathbf{\Theta}[\delta_\zeta\Phi,\Phi] + d\mathbf{X}[\delta_\zeta\Phi,\Phi]$. The corresponding effect on the Noether current is $\mathbf{J}_\zeta \to \mathbf{J}_\zeta + d\mathbf{X}[\delta_\zeta\Phi,\Phi]$ [131].

- From (4.11), it can be seen that the Noether-Wald charge density is also ambiguous up to an exact $(d-2)$-form, $\mathrm{d}\mathbf{Y}$, $\mathbf{Q}^N_\zeta \to \mathbf{Q}^N_\zeta + d\mathbf{Y}$. In this case, by (4.11), we find that there is no effect on the Noether current.

- Since the presymplectic form is given by the exterior derivative of the surface charge $\mathbf{k}_\zeta[\delta\Phi,\Phi]$ in (4.15), we see that the addition of an exact form to this surface charge, $\mathbf{k}_\zeta \to \mathbf{k}_\zeta + d\mathbf{Z}$, would leave $\boldsymbol{\omega}[\delta\Phi,\delta_\zeta\Phi,\Phi]$ unaffected.

4.1.2 Integrability

The local charge variation, $\delta\mathcal{Q}_\zeta[\delta\Phi,\Phi]$ is not necessarily integrable - it may not be an exact differential on the phase space. Despite the notation δ, there may not exist a charge $\mathcal{Q}_\zeta$ that can be built from $\delta\mathcal{Q}_\zeta$ [2]. Formally, the necessary criterion for integrability is that the surface charge satisfy,

$$\delta_1 \int_{\partial\Sigma} \mathbf{k}_\zeta[\delta_2\Phi,\Phi] - \delta_2 \int_{\partial\Sigma} \mathbf{k}_\zeta[\delta_1\Phi,\Phi] \;=\; 0, \tag{4.18}$$

for all variations $\delta_{1,2}\Phi$ belonging to the tangent bundle of $\mathcal{M}$. In other words, if there exists a charge $\mathcal{Q}_\zeta$, then it obeys,

$$(\delta_1\delta_2 - \delta_2\delta_1)\mathcal{Q}_\zeta \;=\; 0. \tag{4.19}$$

This condition may also be written [131]

$$\int_{\partial\Sigma} \zeta \cdot \boldsymbol{\omega}[\delta_1\Phi,\delta_2\Phi,\Phi] \approx 0. \tag{4.20}$$

[2] It is for this reason that this notation may be misleading, and a more appropriate way of writing the charge (which is often used elsewhere in the literature) might be $\delta\!\!\!/\mathcal{Q}_\zeta[\delta\Phi,\Phi]$, to emphasise the potential for non-integrability.

One can most easily see the problem of integrability in the charge variation by examining the formula for the surface charge integrand in terms of the Noether charge and presymplectic potential as in (4.16). Here, the variation of the Noether charge, $\delta\mathbf{Q}^N_\zeta$ is manifestly an exact differential since this is a genuine variation. However, the second term on the right hand side of (4.16) may not itself be integrable. In cases where this term is not integrable, one may exploit the ambiguities noted above to find a suitable counterterm that may be added to render the total charge variation integrable. Several examples of such counterterms have been proposed in the literature, and discussed for example in [58, 83–86], but in general these have been produced on a case-by-case basis. The task of finding a universal counterterm which is compatible with the allowed ambiguities while simultaneously making the total charge variation integrable is, in general, an ongoing problem. In section 7.5 and Appendix C we will explore the case of the Kerr black hole and propose a counterterm for this scenario.

4.1.3 Conservation

Under the assumption of integrability, one can construct the surface charge $\mathcal{Q}_\zeta$ from $\delta\mathcal{Q}_\zeta$ and find that it is conserved if

$$\boldsymbol{\omega}[\delta\Phi, \delta_\zeta\Phi, \Phi] \approx 0, \tag{4.21}$$

where again the fields Φ^i solve the equations of motion and the $\delta\Phi^i$ solve the linearised equations of motion. This naturally means that $\delta\mathcal{Q}_\zeta$ will not depend on the choice $\partial\Sigma$, or more generally that the symplectic form Ω is not dependent on the choice of surface Σ.

4.1.4 Charge algebra

Any two diffeomorphisms ζ, χ form an algebra under the Lie bracket,

$$[L_\zeta, L_\chi] = L_{[\zeta,\chi]}. \tag{4.22}$$

The same should be true of the charges defined with respect to these vector fields. We will soon see that explicit calculations reveal the possibility that we have a central term in the algebra of the charges.

We will consider the case of asymptotic symmetries - those which act non-trivially at infinity and that obey a set of boundary conditions. We will assume that we can choose boundary conditions such that any ‘allowed’ vector field ζ asymptotically solves the Killing equation. If the resulting charge variation, $\delta\mathcal{Q}_\zeta$ is zero, the diffeomorphism ζ is simply a gauge or coordinate transformation and is called trivial. The asymptotic symmetry group is defined as the group of diffeomorphisms which physically change the state of the system and will result in a non-zero charge, $\delta\mathcal{Q}_\zeta$.

$$\text{Asymptotic symmetry group} = \frac{\text{Allowed diffeomorphisms}}{\text{Trivial gauge transformations}}. \tag{4.23}$$

We will now find the algebra of the charges. As in [106], we can pick some reference set of fields, $\bar{\Phi}^i$ and consider the path integral along some curve γ in the phase space, from $\bar{\Phi}^i$ to the field configuration Φ^i. The charge $\mathcal{Q}_\zeta$ (assuming it exists by integrability of $\delta\mathcal{Q}_\zeta$) is then defined as

$$\mathcal{Q}_\zeta[\Phi,\bar{\Phi}] = \int_\gamma\int_{\partial\Sigma}\mathbf{k}_\zeta[\delta\Phi,\Phi] + N_\zeta[\bar{\Phi}]. \tag{4.24}$$

Here $N_\zeta[\bar{\Phi}]$ is the charge associated to the reference field configuration, which may include for example, any counterterms that are required so that $\mathcal{Q}_\zeta$ may be integrable. At this stage, $N_\zeta[\bar{\Phi}]$ is dependent on the theory in question and a universal prescription is unknown.

One can now consider a Lie bracket for any two conserved charges $\mathcal{Q}_\zeta$ and $\mathcal{Q}_\chi$ associated to two diffeomorphisms ζ and χ. The bracket is defined [106],

$$\{\mathcal{Q}_\zeta,\mathcal{Q}_\chi\} = \delta_\zeta\mathcal{Q}_\chi = \int_{\partial\Sigma}\mathbf{k}_\chi[\delta_\zeta\Phi,\Phi]. \tag{4.25}$$

The algebra of charges is then isomorphic to the Lie algebra of diffeomorphisms up to a possible central extension,

$$\{\mathcal{Q}_\zeta, \mathcal{Q}_\chi\} = \mathcal{Q}_{[\chi,\zeta]} + K_{\chi,\zeta}[\bar{\Phi}], \tag{4.26}$$

where

$$K_{\chi,\zeta}[\bar{\Phi}] = \int_{\partial\Sigma} \mathbf{k}_\chi[\delta_\zeta \bar{\Phi}, \bar{\Phi}] - N_{[\zeta,\chi]}[\bar{\Phi}]. \tag{4.27}$$

$K_{\chi,\zeta}[\bar{\Phi}]$ depends only on the reference field configuration, $\bar{\Phi}$ and not on $\delta\Phi$ and furthermore it is a conserved quantity, just like $\mathcal{Q}_\zeta$. This means that it is constant on the phase space and therefore commutes with all charges $\mathcal{Q}_\zeta$, i.e. it is indeed a central term. Later we will see that if a central term is computed that is not constant on the phase space, one must conclude that there will be an obstruction to constructing and integrating well-defined charges with a well-defined algebra, the existence of which has been assumed in arriving at (4.26). Such an obstruction must be removed by again exploiting the ambiguities introduced in deriving the charges in order to add extra terms.

When $K_{\chi,\zeta}[\bar{\Phi}] = 0$, (4.26) expresses general coordinate invariance. However, if the central term does not vanish we see a violation of diffeomorphism symmetry, which is physically unacceptable. It is like an anomaly in many field theories, and must be cancelled by something. We will explore this further in later sections.

4.2 Covariant charges for gravity

The starting point is the Einstein-Hilbert Lagrangian four-form,

$$\mathbf{L} = \frac{\sqrt{-g}}{16\pi} R. \tag{4.28}$$

The only dynamical field, Φ^i, in this configuration is the metric tensor g_{ab} with variation $\delta g^{ab} = h^{ab}$. [3] The variation of the Lagrangian is

$$\delta\mathbf{L} = -\frac{\sqrt{-g}}{16\pi} G_{ab} h^{ab} + d\mathbf{\Theta}[h, g]. \tag{4.29}$$

[3] Note that in this case we choose to vary the metric with respect to indices upstairs.

The presymplectic potential $\mathbf{\Theta} = *\theta$ is a spacetime three-form, where

$$\theta^a[h, g] = -\frac{1}{16\pi}(\nabla_b h^{ab} - \nabla^a h). \tag{4.30}$$

The infinite-dimensional phase space of general relativity has as its tangent vectors the infinitesimal metric perturbations h^{ab} that obey the linearised Einstein equations. Although $\mathbf{\Theta}$ is a three-form in spacetime, it is also a one-form in the phase space. The presymplectic form,

$$\boldsymbol{\omega}[h_1, h_2, g] = \delta_{h_1}\mathbf{\Theta}[h_2, g] - \delta_{h_2}\mathbf{\Theta}[h_1, g] \tag{4.31}$$

obeys $d\boldsymbol{\omega} = 0$ and can therefore be used to define a conserved inner product. $\boldsymbol{\omega}$ is a two-form in the phase space. The linearized charge $\delta\mathcal{Q}$ is then obtained from the presymplectic form $\boldsymbol{\omega}(h_1, h_2; g)$ by the replacement of h_1 with the perturbation generated by a large diffeomorphism $\mathcal{L}_\zeta g$,

$$\delta\mathcal{Q}(\zeta, h; g) = \int_{\Sigma_3} \boldsymbol{\omega}(\mathcal{L}_\zeta g, h; g). \tag{4.32}$$

Assuming the background field equations are satisfied, the presymplectic form is exact and thus reduces to a boundary integral, giving rise to the Iyer-Wald charge,

$$\delta\mathcal{Q}_{IW}(\zeta, h; g) = \frac{1}{16\pi}\int_{\partial\Sigma_3} \mathbf{k}_{IW}, \tag{4.33}$$

as it must in order for diffeomorphisms which vanish on the boundary to have $\delta\mathcal{Q}_{IW} = 0$. Explicitly, writing $\mathbf{k}_{IW} = *F_{IW}$,

$$(F_{IW})_{ab} = \frac{1}{2}\nabla_a\zeta_b h + \nabla_a h^c{}_b\zeta_c + \nabla_c\zeta_a\ h^c{}_b + \nabla_c h^c{}_a\ \zeta_b - \nabla_a h\ \zeta_b - (a \leftrightarrow b), \tag{4.34}$$

where $h = h^a_a$.

As before, it is possible to arrive at this expression by considering the Noether charge. The presymplectic potential (4.30) is a function of the metric variation h^{ab}. When this is due to a diffeomorphism ζ, i.e., $h^{ab} = \mathcal{L}_\zeta g^{ab}$, the presymplectic potential becomes,

$$\theta^a[\mathcal{L}_\zeta g, g] = -\frac{1}{8\pi}(\nabla_b \nabla^{(a}\zeta^{b)} - \nabla^a \nabla_b \zeta^b). \tag{4.35}$$

When the equations of motion are satisfied, the presymplectic potential may be written

$$\theta^a[\mathcal{L}_\zeta g, g] \approx \frac{-1}{16\pi}\nabla_b(\nabla^b \zeta^a - \nabla^a \zeta^b). \tag{4.36}$$

Using

$$d\mathbf{Q}^N_\zeta \approx \mathbf{\Theta} - \zeta \cdot \mathbf{L}, \tag{4.37}$$

this gives rise to a Noether charge [106],

$$(Q^N_\zeta)^{ab} = \frac{-1}{16\pi}(\nabla^b \zeta^a - \nabla^a \zeta^b), \tag{4.38}$$

where $\mathbf{Q}^N_\zeta = *Q^N_\zeta$. The symplectic two-form charge integrand can then be found from

$$\mathbf{k}[h, g] = \delta \mathbf{Q}^N_\zeta[g] - \iota_\zeta \mathbf{\Theta}[h, g], \tag{4.39}$$

which again after some algebra gives rise to the Iyer-Wald surface charge, $\mathbf{k}_{IW}$, given by the explicit expression (4.34).

We can examine the explicit forms of each term in expression (4.39) to get a better understanding of the integrability of $\mathbf{k}[h, g]$. Of course, the Noether term is integrable, but we can see from a second variation of $\mathbf{\Theta}[h, g]$ using (4.30), that the second term on the right hand side is not, in general, integrable. Writing the second variation as $\delta g^{ab} = k^{ab}$ and fixing the gauge such that $h = k = 0$, under antisymmetry of $k \leftrightarrow h$, we get

$$(\delta_2\theta_1 - \delta_1\theta_2)^a = \frac{1}{2}h^{bc}(\nabla^a k_{bc} - \nabla_b k^a_c - \nabla_c k^a_b) - (k \leftrightarrow h). \tag{4.40}$$

Thus it is clear that $\mathbf{\Theta}$ is not in general integrable. The full calculation can be found in Appendix C.

In terms of the local charge variation, $\delta\mathcal{Q}$, Wald and Zoupas [85] noted an ambiguity in the addition of a possible counterterm $\delta\mathcal{Q}_X$ of the general form

$$\delta\mathcal{Q}_X = \frac{1}{16\pi}\int_{\partial\Sigma} \iota_\zeta(*X), \tag{4.41}$$

where X is a spacetime one-form constructed from the geometry and linear in h.[4] X is not a priori fully determined by the considerations of [84, 85], where its precise form is left as an ambiguity. Ultimately one hopes it is fixed by consistency conditions such as integrability and the demand that the charges generate the symmetry via a Dirac bracket as in [28], or in the quantum form by action on a Hilbert space. In practice, the determination of X has been made on a case-by-case basis.

The resulting general form for the linearized charge associated to a diffeomorphism ζ on a surface Σ with boundary $\partial\Sigma$ is [85]

$$\delta\mathcal{Q} = \delta\mathcal{Q}_{IW} + \delta\mathcal{Q}_X. \tag{4.42}$$

The interpretation of $\delta\mathcal{Q}$ is the difference in the charge conjugate to ζ between the configuration g_{ab} and $g_{ab} - h_{ab}$.[5] This is the crucial expression that we must use in order to compute the charge for the Kerr black hole. We will later see how to construct a counterterm $\delta\mathcal{Q}_X$ to arrive at a total charge $\delta\mathcal{Q}$.

4.3 Using ambiguities to simplify the expressions

As derived in section 4.2, the Iyer-Wald charge is given by,

$$\delta\mathcal{Q}_{IW}(\zeta, h; g) = \frac{1}{16\pi}\int_{\partial\Sigma_3} \mathbf{k}_{IW}, \tag{4.43}$$

where explicitly, with $F_{IW} = *k_{IW}$,

$$(F_{IW})_{ab} = \frac{1}{2}\nabla_a\zeta_b h + \nabla_a h^c{}_b\zeta_c + \nabla_c\zeta_a\ h^c{}_b + \nabla_c h^c{}_a\ \zeta_b - \nabla_a h\ \zeta_b - (a \leftrightarrow b). \tag{4.44}$$

[4] $*X$ is often denoted Θ.

[5] The minus sign here arises because the variation of the metric is defined with respect to indices upstairs.

This is the situation when one variation is generated by a diffeomorphism ζ.

It is now worthwhile considering the situation in which both variations, h_1 and h_2 of the presymplectic form are generated by diffeomorphisms ζ and $\tilde{\zeta}$, i.e. in the above expression we set $h = \mathcal{L}_{\tilde{\zeta}} g$. Writing $D = \nabla_a \zeta^a$, $\tilde{D} = \nabla_a \tilde{\zeta}^a$, the surface charge integrand becomes,

$$\begin{aligned}(F)_{ab} = \tilde{D}\nabla_a\zeta_b + \zeta_c\nabla_a\nabla^c\tilde{\zeta}_b + \zeta_c\nabla_a\nabla_b\tilde{\zeta}^c + \nabla_c\zeta_a\nabla^c\tilde{\zeta}_b \\ +\nabla_c\zeta_a\nabla_b\tilde{\zeta}^c + \zeta_a\nabla_b\tilde{D} - \zeta_a\Box\tilde{\zeta}_b - (a \leftrightarrow b).\end{aligned} \tag{4.45}$$

Given the ambiguities described above, we have the freedom to add to this a term of the form, $\nabla^c A_{abc}$, where A is any antisymmetric three-form.

Consider,

$$A_{abc} = \zeta_b\nabla_a\tilde{\zeta}_c + \zeta_a\nabla_c\tilde{\zeta}_b + \zeta_c\nabla_b\tilde{\zeta}_a - (a \leftrightarrow b). \tag{4.46}$$

Then,

$$\begin{aligned}\nabla^c A_{abc} = \nabla^c\zeta_b\nabla_a\tilde{\zeta}_c + \nabla^c\zeta_a\nabla_c\tilde{\zeta}_b + \nabla_c\nabla^c\nabla_b\tilde{\zeta}_a + \zeta_a\Box\tilde{\zeta}_b \\ +\zeta_b\nabla_a\tilde{D} - D\nabla_a\tilde{\zeta}_b - (a \leftrightarrow b).\end{aligned} \tag{4.47}$$

Now adding this divergence of this three-form to the original surface charge we get,

$$\begin{aligned}(F')_{ab} &= (F)_{ab} + \nabla^c A_{abc} \\ &= 2\zeta_c[\nabla_a, \nabla^c]\tilde{\zeta}_b + 2\nabla_c\zeta_a\nabla^c\tilde{\zeta}_b - D\nabla_a\tilde{\zeta}_b + \tilde{D}\nabla_a\zeta_b - (a \leftrightarrow b) \\ &= R_{abcd}\zeta^c\tilde{\zeta}^d + 2\nabla_c\zeta_a\nabla^c\tilde{\zeta}_b - D\nabla_a\tilde{\zeta}_b + \tilde{D}\nabla_a\zeta_b - (a \leftrightarrow b).\end{aligned} \tag{4.48}$$

This new expression for the surface charge is much simpler than the original, (4.45). It is also now manifestly antisymmetric under $(\zeta \leftrightarrow \tilde{\zeta})$ and only involves single derivatives of the vector field. When the vector field is divergence-free, we are left with just two terms. For the simple vector fields considered later on, this new expression will prove very useful.

5 The AdS/CFT correspondence

Holography is the deep connection between two seemingly distinct theories: a gravitational theory in $d+1$ dimensions and a quantum theory without gravity in d dimensions. This profound duality was first established by 't Hooft and Susskind in the 1990s [132, 133]. Since then, the subject has had far reaching applications, firstly for black holes, for which a huge volume of work has been produced, reviewed for example in [134], but also spanning numerous other fields, including condensed matter [135–137], atomic [138, 139] and nuclear [140–145] physics.

The first and most concrete example of holography is the AdS/CFT-correspondence: an equivalence between a theory of gravity in Anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) without gravity in one dimension lower [146–148]. The theory of gravity exists in the bulk of the spacetime and the CFT lives on the conformal boundary of the AdS spacetime. This correspondence was first investigated in the context of string theory, for the case of $\mathrm{AdS}_5 \times \mathrm{S}^5$ dual to a four dimensional maximally supersymmetric Yang-Mills theory [146].

The equivalence of the two different theories means that the same physics can be described by two utterly different descriptions. This means that in the case of black holes, we have the potential to describe the non-perturbative aspects of gravity via a field theoretic model, a tool which has proven to be very powerful.

In this section we will consider the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$. The use of this correspondence by Brown and Henneaux [104] actually predated the discovery of holography [149]. In this work, it was shown that the asymptotic boundary of AdS spacetime contains two copies of the infinite dimensional Virasoro algebra. Brown and Henneaux computed the corresponding central charges which were then used to calculate the entropy of the CFT. It was later shown by Strominger that this entropy is exactly that of the three dimensional black hole in the dual theory, as given by the Bekenstein-Hawking entropy law (1.2). This remarkable fact has suggested a tantalising possibility for a microscopic description of black hole entropy, which could be computed without reliance on string theory. Indeed, this idea has been extended to the case of the extreme Kerr black hole, in the Kerr/CFT correspondence [127]. The main goal of this chapter is to describe how we may use the lessons learned from AdS/CFT to calculate the entropy of the generic Kerr black hole in terms of microstates on the horizon. Before doing so, however, we will recap the methods used in the original Brown and Henneaux AdS_3/CFT_2 case. We will begin by going over the basics of conformal field theory, then explore the symmetries and properties of three-dimensional gravity, before linking the two together via the AdS/CFT correspondence.

5.1 Review of conformal field theory and the Virasoro algebra

We would like to investigate why the symmetries of AdS_3 and a two-dimensional CFT suggest a duality between the two theories. We will begin with a brief review of the basics of two dimensional conformal field theory. Much of this was first described by Belavin, Polyakov and Zamalodchikov in 1984 [150]. For a fuller review of the subject, see e.g. [151]. Here we will follow the outline given in [152].

A conformal field theory is a field theory that is invariant under conformal transformations - those in which under a change of coordinates $x \to \tilde{x}$, the metric changes by

$$g_{ab}(x) \to g_{ab}(\tilde{x}) = \Omega^2(x) g_{ab}(x), \tag{5.1}$$

as explained in the first chapter.

We will here consider only two-dimensional CFTs. In 2D Euclidean spacetime with coordinates (x^1, x^2), consider the complex coordinates,

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2. \tag{5.2}$$

The holomorphic functions are commonly referred to as 'left-moving', while the anti-holomorphic functions are often called 'right-moving'. If the space is flat, then the metric is,

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dzd\bar{z}. \tag{5.3}$$

In two dimensions, there are an infinite number of conformal transformations given by any holomorphic coordinate transformation,

$$z \to z' = f(z), \qquad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}). \tag{5.4}$$

A special case is a translation, $z \to z + a$. The stress-energy tensor of a CFT, T_{ab} is formed from the conserved currents associated to translational invariance. The stress-energy tensor is conserved, $\nabla^a T_{ab} = 0$ and is also traceless at the classical level, $T^a_a = 0$. In the complex coordinates, these conditions mean that $T_{zz} = T(z)$ is a holomorphic function and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$ is an anti-holomorphic function. Under a finite conformal transformation $z \to \tilde{z}(z)$, the stress-energy tensor transforms as,

$$T(z) \to \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T(z) - \frac{c_L}{12} S(\tilde{z}, z)\right], \tag{5.5}$$

where c_L is a constant, also known as the central charge, and $S(\tilde{z}, z)$ is the Schwarzian derivative, defined as

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right) \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}. \tag{5.6}$$

A similar formula applies to the anti-holomorphic part of the stress-energy tensor, with corresponding central charge, c_R.

We will now consider the case of a cylinder, parameterized by the complex coordinate ω [152],

$$\omega = \sigma + i\tau, \qquad \sigma \in [0, 2\pi). \tag{5.7}$$

The cylinder is related to the plane by a conformal transformation,

$$z = e^{-i\omega}. \tag{5.8}$$

On the cylinder, states evolve in time with the Hamiltonian operator,

$$H = \partial_\tau. \tag{5.9}$$

Constant time slices on the cylinder correspond to circles on the plane of constant radius and states evolve radially, governed by the dilatation operator,

$$D = z\partial_z + \bar{z}\partial_{\bar{z}}. \tag{5.10}$$

In moving from one geometry to the other, the stress-energy tensor transforms as

$$T_{cyl}(\omega) = -z^2 T_{plane}(z) + \frac{c_L}{24} \tag{5.11}$$

using (5.6). The Hamiltonian energy is defined,

$$H = \int d\sigma T_{\tau\tau} = -\int d\sigma (T_{\omega\omega} + \bar{T}_{\bar{\omega}\bar{\omega}}). \tag{5.12}$$

If the ground state energy of the theory on the plane is zero, then on the cylinder the ground state energy is the Casimir energy,

$$E = -2\pi \frac{c_L + c_R}{24}. \tag{5.13}$$

We can write the stress tensor on the cylinder as a Fourier expansion,

$$T_{cyl}(\omega) = -\sum_{m=-\infty}^{\infty} L_m e^{im\omega} + \frac{c_L}{24}, \tag{5.14}$$

which after the transformation (5.11) onto the plane, becomes the Laurent expansion,

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \tag{5.15}$$

and similarly for the anti-holomorphic part. Writing this the other way around to find L_m as a contour integral of $T(z)$, we have

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z), \qquad \tilde{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}). \tag{5.16}$$

On the plane, these are the charges associated to the conformal transformations given by $\delta z = z^{n+1}$, $\delta \bar{z} = \bar{z}^{n+1}$, with Noether currents $J(z) = z^{n+1}T(z)$, $\bar{J}(\bar{z}) = \bar{z}^{n+1}\bar{T}(\bar{z})$. These charges are conserved under radial evolution on the plane, or under time evolution on the cylinder.

In the quantum theory, the $L_n, \tilde{L}_n$ become operators for the generation of the conformal transformations $\delta z = z^{n+1}, \delta \bar{z} = \bar{z}^{n+1}$. These generators satisfy a *Virasoro algebra*, given by

$$\begin{aligned}
\big[L_n, L_m\big] &= (m-n)L_{m+n} + \frac{c_L}{12} m(m^2-1)\delta_{m+n,0}, \\
\big[\tilde{L}_n, \tilde{L}_m\big] &= (m-n)\tilde{L}_{m+n} + \frac{c_R}{12} m(m^2-1)\delta_{m+n,0}, \\
\big[L_n, \tilde{L}_m\big] &= 0
\end{aligned} \tag{5.17}$$

where c_L, c_R are the central charges of the algebra. The terms linear in m arise from the Casimir energy (5.13) on the cylinder. Each copy of the Virasoro algebra contains within it an $SL(2,\mathbb{R})$ subalgebra, spanned by the components $\{L_{-1}, L_0, L_1\}$ and $\{\tilde{L}_{-1}, \tilde{L}_0, \tilde{L}_1\}$.

Representations of the Virasoro algebra classify the states of a two dimensional conformal field theory. The emergence of this algebra as the symmetry algebra in other contexts will provide a fundamental link to conformal field theory.

5.2 Three-dimensional gravity

The classical action for (2+1)-dimensional gravity is given by the Einstein-Hilbert action,

$$I_{EH} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g}(R - 2\Lambda), \tag{5.18}$$

where G_N is the Newton constant in three dimensions and Λ is the cosmological constant. The resulting vacuum Einstein equations read

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0, \tag{5.19}$$

and solutions have constant curvature. The Riemann tensor can be written in terms of a trace-part (made up of the Ricci tensor) and a traceless conformal invariant part known as the Weyl tensor. In three dimensions, the Weyl tensor vanishes and the Riemann tensor becomes,

$$R_{abcd} = \Lambda(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{5.20}$$

Since this expression involves no derivatives of the metric, the curvature is locally completely defined and does not vary in any dynamical way. This means that there are no propagating degrees of freedom in three dimensions, and thus there are no gravitational waves. Even though three-dimensional gravity has no local degrees of freedom, there may be global degrees of freedom. For example, three-dimensional spacetimes may have different topologies while remaining locally indistinguishable. We will explore this idea further in the following sections.

5.2.1 AdS_3 spacetimes

Three dimensional Anti-de-Sitter spacetime (AdS_3) is the maximally symmetric solution of Einstein's equations in three dimensions with constant negative curvature.

The metric of AdS_3 is given by,

$$g_{ab}dX^a dX^b = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2, \tag{5.21}$$

which is the induced metric on the hyperboloid,

$$X_a X^a = -\ell^2 \tag{5.22}$$

where the index $a = 0, 1, 2, 3$. The length scale ℓ is a measure of the curvature and the cosmological constant is $\Lambda = -1/\ell^2$. Setting,

$$\begin{aligned} X_0 &= \ell \cosh\rho \cos t, \\ X_1 &= \ell \sinh\rho \sin\phi, \\ X_2 &= \ell \sinh\rho \cos\phi, \\ X_3 &= \ell \cosh\rho \sin t, \end{aligned} \tag{5.23}$$

the metric becomes,

$$ds^2 = \ell^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2). \tag{5.24}$$

This is the AdS_3 metric in global coordinates. Note that although (5.23) defines the coordinate t with period 2π, the AdS_3 spacetime is considered to be given by the metric (5.24) above with $t \in (-\infty, +\infty)$.

AdS$_3$ has six Killing vectors, given by,

$$
\begin{aligned}
\zeta_{-1} &= \frac{1}{2}[\tanh\rho e^{-i(t+\phi)}\partial_t + \coth\rho e^{-i(t+\phi)}\partial_\phi + ie^{-i(t+\phi)}\partial_\rho], \\
\zeta_0 &= \frac{1}{2}[\partial_t + \partial_\phi], \\
\zeta_1 &= \frac{1}{2}[\tanh\rho e^{i(t+\phi)}\partial_t + \coth\rho e^{i(t+\phi)}\partial_\phi - ie^{i(t+\phi)}\partial_\rho], \\
\bar{\zeta}_{-1} &= \frac{1}{2}[\tanh\rho e^{-i(t-\phi)}\partial_t - \coth\rho e^{-i(t-\phi)}\partial_\phi + ie^{-i(t-\phi)}\partial_\rho], \\
\bar{\zeta}_0 &= \frac{1}{2}[\partial_t - \partial_\phi], \\
\bar{\zeta}_{-1} &= \frac{1}{2}[\tanh\rho e^{i(t-\phi)}\partial_t - \coth\rho e^{i(t-\phi)}\partial_\phi - ie^{i(t-\phi)}\partial_\rho].
\end{aligned}
\tag{5.25}
$$

The exponent, $(t+\phi)$ means we can identify the Killing vectors $\zeta_{-1,0,+1}$ with 'left-moving' vector fields, and similarly the $\bar{\zeta}_{-1,0,+1}$ with 'right-moving' vector fields.

The left-moving and right-moving vector fields commute with each other and each form a copy of the $SL(2)$ algebra,

$$i[\zeta_1, \zeta_{-1}] = 2\zeta_0, \quad i[\zeta_1, \zeta_0] = \zeta_1, \quad i[\zeta_{-1}, \zeta_0] = -\zeta_{-1}. \tag{5.26}$$

This means that we are left with the algebra,

$$SL(2,\mathbb{R})_L \oplus SL(2,\mathbb{R})_R. \tag{5.27}$$

Thus global AdS$_3$ has an $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ group of isometries. This is the two-dimensional global conformal group on the plane or cylinder, a feature that will become important in identifying a possible conformal symmetry.

The AdS$_3$ metric is conformal to the metric on the cylinder. This can be seen by making the coordinate transformation $\sinh\rho = \tan\chi$, with $\chi \in [0, \pi/2]$. The resulting metric is

$$ds^2 = \frac{\ell^2}{\cos^2\chi}(-dt^2 + d\chi^2 + \sin^2\chi d\phi^2) \tag{5.28}$$

Ignoring the prefactor, this is just the metric on the infinite solid cylinder, with radius $\chi = \pi/2$. This cylinder defines the conformal boundary of the global AdS spacetime. In terms of the AdS/CFT-correspondence, the dual conformal field theory will live on this conformal boundary.

For future reference, we will also define another useful set of coordinates: *Poincaré coordinates*. These are found by taking the metric (5.24) and making the following coordinate transformations [153]:

$$\begin{aligned} \frac{1}{z} &= \cosh\rho\cos t + \sinh\rho\cos\phi, \\ \tau &= z\cosh\rho\sin t, \\ x &= z\sinh\rho\sin\phi. \end{aligned} \tag{5.29}$$

The resulting Poincaré metric is

$$ds^2 = \frac{\ell^2}{z^2}(-d\tau^2 + dx^2 + dz^2) \tag{5.30}$$

The coordinates $\tau, x \in (-\infty, +\infty)$. The z coordinate behaves as a radial coordinate. The two regions $z < 0$ and $z > 0$ each correspond to one half of the hyperboloid (5.22). The 'Poincaré patch' is usually taken to be the region with $z > 0$. Both regions only cover part of the spacetime.

Writing $u = 1/z$, the metric (5.30) can be written,

$$ds^2 = \ell^2\left(\frac{du^2}{u^2} + u^2(-d\tau^2 + dx^2)\right). \tag{5.31}$$

Both forms of this Poincaré metric will become useful in later sections.

5.2.2 The BTZ black hole

To study black hole solutions, we consider a spacetime that is *asymptotically* AdS_3 - one that approaches AdS_3 at spatial infinity. Here we explore one particular solution, describing the BTZ black hole, originally found by Bañados, Teitelboim and Zanelli [154]. For a fuller review of the subject, see [155]. The metric of the

BTZ black hole is given by,

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2} + r^2(d\phi - N^\phi(r)dt)^2, \tag{5.32}$$

where the lapse function is given by

$$N^2(r) = -8M + \frac{r^2}{\ell^2} + \frac{16J^2}{r^2}, \tag{5.33}$$

and the angular 'dragging' is

$$N^\phi(r) = -\frac{4J}{r^2}. \tag{5.34}$$

The metric admits two Killing vectors ∂_t and ∂_ϕ. If one evaluates the surface charges with respect to these vector fields on the unit circle at infinity, one recovers the total mass and angular momentum of the black hole, the parameters M and J in the lapse function [106],

$$\begin{aligned} \int_{S^1_\infty} \mathbf{k}_{\partial_t}[\delta g, g] &= \delta M, \\ \int_{S^1_\infty} \mathbf{k}_{\partial_\phi}[\delta g, g] &= \delta J. \end{aligned} \tag{5.35}$$

The scalar curvature is $R = -\frac{6}{\ell^2}$ everywhere and thus the metric does not admit a curvature singularity - it is a black hole because one can identify event horizons. Writing the lapse function as,

$$N(r) = \sqrt{\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}}, \tag{5.36}$$

one now sees that, analogously to the Kerr metric, there are inner and outer horizons, the surfaces $\mathcal{H}^\pm$, defined by $r = r_-$ and $r = r_+$ respectively. One can compute the surface gravity, κ, of the future horizon and then the Hawking temperature,

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}. \tag{5.37}$$

The angular velocity of the horizon, Ω_H is given by the angular dragging function,

$$\Omega_H = -N^\phi(r_+) = \frac{4J}{r_+^2} = \frac{r_-}{r_+\ell}. \tag{5.38}$$

Using the second law of thermodynamics one finds that the entropy of the BTZ black hole is,

$$S_{BTZ} = \frac{\pi r_+}{2}. \tag{5.39}$$

It is equal to a quarter of the perimeter of the horizon, the three-dimensional analogue of the Bekenstein-Hawking entropy-area law (1.2). We will soon see that this is also the thermodynamic entropy in the CFT. This relationship was first shown by Strominger in [105].

It is interesting to consider solutions with different values of the mass M. When $M = 0$, we have the massless BTZ black hole. In this limit there is no event horizon and since we do not have a curvature singularity, this solution no longer represents a black hole. When the mass takes the specific value $M = -1/8$, the solution becomes global AdS$_3$. Thus the continuous BTZ black hole spectrum is separated from the global AdS spacetime by a mass gap of $\Delta M = 1/8$ [158]. In between these two values of M, the solutions pick up a conical defect and there is a naked singularity. It has been suggested that the interpretation of this should be as particle-like objects [156].

Every point of the BTZ black hole is locally AdS$_3$, and the BTZ black hole solutions can be shown to be quotients of AdS$_3$ [157], found by making periodic identifications of the coordinates [158].

As shown above, the isometry group of global AdS$_3$ is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, the global conformal group in two dimensions. As explained in section 4.1.4, we will consider instead the asymptotic symmetry group, whose algebra turns out to be an infinite extension of the isometry algebra $SL(2,\mathbb{R}) \oplus SL(2,\mathbb{R})$, as shown by Brown and Henneaux [104]. In two dimensions this is isomorphic to the conformal algebra, which generates the local conformal symmetries of a CFT. The asymptotic symmetry algebra of AdS$_3$ is then given by two copies of the

Witt algebra, the centreless Virasoro algebra, within which there is a subalgebra that is the isometry algebra of global AdS$_3$. We will explore how this arises in this next section.

5.2.3 Brown and Henneaux boundary conditions

It is useful to understand the class of metrics that asymptote to AdS$_3$ at spatial infinity, of which BTZ is one particular example. These can be best described in Fefferman-Graham coordinates, (ρ, x^a). According to the Fefferman-Graham theorem, any asymptotically AdS$_3$ spacetime can be written in the neighbourhood of the boundary as [106],

$$\begin{aligned} ds^2 &= \ell^2 d\rho^2 + e^{2\rho} g_{(0)ab} dx^a dx^b + \mathcal{O}(e^\rho) \\ &= \ell^2 \frac{dr^2}{r^2} + \frac{r^2}{\ell^2} g_{(0)ab} dx^a dx^b + \mathcal{O}(e^\rho) \end{aligned} \tag{5.40}$$

where $g_{(0)ab}$ is the 2D zeroth order boundary metric and in the second line we have set $r = \ell e^\rho$. The spacelike coordinate ρ is defined so that the boundary is at $\rho \to \infty$ and $x^a = (t, \phi)$ are the coordinates on the boundary.

Brown and Henneaux developed a set of boundary conditions to define those metrics that asymptote to AdS$_3$ at spatial infinity [104]. In terms of the coordinates (5.40) given above, these are the Dirichlet boundary conditions,

$$g_{(0)ab} = \eta_{ab}, \tag{5.41}$$

where

$$\eta_{ab} dx^a dx^b = -dt^2 + \ell^2 d\phi^2. \tag{5.42}$$

We can now investigate the metrics which satisfy both the Fefferman-Graham expansion and the Brown-Henneaux choice of boundary conditions, along with the diffeomorphisms that preserve such metrics. Note that other choices of boundary conditions are also possible, see for example [159].

Consider lightcone coordinates,

$$x^{\pm} = \frac{t}{\ell} \pm \phi \tag{5.43}$$

so that the boundary metric becomes,

$$ds_0^2 = g_{(0)ab} dx^a dx^b = -dt^2 + \ell^2 d\phi^2 = -\ell^2 dx^+ dx^-. \tag{5.44}$$

In these coordinates, the most general solution of Einstein's equations which obeys the Brown-Henneaux boundary conditions is,

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \ell^2 \frac{L_-(x^-)}{r} dx^- \right) \left(r dx^- - \ell^2 \frac{L_+(x^+)}{r} dx^+ \right). \tag{5.45}$$

For this metric we can interpret $L_\pm$ as the dynamical fields.

The vector fields which preserve the class of metrics (5.45) are given by [106],

$$\begin{aligned} \zeta^{(+)} &= V^+(x^+)\partial_+ - \frac{1}{2}\partial_+ V^+ \partial_\rho + \int d\rho\, g^{+-} \partial_+ \partial_+ V^+ \partial_-, \\ \zeta^{(-)} &= V^-(x^-)\partial_- - \frac{1}{2}\partial_- V^- \partial_\rho + \int d\rho\, g^{+-} \partial_- \partial_- V^- \partial_+, \end{aligned} \tag{5.46}$$

where V^+ and V^- are left and right-moving conformal Killing vectors on the boundary. The last terms are sub-leading, since $g^{ab} = e^{-2\rho} g_{ab}$, and will not be needed to compute the algebra or the charges. We will therefore neglect these terms from here on.

Writing

$$V_n^+ = e^{inx^+}, \qquad V_n^- = e^{inx^-}, \tag{5.47}$$

the resulting vector fields, $\zeta_n^{\pm}$ satisfy,

$$\begin{aligned}
[\zeta_m^{(+)}, \zeta_n^{(-)}] &= 0, \\
i[\zeta_m^{(+)}, \zeta_n^{(+)}] &= (m-n)\zeta_{m+n}^{(+)}, \\
i[\zeta_m^{(-)}, \zeta_n^{(-)}] &= (m-n)\zeta_{m+n}^{(-)}.
\end{aligned} \tag{5.48}$$

In other words, the two sets of vector fields commute with one another and each obey the Witt algebra, the centreless algebra of diffeomorphisms on the circle. Moreover, the subsets, $\{\zeta_{-1}^{(+)}, \zeta_0^{(+)}, \zeta_1^{(+)}\}$ and $\{\zeta_{-1}^{(-)}, \zeta_0^{(-)}, \zeta_1^{(-)}\}$ each forms a copy of the $SL(2)$ algebra. By comparison with the Killing vectors of the AdS_3 spacetime, we see that this 'asymptotic' algebra contains within it the symmetries of AdS_3.

The associated charges form an algebra that is isomorphic to the centrally extended asymptotic symmetry algebra [160]. The centrally extended asymptotic symmetry algebra of AdS_3 is given by the Virasoro algebra, which is intimately related to conformal field theory. Thus the charge algebra is given by two copies of the Virasoro algebra, exactly as in (5.17).

It is this conformal charge algebra of the asymptotically AdS_3 spacetime that motivates the duality between these seemingly distinct theories: a theory of gravity in three dimensions and a local quantum field theory in two dimensions. We can relate quantities in the two theories using this duality. For example, the linear combination $L_0 - \tilde{L}_0$ of the lowest Virasoro generators, gives rise to rotations on the plane. We therefore associate this to the angular momentum in the AdS spacetime,

$$J = L_0 - \tilde{L}_0. \tag{5.49}$$

Similarly, the combination $L_0 + \tilde{L}_0$ gives the dilatation operator on the plane. The Hamiltonian, or energy of states, generates time translations on the cylinder or radially evolving circles on the plane. This means that we can relate the mass or energy in AdS with the Virasoro generators by

$$M\ell = L_0 - \frac{c_L}{24} + \tilde{L}_0 - \frac{c_R}{24}, \tag{5.50}$$

where the extra terms involving the central charges arise from the Casimir energy (5.13).

We are going to hypothesise that the BTZ black hole is dual to a thermal state in the CFT. We can then hope to use CFT techniques to understand properties of the black hole. Of particular importance is the thermodynamic entropy associated with such a thermal state, given by the Cardy formula.

5.3 Calculating the entropy

In this section we will give a derivation of the Cardy formula and then show how the entropy as calculated by the Cardy formula matches the black hole entropy for the case of the BTZ black hole.

5.3.1 The Cardy formula

Cardy's formula [161] gives the entropy of a conformal field theory. It provides the essential link between the conformal field theory and the gravitational theory since in many contexts the Cardy entropy is found to match the Bekenstein-Hawking entropy for black holes, as is shown in the following sections. States of the CFT are interpreted as microstates of the black hole, which allow for this connection to be made and for a microscopic description of the entropy to be hypothesised.

However, Cardy's entropy formula and the Hawking entropy formula have two very different regions of validity, which calls into question its use in the black hole context. This has been explored in detail in [162], where its range of validity has been shown to be extended.

We will begin by recapping the derivation of Cardy's formula, following [162]. An essential feature of a CFT which allows for this derivation is modular invariance.

We showed in section 5.2.1 that the conformal boundary of asymptotically AdS_3 spacetimes is a cylinder, on which the CFT is hypothesized to live. This cylinder is parameterized by the complex coordinate ω as in (5.7) [152],

$$\omega = \sigma + i\tau, \qquad \sigma \in [0, 2\pi). \tag{5.51}$$

We can instead define this conformal boundary as a torus by taking the parameterization (5.7) of the cylinder but now asking that τ is also periodic, with period β. This quantity is the inverse temperature, $\beta = 1/T$. The CFT now lives on the torus and the AdS spacetime exists within it. The complex number τ is the modular parameter of the torus. For the case of thermal AdS, the modular parameter is given by [153]

$$\tau_{AdS} = \frac{\theta}{2\pi} + \frac{i\beta}{2\pi\ell}. \tag{5.52}$$

Here, θ is the 'angular potential', which is the canonical conjugate of the angular momentum J. Again, ℓ is the AdS length. This modular parameter has been shown to be related to the modular parameter of BTZ by

$$\tau_{BTZ} = -\frac{1}{\tau_{AdS}} \tag{5.53}$$

a result by Maldacena and Strominger [153].

The CFT partition function is defined as a trace over the CFT Hilbert space,

$$Z = \mathrm{Tr} e^{-\beta H + i\theta J}. \tag{5.54}$$

We can use the expressions (5.49) and (5.50) to replace the Hamiltonian H and the angular momentum J in this expression. On the torus, the partition function becomes a function of the modular parameter τ and its complex conjugate $\bar{\tau}$. Using (5.52) we can therefore write the partition function as,

$$Z(\tau, \bar{\tau}) = e^{\frac{i\theta(c_L - c_R)}{24}} \mathrm{Tr}[e^{2\pi i\tau(L_0 - \frac{c_L}{24})} e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \frac{c_R}{24})}]. \tag{5.55}$$

Considering for now only the part involving τ (we can re-insert the part involving

$\bar{\tau}$ at a later stage), the trace can be written,

$$Z(\tau) = \sum_{N\geq 0} \rho(N) e^{2\pi i \tau (N - c_L/24)} \tag{5.56}$$

where $\rho(N)$ is the number of states with energy eigenvalue $L_0 = N$. We will now assume that the modular parameter τ is purely imaginary, $\tau = \frac{i\beta}{2\pi\ell}$, which means that as $\beta \to 0$ in the high-temperature limit, we find that $\tau \to 0$.

This partition function exhibits modular invariance, since when regarding τ as the modular parameter of a torus, the partition function $Z(\tau)$ is invariant under the $SL(2,\mathbb{Z})$ group describing conformal transformations of the torus,

$$Z(\tau) = Z\left(\frac{a\tau + b}{c\tau + d}\right), \qquad ad - bc = 1. \tag{5.57}$$

In particular, we have

$$Z(\tau) = Z\left(\frac{-1}{\tau}\right), \tag{5.58}$$

which means we can compare the partition functions for $\tau \to 0$ and $\tau \to i\infty$ and relate the high and low-temperature behaviours. We can write,

$$Z\left(\frac{-1}{\tau}\right) = e^{\frac{2\pi i}{\tau}\frac{c_L}{24}} \tilde{Z}(-\frac{1}{\tau}), \quad \tilde{Z}(-\tfrac{1}{\tau}) = \mathrm{Tr}[e^{-2\pi i L_0/\tau}]. \tag{5.59}$$

In the high-temperature regime, as $\tau \to 0$, the dominating state is the ground state, where $N = 0$ and $\tilde{Z} \to 1$. This means that we may approximate,

$$\lim_{\tau\to 0} Z(\tau) = \exp(\frac{2\pi i}{\tau}\frac{c_L}{24}) \tag{5.60}$$

and thus the partition function in this regime is controlled by the central charge c_L.

One can calculate an expression for the asymptotic behaviour of $\rho(N)$ using a Laplace transform of $Z(\tau)$,

$$\rho(N) = \int_{-i\infty}^{i\infty} d\tau Z(\tau) e^{-2\pi i \tau (N - \frac{c_L}{24})}. \tag{5.61}$$

In the high temperature limit, using (5.60), this becomes

$$\rho(N) \approx \int_{-i\infty}^{i\infty} d\tau \exp\left[-2\pi i\tau(N - \frac{c_L}{24}) + \frac{2\pi i}{\tau}\frac{c_L}{24}\right]. \tag{5.62}$$

As $\tau \to 0$ at high temperatures, N gets large. When $N \gg c_L/24$, this integral will be dominated by a saddle point at

$$\tau_0 = i\sqrt{\frac{c_L/24}{N - c_L/24}}. \tag{5.63}$$

Using the saddle point approximation results in

$$\rho(N) \approx exp\left[4\pi\sqrt{\frac{c_L}{24}(N - \frac{c_L}{24})}\right]. \tag{5.64}$$

In general, (5.60) will no longer hold when $|\tau|$ becomes of order one, i.e. this approximation is only valid when

$$\frac{N - c_L/24}{c_L/24} \gg 1. \tag{5.65}$$

We can now use the expression for $\rho(N)$ to find the entropy. In the microcanonical ensemble, the entropy is given by the logarithm of the number of states, $S = \log\rho(N)$. As the temperature goes to zero, the only state is the ground state so $N = 1$ and the entropy vanishes. At higher temperatures, N is finite and there will be a non-zero entropy.

Using (5.64) and restoring the anti-holomorphic part, we can arrive at the Cardy formula for the thermodynamic entropy of a CFT,

$$S_{Cardy} = \left[4\pi\sqrt{\frac{c_L}{24}(N_L - \frac{c_L}{24})} + 4\pi\sqrt{\frac{c_R}{24}(N_R - \frac{c_R}{24})}\right]. \tag{5.66}$$

In the canonical ensemble, we can define left and right temperatures,

$$\frac{\partial S_{Cardy}}{\partial N_L}\Big|_{N_R} = \frac{1}{T_L}, \qquad \frac{\partial S_{Cardy}}{\partial N_R}\Big|_{N_L} = \frac{1}{T_R}. \tag{5.67}$$

Therefore,

$$T_L = \frac{1}{2\pi}\sqrt{\frac{N_L - c_L/24}{c_L/24}}, \qquad T_R = \frac{1}{2\pi}\sqrt{\frac{N_R - c_R/24}{c_R/24}}, \tag{5.68}$$

and in terms of these temperatures the Cardy formula becomes,

$$S_{Cardy} = \frac{\pi^2}{3}(c_L T_L + c_R T_R), \tag{5.69}$$

which now holds for $T_{L,R} \gg 1$.

5.3.2 Matching Cardy entropy with black hole entropy

In this section we will show that for the case of BTZ, the Cardy entropy of the dual CFT matches the Bekenstein-Hawking entropy of the black hole. We will begin by recapping the calculation of the well-known central charge of the BTZ black hole, computed at spatial infinity by Brown and Henneaux [160]. We will then exploit the conformal symmetry to use the Cardy formula and find the entropy.

In order to compute the charges using (4.42) we must vary the metric with respect to the fields, $L_\pm$, giving,

$$h_{ab} = \delta g_{ab} = \frac{\partial g_{ab}}{\partial L_+}\delta L_+ + \frac{\partial g_{ab}}{\partial L_-}\delta L_-. \tag{5.70}$$

Assuming at this stage that there are no counterterms so that we may simply use the expression (4.33) to compute the charge variations with respect to the vector fields $\zeta_m^{(\pm)}$, we obtain,

$$\begin{aligned} \delta\mathcal{Q}_m^{(+)} &= \int_{S^1_\infty} \mathbf{k}_{\zeta_m^{(+)}}[\delta g; g] = \frac{\ell}{8\pi}\int_0^{2\pi} d\phi \delta L_+(x^+) e^{imx^+}, \\ \delta\mathcal{Q}_m^{(-)} &= \int_{S^1_\infty} \mathbf{k}_{\zeta_m^{(-)}}[\delta g; g] = \frac{\ell}{8\pi}\int_0^{2\pi} d\phi \delta L_-(x^-) e^{imx^-}. \end{aligned} \tag{5.71}$$

Note that the integrations are performed on the unit circle at infinity. The integrands on the left hand sides of (5.71) are clearly integrable as they involve the variations of $L_\pm$. This justifies the decision to leave out counterterms and we can immediately read off the charges $\mathcal{Q}^{(\pm)}$ from $\delta\mathcal{Q}^{(\pm)}$, as

$$\begin{aligned}
\mathcal{Q}^{(+)}_m &= \frac{\ell}{8\pi}\int_0^{2\pi} d\phi L_+(x^+)e^{imx^+}, \\
\mathcal{Q}^{(-)}_m &= \frac{\ell}{8\pi}\int_0^{2\pi} d\phi L_-(x^-)e^{imx^-}.
\end{aligned} \tag{5.72}$$

By analyzing the charge algebra using the Poisson bracket defined in (4.25), one finds

$$\begin{aligned}
\{\mathcal{Q}^{(+)}_m, \mathcal{Q}^{(-)}_n\} &= 0, \\
i\{\mathcal{Q}^{(+)}_m, \mathcal{Q}^{(+)}_n\} &= (m-n)\mathcal{Q}^{(+)}_{m+n} + m^3\delta_{m+n,0}\frac{\ell}{8}, \\
i\{\mathcal{Q}^{(-)}_m, \mathcal{Q}^{(-)}_n\} &= (m-n)\mathcal{Q}^{(-)}_{m+n} + m^3\delta_{m+n,0}\frac{\ell}{8}.
\end{aligned} \tag{5.73}$$

The central term in the algebra of charges is in general of the form $\frac{c}{12}m^3\delta_{m+n,0}$ with central charge c. One can define the central charge of the $(+)$ sector to be $c = c_L$, and for the $(-)$ sector one can define $c = c_R$. By shifting the zero mode of the charges, they may be put in the form more commonly found in the Virasoro algebra, in which the central extension vanishes for $m = -1, 0, +1$.

Defining,

$$\tilde{\mathcal{Q}}^{(+)}_m = \mathcal{Q}^{(+)}_m + \alpha^{(+)}\delta_{m,0}, \tag{5.74}$$

for some constant $\alpha^{(+)}$, we get

$$i\{\tilde{\mathcal{Q}}^{(+)}_m, \tilde{\mathcal{Q}}^{(+)}_n\} = (m-n)(\tilde{\mathcal{Q}}^{(+)}_{m+n} - \alpha^{(+)}\delta_{m+n,0}) + \frac{c_L}{12}m^3\delta_{m+n,0}. \tag{5.75}$$

Choosing $\alpha^{(+)} = \frac{c_L}{24}$, we get

$$i\{\tilde{\mathcal{Q}}^{(+)}_m, \tilde{\mathcal{Q}}^{(+)}_n\} = (m-n)\tilde{\mathcal{Q}}^{(+)}_{m+n} + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0}, \tag{5.76}$$

which has a centreless subalgebra of AdS_3 exact symmetries. The same can be done for the charges, $\mathcal{Q}_m^{(-)}$, and one again finds a shift of $c_R/24$. These shifts correspond to a shift of the mass from the zero mass BTZ black hole to the spacetime of global AdS_3.

We can find the values of the two central charges, c_L and c_R, from the 'left' and 'right'-moving algebras generated by the charges $\mathcal{Q}^{(\pm)}$ in (5.73) as,

$$c_L = c_R = \frac{3\ell}{2}. \tag{5.77}$$

This is the famous result of Brown and Henneaux. The appearance of central charges in Virasoro algebras is extremely suggestive of a link to conformal field theory. Indeed, if we assume the existence of a two-dimensional conformal field theory living on the unit circle at infinity, one may use the Cardy formula (5.66) to compute the entropy. Writing the zero modes of the BTZ black hole using (5.49) and (5.50), one finds $N_L - c_L/24 = \frac{1}{2}(M\ell + J)$ and $N_R - c_R/24 = \frac{1}{2}(M\ell - J)$. Inserting the central charges $c_{L,R}$ as computed above into the Cardy formula, one finds

$$S_{BH} = 4\pi\sqrt{\frac{\ell}{32}(M\ell + J)} + 4\pi\sqrt{\frac{\ell}{32}(M\ell - J)} = \frac{\pi r_+}{2}. \tag{5.78}$$

This is exactly the entropy (5.39) of the BTZ black hole. Thus there is an exact agreement between the Bekenstein-Hawking entropy of the black hole and the thermodynamic entropy of the CFT, calculated from the Cardy formula. In other words, we hypothesize that the BTZ black hole can itself be described as a 2D thermal CFT, and we use this AdS/CFT correspondence to recover the entropy in the bulk of the spacetime by looking at field theoretic techniques on the boundary. Since the Cardy formula gives the entropy as a measure of the microscopic degeneracy, this gives the BTZ black hole entropy a microscopic interpretation.

One important issue that has been raised with this matching argument concerns the range of validity of the two different entropy formulae. As explained above, the Cardy formula is valid in the high-temperature limit, where N is large. The Bekenstein-Hawking formula on the other hand, is valid in the semiclassical limit, when the area of the black hole is large in Planck units. This occurs when

$$\frac{c}{24}(N - \frac{c}{24}) \gg 1, \tag{5.79}$$

rather than just demanding that the ratio (5.65) is large. This is an ongoing problem, but has motivated research into an extended range of validity of Cardy's formula. One such extension involves using the mass gap between massless BTZ and global AdS, to find a "sparse light spectrum of states" in theories with large central charge [163]. This allows the range of validity to be extended from $T_{L,R} \gg 1$ to $T_{L,R} > 1/2\pi$. More recently, by studying the chiral 'Monster' CFT with $c = 24$, the region of validity of Cardy's formula has been extended further, right the way down to the AdS-scale [162].

Before proceeding to four dimensional examples, it is worth noting that this three-dimensional treatment turns out to be relatively simple. Firstly, the central charge as calculated from (4.33) is a constant, not continuously variable. Secondly, these charges turn out to be integrable. There is therefore no reason to introduce a counterterm.

6 The extreme Kerr black hole

Ultimately, we wish to study the generic Kerr black hole. However, as a simpler intermediate step, we will first consider the extreme case, which exhibits its own set of interesting properties. The Kerr/CFT correspondence [127] explored the possibility that the extreme Kerr black hole itself has a dual CFT description.

The Kerr solution is a two parameter family of solutions to the four dimensional vacuum Einstein equations for a generic rotating black hole. It is labelled by two parameters: M, the mass of the black hole, and J, the angular momentum. The Kerr metric in Boyer-Lindquist coordinates is given by,

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - a dt)^2 - \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2, \quad (6.1)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 + a^2 - 2Mr. \end{aligned} \quad (6.2)$$

a is a rotation parameter, given by

$$a = \frac{J}{M}. \quad (6.3)$$

There are outer and inner event horizons at

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \quad (6.4)$$

The Hawking temperature is given by ,

$$T_H = \frac{(r_+ - M)}{4\pi M r_+}, \quad (6.5)$$

and the angular velocity of the (outer) horizon is given by,

$$\Omega_H = \frac{a}{2Mr_+}. \tag{6.6}$$

At extremality, we have

$$M = a. \tag{6.7}$$

Therefore we see that in this limit, the inner and outer horizons coincide and we have a single event horizon, at

$$r_+ = r_- = \hat{r} = M. \tag{6.8}$$

The Killing horizons are called *degenerate* horizons, in contrast to the *bifurcate* horizons of non-extremal black holes. They have zero surface gravity, as can be seen from the Hawking temperature (6.5) which becomes zero. However, the black hole still has a non-zero and finite entropy, given by

$$S_{BH} = 2\pi M^2. \tag{6.9}$$

We will show in the subsequent sections how this entropy can be reproduced by considering microstates on the horizon of the extreme Kerr black hole. We will begin by understanding the geometry close to the horizon.

6.1 Geometry and conformal symmetries

A useful place to start is by introducing the parameter λ,

$$\lambda = \sqrt{1 - \frac{a^2}{M^2}}, \tag{6.10}$$

a measure of the distance to extremality. At extremality, $\lambda = 0$. The other limit occurs when $\lambda = 1$, describing the case of the Schwarzschild black hole.

To explore the geometry close to extremality, we can expand coordinates in terms of the new parameter λ, and eventually take the limit $\lambda \to 0$. Near the horizon, we must use a co-rotating angular coordinate, defined in relation to the Boyer-Lindquist coordinate as,

$$\hat{\phi} = \phi - \frac{t}{2M} + \mathcal{O}(\lambda). \tag{6.11}$$

We will also define new time and radial coordinates, similarly to [127], as

$$\begin{aligned} T &= \frac{\lambda t}{2M}, \\ R &= \frac{r - r_+}{\lambda M}. \end{aligned} \tag{6.12}$$

In the limit, $\lambda \to 0$, the Kerr metric becomes,

$$ds^2 = 2J\Omega^2(\theta)\left[-R^2 dT^2 + \frac{dR^2}{R^2} + d\theta^2 + \Lambda^2(\theta)(d\hat{\phi} + R dT)^2\right] + \mathcal{O}(\lambda^p), \tag{6.13}$$

where

$$\begin{aligned} \Omega^2(\theta) &= \frac{1}{2}(1 + \cos^2\theta), \\ \Lambda(\theta) &= \frac{2\sin\theta}{1 + \cos^2\theta}. \end{aligned} \tag{6.14}$$

This is the metric of Near Horizon Extreme Kerr, or NHEK [164].

From this metric we can immediately get a sense of an $SL(2)$ symmetry. The first two terms in the square bracket of (6.13) are the same as the metric for AdS_2 in Poincaré coordinates. AdS_2 spacetime has three Killing vectors which form an $SL(2,\mathbb{R})$ algebra. The NHEK geometry has four Killing vectors in total, given by

$$\begin{aligned} \zeta_1 &= \partial_T, \\ \zeta_2 &= \partial_{\hat{\phi}}, \\ \zeta_3 &= T\partial_T - R\partial_R, \\ \zeta_4 &= \left(\frac{1}{2R^2} + \frac{T^2}{2}\right)\partial_T - TR\partial_R - \frac{1}{R}\partial_{\hat{\phi}}. \end{aligned} \tag{6.15}$$

The third, ζ_3 is a scale transformation. We see that ζ_1, ζ_3 and ζ_4 are independent of the coordinate $\hat{\phi}$ and thus the second Killing symmetry, $\partial_{\hat{\phi}}$ commutes with all the others. This means that the symmetry group of NHEK is

$$SL(2,\mathbb{R}) \times U(1). \tag{6.16}$$

All four isometries given in (6.15) are independent of the polar angle θ and therefore act within three-dimensional slices given by the coordinates $(T, R, \hat{\phi})$. The geometry of these slices at any given angle θ looks like warped AdS_3. Periodicity in the angular coordinate $\hat{\phi}$ means we in fact have a quotient of warped AdS_3. At the specific value of θ where $\Omega^2 = \sin\theta$ such that $\Lambda = 1$, the geometry is locally an ordinary AdS_3. This means that here the symmetry algebra is locally $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$, whereas at all other values of θ, the isometry is broken down from $SL(2,\mathbb{R})_L$ to $U(1)$ [127].

6.2 Computation of charges

The way forward in finding surface charges in NHEK is very similar to the approach of Brown and Henneaux in BTZ, as described earlier. Boundary conditions are imposed and then diffeomorphisms are found that preserve these conditions. Charges associated to such diffeomorphisms may then be calculated using the covariant phase space formalism. For the case of BTZ, the appropriate vector fields involved two commuting copies of the Virasoro algebra, and related to a two dimensional dual conformal field theory on the boundary. In the case of NHEK, if we are again to find a dual CFT description, we might also like to look for vector fields that satisfy Virasoro algebras. However, since the symmetry algebra is no longer $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ but $SL(2,\mathbb{R}) \times U(1)$, it is reasonable to expect that we might only find one such Virasoro. This has been done in [127] and updated in [165]. The same applies to the case of near-horizon extreme BTZ, which has a very similar structure to NHEK.

The first step is to determine the boundary conditions. In taking the near-horizon limit of extreme Kerr, the metric (6.13) is no longer asymptotically flat, so it is not clear which boundary conditions are the most appropriate. For each set of boundary conditions there is an associated asymptotic symmetry group of 'allowed' transformations, excluding those that are trivial, as in (4.23).

The boundary conditions used in [127] are,

$$
\begin{aligned}
&\delta g_{TT} = \mathcal{O}(R^2),\\
&\delta g_{T\hat{\phi}} = \delta g_{\hat{\phi}\hat{\phi}} = \mathcal{O}(1),\\
&\delta g_{T\theta} = \delta g_{\hat{\phi}\theta} = \delta g_{\theta\theta} = \delta g_{\hat{\phi}R} = \mathcal{O}(\frac{1}{R}),\\
&\delta g_{TR} = \delta g_{\theta R} = \mathcal{O}(\frac{1}{R^2}),\\
&\delta g_{RR} = \mathcal{O}(\frac{1}{R^3}),
\end{aligned}
\tag{6.17}
$$

and the corresponding symmetry generators are given by

$$\zeta = \varepsilon(\hat{\phi})\partial_{\hat{\phi}} - R\partial_{\hat{\phi}}\varepsilon\partial_R, \tag{6.18}$$

where ε is an arbitrary function of $\hat{\phi}$. We can expand ε in eigenmodes,

$$\varepsilon_n = e^{in\hat{\phi}}. \tag{6.19}$$

In [165] it was proposed that an additional term of the form

$$\frac{b}{R}\partial_{\hat{\phi}}\varepsilon\partial_T \tag{6.20}$$

should be included, where b is some constant, to avoid a situation where the phase space is not smooth. Note that in either case, when $n = 0$, the vector field generates the $U(1)$ rotational isometry.

The vector fields obey a centreless Virasoro algebra under the Lie bracket,

$$i\,[\zeta_m, \zeta_n] = (m-n)\zeta_{m+n}. \tag{6.21}$$

The corresponding charges, $\delta\mathcal{Q}_\zeta$ and their algebra can then be found. Since there is only one Virasoro algebra, we find that there is only one (left-moving) central charge, which can be calculated over surfaces of constant t as $r \to r_+$ and has been found to be $c_L = 12J$, independent of the value of b in (6.20) [127,165].

Notice again that this central charge is constant on the phase space. In [127], the charges $\delta\mathcal{Q}_\zeta$ were also shown to be integrable to quadratic order around the NHEK background. This means that there was no requirement for the addition of counterterms.

6.3 Kerr/CFT

In the Kerr/CFT correspondence [127], the extreme Kerr black hole is hypothesised to be dual to a thermal CFT. The central charge ($c_L = 12J$) is used in the Cardy formula to calculate the entropy of the dual conformal field theory and again find that it matches the black hole entropy. We will here show how this arises.

In order to use the central charge to find the entropy for the black hole, we must use the Cardy formula which involves the thermal temperatures of the dual CFT. The derivation of the temperatures requires a definition of the vacuum. The problem is that for a generic curved spacetime, the vacuum is not unique. It depends on a timelike Killing vector. In the Kerr solution, it is not possible to define a Killing vector which is everywhere timelike, due to the existence of the ergoregion. This means there is no global quantum state which can represent the vacuum (see [166] and references therein). One approach is to consider the Frolov-Thorne vacuum as in [127], and choose a timelike vector field that is well behaved in the near-horizon region. Close to the horizon, one might consider the generator of the horizon, which in Boyer-Lindquist coordinates is given by, $\zeta = \partial_t - \Omega_H \partial_\phi$. This is timelike between two surfaces: the horizon, and the surface at which in order to co-rotate with the black hole an observer must travel at the speed of light. Frolov and Thorne [167] defined the vacuum in this way, by describing it as a density matrix,

$$\rho = exp\left(-\frac{\omega - \Omega_H m}{T_H}\right), \tag{6.22}$$

where the eigenmodes of energy and angular momentum are (ω, m). The next step is to rewrite the density matrix in terms of the near-horizon coordinates. To do this, it is simplest to use the coordinates $(T, R, \theta, \hat{\phi})$ defined above in (6.11) and (6.12). In these coordinates, we have,

$$-i\omega t + im\phi = -\frac{i}{\lambda}(2M\omega - m)T + im\hat{\phi} = -in_R T + in_L\hat{\phi} \tag{6.23}$$

where

$$\begin{aligned} n_L &= m, \\ n_R &= \frac{1}{\lambda}(2M\omega - m) \end{aligned} \tag{6.24}$$

are the left and right charges associated to the Killing vectors $\partial_{\hat{\phi}}$ and ∂_T of the near-horizon region. Writing (6.22) in terms of these quantities, we get

$$\rho = exp\left(-\frac{n_L}{T_L} - \frac{n_R}{T_R}\right), \tag{6.25}$$

where now the left and right temperatures are

$$\begin{aligned} T_L &= \frac{r_+ - M}{2\pi(r_+ - a)}, \\ T_R &= \frac{r_+ - M}{2\pi\lambda r_+}. \end{aligned} \tag{6.26}$$

In the extremal limit these become [127],

$$\begin{aligned} T_L &= \frac{1}{2\pi}, \\ T_R &= 0. \end{aligned} \tag{6.27}$$

This means that for extreme Kerr, the Frolov-Thorne vacuum is thermally populated at temperature $T_L = 1/2\pi$. Since the states of quantum gravity of NHEK were shown above to be dual to the left-moving part of a CFT, the CFT dual of the Frolov-Thorne vacuum must also have this temperature.

Using the Cardy formula for the entropy,

$$S = \frac{\pi^2}{3} c_L T_L, \tag{6.28}$$

with $c_L = 12J$ and $T_L = 1/2\pi$ we recover the Bekenstein-Hawking entropy,

$$S_{BH} = 2\pi J. \tag{6.29}$$

This suggests that the extreme Kerr black hole is dual to a thermal CFT at temperature T_L.

7 The Kerr black hole

We will now proceed to discuss the case of the generic Kerr black hole, given by the metric (6.1). We will begin by recapping the hidden conformal symmetry of Kerr black holes, and then move on to discuss the new approach to the problem and the relevant findings.

We will attempt to build up a microscopic description of the entropy, with methods similar to those used in previous sections. However, finding a dual CFT description for the case of the Kerr black hole involves the consideration of a different viewpoint, in which the boundary of the spacetime is the horizon of the black hole.

Using the horizon as the boundary of spacetime means that our calculations are inherently observer dependent [168]. In [27, 28, 78], it was shown that nontrivial diffeos act on the horizon of a generic 4D Kerr black hole, resulting in additional features known as soft hair. A non-zero central term found in the algebra of the corresponding horizon charges means that an observer outside the horizon will see a violation of diffeomorphism invariance. This is physically unacceptable and must be cancelled by something. We might hypothesize that there is a conformal field theory living on the horizon with a central charge that can do just that. If so, the Cardy formula will tell us the entropy of the CFT. If this entropy turns out to be equal to the Bekenstein-Hawking entropy, then we might conclude that the black hole entropy can be described by microstates on the horizon.

Ultimately we want to follow this procedure to find the horizon charges for the Kerr black hole. In what follows, in order to simplify the computations we will in fact find the charges on a *section* of the horizon, the bifurcation surface. Extending the analysis away from the bifurcation surface to the full black hole horizon is left to future work.

However, before finding the charges for the case of the Kerr black hole, it is worth noting that this shift onto the horizon is possible for the previous case of the BTZ black hole, and produces the same central term. This calculation is done explicitly in Appendix D. The calculations are almost identical to those in the previous sections, but with a slight shift in interpretation.

Similarly, an interesting feature of the geometry of extreme Kerr is that surfaces of constant t, r result in 'internal infinities', or a 'throat'. We can explicitly calculate the central terms on the future horizon by using a different set of coordinates. This calculation can be found in Appendix E.

7.1 Hidden conformal symmetry

For extremal black holes a $SL(2)\times U(1)$ symmetry can be identified. Calculations of surface charges with respect to diffeomorphisms in the NHEK region form the Virasoro algebra with central charge $12J$. This suggests that the extreme Kerr black hole is dual to a 2D CFT. Indeed, using the Cardy formula for the CFT reproduces the Bekenstein-Hawking entropy of the extreme Kerr black hole, which is further evidence for such a dual description. The conformal symmetry arises from the geometry of the spacetime in this case.

Moving to non-extreme geometries is a significantly harder challenge. Away from the extreme limit, the near horizon geometry is just Rindler space, to which one cannot associate a conformal field theory. If extreme Kerr black holes have a dual CFT, then one might expect that finite excitations of the CFT correspond to non-extremal Kerr black holes. However, the back-reaction of the excitations on the geometry appears to destroy the conformal symmetry. The way out from this problem is to look for the conformal symmetry in the phase space rather than the space time. When fields propagate on a space which has a conformal symmetry, the interactions will exhibit conformal invariance. The same will also be true when the solution space of the wave equation for the propagating field has the conformal symmetry, even if the background spacetime does not. A local conformal symmetry is found in [79] to act on the solution space, and we will present here this '*hidden conformal symmetry'*.

7.1.1 Wave equation

Kerr black holes with generic mass M and spin $J \leq M^2$ exhibit a hidden conformal symmetry which acts on low-lying soft modes [79]. The symmetry emerges, not in a near-horizon region of *spacetime*, but in the near-horizon region of *phase space* defined by

$$\omega(r - r_+) \ll 1, \tag{7.1}$$

where ω is the energy of the soft mode, r is the Boyer-Lindquist radial coordinate and $r_+ = M + \sqrt{M^2 - a^2}$, with $a = \frac{J}{M}$, is the location of the outer horizon. This simply states that the soft mode wavelength is large compared to the black hole. One way to see the emergent symmetry is by examining the wave equation for massless scalar fields in the Kerr spacetime. We will now explore how this arises, by following [79].

The Klein-Gordon equation for a massless scalar is

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\Phi) = 0. \tag{7.2}$$

We want to solve this equation in the Kerr spacetime. Using Boyer-Lindquist coordinates, writing the field in Fourier modes,

$$\Phi[t, r, \theta, \phi] = e^{-i\omega t + im\phi}\Phi[r, \theta], \tag{7.3}$$

the wave equation in the Kerr spacetime (6.1) becomes,

$$\begin{aligned} \partial_r(\Delta\partial_r\Phi) + \frac{(2Mr_+\omega - am)^2}{(r - r_+)(r - r_-)}\Phi - \frac{(2Mr_-\omega - am)^2}{(r - r_+)(r - r_-)}\Phi \\ +(r^2 + a^2\cos^2\theta + 2M(r + 2M))\omega^2\Phi + \nabla_{S^2}\Phi = 0, \end{aligned} \tag{7.4}$$

where $r_- = M - \sqrt{M^2 - a^2}$ and $\Delta = (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$. This equation separates into a radial and angular part. The resulting equations remain seemingly rather complicated, but both are dramatically simplified in the following limit,

$$\omega M \ll 1, \tag{7.5}$$

which corresponds to the situation whereby the soft mode wavelength is large compared to the radius of curvature.

In this situation the geometry naturally divides into a 'near' region,

$$r \ll \frac{1}{\omega} \tag{7.6}$$

and a 'far' region,

$$r \gg M. \tag{7.7}$$

One can then solve the equations in the two regions and then match the solutions along a suitable surface in the matching region,

$$M \ll r \ll \frac{1}{\omega}. \tag{7.8}$$

It is important to note that this is not a near region in the sense of near-horizon geometries, but a near region of the phase space. It is this near region of the phase space that exhibits the conformal symmetry. Further discussion about the distinction between the near region of phase space and the near region of spacetime can be found in [79].

By focusing in on the 'near region' (7.6), the angular part of the Klein-Gordon equation reduces to the standard Laplacian on the 2-sphere, which is solved by spherical harmonics. The radial equation is more complicated however, and is given by,

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = \ell(\ell+1)R(r). \tag{7.9}$$

We will write this as,

$$\left[\partial_r \Delta \partial_r + \frac{\alpha(r_+)^2}{(r-r_+)(r_+-r_-)} - \frac{\alpha(r_-)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = \ell(\ell+1)R(r). \tag{7.10}$$

This equation is solved by hypergeometric functions of r, and therefore fall into representations of $SL(2,\mathbb{R})$. This is a first suggestion of a conformal symmetry in the solution space. To understand this conformal symmetry, it is useful to introduce a set of 'conformal' coordinates, defined in [79] by

$$\begin{aligned} w^+ &= \sqrt{\frac{r-r_+}{r-r_-}} e^{2\pi T_R(\phi-\Omega_R t)}, \\ w^- &= \sqrt{\frac{r-r_+}{r-r_-}} e^{2\pi T_L(\phi-\Omega_L t)}, \\ y &= \sqrt{\frac{r_+-r_-}{r-r_-}} e^{\pi T_L(\phi-\Omega_L t)+\pi T_R(\phi-\Omega_R t)}, \end{aligned} \tag{7.11}$$

where T_L, T_R and Ω_L, Ω_R are constants to be determined.

In [79], two sets of vector fields are defined locally, each of which forms an $SL(2,\mathbb{R})$ Lie bracket algebra as follows. The first set is,

$$\begin{aligned} H_1 &= i\partial_+, \\ H_0 &= i(w^+\partial_+ + \frac{1}{2}y\partial_y), \\ H_{-1} &= i(w^{+2}\partial_+ + w^+ y\partial_y - y^2\partial_-), \end{aligned} \tag{7.12}$$

which satisfies,

$$\begin{aligned} [H_0, H_{\pm 1}] &= \mp i H_{\pm 1}, \\ [H_{-1}, H_1] &= -2iH_0. \end{aligned} \tag{7.13}$$

The second set of vector fields, which commute with the first, are defined,

$$\begin{aligned} \bar{H}_1 &= i\partial_-, \\ \bar{H}_0 &= i(w^-\partial_- + \frac{1}{2}y\partial_y), \\ \bar{H}_{-1} &= i(w^{-2}\partial_- + w^- y\partial_y - y^2\partial_+), \end{aligned} \tag{7.14}$$

which satisfies a corresponding $SL(2,\mathbb{R})$ algebra. In fact, the scalar wave equation for angular momentum ℓ can be written in this region [79] in terms of these vector fields as,

$$\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = \ell(\ell+1)\Phi, \tag{7.15}$$

where

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}(H_1H_{-1} + H_{-1}H_1) \tag{7.16}$$

is the $SL(2,\mathbb{R})$ quadratic Casimir. The conformal weights of the field Φ are

$$(h_L, h_R) = (\ell, \ell). \tag{7.17}$$

A suitably modified formula applies to spinning fields.

In terms of the Boyer-Lindquist coordinates, the Casimir is [120]

$$\begin{aligned}\mathcal{H}^2 =& -\frac{r_+ - r_-}{(r-r_+)(4\pi T_R)^2}\left(\partial_\phi + \frac{T_L + T_R}{T_L(\Omega_L - \Omega_R)}(\partial_t + \Omega_R\partial_\phi)\right)^2 \\ &+ \frac{r_+ - r_-}{(r-r_-)(4\pi T_R)^2}\left(\partial_\phi + \frac{T_L - T_R}{T_L(\Omega_L - \Omega_R)}(\partial_t + \Omega_R\partial_\phi)\right)^2 + \partial_r\Delta\partial_r,\end{aligned} \quad . \tag{7.18}$$

In order to match this equation with equation (7.10), we must have,

$$\alpha(r_\pm) = \frac{r_+ - r_-}{4\pi T_R}\left(-m + \frac{T_L \pm T_R}{T_L(\Omega_L - \Omega_R)}(\omega - \Omega_R m)\right). \tag{7.19}$$

These equations admit the following solution,

$$\Omega_R = 0, \quad \Omega_L = \tfrac{a}{2M^2}, \tag{7.20}$$

with temperatures,

$$T_L = \frac{r_+ + r_-}{4\pi a}, \quad T_R = \frac{r_+ - r_-}{4\pi a}. \tag{7.21}$$

Thus the conformal coordinates are given by (7.11) with these definitions of $\Omega_{L,R}$ and $T_{L,R}$.

These coordinates most clearly express the hidden conformal symmetry acting on massless scalar fields in the phase space, as can be seen from the solutions to the wave equation in the background Kerr spacetime. More detail about these conformal coordinates will be discussed in section 7.2.4.

7.1.2 Scattering

Another signal of the symmetry comes from the near region contribution to the soft absorption cross sections[6].

The absorption probability can be written,

$$\begin{aligned} \mathcal{P}_{abs} \sim & \sinh\left(\frac{4\pi M r_+}{r_+ - r_-}(\omega - m\Omega)\right) \left|\Gamma(1+\ell-2iM\omega)\right|^2 \\ & \times \left|\Gamma(1+\ell-i\frac{4M^2}{r_+-r_-}\omega + i\frac{4Mr_+\Omega}{r_+-r_-}m)\right|^2, \end{aligned} \tag{7.22}$$

with (ω, m) the soft mode energy and axial component of angular momentum. If we assume that there is a dual CFT at temperatures (T_L, T_R) as defined in (7.21), a next step may be to rewrite this absorption cross section in terms of these temperatures. We find,

$$\mathcal{P}_{abs} \sim T_L^{2h_L-1} T_R^{2h_R-1} \sinh(\frac{\omega_L}{2T_L} + \frac{\omega_R}{2T_R}) \left|\Gamma(h_L + i\frac{\omega_L}{2\pi T_L})\right|^2 \left|\Gamma(h_R + i\frac{\omega_R}{2\pi T_R})\right|^2. \tag{7.23}$$

Here the left and right soft mode energies are

$$\omega_L = \frac{2M^2}{a}\omega, \qquad \omega_R = \frac{2M^2}{a}\omega - m. \tag{7.24}$$

The left/right temperatures and entropies are thermodynamically conjugate, as follows from

$$\delta S_{BH} = \frac{\omega_L}{T_L} + \frac{\omega_R}{T_R}, \tag{7.25}$$

where $S_{BH} = 2\pi M r_+$ is the Kerr black hole entropy. Inserting $\omega = \delta M$ and $m = \delta J$ into the expression (7.25), along with the definitions of the temperatures, recovers the second law of black hole thermodynamics,

$$T_H \delta S_{BH} = \delta M - \Omega \delta J, \tag{7.26}$$

[6]See [79] for a discussion of the range of validity of this expression.

where the Hawking temperature is

$$T_H = \frac{1}{8\pi}\frac{r_+ - r_-}{mr_+}. \tag{7.27}$$

Equation (7.23) is precisely the well-known formula for the absorption cross section of an energy (ω_L, ω_R) excitation of a 2D CFT at temperatures (T_L, T_R). This motivates the hypothesis that the Kerr black hole is itself a thermal 2D CFT and transforms under a $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ action. Motivated by this, in the spirit of [27, 28], in section 7.5 below we explicitly realize the hidden conformal symmetry in the form of $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ diffeos which act non-trivially on the black hole horizon.

It is worth noting however, that there may also exist, as in the Kerr/CFT context [127], an alternate holographic formulation with a left Virasoro-Kac-Moody symmetry, where the Kac-Moody zero mode generates right-moving translations [169], which surprisingly in some cases provides an alternate explanation for example of formulae like (7.23). Indeed with the exciting recent progress in understanding the underlying warped conformal field theories [170–172] this latter possibility is looking the more plausible for the case of Kerr/CFT. Investigation of hidden Virasoro-Kac-Moody symmetries for generic Kerr black holes is left to future work.

Nevertheless, we are left with the tantalizing possibility that we may be able to find a real microscopic description of the entropy of four dimensional astrophysical black holes in terms of microstates living on the horizon. In the following sections we will explore different approaches to this problem.

7.2 Useful coordinate systems

We have just seen that the Kerr black hole admits a hidden conformal symmetry in the near region of phase space. In a similar manner to the previously discussed cases of BTZ and NHEK, if one assumes the black hole is dual to a two dimensional CFT, one can use the Cardy formula to compute the entropy. For the case of the Kerr black hole, it has been observed that if $c_L = c_R = 12J$, the Cardy formula reproduces the correct Bekenstein-Hawking entropy.

In the following sections we make some headway with finding an actual explanation of this entropy. This involves finding a set of diffeomorphisms on the black hole horizon which obey Virasoro algebras and can display the hidden conformal symmetry. We calculate the corresponding charges and central terms, and discover that we must add a Wald-Zoupas counterterm to remove certain obstructions. We find such a counterterm and the resulting central charges are found to be $c_L = c_R = 12J$.

Before we do this however, we will go through different ways of setting up the problem and the various merits of different coordinate systems.

7.2.1 Boyer-Lindquist coordinates

The Kerr solution is the generic rotating black hole solution to Einstein's equations. As given previously in (6.1), in Boyer-Lindquist coordinates, the metric is

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - adt)^2 - \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2. \quad (7.28)$$

This metric exhibits coordinate singularities at the future and past horizons, $r = r_\pm$ and so is not well adapted to calculations that are performed on these surfaces.

7.2.2 Eddington-Finkelstein coordinates

It is possible to remove the coordinate singularities at $r = r_\pm$ and define coordinates that are smooth for all $r > 0$. One example of coordinates with these properties are the Eddington-Finkelstein coordinates. In terms of calculating horizon charges, these will allow for computations that explicitly take place on the horizon, just as we showed for the case of NHEK in Appendix E.

Advanced Eddington-Finkelstein coordinates are defined,

$$\begin{aligned} v &= t + \int \frac{r^2 + a^2}{\Delta} dr, \\ \hat{\phi} &= \phi + \int \frac{a}{\Delta} dr. \end{aligned} \tag{7.29}$$

The Kerr metric becomes

$$\begin{aligned} ds^2 = &\left(-1 + \frac{2Mr}{\rho^2}\right) dv^2 + 2dvdr - \frac{4Mar\sin^2\theta}{\rho^2} dvd\hat{\phi} - 2a\sin^2\theta dr d\hat{\phi} \\ &+ \rho^2 d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} d\hat{\phi}^2, \end{aligned} \tag{7.30}$$

which is clearly smooth at the horizon.

Similarly, retarded time coordinates are defined by

$$\begin{aligned} u &= t - \int \frac{r^2 + a^2}{\Delta} dr, \\ \tilde{\phi} &= \phi - \int \frac{a}{\Delta} dr \end{aligned} \tag{7.31}$$

and the metric becomes

$$\begin{aligned} ds^2 = &\left(-1 + \frac{2Mr}{\rho^2}\right) du^2 - 2dudr - \frac{4Mar\sin^2\theta}{\rho^2} dud\tilde{\phi} - 2a\sin^2\theta dr d\tilde{\phi} \\ &+ \rho^2 d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2} d\tilde{\phi}^2. \end{aligned} \tag{7.32}$$

To deal with the region $\mathcal{H}^+ \bigcap \mathcal{H}^-$, one needs either Kruskal-type coordinates or conformal coordinates.

7.2.3 Kruskal coordinates

We can find the Kerr metric close to the bifurcation surface in Kruskal-like coordinates. To my knowledge the form of the metric presented here is new and cannot be found elsewhere in the literature. The simple expression found here may be very useful for understanding the bifurcate structure of the Kerr black hole. The calculation and coordinate transformations required to produce such a metric are shown in detail in Appendix F.

Using the advanced and retarded coordinates, v and u defined previously, we can define Kruskal-like coordinates,

$$\begin{aligned} U &= -e^{-\kappa u}, \\ V &= e^{\kappa v}, \end{aligned} \tag{7.33}$$

where U is defined with a minus sign so that U increases as u increases. The surface gravity, κ is defined,

$$\kappa = \frac{r_+ - r_-}{4Mr_+}. \tag{7.34}$$

The co-rotating angular coordinate is

$$d\phi_+ = d\phi - \Omega_+ dt, \tag{7.35}$$

and thus

$$d\phi = d\phi_+ + \frac{1}{2}\Omega_+(du + dv), \tag{7.36}$$

where $\Omega_+ = \frac{a}{2Mr_+}$ is the angular velocity of the horizon.

To leading and subleading order around $r = r_+$, the Kerr metric in the new

'Kruskal' coordinates is,

$$
\begin{aligned}
ds^2 &= 2\rho_+^2 dUdV + \rho_+^2 d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2\theta}{\rho_+^2} d\phi_+^2 \\
&\quad - \frac{a\sin^2\theta}{r_+\rho_+^2}(3r_+^2 + a^2 r_+^2 + a^2 r_+^2 \cos^2\theta - a^4\cos^2\theta)(UdV - VdU)d\phi_+ + \cdots ,
\end{aligned}
\tag{7.37}
$$

where $\rho_+^2 = r_+^2 + a^2\cos^2\theta$.

This new form of the metric will be most useful in understanding behaviour at the bifurcation surface, since it is an expansion in small U, V (see Appendix F for an explanation of this limit). All other versions of the Kerr metric in Kruskal-like coordinates involve much more complicated expressions. See e.g. [173]. The same is true for many other versions of near-horizon Kerr metrics in other coordinate systems.

7.2.4 Conformal coordinates

As presented in section 7.1.1, with the constants as determined by examining the near-region wave equation, conformal coordinates are [79]

$$
\begin{aligned}
w^+ &= \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_R \phi}, \\
w^- &= \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_L \phi - \frac{t}{2M}}, \\
y &= \sqrt{\frac{r_+ - r_-}{r - r_-}} e^{\pi(T_R + T_L)\phi - \frac{t}{4M}}.
\end{aligned}
\tag{7.38}
$$

The past horizon is at $w^+ = 0$, the future horizon at $w^- = 0$ and the bifurcation surface Σ_{bif} at $w^\pm = 0$. The inverse transformation is

$$
\begin{aligned}
\phi &= \frac{1}{4\pi T_R} \ln \frac{w^+(w^+w^- + y^2)}{w^-}, \\
r &= r_+ + 4\pi a T_R \frac{w^+w^-}{y^2}, \\
t &= \frac{M(T_R + T_L)}{T_R} \ln \frac{w^+}{w^-} + \frac{M(T_L - T_R)}{T_R} \ln(w^+w^- + y^2).
\end{aligned}
\tag{7.39}
$$

Under azimuthal identification $\phi \to \phi + 2\pi$ one finds

$$w^+ \sim e^{4\pi^2 T_R} w^+, \quad w^- \sim e^{4\pi^2 T_L} w^-, \quad y \sim e^{2\pi^2 (T_R + T_L)} y. \tag{7.40}$$

This is the same as the identification which turns AdS_3 in Poincaré coordinates into BTZ with temperatures (T_L, T_R) where the $w^\pm$ plane becomes thermal Rindler space [153]. It is for this reason that conformal coordinates are well-adapted to an analysis of 4D black holes, mirroring that of the 3D BTZ black holes. For BTZ we concluded that the black hole is dual to a thermal CFT, and here we also see the possibility that the Kerr black hole is dual to a CFT with temperatures (T_L, T_R). If so, this gives physical meaning to the constants T_L, T_R given in the conformal coordinates and defined by (7.21) as thermal left and right-moving temperatures.

To leading and subleading order around the bifurcation surface, the Kerr metric becomes

$$\begin{aligned} ds^2 = {} & \frac{4\rho_+^2}{y^2} dw^+ dw^- + \frac{16J^2 \sin^2\theta}{y^2 \rho_+^2} dy^2 + \rho_+^2 d\theta^2 \\ & - \frac{2w^+ (8\pi J)^2 T_R (T_R + T_L)}{y^3 \rho_+^2} dw^- dy \\ & + \frac{8w^-}{y^3 \rho_+^2} \Big(-(4\pi J)^2 T_L (T_R + T_L) + (4J^2 + 4\pi J a^2 (T_R + T_L) + a^2 \rho_+^2) \sin^2\theta \Big) dw^+ dy \\ & + \cdots , \end{aligned} \tag{7.41}$$

where corrections are at least second order in (w^+, w^-). The volume element is

$$\varepsilon_{\theta y + -} = \frac{8J \sin\theta \rho_+^2}{y^3} + \cdots . \tag{7.42}$$

In terms of the advanced coordinates, the conformal coordinates become,

$$\begin{aligned} w^+ &= e^{2\pi T_R \hat{\phi}}, \\ w^- &= (r - r_+) e^{2\pi T_L \hat{\phi}} e^{-\frac{v}{2M}} e^{\frac{r}{2M}}, \\ y &= (r_+ - r_-)^{\frac{1}{2}} e^{\pi (T_R + T_L) \hat{\phi}} e^{-\frac{v}{4M}} e^{\frac{r}{4M}}, \end{aligned} \tag{7.43}$$

from which we can clearly see that $\mathcal{H}^+$ is the surface $w^- = 0$, w^+ and y are finite.

Similarly, in terms of the retarded time coordinates,

$$\begin{aligned} w^+ &= \frac{r - r_+}{r - r_-} e^{2\pi T_R \tilde{\phi}}, \\ w^- &= \frac{1}{r - r_-} e^{2\pi T_L \tilde{\phi}} e^{-\frac{u}{2M}} e^{-\frac{r}{2M}}, \\ y &= \frac{r_+ - r_-}{r - r_-} e^{\pi (T_R + T_L)\tilde{\phi}} e^{-\frac{u}{4M}} e^{-\frac{r}{4M}}, \end{aligned} \tag{7.44}$$

and thus on $\mathcal{H}^-$, $w^+ = 0$ and w^- and y are finite.

Note that on the future endpoints of the future and past horizons, $\mathcal{H}^+_+$ and $\mathcal{H}^-_+$, we see that $y \to 0$, but on the past endpoints, $\mathcal{H}^+_-$ and $\mathcal{H}^-_-$, we find that y becomes infinite.

7.3 Heuristic derivation

We will now use the conformal coordinates to give a brief heuristic argument for the existence of a central charge in the Kerr spacetime.

In conformal coordinates, we redefine the w^- coordinate by,

$$w^- = \tilde{w}^- \frac{4M^2 a^2 \sin^2\theta}{\rho_+^4}. \tag{7.45}$$

Holding θ fixed, we may then rewrite the 3-dimensional metric as,

$$ds^2 \;=\; \ell^2 \frac{d\tilde{w}^- dw^+ + dy^2}{y^2} + q_{-y} \frac{w^+}{y^3} d\tilde{w}^- dy + q_{+y} \frac{\tilde{w}^-}{y^3} dw^+ dy, \tag{7.46}$$

where

$$\ell = \frac{4J\sin\theta}{\rho_+} \tag{7.47}$$

and $q_{\pm y}$ depend only on θ (which is held fixed). Thus for fixed angle θ, the Kerr metric in conformal coordinates around the bifurcation surface begins to resemble AdS_3, although it has extra terms.

A BTZ black hole can be specified by the dimensionless quantities T_L, T_R and the dimensionful horizon radius x_+, all of which are determined by the intrinsic and extrinsic geometry of the horizon[7]. In terms of the inner horizon x_- and the AdS_3 radius ℓ, these are

$$T_L = \frac{x_+ + x_-}{2\pi\ell}, \qquad T_R = \frac{x_+ - x_-}{2\pi\ell}. \tag{7.48}$$

Hence ℓ is expressed in terms of these horizon quantities as

$$\ell(T_L, T_R, x_+) = \frac{x_+}{\pi(T_L + T_R)}. \tag{7.49}$$

The central charge is then determined from the horizon geometry via the Brown-Henneaux formula as

$$c = \frac{3\ell}{2} = \frac{3x_+}{2\pi(T_L + T_R)}. \tag{7.50}$$

For Kerr we have,

$$x_+ = \frac{2Mr_+ \sin\theta}{\rho_+}, \qquad \pi(T_R + T_L) = \frac{r_+}{2a}. \tag{7.51}$$

The horizon geometry of a fixed θ-leaf can be realized as that of a BTZ black hole in an AdS_3 with radius as in (7.47),

$$\ell = \frac{4J \sin\theta}{\rho_+}. \tag{7.52}$$

This suggests that if we use it to compute the central charge from the same vector field that we used in BTZ (as done above) we should get

$$c(\theta) = \frac{3\ell}{2} = \frac{6J \sin\theta}{\rho_+}. \tag{7.53}$$

Integrating one finds,

$$c = \int_0^\pi c(\theta)\rho_+ d\theta = 12J. \tag{7.54}$$

This strongly suggests that the Kerr black hole may also have a central charge $c = 12J$. It also suggests that for fixed θ, the appropriate vector field is the same

[7]Up to a choice of normalization for the null generator which must also be specified.

as that used in the case of the BTZ black hole.

The flaw in this naive argument is that the fixed θ-leaves do not fully decouple in the naive Iyer-Wald charge formula. The decoupling can only be achieved after the addition of a counterterm. We will see the constraints on this counterterm and a possible candidate for it in the following sections.

7.4 Vector fields

As suggested by the heuristic derivation of the central charge given above, the vector field in question for the case of the Kerr black hole should resemble that used for BTZ. We therefore consider the vector fields in conformal coordinates,

$$\zeta(\varepsilon) = \varepsilon \partial_+ + \frac{1}{2} \partial_+ \varepsilon \, y \partial_y, \tag{7.55}$$

where ε is any function of w^+. These obey the Lie bracket algebra,

$$[\zeta(\varepsilon), \zeta(\tilde{\varepsilon})] = \zeta(\varepsilon \partial_+ \tilde{\varepsilon} - \tilde{\varepsilon} \partial_+ \varepsilon). \tag{7.56}$$

We wish to restrict ε so that ζ is invariant under 2π azimuthal rotations (7.40). A complete set of such functions is

$$\varepsilon_n = 2\pi T_R (w^+)^{1 + \frac{in}{2\pi T_R}}. \tag{7.57}$$

The corresponding vector fields $\zeta_n \equiv \zeta(\varepsilon_n)$ obey the centreless Vir_R algebra

$$[\zeta_m, \zeta_n] = i(n - m)\zeta_{n+m}. \tag{7.58}$$

The zero mode is

$$\zeta_0 = 2\pi T_R (w^+ \partial_+ + \frac{1}{2} y \partial_y) = \partial_\phi + \frac{2M^2}{a} \partial_t = -i\omega_R, \tag{7.59}$$

where the right moving energy ω_R is defined in (7.24).

Similarly, we can define a 'left-moving' vector field,

$$\begin{aligned}\bar{\zeta}_n &= \bar{\varepsilon}_n \partial_- + \frac{1}{2}\partial_- \bar{\varepsilon}_n y \partial_y, \\ \bar{\varepsilon}_n &= 2\pi T_L (w^-)^{1+\frac{in}{2\pi T_L}},\end{aligned} \tag{7.60}$$

with

$$\bar{\zeta}_0 = -\frac{2M^2}{a}\partial_t = i\omega_L. \tag{7.61}$$

This obeys the centreless Vir_L algebra

$$[\bar{\zeta}_m, \bar{\zeta}_n] = i(n-m)\bar{\zeta}_{n+m}, \tag{7.62}$$

and the two sets of vector fields commute with one another

$$[\zeta_m, \bar{\zeta}_n] = 0. \tag{7.63}$$

Note that the $\mathrm{Vir_L} \otimes \mathrm{Vir_R}$ action maps the 'θ-leaves' of fixed polar angle to themselves.

The Frolov-Thorne vacuum density matrix for a Kerr black hole is (up to normalization)

$$\rho_{FT} = e^{-\frac{\omega}{T_H}+\frac{\Omega m}{T_H}}, \tag{7.64}$$

where $T_H = \frac{r_+ - r_-}{8\pi M r_+}$ and $\Omega = \frac{a}{2Mr_+}$ are the Hawking temperature and angular velocity of the horizon, with ω and m being interpreted here as energy and angular momentum operators. Rewriting this in terms of the eigenvalues of the zero modes, ζ_0 and $\bar{\zeta}_0$, one finds simply

$$\rho_{FT} = e^{-\frac{\omega_R}{T_R}-\frac{\omega_L}{T_L}}. \tag{7.65}$$

This is a restatement of the fact that $\omega_{R,L}$ is thermodynamically conjugate to $T_{R,L}$.

For future reference the only non-zero covariant derivatives of ζ on the bifurcation surface $\Sigma_{\rm bif}$ are

$$\begin{aligned} \nabla_+\zeta^+ = -\Gamma^-_{y-}\zeta^y, \quad \nabla_-\zeta^- = \Gamma^-_{y-}\zeta^y, \\ \nabla_+\zeta^y = \partial_+\zeta^y, \quad \nabla_\theta\zeta^y = \Gamma^y_{\theta y}\zeta^y, \quad \nabla_y\zeta^\theta = \Gamma^\theta_{yy}\zeta^y, \end{aligned} \tag{7.66}$$

while the only non-zero metric deviations on the bifurcation surface are

$$\mathcal{L}_\zeta g_{y+} = g_{yy}\partial_+\zeta^y, \quad \mathcal{L}_\zeta g_{+-} = g_{y-}\partial_+\zeta^y. \tag{7.67}$$

Similar formulae apply to $\bar\zeta$.

It is instructive to consider these vector fields in the Kruskal-like coordinates (7.37). The first vector field, ζ, becomes

$$\begin{aligned} \zeta^U &= -U\frac{2\pi T_L}{T_L+T_R}\left(T_R+\frac{in}{2\pi}\right)V^{\frac{in}{2\pi T_R}}e^{in\phi_+}, \\ \zeta^V &= V\frac{2\pi T_R}{T_L+T_R}\left(T_L-\frac{in}{2\pi}\right)V^{\frac{in}{2\pi T_R}}e^{in\phi_+}, \\ \zeta^{\phi_+} &= \frac{1}{T_L+T_R}\left(T_R+\frac{in}{2\pi}\right)V^{\frac{in}{2\pi T_R}}e^{in\phi_+}. \end{aligned} \tag{7.68}$$

The zero mode is

$$\zeta_0 = \frac{T_R}{T_L+T_R}\left(2\pi T_L(V\partial_V - U\partial_U) + \partial_{\phi_+}\right). \tag{7.69}$$

The second vector field, $\bar\zeta$, becomes

$$\begin{aligned} \bar\zeta^U &= U\frac{2\pi T_L}{T_L+T_R}\left(T_R-\frac{in}{2\pi}\right)U^{\frac{in}{2\pi T_L}}e^{in\phi_+}, \\ \bar\zeta^V &= -V\frac{2\pi T_R}{T_L+T_R}\left(T_L+\frac{in}{2\pi}\right)U^{\frac{in}{2\pi T_L}}e^{in\phi_+}, \\ \bar\zeta^{\phi_+} &= \frac{1}{T_L+T_R}\left(T_L+\frac{in}{2\pi}\right)U^{\frac{in}{2\pi T_L}}e^{in\phi_+}. \end{aligned} \tag{7.70}$$

The zero mode is

$$\bar\zeta_0 = \frac{T_L}{T_L+T_R}\left(2\pi T_R(U\partial_U - V\partial_V) + \partial_{\phi_+}\right). \tag{7.71}$$

ζ preserves the future horizon whereas $\bar{\zeta}$ preserves the past horizon. The vector field ζ can be changed into $\bar{\zeta}$ and vice versa by making the transformation,

$$U \longleftrightarrow V, \qquad T_L \longleftrightarrow T_R. \tag{7.72}$$

7.5 Central charge computation

In this section we construct the linearized covariant charges $\delta\mathcal{Q}_n \equiv \delta\mathcal{Q}(\zeta_n, h; g)$ associated to the diffeos ζ_n acting on the horizon. Formally, the linearized charges are expected to generate the linearized action, via Dirac brackets, of ζ_n on the on-shell linearized fluctuation h around a fixed background g. The formal argument proceeds from the fact that they reduce to the covariant symplectic form with one argument the ζ-transformed perturbation h. However, in practice many subtleties arise when attempting to verify such expectations. Among other things one must reduce, via gauge fixing and the application of the constraints, with careful analyses of zero modes and boundary conditions, to a physical phase space on which the symplectic form is non-degenerate. Various obstructions may arise, such as non-integrability of the charges or violations of associativity which necessitate the addition of boundary counterterms as discussed for example in [58, 83–86].

In the much simpler case of horizon supertranslations of Schwarzschild, it was verified in full detail [28] that the linearized charges $\delta\mathcal{Q}_f$ do indeed generate the linearized symmetries as expected. Moreover, the $\delta\mathcal{Q}_f$ were in this case recently explicitly integrated to the full horizon supertranslation charges $\mathcal{Q}_f$ [58]. The $\delta\mathcal{Q}_n$ of interest here are significantly more complicated than their supertranslation counterparts $\delta\mathcal{Q}_f$. We leave a comprehensive analysis of $\delta\mathcal{Q}_n$ in the style of [28] to future work, and the present analysis should therefore be regarded as a preliminary first step.

The construction of covariant charges was outlined in section 4. The general form for the linearized charge associated to a diffeo ζ on a surface Σ with boundary $\partial\Sigma$ is [85]

$$\delta\mathcal{Q} = \delta\mathcal{Q}_{IW} + \delta\mathcal{Q}_X. \tag{7.73}$$

Here the Iyer-Wald charge is

$$\delta\mathcal{Q}_{IW}(\zeta, h; g) = \frac{1}{16\pi}\int_{\partial\Sigma} *F_{IW}, \tag{7.74}$$

with F_{IWab} explicitly given by (4.34)

$$F_{IWab} = \frac{1}{2}\nabla_a\zeta_b h + \nabla_a h^c{}_b\zeta_c + \nabla_c\zeta_a\ h^c{}_b + \nabla_c h^c{}_a\ \zeta_b - \nabla_a h\ \zeta_b - a \leftrightarrow b, \tag{7.75}$$

where the variation h^{ab} is defined by $g^{ab} \to g^{ab} + h^{ab}$ and $h = h^{ab}g_{ab}$.

The Wald-Zoupas counterterm is

$$\delta\mathcal{Q}_X = \frac{1}{16\pi}\int_{\partial\Sigma}\iota_\zeta(*X), \tag{7.76}$$

as given in (4.41) where X is a spacetime one-form constructed from the geometry. As explained previously, the precise form of this counterterm is not well understood. Our case involves a surface Σ with interior boundary on the far past of the future horizon, namely the bifurcation surface $\Sigma_{\rm bif}$ at $w^\pm = 0$. The boundary charge on $\partial\Sigma = \Sigma_{\rm bif}$ is the black hole contribution to the charge. We will find below consistency conditions that require a nonzero X. A candidate that enables them to be satisfied is simply

$$X = 2dx^a h_a{}^b\Omega_b, \tag{7.77}$$

where Ω_a is the Hájíček one-form,

$$\Omega_a = q_a^c n^b\nabla_c l_b, \tag{7.78}$$

a measure of the rotational velocity of the horizon. Here the null vectors ℓ^a and n^a are both normal to $\Sigma_{\rm bif}$ and normalized such that $\ell\cdot n = -1$. ℓ (n) is taken to be normal to the future (past) horizon. ℓ and n must be invariant under 2π rotations which act in conformal coordinates as (7.40). This is satisfied by

$$\ell \sim y^{\frac{2T_R}{T_R+T_L}}\partial_+, \qquad n \sim y^{\frac{2T_L}{T_R+T_L}}\partial_- \tag{7.79}$$

on $\Sigma_{\rm bif}$[8]. $q_{ab} = g_{ab} + \ell_a n_b + n_a \ell_b$ is the induced metric on $\Sigma_{\rm bif}$.[9]

As a check on the normalization, we note that

$$\delta \mathcal{Q}(\partial_t, \delta_M g; g) = 1. \tag{7.80}$$

Here $\delta_M g$ is the linearized variation of the Kerr metric at fixed J. The Wald-Zoupas term $\delta \mathcal{Q}_X$ does not contribute to this computation.

We are especially interested in the central term in the Virasoro charge algebra. When the charge is integrable *and* there is a well-defined (invertible and associative) Dirac bracket $\{,\}$ on the reduced phase space, or in quantum language when $\mathcal{Q}_m$ is realized as an operator generating the diffeo ζ_n on a Hilbert space, one has, as shown in section 4,

$$\{\mathcal{Q}_n, \mathcal{Q}_m\} = (m-n)\mathcal{Q}_{m+n} + K_{m,n}, \tag{7.81}$$

where the central term is given by

$$K_{m,n} = \delta \mathcal{Q}(\zeta_n, \mathcal{L}_{\zeta_m} g; g). \tag{7.82}$$

Moreover, under these conditions, it has been proven (as reviewed in [106]) that the central term must be constant on the phase space and given, for some constant c_R by

$$K_{m,n} = \frac{c_R m^3}{12}\delta_{m+n}, \tag{7.83}$$

[8]These conditions uniquely fix ℓ and n up to a smooth rescaling under which $X_a \to \partial_a \phi$. We could fix this ambiguity by demanding *e.g.* that Ω be divergence-free on $\Sigma_{\rm bif}$ but this condition will not be relevant at the order to which we work.

[9]See for example [174] for a nice review of hypersurface geometry in the context of black holes.

up to terms which can be set to zero by shifting the charges.

In order to evaluate the charge and the central terms we must specify falloffs for h^{ab} near $\partial\Sigma = \Sigma_{\rm bif}$. One might demand that all components of h^{ab} (which is always required to be on shell) approach finite functions at $\Sigma_{\rm bif}$ at some rate as in [58]. However, this condition is violated by the h^{ab} produced by the large diffeos ζ_n. We accordingly augment the phase space to allow for these pure gauge modes as well as the on-shell non-gauge modes that approach finite values at $\Sigma_{\rm bif}$. [10] These oscillate periodically in the affine time along the null generators and do not approach a definite value at $\Sigma_{\rm bif}$, which is at infinite affine distance from any finite point on the horizon. Were they not pure gauge, such oscillating perturbations would have infinite energy flux and would be physically excluded. In the (non-affine) null coordinate w^+ along the horizon these modes can have poles at $w^+ = 0$. We will find that the charges are nevertheless well-defined and have a smooth $w^+ \to 0$ limit with such pure gauge excitations. Moreover, the emergence of a non-vanishing central term relies on the poles: since ζ is actually tangent to $\Sigma_{\rm bif}$ precisely at $w^+ = 0$, the $\delta\mathcal{Q}_X$ vanishes unless the perturbation produces a w^+-pole in X. In fact, in [58] it was shown that central terms cannot appear in the absence of poles. We will define and compute these central terms by working at small w^+ and then taking the limit. This amounts to approaching $\Sigma_{\rm bif}$ along the future horizon.

To compute the central term we take $\zeta = \zeta_m$ and $h^{ab} = \mathcal{L}_{\zeta_n} g^{ab}$.

As a first step, it is useful to examine the integrability of the Iyer-Wald term. As shown in section 4.2, the Iyer-Wald charge can be written in the form (4.39),

$$\mathbf{k}[h,g] = \delta \mathbf{Q}^N_\zeta[g] - \iota_\zeta \mathbf{\Theta}[h,g], \tag{7.84}$$

where the first term on the right hand side is the variation of the Noether charge density and the second term depends on the presymplectic potential. As

[10] The details of these rates are important for a complete investigation of integrability. We also restrict here to the phase space of fixed J . This is an analogue of fixing the number of branes in string theory, which indeed in some cases is U-dual to the higher-dimensional angular momentum.

explained earlier, the existence of the second term means that the Iyer-Wald charge is not necessarily integrable. However, as shown in Appendix C.1, for the specific class of functions that we are considering, it is possible to demonstrate that this term is integrable when evaluated on the bifurcation surface. This means that integrability arguments do not in themselves demand a counterterm when our specific choice of vector field is used and the bifurcation surface is chosen as the surface on which the charges are evaluated. However, we will see below that we are nevertheless required to add a non-zero Wald-Zoupas counterterm due to other conditions.

To evaluate the central term we will return to the explicit expression for the Iyer-Wald charge integrand, F_{ab}. It turns out that nonzero contributions to $K_{m,n}$ from $\delta\mathcal{Q}_{IW}$ come only from the component F_{IW}^{-y} in the form

$$\frac{1}{16\pi}\int_{\Sigma_{\text{bif}}} d\theta dw^{+}\varepsilon_{\theta+-y}F_{IW}^{-y}. \tag{7.85}$$

The range of $w^{+} \sim e^{4\pi^2 T_R}w^{+}$ goes to zero as Σ_{bif} is approached, so this expression naively vanishes. However, using the relation

$$\lim_{w_0^+\to 0}\int_{w_0^+}^{w_0^+ e^{4\pi^2 T_R}} \frac{dw^+}{w^+} = 4\pi^2 T_R, \tag{7.86}$$

such terms can nevertheless contribute as $\partial_+\zeta^y$ and h^{-y} develop $\frac{1}{w^+}$ poles for $w^+ \to 0$. One finds, after some algebra,

$$F_{IW}^{-y} = -4h_m^{y-}\zeta_n^y\Gamma^{-}_{y-}, \tag{7.87}$$

where

$$h_m^{-y} = g^{+-}\partial_+\zeta_m^y \tag{7.88}$$

has the requisite pole in w^+. Integrating over the sphere gives

$$K_{IWm,n} = 2J\frac{T_R}{T_L+T_R}m^3\delta_{n+m}. \tag{7.89}$$

Temperature dependence of the central term (7.89) violates the theorem [106] that it must be constant on the phase space. Hence there is an obstruction to constructing and integrating the charges $\delta\mathcal{Q}_{IW}$ with well-defined associative Dirac brackets, the existence of which is assumed in the theorem. We seek to remove this obstruction on the phase space of constant J by a suitable choice of X. However, we wish to stress the absence of this obstruction is necessary, but not a priori sufficient, for $\delta\mathcal{Q}$ to exist as an operator on a Hilbert space with all the desired properties. This is left to future investigations. Moreover, we have not shown that the choice of counterterm considered here is unique.

The obstruction is eliminated by including the Wald-Zoupas contribution $K_{Xm,n} = \delta\mathcal{Q}_X(\zeta_n, \mathcal{L}_{\zeta_m} g; g)$, which after integration over $\Sigma_{\rm bif}$ gives

$$K_{Xm,n} = J\frac{T_L - T_R}{T_L + T_R} m^3 \delta_{n+m}. \tag{7.90}$$

Adding terms (7.89) and (7.90) then yields the central charge

$$c_R = 12J. \tag{7.91}$$

7.6 Left movers

In order to compute the left-moving charges on $\Sigma_{\rm bif}$, it is necessary to evaluate (7.73) with $\zeta = \bar{\zeta}_m$ and $\bar{h}^{ab} = \mathcal{L}_{\bar{\zeta}_n} g^{ab}$. Now the relevant contribution to $\bar{K}_{m,n}$ comes only from F^{+y}_{IW}. On the past horizon, the range of $w^- \sim e^{4\pi^2 T_L} w^-$ now goes to zero as $\Sigma_{\rm bif}$ is approached but again one finds the appearance of poles for $w^- \to 0$, coming from terms such as $\partial_- \bar{\zeta}^y$ and $\bar{h}^{+y}$. F^{+y}_{IW} can be evaluated to be,

$$F^{+y}_{IW} = -4\bar{h}^{y+}_m \bar{\zeta}_n \Gamma^+_{+y}, \tag{7.92}$$

where

$$\bar{h}^{+y}_m = g^{+-}\partial_- \bar{\zeta}^y_m \tag{7.93}$$

has a pole in w^-. Integrating over the sphere gives

$$\bar{K}_{IW\,m,n} = 2J\frac{T_L}{T_R + T_L}m^3\delta_{m+n}. \tag{7.94}$$

Since $\Sigma_{\rm bif}$ is being approached from the past horizon, the vector fields ℓ^a and n^a are now defined so that ℓ is normal to the past horizon and n is normal to the future horizon. Again, both are null and satisfy $\ell \cdot n = -1$. An analysis of the periodicities gives

$$\ell \sim y^{\frac{2T_L}{T_R+T_L}}\partial_-, \; n \sim y^{\frac{2T_R}{T_R+T_L}}\partial_+. \tag{7.95}$$

The resulting term involving X integrates to

$$\bar{K}_{X\,m,n} = J\frac{T_R - T_L}{T_R + T_L}m^3\delta_{m+n}. \tag{7.96}$$

The sum of these two terms yields

$$c_L = 12J. \tag{7.97}$$

Without the counterterm, the sum of the charges resulting only from the Iyer-Wald term gives, $c_L + c_R = 24J$. We note that the Wald-Zoupas counterterm $\delta\mathcal{Q}_X$ contributes only to $c_L - c_R$ and not $c_L + c_R$ and hence may be related to the holographic gravitational anomalies discussed in [175]. This setup bears an interesting resemblance to Chern-Simons theory, which might provide a model for the general framework for the counterterm, leaving the enticing possibility that there is somehow a Chern-Simons theory living on the boundary of the horizon.

7.7 Reproducing entropy and discussion of Hilbert spaces

In this section we will show how the central charges derived above can lead to the black hole entropy. We will consider this entropy for the limiting case of the

Schwarzschild black hole and then discuss how the black hole Hilbert space may be contained within the Hilbert space of states outside the black hole.

7.7.1 CFT and Cardy entropy

As outlined in section 5.3.1, the statistical entropy of a CFT is given by Cardy's formula,

$$S_{Cardy} = \frac{\pi^2}{3}(c_L T_L + c_R T_R). \tag{7.98}$$

Using $c_L = c_R = 12J$ as given above and the temperature formulae (7.21) yields the Bekenstein-Hawking area-entropy law for the generic Kerr black hole,

$$S_{BH} = S_{Cardy} = 2\pi M r_+ = \frac{Area}{4}. \tag{7.99}$$

This shows that the entropy of the Kerr black hole can be understood in terms of microstates on the horizon and soft hair really does account for the black hole entropy. It is an inherently holographic picture. While a very illuminating result, this is not a solution to the information paradox but merely a hopeful step in the right direction. In section 9 we will discuss the limitations and further work needed in order that one might properly resolve the information paradox.

7.7.2 Schwarzschild limit

The Schwarzschild black hole is the generic non-rotating black hole. It is the Kerr black hole in the limit that the rotation parameter $a \to 0$. Looking back at the results of the previous sections, using (7.21), one sees that in this limit the left and right temperatures diverge and the central charges go to zero. This means that an analysis of the Schwarzschild black hole, following the steps outlined above, would have failed. However, despite these apparent degeneracies, the final result for the entropy is finite as the rotation becomes zero and does hold for the case of the Schwarzschild black hole. Further work is needed to properly understand how to find the central charges and corresponding Cardy formula in this context.

7.7.3 Discussion of Hilbert spaces

In this section we give a formal argument that, whenever black hole microstates are in representations of large-diffeomorphism-generated Virasoro algebras, as conjectured for Kerr in the previous sections, the black hole Hilbert space must be contained within the Hilbert space of states outside the black hole. The observations apply equally to the case discussed here and to the stringy black holes with near-AdS_3 regions. Our argument is a refined and sharpened version of those made elsewhere from different perspectives and is perhaps in the general spirit, if not the letter, of black hole complementarity.[11]

Consider a hypersurface $\Sigma_{\rm div}$ which divides the black hole spacetime into a black hole region and an asymptotically flat region with a hole. $\Sigma_{\rm div}$ may be taken to be the stretched horizon, the event horizon or in stringy cases the outer boundary of an AdS region: for the purposes of microstate counting the difference will be subleading and the distinction irrelevant. For a scalar field theory on such a fixed geometry it is reasonably well understood how to decompose the full Hilbert space $\mathcal{H}_{\rm full}$ of scalar excitations on a complete spacelike slice which goes through the black hole[12] as a product of 'black hole' and 'exterior' Hilbert spaces $\mathcal{H}_{\rm BH}$ and $\mathcal{H}_{\rm ext}$, following the Minkowski decomposition into the left and right Rindler Hilbert spaces. Roughly speaking, one expects the tensor product factorization,

$$\mathcal{H}_{\rm full} = \mathcal{H}_{\rm ext} \otimes \mathcal{H}_{\rm BH}. \tag{7.100}$$

For full quantum gravity, or even for linearized gravitons, it is not understood how to make such a decomposition. Nevertheless, in the stringy cases if $\Sigma_{\rm div}$ is taken to be the outer boundary of an AdS region, a practical working knowledge of how to proceed is well-established.

Let us nevertheless imagine that we have achieved such a decomposition which makes sense at leading semiclassical order for any of the above choices of $\Sigma_{\rm div}$.

[11] See [176] for a recent review.

[12] We consider here black holes such as those formed in a collapse process with no second asymptotic region, so that complete spacelike slices with only one asymptotic boundary exist.

A state in the full Hilbert space may then be expressed as a sum over product states,[13]

$$|\Psi_{\text{full}}\rangle = \sum_{A,b} c_{Ab} |\Psi^A_{\text{ext}}\rangle |\Psi^b_{\text{BH}}\rangle. \tag{7.101}$$

The existence of such a decomposition is presumed in many discussions of black hole information. Consider a set of diffeos ζ_n, defined everywhere in the spacetime, which all vanish near spatial infinity, but in a neighbourhood of Σ_{div} becomes a pair of Virasoro algebras which act nontrivially on the black hole. Since the diffeos vanish at infinity, the associated full charges must annihilate the full quantum state,

$$\mathcal{Q}(\zeta_n)_{\text{full}} |\Psi_{\text{full}}\rangle = 0. \tag{7.102}$$

On the other hand, beginning with the asymptotic surface integral expression for $\mathcal{Q}_{\text{full}}$ and integrating by parts we have

$$\mathcal{Q}(\zeta_n)_{\text{full}} = \mathcal{Q}(\zeta_n)_{\text{ext}} + \mathcal{Q}(\zeta_n)_{\text{BH}}. \tag{7.103}$$

Equation (7.102) then becomes

$$\sum_{A,b} c_{Ab} \big(\mathcal{Q}(\zeta_n)_{\text{ext}} |\Psi^A_{\text{ext}}\rangle\big) |\Psi^b_{\text{BH}}\rangle = -\sum_{A,b} c_{Ab} |\Psi^A_{\text{ext}}\rangle \mathcal{Q}(\zeta_n)_{\text{BH}} |\Psi^b_{\text{BH}}\rangle. \tag{7.104}$$

By assumption the black hole microstates transform non-trivially under the Virasoro algebra so neither side of the equation vanishes for all n.

In the generic case, absent any extra symmetries such as supersymmetry, we expect $\mathcal{H}_{\text{BH}}$ to be composed of Virasoro representations with highest weight h_k, where each h_k is distinct. A black hole microstate is then uniquely determined by specifying the representation in which it lies and location therein. In that case, (7.104) can be satisfied only if $\mathcal{H}_{\text{ext}}$ contains all the conjugate representations, and the constants c_{Ab} are chosen so that $|\Psi_{\text{full}}\rangle$ is a Virasoro singlet. At first the

[13] Very likely we will actually need an integral over Hilbert spaces corresponding to different boundary conditions on Σ_{div} [177–180] but we suppress this important point for notational brevity.

conclusion that the exterior state should transform under the Virasoro action may seem strange. But at second thought, the exterior region has an inner boundary on which ζ_n necessarily acts non-trivially, so this is entirely plausible.

Given this state of affairs, it follows immediately that the specific black hole microstate in $\mathcal{H}_{\mathrm{BH}}$ is fully determined by complete measurement of the microstate in $\mathcal{H}_{\mathrm{ext}}$: it is the unique element in the conjugate representation which forms a singlet with the exterior state. Instead of (7.100) we therefore have

$$\mathcal{H}_{\mathrm{full}} = \mathcal{H}_{\mathrm{ext}}. \tag{7.105}$$

That is, factorization of the Hilbert space with the inclusion of gravity fails in the most extreme possible way: there are no independent interior black hole microstates at all! This is of course a pleasing conclusion since the independent interior microstates are at the root of the information paradox.

For supersymmetric black holes, Bogomolny bounds enforce degeneracies in the weights h_k and the argument leading to (7.105) no longer works. Nevertheless, one may hope for a related mechanism, perhaps along the lines discussed in [181, 182] using discrete rather than continuous gauge symmetries, preventing an unwanted independent black hole Hilbert space.

8 The Kerr-Newman black hole

In this section we add charge Q and generalize the previous work [78] to the Kerr-Newman black hole. The addition of charge to the black hole is shown to require a minimal modification to this construction which then reproduces the Bekenstein-Hawking entropy for the Kerr-Newman black hole.

A Kerr black hole is characterized by two quantities, the mass M and spin J. It therefore cannot correspond to a general thermal state in a (parity symmetric) 2D CFT, which would be described by three parameters: the central charge $c_L = c_R$ and left and right temperatures T_L and T_R. What happens is that the Kerr black hole corresponds to thermal states with temperatures related by $T_L^2 = T_R^2 + 1$.

Interestingly the addition of charge relaxes the constraint between T_R and T_L which can be independently varied. Moreover, the conformal-coordinate expression for the near horizon geometry in terms of T_R and T_L given in [78] is unchanged, and the analysis proceeds in a nearly identical fashion to that of the neutral black hole. In particular, the macroscopic area law for Kerr-Newman also follows from the assumption of a Cardy formula governing the edge Hilbert space at the horizon.

8.1 Hidden conformal symmetry

We follow closely [108], in which the notion of hidden conformal symmetry was generalized from Kerr to Kerr-Newman black holes.

As explained in section 7.1, hidden conformal symmetry of the 4D Kerr black hole was identified [79] by examining the scalar wave equation of soft modes in the near-horizon region of phase space. In this region the solutions are hypergeometric functions of r which fall into representations of an $SL(2,\mathbb{R})$ conformal symmetry. The scalar wave equation can be written as the Casimir operator of a set of vector fields either with an $SL(2,\mathbb{R})_L$ or $SL(2,\mathbb{R})_R$ Lie bracket algebra. These 'hidden' symmetries are broken by the azimuthal angular identification. This allows for a canonical identification of left and right temperatures T_L and T_R (see formulae below).

Another way to glimpse the symmetry as shown in section 7.1 is from the near region contribution to the soft scalar absorption cross section

$$\mathcal{P}_{abs} \sim T_L^{2h_L-1} T_R^{2h_R-1} \sinh(\frac{\omega_L}{2T_L} + \frac{\omega_R}{2T_R}) \left|\Gamma(h_L + i\frac{\omega_L}{2\pi T_L})\right|^2 \left|\Gamma(h_R + i\frac{\omega_R}{2\pi T_R})\right|^2, \quad (8.1)$$

where for a scalar $h_L = h_R = \ell$ with ℓ the angular momentum and $\omega_{L,R}$ are thermodynamically conjugate to $T_{L,R}$. This precisely matches that of the absorption cross section of an energy (ω_L, ω_R) excitation of a 2D CFT at temperatures (T_L, T_R).

This structure is perhaps a hint that a hidden conformal symmetry, which acts not just on the geometry but on the phase space, is relevant to the structure of all black holes, not just extremal ones. As shown for the case of Kerr in section 7.7.1, with central charges $c_L = c_R = 12J$, the Cardy formula reproduces the entropy of the black hole [79]. It is of interest to try to push this speculative idea further in various directions and make it more explicit.[14]

This hidden conformal symmetry was subsequently shown to also be present in the case of 4D Kerr-Newman black holes [108, 109, 183, 184]. Here, the near-horizon neutral scalar wave equation for soft modes exhibits this same behaviour, with the additional constraint,

$$\omega Q \ll 1. \quad (8.2)$$

The Kerr-Newman entropy is

$$S_{BH} = \pi(r_+^2 + a^2) = \pi(2Mr_+ - Q^2). \quad (8.3)$$

where the outer event horizon r_+ and the inner Cauchy horizon r_- are defined by

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad (8.4)$$

[14]One interesting direction would be to investigate the crossover Kac-Moody structure discussed in [169].

and $a = \frac{J}{M}$. The first law is

$$T_H \delta S_{BH} = \delta M - \Omega \delta J - \Phi \delta Q, \tag{8.5}$$

where T_H is the Hawking temperature, given by

$$T_H = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \tag{8.6}$$

the angular velocity of the horizon Ω is

$$\Omega = \frac{a}{r_+^2 + a^2}, \tag{8.7}$$

and Φ is the electric potential of the Kerr-Newman black hole,

$$\Phi = \frac{Q r_+}{r_+^2 + a^2}. \tag{8.8}$$

The first law may also be written

$$\delta S_{BH} = \frac{\delta E_L}{T_L} + \frac{\delta E_R}{T_R}, \tag{8.9}$$

where

$$\begin{aligned} \delta E_L &= \frac{2M^2 - Q^2}{a}\delta M + \frac{Q(Q^2 - 2M^2)}{2J}\delta Q, \\ \delta E_R &= \frac{2M^2 - Q^2}{a}\delta M - \delta J - \frac{QM}{a}\delta Q, \end{aligned} \tag{8.10}$$

and the left and right temperatures are defined by,

$$\begin{aligned} T_L &= \frac{r_+ + r_-}{4\pi a} - \frac{Q^2}{4\pi M a}, \\ T_R &= \frac{r_+ - r_-}{4\pi a}. \end{aligned} \tag{8.11}$$

For a neutral scalar

$$\delta M = \omega, \quad \delta J = m, \quad \delta Q = 0, \tag{8.12}$$

with ω and m being the soft mode scalar energy and angular momentum operators. The Frolov-Thorne vacuum density matrix for such a scalar is (up to normalization)

$$\rho_{FT} = e^{-\frac{\omega}{T_H}+\frac{\Omega m}{T_H}} = e^{-\frac{\delta E_R}{T_R}-\frac{\delta E_L}{T_L}}. \tag{8.13}$$

The left/right energies are then given in terms of the left/right-moving frequencies by [109],

$$\begin{aligned} \delta E_L = \omega_L &= \frac{2M^2 - Q^2}{a}\omega, \\ \delta E_R = \omega_R &= \frac{2M^2 - Q^2}{a}\omega - m, \end{aligned} \tag{8.14}$$

with (ω, m) the soft mode energy and axial component of angular momentum. Using these modified definitions for Kerr-Newman, one then finds that the soft scalar absorption (8.1), originally derived for Kerr, remains valid.

In [78] and as shown here in earlier sections, this numerological discussion was brought into sharper focus for the case of the Kerr black hole by providing a set of $\mathrm{Vir}_{\mathrm{L}} \otimes \mathrm{Vir}_{\mathrm{R}}$ vector fields which generate the full symmetry. These vector fields were used to compute the central charges in the covariant phase space formalism. Here, the same argument is followed, with minor modifications, for the case of the Kerr-Newman black hole.

8.2 Conformal coordinates

The Kerr-Newman metric in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 = &-\Big(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\Big)dt^2 + \Big(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\rho^2}\Big)\sin^2\theta d\phi^2 \\ &- \Big(\frac{2a^2\sin^2\theta(r^2+a^2-\Delta)}{\rho^2}\Big)d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \end{aligned} \tag{8.15}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 + Q^2 - 2Mr. \tag{8.16}$$

The gauge field A is

$$A = -\frac{Qr}{\rho^2}(dt - a\sin^2\theta d\phi). \tag{8.17}$$

Conformal coordinates are [79]

$$\begin{aligned} w^+ &= \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_R \phi}, \\ w^- &= \sqrt{\frac{r - r_+}{r - r_-}} e^{2\pi T_L \phi - \frac{t}{2M}}, \\ y &= \sqrt{\frac{r_+ - r_-}{r - r_-}} e^{\pi (T_R + T_L)\phi - \frac{t}{4M}}. \end{aligned} \tag{8.18}$$

These are the same as defined in [79] for the case of Kerr, but note the different Q-dependent definitions of the temperatures (8.11) are used here. As before, it can be shown that the past horizon is at $w^+ = 0$, the future horizon at $w^- = 0$ and the bifurcation surface Σ_{bif} is at $w^\pm = 0$. Under azimuthal identification $\phi \to \phi + 2\pi$, the coordinates again have the periodicities,

$$w^+ \sim e^{4\pi^2 T_R} w^+, \quad w^- \sim e^{4\pi^2 T_L} w^-, \quad y \sim e^{2\pi^2 (T_R + T_L)} y. \tag{8.19}$$

Writing the Kerr-Newman metric in conformal coordinates, to leading and subleading order around the bifurcation surface, we get

$$\begin{aligned} ds^2 = &\frac{4\rho_+^2}{y^2} dw^+ dw^- + \frac{16 J^2 \sin^2\theta}{y^2 \rho_+^2} dy^2 + \rho_+^2 d\theta^2 \\ &- \frac{2w^+ (8\pi J)^2 T_R (T_R + T_L)}{y^3 \rho_+^2} dw^- dy \\ &+ \frac{8w^-}{y^3 \rho_+^2}\Big(- (4\pi J)^2 T_L (T_R + T_L) + (4J^2 + 4\pi J a^2 (T_R + T_L) + a^2 \rho_+^2)\sin^2\theta \Big) dw^+ dy \\ &+ \cdots , \end{aligned} \tag{8.20}$$

where corrections are at least second order in (w^+, w^-). This metric takes precisely the same form as the Kerr black hole (as in (7.41)), but again with different definitions of T_L, T_R and hence of w^+, w^-, y.

8.3 Conformal vector fields

Consider the same set of vector fields presented earlier in (7.55), but now with the new coordinate definitions and new temperatures,

$$
\begin{aligned}
\zeta_n &= \varepsilon_n \partial_+ + \frac{1}{2}\partial_+ \varepsilon_n y \partial_y, \quad \varepsilon_n = 2\pi T_R (w^+)^{1+\frac{in}{2\pi T_R}}, \\
\bar{\zeta}_n &= \bar{\varepsilon}_n \partial_- + \frac{1}{2}\partial_- \bar{\varepsilon}_n y \partial_y, \quad \bar{\varepsilon}_n = 2\pi T_L (w^-)^{1+\frac{in}{2\pi T_L}},
\end{aligned}
\tag{8.21}
$$

so that ζ and $\bar{\zeta}$ are invariant under 2π azimuthal rotations (8.19). These vector fields commute with one another and each obey a centreless Virasoro algebra,

$$
[\zeta_m, \zeta_n] = i(n-m)\zeta_{n+m}, \tag{8.22}
$$

and similarly for $\bar{\zeta}$. Their zero modes are

$$
\begin{aligned}
\zeta_0 &= 2\pi T_R (w^+ \partial_+ + \frac{1}{2} y \partial_y) = \partial_\phi + \frac{2M^2 - Q^2}{a}\partial_t = -i\omega_R, \\
\bar{\zeta}_0 &= 2\pi T_L (w^- \partial_- + \frac{1}{2} y \partial_y) = -\frac{2M^2 - Q^2}{a}\partial_t = i\omega_L
\end{aligned}
\tag{8.23}
$$

where the right and left moving energies ω_R, ω_L are defined in (8.14).

8.4 Covariant charges

Here we employ the construction of covariant charges as in section 4, in which we begin with the Lagrangian,

$$
\mathbf{L} = \frac{\sqrt{-g}}{16\pi}(R - F_{ab}F^{ab}), \tag{8.24}
$$

where R is the Ricci scalar and $F = dA$ is the electromagnetic field strength. Upon varying the field strength, δA_a and the metric, $\delta g^{ab} = h^{ab}$ in the Lagrangian, we get the presymplectic potential three-form, $\mathbf{\Theta} = *\theta$, where

$$
\theta[h, g, A, F] = (\theta_G[h, g] + \theta_E[A, F])_a dx^a \tag{8.25}
$$

where θ_G is the gravitational part of the presymplectic potential, as given in (4.30),

$$\theta^a_G[h,g] = -\frac{1}{16\pi}(\nabla_b h^{ab} - \nabla^a h) \tag{8.26}$$

and θ_E is the part arising from the electromagnetic piece of the Lagrangian,

$$\theta^a_E[A,F] = -\frac{1}{4\pi}F^{ab}\delta A_b. \tag{8.27}$$

When the metric variation is due to a diffeo ζ and A has a gauge transformation λ (which will be fixed below), i.e.

$$\begin{aligned} \delta g^{ab} &= h^{ab} = \mathcal{L}_\zeta g^{ab}, \\ \delta A_a &= (\mathcal{L}_\zeta A)_a + \nabla_a \lambda, \end{aligned} \tag{8.28}$$

then provided the background field equations are satisfied, the Noether charge density two-form, $\mathbf{Q_N} = *Q_N$, is defined from (4.38) and gives rise to,

$$\begin{aligned} (Q^{ab}_N)_G &= -\frac{1}{16\pi}(\nabla^a\zeta^b - \nabla^b\zeta^a), \\ (Q^{ab}_N)_E &= -\frac{1}{4\pi}F^{ab}(A_c\zeta^c + \lambda). \end{aligned} \tag{8.29}$$

The general form for the linearized charge associated to a diffeo ζ on a surface Σ with boundary $\partial\Sigma$ is (4.17)

$$\delta\mathcal{Q} = \frac{1}{16\pi}\int_{\partial\Sigma}\mathbf{k}, \tag{8.30}$$

where the symplectic 2-form charge integrand can be found from (4.16). With $\mathbf{k} = *(k_G + k_E)$, the gravitational part is the same as in (4.34),

$$k^{ab}_G = \frac{1}{16\pi}\big[\frac{1}{2}\nabla^a\zeta^b h + \nabla^a h^{cb}\zeta_c + \nabla_c\zeta^a\ h^{bc} + \nabla_c h^{ac}\ \zeta^b - \nabla^a h\ \zeta^b\big] - (a \leftrightarrow b). \tag{8.31}$$

This makes up the Iyer-Wald charge (previously written as F_{IWab}). The part due to the electromagnetic field strength is [185],

$$k_E^{ab} = -\frac{1}{8\pi}\left[(\delta F^{ab} - \frac{h}{2}F^{ab} + 2F^{ad}h_d^b)(A_c\zeta^c + \lambda) + F^{ab}\delta A_c\zeta^c - 2F^{ac}\delta A_c\zeta^b\right] - (a \leftrightarrow b). \tag{8.32}$$

Let us first consider the electromagnetic part of the charge. At this stage we fix the gauge freedom λ with the judicious choice

$$\lambda = -A_c\zeta^c \tag{8.33}$$

so that all but the last two terms in k_E^{ab} vanish. On the future horizon, where $w^- = 0$, the components that contribute to the integral are k_E^{-y} and k_E^{-+}. Since $\zeta^- = 0$, all terms in the calculation involve the components of the electromagnetic field strength where one index is w^-, i.e. either F^{-y}, F^{-+} or the $F^{-\theta}$ components. It is straightforward to compute these components in conformal coordinates and one finds that they are either zero or linear in w^-. Since the metric contains no poles in w^-, these components will vanish on the future horizon. This means that there is no contribution to the charge integral from k_E. Therefore the entire contribution to the charge arises from the gravitational part.

As explained in the previous sections, the construction of these covariant charges involves many subtleties and ambiguities. The Iyer-Wald charge associated to the diffeomorphisms ζ_n is built from the symplectic form when the metric variation is due to this diffeomorphism. This formalism alone is found to be inadequate for the construction of well-defined charges, as integrability and associativity may be violated, necessitating the addition of certain counterterms. It is equally possible that counterterms may arise for the electromagnetic part of the charge, due to similar ambiguities. However, since the electromagnetic part as defined gives zero contribution to the charge, one can not use arguments from integrability or associativity to motivate such an addition.

We may now proceed exactly as before. The general form for the linearized charge associated to a diffeo ζ on a surface Σ with boundary $\partial\Sigma$ is as before [85]

$$\delta\mathcal{Q} = \delta\mathcal{Q}_{IW} + \delta\mathcal{Q}_X, \tag{8.34}$$

where the Iyer-Wald charge is generated from the gravitational part of the presymplectic form as given above (7.73) and the Wald-Zoupas counterterm is of the form (4.41). As shown for the case of Kerr, the choice of counterterm for the gravitational case is not well-understood, but without a counterterm, there is an obstruction to defining integrable charges which canonically generate the symmetry. This obstruction can be eliminated by the same counterterm choice (7.77) given above in which

$$X = 2dx^a h_a{}^b \Omega_b. \tag{8.35}$$

As explained, it has not been shown that this choice is unique in eliminating the obstruction or that the charges so obtained do indeed canonically generate the symmetry. We nevertheless continue to use this formalism to construct the charges for the Kerr-Newman black hole.

Since the Kerr-Newman metric (8.20) is identical to the corresponding metric in Kerr (albeit with different definitions of the temperatures), the calculations of the charges with respect to the same vector fields will be identical. The resulting central terms are therefore,

$$c_L = c_R = 12J. \tag{8.36}$$

8.5 The area law

Using $c_L = c_R = 12J$ as given above, the temperature formulae (8.11) and the Cardy formula (5.69) yields the Bekenstein-Hawking area-entropy law for generic Kerr-Newman black holes,

$$S_{BH} = S_{Cardy} = \pi(2Mr_+ - Q^2) = \frac{Area}{4}. \tag{8.37}$$

9 Resolving the information paradox - further work

The above calculations of the entropy from microstates on the black hole horizon are extremely suggestive. However, the argument presented above is by no means a derivation of the entropy, nor does it resolve the information paradox, which remains one of the greatest mysteries in black hole physics. While we might still be very far from any resolution, the recent progress outlined above has demonstrated a fruitful direction to continue to explore.

One important feature of the above approach that is still not well understood is the precise nature of the Wald-Zoupas counterterm. Ultimately, it must be fully determined by the Dirac brackets. Computing these to find the charges whose brackets generate the symmetries will clinch the argument for the existence and identities of the counterterms. This calculation is left to future work.

Once the appropriate form of the counterterm has been determined, a natural next step is to discover whether this new formulation is universal. As shown, the results hold for both Kerr and Kerr-Newman black hole spacetimes, but beyond this it would be interesting to look at higher dimensional space-times, and to consider allowing for supersymmetry. A supersymmetric extension of the horizon generators for $\mathcal{N} = 1, 2$ and 4 will appear shortly. This will allow some insight into the precise nature of the CFT that exists on the black hole horizon.

The current formulation involves using two Virasoro algebras, one on the past and one on the future horizon. While this fits in nicely with the idea of left and right moving CFTs and allows for application of the Cardy formula, the use of the past horizon is somewhat uncomfortable. Another possibility as suggested above is that on the future horizon, we have both a Virasoro and a Kac-Moody algebra, and together these can reproduce the entropy [169].

In terms of the bigger picture of the information paradox, there remain many unanswered questions, including: Does this entropy account for all the black hole's information? How does information in collapse get encoded into the field theory? How can we reconcile this approach with the species problem? A final resolution of the information paradox requires that these questions be addressed.

10 Conclusion

In this thesis, we have considered the action of large gauge transformations on both null infinity and the black hole horizon.

The first part of this thesis involved extending the known BMS group of symmetry transformations that preserve the structure of null infinity, to allow for conformal symmetry. This resulted in a conformal BMS group, an infinite extension of the flat space conformal group, analogous to the BMS group, an infinite extension of the flat space Poincaré group. Constructing the algebra of this new conformal BMS group involved taking into account an important subtlety: the vector fields that generate the group cause perturbations in the metric, but they are themselves metric dependent. This means that instead of the usual Lie bracket, a modified bracket is required. This modified bracket was developed and the resulting algebra analysed in detail.

The latter part of this thesis involved looking at large gauge transformations in the context of black holes. We began with an overview of the AdS/CFT correspondence, exploring three-dimensional gravity and the BTZ black hole. We gave a brief recap of conformal field theory and showed how the Cardy formula can be used to compute the entropy. We hypothesised that the BTZ black hole is dual to a two-dimensional conformal field theory and showed that the Cardy entropy precisely matches the black hole entropy, computed from Bekenstein-Hawking area law.

We then moved away from the AdS/CFT correspondence to consider four dimensional gravity, exploring the extreme Kerr black hole and then ultimately examining the case of the generic Kerr black hole. By looking at the hidden conformal symmetry in the near region of the phase space, we found motivation for the idea that the Kerr black hole is itself dual to a two-dimensional thermal conformal field theory. We found a set of Virasoro generators which realize this symmetry and act non-trivially on the black horizon. The covariant phase space formalism provides a formula for the Virasoro charges as surface integrals on the horizon. The path taken to reach such formulae however involve many subtleties and ambiguities, meaning that it is possible that one can add additional terms, or counterterms. Integrability and associativity of the charge algebra were shown to require the inclusion of 'Wald-Zoupas' counterterms. A counterterm satisfying the known consistency requirement was constructed and yields central charges $c_L = c_R = 12J$. Assuming the existence of a quantum Hilbert space on which these charges generate the symmetries, as well as the applicability of the Cardy formula, the central charges reproduce the macroscopic area-entropy law for generic Kerr black holes. We have thus shown that it is possible to reproduce the macroscopic area law by using microstates on the horizon. This is not a solution to the information paradox, but hopefully a useful step in the right direction.

A The modified bracket of CBMS

As explained in section 3.1, computing the algebra of the conformal BMS group required a delicate examination of the effect of each vector field on the spacetime, and how this would affect the action of a subsequent transformation. Here is a worked example for the commutator of two different supertranslations, giving

$$[T_1, T_2] = 0. \tag{A.1}$$

We start by considering the action of a supertranslation, generated by the function g,

$$T_1 = g\,\partial_u + \frac{1}{2}D^2 g\,\partial_r - \frac{1}{r}D^A g\,\partial_A\,. \tag{A.2}$$

The ordinary commutator of this supertranslation, together with another supertranslation, T_2, generated by the function f, gives,

$$\begin{aligned}
[T_1, T_2] &= \left[g\,\partial_u + \frac{1}{2}D^2 g\,\partial_r - \frac{1}{r}D^A g\,\partial_A, f\,\partial_u + \frac{1}{2}D^2 f\,\partial_r - \frac{1}{r}D^A f\,\partial_A\right], \\
&= \frac{1}{2r}(D^A f D_A D^2 g - D^A g D_A D^2 f)\,\partial_r \\
&\quad + \frac{1}{2r^2}(D^2 g D^A f - D^2 f D^A g + 2D^B g D_B D^A f - 2D^B f D_B D^A g)\partial_A\,.
\end{aligned} \tag{A.3}$$

This has the form,

$$[T_1, T_2] = \frac{1}{r}A\,\partial_r + \frac{1}{r^2}B^A\partial_A\,, \tag{A.4}$$

where A and B are functions of the two-sphere only.

By considering dimensions, this implies that

$$\mu_2^u = 0\,, \tag{A.5}$$

and we write

$$\mu_2^r = \frac{1}{r}\hat{A}\,, \tag{A.6}$$

and

$$\mu_2^A = \frac{1}{r^2}\hat{B}^A\,. \tag{A.7}$$

Under the action of the first supertranslation the resulting infinitesimal changes to the metric are given by,

$$\begin{aligned}\hat{h}_{uA} &= -\frac{1}{2}D_A(2g + D^2 g)\,,\\ \hat{h}_{AB} &= -r(2D_A D_B g - \gamma_{AB}D^2 g)\,,\end{aligned} \tag{A.8}$$

with all other components zero.

Then, under the action of the second supertranslation, T_2, on the metric there will be extra second order terms, $\hat{K}_{ab}$, given by,

$$\begin{aligned}\hat{K}_{ab} =& \mu_2^c\,\partial_c g_{ab} + T_2^c\,\partial_c \hat{h}_{ab} + \,\partial_a T_2^c \hat{h}_{bc} + \,\partial_a \mu_2^c g_{bc} + \,\partial_b \mu^c g_{ac} + \,\partial_b T_2^c \hat{h}_{ac}\\ &- \frac{1}{2}g_{ab}\,\partial_c\mu_2^c - \frac{1}{2}\hat{h}_{ab}\,\partial_c T_2^c - \frac{1}{2}g_{ab}\Gamma^c_{cd}\mu^d - \frac{1}{2}\hat{h}_{ab}\Gamma^c_{cd}T_2^d - \frac{1}{2}g_{ab}\delta\Gamma^c_{cd}T_2^d\,.\end{aligned} \tag{A.9}$$

The relevant Christoffel symbols and perturbations are given by,

$$\begin{aligned}\Gamma^A{}_{Ar} &= \frac{2}{r}\,,\\ \delta\Gamma^r_{rA} &= \frac{1}{2r}D_A(D^2+2)g\,,\\ \delta\Gamma^A_{AB} &= -\frac{1}{2r}D_B(D^2+2)g\,.\end{aligned} \tag{A.10}$$

Thus, explicitly calculating the second order changes to the metric,

$$\begin{aligned}\hat{K}_{rA} =& 0 = g_{ru}D_A\mu^u + g_{AB}\,\partial_r\mu^B + \,\partial_r T_2^B \hat{h}_{AB},\\ &= -r^2\hat{\gamma}_{AB}\left(\frac{2}{r^3}\hat{B}^A\right) - \frac{1}{r}D^B f(2D_A D_B g - \gamma_{AB}D^2 g)\,,\end{aligned} \tag{A.11}$$

Therefore,

$$\hat{B}_A = -\frac{1}{2}D^B f(2D_A D_B g - \gamma_{AB}D^2 g)\,. \tag{A.12}$$

$$
\begin{aligned}
\hat{K}_{AB} = \mathcal{O}(r) = & \, r^2(D_A\mu_B + D_B\mu_A - \frac{1}{2}\gamma_{AB}(\partial_u\mu^u + \partial_r\mu^r + D_C\mu^C - \frac{2}{r}\mu^r)) \\
& + D_A T_2^C \hat{h}_{BC} + D_B T_2^C \hat{h}_{AC} + D_A T_2^u \hat{h}_{uB} + D_B T_2^u \hat{h}_{uA} \\
& + T_2^r \, \partial_r \hat{h}_{AB} + T_2^C D_C \hat{h}_{AB} - \frac{1}{2}\hat{h}_{AB} D_C T_2^C - \frac{1}{r}\hat{h}_{AB} T_2^r \\
& - \frac{1}{2} r^2 \gamma_{AB} \delta\Gamma^c_{cd} T_2^d \, .
\end{aligned} \tag{A.13}
$$

Since

$$
\hat{h}^A_A = 0 \, , \tag{A.14}
$$

then,

$$
\begin{aligned}
\hat{K}^A_A = & \, r^2 D_A \mu^A - r^2 \, \partial_r \mu^r + 2r\mu^r + 2D^A T_2^B \hat{h}_{AB} + 2D^A T_2^u \hat{h}_{uA} - r^2 \delta\Gamma^c_{cd} T_2^d \, , \\
= & \, 2r\mu^r - r^2 \, \partial_r \mu^r + r^2 D_A \mu^A + 2D^A D^B f (2D_A D_B g - \gamma_{AB} D^2 g) \\
& - D^A f D_A (D^2 + 2) g, \\
= & \, 3\hat{A} - \frac{1}{2} D^A D^B f (2D_A D_B g - \gamma_{AB} D^2 g) - \frac{1}{2} D^B f (2D^2 D_B g - D_B D^2 g) \\
+ & \quad 2D^A D^B f (2D_A D_B g - \gamma_{AB} D^2 g) - D^A f D_A (D^2 + 2) g, \\
= & \, 3\hat{A} + 3D^A D^B f D_A D_B g - \frac{3}{2} D^2 f \, D^2 g - \frac{3}{2} D^B f \, D_B D^2 g - 3D^A f D_A g \, .
\end{aligned} \tag{A.15}
$$

Since

$$
\partial_r \left(\det \left(\frac{g_{AB}}{r^2} \right) \right) = 0 \, , \tag{A.16}
$$

we have,

$$
\hat{A} = \frac{1}{2} D^B f D_B D^2 g - D_A D_B f D^A D^B g + \frac{1}{2} D^2 f D^2 g + D^A f D_A g \, . \tag{A.17}
$$

Thus,

$$
\begin{aligned}
\mu_2^u &= 0\,, \\
\mu_2^r &= \frac{1}{r}\left(\frac{1}{2}D^B f D_B D^2 g - D_A D_B f D^A D^B g + \frac{1}{2}D^2 f D^2 g + D^A f D_A g\right)\,, \\
\mu_2^A &= \frac{1}{2r^2} D^B f (2D_A D_B g - \gamma_{AB} D^2 g)\,.
\end{aligned}
\tag{A.18}
$$

When we perform the same set of calculations using first the action of T_2, followed by T_1, we get the same results for μ_1^a, with $f \leftrightarrow g$.

Therefore, we can calculate,

$$
\delta\mu^a = \mu_2^a - \mu_1^a\,, \tag{A.19}
$$

to find,

$$
\begin{aligned}
\delta\mu^u &= 0\,, \\
\delta\mu^r &= \frac{1}{2r}(D^B f D_B D^2 g - D^B g D_B D^2 f)\,, \\
\delta\mu^A &= \frac{1}{2r^2}(D_B g(2D^A D^B f - \gamma^{AB} D^2 f) - D_B f(2D^A D^B g - \gamma^{AB} D^2 g)) \\
&= \frac{1}{2r^2}(D^2 g D^A f - D^2 f D^A g + 2D^B g D_B D^A f - 2D^B f D_B D^A g)\,.
\end{aligned}
\tag{A.20}
$$

These terms exactly cancel those arising from the ordinary commutator, and so upon subtracting these off, we find that,

$$
[T_1, T_2] = 0\,. \tag{A.21}
$$

B Proof of surface charge formula

In this section we will prove the following formula, as given in (4.16),

$$
\mathbf{k}_\zeta[\delta\Phi, \Phi] \;=\; \delta\mathbf{Q}_\zeta^N[\delta\Phi, \Phi] - \iota_\zeta \boldsymbol{\Theta}[\delta\Phi, \Phi]. \tag{B.1}
$$

This proof follows that given in [131], with intermediate steps stem from [186].

The starting point is the Noether current, $\mathbf{J}_\zeta$, defined in (4.10), as

$$\mathbf{J}_\zeta = \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \zeta \cdot \mathbf{L}. \tag{B.2}$$

Variation of the current gives

$$\delta \mathbf{J}_\zeta = \delta \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \zeta \cdot \delta \mathbf{L}, \tag{B.3}$$

since the vector field ζ is not a dynamical variable and so is not affected by the variation. Using (4.9) we have,

$$\begin{aligned} \delta \mathbf{L} &= \mathbf{E}_\Phi \delta \Phi + d\mathbf{\Theta}[\delta \Phi, \Phi] \\ &\approx d\mathbf{\Theta}[\delta \Phi, \Phi], \end{aligned} \tag{B.4}$$

where the second line follows when the equations of motion are satisfied and $\mathbf{E}_\Phi = 0$. Cartan's "magic formula" tells us that $\delta_\zeta \mathbf{\Theta} = \zeta \cdot d\mathbf{\Theta} + d(\zeta \cdot \mathbf{\Theta})$ and therefore,

$$\begin{aligned} \delta \mathbf{J}_\zeta &\approx \delta \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \zeta \cdot d\mathbf{\Theta}[\delta \Phi, \Phi], \\ &= \delta \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \delta_\zeta \mathbf{\Theta}[\delta \Phi, \Phi] + d(\zeta \cdot \mathbf{\Theta}). \end{aligned} \tag{B.5}$$

Since $\delta \mathbf{J}_\zeta = d\delta \mathbf{Q}_\zeta^N$, we find

$$\delta \mathbf{\Theta}[\delta_\zeta \Phi, \Phi] - \delta_\zeta \mathbf{\Theta}[\delta \Phi, \Phi] = d(\delta \mathbf{Q}_\zeta^N - \zeta \cdot \mathbf{\Theta}). \tag{B.6}$$

The left hand side of this expression can be identified as $\boldsymbol{\omega}[\delta_\zeta \Phi, \delta \Phi, \Phi]$. Since

$$\boldsymbol{\omega}[\delta \Phi, \delta_\zeta \Phi, \Phi] \approx d\mathbf{k}_\zeta[\delta \Phi, \Phi], \tag{B.7}$$

we see that

$$\mathbf{k}_\zeta[\delta \Phi, \Phi] \;=\; \delta \mathbf{Q}_\zeta^N[\delta \Phi, \Phi] - \zeta \cdot \mathbf{\Theta}[\delta \Phi, \Phi] \tag{B.8}$$

and thus we arrive at our original expression (4.16).

C Integrability of $\iota_\zeta\theta$

In section 4 we derived an expression for the charge variation, $\mathcal{Q}_\zeta$ in terms of a Noether charge and the presymplectic form (4.39). This charge variation is not, in general integrable, since although the Noether term is exact, the term arising from the presymplectic potential is not. Here we will show this explicitly. Ignoring factors, the presymplectic potential θ is given for general relativity by

$$\begin{aligned}\theta^a &= \nabla_b h^{ab} - \nabla^a h \\ &= \partial_b h^{ab} + \Gamma^a_{bc} h^{bc} + \Gamma^b_{bc} h^{ac} - g^{ab}\partial_b(g_{cd}h^{cd}).\end{aligned} \tag{C.1}$$

If we now vary this with $\delta g^{ab} = k^{ab}$, $\delta g_{ab} = -k_{ab}$, $\delta\Gamma^a_{bc} = \frac{1}{2}(\nabla^a k_{bc} - \nabla_b k^a_c - \nabla_c k^a_b)$ and $\delta h^{ab} = 0$, we get,

$$\begin{aligned}\delta\theta^a &= \delta\Gamma^a_{bc} h^{bc} + \delta\Gamma^b_{bc} h^{ac} - k^{ab}\partial_b(g_{cd}h^{cd}) + g^{ab}\partial_b(k_{cd}h^{cd}) \\ &= \frac{1}{2}h^{bc}(\nabla^a k_{bc} - \nabla_b k^a_c - \nabla_c k^a_b) - \frac{1}{2}h^{ac}\nabla_c k - k^{ac}\nabla_c h + \nabla^a(k_{cd}h^{cd}).\end{aligned} \tag{C.2}$$

We fix the gauge such that $h = k = 0$ and so under antisymmetry of $k \leftrightarrow h$, we get

$$(\delta_2\theta_1 - \delta_1\theta_2)^a = \frac{1}{2}h^{bc}(\nabla^a k_{bc} - \nabla_b k^a_c - \nabla_c k^a_b) - (k \leftrightarrow h) \tag{C.3}$$

Since this does not identically vanish, we must conclude that θ is not in general integrable.

C.1 Our special case

In this section we will demonstrate that, for the specific class of functions considered in section 7.4, when evaluated on the bifurcation surface, the charges turn out to be integrable.

The potentially non-integrable term as shown above comes from the presymplectic potential. The antisymmetrized variation of θ^- is (C.3)

$$(\delta_2\theta_1 - \delta_1\theta_2)^- \;=\; \frac{1}{2}h^{bc}(\nabla^- k_{bc} - \nabla_b k_c^- - \nabla_c k_b^-) - (k \leftrightarrow h). \tag{C.4}$$

In our case, metric perturbations are generated by the vector field ζ, given by (7.55), i.e. $\delta g^{ab} = h^{ab} = (\mathcal{L}_\zeta g)^{ab}$. To first order in w^+ there are three non-zero components of h^{ab}. These are:

$$h^{+y}, h^{-y}, h^{yy}. \tag{C.5}$$

This means that the variation of θ^- with our specific perturbation h^{ab} is explicitly,

$$\begin{aligned}(\delta_2\theta_1 - \delta_1\theta_2)^- = {}& h^{-y}(\nabla^- k_{-y} - \nabla_- k_y^- - \nabla_y k_-^-)\\ &+\; h^{+y}(\nabla^- k_{+y} - \nabla_+ k_y^- - \nabla_y k_+^-)\\ &+\; \frac{1}{2}h^{yy}(\nabla^- k_{yy} - \nabla_y k_y^- - \nabla_y k_y^-)\\ &-\; (h \leftrightarrow k)\end{aligned} \tag{C.6}$$

We can now go through line by line, noting that the only non-zero components of h_{ab} to first order in w^+ are,

$$h_{+y}, \quad h_{-y}, \quad h_{+-}. \tag{C.7}$$

The first line gives,

$$\begin{aligned}(1) &= h^{-y}(\nabla^- k_{-y} - \nabla_- k_y^- - \nabla_y k_-^-) - (h \leftrightarrow k)\\ &= h^{-y}(g^{+-}(\partial_+ k_{-y} - \Gamma^+_{+y} k_{-+}) - (\partial_- k_y^- - \Gamma^-_{-y} k_-^- + \Gamma^-_{-y} k_y^y)\\ &\quad - (\partial_y k_-^- - \Gamma^-_{-y} k_-^- + \Gamma^-_{y-} k_-^-) - (h \leftrightarrow k)\\ &= h^{-y}(g^{+-}(\partial_+ k_{-y} - \Gamma^+_{+y} k_{+-} - \partial_- k_{+y} + \Gamma^-_{-y} k_{+-})\\ &\quad - \partial_y(g^{+-} k_{+-}) - (h \leftrightarrow k)\end{aligned} \tag{C.8}$$

Now,

$$h^{-y} = g^{+-}\partial_+\zeta^y \tag{C.9}$$

and

$$k_{+-} = g_{-y}\partial_+\tilde{\zeta}^y \tag{C.10}$$

Thus the combination

$$h^{-y}k_{+-} - (\zeta \leftrightarrow \tilde{\zeta}) = 0, \tag{C.11}$$

so terms of this form in (C.8) vanish, since they are multiplied simply by connection components which are functions only of the coordinates. We also have $\partial_- k_{+y} = 0$ and so

$$(1) \quad = \quad g^{+-}h^{-y}\partial_+ k_{-y} - (h \leftrightarrow k). \tag{C.12}$$

Now,

$$\begin{aligned} k_{-y} &= \tilde{\zeta}^+\partial_+ g_{-y} + \tilde{\zeta}^y\partial_y g_{-y} + g_{-y}\partial_y\tilde{\zeta}^y \\ &= \tilde{\zeta}^+\partial_+ g_{-y} - \frac{2}{y}\tilde{\zeta}^y g_{-y}. \end{aligned} \tag{C.13}$$

Thus

$$\begin{aligned} \partial_+ k_{-y} &= (\partial_+\tilde{\zeta}^+ - \frac{2}{y}\tilde{\zeta}^y)\partial_+ g_{-y} - \frac{2}{y}g_{-y}\partial_+\tilde{\zeta}^y \\ &= -\frac{2}{y}g_{-y}\partial_+\tilde{\zeta}^y. \end{aligned} \tag{C.14}$$

Therefore,

$$\begin{aligned} (1) &= g^{+-}h^{-y}\partial_+ k_{-y} - (h \leftrightarrow k) \\ &= g^{+-}g^{+-}\partial_+\zeta^y(-\frac{2}{y}g_{-y}\partial_+\tilde{\zeta}^y) - (\zeta \leftrightarrow \tilde{\zeta}) \\ &= 0, \end{aligned} \tag{C.15}$$

since this is symmetric in $\zeta, \tilde{\zeta}$.

The second line gives,

$$\begin{aligned}(2) &= h^{+y}(\nabla^{-}k_{+y} - \nabla_{+}k_{y}^{-} - \nabla_{y}k_{+}^{-}) - (h \leftrightarrow k) \\ &= -h^{+y}(\partial_{y}k_{+}^{-} + \Gamma^{-}_{y-}k_{+}^{-} - \Gamma^{+}_{+y}k_{+}^{-}) \\ &= 0, \end{aligned} \tag{C.16}$$

since $k_{++} = 0$ to first order on the bifurcation surface.

The third line gives,

$$\begin{aligned}(3) &= \frac{1}{2}h^{yy}(\nabla^{-}k_{yy} - 2\nabla_{y}k_{y}^{-}) - (h \leftrightarrow k) \\ &= \frac{1}{2}h^{yy}(-2g^{-+}\Gamma^{+}_{+y}k_{+y} - 2(\partial_{y}(g^{-+}k_{+y}) + \Gamma^{-}_{-y}k_{y}^{-} - \Gamma^{y}_{yy}k_{y}^{-})) - (h \leftrightarrow k). \end{aligned} \tag{C.17}$$

Every term in (C.17) is, to first order on the bifurcation surface, of the form

$$h^{yy}k_{+y}, \tag{C.18}$$

multiplied by some function of the coordinates. But,

$$h^{yy} = -2g^{+y}\partial_{+}\zeta^{y}, \tag{C.19}$$

and

$$k_{+y} = g_{yy}\partial_{+}\tilde{\zeta}^{y}, \tag{C.20}$$

so

$$h^{yy}k_{+y} - (h \leftrightarrow k) = -2g^{+y}\partial_{+}\zeta^{y}g_{yy}\partial_{+}\tilde{\zeta}^{y} - (\zeta \leftrightarrow \tilde{\zeta}) = 0. \tag{C.21}$$

This means that although θ^{-} is in general not integrable, for the specific functions that we are considering, we find, on the bifurcation surface,

$$(\delta_{2}\theta_{1} - \delta_{1}\theta_{2})^{-} = 0. \tag{C.22}$$

D BTZ charges on the horizon

We will consider a similar computation of the charges to that in section 5.3.2 but now the calculation will explicitly take place on the horizon of the black hole.

It is worth noting that although the original calculation of the BTZ central charge by Brown and Henneaux was performed at infinity, an extension away from infinity is possible. Although the integrals in (5.71) are taken over S^1_∞, the integrands are independent of the radial coordinate, and thus we may in fact perform this integral at other points in the spacetime and achieve the same result.

Nonetheless, we will use a different coordinate system, that will be more aligned with the methods used for the Kerr black hole. We consider the AdS_3 spacetime in Poincaré coordinates, with the line element

$$ds^2 = l^2 \, \frac{dw^+ dw^- + dy^2}{y^2} \tag{D.1}$$

where $w^\pm$ are unrestricted and $y \geq 0$. These coordinates do not cover the entirety of AdS_3. The BTZ black hole is the quotient of AdS_3 found by making the identifications

$$\begin{aligned} w^+ &\rightarrow e^{4\pi^2 T_R} w^+, \\ w^- &\rightarrow e^{4\pi^2 T_L} w^-, \\ y &\rightarrow e^{2\pi^2 (T_L + T_R)} y. \end{aligned} \tag{D.2}$$

The parameters T_L and T_R can be related to the mass and angular momentum of the black hole and l by

$$M = \frac{\pi^2}{2} T_L T_R \tag{D.3}$$

and

$$J = \frac{\pi^2 l}{4} (T_L^2 - T_R^2). \tag{D.4}$$

The past and future (outer) horizons are the surfaces $w^+ = 0$ and $w^- = 0$

respectively and spatial infinity is at $y = 0$.

T_L and T_R are related to r_+ and r_- by

$$T_L = \frac{r_+ + r_-}{2\pi l} \tag{D.5}$$

and

$$T_R = \frac{r_+ - r_-}{2\pi l} \tag{D.6}$$

Armed with the diffeomorphism found previously for BTZ, we can now consider the following vector field in these coordinates,

$$\begin{aligned} \zeta_n &= \varepsilon_n(w^+)\partial_+ + \frac{1}{2}y\partial_+\varepsilon_n\partial_y, \\ \varepsilon_n &= 2\pi T_R (w^+)^{1+\frac{in}{2\pi T_R}}. \end{aligned} \tag{D.7}$$

The Lie bracket is again the centreless Virasoro algebra. Another vector field defined,

$$\begin{aligned} \tilde{\zeta}_n &= \tilde{\varepsilon}_n(w^-)\partial_- + \frac{1}{2}y\partial_-\tilde{\varepsilon}_n\partial_y, \\ \tilde{\varepsilon}_n &= 2\pi T_L (w^-)^{1+\frac{in}{2\pi T_L}}, \end{aligned} \tag{D.8}$$

also obeys a centreless Virasoro algebra and commutes with the first set of vector fields. These diffeomorphisms are now used to calculate the central charges c_L and c_R for the BTZ black hole on the horizons, $\mathcal{H}^{\pm}$.

In order to compute the charges and any central terms, following the covariant phase space formalism explained in section 4, we need to calculate the integral,

$$\delta\mathcal{Q}_{IW}(\zeta, h; g) = \frac{1}{16\pi}\int_{\partial\Sigma_3} \mathbf{k}_{IW}, \tag{D.9}$$

where the integral is computed over the future horizon.

If the charges contain non-zero central terms in their algebra, we will find that (D.9) will be non-vanishing. The central terms in the algebra between the charges $\mathcal{Q}_m, \mathcal{Q}_n$, built from the vector fields $\zeta_n, \bar{\zeta}_m$ will be of the form,

$$K_{\zeta,\bar{\zeta}}[\bar{\Phi}] = \int_{\partial\Sigma} \mathbf{k}_\zeta[\delta_{\bar{\zeta}}\bar{\Phi}, \bar{\Phi}], \tag{D.10}$$

assuming at this stage that there are no additional counterterms.

The general formula for the central term is

$$K_{\zeta_m,\bar{\zeta}_n} = \frac{c}{12} m^3 \delta_{m+n,0} \tag{D.11}$$

up to terms that can be set to zero by shifting the zero modes. This means that in order to determine the value of the central charge, c, it will be sufficient to look only at those terms in (4.34) which could contribute to cubics in m.

To evaluate the central term we will return to the explicit expression for the Iyer-Wald charge integrand, F_{ab},

$$(F_{IW})_{ab} = \frac{1}{2}\nabla_a \zeta_b h + \nabla_a h^c{}_b \zeta_c + \nabla_c \zeta_a \; h^c{}_b + \nabla_c h^c{}_a \; \zeta_b - \nabla_a h \; \zeta_b - (a \leftrightarrow b). \tag{D.12}$$

As explained in section 4.3, when the variation h is also generated by a vector field $\bar{\zeta}$, we can add the divergence of a three-form to simplify the expression and end up with an equivalent expression,

$$\begin{aligned}(F')_{ab} &= (F)_{ab} + \nabla^c A_{abc} \\ &= R_{abcd}\zeta^c\bar{\zeta}^d + 2\nabla_c\zeta_a\nabla^c\bar{\zeta}_b - D\nabla_a\bar{\zeta}_b + \bar{D}\nabla_a\zeta_b - (a \leftrightarrow b),\end{aligned} \tag{D.13}$$

where $D = \nabla_a \zeta^a$.

The vector field involves one derivative of ε_m, and therefore to find a cubic term in the central charge we must have at least one derivative of the vector field. This means that the term involving the Riemann tensor in (D.13) cannot possibly contribute to the central charge c in (D.11). The vector field is also divergence free, which means that we are left with an extremely simple expression for the only terms that can contribute to the cubics of c:

$$(F_{cubics})_{ab} = 2\nabla_c \zeta_a \nabla^c \bar{\zeta}_b - 2\nabla_c \zeta_b \nabla^c \bar{\zeta}_a. \tag{D.14}$$

We will first use the right-moving vector fields, ζ and consider an integral over the bifurcation surface, where $w^+ = w^- = 0$. This surface is explored in section 7.5 for the case of Kerr. It turns out that nonzero contributions to $K_{m,n}$ from $\delta \mathcal{Q}_{IW}$ come only from the component F_{IW}^{-y} in the form

$$\frac{1}{16\pi} \int d\theta dw^+ \varepsilon_{\theta+-y} F_{IW}^{-y}. \tag{D.15}$$

This integral is discussed in more detail in section 7.5. We have,

$$\begin{aligned} F^{-y} &= 2\nabla_c \zeta^- \nabla^c \bar{\zeta}^y - (\zeta \leftrightarrow \bar{\zeta}) \\ &= 2g^{+-} \nabla_- \zeta^- \nabla_+ \bar{\zeta}^y - (\zeta \leftrightarrow \bar{\zeta}) \\ &= 2g^{+-} \Gamma^-_{-y} \zeta^y \partial_+ \bar{\zeta}^y - (\zeta \leftrightarrow \bar{\zeta}) \\ &= -\frac{y^3}{\ell^2} \varepsilon' \bar{\varepsilon}'' - (\zeta \leftrightarrow \bar{\zeta}). \end{aligned} \tag{D.16}$$

Since

$$\varepsilon_{\theta+-y} = \frac{\ell^3}{2y^3} \tag{D.17}$$

after performing the θ integral in (D.15) and integrating by parts over w^+, we have

$$K_{m,n} = \frac{\ell}{8} \int_1^{e^{4\pi^2 T_R}} dw^+ \varepsilon_m \bar{\varepsilon}_n'''. \tag{D.18}$$

Inserting the mode expansions for ε_n, we get a pole in w^+, a feature which, as we discover during the discussion for the Kerr case, turns out to be essential. We can integrate to find the cubic term and the result is

$$c_R = \frac{3\ell}{2}. \tag{D.19}$$

Repeating the calculation for the other, left-moving vector field, $\tilde{\zeta}$, we find $c_L = 3\ell/2$. Thus we obtain the same charges as the original computation by Brown and Henneaux but now this calculation has explicitly been done on the bifurcation surface, a section of the horizon.

We can now imagine that there is a conformal field theory that exists on the *horizon* of the BTZ black hole. We can again use the Cardy formula, to find the entropy of the CFT, yielding exactly the Bekenstein-Hawking entropy, as we discovered earlier in section 5.3.2. While the extension onto the horizon may have been computationally almost trivial, its implications are profound. It suggests that the entropy of the BTZ black hole can be completely recovered by looking at microstates on the horizon.

E NHEK charge computation on the horizon

In order to explicitly compute the charges on the horizon of NHEK, we will define the NHEK geometry in advanced coordinates, by setting,

$$\begin{aligned} v &= T - \frac{1}{R}, \\ \tilde{\phi} &= \hat{\phi} - \ln R. \end{aligned} \tag{E.1}$$

In these new $(v, R, \theta, \tilde{\phi})$ coordinates, the metric (6.13) then becomes,

$$ds^2 = 2J\Omega^2 \left(-R^2 dv^2 + 2dvdR + d\theta^2 + \Lambda^2 (d\tilde{\phi} + Rdv)^2 \right). \tag{E.2}$$

The vector field (6.18) is (excluding the extra term (6.20))

$$\zeta_n^a = \left(-\frac{\varepsilon_n'}{R}, -R\varepsilon_n', 0, \varepsilon_n + \varepsilon_n' \right), \tag{E.3}$$

where we now define

$$\varepsilon_n = e^{im\tilde{\phi}}, \qquad \varepsilon_n' = \partial_{\tilde{\phi}} \varepsilon_n. \tag{E.4}$$

Just as for the case of BTZ, here the vector field is divergence free. It also involves just one derivative of ε, and therefore we are again left with an extremely simple expression for the only terms that can contribute to the cubics of c:

$$(F_{cubics})_{ab} = 2\nabla_c\zeta_a\nabla^c\tilde{\zeta}_b - 2\nabla_c\zeta_b\nabla^c\tilde{\zeta}_a. \tag{E.5}$$

Evaluating this on a surface of fixed v, R, and integrating over the angular coordinates results in a central charge,

$$c_L = 12J. \tag{E.6}$$

Again, we find the same result but on the horizon of the black hole. Thus, using the Cardy formula again reproduces the entropy of the extreme Kerr black hole.

F Kerr metric in Kruskal-like coordinates

The metric is

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - adt)^2 - \frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2. \tag{F.1}$$

Using the advanced and retarded coordinates, v and u defined in (7.29) and (7.31), we have

$$\begin{aligned} dt &= \frac{1}{2}(du+dv), \\ dr &= \frac{\Delta}{2(r^2+a^2)}(dv-du). \end{aligned} \tag{F.2}$$

The co-rotating angular coordinate is

$$d\phi_+ = d\phi - \Omega_+ dt, \tag{F.3}$$

and thus

$$d\phi = d\phi_+ + \frac{1}{2}\Omega_+(du+dv), \tag{F.4}$$

where $\Omega_+ = \frac{a}{2Mr_+}$ is the angular velocity of the horizon.

The metric becomes,

$$
\begin{aligned}
ds^2 = \frac{\Delta\rho^2}{r(r^2+a^2)}(dv-du)^2 \quad &+ \quad \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)d\phi_+ + F[r](du+dv)\right)^2 \\
&- \quad \frac{\Delta}{\rho^2}\left(G[\theta](du+dv) - a\sin^2\theta d\phi_+\right)^2, \qquad \text{(F.5)}
\end{aligned}
$$

where the functions F and G are defined,

$$
\begin{aligned}
F[r] &= \frac{a}{4Mr_+}(r^2 - r_+^2), \\
G[\theta] &= \frac{\rho_+^2}{2(r_+^2+a^2)}, \qquad \text{(F.6)}
\end{aligned}
$$

where

$$
\rho_+^2 = r_+^2 + a^2\cos^2\theta. \qquad \text{(F.7)}
$$

Expanding the metric close to the horizon, to leading and subleading orders around $r = r_+$, the metric becomes,

$$
\begin{aligned}
ds^2 = -\ &\frac{(r-r_+)(r_+-r_-)\rho_+^2}{2(r_+^2+a^2)^2}du dv + \rho_+^2 d\theta^2 + \frac{(r_+^2+a^2)^2\sin^2\theta}{\rho_+^2}d\phi_+^2 \\
+\ &\frac{a\sin^2\theta(r-r_+)}{\rho_+^2(r_+^2+a^2)}\left(3r_+^3 + a^2 r_+ + 2a^2(r_+-M)\cos^2\theta\right)(du+dv)d\phi_+. \qquad \text{(F.8)}
\end{aligned}
$$

Define Kruskal coordinates,

$$
\begin{aligned}
U &= -e^{-\kappa u}, \\
V &= e^{\kappa v}, \qquad \text{(F.9)}
\end{aligned}
$$

where the minus sign in the definition of U is there so that U increases as u increases. The surface gravity, κ is defined,

$$
\kappa = \frac{r_+ - r_-}{4Mr_+}. \qquad \text{(F.10)}
$$

The inverse transformation,

$$\begin{aligned} u &= -\frac{1}{\kappa}\ln(-U), \\ v &= \frac{1}{\kappa}\ln(V) \end{aligned} \tag{F.11}$$

defines the exterior region, where $V > 0, U < 0$.

In order to specify the metric in these coordinates in as simple a way as possible, one must perform a trick as follows. We have,

$$\begin{aligned} v, u &= t \pm \int \frac{(r^2+a^2)}{(r-r_+)(r-r_-)} dr, \\ &= t \pm \big[r + \frac{2Mr_+}{r_+ - r_-}\ln(\frac{r-r_+}{r_+-r_-}) - \frac{2Mr_-}{r_+-r_-}\ln(\frac{r-r_-}{r_+-r_-}) + \ln C \big] \end{aligned} \tag{F.12}$$

where C is some arbitrary constant of integration.

Now consider the product,

$$\begin{aligned} UV &= e^{\kappa(v-u)} \\ &= -\exp\big[\frac{r_+ - r_-}{2Mr_+}\big(r + \frac{2Mr_+}{r_+-r_-}\ln(\frac{r-r_+}{r_+-r_-}) \\ &\quad - \frac{2Mr_-}{r_+-r_-}\ln(\frac{r-r_-}{r_+-r_-}) + \ln C\big)\big] \\ &= -De^{\frac{r(r_+-r_-)}{r_+^2+a^2}} \frac{r-r_+}{r_+-r_-}\left(\frac{r-r_-}{r_+-r_-}\right)^{-\left(\frac{2Mr_-}{r_+^2+a^2}\right)}, \end{aligned} \tag{F.13}$$

where the constant C has been absorbed into the new constant D.

Thus, near $r = r_+$, the term in brackets is approximately 1 and we get,

$$UV \approx -De^{\frac{r_+(r_+-r_-)}{r_+^2+a^2}} \frac{r-r_+}{r_+-r_-}. \tag{F.14}$$

We can choose the constant D to be,

$$D = e^{-\frac{r_+(r_+-r_-)}{r_+^2+a^2}}. \tag{F.15}$$

Therefore

$$UV \approx -\frac{r - r_+}{r_+ - r_-}. \tag{F.16}$$

Then, the coefficient of the $dudv$ metric component is,

$$\begin{aligned}
&-\frac{(r - r_+)(r_+ - r_-)\rho_+^2}{2(r_+^2 + a^2)^2} dudv \\
&\approx -\frac{(r - r_+)(r_+ - r_-)\rho_+^2}{2(r_+^2 + a^2)^2} \frac{1}{\kappa^2} \frac{dUdV}{UV} \\
&\approx \rho_+^2 \frac{8M^2 r_+^2}{(r_+^2 + a^2)^2} dUdV \\
&\approx 2\rho_+^2 dUdV
\end{aligned} \tag{F.17}$$

This means that the zeroth order metric is

$$ds_0^2 \;=\; 2\rho_+^2 dUdV + \rho_+^2 d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2\theta}{\rho_+^2} d\phi_+^2 \tag{F.18}$$

and hence

$$\sqrt{-g} = (r_+^2 + a^2)\rho_+^2 \sin\theta. \tag{F.19}$$

The first order correction is

$$-\frac{a \sin^2\theta}{r_+ \rho_+^2}(3r_+^2 + a^2 r_+^2 + a^2 r_+^2 \cos^2\theta - a^4 \cos^2\theta)(UdV - VdU)d\phi_+. \tag{F.20}$$

Therefore, to leading and subleading order around $r = r_+$, the Kerr metric in Kruskal-like coordinates is,

$$\begin{aligned}
ds^2 = {}& 2\rho_+^2 dUdV + \rho_+^2 d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2\theta}{\rho_+^2} d\phi_+^2 \\
&- \frac{a \sin^2\theta}{r_+ \rho_+^2}(3r_+^2 + a^2 r_+^2 + a^2 r_+^2 \cos^2\theta - a^4 \cos^2\theta)(UdV - VdU)d\phi_+ + \cdots .
\end{aligned} \tag{F.21}$$

To my knowledge this metric does not appear elsewhere in the literature. All other versions of the Kerr metric in Kruskal coordinates involve much more complicated expressions. See eg [173].

References

[1] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, no. 6, 061102 (2016) [arXiv:1602.03837 [gr-qc]].

[2] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31**, 161 (1973).

[3] J. D. Bekenstein, "Black holes and the second law," Lett. Nuovo Cim. **4**, 737 (1972).

[4] S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43**, 199 (1975) Erratum: [Commun. Math. Phys. **46**, 206 (1976)].

[5] S. W. Hawking, "Black hole explosions," Nature **248**, 30 (1974).

[6] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973).

[7] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. B **379**, 99 (1996) [hep-th/9601029].

[8] L. Susskind and J. Lindesay, "An introduction to black holes, information and the string theory revolution: The holographic universe," Hackensack, USA: World Scientific (2005) 183 p

[9] S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys. Rev. D* **14**, *2460 (1976)*

[10] R. J. Adler, P. Chen and D. I. Santiago, "The Generalized uncertainty principle and black hole remnants," Gen. Rel. Grav. **33**, 2101 (2001) [gr-qc/0106080].

[11] S. B. Giddings, "Black holes and massive remnants," Phys. Rev. D **46**, 1347 (1992) [hep-th/9203059].

[12] Y. Aharonov, A. Casher and S. Nussinov, "The Unitarity Puzzle and Planck Mass Stable Particles," Phys. Lett. B **191**, 51 (1987).

[13] M. K. Parikh and F. Wilczek, "Hawking radiation as tunneling," Phys. Rev. Lett. **85**, 5042 (2000) [hep-th/9907001].

[14] B. Zhang, Q. y. Cai, L. You and M. s. Zhan, "Hidden Messenger Revealed in Hawking Radiation: A Resolution to the Paradox of Black Hole Information Loss," Phys. Lett. B **675**, 98 (2009) [arXiv:0903.0893 [hep-th]].

[15] I. Y. Park, "On the pattern of black hole information release," Int. J. Mod. Phys. A **29**, 1450047 (2014) [arXiv:1301.6320 [hep-th]].

[16] J. Preskill, "Do black holes destroy information?," In *Houston 1992, Proceedings, Black holes, membranes, wormholes and superstrings* 22-39, and Caltech Pasadena - CALT-68-1819 (92,rec.Oct.) 17 p [hep-th/9209058].

[17] R. D. Carlitz and R. S. Willey, "The Lifetime of a Black Hole," Phys. Rev. D **36**, 2336 (1987).

[18] J. Smolin and J. Oppenheim, "Information locking in black holes," Phys. Rev. Lett. **96**, 081302 (2006) [hep-th/0507287].

[19] S. R. Coleman, J. Preskill and F. Wilczek, "Quantum hair on black holes," Nucl. Phys. B **378**, 175 (1992) [hep-th/9201059].

[20] S. W. Hawking, Phys. Rev. D **37**, 904 (1988).

[21] S. W. Hawking and R. Laflamme, Phys. Lett. B **209**, 39 (1988).

[22] L. Susskind, L. Thorlacius and J. Uglum, "The Stretched horizon and black hole complementarity," Phys. Rev. D **48**, 3743 (1993) [hep-th/9306069].

[23] L. Susskind, "String theory and the principles of black hole complementarity," Phys. Rev. Lett. **71**, 2367 (1993) [hep-th/9307168].

[24] S. D. Mathur, “What the information paradox is not,” arXiv:1108.0302 [hep-th].

[25] S. B. Giddings, “The Black hole information paradox,” hep-th/9508151.

[26] G. T. Horowitz and A. Strominger, “Counting states of near extremal black holes,” Phys. Rev. Lett. **77**, 2368 (1996)

[27] S. W. Hawking, M. J. Perry and A. Strominger, “Soft Hair on Black Holes,” Phys. Rev. Lett. **116**, no. 23, 231301 (2016) [arXiv:1601.00921 [hep-th]].

[28] S. W. Hawking, M. J. Perry and A. Strominger, “Superrotation Charge and Supertranslation Hair on Black Holes,” JHEP **1705**, 161 (2017) [arXiv:1611.09175 [hep-th]].

[29] E. E. Flanagan and D. A. Nichols, “Conserved charges of the extended Bondi-Metzner-Sachs algebra,” Phys. Rev. D **95**, no. 4, 044002 (2017) [arXiv:1510.03386 [hep-th]].

[30] A. Averin, G. Dvali, C. Gomez and D. Lust, “Gravitational Black Hole Hair from Event Horizon Supertranslations,” JHEP **1606**, 088 (2016) [arXiv:1601.03725 [hep-th]].

[31] G. Compère and J. Long, “Vacua of the gravitational field,” JHEP **1607**, 137 (2016) [arXiv:1601.04958 [hep-th]].

[32] M. M. Sheikh-Jabbari, “Residual Diffeomorphisms and Symplectic Softs Hairs: The Need to Refine Strict Statement of Equivalence Principle,” arXiv:1603.07862 [hep-th].

[33] J. E. Baxter, “On the global existence of hairy black holes and solitons in anti-de Sitter Einstein-Yang-Mills theories with compact semisimple gauge groups,” Gen. Rel. Grav. **48**, no. 10, 133 (2016) [arXiv:1604.05012 [gr-qc]].

[34] G. Compère, “Bulk supertranslation memories: a concept reshaping the vacua and black holes of general relativity,” arXiv:1606.00377 [hep-th].

[35] P. Mao, X. Wu and H. Zhang, "Soft hairs on isolated horizon implanted by electromagnetic fields," arXiv:1606.03226 [hep-th].

[36] A. Averin, G. Dvali, C. Gomez and D. Lust, "Goldstone origin of black hole hair from supertranslations and criticality," Mod. Phys. Lett. A **31**, no. 39, 1630045 (2016) [arXiv:1606.06260 [hep-th]].

[37] V. Cardoso and L. Gualtieri, "Testing the black hole 'no-hair' hypothesis," Class. Quant. Grav. **33**, no. 17, 174001 (2016) [arXiv:1607.03133 [gr-qc]].

[38] M. Mirbabayi and M. Porrati, "Dressed Hard States and Black Hole Soft Hair," Phys. Rev. Lett. **117**, no. 21, 211301 (2016) [arXiv:1607.03120 [hep-th]].

[39] D. Grumiller, A. Perez, S. Prohazka, D. Tempo and R. Troncoso, "Higher Spin Black Holes with Soft Hair," arXiv:1607.05360 [hep-th].

[40] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, "Extended Symmetries at the Black Hole Horizon," arXiv:1607.05703 [hep-th].

[41] B. Gabai and A. Sever, "Large gauge symmetries and asymptotic states in QED," JHEP **1612**, 095 (2016) [arXiv:1607.08599 [hep-th]].

[42] C. Gomez and M. Panchenko, "Asymptotic dynamics, large gauge transformations and infrared symmetries," arXiv:1608.05630 [hep-th].

[43] D. He and Q. y. Cai, "Gravitational correlation, black hole entropy and information conservation," Sci. China Phys. Mech. Astron. **60**, no. 4, 040011 (2017) [arXiv:1609.05825 [hep-th]].

[44] F. Tamburini, M. De Laurentis, I. Licata and B. Thidé, "Twisted soft photon hair implants on Black Holes," Entropy **19**, no. 9, 458 (2017) [arXiv:1702.04094 [gr-qc]].

[45] M. Ammon, D. Grumiller, S. Prohazka, M. Riegler and R. Wutte, "Higher-Spin Flat Space Cosmologies with Soft Hair," JHEP **1705**, 031 (2017) [arXiv:1703.02594 [hep-th]].

[46] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, “Soft gravitons and the memory effect for plane gravitational waves,” Phys. Rev. D **96**, no. 6, 064013 (2017) [arXiv:1705.01378 [gr-qc]].

[47] R. Bousso and M. Porrati, “Soft Hair as a Soft Wig,” Class. Quant. Grav. **34**, no. 20, 204001 (2017) [arXiv:1706.00436 [hep-th]].

[48] A. Strominger, “Black Hole Information Revisited,” arXiv:1706.07143 [hep-th].

[49] M. Hotta, Y. Nambu and K. Yamaguchi, “Soft-Hair-Enhanced Entanglement Beyond Page Curves in a Black-hole Evaporation Qubit Model,” Phys. Rev. Lett. **120**, no. 18, 181301 (2018) [arXiv:1706.07520 [gr-qc]].

[50] R. K. Mishra and R. Sundrum, “Asymptotic Symmetries, Holography and Topological Hair,” JHEP **1801**, 014 (2018) [arXiv:1706.09080 [hep-th]].

[51] C. Gomez and S. Zell, “Black Hole Evaporation, Quantum Hair and Supertranslations,” Eur. Phys. J. C **78**, no. 4, 320 (2018) [arXiv:1707.08580 [hep-th]].

[52] D. Grumiller, P. Hacker and W. Merbis, “Soft hairy warped black hole entropy,” JHEP **1802**, 010 (2018) [arXiv:1711.07975 [hep-th]].

[53] A. Chatterjee and D. A. Lowe, “BMS symmetry, soft particles and memory,” Class. Quant. Grav. **35**, no. 9, 094001 (2018) [arXiv:1712.03211 [hep-th]].

[54] C. S. Chu and Y. Koyama, “Soft Hair of Dynamical Black Hole and Hawking Radiation,” JHEP **1804**, 056 (2018) [arXiv:1801.03658 [hep-th]].

[55] J. Kirklin, “Localisation of Soft Charges, and Thermodynamics of Softly Hairy Black Holes,” Class. Quant. Grav. **35**, no. 17, 175010 (2018) [arXiv:1802.08145 [hep-th]].

[56] B. Cvetković and D. Simić, "Near horizon OTT black hole asymptotic symmetries and soft hair," Chin. Phys. C **43**, no. 1, 013109 (2019) arXiv:1804.00484 [hep-th].

[57] D. Grumiller and M. M. Sheikh-Jabbari, "Membrane Paradigm from Near Horizon Soft Hair," Int. J. Mod. Phys. D **27**, no. 14, 1847006 (2018) [arXiv:1805.11099 [hep-th]].

[58] V. Chandrasekaran, . . Flanagan and K. Prabhu, "Symmetries and charges of general relativity at null boundaries," JHEP **1811**, 125 (2018) [arXiv:1807.11499 [hep-th]].

[59] A. Averin, "Schwarzschild/CFT from soft black hole hair?," JHEP **1901**, 092 (2019) [arXiv:1808.09923 [hep-th]].

[60] S. Choi and R. Akhoury, "Soft Photon Hair on Schwarzschild Horizon from a Wilson Line Perspective," JHEP **1812**, 074 (2018) [arXiv:1809.03467 [hep-th]].

[61] L. Donnay, G. Giribet, H. A. Gonzlez and A. Puhm, "Black hole memory effect," Phys. Rev. D **98**, no. 12, 124016 (2018) [arXiv:1809.07266 [hep-th]].

[62] G. Compère and J. Long, "Classical static final state of collapse with supertranslation memory," Class. Quant. Grav. **33**, no. 19, 195001 (2016) [arXiv:1602.05197 [gr-qc]].

[63] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," Phys. Rev. Lett. **116**, no. 9, 091101 (2016) [arXiv:1511.08687 [hep-th]].

[64] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," Proc. Roy. Soc. Lond. A **269**, 21 (1962).

[65] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," Proc. Roy. Soc. Lond. A **270**, 103 (1962).

[66] G. Barnich and C. Troessaert, "Supertranslations Call for Superrotations," *PoS (2010) 010, arXiv:1102.4632 [gr-qc].* [Ann. U. Craiova Phys.21,S11(2011)].

[67] G. Barnich and C. Troessaert, "Symmetries of Asymptotically Flat 4-Dimensional Spacetimes at Null Infinity Revisited," *Phys. Rev. Lett.* **105** (2010) 111103, arXiv:0909.2617 [gr-qc].

[68] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT Correspondence," *JHEP* **05** (2010) 062, arXiv:1001.1541 [hep-th].

[69] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP* **1407** (2014) 152, arXiv:1312.2229 [hep-th].

[70] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS Supertranslations and Weinberg's Soft Graviton Theorem," *JHEP* **05** (2015) 151, arXiv:1401.7026 [hep-th].

[71] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," *JHEP* **1601***, 086 (2016), arXiv:1411.5745 [hep-th].*

[72] S. Pasterski, A. Strominger and A. Zhiboedov, "New Gravitational Memories," JHEP **1612**, 053 (2016)

[73] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," arXiv:1404.4091 [hep-th].

[74] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, "Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix," *JHEP* **1408***, 058 (2014), arXiv:1406.3312 [hep-th]*

[75] M. Campiglia and A. Laddha, "Asymptotic symmetries of gravity and soft theorems for massive particles," JHEP **1512**, 094 (2015)

[76] M. Campiglia and A. Laddha, “Asymptotic symmetries and subleading soft graviton theorem,” Phys. Rev. D **90**, no. 12, 124028 (2014)

[77] S. Weinberg, “Infrared Photons and Gravitons,” *Phys. Rev.* **140** (1965) B516–B524.

[78] S. Haco, S. W. Hawking, M. J. Perry and A. Strominger, “Black Hole Entropy and Soft Hair,” arXiv:1810.01847 [hep-th].

[79] A. Castro, A. Maloney and A. Strominger, “Hidden Conformal Symmetry of the Kerr Black Hole,” Phys. Rev. D **82**, 024008 (2010) [arXiv:1004.0996 [hep-th]].

[80] C. Crnkovic and E. Witten, “Covariant Description Of Canonical Formalism In Geometrical Theories,” In *Hawking, S.W. (ed.), Israel, W. (ed.): Three hundred years of gravitation*, 676-684 (1986).

[81] G. J. Zuckerman, “Action Principles And Global Geometry,” Conf. Proc. C **8607214**, 259 (1986).

[82] J. D. Brown and J. W. York, Jr., “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D **47**, 1407 (1993) [gr-qc/9209012].

[83] J. Lee and R. M. Wald, “Local symmetries and constraints,” J. Math. Phys. **31**, 725 (1990).

[84] V. Iyer and R. M. Wald, “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” Phys. Rev. D **52**, 4430 (1995) [gr-qc/9503052].

[85] R. M. Wald and A. Zoupas, “A General definition of ‘conserved quantities’ in general relativity and other theories of gravity,” Phys. Rev. D **61**, 084027 (2000) [gr-qc/9911095].

[86] G. Barnich and F. Brandt, Nucl. Phys. B **633**, 3 (2002) [hep-th/0111246].

[87] R. Sachs, "Asymptotic Symmetries in Gravitational Theory," *Phys. Rev.* **128** (1962) 2851–2864.

[88] R. Penrose, "Asymptotic Properties of Fields and Space-Times," *Phys. Rev. Lett.* **10** (1963) 66–68.

[89] E. T. Newman and R. Penrose, "Note on the Bondi-Metzner-Sachs Group," *J. Math. Phys.* **7** (1966) 863–870.

[90] S. W. Hawking, M. J. Perry and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," arXiv:1611.09175 [hep-th].

[91] R. M. Wald and A. Zoupas, "A General Definition of 'Conserved Quantities' in General Relativity and other Theories of Gravity," *Phys. Rev.* **D61** (2000) 084027, arXiv:gr-qc/9911095 [gr-qc].

[92] G. Barnich and C. Troessaert, "BMS Charge Algebra," *JHEP* **12** (2011) 105, arXiv:1106.0213 [hep-th].

[93] V. Lysov, S. Pasterski, and A. Strominger, "Low's Subleading Soft Theorem as a Symmetry of QED," *Phys. Rev. Lett.* **113** (2014) 111601, arXiv:1407.3814 [hep-th].

[94] F. E. Low, "Scattering of Light of Very Low Frequency by Systems of Spin 1/2," *Phys. Rev.* **96** (1954) 1428–1432.

[95] R. Penrose, "Zero rest mass fields including gravitation: Asymptotic behavior," *Proc. Roy. Soc. Lond. A* **284**, *159 (1965).*

[96] L. Borsten and M. J. Duff, "Gravity as the square of Yang-Mills?," *Phys. Scripta* **90**, *108012 (2015), arXiv:1602.08267 [hep-th]*

[97] C. Duval, G. W. Gibbons, P. A. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class. Quant. Grav.* **31** *(2014) 092001, arXiv:1402.5894 [gr-qc].*

[98] C. Duval, G. W. Gibbons and P. A. Horvathy, "Conformal Carroll groups," *J. Phys.* **A 47** *(2014) 335204., arXiv:1403.4213 [hep-th].*

[99] M. A. Awada, G. W. Gibbons, and W. T. Shaw, "Conformal Supergravity, Twistors and the Super BMS Group," *Annals Phys.* **171** (1986) 52.

[100] M. Irakleidou and I. Lovrekovic, "Asymptotic Symmetry Algebra of Conformal Gravity," arXiv:1611.01810 [hep-th].

[101] E. Newman and R. Penrose, "An Approach to gravitational radiation by a method of spin coefficients," *J. Math. Phys.* **3**, *566 (1962).*

[102] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," *JHEP* **1407** (2014) 151, arXiv:1308.0589 [hep-th].

[103] C. Cheung, A. de la Fuente, and R. Sundrum, "4D Scattering Amplitudes and Asymptotic Symmetries from 2D CFT," JHEP **1701**, 112 (2017), arXiv:1609.00732 [hep-th].

[104] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104**, 207 (1986).

[105] A. Strominger, "Black hole entropy from near horizon microstates," JHEP **9802**, 009 (1998) [hep-th/9712251].

[106] A. Fiorucci and G. Compère, "Advanced Lectures in General Relativity," arXiv:1801.07064 [hep-th].

[107] C. M. Chen and J. R. Sun, "Hidden Conformal Symmetry of the Reissner-Nordstrom Black Holes," JHEP **1008**, 034 (2010) [arXiv:1004.3963 [hep-th]].

[108] Y. Q. Wang and Y. X. Liu, "Hidden Conformal Symmetry of the Kerr-Newman Black Hole," JHEP **1008**, 087 (2010) [arXiv:1004.4661 [hep-th]].

[109] B. Chen and J. Long, "Real-time Correlators and Hidden Conformal Symmetry in Kerr/CFT Correspondence," JHEP **1006**, 018 (2010) [arXiv:1004.5039 [hep-th]].

[110] M. Becker, S. Cremonini and W. Schulgin, "Correlation Functions and Hidden Conformal Symmetry of Kerr Black Holes," JHEP **1009**, 022 (2010) [arXiv:1005.3571 [hep-th]].

[111] H. Wang, D. Chen, B. Mu and H. Wu, "Hidden conformal symmetry of extreme and non-extreme Einstein-Maxwell-Dilaton-Axion black holes," JHEP **1011**, 002 (2010) [arXiv:1006.0439 [gr-qc]].

[112] I. Agullo, J. Navarro-Salas, G. J. Olmo and L. Parker, "Hawking radiation by Kerr black holes and conformal symmetry," Phys. Rev. Lett. **105**, 211305 (2010) [arXiv:1006.4404 [hep-th]].

[113] B. Chen and J. Long, "Hidden Conformal Symmetry and Quasi-normal Modes," Phys. Rev. D **82**, 126013 (2010) [arXiv:1009.1010 [hep-th]].

[114] M. R. Setare and V. Kamali, "Hidden Conformal Symmetry of Extremal Kerr-Bolt Spacetimes," JHEP **1010**, 074 (2010) [arXiv:1011.0809 [hep-th]].

[115] M. Cvetic, G. W. Gibbons and C. N. Pope, "Universal Area Product Formulae for Rotating and Charged Black Holes in Four and Higher Dimensions," Phys. Rev. Lett. **106**, 121301 (2011)

[116] D. A. Lowe, I. Messamah and A. Skanata, "Scaling dimensions in hidden Kerr/CFT," Phys. Rev. D **84**, 024030 (2011) [arXiv:1105.2035 [hep-th]].

[117] M. Cvetic and F. Larsen, "Conformal Symmetry for General Black Holes," JHEP **1202**, 122 (2012) [arXiv:1106.3341 [hep-th]].

[118] M. Cvetic and G. W. Gibbons, "Conformal Symmetry of a Black Hole as a Scaling Limit: A Black Hole in an Asymptotically Conical Box," JHEP **1207**, 014 (2012) [arXiv:1201.0601 [hep-th]].

[119] M. Cvetic and F. Larsen, "Conformal Symmetry for Black Holes in Four Dimensions," JHEP **1209**, 076 (2012) [arXiv:1112.4846 [hep-th]].

[120] G. Compère, "The Kerr/CFT correspondence and its extensions," Living Rev. Rel. **15**, 11 (2012) [Living Rev. Rel. **20**, no. 1, 1 (2017)] [arXiv:1203.3561 [hep-th]].

[121] A. Virmani, "Subtracted Geometry From Harrison Transformations," JHEP **1207**, 086 (2012) [arXiv:1203.5088 [hep-th]].

[122] M. R. Setare and H. Adami, "Near Horizon Symmetry and Entropy Formula for Kerr-Newman (A)dS Black Holes," JHEP **1804**, 133 (2018) [arXiv:1802.04665 [gr-qc]].

[123] H. Gonzalez, D. Grumiller, W. Merbis and R. Wutte, "New entropy formula for Kerr black holes," EPJ Web Conf. **168**, 01009 (2018) [arXiv:1709.09667 [hep-th]].

[124] S. Carlip, "Black hole entropy from horizon conformal field theory," Nucl. Phys. Proc. Suppl. **88**, 10 (2000) [gr-qc/9912118].

[125] S. Carlip, "Symmetries, Horizons, and Black Hole Entropy," Gen. Rel. Grav. **39**, 1519 (2007) [Int. J. Mod. Phys. D **17**, 659 (2008)] [arXiv:0705.3024 [gr-qc]].

[126] S. Carlip, "Black Hole Entropy from Bondi-Metzner-Sachs Symmetry at the Horizon," Phys. Rev. Lett. **120**, no. 10, 101301 (2018) [arXiv:1702.04439 [gr-qc]].

[127] M. Guica, T. Hartman, W. Song and A. Strominger, "The Kerr/CFT Correspondence," Phys. Rev. D **80**, 124008 (2009) [arXiv:0809.4266 [hep-th]].

[128] K. Hajian, M. M. Sheikh-Jabbari and H. Yavartanoo, "Extreme Kerr black hole microstates with horizon fluff," Phys. Rev. D **98**, no. 2, 026025 (2018) [arXiv:1708.06378 [hep-th]].

[129] G. Barnich and G. Compere, "Surface charge algebra in gauge theories and thermodynamic integrability," J. Math. Phys. **49**, 042901 (2008) [arXiv:0708.2378 [gr-qc]].

[130] A. Seraj, "Conserved charges, surface degrees of freedom, and black hole entropy," arXiv:1603.02442 [hep-th].

[131] K. Hajian, "On Thermodynamics and Phase Space of Near Horizon Extremal Geometries," arXiv:1508.03494 [gr-qc].

[132] G. 't Hooft, "Dimensional reduction in quantum gravity," Conf. Proc. C **930308**, 284 (1993) [gr-qc/9310026].

[133] L. Susskind, "The World as a hologram," J. Math. Phys. **36**, 6377 (1995)

[134] J. de Boer, Les Houches Lect. Notes **97**, 163 (2015).

[135] C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, "Quantum critical transport, duality, and M-theory," Phys. Rev. D **75**, 085020 (2007)

[136] S. A. Hartnoll, P. K. Kovtun, M. Muller and S. Sachdev, "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes," Phys. Rev. B **76**, 144502 (2007)

[137] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. **101**, 031601 (2008) [arXiv:0803.3295 [hep-th]].

[138] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," Phys. Rev. Lett. **101**, 061601 (2008) [arXiv:0804.4053 [hep-th]].

[139] D. T. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry," Phys. Rev. D **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]].

[140] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. G. Yaffe, "Energy loss of a heavy quark moving through N=4 supersymmetric Yang-Mills plasma," JHEP **0607**, 013 (2006) [hep-th/0605158].

[141] S. S. Gubser, "Drag force in AdS/CFT," Phys. Rev. D **74**, 126005 (2006) [hep-th/0605182].

[142] C. P. Herzog, "Energy Loss of Heavy Quarks from Asymptotically AdS Geometries," JHEP **0609**, 032 (2006) [hep-th/0605191].

[143] J. J. Friess, S. S. Gubser and G. Michalogiorgakis, "Dissipation from a heavy quark moving through N=4 super-Yang-Mills plasma," JHEP **0609**, 072 (2006) [hep-th/0605292].

[144] H. Liu, K. Rajagopal and U. A. Wiedemann, "Wilson loops in heavy ion collisions and their calculation in AdS/CFT," JHEP **0703**, 066 (2007) [hep-ph/0612168].

[145] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007) [arXiv:0704.0240 [hep-th]].

[146] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)]

[147] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998)

[148] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998)

[149] E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].

[150] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory,” Nucl. Phys. B **241**, 333 (1984).

[151] M. R. Gaberdiel, Rept. Prog. Phys. **63**, 607 (2000) [hep-th/9910156].

[152] D. Tong, “String Theory,” arXiv:0908.0333 [hep-th].

[153] J. M. Maldacena and A. Strominger, “AdS(3) black holes and a stringy exclusion principle,” JHEP **9812**, 005 (1998) [hep-th/9804085].

[154] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. **69**, 1849 (1992) [hep-th/9204099].

[155] S. Carlip, “The (2+1)-Dimensional black hole,” Class. Quant. Grav. **12**, 2853 (1995) [gr-qc/9506079].

[156] S. Deser and R. Jackiw, “Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature,” Annals Phys. **153**, 405 (1984).

[157] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, “Warped AdS(3) Black Holes,” JHEP **0903**, 130 (2009) [arXiv:0807.3040 [hep-th]].

[158] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, “Geometry of the (2+1) black hole,” Phys. Rev. D **48**, 1506 (1993) Erratum: [Phys. Rev. D **88**, 069902 (2013)] 10.1103/PhysRevD.88.069902 [gr-qc/9302012].

[159] G. Compre, W. Song and A. Strominger, “New Boundary Conditions for AdS3,” JHEP **1305**, 152 (2013) [arXiv:1303.2662 [hep-th]].

[160] J. D. Brown and M. Henneaux, J. Math. Phys. **27**, 489 (1986).

[161] J. L. Cardy, “Operator Content of Two-Dimensional Conformally Invariant Theories,” Nucl. Phys. B **270**, 186 (1986).

[162] P. De Lange, A. Maloney and E. Verlinde, “Monstrous Product CFTs in the Grand Canonical Ensemble,” arXiv:1807.06200 [hep-th].

[163] T. Hartman, C. A. Keller and B. Stoica, "Universal Spectrum of 2d Conformal Field Theory in the Large c Limit," JHEP **1409**, 118 (2014) [arXiv:1405.5137 [hep-th]].

[164] J. M. Bardeen and G. T. Horowitz, "The Extreme Kerr throat geometry: A Vacuum analog of AdS(2) x S**2," Phys. Rev. D **60**, 104030 (1999) [hep-th/9905099].

[165] G. Compre, K. Hajian, A. Seraj and M. M. Sheikh-Jabbari, "Wiggling Throat of Extremal Black Holes," JHEP **1510**, 093 (2015) [arXiv:1506.07181 [hep-th]].

[166] G. Duffy and A. C. Ottewill, "The Renormalized stress tensor in Kerr space-time: Numerical results for the Hartle-Hawking vacuum," Phys. Rev. D **77**, 024007 (2008) [gr-qc/0507116].

[167] V. P. Frolov and K. S. Thorne, "Renormalized Stress - Energy Tensor Near the Horizon of a Slowly Evolving, Rotating Black Hole," Phys. Rev. D **39**, 2125 (1989).

[168] G. W. Gibbons and S. W. Hawking, "Cosmological Event Horizons, Thermodynamics, and Particle Creation," Phys. Rev. D **15**, 2738 (1977).

[169] S. Detournay, T. Hartman and D. M. Hofman, "Warped Conformal Field Theory," Phys. Rev. D **86**, 124018 (2012) [arXiv:1210.0539 [hep-th]].

[170] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, "Modular covariance and uniqueness of $J\bar{T}$ deformed CFTs," arXiv:1808.08978 [hep-th].

[171] A. Bzowski and M. Guica, "The holographic interpretation of $J\bar{T}$-deformed CFTs," arXiv:1803.09753 [hep-th].

[172] M. Guica, "An integrable Lorentz-breaking deformation of two-dimensional CFTs," arXiv:1710.08415 [hep-th].

[173] B. Carter, "Black holes equilibrium states," (1973)

[174] E. Gourgoulhon and J. L. Jaramillo, "New theoretical approaches to black holes," New Astron. Rev. **51**, 791 (2008) [arXiv:0803.2944 [astro-ph]].

[175] P. Kraus and F. Larsen, "Holographic gravitational anomalies," JHEP **0601**, 022 (2006) [hep-th/0508218].

[176] D. Harlow, "Jerusalem Lectures on Black Holes and Quantum Information," Rev. Mod. Phys. **88**, 015002 (2016) [arXiv:1409.1231 [hep-th]].

[177] W. Donnelly and A. C. Wall, "Entanglement entropy of electromagnetic edge modes," Phys. Rev. Lett. **114**, no. 11, 111603 (2015) [arXiv:1412.1895 [hep-th]].

[178] W. Donnelly and A. C. Wall, "Geometric entropy and edge modes of the electromagnetic field," arXiv:1506.05792 [hep-th].

[179] A. Blommaert, T. G. Mertens, H. Verschelde and V. I. Zakharov, JHEP **1808**, 196 (2018) [arXiv:1801.09910 [hep-th]].

[180] D. Harlow, "Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture," JHEP **1601**, 122 (2016) [arXiv:1510.07911 [hep-th]].

[181] J. H. Schwarz, "Can string theory overcome deep problems in quantum gravity?," Phys. Lett. B **272**, 239 (1991).

[182] A. Strominger, "Statistical hair on black holes," Phys. Rev. Lett. **77**, 3498 (1996) [hep-th/9606016].

[183] D. Chen, P. Wang and H. Wu, "Hidden conformal symmetry of rotating charged black holes," Gen. Rel. Grav. **43**, 181 (2011) [arXiv:1005.1404 [gr-qc]].

[184] C. M. Chen, Y. M. Huang, J. R. Sun, M. F. Wu and S. J. Zou, "Twofold Hidden Conformal Symmetries of the Kerr-Newman Black Hole," Phys. Rev. D **82**, 066004 (2010) [arXiv:1006.4097 [hep-th]].

[185] K. Hajian and M. M. Sheikh-Jabbari, "Solution Phase Space and Conserved Charges: A General Formulation for Charges Associated with Exact

Symmetries," Phys. Rev. D **93**, no. 4, 044074 (2016) [arXiv:1512.05584 [hep-th]].

[186] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50**, 846 (1994)